Topic 12: Capacitors

Advanced Placement Physics C

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Last Updated: April 13, 2021

Olympiads School

Parallel-Plate Capacitors

Electric Field and Electric Potential Difference

Recall that the relationship between electrostatic force ($\boldsymbol{F}_q$) and electric potential energy (U_q) can be expressed using definition of mechanical work and the fundamental theorem of calculus:

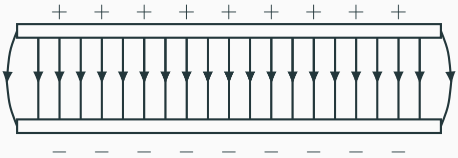
$$\Delta U_q = -\int \boldsymbol{F}_q \cdot \mathrm{d}\boldsymbol{r} \qquad \boldsymbol{F}_q = -\nabla U_q = -\frac{\partial U_q}{\partial r}\hat{\boldsymbol{r}}$$

Dividing both sides of the equations by q, we get the relationship between electric field ($\boldsymbol{E}$), electric potential (V) and electric potential difference (ΔV):

$$\Delta V = -\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} \qquad \boxed{\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{\boldsymbol{r}}}$$

This relationship holds regardless of the charge configuration.

Electric Field and Electric Potential Difference



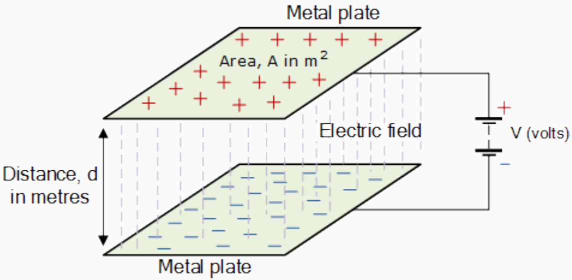
Recall that for two charged parallel plates, the electric field is uniform, and the relationship between electric field and potential difference simplifies to:

$$\boxed{E = \frac{\Delta V}{d}} \quad \text{or} \quad \boxed{\Delta V = Ed}$$

Quantity	**Symbol**	**SI Unit**
Electric field intensity	E	N/C
Potential difference between plates	ΔV	V
Distance between plates	d	m

Capacitors

Capacitors is a device that stores energy in an electric field. The simplest form of a capacitor is a set of closely spaced parallel plates:



When the plates are connected to a battery, the battery transfer charges to the plates until the voltage V equals the battery terminals. After that, one plate has charge $+Q$; the other has $-Q$.

Parallel-Plate Capacitors

As we have seen already, the (uniform) electric field between two parallel plates is proportional to the charge density σ, which is the charge Q divided by the area of the plates A:

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$$

Substituting this into the relationship between the plate voltage V and electric field, we find the relationship between the charges across the plates and the voltage:

$$\Delta V = Ed = \frac{Qd}{A\epsilon_0} \quad \longrightarrow \quad \boxed{Q = \left[\frac{A\epsilon_0}{d}\right]\Delta V}$$

Parallel-Plate Capacitors

Since area A, distance of separation d and the vacuum permittivity ϵ_0 are all constants, the relationship between charge Q and voltage ΔV is *linear*. And the constant is called the **capacitance** C, defined as:

$$\boxed{C = \frac{Q}{\Delta V}}$$

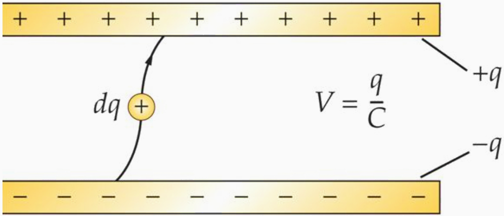
For parallel plates:

$$\boxed{C = \frac{A\epsilon_0}{d} \quad \text{parallel plate}}$$

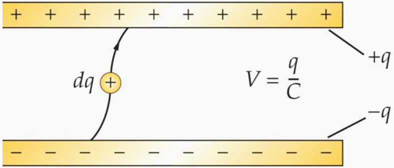
The unit for capacitance is a **farad** (named after Michael Faraday), where $1\,\mathrm{F} = 1\,\mathrm{C/V}$.

Storage of Electrical Energy

When charging up a capacitor, imagine positive charges moving from the negatively charged plate to the positively charged plate



Storage of Electrical Energy



In the beginning—when the plates aren't charged—moving an infinitesimal charge dq across the plates, the infinitesimal work done dU is very small and related to the capacitance by:

$$dU = Vdq = \frac{q}{C}dq$$

As the electric field begins to form between plates, more and more work is required to move the charges.

Storage of Electrical Energy

To fully charge the plates, the total work U_c is the integral:

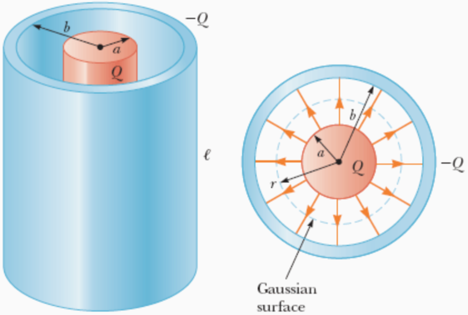
$$U_c = \int \mathrm{d}U = \int_0^Q \frac{q}{C}\mathrm{d}q = \frac{1}{2}\frac{Q^2}{C}$$

The work done is stored as a potential energy inside the capacitor. There are different ways to express U_c using definition of capacitance:

$$\boxed{U_c = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2}$$

Cylindrical Capacitors

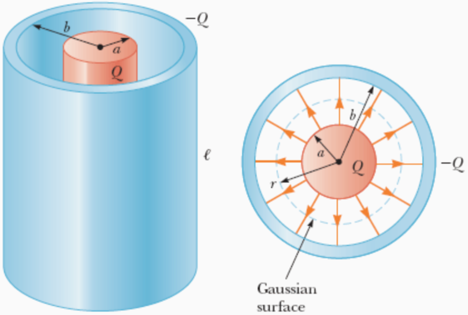
Cylindrical Capacitors



Not all capacitors are parallel plates. Cylindrical capacitors are also popular.

- The capacitor has length ℓ which is much larger than the radii of the inner & outer cylinders (a, b)
- Inner cylinder has total charge Q
- Outer cylinder has total charge $-Q$
- Inside the capacitor, the electric field in the radial direction
- Outside of the capacitor, there is no electric field

Cylindrical Capacitors: Electric Field



Use Gauss's law to find the electric field between the cylinders, by placing a Gaussian surface of radius r between the cylinders:

$$\oint \mathbf{E} \cdot d\mathbf{A} = 2\pi r L E = \frac{Q}{\epsilon_0}$$

which gives the expression:

$$E = \frac{Q}{2\pi r L \epsilon_0} \quad \text{or} \quad E = \frac{\lambda}{2\pi r \epsilon_0}$$

where $\lambda = Q/L$ is the linear charge density

Cylindrical Capacitors: Voltage Across the Cylinders

Integrating the eclectic field to get voltage across the plates:

$$\Delta V = -\int_a^b E dr = \frac{Q}{2\pi L \epsilon_0} \int_b^a \frac{1}{r} dr = \frac{Q}{2\pi L \epsilon_0} \ln\left[\frac{b}{a}\right]$$

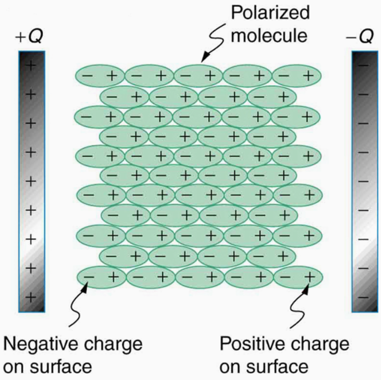
Like the parallel plate, the relationship between voltage and charge is still linear, but in this case, the capacitance is defined as:

$$C = \frac{Q}{\Delta V} = \frac{2\pi L \epsilon_0}{\ln(b/a)} \quad \longrightarrow \quad \boxed{\frac{C}{L} = \frac{2\pi \epsilon_0}{\ln(b/a)} \quad \text{cylindrical}}$$

The capacitance is generally expressed by C/L (unit F/m). Like the parallel-plate, the capacitance of the cylindrical capacitor also only depends on the geometry (i.e. the raidii a and b) and the permittiviy.

Practical Capacitors

Practical Capacitors



- Capacitors (both parallel-plate and cylindrical) are very common in electric circuits, but the vacuum between the plates is not very effective
- Instead, a non-conducting **dielectric** material is inserted between the plates
- When the plates are charged, the electric field of the plates polarizes the dielectric.
- The polarization produces an electric field that opposes the field from the plates, therefore reduces the effective voltage, and increasing the capacitance

Dielectric Constant

If electric field without dielectric is E_0, then E in the dielectric is reduced by κ, the **dielectric constant**:

$$\boxed{\kappa = \frac{E_0}{E}}$$

The capacitance of the plates with the dielectric is now amplified by the same factor κ:

$$\boxed{C = \kappa C_0}$$

We can also view the dielectric as something that increases the *effective permittivity*:

$$\boxed{\epsilon = \kappa \epsilon_0}$$

Dielectric Constant

The dielectric constants of commonly used materials are:

Material	κ
Air	1.000 59
Bakelite	4.9
Pyrex glass	5.6
Neoprene	6.9
Plexiglas	3.4
Polystyrene	2.55
Water (20 °C)	80

Notes About Storage of Electric Energy

The work done (i.e. the energy stored in the capacitor) is inversely proportional to the capacitance:

$$\mathrm{d}U = V\mathrm{d}q = \frac{q}{C}\mathrm{d}q$$

- The presence of a dielectric *increases* the capacitance; therefore the work (and potential energy stored) to move the charge dq *decreases* with the dielectric constant κ
- After the capacitor is charged, removing the dielectric material from the capacitor plates will require additional work.

Capacitors in Electric Circuits

Capacitors are an important part of an electric circuits because it stores energy in the electric field

- Denoted by this symbol (with reference to the parallel-plate capacitor):

- Act like a voltage source
- Unlike a battery, the voltage increases or decreases as the charge across the capacitor plates change.