Review exercises on probabilistic graphical models
AMMI 2018-2019

These exercises are not meant to provide an exhaustive coverage of the material to review for the final exam. Also, all these exercises should not be taken as representative of the difficulty of the questions posed at the exam, although several questions of the exam are likely to have a similar level.

Factorization and exponential families

Let $G = (V, E)$ be a directed graph, Π_j the set of parents of node j in G and

$$p(x) = \prod_{j=1}^{d} p(x_j \mid x_{\Pi_j})$$

a probability distribution that factorizes in that graph on a discrete random variable $X = (X_1, X_2, \ldots, X_d)$. Assume that X_j takes values in $\{1, \ldots, K_j\}$. Write the distribution in exponential form.

Marginalization

Let $G = (V, E)$ be an undirected graph, with $|V| = d$. Let $p(x_V)$ be a distribution that factorizes according to that graph G. Consider the distribution $p(x_{V\backslash\{i\}}) = \sum_{x_i} p(x_V)$ on the $d-1$ nodes in $V\backslash\{i\}$.

- Assume that i had only a single neighbor. What is the undirected graph with nodes $V' = V\backslash\{i\}$ that has the smallest possible number of nodes and such that $p(x_{V\backslash\{i\}})$ factorizes according to that graph?
- Answer the same question but for a general node i.

Directed vs undirected graph

Consider the undirected graph $G = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \big\{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\big\}$. Let $\mathcal{L}(G)$ be the set of distributions that factorize according to G

1. Draw the graph
2. Prove that there exists no directed graph G' on 4 nodes such that $\mathcal{L}(G) = \mathcal{L}(G')$

Gaussian Markov Chain

The following exercise does not require to do any tedious calculations. If you get into complicated calculations, it means you have the wrong approach...

Assume that the ε_i are i.i.d. with $\varepsilon_i \sim \mathcal{N}(0, 1)$ and let $X_1 = \varepsilon_1,\ X_2 = \rho X_1 + \varepsilon_2,\ X_3 = \rho X_2 + \varepsilon_3$.

(a) What is the precision matrix of the joint distribution of (X_1, X_2, X_3)?

(b) Compute $\mathbb{E}[X_2 \mid X_1, X_3]$ and $\mathrm{Var}(X_2 \mid X_1, X_3)$.