M311 Calculus III Recitation

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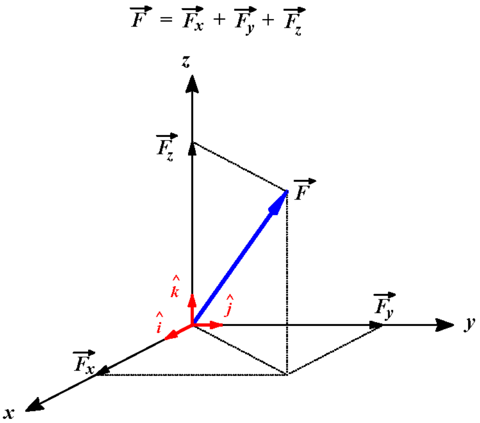

$\vec{F} = \vec{F_x} + \vec{F_y} + \vec{F_z}$
z
$\vec{F_z}$
$\vec{F}$
$\hat{k}$
$\hat{j}$
$\hat{i}$
$\vec{F_y}$
y
$\vec{F_x}$
x

Suppose $a = \langle 2, 3, 0\rangle$ and $b = \langle -1, 0, 1\rangle$. Then

$$\begin{aligned} |a + 2b| &= |\langle 2, 3, 0\rangle + 2\langle -1, 0, 1\rangle| \\ &= |\langle 2, 3, 0\rangle + \langle -2, 0, 2\rangle| \\ &= |\langle 0, 3, 2\rangle| \\ &= \sqrt{0^2 + 3^2 + 2^2} \\ &= \sqrt{13} \end{aligned}$$

Dot products are a special case of matrix multiplication. Suppose $a = \langle 2, 3, 0\rangle$ and $b = \langle -1, 0, 1\rangle$. Then

$$a \cdot b = 2(-1) + 3(0) + 0(1) = -2$$

Compare to $(2, 3, 0)(-1, 0, 1)^T$ where these are now thought of as matrices and T denotes transpose.

- Dot product gives a computationally conducive way to get a handle on angles between vectors.
- $a \cdot b = |a||b| \cos\theta$ where θ is the angle between the two vectors.
- In particular, since $\cos\theta = 0$ if and only if $\theta = \pi/2 + k\pi$, we can conclude that $a \perp b \Leftrightarrow a \cdot b = 0$.
- Note: dot product takes two vectors and gives a number.
- In the previous example, since the dot product was nonzero, we can conclusively say that the vectors were not orthogonal.

Computing unit vectors perpendicular to $j - k$ and $i + j$:

$$(j - k) \times (i + j) = i - j - k$$

However, this vector is not a unit vector! We must normalize by dividing by the magnitude, in this case, $1/\sqrt{3}$.

Finding the area of a parallelogram given the corner points.

1. Plot the vectors and draw in the parallelogram
2. Identify two vectors that determine the parallelogram
3. Compute the magnitude of the cross product.

Finding a vector orthogonal to the plane given three points on the plane.

1. The plane through P, Q, R must contain the vectors formed by the points, such as PQ and PR.
2. Find a vector orthogonal to both of these vectors with the corss product.
3. This vector must then be orthogonal to the plane itself.

Find equation of line containing $P_0 = (2, 1, 0)$ and direction perpendicular to $i + j$ and $j + k$.

1. Compute a vector orthogonal to both
2. Say the vector is (a, b, c). Then the parametric equation is $x = 2 + at, y = 1 + bt, z = ct$.

Line segment from $r_0 = \langle 6, -1, 9\rangle$ to $r_1 = \langle 7, 6, 0\rangle$:

$$\begin{aligned} r(t) &= (1-t)r_0 + tr_1 \\ &= (1-t)\langle 6, -1, 9\rangle + t\langle 7, 6, 0\rangle \\ &= \langle 6, -1, 9\rangle - t\langle 6, -1, 9\rangle + t\langle 7, 6, 0\rangle \\ &= \langle 6, -1, 9\rangle + t\langle 1, 7, -9\rangle \end{aligned}$$

How do you determine whether two lines are parallel given their direction vectors?

1. Say we have two lines with direction vectors v_1 and v_2.
2. Determine whether there exists a constant such that $v_1 = cv_2$.
3. If there is, then the two lines are parallel. If not, it may be possible that the lines intersect or are skew.

- The equation of a plane depends heavily on the normal vector.
- Main goal: find the normal vector
- Our strategy to doing so involves taking two cross products of vectors.
- Therefore, in all problems, get such vectors, either on the plane, parallel to the plane, etc.

- Note that both problems have one coordinate equal to $\cos t$ and another coordinate has $\sin t$.
- Use $\sin^2 t + \cos^2 t = 1$
- Incorporate the third coordinate

- Statistics on Exam 1
- The cumulative grades on Canvas are not correct.
- As a result, if you want to know your current grade in the class, you'll need to compute it.

- Mass to charge ratio is a quantity in electrodynamics with wide applications
- Upshot: Ions' motions are completely determined by the ratio and the electromagnetic forces
- $F = q(E + v \times B)$ the force on an ion in electromagnetic fields
- Combined with Newton's Law $F = ma$ gives $(m/q)a = E + v \times B$
- q is the charge of the ion, v the velocity, E, B are the electric and magnetic fields resp.

Suppose that $f : \mathbf{R}^n \to \mathbf{R}$. For example, say f is a function of two variables x, y. Then we can talk about the (partial) derivatives wrt x or y.

- Partial derivative wrt x is denoted as f_x
- It is defined as the change in f when all other variables, which in this case is just y, is treated as a constant.
- To compute it, we should think of all other variables as a constant then take the derivative as we would in single variable calculus.

Given $f(x, y) = \sqrt{3x + 4y}$, compute all first order partials.
Solution:

$$f_x = (3x+4y)_x^{1/2} = \frac{1}{2}(3x+4y)^{-1/2} \cdot (3x+4y)_x = \frac{3}{2}(3x+4y)^{-1/2}$$

All the variables are being square rooted, so we first apply the Chain Rule, which then focuses the derivative on the variables themselves. Remember, in this case, y is treated as a constant, so $4y$ is also a constant and therefore, zeros out.

Compute first order partials of $f(x,y) = x\sin(xy)$

$$f_y = x(\sin(xy))_y = x\cos(xy)\cdot(xy)_y = x^2\cos(xy)$$

In this case x is treated as a constant, so we focus on the sin term and Chain Rule as before.

$$f_x = xy\cos(xy) + \sin(xy)$$

Now, x is a variable so it appears in both the first factor and also the sin term! This means we must use the Product Rule.

- As in single variable calculus, we will have reason to take second and higher order partials (actually in this class, usually just second).
- The idea is the same as in one variable: we take one derivative at a time and keep track of each derivative.
- The twist is that there are multiple variables.
- Notation: f_{xy} means take the derivative wrt x first and then take derivative wrt y.
- Most of the time, $f_{xy} = f_{yx}$.
- Start memorizing this formula now: $f_{xx}f_{yy} - f_{xy}^2$

Compute second mixed partials of

$$f(x,y) = \log(x+2y)$$

We need both first order partials: $f_x = (x+2y)^{-1}$ and $f_y = 2(x+2y)^{-1}$. Now,

$$f_{xy} = \left((x+2y)^{-1}\right)_y = -\frac{(x+2y)_y}{(x+2y)^2} = -\frac{2}{(x+2y)^2}$$

and

$$f_{yx} = \left(2(x+2y)^{-1}\right)_x = -\frac{2(x+2y)_x}{(x+2y)^2} = -\frac{2}{(x+2y)^2}$$

- Derivative of one variable function gives numbers corresponding to slopes of tangent lines.
- Gradient generalizes this idea.
- ∇f is a vector whose components are the partial derivatives
- It gives the direction and magnitude of the fastest increase of the function.
- Note: the gradient is the one-dimensional (but still vector) version of the Jacobian.
- Gradient satisfies all the properties of a derivative: linearity, product and chain rule

- Equivalent definition of gradient: given any vector v, it is the unique vector field such that $D_v f = \nabla f \cdot v$
- $f(x) = f(a) + \nabla f(a)(x-a) + O(x^2)$
- The gradient is orthogonal to the level sets.
- Appears in formulations of Maxwell's equations (Laws of Electrodynamics)

Vector field is an assignment of a vector to each point in space. Contrast with functions we've been studying, which could be considered scalar fields, where for each point in space, we assign a scalar instead of a vector.
Examples:

- A weather chart displaying the strength and direction of winds superimposed on a map.
- Maxwell's equations tell us the force as a vector experienced by an ion at any point. This gives a vector field which is the electromagnetic field.
- Gravitational fields caused by celestial objects: each point gives the gravitational force as a vector. A spherical object would have all vectors pointing towards center.

- Can construct vector fields from our functions using ∇f
- This would then be a vector field where each point gives the tangent vector to f.
- Such vector fields are extremely important and called "conservative".
- The work done in a conservative force field depends only on the start and end points.
- Conservative fields are irrotational (zero curl).

Draw the gradient field of $f(x,y) = x^2 + y^2$ superimposed on its contour graph.

- Recall that gradient points in direction of maximal increase.
- We can use this idea to iteratively find local min/max.
- Why use iterative methods at all? Why not set derivatives equal to zero and find all critical points right away!
- This can be inefficient or not computationally feasible in the presence of large datasets
- We may not always have explicit equations for our optimization problem but can computationally and locally get the gradients.

- In one dimension (one integral), we have length. We can multiply this length by number(s) to get area, but this is effectively increasing the dimension by adding a second integral.
- Two integrals give area. Three gives volume, etc.
- Similarly, we can compute volume with two integrals with an integrand other than 1.
- Area can be thought of as a special case of volume.
- For example, a square has area x^2 while a rectangular box with a square base has volume x^2h. However, if you let the latter have height 1, which incidentally, is the integrand in these computations, then you recover x^2.

- The hardest part about triple integrals is determining the limits of the integrals.
- For this discussion, we stick with Cartesian coordinates.
- In which order should we integrate?
- If you pick z first, can you easily parameterize the other variables in terms of z?
- For example, if the shape is sitting on the xy- plane and its height is determined completely by $z = f(x, y)$ then great! Notice I've basically cheated here.
- A negative example could be a pyramid sitting on the xy-plane: its height could be bounded by multiple planes so trying to parameterize in terms of z would require you to partition.

- At this step, we've determined which variable goes inside our innermost integral, say z.
- For the next two variables, we project onto the appropriate plane, in this case, xy-plane.
- At this point, it becomes a double integral problem.
- Set up the double integral as you would have last week.
- Wrap this double integral outside the innermost integral from the first step.

- When performing a change of variables, there are several components to consider.
- First, you should obviously decide what your change of variables will be.
- Afterwards, you must determine what the new limits are.
- Because we have multiple integrals, we cannot take antiderivatives and undo the substitution at the end.
- Remember when we did u-subs in single variable, we needed a du.
- The analogue of that in multiple variables is the Jacobian.

$$\iint dxdy = \iint r \;\, drd\theta$$

We convert to polar coordinates from Cartesian using $x = r\cos\theta$ and $y = r\sin\theta$. Then, $dxdy = |J| \;\, drd\theta$ where

$$J = \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and so

$$\det J = r\cos^2\theta + r\sin^2\theta = r$$