Kinematics of Rectilinear Motion

Dr. Timothy Leung

Kinematics is a discipline with in mechanics for describing the motion of points, bodies (objects), and systems of bodies (groups of objects). It is the mathematical representation of the relationship between *position*, *displacement*, *distance*, *velocity*, *speed* and *acceleration*. Note that kinematics does *not* deal with what causes motion. In high-school level[1] physics courses, kinematics problem usually deals with motion under constant acceleration. However, in the AP Physics C, we need a fuller understanding using calculus.

1 Position

Position is a vector describing the location of an object in a coordinate system. For rectinlinear motion, the preferred coordinate system is the *cartesian* system.[2]. The origin of the coordinate system is called "reference point". In the IJK notation for rectilinear motion, we can express position of an object by its x, y and z components (i.e. their x, y and z coordinates). The SI unit for position is *meters* (m).

$$\mathbf{x}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$$

If the object is in motion, then the position vector is a function of time t.

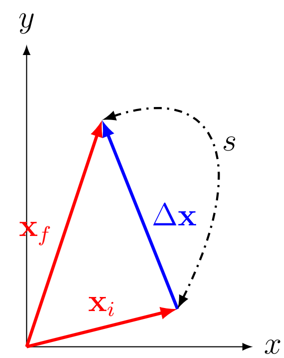

Figure 1: Position, displacement and distance in a Cartesian coordinate system.

2 Displacement

Displacement is the change in position from $\mathbf{x}_i$ to $\mathbf{x}_f$ within the same coordinate system, whenever an object moves.

$$\Delta\mathbf{x} = \mathbf{x}_f - \mathbf{x}_i = (x_f - x_i)\hat{\boldsymbol{\imath}} + (y_f - y_i)\hat{\boldsymbol{\jmath}} + (z_f - z_i)\hat{\boldsymbol{k}}$$

It it illustrated in Fig. 1. Like position (not surprisingly), the SI unit for displacement is also *meters*. The use of IJK notation makes vector addition and subtraction less prone to errors. Note that since "reference point" is the origin of the coordinate system, i.e. $\mathbf{x}_{\text{ref}} = \mathbf{0}$, any position vector $\mathbf{x}$ is also its displacement from the reference point.

[1] Grades 11 and 12 in the provincial curriculum

[2] For circular motions, we will use the *polar coordinate system* in 2D, or *cylindrical coordinate system* or *spherical coordinate system* in 3D

3 Distance

Distance s is a quantity that is *similar* (and related) to displacement. It is the *length of the path* taken when an object moves from position $\mathbf{x}_i$ to position $\mathbf{x}_f$, as shown in Fig. 1. Unlike displacement, however, distance is a scalar quantity that is always positive: $s \geq 0$, i.e. you can never walk a *negative* distance to the store. Because the path is not always a straight line, therefore while the magnitude of the displacement vector is also a scalar, it is not necessarily the same as distance:

$$s \geq |\Delta \mathbf{x}|$$

4 Instantaneous & Average Velocity

Velocity is a quantity used to describe how *fast* an object is moving. If position $\mathbf{x}(t)$ is differentiable in time t, then its **instantaneous velocity** $\mathbf{v}(t)$ can be found at any time t by differentiating $\mathbf{x}$ with respect to t. The SI unit for velocity is *meters per second* (m/s):

$$\boxed{\mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt}} \tag{1}$$

Since position $\mathbf{x}(t)$ has $x(t)$, $y(t)$ and $z(t)$ components along the (linearly independent) $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ directions, we can take the time derivative of every component to obtain the velocity components v_x, v_y and v_z in those directions:

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt} = \frac{dx}{dt}\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\hat{\boldsymbol{\jmath}} + \frac{dz}{dt}\hat{\boldsymbol{k}} = v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}}$$

By the fundamental theorem of calculus, if instantaneous velocity $\mathbf{v}(t)$ is the time derivative of position $\mathbf{x}(t)$ with respect to time t, then $\mathbf{x}(t)$ is the time integral of $\mathbf{v}(t)$:

$$\boxed{\mathbf{x}(t) = \int \mathbf{v}(t)dt + \mathbf{x}_0} \tag{2}$$

The constant of integration $\mathbf{x}_0 = \mathbf{x}(0)$ is the object's *initial position* at $t = 0$. As was the case in differentiation, we can integrate each component to get $\mathbf{x}$:

$$\mathbf{x}(t) = \left(\int u\hat{\boldsymbol{\imath}} + \int v\hat{\boldsymbol{\jmath}} + \int w\hat{\boldsymbol{k}}\right) dt + \mathbf{x}_0$$

The **average velocity** ($\overline{\mathbf{v}}$)[3]of an object is the change in position $\Delta\mathbf{x}$ over a finite time interval Δt:

$$\boxed{\overline{\mathbf{v}} = \frac{\Delta \mathbf{x}}{\Delta t}} \tag{3}$$

[3]For *time averages*, the convention amongst *most* physicists is to write a bar over the quantity, as we have done here. In contrast, for *ensemble averages*, e.g. the average speeds of many particles, we use the notation $\langle v \rangle$.

Like instantaneous velocity, we can find the x, y and z components of average velocity by separating components in each direction:

$$\overline{\mathbf{v}} = \frac{\Delta x}{\Delta t}\hat{\boldsymbol{\imath}} + \frac{\Delta y}{\Delta t}\hat{\boldsymbol{\jmath}} + \frac{\Delta z}{\Delta t}\hat{\boldsymbol{k}} = \overline{u}\hat{\boldsymbol{\imath}} + \overline{v}\hat{\boldsymbol{\jmath}} + \overline{w}\hat{\boldsymbol{k}}$$

5 Instantaneous & Average Speed

Instantaneous speed $v(t)$ is the rate of change of distance with respect to time.[4] Like velocity, the unit for speed is also m/s:

$$\boxed{v(t) = \frac{ds}{dt}}$$

Since distance is a scalar quantity, so too is speed. As distance of any path must always be positive $s > 0$, instantaneous speed must also be positive. Instantaneous speed v is the magnitude of the instantaneous velocity vector $\mathbf{v}$. Likewise, **average speed** ($\overline{v}$) is similar to average velocity: it is the distance travelled over a finite time interval.[5]

$$\boxed{\overline{v} = \frac{s}{\Delta t}} \tag{4}$$

6 Instantaneous and Average Acceleration

In the same way that velocity is the rate of change in position with respect to time, **instantaneous acceleration** $\mathbf{a}(t)$ is the rate of change in velocity with respect to time, and the second time derivative of position. The SI unit for acceleration is *meters per second squared* (m/s^2):

$$\boxed{\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{x}(t)}{dt^2}} \tag{5}$$

Although in grades 11 and 12 physics courses, students deal almost exclusively with constant cccel-eration, in AP Physics, it must be understood that acceleration can also vary with time, and that calculus must be used in many cases. Again, by the fundamental theorem of calculus, instantaneous velocity $\mathbf{v}(t)$ is the time integral of instantaneous acceleration $\mathbf{a}(t)$:

$$\boxed{\mathbf{v}(t) = \int \mathbf{a}(t)dt + \mathbf{v}_0} = \left(\int a_x\hat{\boldsymbol{\imath}} + \int a_y\hat{\boldsymbol{\jmath}} + \int a_z\hat{\boldsymbol{k}}\right) dt + \mathbf{v}_0 \tag{6}$$

where $\mathbf{v}_0$ is the initial velocity at $t = 0$.

[4]It is regrettable that both velocity and speed use the symbol v, but *c'est la vie*.

[5]It should be obvious that unlike instantaneous speed, average speed is not the magnitude of the average velocity.

7 Acceleration as Functions of Position and Velocity

Since, according to the second law of motion, acceleration is proportional to the net force ($F = ma$), therefore there are instances where acceleration is often expressed as functions of other motion quantities rather than time. For example:

- **Gravitational force**[6] or **electrostatic force**[7] are both inversely proportional to the square of the distance (called the inverse-square law), and therefore acceleration is best express by this complicated differential equation:

  $$a(x) = \frac{A}{x^2} \quad \text{or} \quad \frac{d^2x}{dt^2} = \frac{A}{x^2}$$

  The solution will likely require numerical integration.

- **Spring force** is proportional to displacement[8], and therefore acceleration is expressed as:

  $$a(x) = -bx \quad \text{or} \quad \frac{d^2x}{dt^2} = -bx$$

  The solution to this *second-order ordinary differential equation with constant coefficient* is a sinusoidal function, i.e. $x(t) = A\sin(\omega t + \phi)$, and the motion is a *simple harmonic motion* that will be studied in a later topic.

- **Damping force** is usually proportional to velocity, leading to an expression for acceleration:

  $$a(v) = -cv \quad \text{or} \quad \frac{dv}{dt} = -cv$$

  This time, the equation is a *first-order ordinary differential equation* that can be solved by separating the dt term and v terms and then integrating, and the expression for velocity is an exponential function:

  $$\frac{dv}{dt} = -cv \quad \rightarrow \quad \int \frac{dv}{v} = -\int c dt \quad \rightarrow \quad \ln(v) = -ct + C \quad \rightarrow \quad v(t) = v_0 e^{-ct}$$

  Once the velocity expression is obtained, the expression for $x(t)$ and $a(t)$ can also easily be obtained by integrating and differentiating.

- **Aerodynamic forces** such as drag and lift[9], which are proportional to the square of the velocity, leading to an expression for acceleration

  $$a(v) = -kv^2 \quad \text{or} \quad \frac{dv}{dt} = -kv^2$$

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[6]Newton's law of universal gravitation: $F_g = \dfrac{Gm_1m_2}{r^2}$

[7]Coulomb's law: $F_q = \dfrac{kq_1q_2}{r^2}$

[8]By Hooke's law $\mathbf{F}_s = -k\mathbf{x}$, where k is the spring constant that describes the stiffness of the spring

[9]The equations for lift and drag forces are $L = \frac{1}{2}\rho v^2 C_L A_{\text{ref}}$ and $D = \frac{1}{2}\rho v^2 C_D A_{\text{ref}}$ respectively, where ρ is the density of the fluid, A_{ref} is the reference area, C_L is the lift coeffcient and C_D is the drag coefficient

Not surprisingly, the process of solving the problem is similar to that of the damping function, but this time, the solution is a hyperbolic function:

$$\frac{dv}{dt} = -kv^2 \quad \rightarrow \quad \int \frac{dv}{v^2} = -\int kdt \quad \rightarrow \quad -\frac{1}{v} = -kt + C \quad \rightarrow \quad v(t) = \frac{1}{ct + C}$$

In practice, multiple forces may act on an object, and each of them will be functions of other motion quantities, and therefore the solution may require solving more complex differential equations (although this is highly unlikely in AP Physics).

8 Special Notation When Differentiating With Time

Physicists and engineers often use a special notation when the derivative is taken with respect to *time* (and not spatial derivatives), by writing a dot above the variable for *first* derivative, and *two* dots for *second* derivative, etc. For example, velocity is $\mathbf{v}(t) = \dot{\mathbf{x}}$ while acceleration is $\mathbf{a}(t) = \dot{\mathbf{v}} = \ddot{\mathbf{x}}$. This notation will be used occasionally in this course when it is convenient to do so.

9 Higher Derivatives

For those who are curious about higher derivatives, the time derivative of acceleration is called **jerk** $\mathbf{j}(t)$ with a unit of m/s^3:

$$\mathbf{j}(t) = \frac{d\mathbf{a}}{dt} = \frac{d^2\mathbf{v}}{dt^2} = \frac{d^3\mathbf{x}}{dt^3} \tag{7}$$

The measurement of jerk is used in many sensors, for example, in accelerometers in airbags to determine if the acceleration of a car is under normal operation (small j value) or if a crash is in progress (high j value). The time derivative of jerk is **jounce**, or **snap**, with unit of m/s^4:

$$\mathbf{s}(t) = \frac{d\mathbf{j}}{dt} = \frac{d^2\mathbf{a}}{dt^2} = \frac{d^3\mathbf{v}}{dt^3} = \frac{d^4\mathbf{x}}{dt^4} \tag{8}$$

The next two derivatives of snap is facetiously called **crackle** and **pop**[10], but these higher derivatives are rarely used, and will *not* be used in in AP Physics.

10 Kinematic Equations for Constant Acceleration

Although kinematic problems in AP Physics often require calculus[11], basic kinematic equations for *constant* acceleration are still a very powerful tool. For constant acceleration **a**, velocity can be

[10] As in the cartoon mascots for Kellogg's rice crispies

[11] Unlike your AP Calculus exams, the differentiation/integration in AP Physics will be fairly straightforward

obtained by integrating in time:

$$\mathbf{v}(t) = \int \mathbf{a}dt \quad \rightarrow \quad \boxed{\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t} \tag{9}$$

where $\mathbf{v}_0$ is the initial velocity at $t = 0$. Integrating again for the position vector:

$$\mathbf{x}(t) = \int \mathbf{v}dt \quad \rightarrow \quad \boxed{\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2} \tag{10}$$

where $\mathbf{x}_0$ is the initial position at $t = 0$. The position vector is quadratic in time.

The derivation of the last equation is slightly more laborious. From Eqs. 9 and 10, if acceleration is constant, then both the the velocity and position vectors are continuously differentiable. In a one-dimensional problem[12], the differentiation can be expressed as:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$$
$$a = v\frac{dv}{dx}$$

Multiplying both sides by dx and integrating, we have:

$$\int_{x_0}^{x} a dx = \int_{v_0}^{v} v dv$$
$$a(x - x_0) = \frac{1}{2}\left(v^2 - v_0^2\right)$$

or in the more familiar form:

$$\boxed{v^2 = v_0^2 + 2a(x - x_0)} \tag{11}$$

Eqs. 9, 10 and 11 are provided in the AP Exam equation sheet.

[12] for simplicity