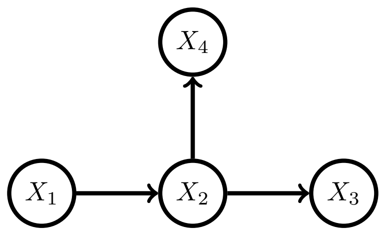
PGM Quizz – Solutions

February 2019



1. If the distribution of $X = (X_1, X_2, X_3, X_4)$ factorizes according to the graph above, which of the following statements are true :

Notice that the given graph is a directed tree. Hence, it is equivalent to an undirected tree of the same shape, or any change of direction in the edges that does not introduce a v-structure.

a) $p(x_1, x_2, x_3, x_4) = p(x_1 \mid x_2)\, p(x_2 \mid x_3, x_4)\, p(x_3)\, p(x_4)$

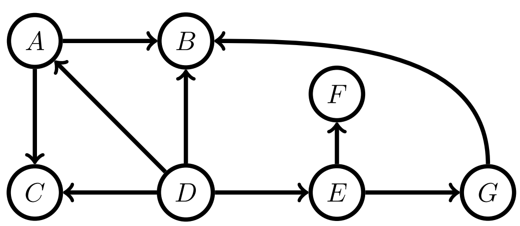
False : The decomposition has a v-structure in X_2.

b) $p(x_1, x_2, x_3, x_4) = p(x_1)\, p(x_2 \mid x_1)\, p(x_3, x_4 \mid x_2)$

True : $p(x_3, x_4|x_2)$ is more general than $p(x_3|x_2)p(x_4|x_2)$. Remark : A probability distribution written in this form is **not guaranteed** to factorize under the graph above.

c) $p(x_1, x_2, x_3, x_4) = p(x_2)\, p(x_1 \mid x_2)\, p(x_3 \mid x_2)\, p(x_4 \mid x_2)$

True : $p(x_2)p(x_1|x_2) = p(x_1, x_2) = p(x_2|x_1)p(x_1)$.



2. Which ones of the following orderings is a correct topological ordering for the above graph :

A topological order σ satisfies :

$$\forall (u, v) \in V^2 \ \big(u \in \text{ancestors}(v) \implies \sigma(u) < \sigma(v)\big). \qquad (A)$$

This property is equivalent to the following :

$$\forall v \in V, \big(u \in \text{parents}(v) \implies \sigma(u) < \sigma(v)\big). \qquad (B)$$

(A) implies (B) since by definition parents$(u) \subset$ ancestors(v).

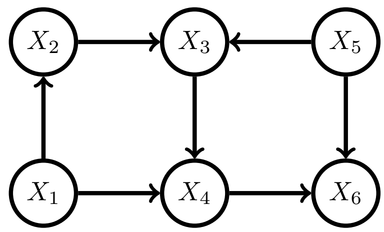
Moreover, (B) implies (A) by noticing that $v_k \in$ ancestors(v_0) implies the existence of a path $v_0 \to v_1 \to \ldots \to v_{k-1} \to v_k$, satisfying $\forall i$ such that $0 \le i < k,\ v_{i+1} \in$ parents(v_i). We can then apply our property (B) along this path, giving :

$$\sigma(v_0) > \sigma(v_1) > \ldots > \sigma(v_{k-1}) > \sigma(v_k).$$

This implies $\sigma(v_0) > \sigma(v_k)$ by transitivity.

This property makes it easy to check if an ordering is topological. Each node must appear in the order before its parents.

a) D, A, E, C, G, F, B
True
b) D, E, F, G, A, C, B
True
c) D, E, A, C, B, F, G
False : B appears before G in the ordering, yet G is a parent of B.



3. If the distribution of X factorizes according to the above graph, which ones of the following propositions are true :

a) X_1 and X_3 are conditionally independent given X_2
True : Observing X_2 blocks the path $X_1 \to X_2 \to X_3$. The two v-structures $X_1 \to X_4 \leftarrow X_3$ and $x_4 \to X_6 \leftarrow X_5$ are unobserved, hence, blocking.
b) X_2 and X_5 are conditionally independent given X_4 and X_1
False : X_4 is a child of X_3, so we can go through the V-structure in X_3 from X_2 to X_5.
c) X_1 and X_5 are conditionally independent given X_2 and X_4
False : Observing X_4 allows to go through the v-structure $X_1 \to X_4 \leftarrow X_3$ and then to X_5.

4. When running the sum-product algorithm on an graphical model which is an undirected tree with 5 nodes, how many messages must be sent to perform marginal inference ?
A tree on n nodes necessarily has $n-1$ edges. Two messages are sent on every edge. $2.(5-1) = 8$.
a) 8
True
b) 10
False
c) This depends on the shape of the tree.
False, the number of edges doesn't depend on the shape of the tree.

5. Which of the following statements are true ?
a) If X satisfies the Global Markov property with respect to an undirected graph $G = (V, E)$ and if A, B and S are three subsets of V, then $X_A \perp\!\!\!\perp X_B$ implies $X_A \perp\!\!\!\perp X_B \mid X_S$.
True. In an undirected graph, $X_A \perp\!\!\!\perp X_B$ implies that A and B are in different connected components. Conditioning on any subset S will only remove nodes, which cannot add connections between the connected components of A and B.
b) If X has a distribution that factorizes with respect to the directed graph $G' = (V, E')$, and if A, B and S are three subsets of V, then $X_A \perp\!\!\!\perp X_B$ implies $X_A \perp\!\!\!\perp X_B \mid X_S$.
False. For a V-structure $X_1 \to X_2 \leftarrow X_3$, we have that $X_1 \perp\!\!\!\perp X_3$, yet we know that $X_1 \perp\!\!\!\perp X_3 | X_2$ is not true.

6. Prove mathematically that if $X_1 \perp\!\!\!\perp (X_2, X_3) \mid X_4$, then $X_1 \perp\!\!\!\perp X_2 \mid X_4$.
We have $p(x_1, x_2, x_3 | x_4) = p(x_1|x_4)p(x_2, x_3|x_4)$.

$$p(x_1, x_2|x_4) = \sum_{x_3} p(x_1, x_2, x_3|x_4) = \sum_{x_3} p(x_1|x_4)p(x_2, x_3|x_4) = p(x_1|x_4)p(x_2|x_4).$$

7. Draw a directed graph G on three nodes, with at most 2 directed edges, so that the following distribution factorizes according to G.

$$p(x_1, x_2, x_3) = \frac{p(x_2 \mid x_1)\, p(x_2 \mid x_3)\, p(x_1)\, p(x_3)}{p(x_2)} \tag{1}$$

$$= p(x_1|x_2)p(x_2|x_3)p(x_3) \quad \text{by Bayes rule.} \tag{2}$$

This justifies the graph : $X_1 \to X_2 \to X_3$.