# Class 10: Harmonic Motion

Advanced Placement Physics C

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Olympiads School

# Hooke’s Law

# Hooke's Law

Hooke's law for an ideal spring relates the **spring force** $\vec{F}_e$ exerted by a compressed or stretched spring onto another object to the **spring constant** $k$ (the stiffness of the spring, also called *Hooke's constant*, *force constant*, or the *spring rate*) of the spring, and the spring's displacement $\vec{x}$:

$$\boxed{\vec{F}_s = -k\vec{x}}$$

| Quantity | Symbol | SI Unit |
|---|---|---|
| Spring force | $\vec{F}_e$ | N |
| Spring constant | $k$ | N/m |
| Amount of extension/compression | $\vec{x}$ | m |

# Elastic Potential Energy

Applying Hooke's law in the work equation gives the amount of **elastic potential energy** stored in the spring when it is compressed or stretched:

$$W = \int_{x_1}^{x_2} \vec{F}_s \cdot \mathrm{d}\vec{x} = -\int_{x_1}^{x_2} kx\mathrm{d}x = -\frac{1}{2}kx^2\Big|_{x_1}^{x_2} = -\Delta U_e$$

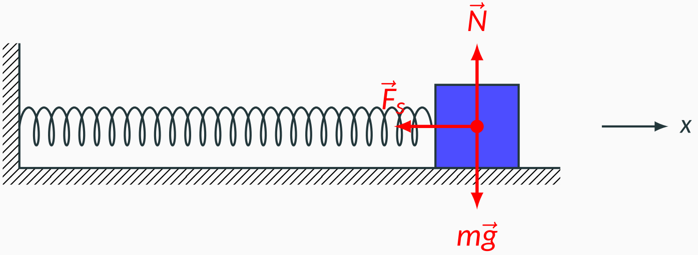
where elastic potential energy is defined as

$$\boxed{U_e = \frac{1}{2}kx^2}$$

# Spring-Mass

# Horizontal Spring-Mass System

Consider the forces acting on a mass connected horizontally to a spring without friction




$m\vec{g}$ and $\vec{N}$ cancel out, so net force is due only to spring force $\vec{F}_s = -k\vec{x}$ along the $x$-axis. This is true both when the spring is in compression or extension. (The spring is in extension in the diagram.)

# Horizontal Spring-Mass System

Applying second law of motion in the *x*-direction:

$$\sum F = F_s = ma \quad \longrightarrow \quad -kx = m\ddot{x}$$

This is a *second-order ordinary differential equation with constant coefficients*. In standard form:

$$\ddot{x} + \frac{k}{m}x = 0$$

# Mass on a Spring

The solution to the equation is a function $x(t)$ where the second time derivative $\ddot{x}$ looks like $x$ but with a negative sign

$$\ddot{x} + \frac{k}{m}x = 0$$

The obvious choices are the trigonometric functions $\sin(t)$ and $\cos(t)$. Starting with this general form:

$$x(t) = A\cos(\omega_0 t - \phi)$$

Cosine is usually preferred over sine because $\cos(0) = 1$, consistent with the fact that oscillations usually begin at maximum amplitude $A$ at $t = 0$, and therefore $\phi = 0$. Mathematically, the two functions only differ in $\phi$

# Mass on a Spring

$$\boxed{\ddot{x} + \frac{k}{m}x = 0}$$

Starting with the general form and take the time derivatives to obtain the velocity and acceleration of the mass:

$$\begin{aligned} x(t) &= A\cos(\omega_0 t - \phi) \\ v(t) &= -A\omega_0 \sin(\omega_0 t - \phi) \\ a(t) &= -A\omega_0^2 \cos(\omega_0 t - \phi) = -\omega_0^2 x \end{aligned}$$

where $\omega_0$ is the angular frequency of the oscillation, $A$ is the amplitude, and $\phi$ is the phase shift

# Mass on a Spring: Angular Frequency

Substituting expressions of $x(t)$ and $a(t) = \ddot{x}$ back into the ODE, we find that the solution is satisfied if $\omega_0$ is related to the spring constant and mass by:

$$\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$$

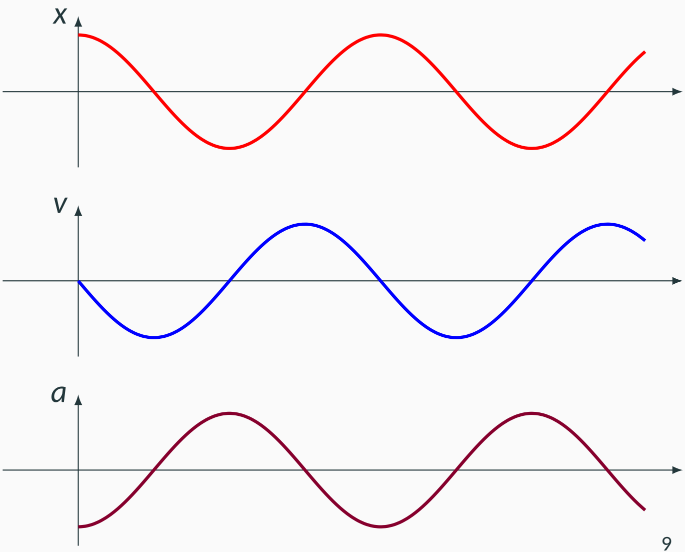
The angular frequency for the (undamped) simple harmonic oscillator is called the **natural frequency**. The period ($T_0$) and frequency ($f_0$) of the undamped harmonic oscillator are then given by:

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T_0 = \frac{1}{f_0} = 2\pi\sqrt{\frac{m}{k}}$$

Angular frequency, frequency, and period do not depend on amplitude $A$

# Displacement, Velocity and Acceleration

$$x(t) = A\cos(\omega_0 t - \phi)$$
$$v(t) = -A\omega_0 \sin(\omega_0 t - \phi)$$
$$a(t) = -A\omega_0^2 \cos(\omega_0 t - \phi) = -\omega_0^2 x$$



# Side Note #1

**Side Note #1:** For anyone who has more experiences with calculus, you will know that the solution to any ODE is the linear combination of *all* possible solutions, i.e.:

$$x(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t)$$

Where $c_1$ and $c_2$ are constant coefficients based on the initial conditions (position and velocity). Trigonometric identities can be used to show that the solution form shown in previous slides is identical.

# Side Note #2

**Side Note #2:** Another function that can be tried as a solution to the ODE is the exponential function, where its derivatives is related to the function itself. However, the second derivative of $e^{\omega t}$ does not have the negative sign that is needed to solve the problem.

$$x(t) = e^{\omega t} \quad \rightarrow \quad \ddot{x} = \omega^2 e^{\omega t} = \omega^2 x$$

But if the exponential function is *imaginary*:

$$x(t) = e^{i\omega t}$$

Then...

# Side Note #2

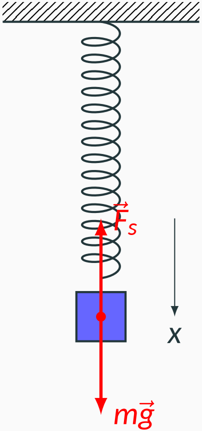
The second derivative *does* in fact have the negative sign:

$$x = e^{i\omega t}$$
$$\dot{x} = i\omega e^{i\omega t}$$
$$\ddot{x} = i^2\omega^2 e^{i\omega t} = -\omega^2 e^{i\omega t}$$

This should not come as a surprise, since the complex exponential function and the sinusoidal functions are related:

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$$

# Vertical Spring-Mass System



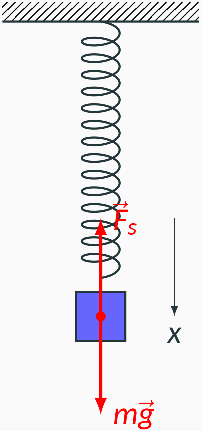

For a vertical spring-mass system, we must consider the weight as well:

$$mg - kx = m\ddot{x}$$

But since $mg$ is a constant, the only change is the addition of a constant $B$ in the expression of $x(t)$:

$$x(t) = A\cos(\omega_0 t - \phi) + B$$
$$v(t) = -A\omega_0 \sin(\omega_0 t - \phi)$$
$$a(t) = -A\omega_0^2 \cos(\omega_0 t - \phi)$$

# Vertical Spring-Mass System




$B$ is found by substituting $x$ and $\ddot{x}$ into the ODE. It is stretching of the spring due to its weight:

$$B = \frac{mg}{k}$$

Angular frequency (natural frequency) remains the same as the horizontal case:

$$\omega_0 = \sqrt{\frac{k}{m}}$$

# Conservation of Energy in a Spring-Mass System

In the spring-mass systems, if there are no frictional losses, then the only forces doing work are the spring force (horizontal and vertical) and gravity (vertical). Both forces are *conservative*, therefore the total mechanical energy is conserved:

$$K_1 + U_{e1} + U_{g1} = K_2 + U_{e2} + U_{g2}$$

For the horizontal spring-mass system, the total energy of the simple harmonic oscillator is:

$$\boxed{E_T = \frac{1}{2}kA^2}$$

# Simple Example

**Example 2:** A mass suspended from a spring is oscillating up and down. Consider the following two statements:

1. At some point during the oscillation, the mass has zero velocity but it is accelerating
2. At some point during the oscillation, the mass has zero velocity and zero acceleration.

(A) Both occur at some time during the oscillation

(B) Neither occurs during the oscillation

(C) Only (1) occurs

(D) Only (2) occurs

# Another Example

**Example 3:** An object of mass 5 kg hangs from a spring and oscillates with a period of 0.5 s. By how much will the equilibrium length of the spring be shortened when the object is removed.

(A) 0.75 cm

(B) 1.50 cm

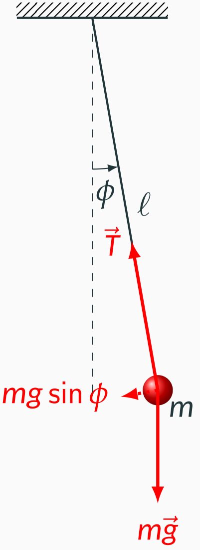
(C) 3.13 cm

(D) 6.20 cm

# Simple Pendulum

# What About a Simple Pendulum?

- Pendulums also exhibit oscillatory motion
- In a **simple pendulum**, all of the mass is concentrated at the end point
- There are two forces acting on the mass: weight $mg$ and tension $T$
- It has already been shown previously that when the mass is deflected by an angle $\phi$, the tangential force is $F_t = -mg \sin \phi$
- No need to worry about the radial direction; it does not have to do with the restoring force



# The Simple Pendulum

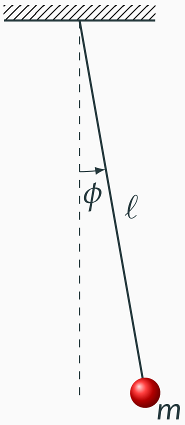
Substitute $F_t$ into second law of motion, and cancelling $m$:

$$F_t = ma_t \quad \rightarrow \quad -g \sin\phi = \ell\ddot{\phi}$$

Solving this ODE in its present form is difficult because of the $\sin\phi$ term. However, the series expansion of the sine function:

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \cdots$$

shows that for small angles, $\sin\phi \approx \phi$

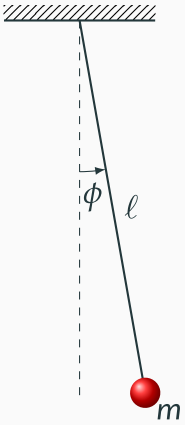


# The Simple Pendulum

For small angles of $\phi$, the ODE reduces to the same form as the spring-mass system

$$\ddot{\phi} + \frac{g}{\ell}\phi = 0$$

So how small is "small angle"? That depends on what tolerance (the number of significant figures) is needed in the answer.



# Ordinary Differential Equation for the Pendulum

The solution for $\phi(t)$ is a sinusoidal function, like the spring-mass system:

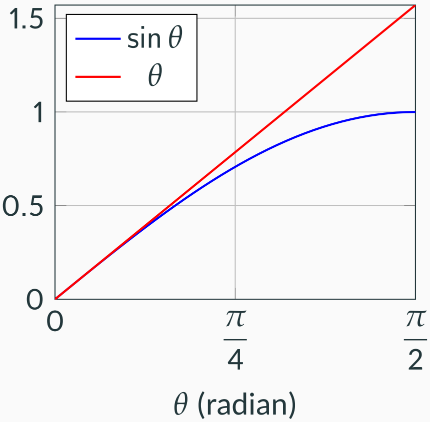
$$\boxed{\phi(t) = \Phi \cos(\omega_0 t - \beta)}$$

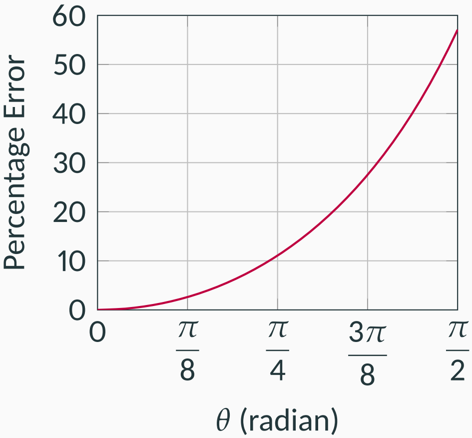
where $\Phi$ is the maximum deflection (amplitude), and angular frequency (natural frequency) of the oscillation $\omega_0$ is given by:

$$\boxed{\omega_0 = \sqrt{\frac{g}{\ell}}}$$

and $\beta$ is a phase shift based on the initial condition of the pendulum. If the oscillation begins at amplitude, then $\beta = 0$.

# How Good is the Small Angle Approximation?




What the maximum angle of deflection can be depends on the accuracy the answer requires.

# Pendulum Example Problem

**Example:** A simple pendulum consists of a mass $m$ attached to a light string of length $l$. If the system is oscillating through small angles, which of the following is true

(A) The frequency is independent of the acceleration due to gravity, $g$.

(B) The period depends on the amplitude of the oscillation.

(C) The period is independent of the mass $m$.

(D) The period is independent of the length $l$.

# A Pendulum Example

**Example:** A bucket full of water is attached to a rope and allowed to swing back and forth as a pendulum from a fixed support. The bucket has a hole in its bottom that allows water to leak out. How does the period of motion change with the loss of water?

(A) The period does not change.
(B) The period continuously decreases.
(C) The period continuously increases.
(D) The period increases to some maximum and then decreases again.

# Think About $g$

**Example:** A little girl is playing with a toy pendulum while riding in an elevator. Being an astute and educated young lass, she notes that the period of the pendulum is $T = 0.5\,\text{s}$. Suddenly the cables supporting the elevator break and all of the brakes and safety features fail simultaneously. The elevator plunges into free fall. The young girl is astonished to discover that the pendulum has:

(A) continued oscillating with a period of $0.5\,\text{s}$.

(B) stopped oscillating entirely.

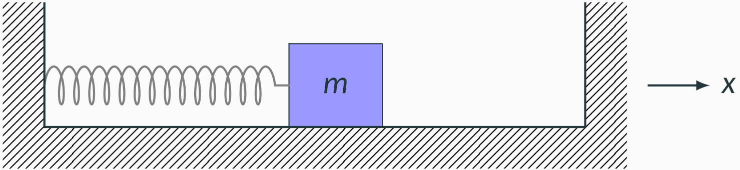
(C) decreased its rate of oscillation to have a longer period.

(D) increased its rate of oscillation to have a lesser period.

# Damped Oscillation

# It's Never Perfect

In reality, there are friction, or drag, or other damping forces present in the spring-mass system, represented schematically by the shock absorber:



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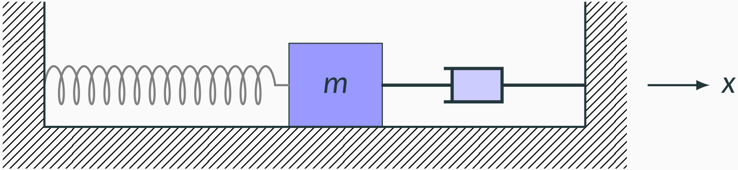

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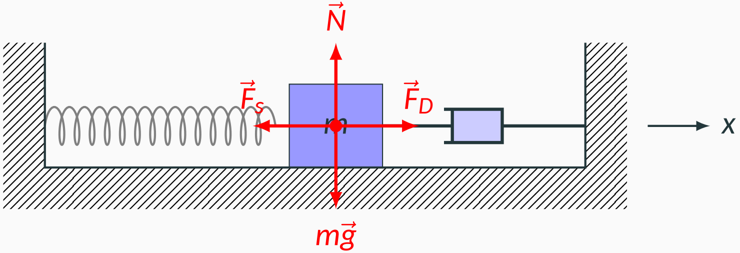

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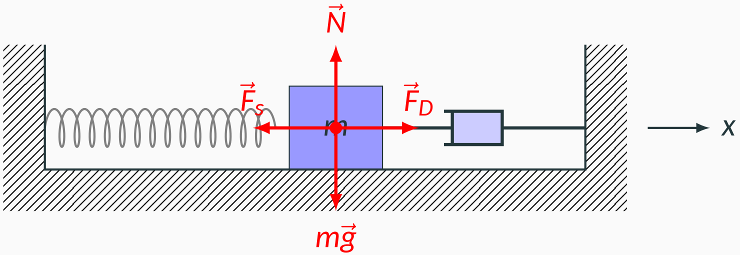



The damping force is typically related to velocity, in the opposite direction:

$$\vec{F}_D = -b(\vec{v})^n$$

where $b$ is a positive constant. In the simplest case is to use $n = 1$ to represent viscous effects. (For kinetic friction, $n = 0$; for aerodynamic drag, $n = 2$.)

# Damped Oscillator

The 2nd-order ODE is obtained by applying second law of motion, this time with the additional term from the damping force:

$$\sum F = F_s + F_D = ma \quad \rightarrow \quad -kx - b\dot{x} = m\ddot{x}$$

Arranging into standard form:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

The solution to this ODE is still relatively straightforward (but not as easy).

# Damped Oscillator

The solution to this ODE[1] has both an exponential decay term and a sinusoidal term:

$$x(t) = A_0 e^{-\frac{b}{2m}t} \cos(\omega t + \phi)$$

where $A_0$ is the initial amplitude of the damped oscillator, and the "natural frequency" for the damped oscillator is given by:

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

Note that $\omega < \omega_0$ (natural frequency decreases) because of the damping factor $b$.

[1]This is still a standard problem in calculus.

# Critical Damping

**Critical damping** occurs when the natural frequency $\omega$ is zero, i.e.:

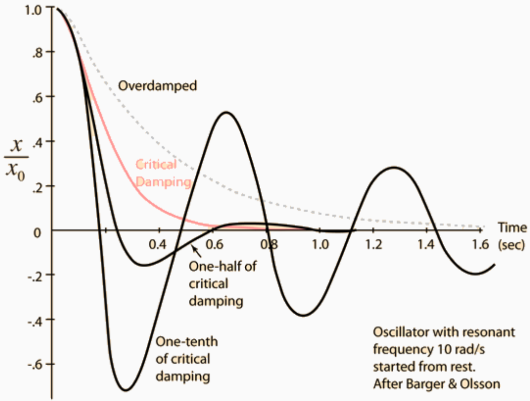
$$\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = 0$$

which corresponds to a damping constant:

$$b_c = 2m\omega_0 = 2\sqrt{km}$$

- A critically damped system returns to its equilibrium position in the shortest time with *no* oscillation
- When $b > b_c$, the system is **over-damped**
- Critical or near-critical damping is desired in many engineering designs (e.g. shock absorbers on car suspensions)

# Comparing Damped System




The motion of the damped oscillator is not strictly periodic.

# Energy in a Damped System

The non-conservative damping force dissipates energy from the oscillator at a rate of:

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \vec{F}_D \cdot \vec{v} = -bv^2$$

As velocity relate to energy by: $(v_{av})^2 = E/m$, power dissipation is a first-order linear ODE:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{b}{m}E$$

The solution to the ODE shows the total amount of energy decreases exponentially with time:

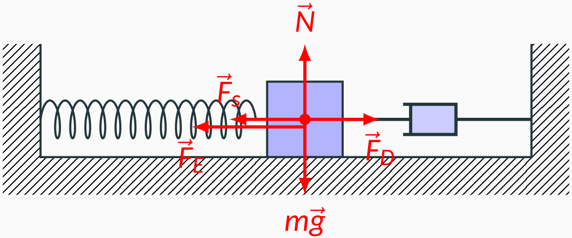
$$E(t) = E_0 e^{-\frac{b}{m}t}$$

# Driven Oscillation

# Forced Harmonic Motion

To keep a damped system going, energy must be added into the system. Assuming that the system is subjected to an external force ($F_E$) that is harmonic with time, with a driving frequency $\omega_E$:

$$F_E = F\cos(\omega_E t)$$




In general, the driving frequency $\omega_E$ does not have to be related to the natural frequency $\omega_0$ or damped frequency $\omega$.

# Forced Harmonic Motion

Again, the second-order ordinary differential equation is obtained by applying the second law of motion:

$$\sum F = -kx - bv + F\cos(\omega_E t) = ma$$

Rearranging the terms gives a similar ODE to the damped case, but with the additional external force term on the right-hand side:

$$m\ddot{x} + b\dot{x} + kx = F\cos(\omega_E t)$$

# Forced Harmonic Motion

$$m\ddot{x} + b\dot{x} + kx = F\cos(\omega_E t)$$

The solution to this ODE has two components:

- A **transient solution** that is identical to that of the damped oscillator
  - Obtained by setting the external force term to zero
  - Depends on the initial condition
  - Solution becomes negligible over time because of exponential-decay
- A **steady-state solution** which does not depend on the initial condition

# Forced Harmonic Motion

Solving for the steady-state solution will be left as a difficult calculus exercise[2], but it can be shown that the solution is a harmonic motion at the driving frequency $\omega_E$ of the external force:

$$\boxed{x(t) = A\cos(\omega_E t - \phi)}$$

where the amplitude of the oscillation $A$ and phase shift $\phi$ are given by:

$$A = \frac{F}{\sqrt{m^2(\omega_0^2 - \omega_E^2)^2 + b^2\omega_E^2}} \qquad \tan\phi = \frac{b\omega_E}{m(\omega_0^2 - \omega_E^2)}$$

[2] usually taught in 2nd-year university level ODE course

# Resonance

**Resonance** is caused by in-phase excitation at natural frequency. This means that:

- The frequency of the driving force is *approximately* the natural frequency of the damped oscillator:

$$\omega_E = \sqrt{\omega_0 - \frac{b}{2m}} \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

  For a lightly damped system, this means that $\omega_E \approx \omega$
- The driving force follows the motion of the oscillator.

# Resonance

$$A = \frac{F}{\sqrt{m^2(\omega_0^2 - \omega_E^2)^2 + b^2\omega_E^2}}$$

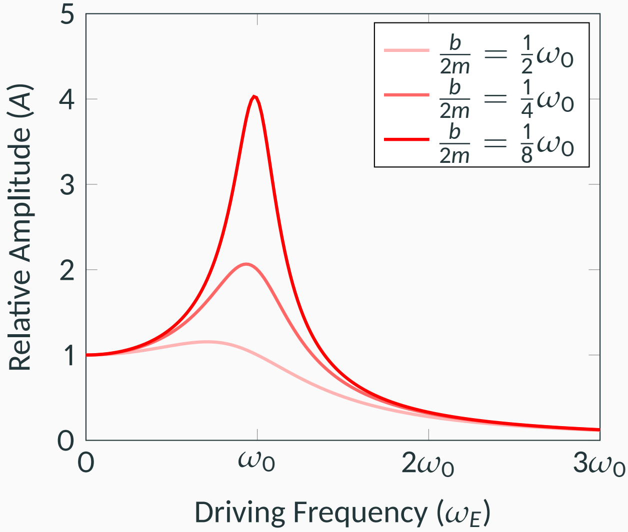
Maximum $A$ occurs when the denominator is the smallest, which can be related to the external frequency by setting the derivative with respect to $\omega_E$ to 0:

$$\frac{\mathrm{d}}{\mathrm{d}\omega_E}\left[m^2(\omega_0^2 - \omega_E^2)^2 + b^2\omega_E^2\right] = 0$$

It is a simple exercise to see that the minimum value occurs at the natural frequency of the damped oscillator.

$$\omega_E = \sqrt{\omega_0 - \frac{b}{2m}}$$

# Resonance




Plotting amplitude $A$ as a function of driving frequency $\omega$ shows that:

- For a lightly damped system (i.e. small $b$), resonance response is highest when $\omega_E \approx \omega \approx \omega_0$
- The smaller the damping constant $b$, the higher and narrower the peak is

# Resonance

$$\boxed{\tan\phi = \frac{b\omega_E}{m(\omega_0^2 - \omega_E^2)}}$$

When $\omega_E = \omega_0$ is substituted into the phase shift expression, the right-hand side becomes undefined. From this, we obtain a phase shift of $\phi = \pi/2$. Taking derivative of $x(t)$ for velocity $v(t)$, and substituting $\phi = \pi/2$:

$$v(t) = \dot{x} = -A\omega_E \sin(\omega_E t - \frac{\pi}{2}) = A\omega_E \cos(\omega_E t)$$

# Resonance

At resonance, the object is always moving in the same direction as the driving force:

$$v(t) = A\omega_E \cos(\omega_E t)$$
$$F_E(t) = F \cos(\omega_E t)$$

This also makes sense from a work-energy perspective, because now the external force is doing positive work to the system.