Class 5: Center of Mass

Advanced Placement Physics C

Dr. Timothy Leung

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Olympiads School

Center of Mass

Finding an object's center of mass is important, because

- The laws of motion are formulated by treating an objects as point masses (for real-life objects, we let the forces apply to the center of mass)
- Objects can have *rotational* motion in addition to *translational* motion as well (we will examine that a bit more in a very-important topic later)

Start with a Definition

The **center of mass** (“CM”) is the *weighted average of the masses in a system*. The “system” may be:

- A collection of individual particles
- A continuous distribution of mass with constant density. In this case, CM is also the geometric center (**centroid**) of the object
- A continuous distribution of mass with varying density
- If the masses are inside of a gravitational field, then the CM is also its **center of gravity** (“CG”)

Simple Example

We start with a very simple example: there are two equal masses m along the x-axis, located at x_1 and x_1. What is the center of mass of the system?

The answer is simple: the half way point between them:

$$x_{cm} = \frac{x_1 + x_2}{2}$$

Multiply both numerator and denominator by mass m (for generalization later), the equation becomes:

$$x_{cm} = \frac{mx_1 + mx_2}{2m}$$

Slightly More Challenging

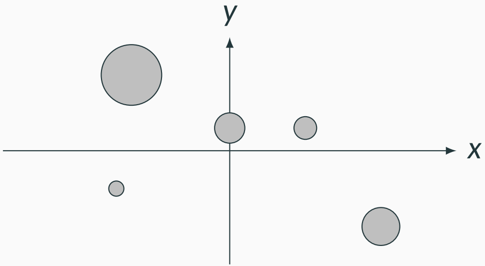
What if one of the masses are increased to $2m$? This is still not a difficult problem; you can still *guess* the right answer without knowing the equation for center of mass.

The answer is still simple. The CM is no longer half way between the two masses, but now $\frac{1}{3}$ the total distance from the larger masses. We can show using a weighted average:

$$x_{cm} = \frac{mx_1 + (2m)x_2}{m + 2m}$$

Complicating Things Further

The weighted average concept can now be applied to cases when there are masses in 2D or 3D:



An Equation Helps

The center of mass is defined for discrete number of masses with the weighted average:

$$\boxed{\vec{x}_{cm} = \frac{\sum \vec{x}_i m_i}{\sum m_i}}$$

Quantity	**Symbol**	**SI Unit**
Position of center of mass (vector)	$\vec{x}_{cm}$	m
Position of point mass *i* (vector)	$\vec{x}_i$	m
Point mass *i*	m_i	kg

In components:

$$x_{cm} = \frac{\sum x_i m_i}{\sum m_i} \qquad y_{cm} = \frac{\sum y_i m_i}{\sum m_i} \qquad z_{cm} = \frac{\sum z_i m_i}{\sum m_i}$$

An Example

Example 1: Consider the following masses and their coordinates which make up a "discrete mass" rigid body"

$$\begin{aligned} m_1 &= 5.0\,\text{kg} & \vec{x}_1 &= 3\hat{\imath} - 2\hat{k} \\ m_2 &= 10.0\,\text{kg} & \vec{x}_2 &= -4\hat{\imath} + 2\hat{\jmath} + 7\hat{k} \\ m_3 &= 1.0\,\text{kg} & \vec{x}_3 &= 10\hat{\imath} - 17\hat{\jmath} + 10\hat{k} \end{aligned}$$

What are the coordinates for the center of mass of this system?

Continuous Mass Distribution

When the number of masses approaches infinity, this becomes a continuous distribution of mass. Taking the limit of masses $N \to \infty$ gives the integral form of our equation:

$$\boxed{\vec{x}_{cm} = \frac{\int \vec{x}\mathrm{d}m}{\int \mathrm{d}m}}$$

What is the infinitesimal mass dm then?

Densities

Linear density (for 1D problems)

$$\gamma = \frac{\mathrm{d}m}{\mathrm{d}L} \quad \rightarrow \quad \mathrm{d}m = \gamma \mathrm{d}L$$

Surface area density (for 2D problems)

$$\sigma = \frac{\mathrm{d}m}{\mathrm{d}A} \quad \rightarrow \quad \mathrm{d}m = \sigma \mathrm{d}A$$

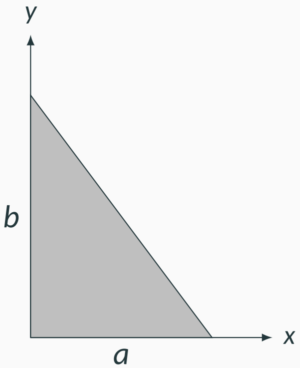
Volume density (for 3D problems)

$$\rho = \frac{\mathrm{d}m}{\mathrm{d}V} \quad \rightarrow \quad \mathrm{d}m = \rho \mathrm{d}V$$

The densities do not have to be constant

An Example with Integrals

Example 2: A triangular plate is placed in a Cartesian coordinate system with two of its edges along the x and y-axis. The length of the edges along the axes are a and b respectively. Assuming that the surface area density σ is uniform, determine the coordinate of its center of mass.



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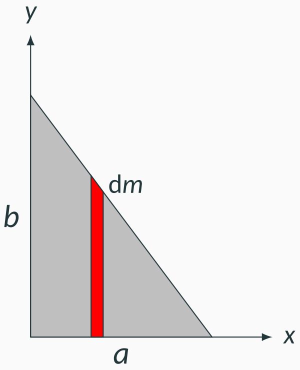
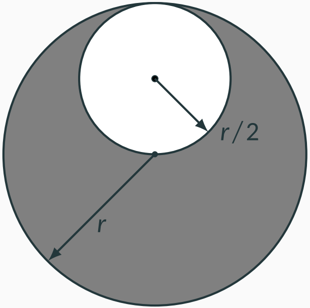


Symmetry

- Any plane of symmetry, mirror line, axis of rotation, point of inversion *must* contain the center of mass.
- Caveat: only works if the density distribution is also symmetric
- Again: if density is uniform, CM is also geometric center (centroid)

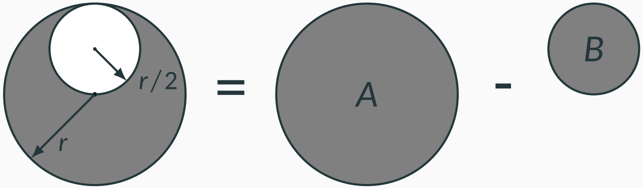
“Negative Mass”

- Where there is a “hole” in the geometry, treat it as having negative mass density $-\sigma$ in that region.
- Negative masses don’t exist, so this is really just a trick.
- **Example:** What is the center of mass of this shape?



Negative Mass Example

- This is how we would think of it:



- Let the origin of the coordinate system to located at the center of A
- Based on symmetry: $x_{cm} = 0$; only have to find y-coordinate.

$$y_{cm} = \frac{\sum y_i m_i}{\sum m_i} = \frac{m_A(0) + m_B(r/2)}{m_A + m_B} = \frac{-\sigma\pi\left(r/2\right)^2\left(r/2\right)}{\sigma\pi r^2 - \sigma\pi\left(r/2\right)^2} = \frac{-}{6}$$

Velocity, Acceleration and Momentum

Take time derivative of the equation for $\vec{x}_{cm}$ to get the velocity at the CM:

$$\vec{v}_{cm} = \frac{\mathrm{d}\vec{x}_{cm}}{\mathrm{d}t} = \frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \vec{x}\mathrm{d}m\right) = \frac{1}{m}\int \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\mathrm{d}m = \frac{\int \vec{v}\mathrm{d}m}{m}$$

The integral in the numerator is the sum of the momentum of all the masses in the system ($\vec{p}_{\text{net}}$) which means that we have

$$\vec{p}_{\text{net}} = m\vec{v}_{cm}$$

Taking the derivative of $\vec{p}_{\text{net}}$ relates force and acceleration at the CM as well:

$$\vec{F}_{\text{net}} = \frac{\mathrm{d}\vec{p}_{\text{net}}}{\mathrm{d}t} = m\frac{\mathrm{d}\vec{v}_{cm}}{\mathrm{d}t} = m\vec{a}_{cm}$$