

# M311 Calculus III Recitation

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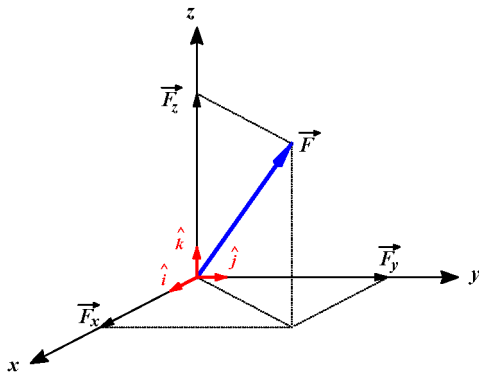
Indiana University

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# Projections

## Section 12.1

$$\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$$



# Vector Operations

## Section 12.2

Suppose  $a = \langle 2, 3, 0 \rangle$  and  $b = \langle -1, 0, 1 \rangle$ . Then

$$\begin{aligned}|a + 2b| &= |\langle 2, 3, 0 \rangle + 2\langle -1, 0, 1 \rangle| \\&= |\langle 2, 3, 0 \rangle + \langle -2, 0, 2 \rangle| \\&= |\langle 0, 3, 2 \rangle| \\&= \sqrt{0^2 + 3^2 + 2^2} \\&= \sqrt{13}\end{aligned}$$

# Dot Product

## Section 12.3

Dot products are a special case of matrix multiplication.

Suppose  $a = \langle 2, 3, 0 \rangle$  and  $b = \langle -1, 0, 1 \rangle$ . Then

$$a \cdot b = 2(-1) + 3(0) + 0(1) = -2$$

Compare to  $(2, 3, 0)(-1, 0, 1)^T$  where these are now thought of as matrices and  $T$  denotes transpose.

# Dot Product

## Important properties and uses

- Dot product gives a computationally conducive way to get a handle on angles between vectors.
- $a \cdot b = |a||b| \cos \theta$  where  $\theta$  is the angle between the two vectors.
- In particular, since  $\cos \theta = 0$  if and only if  $\theta = \pi/2 + k\pi$ , we can conclude that  $a \perp b \Leftrightarrow a \cdot b = 0$ .
- Note: dot product takes two vectors and gives a number.
- In the previous example, since the dot product was nonzero, we can conclusively say that the vectors were not orthogonal.

# Cross Product

## Problem 20

Computing unit vectors perpendicular to  $j - k$  and  $i + j$ :

$$(j - k) \times (i + j) = i - j - k$$

However, this vector is not a unit vector! We must normalize by dividing by the magnitude, in this case,  $1/\sqrt{3}$ .

# Cross Product

## Problem 27

Finding the area of a parallelogram given the corner points.

- 1 Plot the vectors and draw in the parallelogram
- 2 Identify two vectors that determine the parallelogram
- 3 Compute the magnitude of the cross product.

# Cross Product

## Problem 29

Finding a vector orthogonal to the plane given three points on the plane.

- 1 The plane through  $P, Q, R$  must contain the vectors formed by the points, such as  $PQ$  and  $PR$ .
- 2 Find a vector orthogonal to both of these vectors with the cross product.
- 3 This vector must then be orthogonal to the plane itself.



# Lines and Planes

## Problem 10

Find equation of line containing  $P_0 = (2, 1, 0)$  and direction perpendicular to  $i + j$  and  $j + k$ .

- ① Compute a vector orthogonal to both
- ② Say the vector is  $(a, b, c)$ . Then the parametric equation is  $x = 2 + at, y = 1 + bt, z = ct$ .

# Lines and Planes

## Problem 17

Line segment from  $r_0 = \langle 6, -1, 9 \rangle$  to  $r_1 = \langle 7, 6, 0 \rangle$ :

$$\begin{aligned} r(t) &= (1 - t)r_0 + tr_1 \\ &= (1 - t)\langle 6, -1, 9 \rangle + t\langle 7, 6, 0 \rangle \\ &= \langle 6, -1, 9 \rangle - t\langle 6, -1, 9 \rangle + t\langle 7, 6, 0 \rangle \\ &= \langle 6, -1, 9 \rangle + t\langle 1, 7, -9 \rangle \end{aligned}$$

# Lines and Planes

## Parallel (Problem 20)

How do you determine whether two lines are parallel given their direction vectors?

- 1 Say we have two lines with direction vectors  $v_1$  and  $v_2$ .
- 2 Determine whether there exists a constant such that  $v_1 = cv_2$ .
- 3 If there is, then the two lines are parallel. If not, it may be possible that the lines intersect or are skew.

# Lines and Planes

## Finding equations of planes

- The equation of a plane depends heavily on the normal vector.
- Main goal: find the normal vector
- Our strategy to doing so involves taking two cross products of vectors.
- Therefore, in all problems, get such vectors, either on the plane, parallel to the plane, etc.

# Vector Functions

Problems 12, 21, and 27

- Note that both problems have one coordinate equal to  $\cos t$  and another coordinate has  $\sin t$ .
- Use  $\sin^2 t + \cos^2 t = 1$
- Incorporate the third coordinate

# Exam 1

## Remarks

- Statistics on Exam 1
- The cumulative grades on Canvas are not correct.
- As a result, if you want to know your current grade in the class, you'll need to compute it.

# Applications

## Mass-to-charge ratio

- Mass to charge ratio is a quantity in electrodynamics with wide applications
- Upshot: Ions' motions are completely determined by the ratio and the electromagnetic forces
- $F = q(E + v \times B)$  the force on an ion in electromagnetic fields
- Combined with Newton's Law  $F = ma$  gives  $(m/q)a = E + v \times B$
- $q$  is the charge of the ion,  $v$  the velocity,  $E, B$  are the electric and magnetic fields resp.

# Partial Derivatives

## Notation and setup

Suppose that  $f : \mathbf{R}^n \rightarrow \mathbf{R}$ . For example, say  $f$  is a function of two variables  $x, y$ . Then we can talk about the (partial) derivatives wrt  $x$  or  $y$ .

- Partial derivative wrt  $x$  is denoted as  $f_x$
- It is defined as the change in  $f$  when all other variables, which in this case is just  $y$ , is treated as a constant.
- To compute it, we should think of all other variables as a constant then take the derivative as we would in single variable calculus.



# Partial Derivatives

Preliminary example: 14.3.18

Given  $f(x, y) = \sqrt{3x + 4y}$ , compute all first order partials.

Solution:

$$f_x = (3x + 4y)_x^{1/2} = \frac{1}{2}(3x + 4y)^{-1/2} \cdot (3x + 4y)_x = \frac{3}{2}(3x + 4y)^{-1/2}$$

All the variables are being square rooted, so we first apply the Chain Rule, which then focuses the derivative on the variables themselves. Remember, in this case,  $y$  is treated as a constant, so  $4y$  is also a constant and therefore, zeros out.

Compute first order partials of  $f(x, y) = x \sin(xy)$

$$f_y = x(\sin(xy))_y = x \cos(xy) \cdot (xy)_y = x^2 \cos(xy)$$

In this case  $x$  is treated as a constant, so we focus on the sin term and Chain Rule as before.

$$f_x = xy \cos(xy) + \sin(xy)$$

Now,  $x$  is a variable so it appears in both the first factor and also the sin term! This means we must use the Product Rule.

# Higher Order Derivatives

## Setup

- As in single variable calculus, we will have reason to take second and higher order partials (actually in this class, usually just second).
- The idea is the same as in one variable: we take one derivative at a time and keep track of each derivative.
- The twist is that there are multiple variables.
- Notation:  $f_{xy}$  means take the derivative wrt  $x$  first and then take derivative wrt  $y$ .
- Most of the time,  $f_{xy} = f_{yx}$ .
- Start memorizing this formula now:  $f_{xx}f_{yy} - f_{xy}^2$

Compute second mixed partials of

$$f(x, y) = \log(x + 2y)$$

We need both first order partials:  $f_x = (x + 2y)^{-1}$  and  $f_y = 2(x + 2y)^{-1}$ . Now,

$$f_{xy} = ((x + 2y)^{-1})_y = -\frac{(x + 2y)_y}{(x + 2y)^2} = -\frac{2}{(x + 2y)^2}$$

and

$$f_{yx} = (2(x + 2y)^{-1})_x = -\frac{2(x + 2y)_x}{(x + 2y)^2} = -\frac{2}{(x + 2y)^2}$$

# Gradient

## Definition

- Derivative of one variable function gives numbers corresponding to slopes of tangent lines.
- Gradient generalizes this idea.
- $\nabla f$  is a vector whose components are the partial derivatives
- It gives the direction and magnitude of the fastest increase of the function.
- Note: the gradient is the one-dimensional (but still vector) version of the Jacobian.
- Gradient satisfies all the properties of a derivative: linearity, product and chain rule

# Gradients

## Related concepts

- Equivalent definition of gradient: given any vector  $v$ , it is the unique vector field such that  $D_v f = \nabla f \cdot v$
- $f(x) = f(a) + \nabla f(a)(x - a) + O(x^2)$
- The gradient is orthogonal to the level sets.
- Appears in formulations of Maxwell's equations (Laws of Electrodynamics)