

# Class 5: Center of Mass

## Advanced Placement Physics C

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Finding an object's center of mass is important, because

- The laws of motion are formulated by treating an objects as point masses (for real-life objects, we let the forces apply to the center of mass)
- Objects can have *rotational* motion in addition to *translational* motion as well (we will examine that a bit more in a very-important topic later)

## Start with a Definition

The **center of mass** (“CM”) is the *weighted average of the masses in a system*. The “system” may be:

- A collection of individual particles
- A continuous distribution of mass with constant density. In this case, CM is also the geometric center (**centroid**) of the object
- A continuous distribution of mass with varying density
- If the masses are inside of a gravitational field, then the CM is also its **center of gravity** (“CG”)

## Simple Example

Start with a very simple example: two equal masses  $m$  along the  $x$ -axis, located at  $x_1$  and  $x_2$ . What is the center of mass of the system?



The answer is simple: the half way point between them:

$$x_{\text{cm}} = \frac{x_1 + x_2}{2}$$

Multiply both numerator and denominator by mass  $m$  (for generalization later), the equation becomes:

$$x_{\text{cm}} = \frac{mx_1 + mx_2}{2m}$$

## Slightly More Challenging

What if one of the masses are increased to  $2m$ ? This is still not a difficult problem; you can still *guess* the right answer without knowing the equation for center of mass.

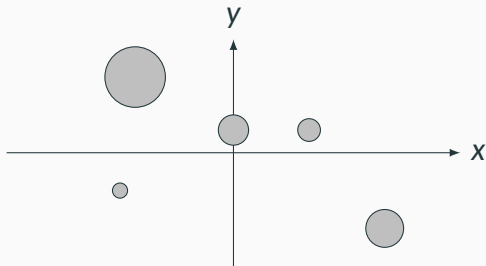


The answer is still simple. The CM is no longer half way between the two masses, but now  $\frac{1}{3}$  the total distance from the larger masses. We can show using a weighted average:

$$x_{cm} = \frac{mx_1 + (2m)x_2}{m + 2m}$$

## Complicating Things Further

The weighted average concept can now be applied to cases when there are masses in 2D or 3D:



# An Equation Helps

The center of mass is defined for discrete number of masses with the weighted average:

$$\vec{x}_{\text{cm}} = \frac{\sum \vec{x}_i m_i}{\sum m_i}$$

Quantity	Symbol	SI Unit
Position of center of mass (vector)	$\vec{x}_{\text{cm}}$	m
Position of point mass $i$ (vector)	$\vec{x}_i$	m
Point mass $i$	$m_i$	kg

In components:

$$x_{\text{cm}} = \frac{\sum x_i m_i}{\sum m_i} \quad y_{\text{cm}} = \frac{\sum y_i m_i}{\sum m_i} \quad z_{\text{cm}} = \frac{\sum z_i m_i}{\sum m_i}$$

## An Example

**Example:** Consider the following masses and their coordinates which make up a “discrete mass” rigid body”

$$m_1 = 5.0 \text{ kg}$$

$$\vec{x}_1 = 3\hat{i} - 2\hat{k}$$

$$m_2 = 10.0 \text{ kg}$$

$$\vec{x}_2 = -4\hat{i} + 2\hat{j} + 7\hat{k}$$

$$m_3 = 1.0 \text{ kg}$$

$$\vec{x}_3 = 10\hat{i} - 17\hat{j} + 10\hat{k}$$

What are the coordinates for the center of mass of this system?



# Continuous Mass Distribution

When the number of masses approaches infinity, this becomes a continuous distribution of mass. Taking the limit of masses  $N \rightarrow \infty$  gives the integral form of our equation:

$$\vec{x}_{\text{cm}} = \frac{\int \vec{x} dm}{\int dm}$$

What is the infinitesimal mass  $dm$  then?

# Densities

Linear density (for 1D problems)

$$\gamma = \frac{dm}{dL} \rightarrow dm = \gamma dL$$

Surface area density (for 2D problems)

$$\sigma = \frac{dm}{dA} \rightarrow dm = \sigma dA$$

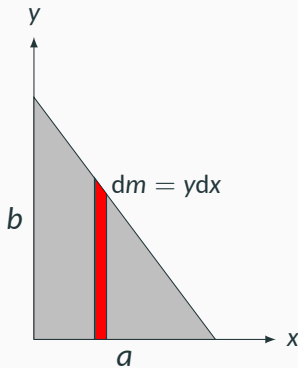
Volume density (for 3D problems)

$$\rho = \frac{dm}{dV} \rightarrow dm = \rho dV$$

The densities do not have to be constant

## An Example with Integrals

**Example 2:** A triangular plate is placed in a Cartesian coordinate system with two of its edges along the  $x$  and  $y$ -axis. The length of the edges along the axes are  $a$  and  $b$  respectively. Assuming that the surface area density  $\sigma$  is uniform, determine the coordinate of its center of mass.

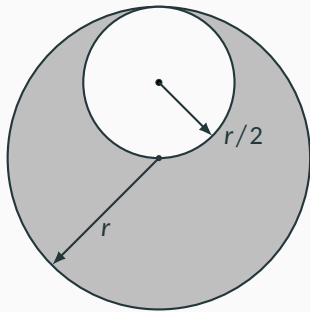


# Symmetry

- Any plane of symmetry, mirror line, axis of rotation, point of inversion *must* contain the center of mass.
- Caveat: only works if the density distribution is also symmetric
- Again: if density is uniform, CM is also geometric center (centroid)

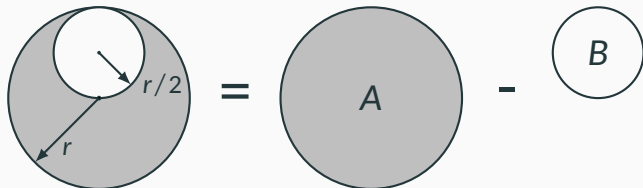
## “Negative Mass”

- Where there is a “hole” in the geometry, treat it as having negative mass density  $-\sigma$  in that region.
- Negative masses don’t exist, so this is really just a trick.
- **Example:** What is the center of mass of this shape?



## Negative Mass Example

- This is how we would think of it:



- Let the origin of the coordinate system to be located at the center of  $A$
- Based on symmetry:  $x_{\text{cm}} = 0$ ; only have to find  $y$ -coordinate.

$$y_{\text{cm}} = \frac{\sum y_i m_i}{\sum m_i} = \frac{m_A(0) + m_B(r/2)}{m_A + m_B} = \frac{-\sigma\pi (r/2)^2 (r/2)}{\sigma\pi r^2 - \sigma\pi (r/2)^2} = \frac{-r}{6}$$

## Velocity of the Center of Mass

Take time derivative of the equation for  $\vec{x}_{\text{cm}}$  to get the velocity at the center of mass:

$$\vec{v}_{\text{cm}} = \frac{d\vec{x}_{\text{cm}}}{dt} = \frac{1}{m} \frac{d}{dt} \left( \int \vec{x} dm \right) = \frac{1}{m} \int \frac{d\vec{x}}{dt} dm = \frac{\int \vec{v} dm}{m}$$

Or, in the form that is familiar, the velocity at the center of mass is the weighted sum of the velocities of the distribution of mass:

$$\vec{v}_{\text{cm}} = \frac{\int \vec{v} dm}{m}$$

# Velocity and Momentum

We can also rearrange the equation for the velocity of the center of mass to relate it to momentum, because the term  $\int \vec{v} dm$  is the net momentum of the mass distribution  $p_{\text{net}}$ :

$$\vec{v}_{\text{cm}} = \frac{\int \vec{v} dm}{m} \longrightarrow \vec{p}_{\text{net}} = m\vec{v}_{\text{cm}}$$

During a collision, there is no change in the net momentum<sup>1</sup>, the center of mass will continue to move at the same velocity before/after the collision, as if the collision never occurred.

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<sup>1</sup>Because there are no external forces



# Acceleration of the Center of Mass

Finding the rate of change of the net momentum (i.e applying the 2nd law of motion to this distribution of masses):

$$\frac{d\vec{p}_{\text{net}}}{dt} = \frac{d}{dt}(m\vec{v}_{\text{cm}})$$

If the system mass is constant, then this equation reduces to:

$$\frac{d\vec{p}_{\text{net}}}{dt} = m \frac{d\vec{v}_{\text{cm}}}{dt} \longrightarrow \boxed{\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}}}$$

We can see that when a net force is applied to an object, the object's acceleration is evaluated at the center of mass.