Class 5: Center of Mass

Advanced Placement Physics C

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Fall 2021

Olympiads School

Center of Mass

Finding an object's center of mass is important, because

- The laws of motion are formulated by treating an objects as point masses (for real-life objects, we let the forces apply to the center of mass)
- Objects can have *rotational* motion in addition to *translational* motion as well (we will examine that a bit more in a very-important topic later)

Start with a Definition

The **center of mass** ("CM") is the *weighted average of the masses in a system*. The "system" may be:

- A collection of individual particles
- A continuous distribution of mass with constant density. In this case, CM is also the geometric center (**centroid**) of the object
- A continuous distribution of mass with varying density
- If the masses are inside of a gravitational field, then the CM is also its **center of gravity** ("CG")

Simple Example

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Answer: the half way point between the two masses!

Slightly More Challenging

- What if one of the masses are increased to $2m$?
- This is still not a terribly difficult problem; you can still *guess* the right answer without knowing the equation for center of mass.

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The answer is still simple. The CM is no longer half way between the two masses, but now $\frac{1}{3}$ the total distance from the larger masses.

Complicating Things Further

If we increase the number of point masses along the *x*-axis, our problem can become much more complicated (although still not devastatingly so)
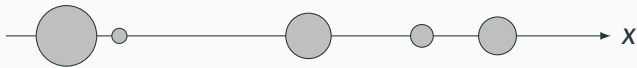
Difficulties really arises when there are many masses in the system in 2D or 3D:
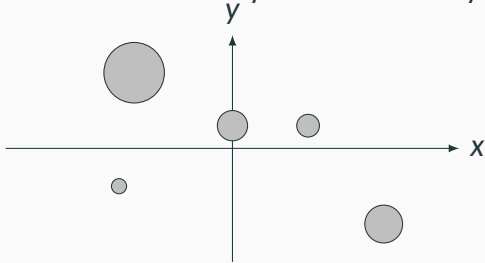
An Equation Helps

The center of mass is defined for discrete number of masses as:

$$\vec{x}_{CM} = \frac{\sum \vec{x}_i m_i}{\sum m_i}$$

Quantity	Symbol	SI Unit
Position of center of mass (vector)	$\vec{x}_{CM}$	m
Position of point mass i (vector)	$\vec{x}_i$	m
Point mass i	m_i	kg

In components:

$$x_{CM} = \frac{\sum x_i m_i}{\sum m_i} \qquad y_{CM} = \frac{\sum y_i m_i}{\sum m_i} \qquad z_{CM} = \frac{\sum z_i m_i}{\sum m_i}$$

An Example

Example 1: Consider the following masses and their coordinates which make up a "discrete mass" rigid body"

$$m_1 = 5.0 \, \text{kg} \qquad\qquad \vec{x}_1 = 3\hat{\imath} - 2\hat{k}$$
$$m_2 = 10.0 \, \text{kg} \qquad\qquad \vec{x}_2 = -4\hat{\imath} + 2\hat{\jmath} + 7\hat{k}$$
$$m_3 = 1.0 \, \text{kg} \qquad\qquad \vec{x}_3 = 10\hat{\imath} - 17\hat{\jmath} + 10\hat{k}$$

What are the coordinates for the center of mass of this system?

Continuous Mass Distribution

In general, objects are not a discrete collection of point masses, but a continuous distribution of mass. Therefore, we take the limit of when the number of masses approaches ∞:

$$\vec{x}_{CM} = \lim_{n \to \infty} \left(\frac{\sum_{i=1}^{n} \vec{x}_i m_i}{\sum_{i=1}^{n} m_i} \right)$$

This gives us an integral form of our equation:

$$\boxed{\vec{x}_{CM} = \frac{\int \vec{x} \, dm}{\int dm}}$$

Densities

- Linear density (for 1D problems)

$$\gamma = \frac{dm}{dL} \quad \rightarrow \quad dm = \gamma dL$$

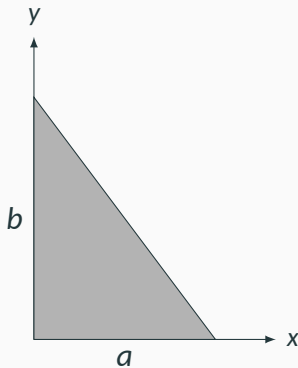- Surface area density (for 2D problems)

$$\sigma = \frac{dm}{dA} \quad \rightarrow \quad dm = \sigma dA$$

- Volume density (for 3D problems)

$$\rho = \frac{dm}{dV} \quad \rightarrow \quad dm = \rho dV$$

Example 2: A triangular plate is placed in a Cartesian coordinate system with two of its edges along the x and y-axis. The length of the edges along the axes are a and b respectively. Assuming that the surface area density σ is uniform, determine the coordinate of its center of mass.

An Example with Integrals

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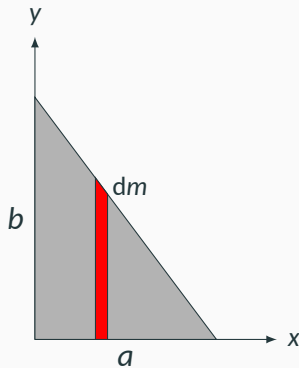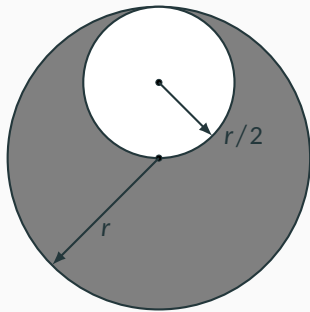
Symmetry

- Any plane of symmetry, mirror line, axis of rotation, point of inversion *must* contain the center of mass.
- Caveat: only works if the density distribution is also symmetric
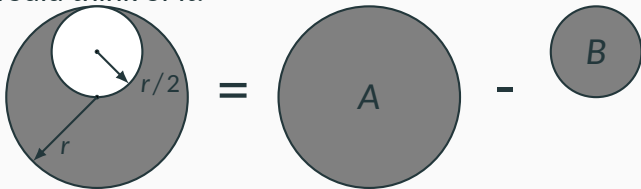- Again: if density is uniform, CM is also geometric center (centroid)

"Negative Mass"

- Where there is a "hole" in the geometry, treat it as having negative mass density $-\sigma$ in that region.
- Negative masses don't exist, so this is really just a trick.
- **Example:** What is the center of mass of this shape?

Negative Mass Example
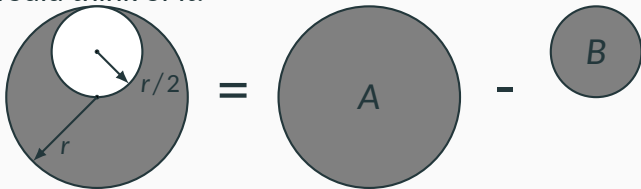
- This is how we would think of it:

- Let the origin of the coordinate system to located at the center of A
- Based on symmetry: $x_{CM} = 0$; only have to find y-coordinate.
- Sum our weighted average:

$$y_{CM} = \frac{\sum y_i m_i}{\sum m_i} = \frac{m_A(0) + m_B(r/2)}{m_A + m_B} = \frac{-\sigma\pi\left(r/2\right)^2\left(r/2\right)}{\sigma\pi r^2 - \sigma\pi\left(r/2\right)^2}$$

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Velocity, Acceleration and Momentum

Take time derivative of the equation for $\vec{x}_{CM}$ to get the velocity at the CM:

$$\vec{v}_{CM} = \frac{d\vec{x}_{CM}}{dt} = \frac{1}{m}\frac{d}{dt}\left(\int \vec{x}dm\right) = \frac{1}{m}\int \frac{d\vec{x}}{dt}dm = \frac{\int \vec{v}dm}{m}$$

The integral in the numerator is the sum of the momentum of all the masses in the system ($\vec{p}_{net}$) which means that we have

$$\vec{p}_{net} = m\vec{v}_{CM}$$

Taking the derivative of $\vec{p}_{net}$ relates force and acceleration at the CM as well:

$$\vec{F}_{net} = \frac{d\vec{p}_{net}}{dt} = m\frac{d\vec{v}_{CM}}{dt} = m\vec{a}_{CM}$$