Basic Vector and Calculus That You Need to Know

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Both AP Physics C exams (Mechanics and Electricity & Magnetism) are calculus based, and will use vectors extensively. Students should be familiar with the material in this handout, however, it is likely that the calculus and vector operations used in the exams will be much simpler. If these concepts are difficult, this should be a good time to grab a calculus textbook and review them.

1 Vectors

Vectors are used extensively in physics. They are an integral part of a larger discipline within mathematics called **linear algebra**. For the purpose of AP Physics, it is sufficient to think of vectors as "a number with a direction".

1.1 Notation

In keeping with the convention used in *most* technical journals and university-level textbooks, vectors are *printed* (e.g. on the slides and handouts) using a bold face font in this course:

$$\mathbf{v} \quad \mathbf{F}_g \quad \mathbf{p} \quad \mathbf{I}$$

while the "arrow on top" notation is used when *writing* (e.g. on the blackboard)[1]:

$$\vec{v} \quad \vec{F}_g \quad \vec{p} \quad \vec{I}$$

The magnitude of vectors are expressed either with the absolute-value symbol:

$$|\mathbf{v}| \quad |\mathbf{F}_g| \quad |\mathbf{p}| \quad |\mathbf{I}|$$

or as a scalar quantity (afterall, the magnitude of a vector is indeed a scalar with a positive value):

$$v \quad F_g \quad p \quad I$$

[1] Although this format is still used in *some* introductory level physics textbooks in universities

1.2 Writing Vectors

In Grades 11 and 12 Physics, vectors are usually written by separating the magnitude from the direction. For example, a velocity vector are usually written as:

$$\mathbf{v} = 4.5\,\mathrm{m/s}\ [\mathrm{N}\ 55^\circ\ \mathrm{E}]$$

This approach is based on the **polar coordinate system**, which is the preferred coordinate system for circular motion. In general, this approach is very intuitive for describing *one* vector in two dimensions (that's why it is used extensively in high-school level physics courses), but it is more complicated when extended into 3D; the coordinate system needs to be extended to **spherical coordinate system** or the **cylindrical coordinate system**. Moreover, it is difficult to perform vector arithmetic for *rectilinear* motion.

Intead, for rectilinear motion, vectors in 2D/3D Cartesian space are generally written in their x, y & z components using the **IJK notation**:

$$\mathbf{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}$$

The vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ are **basis vectors** indicating the directions of the x, y and z axes. Basis vectors are **unit vectors** (i.e. length 1). Note that the IJK notation does not give the magnitude of the vector, which needs to be calculated:

$$A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

1.3 Vector Addition and Subtraction

Adding and subtracting vectors is straightforward:

$$\mathbf{A} \pm \mathbf{B} = (A_x \pm B_x)\hat{\boldsymbol{\imath}} + (A_y \pm B_y)\hat{\boldsymbol{\jmath}} + (A_z \pm B_z)\hat{\boldsymbol{k}}$$

All that is required is to add or subtract each component of the vector in the $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ directions.

1.4 Dot Product

The vector **dot product** (or **inner product** for general vectors) is the *scalar* multiplication of two vectors. This vector operation had been used throughout Grades 11 and 12 Physics courses (although without explicitly referring using this notation), for example, when calculating mechanical work. It is determined by the magnitude of the two vectors and the cosine of the angle θ between them:

$$C = \mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} = |\mathbf{A}||\mathbf{B}|\cos\theta$$

In the dot product, C is the *projection* of the vector $\mathbf{A}$ onto $\mathbf{B}$, or the component of $\mathbf{A}$ along $\mathbf{B}$. Note that $\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\imath}} = 1$, $\hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{\jmath}} = 1$, and $\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}} = 1$. For general vectors written in IJK notation, where the magnitude and direction of vectors are not immediately known, the dot product is the sum of the products of individual components of $\mathbf{A}$ and $\mathbf{B}$:

$$C = \mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

1.5 Cross Products

The vector **cross product** is the *vector multiplication* of two vectors:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

The magnitude of the cross product is determined by the magnitude of **A** and **B** and the angle θ between them:

$$C = AB \sin\theta$$

The cross product **C** is perpendicular to *both* **A** and **B**; its direction given by the right hand rule. Cross products are used extensively in rotational motion and in electromagnetism. Note that unlike the dot

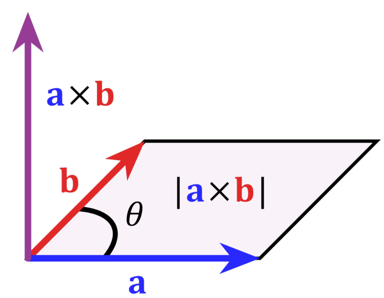

Figure 1: Vector cross product.

product, the order of the cross product is important. (This is why you have to get the right hand rule correctly.)

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

In general, the cross product of any two vectors in 3D space is the determinant of this 3×3 matrix:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_z B_y - B_y A_x)\hat{k}$$

although such notation is extremely rare in any AP exams. Most cross product applications in AP C exams are much simpler, so we only have to remember the circle shown in Figure 2.

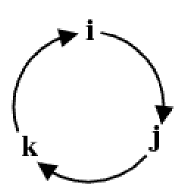

Figure 2: Cross product circle that you will likely see in Physics C exams

The direction of the arrow gives the index of the cross product (e.g. $\hat{\imath} \times \hat{\jmath} = \hat{k}$); going against the direction of the arrow gives the negative of the next index (e.g. $\hat{k} \times \hat{\jmath} = -\hat{\imath}$)

2 Calculus

We cannot learn physics properly without calculus (you got away with it for long enough in grades 11 and 12!). Calculus was a mathematical tool that was "invented" so that we can understand motion, especially non-constant velocities and accelerations. Even in your calculus class(es), you may have already noticed that a lot of the word problems are really physics problems. In this course, both forms of calculus will be used:

- **Differential Calculus**
  - How quickly something is changing ("rate of change" of a quantity)
  - Math: slopes of functions
  - Physics: how quickly a physical quantity is changing in time and/or space
  - Examples: velocity (how quickly position changes with time), acceleration (how quickly velocity changes with time), power (how quickly work is done), electric fields (how electric potential changes in space)
- **Integral Calculus**
  - The opposite of differentiation
  - We use it to compute the area under a curve, or
  - Summation of many small terms
  - Examples: area under the $\mathbf{v}$-t graph (displacement), area under the $\mathbf{F}$-t graph (impulse), area under the $\mathbf{F}$-d graph (work)

2.1 Derivative

For any arbitrary function $f(x)$, the derivative with respect to x is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The "limit as h approaches 0" is the mathematical way of making h a very small non-zero number.

2.2 Basic Rules for Differentiation

The derivative of a constant C with respect to any variable is zero. This should be obvious, since the slope of any function $f(x) = C$ is zero.

$$\frac{dC}{dx} = 0$$

A constant multiple a of any function f can be factored outside the derivative:

$$\frac{d}{dx}(af) = a\frac{df}{dx}$$

The derivative of a sum of two functions is the sum of the derivatives of the functions:

$$\frac{d}{dt}\left(f(t) + g(t)\right) = \frac{df}{dt} + \frac{dg}{dt}$$

Power Rule:

$$\frac{d}{dt}\left(t^n\right) = nt^{n-1} \quad \text{for} \quad n \neq 0$$

Product Rule:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

Chain Rule:

$$\frac{d}{dx}f\left(g(x)\right) = f'(g(x))g'(x)$$

Quotient Rule is rarely used in physics tests in AP or first-year university, but you should remember it anyway:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

2.3 Elementary Derivatives

When studying **harmonic motion** and **circular motion**, trigonometric and exponential functions are often used. We will also find out the relationship between complex exponential functions and sine/cosine functions.

$$\frac{d}{dt}\sin t = \cos t$$
$$\frac{d}{dt}\cos t = -\sin t$$

And the exponential function:

$$\frac{d}{dt}e^{at} = ae^{at}$$

2.4 Partial Derivatives

Some functions have many variables (multi-variable function). For example, gravitational potential energy U_g has three variables: masses m_1 and m_2 and the distance r between them:

$$U_g(m_1, m_2, r) = -\frac{Gm_1m_2}{r}$$

Differentiating with respect to one variable while holding others constant gives its **partial derivative**. (We use the ∂ symbol). For example, the partial derivative of U_g with respect to r is

$$\frac{\partial U_g}{\partial r} = \frac{Gm_1 m_2}{r^2}$$

In case you have not noticed: the derivative is the is the relationship between gravitational potential energy U_g and the magnitude of the gravitational force F_g.

2.5 Integration

If $F(x)$ is the anti-derivative of $f(x)$, they are related this way:

$$\frac{d}{dx}F(x) = f(x) \quad \longrightarrow \quad F(x) = \int f(x)dx$$

The mathematical proof is the **fundamental theorem of calculus**.

2.6 Common Integrals in Physics

Integration, while often necessary, can be very daunting, but integrals in AP Physics C are generally straightforward. These rules should help in most cases.

Power rule in reverse:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

Natural logarithm:

$$\int \frac{1}{x}dx = \ln|x| + C$$

Sines and cosines:

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

2.7 Definite and Indefinite Integrals

Integrals can be either **indefinite** or **definite**. An "indefinite" integral is another function, e.g. position $\mathbf{x}(t)$ as a function of time is found by integrating velocity $\mathbf{v}(t)$:

$$\mathbf{x}(t) = \int \mathbf{v}(t)dt = \cdots + \mathbf{C}$$

A **constant of integration C** is added to the integral $\mathbf{x}(t)$. It is obtained through applying "initial condition" to the problem. On the other hand, a **definite integral** has lower and upper bounds. For example, given $\mathbf{v}(t)$, the displacement between t_1 and t_2 can be found by integrating between these limits:

$$\Delta\mathbf{x} = \int_{t_0}^{t_1} \mathbf{v}(t)dt$$

Once we have computed the integral, we evaluate the limits:

$$\Delta\mathbf{x} = \mathbf{x}(t)\Big|_{t_0}^{t_1} = \mathbf{x}(t_1) - \mathbf{x}(t_0) = \mathbf{x}_1 - \mathbf{x}_0$$

The constant of integration **C** cancels when we evaluate the upper and lower bounds.