Some basic definition and concepts from Graph Theory used in Graphical Model Theory

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- V is a finite set of nodes
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Edges in undirected graphs are undirected

Directed edges are pairs of nodes $\{u, v\} \in E \subset \binom{V}{2}$

Clearly $(u, v) \neq (v, u)$ and $\{u, v\} = \{v, u\}$.

Remarks on graph definitions

Remark In this course we will only consider graphs with no self-edges ($\{v, v\}$ or (v, v))

Outline

Concepts in undirected graphs: Neighbors and cliques

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Clique

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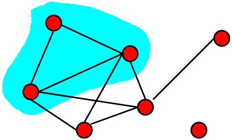

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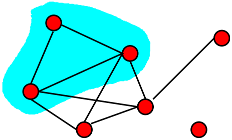
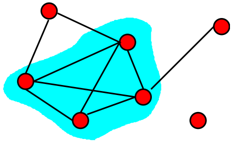


Maximal clique

A clique that is not contained in a larger clique.



Concepts in undirected graphs: Paths and cycles

Path

A sequence of distinct nodes $(v_0, v_1, \ldots, v_k)$ s.t. $\forall i$, $\{v_{i-1}, v_i\} \in E$.

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Cycle

A sequence of nodes $(v_0, v_1, \ldots, v_{k-1}, v_0)$ s.t. $(v_0, v_1, \ldots, v_{k-1})$ is a path and $\{v_{k-1}, v_0\} \in E$.

Concepts in undirected graphs: Connectedness

The relation $a \sim_G b$ defined by "there exists a path between a and b" is an *equivalence relation*[1].

[1] A binary relation which is reflexive, symmetric and transitive.

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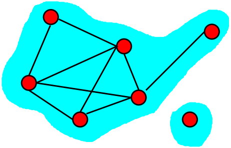

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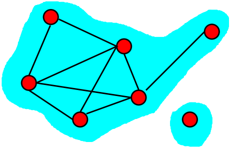

Connected graph

A graph is connected iff it has a single connected component.

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Other concepts in undirected graph

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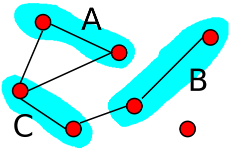
Separation

Let A, B, S three disjoint subsets of V.
S separates A from B iff

- all paths from $a \in A$ to $b \in B$ go through S
- equivalently: any connected component K the graph induced on $V \setminus S$ is such that either $K \cap A = \varnothing$ or $K \cap B = \varnothing$



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Directed Acyclic Graph (DAG)

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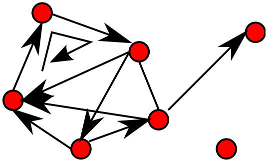
Counterexample:



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$$\forall i, \quad \text{either} \quad (v_{i-1}, v_i) \in E \quad \text{or} \quad (v_i, v_{i-1}) \in E.$$

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Descendant

v is a (strict) descendant of u iff u is a (strict) ancestor of v.

Topological order for a DAG

A topological order is a *total order* $\prec$ on V compatible with the partial order $\prec_G$ in the sense that

$$u \prec_G v \;\Rightarrow\; u \prec v$$

In other words, if $v_1, v_2, \ldots, v_n$ are in topological order the ancestors of v_i are among $(v_j)_{j \leq i}$.

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Proposition

A topological order always exists.

Proof: Induction by removing a maximal element.

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Forest vs trees, etc

- In graph theory, a forest is a disjoint union of trees (both in the directed and undirected case)
- In this course, we will often not make the distinction and use the word tree even for a graph which is a forest, because the same theory applies to both.
- More generally, when a graph has several connected components, we will be able to treat one component at a time for all relevant task of graphical model theory.