Class 7: Rotational Motion of a Rigid Body

Advanced Placement Physics C

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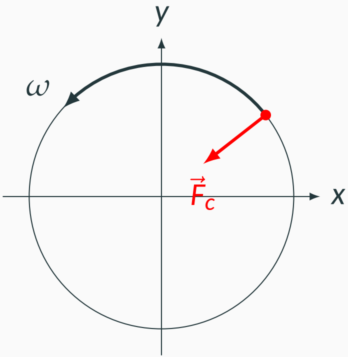
Olympiads School

Introduction

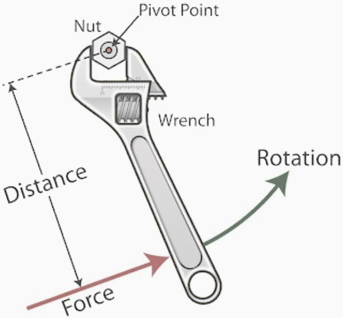
Uniform Circular Motion

Consider the uniform circular motion of an object with (constant) angular velocity $\vec{\omega}$. If the rotation is counterclockwise, the direction of $\vec{\omega}$ is *out of the page*; if rotation is clockwise, $\vec{\omega}$ is *into the page*.



- Centripetal force $\vec{F}_c$ is always perpendicular to the motion of the object
- $\vec{F}_c$ does not do any mechanical work
- Therefore, angular velocity $\vec{\omega}$ remains constant
- **The "rotational state" of the object does not change**
- Rotation of an object is not determined by merely what forces are acting it

Turning A Wrench



Similarly, when tightening/loosening a bolt by turning a wrench,

- When the nut turns, its “rotational state” changes
- The applied force has to be directed at a distance away from the bolt
- How easy to turn the nut depends on *both* the distance and the force

Torque

Recall the second law of motion for objects with constant mass:

$$\vec{F}_{\text{net}} = m\vec{a}$$

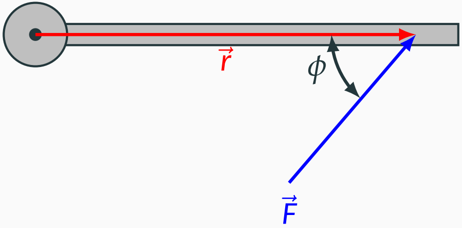
Is it also true for *rotational* motion? If a net force $\vec{F}_{\text{net}}$ causes the center of mass of an object to begin to accelerate, what causes a mass to rotate?

What is Torque?

Torque (or **moment**) is the tendency for a force to change the rotational motion of a body.

- A force $\vec{F}$ acting at a point some distance $\vec{r}$ (called the **moment arm**) from a **fulcrum** (or **pivot**) at an angle ϕ between $\vec{F}$ and $\vec{r}$
- e.g. the force to twist a screw

In the example below, a force $\vec{F}$ is applied $\vec{r}$ away from the pivot at an angle ϕ. This generates a torque around the pivot.



Torque

Torque $\vec{\tau}$ is defined as the cross product of the force $\vec{F}$ and the **moment arm** $\vec{r}$. The unit for torque is a **newton meter** ($\mathrm{N} \cdot \mathrm{m}$).

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

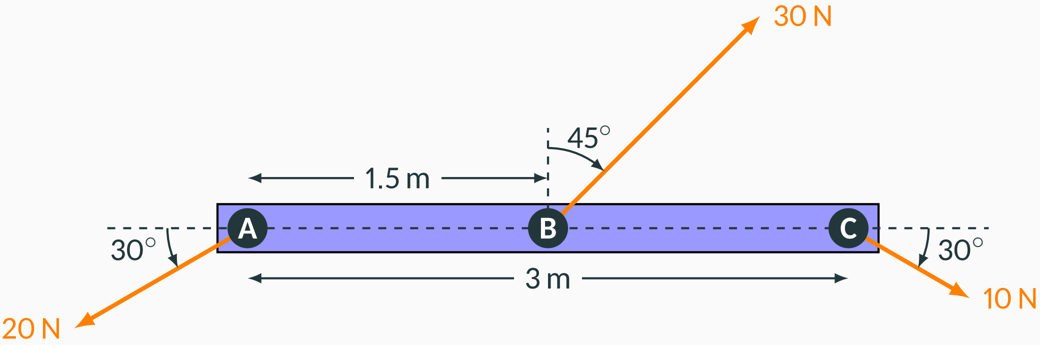
Its magnitude can be calculated in scalar form using the angle ϕ between $\vec{F}$ and $\vec{r}$:

$$\boxed{\tau = Fr \sin \phi}$$

Quantity	Symbol	SI Unit
Torque	$\vec{\tau}$	$\mathrm{N} \cdot \mathrm{m}$
Applied force	$\vec{F}$	N
Moment arm (from fulcrum to force)	$\vec{r}$	m
Angle between force and moment arm	ϕ	(no units)

Example Problem

Example: Find the net torque on point C.

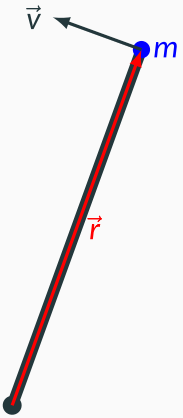


Angular Momentum

Angular Momentum

Consider a mass m connected to a massless beam rotates with velocity $\vec{v}$ at a position $\vec{r}$ from the center (shown on the right). It has an **angular momentum** ($\vec{L}$), defined as:

$$\boxed{\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})}$$



Expanding the term with $\vec{v} = \vec{\omega} \times \vec{r}$, the expression for angular momentum can now be expressed in quantities related to rotations:

$$\boxed{\vec{L} = m(\vec{r} \times \vec{v}) = m(\vec{r} \times (\vec{\omega} \times \vec{r})) = mr^2\vec{\omega}}$$

Or in scalar form:

$$\boxed{L = rmv = mr^2\omega}$$

The unit for angular momentum is a **kilogram meter squared per second** ($\mathrm{N} \cdot \mathrm{m}^2/\mathrm{s}$).

Moment of Inertia

Look again at the definition of angular momentum:

$$\vec{L} = \underbrace{mr^2}_{I}\,\vec{\omega}$$

The quantity $I = mr^2$ is called the **moment of inertia** with a unit of **kilogram meter squared** ($\mathrm{kg} \cdot \mathrm{m}^2$), and

$$\boxed{\vec{L} = I\vec{\omega}}$$

Momentum of inertia can be considered to be an object's "rotational mass"

Moment of Inertia

For a *single particle* of m rotating at a distance r from the pivot:

$$\boxed{I = mr^2}$$

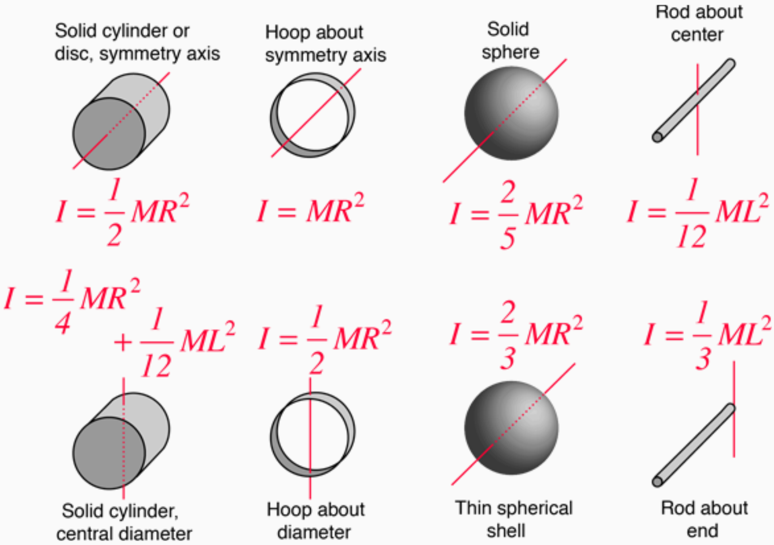
For a *collection of particles* rotating at ω, each of mass m_i at distance r_i from the pivot:

$$\boxed{I = \sum m_i r_i^2}$$

For a *continuous distribution of mass* rotating about a pivot, integral calculus is need to calculate the momentum of inertia:

$$\boxed{I = \int r^2 \mathrm{d}m}$$

Moment of Inertia



Angular Momentum and Moment of Inertia

Linear and angular momentum have very similar expressions

$$\vec{p} = m\vec{v}$$
$$\vec{L} = I\vec{\omega}$$

Just as $\vec{p}$ describes the overall *translational* state of motion of a physical system, $\vec{L}$ describes its overall *rotational* state

Laws of Motion

Equilibrium: First Law of Motion

An object is in **translational equilibrium** is when the net force acting on it is zero:

$$\vec{F}_{\text{net}} = \vec{0}$$

This does *not* mean that the object has no translational motion; it just means that the object's overall *transtational state* is not changing, i.e. momentum $\vec{p}$ is constant. For constant mass m, this means that $\vec{a} = \vec{0}$.

Equilibrium: First Law of Motion

Likewise, an object is in **rotational equilibrium** when the net torque acting on it is zero:

$$\vec{\tau}_{\text{net}} = \vec{0}$$

This does *not* mean that the object has no rotational motion; it just means that the object's overall *rotational state* is not changing, i.e. angular momentum $\vec{L}$ is constant. For constant moment of inertia I, this means that $\vec{\alpha} = \vec{0}$.

Second Law of Motion for Rotational Motion

The net torque is the time rate of change of angular momentum:

$$\vec{\tau}_{\text{net}} = \vec{r} \times \vec{F}_{\text{net}} = \vec{r} \times \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}(\vec{r} \times \vec{p})}{\mathrm{d}t} \quad \longrightarrow \quad \boxed{\vec{\tau}_{\text{net}} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}}$$

- If the net torque on a system is zero, then the rate of change of angular momentum is zero, and we say that the angular momentum is conserved.
- e.g. When an ice skater starts to spin and draws his arms inward. Since angular momentum is conserved, a decrease in r means an increase in ω.

Second Law of Motion for Translational Motion

For translational motion, the general form of the first and second laws of motion states that the net force is rate of change of the object's momentum:

$$\vec{F}_{\text{net}} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

For objects with constant mass, this reduces to the more familiar form:

$$\vec{F} = m\vec{a}$$

Second Law of Motion for Rotational Motion

Likewise, the second law of motion for rotational motion has a similar form, but with torque $\vec{\tau}$ replacing force $\vec{F}$, and angular momentum $\vec{L}$ replacing linear momentum $\vec{p}$:

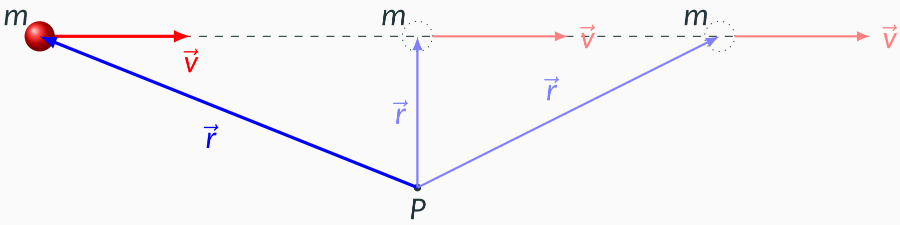
$$\boxed{\vec{\tau}_{\text{net}} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}}$$

For objects with constant momentum of inertia I, this reduces to:

$$\boxed{\vec{\tau}_{\text{net}} = I\vec{\alpha}}$$

But there is no rotational motion, is there?

Even when there is no apparent rotational motion, it does not necessarily mean that angular momentum is zero! In this case, mass m travels along a straight path at constant velocity (uniform motion), but the angular momentum around point P is not zero:



Since there is no force and no torque acting on the object, both the linear momentum ($\vec{p} = m\vec{v}$) and angular momentum ($\vec{L} = \vec{r} \times \vec{v}$) are constant.

Example Problem

Example: A skater extends her arms (both arms!), holding a 2 kg mass in each hand. She is rotating about a vertical axis at a given rate. She brings her arms inward toward her body in such a way that the distance of each mass from the axis changes from 1 mto 0.50 m. Her rate of rotation (neglecting her own mass) will?

Example Problem

Example: A 1 kg mass swings in a vertical circle after having been released from a horizontal position with zero initial velocity. The mass is attached to a massless rigid rod of length 1.5 m. What is the angular momentum of the mass, when it is in its lowest position?