Topic 13: Circuit Analysis

Advanced Placement Physics

Dr. Timothy Leung

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Olympiads School

Electric Current

Current

We define conventional current as the rate at which charge carriers pass through a point in a circuit:

$$\boxed{I(t) = \frac{dQ}{dt}}$$

We can also think of the current as the flux of "charge density" flowing past a surface, i.e.:

$$\boxed{I(t) = \int \mathbf{J}(t) \cdot d\mathbf{A}}$$

$d\mathbf{A}$ is defined the same way as fluxes: its magnitude is the infinitesimal area dA and the direction is the outward normal from the surface.

Current

- We now know that in a wire, instead of positive charges flowing in one direction, we have, in fact, electrons (negative charges) flowing in the opposite direction
- As charges move through a conductor, they will lose potential energy
- How much energy it loses depends on the resistance of the material

Resistors

Resistivity

The resistivity of a material is proportional to the electric field and current density:

$$\boxed{\mathbf{E} = \rho \mathbf{J}} \qquad \text{Scalar:} \qquad \boxed{\rho = \left| \frac{E}{J} \right|}$$

Quantity	Symbol	SI Unit
Electric field	$\mathbf{E}$	N/C
Charge density	$\mathbf{J}$	A/m^2
Resistivity	ρ	Ω/m

Resistivity

$$\boxed{\mathbf{E} = \rho \mathbf{J}} \qquad \text{Scalar:} \quad \boxed{\rho = \left|\frac{E}{J}\right|}$$

- In a conductor, because the electrons are free to move, the electric field tend to be very small, and the resistivity is low.
- In a dielectric (non conducting material), electrons cannot move easily–only polarize themselves–the electric field are generally strong, and the resistivity is higher.

Resistance of a Conductor

The resistance of a conductor is proportional to the resistivity ρ and its length L, and inversely proportional to the cross-sectional area A:

$$\boxed{R = \rho \frac{L}{A}}$$

Quantity	**Symbol**	**SI Unit**
Resistance	R	Ω
Resistivity	ρ	$\Omega\,\mathrm{m}$ (ohm metres)
Length of conductor	L	m (metres)
Cross-sectional area	A	m^2 (square metres)

Resistance of a Conductor

$$\boxed{R = \rho \frac{L}{A}}$$

Gauge	Diameter (mm)	R/L ($10^{-3}\,\Omega/\mathrm{m}$)
0	9.35	0.31
10	2.59	2.20
14	1.63	8.54
18	1.02	21.90
22	0.64	51.70

Material	Resistivity ρ ($\Omega\,\mathrm{m}$)
silver	1.6×10^{-8}
copper	1.7×10^{-8}
aluminum	2.7×10^{-8}
tungsten	5.6×10^{-8}
Nichrome	100×10^{-8}
carbon	3500×10^{-8}
germanium	0.46
glass	10^{10} to 10^{14}

Ohm’s Law

Ohm's Law

The electric potential difference V across a "load" (resistor) equals the product of the current I through the load and the resistance R of the load.

$$\boxed{V = IR}$$

Quantity	**Symbol**	**SI Unit**
Potential difference	V	V
Current	I	A
Resistance	R	Ω

A resistor is considered "ohmic" if it obeys Ohm's law

Power Dissipated by a Resistor

Power is the rate at which work W is done, and from electrostatics, the change in electric potential energy ΔE_q (i.e. the work done!) is proportional to the amount of charge q and the voltage V. This gives a very simple expression for power through a resistor:

$$P = \frac{dW}{dt} = \frac{d(qV)}{dt} = \left(\frac{dq}{dt}\right) V \rightarrow \boxed{P = IV}$$

Quantity	**Symbol**	**SI Unit**
Power through a resistor	P	W
Current through a resistor	I	A
Voltage across the resistor	V	V

Other Equations for Power

When we combine Ohm's Law ($V = IR$) with power equation, we get two additional expressions for power through a resistor:

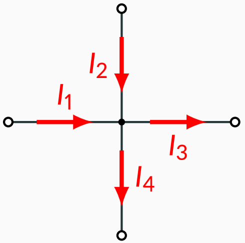
$$\boxed{P = \frac{V^2}{R}} \qquad \boxed{P = I^2 R}$$

Quantity	Symbol	SI Unit
Power	P	W
Voltage	V	V
Resistance	R	Ω
Current	I	A

Kirchkoff’s Laws

Kirchkoff's Current Law

The electric current that flows into any junction in an electric circuit must be equal to the current which flows out.



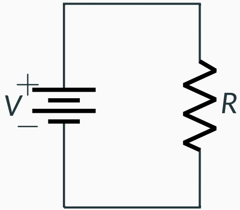
e.g. if there are 4 paths to the junction at the center, with I_1 and I_2 going into the junction, and I_3 and I_4 coming out, then the current law says that

$$I_1 + I_2 - I_3 - I_4 = 0$$

Basically, it means that there cannot be any accumulation of charges anywhere in the circuit. The law is a consequence of conservation of energy.

Kirchkoff's Voltage Law

The voltage changes around any closed loop in the circuit must sum to zero, no matter what path you take through an electric circuit.



Assume that the current flows clockwise and we draw a clockwise loop, we get

$$V - V_R = 0 \quad \rightarrow \quad V - IR = 0$$

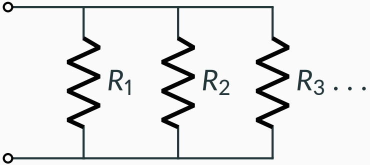
If I incorrectly guess that I flows counterclockwise, I will still have a similar expression

$$-V_R - V = 0 \quad \rightarrow \quad -V - IR = 0$$

When solving for I, we get a negative number, indicating that my guess was in the wrong direction.

Resistors in Circuits

Resistors in Parallel



From the current law, we know that the total current is the current through all the resistors, which we can rewrite in terms of voltage and resistance using Ohm's law:

$$I = I_1 + I_2 + I_3 \cdots = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} \cdots$$

But since we also know that $V_1 = V_2 = V_3 = \cdots = V$ from the voltage law, we can re-write as

$$I = \frac{V}{R_{\mathrm{eq}}} = V \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \cdots \right)$$

Resistors in Parallel

Through applying Ohm's Law and Kirchkoff's laws, we find the equivalent resistance of a parallel circuit, which we have known since Grade 9: The inverse of the equivalent resistance for resistors connected in parallel is the sum of the inverses of the individual resistances.

$$\boxed{\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}}$$

Quantity	Symbol	SI Unit
Equivalent resistance in parallel	R_p	Ω
Resistance of individual loads	$R_{1,2,3,\cdots,N}$	Ω

Resistors in Series

The analysis for resistors in series is similar (but easier). From the current law, the current through each resistor is the same:

$$I_1 = I_2 = I_3 = \cdots = I$$

And the total voltage drop across all resistor is therefore:

$$V = V_1 + V_2 + V_3 + \cdot = I\big(R_1 + R_2 + R_3 + \cdots\big)$$

Resistors in Series

Again, through applying Ohm's Law and Kirchkoff's laws, we find that when resistors are connected in series: the equivalent resistance of loads is the sum of the resistances of the individual loads.

$$\boxed{R_s = \sum_{i=1}^{N} R_i}$$

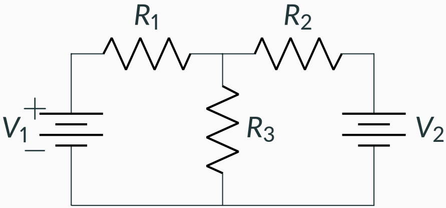
Quantity	Symbol	SI Unit
Equivalent resistance in series	R_s	Ω
Resistance of individual loads	$R_{1,2,3,\cdots,N}$	Ω

Tips for Solving “Simple” Circuit Problems

1. Identify groups of resistors that are in parallel or in series, and find their equivalent resistance.
2. Gradually reduce the entire circuit to one voltage source and one resistor.
3. Using Ohm’s law, find the current out of the battery.
4. Using Kirchkoff’s laws, find the current through each of the resistors.

Circuits Aren't Always Simple

Some of these problems require you to solve a system of linear equations. The following is a simple example with two voltage sources:



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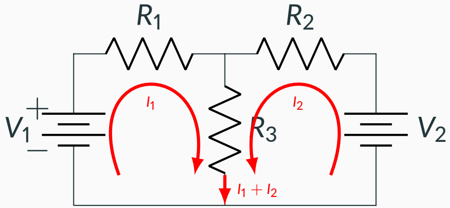
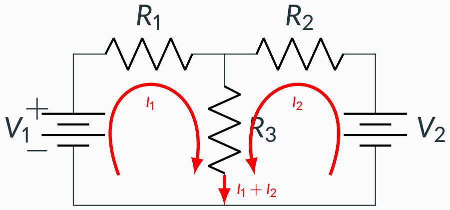


In this case, we have to draw two loops of current.

A More Difficult Example



We split the circuit into two loops, and apply Kirchkoff's voltage in both:

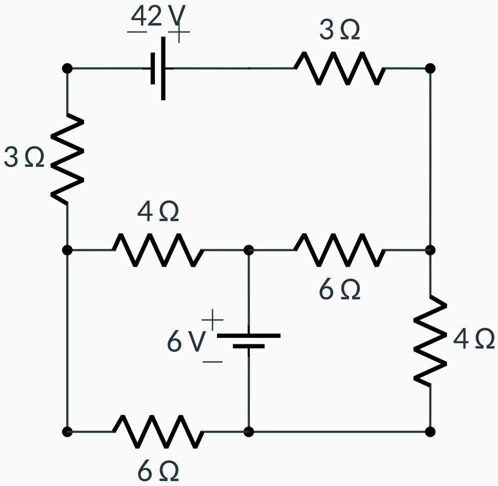
$$V_1 - I_1 R_1 - (I_1 + I_2) R_3 = 0$$
$$V_2 - I_2 R_2 - (I_1 + I_2) R_3 = 0$$

Two equations, two unknowns (I_1 and I_2). We can subtract (2) from (1), then solve for I_1 and I_2:

$$I_1 = \frac{V_1 - I_2 R_3}{R_1 + R_3} \qquad I_2 = \frac{\left[V_2 - \frac{(V_1 - V_2) R_3}{R_1}\right]}{\left[R_2 + \frac{(R_1 + R_2) R_3}{R_1}\right]}$$

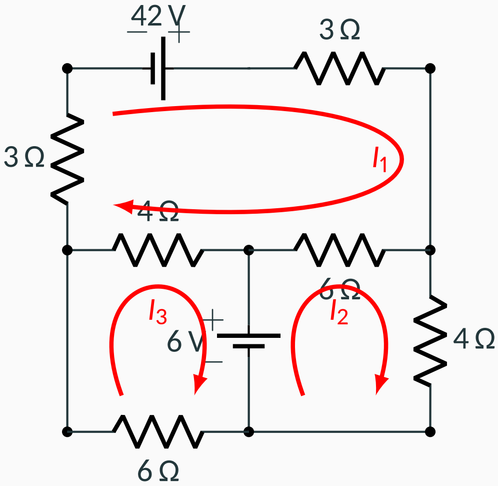
(Try this at home as an exercise.)

This is As Difficult As It'll Get



- To solve this problem, we define a few “loops” around the circuit: one on top, one on bottom left, and one on bottom right.

This is As Difficult As It'll Get



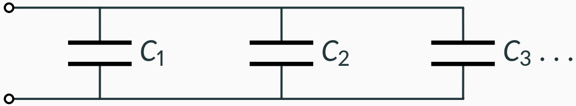
- To solve this problem, we define a few "loops" around the circuit: one on top, one on bottom left, and one on bottom right.
- Apply the voltage law in the loops. For example, in the lower left:

$$4(I_1 - I_3) - 6 - 6I_3 = 0$$

- Solve the linear system to find the current. If the current that you worked out is negative, it means that you have the direction wrong.

Capacitors in Circuit

Capacitors in Parallel



From the voltage law, we know that the voltage across all the capacitors are the same, i.e. $V_1 = V_2 = V_3 - \cdots = V$. We can express the total charge Q_{tot} stored across all the capacitors in terms of capacitance and this common voltage V:

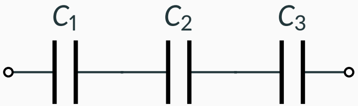
$$Q_{\mathrm{tot}} = Q_1 + Q_2 + Q_3 + \cdots = C_1 V + C_2 V + C_3 V + \cdots$$

Factoring out V from each term gives us the equivalent capacitance:

$$\boxed{C_p = \sum_i C_i}$$

Capacitors in Series

Likewise, we can do a similar analysis to capacitors connected in series.



The total voltage across these capacitors are the sum of the voltages across each of them, i.e. $V_{\mathrm{tot}} = V_1 + V_2 + V_3 + \cdots$

We recognize that the charge stored on all the capacitors must be the same! We can then write the total voltage in terms of capacitance and charge:

$$V_{\mathrm{tot}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \cdots$$

Capacitors in Series

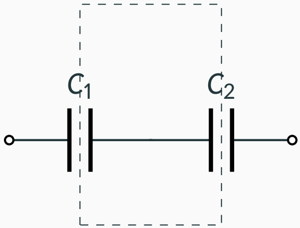
The inverse of the equivalent capacitance for N capacitors connected in series is the sum of the inverses of the individual capacitance.

$$\boxed{\frac{1}{C_s} = \sum_i \frac{1}{C_i}}$$

Make sure we don't confuse ourselves with resistors.

How Do We Know That Charges Are The Same?

It's simple to show that the charges across all the capacitors are the same

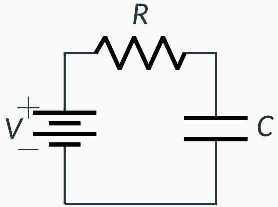


The capacitor plates and the wire connecting them are really one piece of conductor. There is nowhere for the charges to leave the conductor, therefore when charges are accumulating on C_1, C_2 must also have the same charge because of conservation of charges.

R-C Circuits

Circuits with Resistors and Capacitors

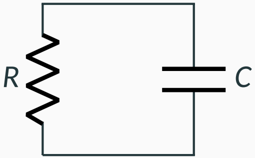
Now that we have seen how resistors and capacitors behave in a circuit, we can look into combining them in to an “R-C circuit”.



The simplest form is a resistor and capacitor connected in series, and then connect to a voltage source. Because of the nature of capacitors, the current through the circuit will not be steady as were the case with only resistors.

Discharging a Capacitor

We start the analysis with something even simpler. There is no voltage source, and the capacitor is already charged to $V_c = Q_{\mathrm{tot}}/C$. What happens when the current begin to flow?



As current starts to flow, the charge on the capacitor decrease. Over time the current decreases, until the capacitor is fully discharged, and current stops flowing.

Apply the voltage law for the circuit, and substitute the definition of current $I = dQ/dt$ and the voltage across a capacitor $V_c = Q/C$:

$$V_c - IR = 0 \quad \rightarrow \quad -\frac{Q}{C} - R\frac{dQ}{dt} = 0$$

Discharging a Capacitor

Separating the Q terms on the left side of the equation, and leaving everything else on the right side, we get:

$$\frac{dQ}{Q} = \frac{-dt}{RC}$$

which we can now integrate and "exponentiate":

$$\int \frac{dQ}{Q} = \int \frac{-dt}{RC} \;\rightarrow\; \ln Q = \frac{-t}{RC} + K \;\rightarrow\; Q = e^K e^{-t/RC}$$

The constant of integration K is the initial charge on the capacitor Q_{tot}:

$$e^K = Q_{\mathrm{tot}}$$

Discharging a Capacitor

The expression of charge across the capacitor is time-dependent:

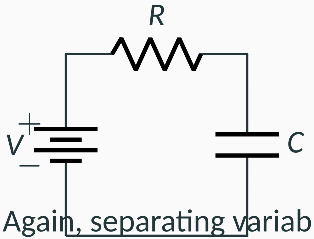
$$\boxed{Q(t) = Q_0 e^{-t/\tau}}$$

where $Q_0 = Q_{\mathrm{tot}}$ is the initial charge on the capacitor, and $\tau = RC$ is called the **time constant**. Taking the time derivative of $Q(t)$ gives us the current through the circuit:

$$\boxed{I(t) = \frac{dQ}{dt} = I_0 e^{-t/\tau}}$$

where the initially current at $t = 0$ is given by $I_0 = Q_{\mathrm{tot}}/\tau = Q_{\mathrm{tot}}/RC$.

Charging a Capacitor



In charging up the capacitor, we go back to our original circuit, and apply the voltage law, then substitute the expression for current and voltage across the capacitor:

$$V - R\frac{dQ}{dt} - \frac{Q}{C} = 0$$

Again, separating variables, and integrating, we get:

$$\int \frac{dQ}{CV - Q} = \int \frac{-dt}{RC} \quad \rightarrow \quad \ln(CV - Q) = \frac{-t}{RC} + K$$

Charging a Capacitor

"Exponentiating" both sides, we have

$$CV - Q = e^K e^{-t/RC}$$

To find the constant of integration K, we note that at $t = 0$, the charge across the capacitor is 0, and we get

$$e^K = CV = Q_{\mathrm{tot}}$$

which is the charge stored in the capacitor at the end. Substitute this back into the equation:

$$\boxed{Q(t) = Q_{\mathrm{tot}}\left(1 - e^{-t/RC}\right)}$$

Capacitors

$$\boxed{Q(t) = Q_{\text{tot}}(1 - e^{-t/\tau})}$$

Charging a capacitor has the same time constant $\tau = RC$ as during discharge. We can also differentiate to find the current through the circuit; it is identical to the equation for discharge:

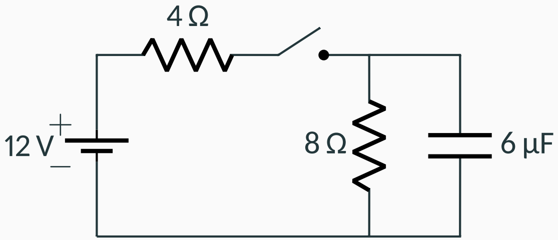
$$\boxed{I(t) = \frac{dQ}{dt} = I_0 e^{-t/\tau}}$$

where the initial current is given by $I_o = Q_{\text{tot}}/\tau = V/R$. This makes sense because $V_C(t=0) = 0$, and all of the energy must be dissipated through the resistor. Similarly at $I(t=\infty) = 0$.

Two Small Notes

1. When a capacitor is uncharged, there is no voltage across the plate, it acts like a short circuit.
2. When a capacitor is charged, there is a voltage across it, but no current flows *through* it. Effectively it acts like an open circuit.

A Slightly More Difficult Problem



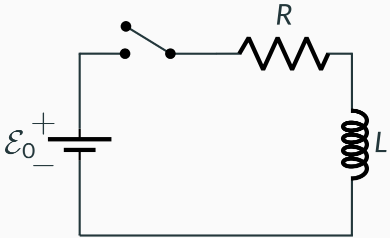
Example: The capacitor in the circuit is initially uncharged. Find the current through the battery

1. Immediately after the switch is closed
2. A long time after the switch is closed

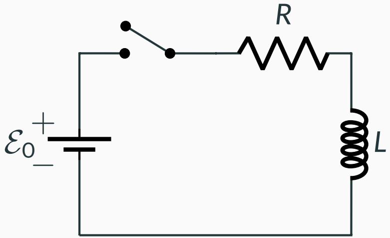
LR Circuits

Circuits with Inductors

- Coils and solenoids in circuits are known as “inductors” and have large self inductance L
- Self inductance prevents currents rising and falling instantaneously
- A basic circuit containing a resistor and an inductor is called an ***LR* circuit**:



Analyzing LR Circuits



Applying Kirchkoff's voltage law:

$$\mathcal{E}_0 - IR - L\frac{dI}{dt} = 0$$

We follow the same procedure as charging a capacitor to find the time dependent current:

$$I = \frac{\mathcal{E}_0}{R}\left(1 - e^{-Rt/L}\right)$$

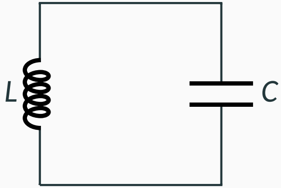
The time constant for an *LR* circuit is

$$\tau = \frac{L}{R}$$

LC Circuit

LC Circuit

The final type of circuit in AP Physics is the LC circuit. In its simplest form, the circuit has an inductor and capacitor connected in series:



We apply the Kirchkoff's voltage law:

$$-V_L - V_C = 0 \quad \rightarrow \quad L\frac{dI}{dt} + \frac{Q}{C} = 0$$

LC Circuits

Since both terms are continuously differentiable, we can differentiate both sides of the equation, which gives:

$$L\frac{d^2I}{dt^2} + \frac{1}{C}\underbrace{\frac{dQ}{dt}}_{I} = 0$$

In fact, the above equation a second-order ordinary differential equation with constant coefficients. The solution to such an eqauation is the simple harmonic motion.

$$\frac{d^2I}{dt^2} + \frac{1}{LC}I = 0$$

LC Circuit

The current inside of an LC circuit is given by:

$$I(t) = I_0 \sin(\omega t + \varphi) \quad \text{where} \quad \omega = \frac{1}{\sqrt{LC}}$$