Class 4: Momentum, Impulse and Collisions

Advanced Placement Physics

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Fall 2021

Olympiads School

Momentum

Momentum

Momentum (or **translational momentum**, or **linear momentum**) is a quantity of motion defined as:

$$\boxed{\vec{p} = m\vec{v}}$$

Quantity	Symbol	SI Unit
Momentum	$\vec{p}$	$\mathrm{kg \cdot m/s}$
Mass	m	kg
Velocity	$\vec{v}$	$\mathrm{m/s}$

For rotational motion of a rigid body, there is also **angular momentum** which will be studied in a later topic.

General Form of Second Law of Motion

Taking the time derivative of the momentum vector from an inertial frame of reference using the chain rule:

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}\vec{v} = m\vec{a} + \dot{m}\vec{v}$$

For constant mass m (i.e. $\dot{m} = 0$), this right-hand-side reduces to the familiar form of the second law of motion, $m\vec{a}$.

First & Second Laws of Motion: General Form

The general form of the first and second law—now accounted for changing mass—is that

I. **The momentum of an object or a system of objects remains constant until a net external force acts on it.**
II. **The net external force acting on an object is the rate of change of its momentum.**

Summarizing this into a single equation:

$$\boxed{\vec{F}_{\text{net}}(t) = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}}$$

Like the "special case" from the dynamics section, this equation is only applicable from an inertial frame of reference.

Impulse

Impulse

Impulse $\vec{J}$ is defined as the time integral of force $\vec{F}$:

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}(t)\mathrm{d}t$$

We can calculate the force generated by

- any of the forces acting on the object, or
- the net force (called the **net impulse**)

over the time interval beteen t_1 and t_2. Since $\vec{F}$, $\vec{p}$ and $\vec{J}$ are all vectors, so the integral can be evaluated in each of the $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ directions, e.g. for the $\hat{\imath}$ direction:

$$J_x = \int_{t_1}^{t_2} F_x \mathrm{d}t$$

Impulse-Momentum Theorem

Rearranging the variables in the general form of the second law of motion:

$$\vec{F}_{\text{net}} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \rightarrow \vec{F}_{\text{net}}\mathrm{d}t = \mathrm{d}\vec{p}$$

Integrating both sides, we get the **impulse-momentum theorem**:

$$\int_{t_1}^{t_2} \vec{F}_{\text{net}}\mathrm{d}t = \int_{p_1}^{p_2} \mathrm{d}\vec{p} \quad \rightarrow \quad \boxed{\vec{J}_{\text{net}} = \Delta\vec{p}}$$

The **net impulse** is equalled to the change in momentum of the object.

Average Force

Average force $\vec{F}_{\text{avg}}$ is the time-averaged force vector that gets the same impulse. It is used extensively in introductory physics courses to avoid integration:

$$\vec{J} = \int_{t_1}^{t_2} \vec{F} \mathrm{d}t = \vec{F}_{\text{avg}} \Delta t$$

Impulse: An Example

Example 1: Jim pushes a box with mass 1.0 kg with a 5.0 N force for 10 s while the box stays on the same place. Find the impulse of the pushing force, friction force, the gravitational force, and the net force.

Rocket Propulsion Problem

Example 2: A rocket generates a thrust force by ejecting hot gases from an engine. If it takes 1 ms to combust 1.0 kg of fuel, ejecting it at a speed of 1000 m / s, what thrust is generated?

A. 1000 N
B. 10 000 N
C. 100 000 N
D. 1 000 000 N

Another Space Example

Example 3: A rocket for mining the asteroid belt is designed like a large scoop. It is approaching asteroids at a velocity of 10^4 m/s. The asteroids are much smaller than the rocket. If the rocket scoops asteroids at a rate of 100 kg/s, what thrust (force) must the rocket's engine provide in order for the rocket to maintain constant velocity? Ignore any variation in the rocket's mass due to the burning fuel.

A. 10^3 N
B. 10^6 N
C. 10^9 N
D. 10^{12} N

Collisions

Conservation of Momentum

- From the third law of motion, we know that the action and reaction forces between two objects are always equal in magnitude and in opposite direction. Thus, their total impulse would be zero.
- When there is no external force, the momentum of the total system will always be constant:

$$\boxed{\sum_i \vec{p}_i = \sum_i \vec{p}_i'}$$

Classifications of Collisions

- Elastic Collision:
    - Total kinetic energy is conserved
    - Momentum is conserved
- Inelastic collision:
    - Kinetic energy is **not** conserved
    - Momentum is conserved
- Completely inelastic collision:
    - "Perfectly inelastic collision"
    - A special case of inelastic collision
    - The objects move together after the collision
    - Kinetic energy is **not** conserved
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Before We Dive Into Some Exercises

The most typical applications of momentum conservation are collision and explosions

- **Collision: A hits B**
  - Regardless of whether they move together or not afterwards, momentum is conserved
  - Head-on collisions are usually 1D
  - Glancing collisions are usually 2D or 3D
- **Explosion: A explodes and becomes B and C (and D and E...)**
  - A perfectly inelastic collision in reverse
  - Total momentum of B and C (and D and E...) is the same as A in the beginning
  - Usually a 2D or 3D problem

Collision Problem

Example 5: Two objects with equal mass are heading toward each other with equal speeds, undergo a head-on collision. Which one of the following statement is correct?

A. Their final velocities are zero
B. Their final velocities may be zero
C. Each must have a final velocity equal to the other's initial velocity
D. Their velocities must be reduced in magnitude

Conservation of Momentum Example

Example 6: Two astronauts, each of mass 75 kg, are floating next to each other in space, outside the space shuttle. One of them pushes the other through a distance of 1.0 m (about an arm's length) with a force of 300 N. What is the final relative velocity of the two?

A. 2.0 m/s
B. 2.83 m/s
C. 4.0 m/s
D. 16.0 m/s

Glancing Collision

Example 7: A billiard ball of mass 0.155 kg (“cue ball”) moves with a velocity of 1.25 m/s toward a stationary billiard ball (“eight ball”) of identical mass and strikes it with a glancing blow. The cue ball moves off at an angle of 29.7° clockwise from its original direction, with a speed of 0.956 m/s. What is the final velocity of the eight ball?

Elastic Collisions

Elastic Collision Problems

In elastic collisions, *both* momentum and kinetic energy is conserved. In a 1D collision, both equations below have to be satisfied:

$$\sum m_i v_i = \sum m_i v_i'$$
$$\sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i v_i'^2$$

How kinetic energy is conserved: In an elastic collision, energy is first converted into a potential energy (e.g. elastic potential energy in a spring), and then all the energy is released back as kinetic energy.

Conservation of Momentum & Energy in Elastic Collisions

For collision of two objects, the conservation of momentum equation can be expressed as:

$$\boxed{m_1(v_1 - v_1') = m_2(v_2' - v_2)} \tag{1}$$

By moving m_1 terms to the left, and m_2 terms to the right. Likewise, the conservation of energy can also be arranged as:

$$\boxed{m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2)} \tag{2}$$

By multiplying every term by 2, and again, moving m_1 terms to the left, and v_2 terms to the right.

Conservation of Momentum & Energy in Elastic Collisions

Dividing the equations (2) by (1) from the last slide, we get:

$$\frac{(2)}{(1)} \quad \rightarrow \quad \frac{m_1(v_1^2 - v_1'^2)}{m_1(v_1 - v_1')} = \frac{m_2(v_2'^2 - v_2^2)}{m_2(v_2' - v_2)}$$

m_1 and m_2 terms cancel out, while the terms in the numerator can be expanded as the difference of two squares which is then simplified:

$$\frac{(v_1 + v_1')(v_1 - v_1')}{(v_1 - v_1')} = \frac{(v_2' + v_2)(v_2' - v_2)}{(v_2' - v_2)}$$

Leading to the final expression, which is substituted back into (1)

$$v_1 + v_1' = v_2 + v_2'$$

Final Velocities in an Elastic Collision

When two objects 1 and 2 of mass m_1 and m_2 and collide elastically, their final velocities will be determined by the initial velocities v_1 and v_2:

$$v_1' = \frac{m_2 - m_1}{m_2 + m_1} v_1 + \frac{2m_2}{m_2 + m_1} v_2$$
$$v_2' = \frac{2m_2}{m_2 + m_1} v_1 + \frac{m_2 - m_1}{m_2 + m_1} v_2$$

These equations are *not* provided in the AP exam equation sheet, which means that we are more interested in the behavior qualitatively rather than quantitatively.

Special Cases

If both objects have equal mass ($m_1 = m_2 = m$) and the second object is initially at rest ($v_2 = 0$), then the equations simplifies to

$$v_1' = \frac{v_1(m - m) + 2mv_2}{m + m} = 0$$

$$v_2' = \frac{v_2(m - m) + 2mv_1}{m + m} = v_1$$

All the momentum and energy from m_1 is transferred to m_2. Object 1 stops all together, while object 2 continues with the initial momentum and velocity of Object 1.

Special Cases

Another special case is when $m_1 \gg m_2$ and $v_2 = 0$ (i.e. a large object colliding with a small stationary object) then we can effectively "ignore" m_2:

$$v_1' = \frac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} \approx \frac{m_1v_1}{m_1} = v_1$$
$$v_2' = \frac{v_2(m_2 - m_1) + 2m_1v_1}{m_1 + m_2} \approx \frac{2m_1v_1}{m_1} = 2v_1$$

Object 1 continues to move like nothing happened, but object 2 is pushed to move at *twice* the initial speed of object 1.

Special Cases

In the reverse case, if $m_1 \ll m_2$, and $v_2 = 0$ (a small object colliding with a large stationary object), then we can "ignore" the m_1 term:

$$v_1' = \frac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} \approx \frac{-m_2v_1}{m_2} = -v_1$$

$$v_2' = \frac{v_2(m_2 - m_1) + 2m_1v_1}{m_1 + m_2} \approx 0$$

Object 1 bounces off object 2, and travels in the opposite direction with the same velocity magnitude, while object 2 does not move.

Elastic Collision Example

Example 8: Blocks A and B have the same mass; A hits B with a speed of $5.0\,\mathrm{m/s}$ while B is initially at rest. If the collision is elastic, what would be the final speed of these two objects?

Elastic Collision Example

Example 9: Blocks A and B with the same mass; A has a velocity 3.0 m / s to the east while B has 2 m / s to the west. If the collision is elastic, after the collision, what would the velocity of the two blocks be?

Elastic Collision Example

Example 10: Throw a ball to a really big wall, when the ball reaches the wall, it has a velocity $10\,\mathrm{m/s}$ toward the wall. If the collision is elastic, what would the final velocity of the ball be?

Elastic Collision Example

Example 11: Throw a ball with a velocity $4.0\,\mathrm{m/s}$ toward a train with a velocity $40\,\mathrm{m/s}$ toward the ball. If the collision is elastic, what would the final velocity of the ball be?

Inelastic Collision: Calculating Energy Loss

Example 12: Two blocks A and B with mass 2.0 kg, block A hits B with velocity 4.0 m / s while B is at rest.

(a) Suppose the collision is completely inelastic, what would the final velocity of A and B be?

(b) What is the loss of energy?