# Classs 18: Circuit Analysis

## AP Physics C


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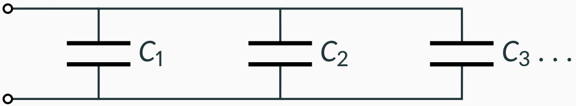
Olympiads School

# Capacitors in Circuit

# Capacitors in Parallel




From the voltage law, we know that the voltage across all the capacitors are the same, i.e. $V_1 = V_2 = V_3 - \cdots = V$. We can express the total charge $Q_{\text{tot}}$ stored across all the capacitors in terms of capacitance and this common voltage $V$:

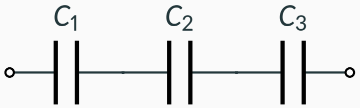
$$Q_{\text{tot}} = Q_1 + Q_2 + Q_3 + \cdots = C_1V + C_2V + C_3V + \cdots$$

Factoring out $V$ from each term gives us the equivalent capacitance:

$$\boxed{C_p = \sum_i C_i}$$

# Capacitors in Series

Likewise, we can do a similar analysis to capacitors connected in series.




The total voltage across these capacitors are the sum of the voltages across each of them, i.e. $V_{tot} = V_1 + V_2 + V_3 + \cdots$

The charge stored on all the capacitors must be the same! The total voltage in terms of capacitance and charge is:

$$V_{tot} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \cdots$$

# Equivalent Capacitance in Series

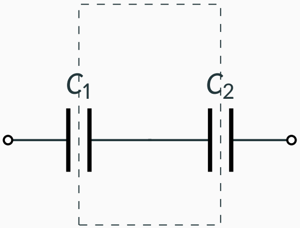
The inverse of the equivalent capacitance for $N$ capacitors connected in series is the sum of the inverses of the individual capacitance.

$$\boxed{\frac{1}{C_s} = \sum_i \frac{1}{C_i}}$$

Make sure we don't confuse ourselves with resistors.

# How Do We Know That Charges Are The Same?

It's simple to show that the charges across all the capacitors are the same

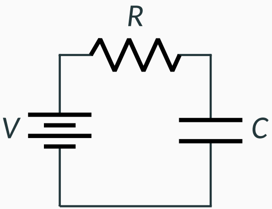



The capacitor plates and the wire connecting them are really one piece of conductor. There is nowhere for the charges to leave the conductor, therefore when charges are accumulating on $C_1$, $C_2$ must also have the same charge because of conservation of charges.

# R-C Circuits

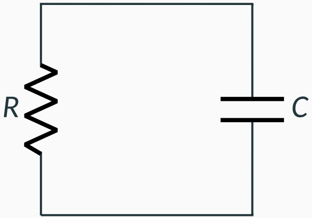
# Circuits with Resistors and Capacitors

An **RC circuit** is one that has both resistors and capacitors. The simplest form is a resistor and capacitor connected in series, and then connect to a voltage source.




Because of the nature of capacitors, the current through the circuit will not be steady as were the case with only resistors.

# Discharging a Capacitor




The analysis starts with something simpler. There is no voltage source, and the capacitor is already charged to $V_c = Q_{\text{tot}}/C$. What happens when the current begin to flow?

As current starts to flow, the charge on the capacitor decreases. Over time the current decreases, until the capacitor is fully discharged, and current stops flowing.

Now we apply the voltage law for the circuit. In this case, as the current flow in the circuit *decreases* the total charge in the capacitor, $I = -\mathrm{d}Q/\mathrm{d}t$, while the voltage across a capacitor is $V_c = Q/C$:

$$V_c - IR = 0 \quad \rightarrow \quad \frac{Q}{C} + R\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

# Discharging a Capacitor

Separating the variable gives the first-order linear differential equation:

$$\frac{\mathrm{d}Q}{Q} = \frac{-\mathrm{d}t}{RC}$$

which we can now integrate and "exponentiate":

$$\int \frac{\mathrm{d}Q}{Q} = \int \frac{-\mathrm{d}t}{RC} \;\rightarrow\; \ln Q = \frac{-t}{RC} + K \;\rightarrow\; Q = e^{K} e^{-t/RC}$$

The constant of integration $K$ is the initial charge on the capacitor $Q_{\text{tot}}$:

$$e^{K} = Q_{\text{tot}}$$

# Discharging a Capacitor

The expression of charge across the capacitor is time-dependent:

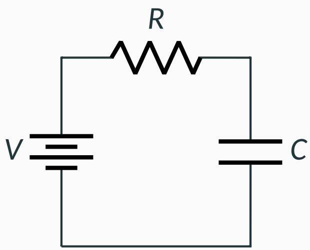
$$\boxed{Q(t) = Q_0 e^{-t/\tau}}$$

where $Q_0 = Q_{\text{tot}}$ is the initial charge on the capacitor, and $\tau = RC$ is called the **time constant**. Taking the time derivative of $Q(t)$ gives us the current through the circuit:

$$\boxed{I(t) = \frac{dQ}{dt} = I_0 e^{-t/\tau}}$$

where the initially current at $t = 0$ is given by $I_0 = Q_{\text{tot}}/\tau$.

# Charging a Capacitor




In charging up the capacitor, we go back to our original circuit, and apply the voltage law, then substitute the expression for current and voltage across the capacitor:

$$V - R\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{Q}{C} = 0$$

Again, separating variables, and integrating, we get:

$$\int \frac{\mathrm{d}Q}{CV - Q} = \int \frac{\mathrm{d}t}{RC} \quad \rightarrow \quad -\ln(CV - Q) = \frac{t}{RC} + K$$

# Charging a Capacitor

"Exponentiating" both sides, we have

$$CV - Q = e^K e^{-t/RC}$$

To find the constant of integration $K$, we note that at $t = 0$, the charge across the capacitor is 0, and we get

$$e^K = CV = Q_{\text{tot}}$$

which is the charge stored in the capacitor at the end. Substitute this back into the equation:

$$\boxed{Q(t) = Q_{\text{tot}}\left(1 - e^{-t/RC}\right)}$$

# Capacitors

$$\boxed{Q(t) = Q_{\text{tot}}(1 - e^{-t/\tau})}$$

Charging a capacitor has the same time constant $\tau = RC$ as during discharge. We can also differentiate to find the current through the circuit; it is identical to the equation for discharge:

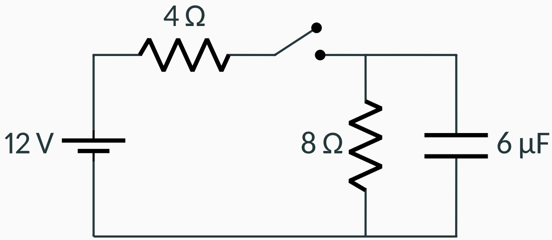
$$\boxed{I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t} = I_0 e^{-t/\tau}}$$

where the initial current is given by $I_o = Q_{\text{tot}}/\tau = V/R$. This makes sense because $V_C(t = 0) = 0$, and all of the energy must be dissipated through the resistor. Similarly at $I(t = \infty) = 0$.

# Two Small Notes

1. When a capacitor is uncharged, there is no voltage across the plate, it acts like a short circuit.
2. When a capacitor is charged, there is a voltage across it, but no current flows *through* it. Effectively it acts like an open circuit.

# A Slightly More Difficult Problem



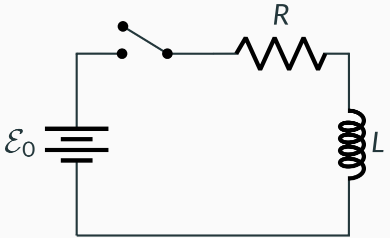

**Example:** The capacitor in the circuit is initially uncharged. Find the current through the battery

1. Immediately after the switch is closed
2. A long time after the switch is closed

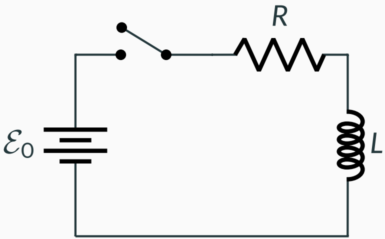
# LR Circuits

# Circuits with Inductors

- Coils and solenoids in circuits are known as "inductors" and have large self inductance $L$
- Self inductance prevents currents rising and falling instantaneously
- A basic circuit containing a resistor and an inductor is called an ***LR* circuit**:



# Analyzing LR Circuits




Applying Kirchhoff's voltage law:

$$\mathcal{E}_0 - IR - L\frac{\mathrm{d}I}{\mathrm{d}t} = 0$$

Following the same procedure as charging a capacitor, the time-dependent current is found to be:

$$I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-t/\tau}\right)$$

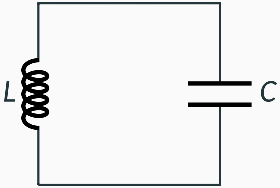
Where the time constant $\tau$ is defined as:

$$\tau = \frac{L}{R}$$

# LC Circuit

# LC Circuit

The final type of circuit in AP Physics is the LC circuit. In its simplest form, the circuit has an inductor and capacitor connected in series:




We apply the Kirchhoff's voltage law:

$$-V_L - V_C = 0 \quad \rightarrow \quad L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{Q}{C} = 0$$

# LC Circuits

Since both terms are continuously differentiable, we can differentiate both sides of the equation, which gives:

$$L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{1}{C}\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

In fact, the above equation a second-order ordinary differential equation with constant coefficients.

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{1}{LC}I = 0$$

The solution to such an equation is the simple harmonic motion.

$$I(t) = I_0 \sin(\omega t + \varphi) \quad \text{where} \quad \omega = \frac{1}{\sqrt{LC}}$$