Topic 8: Harmonic Motion

Advanced Placement Physics C Mechanics1

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July 2020

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Hooke’s Law

Hooke's Law

Hooke's law for an ideal spring relates the force exerted by a compressed/stretched) spring onto another object (**spring force** $\mathbf{F}_s$) to the stiffness of the spring (**spring constant** k) and spring displacement $\mathbf{x}$:

$$\boxed{\mathbf{F}_s = -k\mathbf{x}}$$

Quantity	Symbol	SI Unit
Spring force	$\mathbf{F}_s$	N
Spring constant	k	N/m
Amount of extension/compression	$\mathbf{x}$	m

Spring constant is also called **Hooke's constant** or **force constant**.

Elastic Potential Energy

Applying Hooke's law in the work equation gives the amount of **elastic potential energy** stored in the spring when it is compressed or stretched:

$$W = \int_{x_1}^{x_2} \mathbf{F}_e \cdot d\mathbf{x} = -\int_{x_1}^{x_2} kx dx = -\frac{1}{2}kx^2\bigg|_{x_1}^{x_2} = -\Delta U_e$$

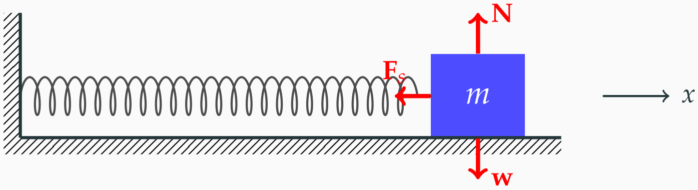
where elastic potential energy is defined as

$$\boxed{U_e = \frac{1}{2}kx^2}$$

Spring-Mass

Mass on a Spring

Consider the forces acting on a mass connected horizontally to a spring



$\mathbf{w}$ and $\mathbf{N}$ cancel out, so net force is due only to spring force $\mathbf{F}_s = -k\mathbf{x}$. This is true both when the spring is in compression or extension. (In the diagram above, the spring is in extension.)

Mass on a Spring

Applying second law of motion in the x-direction:

$$\sum F = F_s = ma \quad \longrightarrow \quad -kx = m\frac{d^2x}{dt^2}$$

This equation is called a *second-order ordinary differential equation with constant coefficients*. It is generally written in the standard form

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

Mass on a Spring

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

- The solution to the equation is a function $x(t)$ where the second time derivative $\ddot{x}$ looks like x but with a negative sign
- The obvious choices are the trigonometric functions $\sin(t)$ and $\cos(t)$
- Starting with this general form:

$$x(t) = A\cos(\omega t - \phi)$$

$\cos$ is usually preferred over $\sin$ because $\cos(0) = 1$, and oscillations usually begin at maximum amplitude A at $t = 0$. Mathematically, the two functions only differ in ϕ

Mass on a Spring

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

Starting with the general form and take the time derivatives to obtain the velocity and acceleration of the mass:

$$\begin{aligned} x(t) &= A\cos(\omega t - \phi) \\ v(t) &= -A\omega\sin(\omega t - \phi) \\ a(t) &= -A\omega^2\cos(\omega t - \phi) = -\omega^2 x \end{aligned}$$

where ω is the **angular frequency**, A is the amplitude of the oscillation and ϕ is a **phase shift** (or **phase constant**)

Mass on a Spring

Substituting expressions of $x(t)$ and $a(t) = \ddot{x}$ back into the ODE, we find that the solution is satisfied if ω is related to the spring constant and mass by:

$$\omega = \sqrt{\frac{k}{m}}$$

The angular frequency for the (undamped) simple harmonic oscillator is called the **natural frequency**.

Mass on a Spring

The period T and frequency f of the simple harmonic motion are given by:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$$

Angular frequency, frequency and period do not depend on amplitude A

Side Note #1

Side Note #1: The solution to any ODE is the linear combination of *all* possible solutions, i.e.:

$$x(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t)$$

Where c_1 and c_2 are constant coefficients based on the initial conditions. Trigonometric identities can be used to show that the solution form shown in previous slides is identical.

Side Note #2

Side Note #2: Another function that can be tried as a solution to the ODE is the exponential function, where its derivatives is related to the function itself, e.g.:

$$x(t) = e^{\omega t} \quad \rightarrow \quad \frac{dx}{dt} = \omega e^{\omega t} = \omega x(t)$$

However, the second derivative of $e^{\omega t}$ does not have the negative sign that is needed to solve the problem. But if the exponential function is *imaginary*:

$$x(t) = e^{i\omega t}$$

Then...

Mass on a Spring

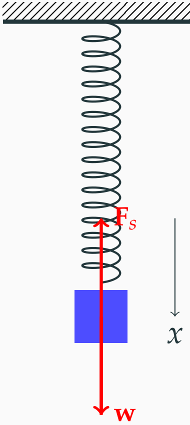
The 2nd derivative *will* in fact have the negative sign:

$$x = e^{i\omega t}$$
$$\dot{x} = i\omega e^{i\omega t}$$
$$\ddot{x} = i^2\omega^2 e^{i\omega t} = -\omega^2 e^{i\omega t}$$

This should not come as a surprise, since the complex exponential function and the sinusoidal functions are related:

$$e^{it} = \cos(t) + i\sin(t)$$

Vertical Spring-Mass System



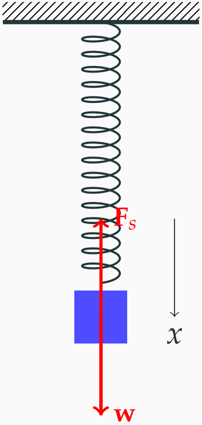
For a vertical spring-mass system, we must consider the weight of the mass as well:

$$mg - kx = m\frac{d^2x}{dt^2}$$

But since $\mathbf{w} = m\mathbf{g}$ is a constant, the only change is the addition of a constant B in the expression of $x(t)$:

$$\begin{aligned} x(t) &= A\cos(\omega t - \phi) + B \\ v(t) &= -A\omega\sin(\omega t - \phi) \\ a(t) &= -A\omega^2\cos(\omega t - \phi) \end{aligned}$$

Vertical Spring-Mass System



The constant B is found by substituting x and $\ddot{x}$ into the ODE. It is stretching of the spring due to its weight:

$$B = \frac{mg}{k}$$

Angular frequency (i.e. natural frequency) remains the same as the horizontal case:

$$\omega = \sqrt{\frac{k}{m}}$$

Conservation of Energy in a Spring-Mass System

In the spring-mass systems, if there are no frictional losses, then the only forces doing work are the spring force (horizontal and vertical) and gravity (vertical). Both forces are *conservative*, therefore the total mechanical energy is conserved:

$$K_1 + U_{e,1} + U_{g,1} = K_2 + U_{e,2} + U_{g,2}$$

For the horizontal spring-mass system, the total energy of the simple harmonic oscillator is:

$$\boxed{E_T = \frac{1}{2}kA^2}$$

Simple Example

Example 2: A mass suspended from a spring is oscillating up and down. Consider the following two statements:

1. At some point during the oscillation, the mass has zero velocity but it is accelerating
2. At some point during the oscillation, the mass has zero velocity and zero acceleration.

(a) Both occur at some time during the oscillation
(b) Neither occurs during the oscillation
(c) Only (1) occurs
(d) Only (2) occurs

Another Example

Example 3: An object of mass 5 kg hangs from a spring and oscillates with a period of 0.5 s. By how much will the equilibrium length of the spring be shortened when the object is removed.

(a) 0.75 cm

(b) 1.50 cm

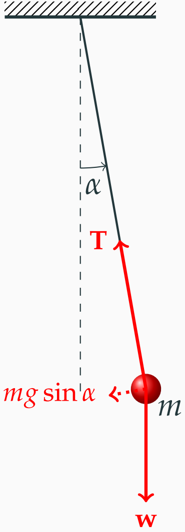
(c) 3.13 cm

(d) 6.20 cm

Simple Pendulum

What About a Simple Pendulum?

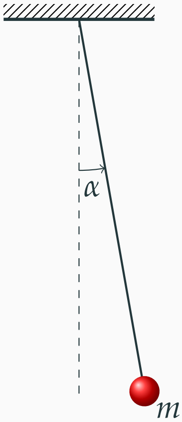
- Pendulums also exhibit oscillatory motion
- For a *simple* pendulum, there are two forces acting on the mass: weight $w = mg$ and tension T
- It has already been shown in the previous topic on circular motion that when the mass is deflected by an angle α, the force in the angular direction is $F_\theta = -mg\sin\alpha$
- No need to worry about the radial direction because it does not have to do with the restoring force



The Simple Pendulum

Substitute F_α into second law of motion, and cancelling mass term, we get:

$$F_\alpha = ma_\alpha \quad \longrightarrow \quad -g\sin\alpha = L\frac{d^2\alpha}{dt^2}$$



Solving this ODE in its present form is difficult because of the $\sin\alpha$ term. However, the series expansion of the sine function

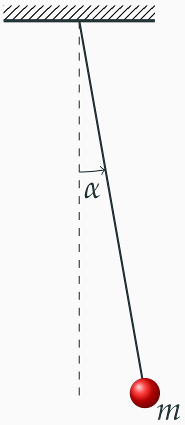
$$\sin(\alpha) = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \cdots$$

shows that for small angles ($\alpha < 15^\circ$), $\sin\alpha \approx \alpha$

The Simple Pendulum

For small angles of α, the ODE reduces to the same form as the spring-mass system!

$$\frac{d^2\alpha}{dt^2} + \frac{g}{L}\alpha = 0$$



Ordinary Differential Equation for the Pendulum

The solution for $\alpha(t)$ is very similar to the spring-mass system:

$$\boxed{\alpha(t) = A\cos(\omega t - \phi)}$$

where A is the maximum deflection (amplitude), and angular frequency of the oscillation ω is given by:

$$\boxed{\omega = \sqrt{\frac{g}{L}}}$$

and ϕ is a phase shift based on the initial condition of the pendulum.

A Pendulum Example

Example: A bucket full of water is attached to a rope and allowed to swing back and forth as a pendulum from a fixed support. The bucket has a hole in its bottom that allows water to leak out. How does the period of motion change with the loss of water?

(a) The period does not change.

(b) The period continuously decreases.

(c) The period continuously increases.

(d) The period increases to some maximum and then decreases again.

Think About g

Example: A little girl is playing with a toy pendulum while riding in an elevator. Being an astute and educated young lass, she notes that the period of the pendulum is $T = 0.5\,\text{s}$. Suddenly the cables supporting the elevator break and all of the brakes and safety features fail simultaneously. The elevator plunges into free fall. The young girl is astonished to discover that the pendulum has:

(a) continued oscillating with a period of 0.5 s.

(b) stopped oscillating entirely.

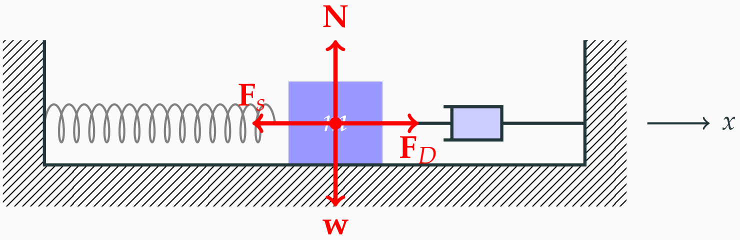
(c) decreased its rate of oscillation to have a longer period.

(d) increased its rate of oscillation to have a lesser period.

Damped Oscillation

It’s Never Perfect

In reality, there are friction, or drag, or other damping forces present in the spring-mass system, represented schematically like the shock absorber:



The damping force is typically related to velocity, in the opposite direction:

$$\mathbf{F}_D = -b\mathbf{v}^n$$

In the simplest case is to use $n = 1$ to represent viscous effects.

Damped Oscillator

The 2nd-order ODE is obtained by applying second law of motion:

$$\sum F = F_s + F_D = ma \quad \rightarrow \quad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

Arranging into standard form:

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

The solution to this ODE is still relatively straightforward (but not easy).

Damped Oscillator

The solution to the ODE is a standard problem in calculus class. It has both an exponential decay term and a sinusoidal term:

$$x(t) = A_0 e^{-\frac{b}{2m}t} \cos(\omega' t + \phi)$$

where A_0 is the initial amplitude of the damped oscillator, and angular frequency ω' now given by:

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Angular frequency ω' of the damped oscillator differs from the undamped case ω_0 depending on the damping factor b.

Critical Damping

Critical damping occurs when the angular frequency ω' term is zero:

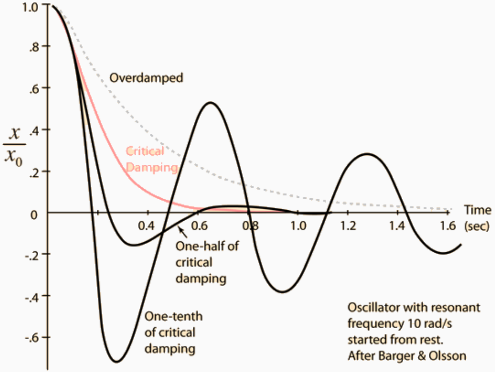
$$\omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = 0$$

which occurs when the damping constant is:

$$b_c = 2m\omega_0$$

- A critically damped system returns to its equilibrium position in the shortest time with *no* oscillation
- When $b > b_c$, the system is **over-damped**
- Critical or near-critical damping is desired in many engineering designs (e.g. shock absorbers on car suspensions)

Comparing Damped System



The motion of the damped oscillator is not strictly periodic.

Energy in a Damped System

The non-conservative damping force dissipates energy from the oscillator at a rate of:

$$P = \frac{dE}{dt} = \mathbf{F}_D \cdot \mathbf{v} = -bv^2$$

As velocity relate to energy by: $(v_{\mathrm{av}})^2 = E/m$, power dissipation is a first-order linear ODE:

$$\frac{dE}{dt} = -\frac{b}{m}E$$

The solution to the ODE shows the total amount of energy decreases exponentially with time:

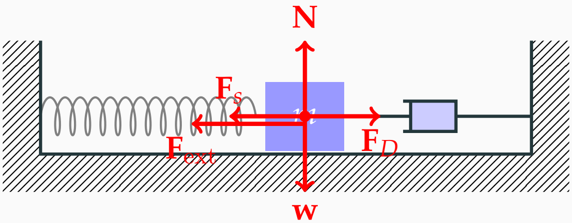
$$E(t) = E_0 e^{-\frac{b}{m}t}$$

Driven Oscillation

Forced Harmonic Motion

To keep a damped system going, energy must be added into the system. Assuming that the system is subjected to an external force that is harmonic with time, with a driving frequency ω:

$$F_{\text{ext}} = F_0 \cos(\omega t)$$



Forced Harmonic Motion

Again, the second-order ordinary differential equation is obtained by applying the second law of motion:

$$\sum F = -kx - bv + F_0 \cos(\omega t) = ma$$

Rearranging the terms gives a similar ODE to the damped case, but with the additional external force term on the right-hand side:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos(\omega t)$$

Forced Harmonic Motion

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos(\omega t)$$

The solution to this ODE has two components:

- A **transient solution** that is identical to that of the damped oscillator
  - Obtained by setting the external force term to zero
  - Depends on the initial condition
  - Solution becomes negligible over time because of exponential-decay
- A **steady-state solution** which does not depend on the initial condition

Forced Harmonic Motion

Solving for the steady-state solution will be left as a difficult calculus exercise, but it can be shown that the solution is a harmonic motion at the driving frequency ω of the external force:

$$\boxed{x(t) = A\cos(\omega t - \phi)}$$

where the amplitude of the oscillation A and phase shift ϕ are given by:

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} \qquad \tan\phi = \frac{b\omega}{m(\omega_0^2 - \omega^2)}$$

Resonance

Resonance is caused by in-phase excitation at natural frequency. This means that:

- The frequency of the driving force is same as the natural frequency of the oscillator

$$\omega = \omega_0 \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

- The driving force follows the motion of the oscillator.

Resonance

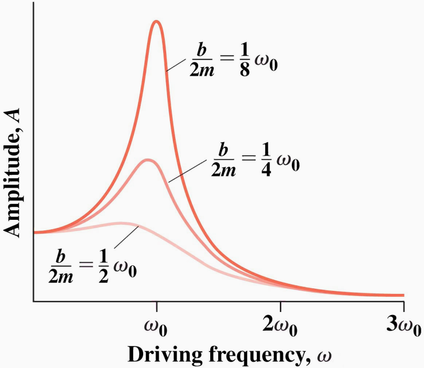
Looking at the expression for amplitude of the oscillation:

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}$$

Amplitude is at maximum when the frequency of the driving force ω is equal to the natural frequency ω_0, with a maximum value of:

$$A_{\max} = \frac{F_0}{b\omega}$$

Resonance



Plotting amplitude A as a function of driving frequency ω shows that:

- Resonance response is highest when $\omega = \omega_0$, which we know already
- The smaller the damping constant b, the higher and narrower the peak is

Resonance

$$\boxed{\tan\phi = \frac{b\omega}{m(\omega_0^2 - \omega^2)}}$$

When $\omega = \omega_0$ is substituted into the phase shift expression, the right-hand side becomes undefined. From this, we obtain a phase shift of $\phi = \pi/2$. Taking derivative of $x(t)$ for velocity $v(t)$, and substituting $\phi = \pi/2$:

$$v(t) = \dot{x} = -A\omega\sin(\omega t - \frac{\pi}{2}) = A\omega\cos(\omega t)$$

Resonance

At resonance, the object is always moving in the same direction as the driving force:

$$v(t) = A\omega \cos(\omega t)$$
$$F_{\text{ext}}(t) = F_0 \cos(\omega t)$$