Topic 3: Work and Energy

Advanced Placement Physics C

Dr. Timothy Leung

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Olympiads School

Work and Energy

We start with some definition at are (unfortunately) not very useful:

- **Energy** is the ability to do work.
- **Work** is the mechanism in which energy is transformed.

Luckily, we can also use equations to define these concepts.

Work

Work

Mechanical work dW is performed when a force $\boldsymbol{F}$ displaces an object by $d\boldsymbol{x}$. If a varying force is applied to move an object from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ along a path, then the total work done by the force is defined by the integral:

$$W = \int \boldsymbol{F}(\boldsymbol{x}) \cdot d\boldsymbol{x}$$

- Work is a scalar quantity
- No work done if the force is perpendicular to displacement, when $\boldsymbol{F} \cdot d\boldsymbol{x} = 0$ (i.e. the force did not cause the displacement)
- No work done if no displacement ($d\boldsymbol{x} = \mathbf{0}$)
- Work can be positive or negative depending on the dot product
- When there are multiple forces acting on an object, we can compute the work done by each *each* force

Work by Constant Force

For a constant force, if the object moves along straight path, the integral simplifies to just the dot product of the two vectors:

$$\boxed{W = \boldsymbol{F} \cdot \Delta\boldsymbol{x}}$$

Or in the scalar form that is more familiar in Grades 11/12 Physics that avoid vector notations:

$$\boxed{W = F\Delta x \cos\theta}$$

where θ is the angle between the force and displacement vectors

Definition of Work

Work done by a force

- The work done by *one specific force*
- Example: A boy pushes a cart forward. The "work done by the boy" is the work done by the applied force.

Work done on an object

- There may be more than one force acting on an object
- The *sum* of all the work done on the object by each force
- The work done by the net force
- Also called the **net work** W_{net}

Kinetic Energy

Kinetic Energy

When a net force on an object (with constant mass) accelerates it, the resulting amount of work done on the object (net work W_{net}) is given by:

$$W_{\text{net}} = \int \boldsymbol{F}_{\text{net}} \cdot \mathrm{d}\boldsymbol{x} = \int m\boldsymbol{a} \cdot \mathrm{d}\boldsymbol{x} = m \int \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{x}$$

Both $\boldsymbol{v}$ and $\boldsymbol{x}$ are continuous functions in time, we can switch the order of differentiation. Also, since $\boldsymbol{v}$ and $d\boldsymbol{v}$ must be in the same direction, the dot product is trivial: $\boldsymbol{v} \cdot d\boldsymbol{v} = v dv$

$$= m \int \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{v} = m \int \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{v} = m \int_{v_1}^{v_2} v\mathrm{d}v$$

Kinetic Energy

This integral, when integrated from v_1 (initial *speed*) to v_2 (final *speed*), becomes:

$$= m\int_{v_1}^{v_2} v dv = \frac{1}{2}mv^2\Big|_{v_1}^{v_2} = \Delta K$$

where K is defined as the **translational kinetic energy**:

$$\boxed{K = \frac{1}{2}mv^2}$$

Later in the course we will discuss *rotational* kinetic energy.

Work and Kinetic Energy

In fact, the *definition* of kinetic energy came from this integration, in that work equals to the change in *something*, and we define that as kinetic energy. This is the **work-energy theorem**:

$$\boxed{W_{net} = \Delta K}$$

- ΔK can be positive or negative depending on the dot product
- When multiple forces acting on an object; each force can add or remove kinetic energy from an object
- Therefore we use the "net" amount of work done in the above equation

Example

Example 1: A force $\boldsymbol{F} = 4.0x\hat{\imath}$ (in newtons) acts on an object of mass 2.0 kg as it moves from $x = 1.0$ to $x = 5.0$ m. Given that the object is at rest at $x = 1$,

(a) Calculate the net work

(b) What is the final speed of the object?

Potential Energy

Gravitational Force & Gravitational Potential Energy

The gravitational force (weight) of an object is defined as:

$$\boldsymbol{w} = m\boldsymbol{g}$$

Near Earth's surface, we assume that $\boldsymbol{g} = -g\hat{\jmath} \approx -10\hat{\jmath}\ \mathrm{m/s^2}$ is constant. The work done to move an object from height h_1 to h_2 is therefore:

$$W = \int \boldsymbol{w} \cdot \mathrm{d}\boldsymbol{h} = \int_{h_1}^{h_2} -mg\hat{\jmath} \cdot \mathrm{d}h\hat{\jmath} = -mgh\Big|_{h_1}^{h_2} = -\Delta U_g$$

where U_g is defined as the **gravitational potential energy**:

$$\boxed{U_g = mgh}$$

where $h = 0$ (and therefore $U_g = 0$) is called the **reference level**.

Spring Force & Elastic Potential Energy

The spring force $\boldsymbol{F_e}$ is the force that a compressed/stretched spring exerts on the object connected to it. It obeys Hooke's law:

$$\boldsymbol{F_e} = -k\boldsymbol{x}$$

We can find the work done to displace a spring:

$$W = \int \boldsymbol{F_e} \cdot \mathrm{d}\boldsymbol{x} = -k \int x\mathrm{d}x = -\frac{1}{2}kx^2\Big|_{x_1}^{x_2} = -\Delta U_e$$

where U_e is defined as the **elastic potential energy**:

$$\boxed{U_e = \frac{1}{2}kx^2}$$

Conservative Forces

Gravitational force, spring force, electrostatic force (later in the course) are called **conservative forces**

- The work done by these forces relate to a change of another quantity called *potential energy*
- Since the potential energy is evaluated at the end points, the work done by a conservative force is *path independent*

Conservative Forces

Potential energies are obtained by integrating the work done by the conservative forces, so by the fundamental theorem of calculus, these forces are therefore the negative gradient of the potential energies:

$$\boldsymbol{F} = -\nabla U = -\frac{\partial U}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial U}{\partial y}\hat{\boldsymbol{\jmath}} - \frac{\partial U}{\partial z}\hat{\boldsymbol{k}}$$

The direction of a conservative force *always* decreases the potential energy. In one-dimension:

$$\boxed{F = -\frac{\mathrm{d}U}{\mathrm{d}x}}$$

Pay attention to the negative sign. Students often forget it.

Work and Potential Energy

The expressions for potential energies also come from integrating the work equation, in that work equals to the change in *something*, and we called that potential energy. Therefore:

$$\boxed{W_c = -\Delta U}$$

- ΔU can be positive or negative depending on the direction of the (conservative) force
- Positive work *decreases* the related potential energy
- Negative work *increases* the related potential energy

Conservation of Mechanical Energy

Positive work done by conservative forces on an object does two things:

1. Decrease its potential energy, while
2. Increase its kinetic energy by the same amount

Mathematically, this shows that mechanical energy must *always* be conserved when there are only conservative forces:

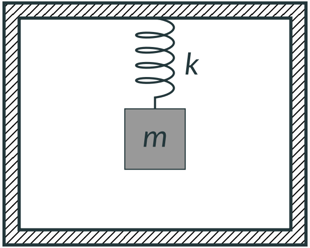
$$W_c = -\Delta U = \Delta K \quad \longrightarrow \quad \Delta K + \Delta U = 0$$

That's why those forces are called conservative forces!

Conservation of Energy

Isolated Systems and the Conservation of Energy

An **isolated system** is a system of objects that does not interact with the surrounding. Think of an isolated system as a bunch of objects inside an insulated box.

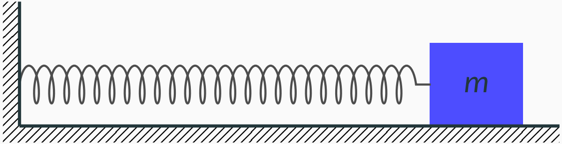


Isolated Systems and Conservation of Energy

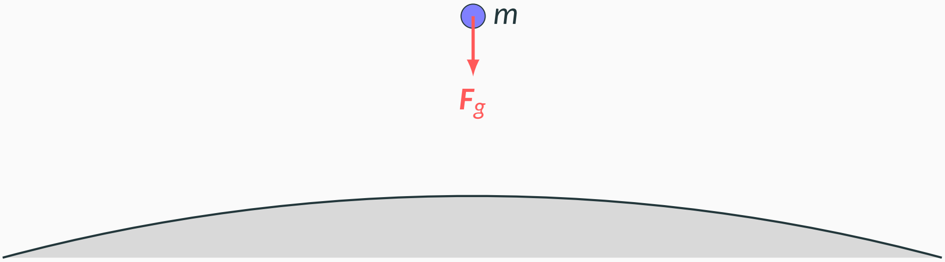
- Since the system is isolated from the surrounding environment, the environment can't do any work on it, by definition!
- Likewise, the energy inside the system cannot escape either
- Therefore energy of the system is conserved
- There are *internal* force inside the system that is doing work, but the work only converts kinetic energy into potential energies, and vice versa.

Example: Mass sliding on a spring

- Assuming that there is no friction in any part of the system
- The isolated system consists of the mass and the spring
- Energies:
  - Kinetic energy of the mass
  - Elastic potential energy stored in the spring

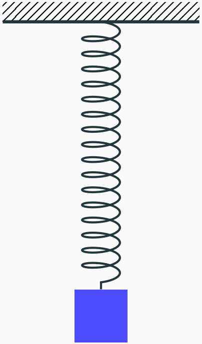


Example: Gravity



- The isolated system consists of the mass and Earth
- Assuming no friction
- Energies:
  - Kinetic energy of the mass
  - Gravitational potential energy of the mass

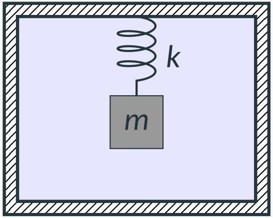
Example: Vertical Spring-Mass System



- The system consists of a mass, a spring and Earth
- Energies:
  - Kinetic energy of the mass
  - Gravitational potential energy of the mass
  - Elastic potential energy stored in the spring
- The total energy of the system is conserved if there is no friction

What if there is friction?

Energy is always conserved as long as your system is defined properly



- The system consists of a mass, a spring, Earth and all the air particles inside the box
- As the mass vibrates, friction with air slows it down
- While the mass loses energy, the temperature of the air rises due to friction
- Energies:
    - Kinetic and gravitational potential energies of the mass
    - Elastic potential energy stored in the spring
    - Kinetic energy of the vibration of the air molecules
- Total energy is conserved even as the mass stops moving

Conservation of Energy

If *only* conservative forces are doing work, mechanical energy (i.e. $K + U$) is always conserved:

$$\boxed{K + U = K' + U'}$$

When non-conservative forces are also doing work, instead of *trying* to isolate the system, we can calculate the work done by them W_{nc} and add it to the total energy of the system

$$\boxed{K + U + W_{nc} = K' + U'}$$

Work By Non-Conservative Force

Some examples of non-conservative forces:

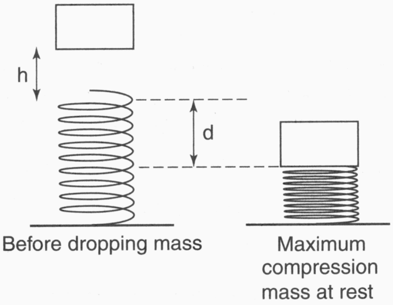
- Work done by these forces are *usually* negative because they oppose the direction of motion
  - Drag (fluid resistance)
  - Friction[1]
- The work done by these forces may be positive or negative, depending on the problem
  - Applied force
  - Tension force
  - Normal force

Note that the work-kinetic energy theorem still applies when non-conservative forces are present

[1] but sometimes it can also do positive work too.

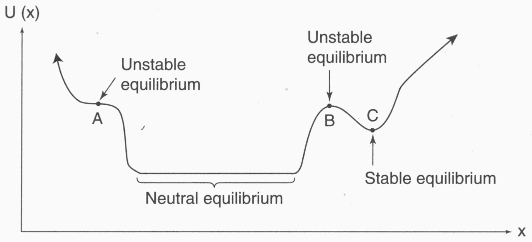
Example

Example 2: A mass m is dropped from a height of h above the equilibrium position of a spring. Set up the equation that determines the spring's compression d when the object is instantaneously at rest.



Energy Diagrams

- Plots of potential energy (U) vs. position for a conservative force



- If more than one conservative force, they can be combined into one graph
- Where slope is zero means no force acting on it: it is in a state of **equilibrium**
- An object placed at an equilibrium point with $K = 0$ will remain there

Power & Efficiency

Power

Power is the *rate* at which work is done, i.e. the rate at which energy is being transformed:

$$\boxed{P(t) = \frac{\mathrm{d}W}{\mathrm{d}t}} \qquad \boxed{\overline{P} = \frac{W}{\Delta t}}$$

Quantity	**Symbol**	**SI Unit**
Instantaneous and average power	P, $\overline{P}$	W
Work done	W	J
Time interval	Δt	s

In engineering, power is often more critical than the actual amount of work done.

Power

If a constant force is used to push an object at a constant velocity, the power produced by the force is:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{F} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \quad \rightarrow \quad \boxed{P = \boldsymbol{F} \cdot \boldsymbol{v}}$$

Application: aerodynamics

- When an object moves through air, the applied force must overcome air resistance (drag force), which is proportional with v^2
- Therefore "aerodynamic power" must scale with v^3 (i.e. doubling your speed requires $2^3 = 8$ times more power)
- Important when aerodynamic forces dominate

Efficiency

Effieiency is the ratio of useful energy or work output to the total energy or work input

$$\boxed{\eta = \frac{E_o}{E_i} \times 100\,\%} \qquad \boxed{\eta = \frac{W_o}{W_i} \times 100\,\%}$$

Quantity	Symbol	SI Unit
Useful output energy	E_o	J
Input energy	E_i	J
Useful output work	W_o	J
Input work	W_i	J
Efficiency	η	no units

Efficiency is always $0 \leq \eta \leq 100\,\%$