Class 15: Gauss's Law

Advanced Placement Physics C

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Olympiads School

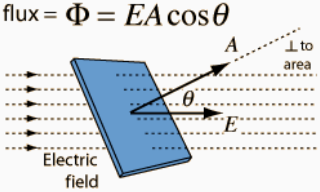
Gauss’s Law

Flux

Flux is an important concept in many disciplines in physics. The flux of a vector quantity $\vec{X}$ is the amount of that quantity flowing through a surface. In integral form:

$$\Phi = \int \vec{X} \cdot d\vec{A} \quad \text{or} \quad \Phi = \int (\vec{X} \cdot \hat{n}) dA$$

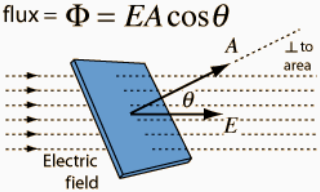
The direction of the infinitesimal area $d\vec{A}$ is **outward normal** to the surface.



Flux

Φ can be something physical, like water, or bananas, or something abstract, like electric field (which is what we are interested in). We can compute a flux as long as there is a vector field i.e. $\vec{X} = \vec{X}(x, y, z)$. In the case of **electric flux**, the quantity $\vec{X}$ is just the electric field, i.e.:

$$\Phi_E = \int \vec{E} \cdot d\vec{A}$$



Electric Flux and Gauss's Law

Gauss's law tells us that if we have a closed surface (think of the surface of a balloon), the total electric flux is very well defined:

$$\boxed{\Phi_E = \oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{\mathrm{encl}}}{\epsilon_0}}$$

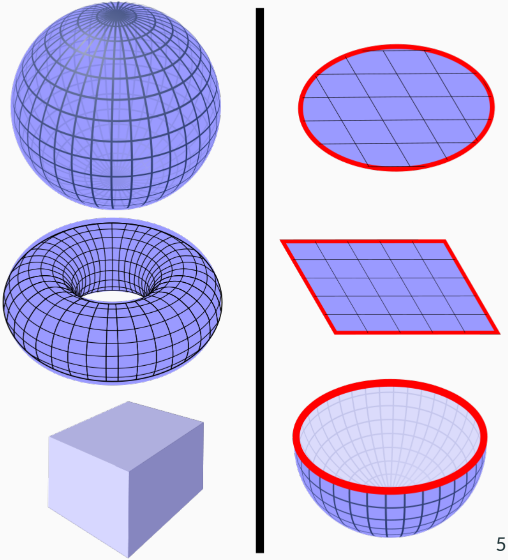
where

- Q_{encl} is the charge enclosed by the surface
- $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$ is the permittivity of free space

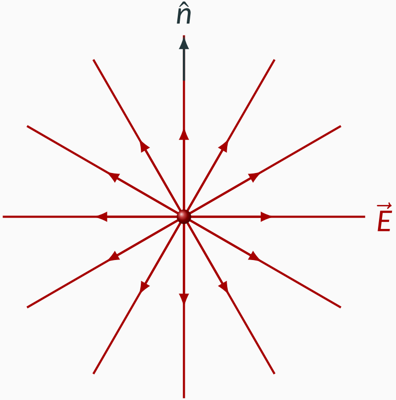
That closed surface is called a **Gaussian surface**

Closed Surfaces

A **closed surface** is one that does not have a boundary, like the sphere, toroid, and cube on the left.



Electric Field from a Positive Point Charge



By symmetry, electric field lines must be radially outward from the charge, so the integral reduces to:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = EA = \frac{q}{\epsilon_0}$$

Since area of a sphere is $A = 4\pi r^2$, we recover Coulomb's law and the magnitude of the electric field from a point charge:

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} = \frac{kq}{r^2}$$

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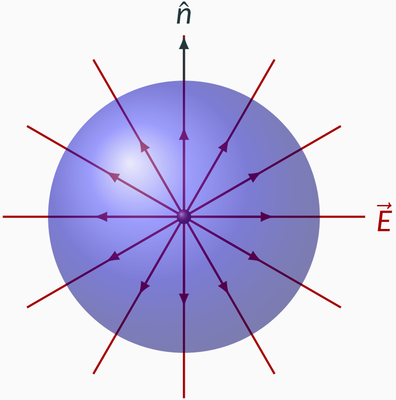

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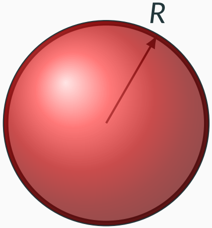
Uniformly-Charged Thin Shell

For a uniformly-charged spherical thin shell with radius *R* and a total charge of *Q*.



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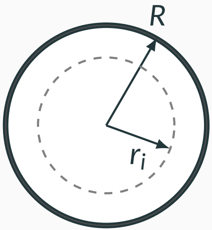

Inside the shell ($r_i < R$), there is no enclosed charge, therefore the electric field must be zero:

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{q_{\text{encl}}}{\epsilon_0} = 0 \quad \rightarrow \quad E = 0$$

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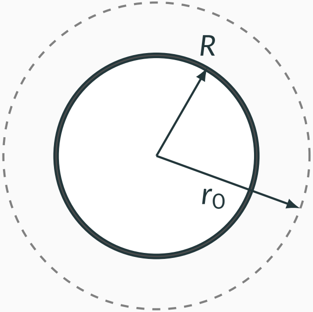

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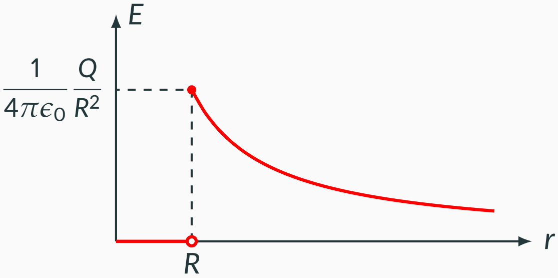
Outside the shell ($r_o > R$), the enclosed charge is Q, and the electric field is given by the same equation as the point charge:

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q}{\epsilon_0} \rightarrow E = \frac{Q}{\epsilon_0 A} = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r^2}\right]$$

Uniformly-Charged Thin Shell

The electric field strength E can be plotted as a function of the distance r from the center of the shell:



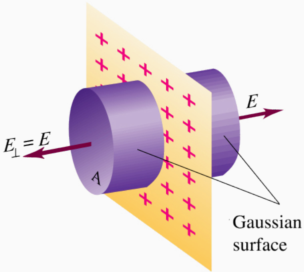
- This graph may be familiar because the graph for the gravitational field strength inside a uniform shell is exactly the same
- Replace ϵ_0 with $4\pi G$

Worksheet Examples

Follow the worksheet for these examples:

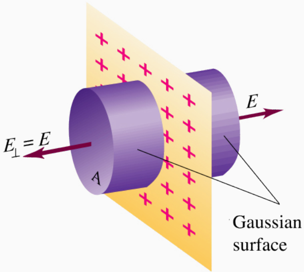
- Uniformly-charged sphere
- Charged sphere with variable charge density
- Infinitely-long charge rod with uniform density

Electric Field Near an Infinite Plane of Charge



- Charge density (charge per unit area) σ
- By symmetry, $\vec{E}$ must be perpendicular to the plane
- Our Gaussian surface is a cylinder shown in the left with an area A; the height of the cylinder is unimportant
- Nothing "flows out" of the side of the cylinder, only at the ends
- The total flux is $\Phi_E = E(2A)$
- The enclosed charge is $Q_{\text{encl}} = \sigma A$

Electric Field Near an Infinite Plane of Charge



Gauss's law simplifies to:

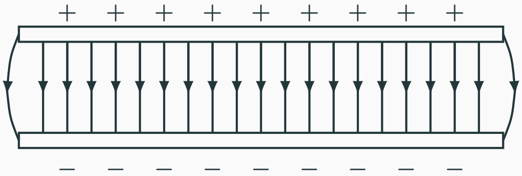
$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{encl}}}{\epsilon_0} \rightarrow E(2A) = \frac{\sigma A}{\epsilon_0}$$

Solving for E, we get:

$$\boxed{E = \frac{\sigma}{2\epsilon_0}}$$

- E is a constant
- Independent of distance from the plane
- Both sides of the plane are the same

Electric Field Between Parallel Charged Plates



- Two plates, each producing an electric field pointing in the same direction
- The total electric field is twice the value of *one* infinite plane, pointing from the positively charged plate toward the negatively charged plate

$$\boxed{E = \frac{\sigma}{\epsilon_0}}$$

- $\vec{E}$ outside the plates is very low (close to zero), except for fringe effects at the edges of the plates

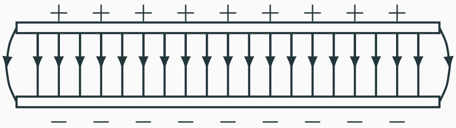
Electric Field and Electric Potential Difference

Recall the relationship between electric field ($\vec{E}$) and electric potential difference (V):

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r}$$

This relationship holds regardless of the charge configuration.

Electric Field and Electric Potential Difference



In the case of two parallel plates, the electric field is uniform, and the relationship simplifies to:

$$\boxed{E = \frac{\Delta V}{d}}$$

Quantity	**Symbol**	**SI Unit**
Electric field intensity	E	N/C
Electric potential difference between plates	ΔV	V
Distance between plates	d	m