

# Topic 6: Circular Motion

## Advanced Placement Physics C

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Olympiads School

# Review of Circular Motion

In a **circular motion**, an object of mass  $m$  moves in a circular path about a fixed center. In Grade 12 Physics, you should have studied *uniform* circular motion, where:

- the object's speed (magnitude of velocity) is constant
- the object's **centripetal acceleration** is toward the center
- the object's acceleration is caused by a **centripetal force**

# Polar Coordinates

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# Polar Coordinate System in 2D

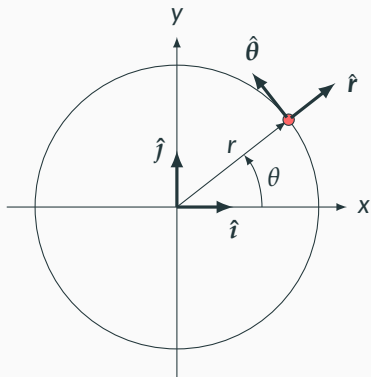
In the Cartesian coordinate system, an object's position is described by its  $x$  and  $y$  coordinates:

$$\mathbf{x}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$$

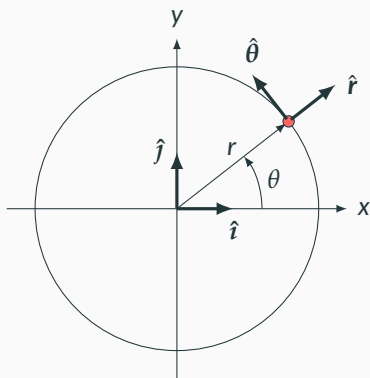
For circular motion or general rotational motion, the **polar coordinate system** is preferred. The position of an object is described by:

$$\mathbf{r}(t) = r(t)\hat{\mathbf{r}} + \theta(t)\hat{\boldsymbol{\theta}}$$

- $r$  is distance from the origin
- $\theta$  is the standard angle, measured counter clockwise from the  $x$  axis in *radians*



# Polar Coordinate System in 2D

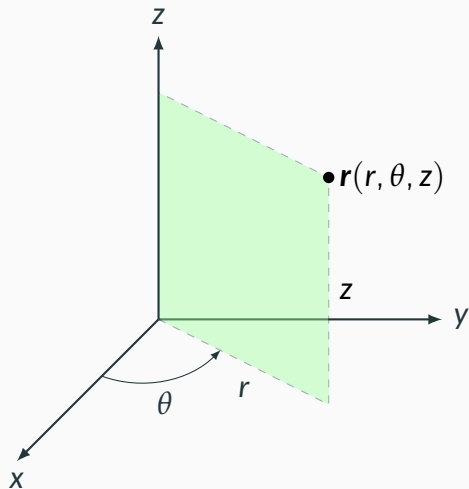


- Like the Cartesian system, the polar coordinate system is also right-handed
- Both basic vectors  $\hat{r}$  (radial direction) and  $\hat{\theta}$  (angular direction) rotate as the object moves
- Simpler way is to think of position as just two parameters, which is *exactly* how position vectors are expressed in Grade 11/12 Physics: magnitude ( $r$ ) and direction ( $\theta$ )!
- Cartesian and polar coordinates are related by:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

# Cylindrical Coordinates in 3D

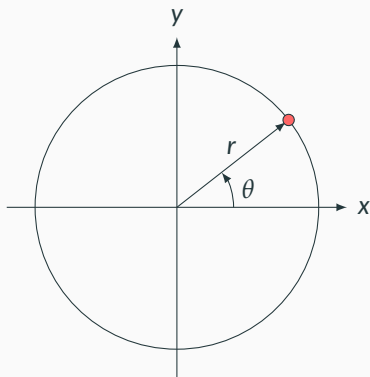


One way to extend the coordinate system into 3D is the **cylindrical coordinate system**. Note that the discussions for this topic focuses on  $xy$  plane. Since the  $z$ -axis is linearly independent of the  $xy$  plane, motion along that direction is independent.

# Rigid-Body Circular Motion

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# Angular Position and Angular Velocity



For a constant  $r$ , the **angular position**  $\theta$  determines an object's position as a function of time:

$$\theta = \theta(t)$$

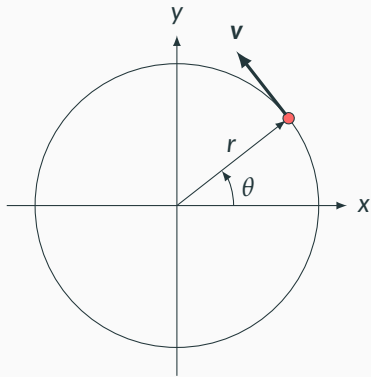
**Angular velocity**  $\omega$  (or **angular frequency**) is its time derivative:

$$\omega(t) = \frac{d\theta}{dt} = \dot{\theta}$$

$\theta$  is measured in radians, and  $\omega$  in rad/s



# Velocity and Angular Velocity

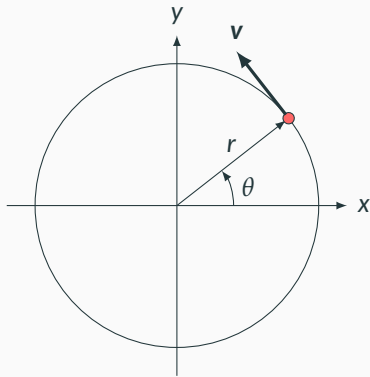


The velocity of the object in circular motion is related to the angular velocity (or angular frequency) by:

$$\mathbf{v} = r\boldsymbol{\omega}$$

- The direction of  $\mathbf{v}$  is tangent to circle, along  $\hat{\boldsymbol{\theta}}$ , and therefore  $\perp$  to  $\hat{\mathbf{r}}$
- If  $\omega > 0$ , the motion is counter-clockwise
- If  $\omega < 0$ , the motion is clockwise

# Velocity and Angular Velocity



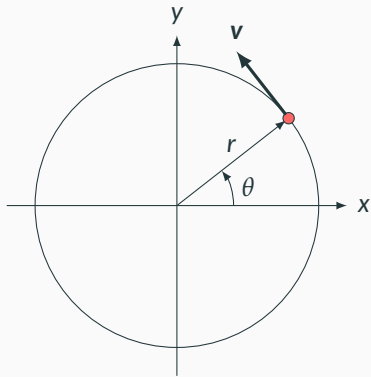
The velocity of the object in circular motion is more properly related to the angular velocity using this vector cross product:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

- $\boldsymbol{\omega}$ : out of the page if motion is counter-clockwise
- $\boldsymbol{\omega}$ : into the page if motion is clockwise

Visualizing  $\boldsymbol{\omega}$  takes practice, but this vector notation is mathematically rigorous and consistent

# Period & Frequency



For constant angular velocity  $\omega$  (uniform circular motion), the motion is periodic. Its **frequency** and **period** are given by:

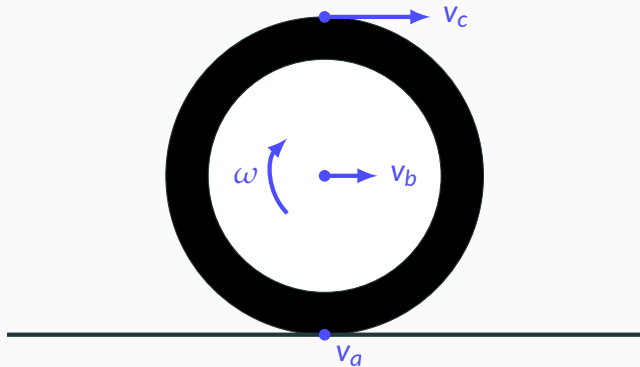
$$f = \frac{\omega}{2\pi} \quad T = \frac{2\pi}{\omega} \quad f = \frac{1}{T}$$

$T$  is in **seconds** (s) and  $f$  is in **hertz** (Hz)

## Rotating Object Without Slipping

A tire with radius  $r$  rolls along the road with an angular velocity  $\omega$  *without slipping*. (This is a very common case for analysis.) What is its velocity  $v$

- a. at the contact between the ground and the tire?
- b. at the center?
- c. at the top of the tire?



# Angular Acceleration

The time derivative of  $\omega$  is **angular acceleration**, which has a unit of  $\text{rad/s}^2$ :

$$\alpha = \dot{\omega} = \ddot{\theta}$$

Similar to the relationship between velocity and angular velocity, **tangential acceleration**  $a_t$  is related to angular acceleration  $\alpha$  by the radius  $r$ :

$$a_t(t) = \dot{v} = r\dot{\omega} = r\alpha$$

For *uniform* circular motion,  $\omega$  is constant, and therefore  $a_t = 0$

## With Calculus

Relationship between angular position and angular velocity:

$$\omega(t) = \frac{d\theta}{dt} \quad \theta(t) = \int \omega(t)dt + \theta_0$$

Relationship between angular velocity and angular acceleration:

$$\alpha(t) = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \quad \omega(t) = \int \alpha(t)dt + \omega_0$$

The relationships are the same as in rectilinear motion.

## Kinematics in the Angular Direction

For constant  $\alpha$ , the kinematic equations are just like in rectilinear motion:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \theta_0 + \frac{\omega_0 + \omega}{2}t$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

If  $\alpha$  is *not* constant, integration will be required.

## A Simple Example

**Example 1:** An object moves in a circle with angular acceleration  $3.0 \text{ rad/s}^2$ . The radius is  $2.0 \text{ m}$  and it starts from rest. How long does it take for this object to finish a circle?



## Centripetal Acceleration & Centripetal Force

There is also a component of acceleration toward the center of the motion, called the **centripetal acceleration**  $a_c$ :

$$\mathbf{a}_c = -\frac{v^2}{r}\hat{\mathbf{r}} = -(\omega^2 r)\hat{\mathbf{r}}$$

(The negative sign is because  $\hat{\mathbf{r}}$  is radially outward from the center.) The force that causes the centripetal acceleration is called the **centripetal force**:

$$\mathbf{F}_c = m\mathbf{a}_c = -\frac{mv^2}{r}\hat{\mathbf{r}}$$

# Centripetal Acceleration for Uniform Circular Motion

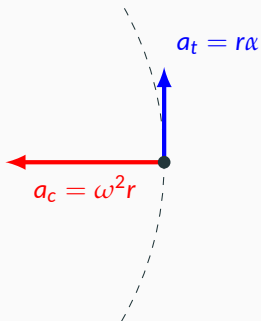
In uniform circular motion ( $\alpha = 0$ ) problems where the period or frequency are known, the speed of the object is:

$$v = \omega r = 2\pi r f = \frac{2\pi r}{T}$$

Centripetal acceleration can therefore be expressed based on  $T$  or  $f$ :

$$\mathbf{a}_c = -\frac{v^2}{r}\hat{\mathbf{r}} \quad \rightarrow \quad \boxed{\mathbf{a}_c = -\frac{4\pi^2 r}{T^2}\hat{\mathbf{r}} = -4\pi^2 r f^2 \hat{\mathbf{r}}}$$

# Acceleration: The General Case



In general circular motion, there are two components of acceleration:

- **Centripetal acceleration**  $a_c$  depends on radius of curvature  $r$  and instantaneous speed  $v$ . The direction of the acceleration is toward the center of the circle.
- **Tangential acceleration**  $a_t$  depends on radius  $r$  and angular acceleration  $\alpha$ . The direction of the acceleration is tangent to the circle

Most of the cases in AP Physics are uniform circular motion.

# How to Solve Circular Motion Problems

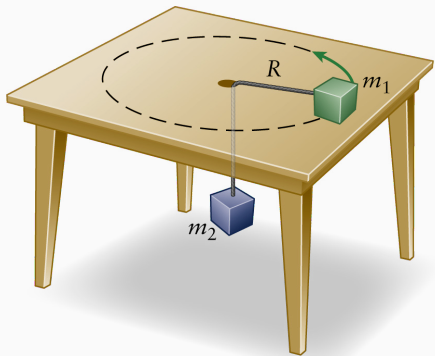
The condition for circular motion is the second law of motion:

$$F_c = \sum F = ma_c$$

The forces that generate the centripetal force comes from the free-body diagram. It may include:

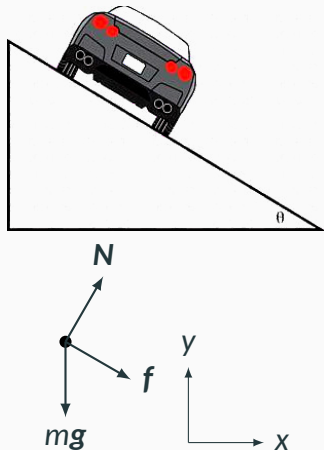
- Gravity
- Friction
- Normal force
- Tension
- Etc.

## Example: Horizontal Motion



**Example 2:** In the figure on the left, a mass  $m_1 = 3.0$  kg is rolling around a frictionless table with radius  $R = 1.0$  m. with a speed of  $2.0$  m/s. What is the mass of the weight  $m_2$ ?

# Banked Curves on Highways and Racetracks



No motion in the  $y$  direction, i.e. no net force:

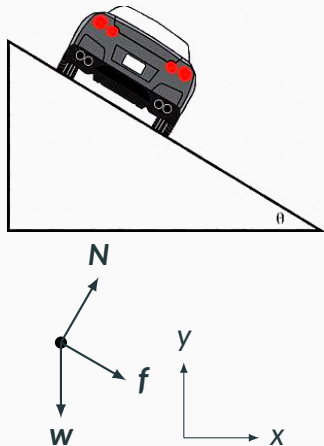
$$\sum F_y = N \cos \theta - f \sin \theta - w = 0$$

Net force in the  $x$  direction is the centripetal force:

$$\sum F_x = N \sin \theta + f \cos \theta = \frac{mv^2}{r}$$

Friction force  $f$  may be static or kinetic.

# Banked Curves on Highways and Racetracks

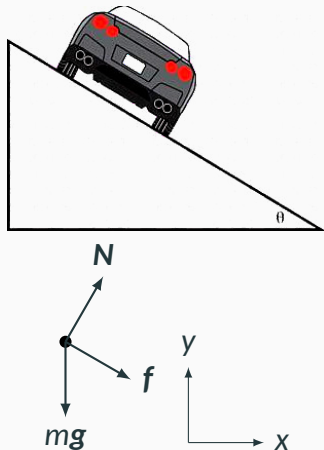


For analysis, use the simplified equation for friction  $f = \mu N$  (i.e. assume either kinetic friction or maximum static friction), and weight  $w = mg$ , the equations on the previous slides can be arranged as:

$$N (\cos \theta - \mu \sin \theta) = mg$$

$$N (\sin \theta + \mu \cos \theta) = \frac{mv^2}{r}$$

# Banked Curves on Highways and Racetracks



Dividing the two equations removes both the normal force and mass terms:

$$\frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta} = \frac{v^2}{rg}$$

The *maximum* velocity  $v_{\max}$  can be expressed as:

$$v_{\max} = \sqrt{rg \frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta}}$$

Note that  $v_{\max}$  does not depend on mass.



# Banked Curves on Highways and Racetracks

In the limit of  $\mu = 0$  (frictionless case), the equation reduces to:

$$v_{\max} = \sqrt{rg \tan \theta}$$

And in the limit of a flat roadway with no banking ( $\theta = 0$ ,  $\sin \theta = 0$  and  $\cos \theta = 1$ ), the equation reduces to:

$$v_{\max} = \sqrt{\mu rg}$$

# Vertical Circles

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# Vertical Circles

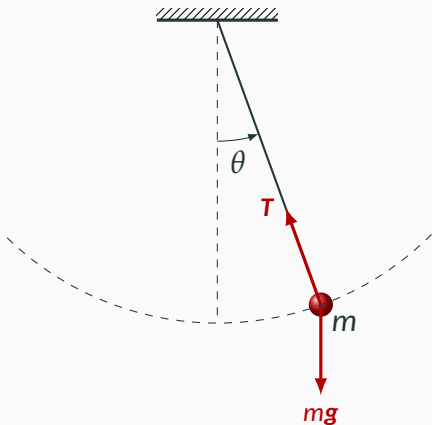
Circular motion with a horizontal path is straightforward. However, for vertical motion:

- Generally difficult to solve by dynamics and kinematics
- Instead, use conservation of energy to solve for  $v$
- Then use the equation for centripetal force to find other forces

**Remember:** If it is impossible to get the required centripetal force, then it could not continue the circular motion

# What About a Pendulum?

A simple pendulum is also like a vertical circular motion problem.



- There are two forces act on the pendulum: weight  $w = mg$ , and tension  $T$
- Speed of the pendulum at any height is found using conservation of energy
  - Tension  $T$  is always  $\perp$  to motion, therefore it doesn't do any work
  - Work is done by gravity (a conservative force) alone
- Tangential and centripetal accelerations are based on the net force along the angular and radial directions

# Simple Pendulum

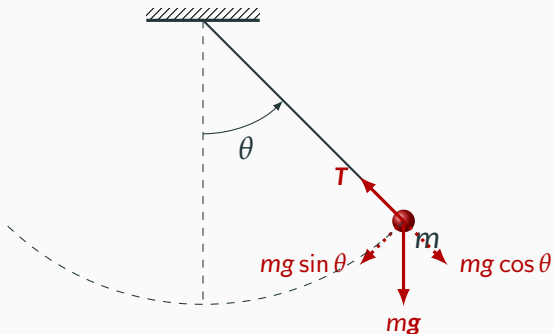
At the top of the swing, velocity  $v$  is zero, therefore:

Centripetal acceleration is also zero:

$$a_c = \frac{v^2}{r} = 0$$

and therefore the net force along the radial direction  $\hat{r}$  is zero. The tension force  $T$  can be calculated:

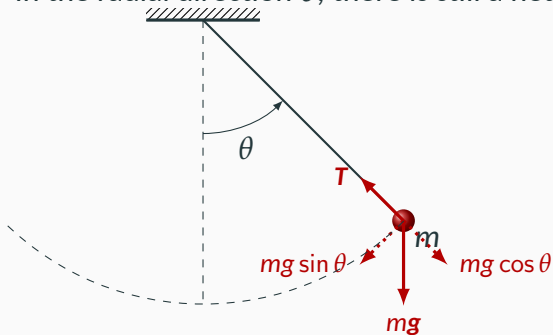
$$T = mg \cos \theta$$



At the highest point when  $\theta$  is largest, tension is the lowest.

# Simple Pendulum

In the radial direction  $\hat{\theta}$ , there is still a net force of  $mg \sin \theta$ , therefore:



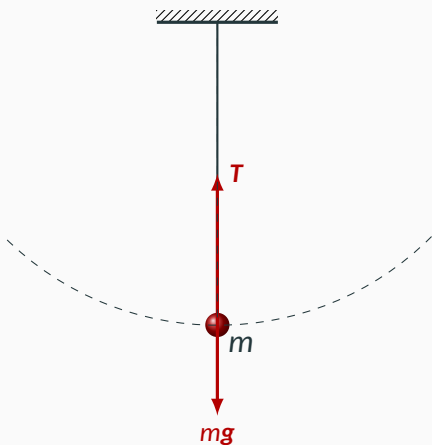
There is a tangential acceleration along  $\hat{\theta}$ , with a magnitude of:

$$a_t = g \sin \theta$$

This is the same acceleration as an object sliding down a frictionless ramp at an angle of  $\theta$ .

# Simple Pendulum

At the bottom of the swing, the velocity is at its maximum value,



- Maximum centripetal acceleration:

$$a_c = \frac{v^2}{r}$$

- No tangential acceleration:

$$a_t = 0$$

- At the lowest point, tension is the highest:

$$T = w + F_c = m \left( g + \frac{v^2}{r} \right)$$

## Example Problem

**Example 4:** You are playing with a yo-yo with a mass of 225 g. The full length of the string is 1.2 m. You decide to see how slowly you can swing it in a vertical circle while keeping the string fully extended, even when the yo-yo is at the top of its swing.

- A. Calculate the minimum speed at which you can swing the yo-yo while keeping it on a circular path.
- B. Find the tension in the string when the yo-yo is at the side and at the bottom of its swing.



## Example Problem

**Example 5:** A cord is tied to a pail of water, and the pail is swung in a vertical circle of 1.0 m. What must be the minimum velocity of the pail be at its highest point so that no water spills out?

- (a) 3.1 m/s
- (b) 5.6 m/s
- (c) 20.7 m/s
- (d) 100.5 m/s

## Example: Roller Coaster

**Example 6:** A roller coaster car is on a track that forms a circular loop, of radius  $R$ , in the vertical plane. If the car is to maintain contact with the track at the top of the loop (generally considered to be a good thing), what is the minimum speed that the car must have at the bottom of the loop. Ignore air resistance and rolling friction.

- (a)  $\sqrt{2gR}$
- (b)  $\sqrt{3gR}$
- (c)  $\sqrt{4gR}$
- (d)  $\sqrt{5gR}$

## Example

**Example 7:** A stone of mass  $m$  is attached to a light strong string and whirled in a *vertical* circle of radius  $r$ . At the exact bottom of the path, the tension of the string is three times the weight of the stone. The stone's speed at that point is given by:

(a)  $2\sqrt{gR}$

(b)  $\sqrt{2gR}$

(c)  $\sqrt{3gR}$

(d)  $4gR$