Topic 7: Rotational Motion of a Rigid Body

Advanced Placement Physics C

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Olympiads School

Torque

Torque and Rotational Equilibrium

Let’s consider this question:

> Two people stand on a board of uniform density. One person has a mass of 50 kg and stands 10 m away from the fulcrum (pivot). The second person has a mass of 65 kg. How far away from the fulcrum would the second person have to stand for the system to have to be in equilibrium?

Equation of Motion

Recall the second law of motion for objects with constant mass:

$$\mathbf{F}_{net} = m\mathbf{a}$$

Is it also true for *rotational* motion? If a net force $\mathbf{F}_{net}$ causes the center of mass to accelerate (linearly), what causes a mass to rotate?

To answer this, we need to introduce a few concepts first...

Torque

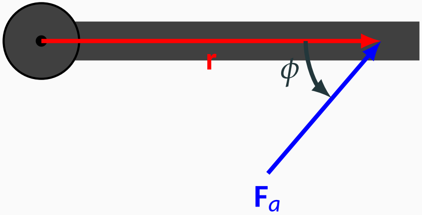
I have a rod on a table, and with my fingers, I push the two ends of the rod with equal force F. **What happens?**

$\mathbf{F}_{net} = \mathbf{0}$, therefore $\mathbf{a} = \mathbf{0}$. But (obviously) it won't stay still either!

What is Torque?

Torque (or **moment**) is the tendency for a force to change the rotational motion of a body.

- A force $\mathbf{F}_a$ acting at a point some distance $\mathbf{r}$ (called the **moment arm**) from a **fulcrum** (or **pivot**) at an angle ϕ between $\mathbf{F}_a$ and $\mathbf{r}$
- e.g. the force to twist a screw



Torque

In scalar form, we can express torque τ as the force $\mathbf{F}_a$, the **moment arm r** and the angle ϕ between $\mathbf{F}_a$ and $\mathbf{r}$:

$$\boxed{\tau = rF_a \sin\phi}$$

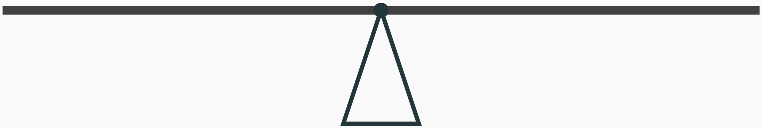
In vector form, we use the cross-product:

$$\boxed{\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}_a}$$

Quantity	Symbol	SI Unit
Torque	$\boldsymbol{\tau}$	N m
Applied force	$\mathbf{F}_a$	N
Moment arm (from fulcrum to force)	$\mathbf{r}$	m
Angle between force and moment arm	ϕ	(no units)

Torque

Going back to the example question:



Torque

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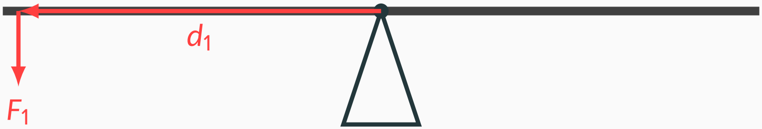

- F_1 will rotate the board counter clockwise

Torque

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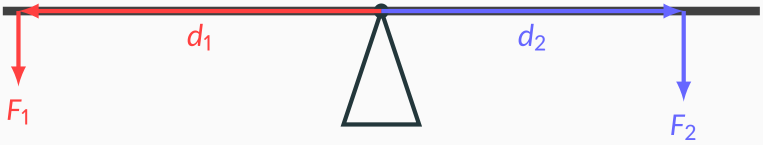

- F_1 will rotate the board counter clockwise
- F_2 will rotate the board clockwise

Torque

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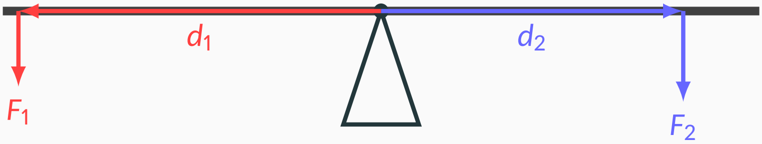


- F_1 will rotate the board counter clockwise
- F_2 will rotate the board clockwise
- The beam will remain static (in equilibrium) if

$$F_1 d_1 = F_2 d_2$$

Rotational Equilibrium: First Law of Motion

An object is in **translational equilibrium** is when the force acting it is zero, and therefore the acceleration of its center of mass (as discussed in Topic 5) is zero:

$$\mathbf{F} = \mathbf{0}$$

Having no net force does *not* mean that the object has no translational motion; it just means that the object's overall *transtational state* is not changing, i.e. the translational momentum $\mathbf{p}$ is constant.

Rotational Equilibrium: First Law of Motion

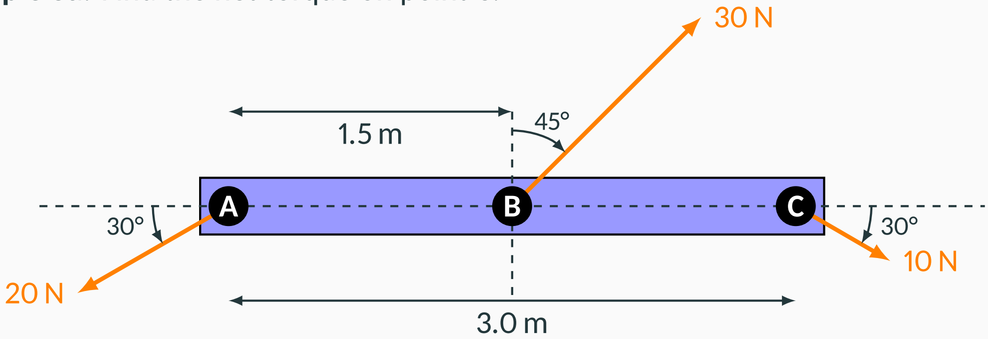
Likewise, an object is in **rotational equilibrium** when the net torque acting on it is zero:

$$\boldsymbol{\tau} = \mathbf{0}$$

Having no net torque does *not* mean that the object has no rotational motion; it just means that the object's overall *rotational state* is not changing, i.e. $\alpha = \mathbf{0}$, or that the **angular momentum L** is constant.

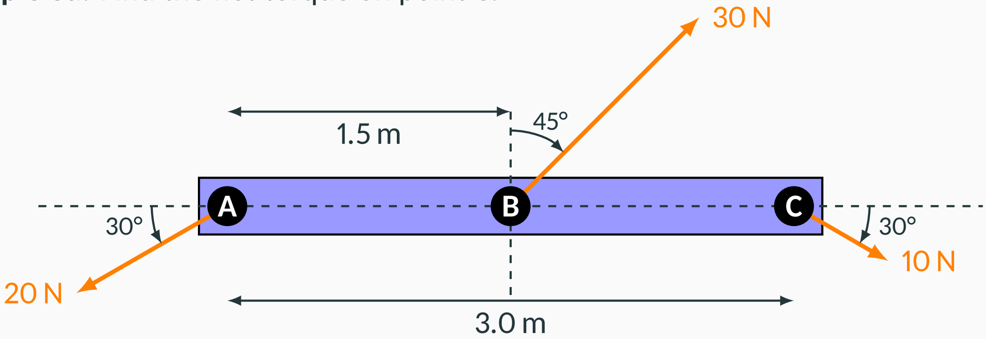
Example Problem

Example 8a: Find the net torque on point C.



Example Problem

Example 8a: Find the net torque on point C.



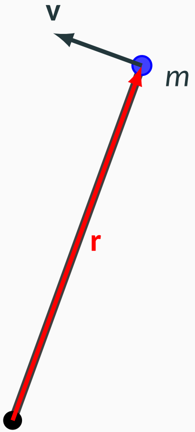
Example 8b: Now find the net torque on A.

Angular Momentum

Angular Momentum

Consider a mass m connected to a massless beam rotates with speed v at a distance r from the center (shown on the right). It has an **angular momentum** (**L**), defined as:

$$\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v})} \quad \text{or} \quad \boxed{L = rmv}$$



The direction of **L** depends on the direction of rotation. Expanding the terms:

$$\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) = m\mathbf{r} \times (\omega \times \mathbf{r}) = mr^2\omega$$

Which gives us:

$$\boxed{\mathbf{L} = I\omega}$$

The quantity I is called the **moment of inertia**.

Moment of Inertia

A single particle:

$$\boxed{I = r^2 m}$$

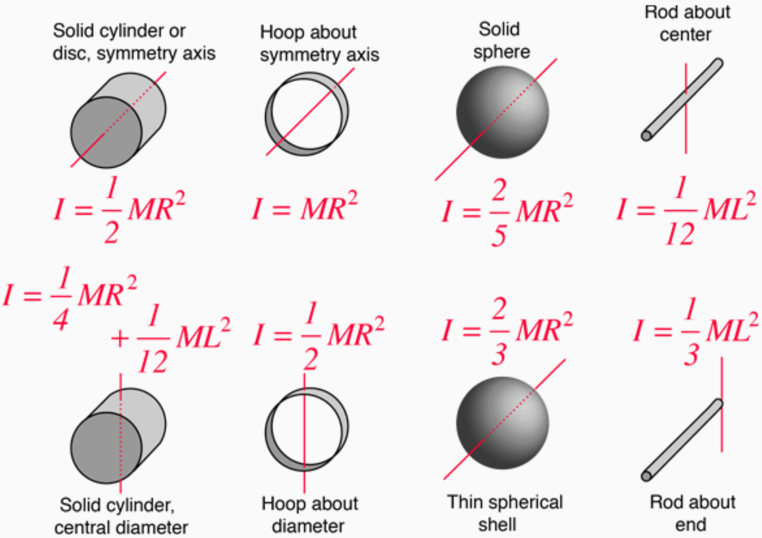
A collection of particles:

$$\boxed{I = \sum r_i^2 m_i}$$

Continuous distribution of mass:

$$\boxed{I = \int r^2 dm}$$

Moment of Inertia



Angular Momentum and Moment of Inertia

Linear and angular momentum have very similar expressions

$$\mathbf{p} = m\mathbf{v} \qquad\qquad \mathbf{L} = I\boldsymbol{\omega}$$

- Just as **p** describes the overall *translational* state of a physical system, **L** describes its overall *rotational* state
- Momentum of inertia *I* can be considered to be an object's "rotational mass"

Second Law of Motion for Rotational Motion

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt} \longrightarrow \boxed{\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}}$$

- If the net torque on a system is zero, then the rate of change of angular momentum is zero, and we say that the angular momentum is conserved.
- e.g. When an ice skater starts to spin and draws his arms inward. Since angular momentum is conserved, a decrease in r means an increase in ω.

Second Law of Motion for Rotational Motion

The second law of motion for rotational motion has a very similar form to translational motion:

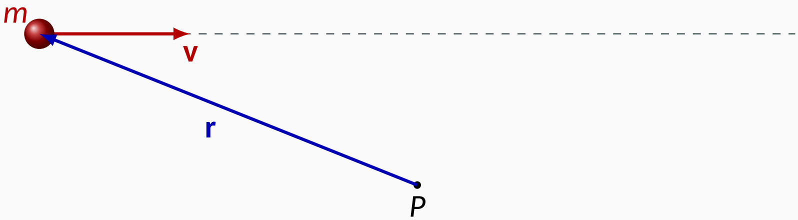
$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

For objects with constant mass (translational motion) or constant moment of inertia (rotational motion), the second law reduces to:

$$\mathbf{F} = m\mathbf{a} \qquad \boldsymbol{\tau} = I\boldsymbol{\alpha}$$

But there is no rotational motion, is there?

Even when there is no apparent rotational motion, it does not mean that angular momentum is zero! In this case, mass m travels along a straight path at constant velocity (uniform motion), but the angular momentum around point P is not zero:



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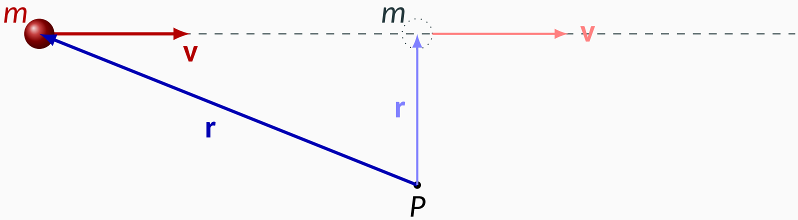

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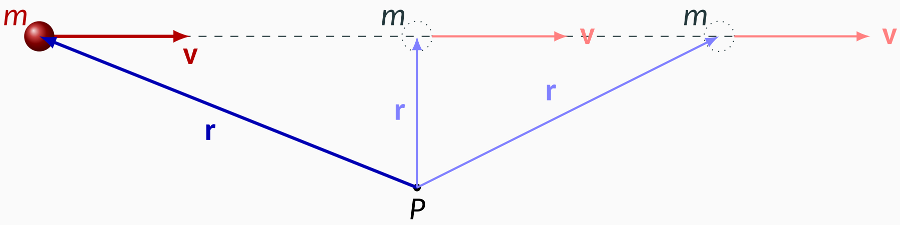

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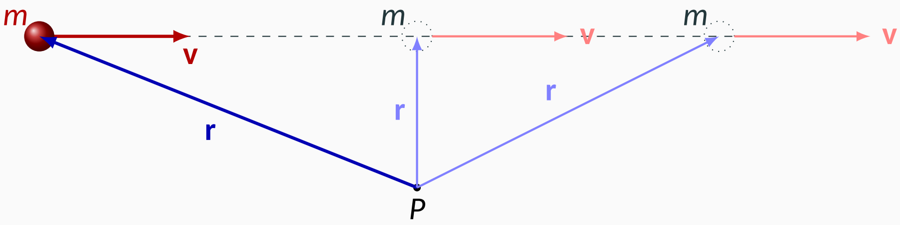


Since there is no force and no torque acting on the object, both the linear momentum ($\mathbf{p} = m\mathbf{v}$) and angular momentum ($\mathbf{L} = \mathbf{r} \times \mathbf{v}$) are constant.

Example Problem

Example 9: A skater extends her arms (both arms!), holding a 2.0 kg mass in each hand. She is rotating about a vertical axis at a given rate. She brings her arms inward toward her body in such a way that the distance of each mass from the axis changes from 1.0 m to 0.50 m. Her rate of rotation (neglecting her own mass) will?

Last Example

Example 10: A 1.0 kg mass swings in a vertical circle after having been released from a horizontal position with zero initial velocity. The mass is attached to a massless rigid rod of length 1.5 m. What is the angular momentum of the mass, when it is in its lowest position?

Solving Rotational Problems

When solving for rotational problems like the ones described in the previous sections:

- Draw a free-body diagram to account for all forces
- The direction of friction force is not always obvious
- The magnitude of any static friction force cannot be assumed to be at maximum.
- If the object is to change its rotational state, there must be a net torque causing it.

Solving Rotational Problems

Once the free-body diagram is complete

- Breaks down the *forces* into $\hat{\imath}$, $\hat{\jmath}$ and $\hat{\mathbf{k}}$ components
- We have now three equations for translation, but it is likely that only *one* direction will have forces:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z$$

- And three equations for rotation, and torque is only applied in one direction (likely $\hat{\mathbf{k}}$):

$$\sum \tau_x = I_x\alpha_x \qquad \sum \tau_y = I_y\alpha_y \qquad \sum \tau_z = I_z\alpha_z$$

Solving Rotational Problems

For rotational motion dynamics equation:

1. Relate the force(s) that causes rotational motion to the net torque

$$\tau = Fr$$

2. Substitute the expression for momentum of inertia (which has both mass and radius terms in it) into the equation for rotational motion
3. Relate angular acceleration to linear acceleration, if applicable:

$$\alpha = \frac{a}{R}$$

Now there are two equations with force and acceleration terms. See handout

Rotational Kinetic Energy

Rotational Kinetic Energy

To find the kinetic energy of a rotating system of particles (discrete number of particles, or continuous mass distribution), we sum (or integrate) the kinetic energy of the individual particles:

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

$$K = \int \frac{1}{2} v^2 dm = \frac{1}{2}\left(\int r^2 dm\right)\omega^2$$

It's no surprise that in both case, rotational kinetic energy is given by:

$$\boxed{K = \frac{1}{2} I \omega^2}$$

Kinetic Energy of a Rotating System

The total kinetic energy of a rotating system is the sum of its translational and rotational kinetic energies at its center of mass:

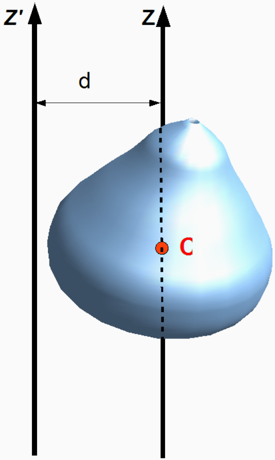
$$\boxed{K = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2}$$

In this case, I_{CM} is calculated at the center of mass. For simple problems, we only need to compute rotational kinetic energy at the pivot:

$$\boxed{K = \frac{1}{2}I_P\omega^2}$$

In this case, the I_P is calculated at the pivot. **IMPORTANT:** $I_{CM} \neq I_P$

Parallel Axis Theorem



The **parallel axis theorem** relates the moment of inertia of an object along two different but parallel axis by:

$$\boxed{I = I_{CM} + md^2}$$