Projectile Motion

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1 General Projectile Motion

A **projectile** is an object that is launched with an initial velocity of $\mathbf{V}_0$ at an angle θ with the horizontal. It travels along a parabolic trajectory and accelerates only due to gravity, as shown in Figure 1.

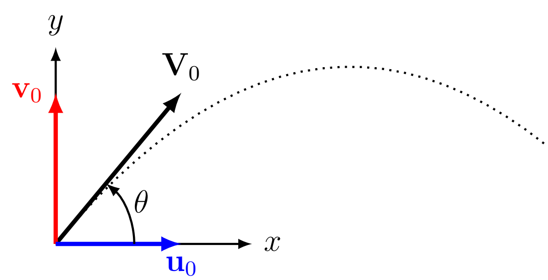

Figure 1: The parameters defining the motion of a projectile.

In general, when solving a projectile motion project:

- the x-axis ($\hat{\boldsymbol{\imath}}$ direction) is defined as the *horizontal* direction, with the (+) direction pointing forward
- the y-axis ($\hat{\boldsymbol{\jmath}}$ direction) is the *vertical* direction, with the (+) direction pointing upwards
- the origin of the coordinate system is where the projectile is launched
- the launch angle is (+) if it is above the horizontal, and (−) if it is below

This set up is consistent with the standard right-handed Cartesian coordinate system. The initial velocity $\mathbf{V}_0$ can be resolved into its $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components, $\mathbf{u}_0$ and $\mathbf{v}_0$:

$$\mathbf{V}_0 = \mathbf{u}_0 + \mathbf{v}_0 = [V_0 \cos\theta]\,\hat{\boldsymbol{\imath}} + [V_0 \sin\theta]\,\hat{\boldsymbol{\jmath}} \tag{1}$$

where $V_0 = |\mathbf{V}_0|$.

1.1 Motion in the Horizontal Direction

There is no acceleration (i.e. $a_x = 0$) along the $\hat{\boldsymbol{\imath}}$ direction, therefore horizontal velocity is a constant $u = u_0$. The kinematic equations are reduce to a single equation that relates the horizontal position (displacement) x as a function of time:

$$x(t) = u_0 t = [V_0 \cos\theta]\, t \tag{2}$$

1.2 Motion in the Vertical Direction

There is a constant acceleration due to gravity alone along the $\hat{\boldsymbol{\jmath}}$ direction, i.e. $a_y = -g$. (Acceleration is *negative* due to the fact that the positive y-axis points upwards.) The kinematic equations along the vertical direction are therefore:

$$y(t) = [V_0 \sin\theta]\, t - \frac{1}{2} g t^2 \tag{3}$$

$$v(t) = [V_0 \sin\theta] - gt \tag{4}$$

$$[v(y)]^2 = V_0^2 \sin^2\theta - 2gy \tag{5}$$

1.3 Solving Projectile Motion Problem

For most projectile motion problems, Eqs. 2 and 3 are the most often used. Because $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions are orthogonal[1], horizontal and vertical displacements $x(t)$ and $y(t)$ are independent of each other. However, there are variables that are shared in both directions, namely:

- Time t
- Launch angle θ
- Initial speed V_0

When solving for most projectile motion problems, there likely will be two unknowns that need to be solved (although you may not be explicitly told what one of them is), requiring two equations (the x and y kinematic equations). In some rare cases, if an object lands on an incline, there will be a third equation describing the relationship between x and y of the incline.

2 Parabolic Path

We can easily see that the path of a projectile is entirely parabolic. When solving for the flight time in the x direction (Eq. 2), we get:

$$t = \frac{x}{V_0 \cos\theta} \tag{6}$$

Substituting Eq. 6 into Eq. 3, we have

$$\begin{aligned} y &= (V_0 \sin\theta)\, t - \frac{1}{2} g t^2 \\ &= (V_0 \sin\theta) \left(\frac{x}{V_0 \cos\theta} \right) - \frac{1}{2} g \left(\frac{x}{V_0 \cos\theta} \right)^2 \\ y &= (\tan\theta)\, x - \left(\frac{g}{2V_0^2 \cos^2\theta} \right) x^2 \qquad (7) \end{aligned}$$

It is clear that the y is quadratic in x, and therefore the path is a parabola.

[1] In proper language of mathematics, they are therefore *linearly independent*.

3 Projectiles with Symmetric Trajectories

A **symmetric trajectory** is a special case of projectile motion where an object is launched at an angle of θ (between $0°$ and $90°$) above the horizontal[2] and then lands at the same height. Examples may include hitting a golf ball toward the hole, or shooting a bullet toward a horizontal target[3] [4]

3.1 Total Time of Flight T

We apply the kinematic equation first in the $\hat{\boldsymbol{\jmath}}$ direction. When the object lands at the same height at time T, the final displacement is $y(T) = 0$:

$$y(T) = V_0 \sin\theta T - \frac{1}{2} g T^2 = 0$$

Solving for T we have:

$$\boxed{T = \frac{2V_0 \sin\theta}{g}} \tag{8}$$

Not surprisingly, a projectile will stay in the air the longest when it is launched at $\theta = 90°$.[5] Mathematically, there are actually a second (trivial[6]) solution, at $T = 0$. Of course, it just means that the project has zero vertical displacement at the moment it is launched.

3.2 Maximum Height H

We apply the kinematic equation (Eq. 5) in the $\hat{\boldsymbol{\jmath}}$ direction. Recognizing that at maximum height $y = H$, the vertical component of velocity is zero $v(H) = 0$:

$$v^2 = V_0^2 \sin^2\theta - 2gH = 0$$

Solving for H, we get the maximum height equation:

$$\boxed{H = \frac{V_0^2 \sin^2\theta}{2g}} \tag{9}$$

The maximum height also (not surprisingly) has a maximum value at $\theta = 90°$.

3.3 Range R

We substitute the expression for total time of flight T from Eq. 8 into the t term, then apply the kinematic equation in the $\hat{\boldsymbol{\imath}}$ irection to compute $R = x(T)$ for any given launch angle and initial speed:

$$R = x(T) = u_0 T = V_0 \cos\theta \left[\frac{2V_0 \sin\theta}{g}\right] \tag{10}$$

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[2]This may be obvious, but angles *below* the horizontal will never have a symmetric trajectory.

[3]Shooting a bullet toward a horizontal target always require an upward angle because of gravity.

[4]Note that the equations for symmetric trajectory are *not* included in the AP Exam equation sheet; if you need these equations during the exams, you will need to derive them yourself. Thankfully, the derivation is striaghtforward.

[5]As a fun exercise, for a known initial speed V_0, you can plot t vs. $\sin\theta$ to find the acceleration due to gravity g!

[6]It means *unimportant*

Using the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$, Eq. 10 simplifies to:

$$\boxed{R = \frac{V_0^2 \sin(2\theta)}{g}} \tag{11}$$

It is obvious that for any given initial speed V_0, the maximum range R_{max} occurs at an angle where $\sin(2\theta) = 1$ (i.e. $\theta = 45°$), with a value of

$$\boxed{R_{\text{max}} = \frac{V_0^2}{g}} \tag{12}$$

Also, for a known initial speed V_0 and range R we can compute the launch angle θ:

$$\theta_1 = \frac{1}{2}\sin^{-1}\left(\frac{gR}{V_0^2}\right)$$

This angle is labelled θ_1 because it is *not* the only angle that can reach this range. Recall that for any angle $0 < \phi_1 < 180°$, there is also another angle $\phi_2 = 180° - \phi_1$ where sine of the angles has the same value:

$$\sin\phi = \sin(180° - \phi)$$

Which means that for any θ_1, there is also another angle θ_2 where $2\theta_2 = 180° - 2\theta_1$, or simply:

$$\boxed{\theta_1 + \theta_2 = 90°}$$