Topic 3: Work and Energy

Advanced Placement Physics C

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Olympiads School

Work and Energy

We start with some definition at are (unfortunately) not very useful:

- **Energy** is the ability to do work.
- **Work** is the mechanism in which energy is transformed.

Luckily, we can also use equations to define these concepts.

Work

Work

Mechanical work $\mathrm{d}W$ is performed when a force $\boldsymbol{F}$ displaces an object by $\mathrm{d}\boldsymbol{x}$. If a varying force is applied to move an object from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ along a path, then the total work done by the force is defined by the integral:

$$W = \int_{x_1}^{x_2} \boldsymbol{F}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{x}$$

- Work is a scalar quantity
- No work done if the force is perpendicular to displacement, when $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} = 0$ (i.e. the force did not cause the displacement)
- No work done if no displacement ($\mathrm{d}\boldsymbol{x} = \boldsymbol{0}$)
- Work can be positive or negative depending on the dot product
- When there are multiple forces acting on an object, we can compute the work done by each *each* force

Work by Constant Force

For a constant force, if the object moves along straight path, the integral simplifies to just the dot product of the two vectors:

$$\boxed{W = \boldsymbol{F} \cdot \Delta\boldsymbol{x}}$$

Or in the scalar form that is more familiar in Grades 11/12 Physics that avoid vector notations:

$$\boxed{W = F\Delta x \cos\theta}$$

where θ is the angle between the force and displacement vectors

Definition of Work

Work done by a force

- The work done by *one specific force*
- Example: A boy pushes a cart forward. The “work done by the boy” is the work done by the applied force.

Work done on an object

- There may be more than one force acting on an object
- The *sum* of all the work done on the object by each force
- The work done by the net force
- Also called the **net work** W_{net}

Kinetic Energy & Work-Energy Theorem

Kinetic Energy

When a net force on an object (with constant mass) accelerates it, the resulting amount of work done on the object (net work W_{net}) is given by:

$$W_{\text{net}} = \int_{x_1}^{x_2} \boldsymbol{F}_{\text{net}} \cdot \mathrm{d}\boldsymbol{x} = \int_{x_1}^{x_2} m\boldsymbol{a} \cdot \mathrm{d}\boldsymbol{x} = m \int_{x_1}^{x_2} \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{x}$$

Both $\boldsymbol{v}$ and $\boldsymbol{x}$ are continuous functions in time, we can switch the order of differentiation.

$$= m \int \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{v} = m \int \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{v} = m \int_{v_1}^{v_2} v \mathrm{d}v$$

Kinetic Energy

This integral, when integrated from v_1 (initial *speed*) to v_2 (final *speed*), becomes:

$$= m \int_{v_1}^{v_2} v dv = \frac{1}{2} m v^2 \bigg|_{v_1}^{v_2} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = \Delta K$$

where K is defined as the **translational kinetic energy**:

$$\boxed{K = \frac{1}{2} m v^2}$$

Later in the course we will discuss *rotational* kinetic energy.

Work and Kinetic Energy

In fact, the *definition* of kinetic energy came from this integration, in that work equals to the change in *something*, and we define that as kinetic energy. This is the **work-energy theorem**:

$$\boxed{W_{\text{net}} = \Delta K}$$

- ΔK can be positive or negative depending on the dot product
- When multiple forces acting on an object; each force can add or remove kinetic energy from an object
- Therefore we use the “net” amount of work done in the above equation

Example

Example 1: A force $\boldsymbol{F} = 4.0x\hat{\imath}$ (in newtons) acts on an object of mass 2.0 kg as it moves from $x = 1.0$ to $x = 5.0$ m. Given that the object is at rest at $x = 1$,

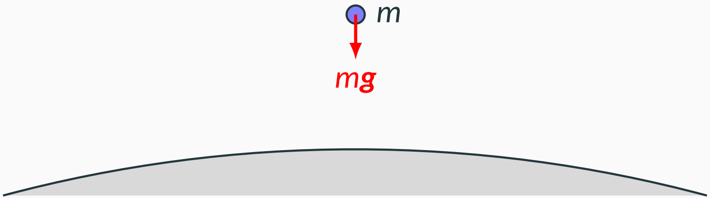
(a) Calculate the net work

(b) What is the final speed of the object?

Potential Energy

Gravitational Force & Gravitational Potential Energy

Consier an object that is free-falling under the force of gravity over a distance of Δx:

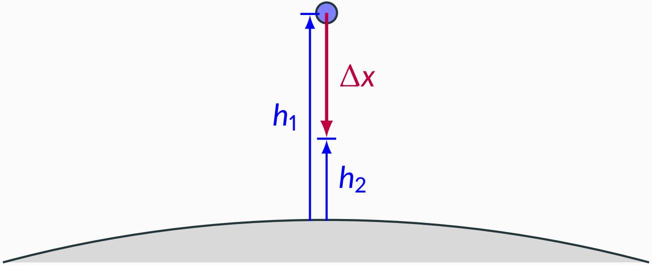


- Assuming that $\Delta \boldsymbol{x}$ is small, $\boldsymbol{g}$ can be considered to be constant
- The work done by the gravity (W_g) is *positive*, and therefore, there is an increase in kinetic energy. The object speeds up.

$$W_g = mg\Delta x = \Delta K > 0$$

Gravitational Potential Energy

The work done by gravity can also be expressed in terms of the change in height. Using ground as the reference level (i.e. $h = 0$):



$$W_g = mg(h_2 - h_1)$$
$$= -mg(h_2 - h_1) = -(mgh_2 - mgh_1)$$

Gravitational Potential Energy

Defining the **gravitational potential energy** U_g as:

$$\boxed{U_g = mgh}$$

The the work done by gravity can be related to this potential energy by:

$$\boxed{W_g = -\Delta U_g}$$

- *Positive* work decreases gravitational potential energy, while
- *Negative* work increases gravitational potential energy
- W_g depends on the end points h_1 and h_2, but not *how* it went from $h_1 \rightarrow h_2$

Spring Force & Elastic Potential Energy

The spring force $\boldsymbol{F}_s$ is the force that a compressed/stretched spring exerts on the object connected to it. An *ideal* spring obeys Hooke's law:

$$\boldsymbol{F}_s = -k\boldsymbol{x}$$

The work done by the spring force as it pushes any masses that are connected to a compressed/stretched spring is therefore:

$$\begin{aligned} W = \int_{x_1}^{x_2} \boldsymbol{F}_s \cdot \mathrm{d}\boldsymbol{x} &= -k \int_{x_1}^{x_2} x \mathrm{d}x \\ &= -\frac{1}{2}kx^2 \Big|_{x_1}^{x_2} = -\left[\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right] \end{aligned}$$

Elastic Potential Energy

Defining the **elastic potential energy** U_e as:

$$\boxed{U_e = \frac{1}{2}kx^2}$$

The the work done by the spring force can be related to the elastic potential energy by:

$$\boxed{W_s = -\Delta U_e}$$

- *Positive* work by the spring decreases spring potential energy, while
- *Negative* work by the spring increases spring potential energy
- W_s depends on the end points x_1 and x_2, but not *how* it went from $x_1 \rightarrow x_2$

Conservative Forces

Gravitational force $\boldsymbol{F}_g$, spring force $\boldsymbol{F}_s$, electrostatic force $\boldsymbol{F}_q$ (studied in Physics 12, and later in the course), magnetic force $\boldsymbol{F}_m$, and nuclear forces are **conservative forces**

- The work done by these forces relate to a change of a related potential energy
    - Positive work decreases this related potential energy,
    - Negative work increases this related potential energy
- The work done by a conservative force is *path independent* (depends only on end points)

Conservative Forces

By the fundamental theorem of calculus, any conservative forces $\boldsymbol{F}$ must be the negative gradient of the potential energies:

$$\boldsymbol{F} = -\nabla U = -\frac{\partial U}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial U}{\partial y}\hat{\boldsymbol{\jmath}} - \frac{\partial U}{\partial z}\hat{\boldsymbol{k}}$$

In one-dimension:

$$\boxed{F = -\frac{\mathrm{d}U}{\mathrm{d}x}}$$

The direction of a conservative force *always* decreases the potential energy. (Pay attention to the negative sign. Students often forget it.)

Conservation of Mechanical Energy

Positive work done by conservative forces on an object does two things:

1. Decrease its potential energy, while
2. Increase its kinetic energy by the same amount

Mathematically, this shows that mechanical energy must *always* be conserved when there are only conservative forces:

$$W_c = -\Delta U = \Delta K \quad \longrightarrow \quad \Delta K + \Delta U = 0$$

That's why those forces are called conservative forces!

Non-Conservative Forces

Examples of Non-Conservative Force

The majority of forces are **non-conservative**. Here are some examples:

- Work done by these forces are *usually* negative because they oppose the direction of motion
  - Drag (fluid resistance)
  - Friction[1]
- The work done by these forces may be positive or negative, depending on the problem
  - Applied force
  - Tension force
  - Normal force

The work-energy theorem still applies for non-conservative forces

[1] but sometimes it can also do positive work too.

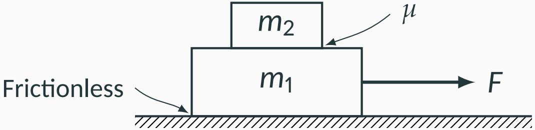
Work by Non-Conservative Forces

The work done by non-conservative forces differs from conservative forces in that:

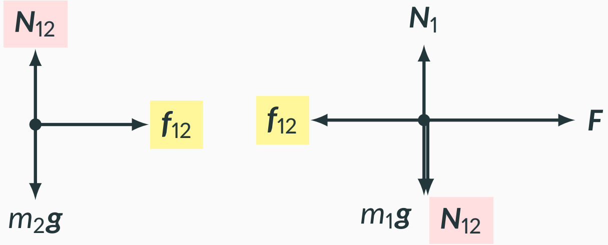
- There is no related potential energies: the work done by a non-conservative force transform energy from one form of kinetic energy to another
- The work is path dependent

Work by Friction, an Illustration

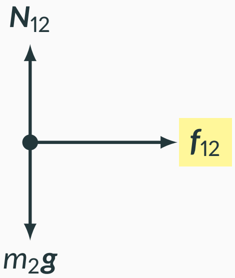
Work by frictional force—and other non-conservative forces—is illustrated below. Two blocks (m_1 and m_2), stacked vertically, move to the right by external force.



The FBDs of the blocks are shown below. (The forces highlighted in the same color are action-reaction pairs.)



Work Done By Friction

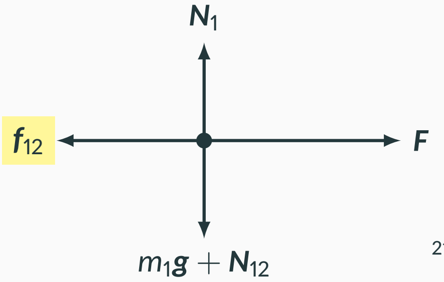


For the top block m_2, when it moves to the right

- Friction between the blocks $\boldsymbol{f}_{12}$ is the only force doing work
- The work done by $\boldsymbol{f}_{12}$ is positive
- Mass m_2 gains kinetic energy

For the bottom block m_1, when it moves to the right

- Applied force $\boldsymbol{F}$ does positive work on m_1, while
- Friction between the blocks $\boldsymbol{f}_{12}$ does negative work
- Therefore $\boldsymbol{f}_{12}$ decreases the kinetic energy



Work by Friction

The work done by friction is

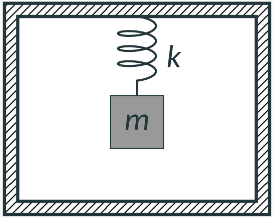
- positive on one object
- negative on another

Therefore, work by non-conservative forces transforms energy from the kinetic energy of one object into the kinetic energy of another object.

Conservation of Energy

Isolated Systems and the Conservation of Energy

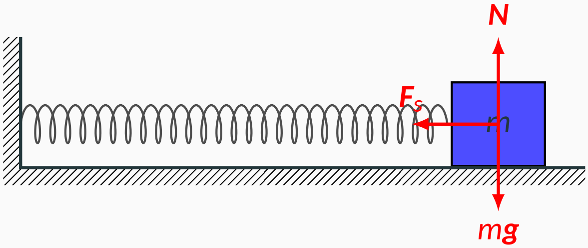
An **isolated system** is a system of objects that does not interact with the surrounding. Think of an isolated system as a bunch of objects inside an insulated box.



Since the system is isolated from the surrounding environment, the environment can't do any work on it. Likewise, the energy inside the system cannot escape either.

Example: Horizontal Spring-Mass System

Assuming that there are no friction, drag or other damping forces present, a horizontal spring-mass system is a closed system:

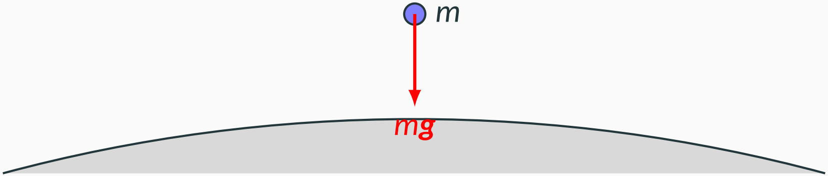


- The only force doing work is the spring force (conservative!) on the mass.
- The sum of the kinetic energy of the mass (K) and the elastic potential energy stored in the spring (U_e) is constant

$$K + U_e = \text{constant}$$

Example: Gravity

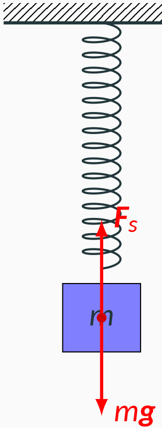
Assuming that there is no friction and drag, a free-falling object forms a closed system with Earth:



- The only force doing work is gravity (conservative!) on the mass
- The sum of the kinetic energy of the mass (K) and the gravitational potential energy stored between the tow masses (U_g) is constant

$$K + U_g = \text{constant}$$

Example: Vertical Spring-Mass System



Assuming that there are no friction, drag or other damping forces in the spring, the vertical spring-mass system (consists of the mass, the spring and Earth) is a closed system.

- Both gravity and spring force are doing work
- The sum of the kinetic energy of the mass (K), the gravitational potential energy stored between the mass and Earth (U_g), and the elastic potential energy stored in the spring (U_e) is constant.

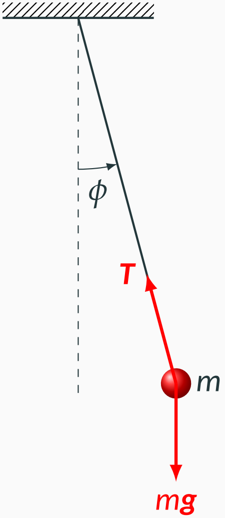
$$K + U_g + U_e = \text{constant}$$

Simple Pendulum System

Assuming that there are no friction, drag or other damping forces in the spring, the simple pendulum system (consists of the mass and Earth) is a closed system.

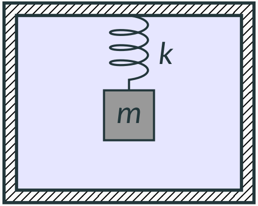
- Gravity ($m\boldsymbol{g}$), which is conservative, is the only force that does work
- Tension ($\boldsymbol{T}$) does not do work on the pendulum because it is perpendicular to the motion of the pendulum bob
- The sum of the kinetic energy of the mass (K), the gravitational potential energy stored between the mass and Earth (U_g) is constant:

$$K + U_g = \text{constant}$$



What if there is friction?

Energy is always conserved as long as your system is defined properly. In this case, the system consists of a mass, a spring, Earth and all the air molecules inside the box:

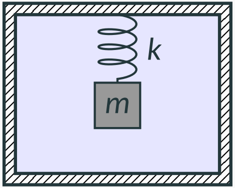


The energies of this system include

- Kinetic energy of the mass (K)
- Gravitational potential energy (U_g) between the mass and Earth
- Elastic potential energy (U_e) stored in the spring
- Thermal (kinetic) energy (K_{air}) of the vibration of the air molecules

Conservation of Energy with Non-Conservative Forces

As the mass vibrates, friction and drag with air slows it down, while the temperature of the air rises due to friction and drag. Total energy is conserved even as the mass stops moving



$$K + K_{\text{air}} + U_g + U_e = \text{constant}$$

Non-conservative forces doing work are *internal* to the system, and therefore energy is still conserved. (Work done by friction transform from the kinetic energy of the mass to the kinetic energy of the air molecules.)

Conservation of Energy

If *only* conservative forces are doing work, mechanical energy (i.e. $K + U$) is always conserved:

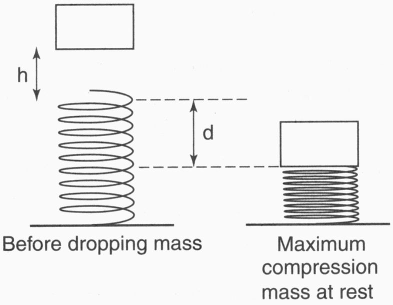
$$\boxed{K + U = K' + U'}$$

When external non-conservative forces are also doing work, instead of *trying* to isolate the system, we can instead calculate the work done by them W_{nc} and add it to the total energy of the system

$$\boxed{K + U + W_{nc} = K' + U'}$$

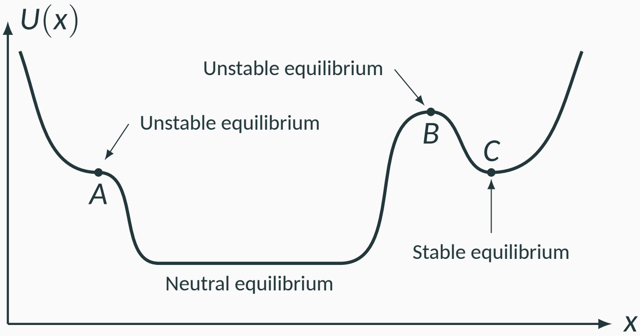
Example

Example 2: A mass *m* is dropped from a height of *h* above the equilibrium position of a spring. Set up the equation that determines the spring's compression *d* when the object is instantaneously at rest.



Energy Diagrams

Plot of potential energy (U) vs. position (x) for a conservative force



Power & Efficiency

Power

Power is the *rate* at which work is done, i.e. the rate at which energy is being transformed:

$$\boxed{P(t) = \frac{\mathrm{d}W}{\mathrm{d}t}} \qquad \boxed{\overline{P} = \frac{W}{\Delta t}}$$

Quantity	Symbol	SI Unit
Instantaneous and average power	P, $\overline{P}$	W
Work done	W	J
Time interval	Δt	s

In engineering, power is often more critical than the actual amount of work done.

Power

If a constant force is used to push an object at a constant velocity, the power produced by the force is:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{F} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \quad \rightarrow \quad \boxed{P = \boldsymbol{F} \cdot \boldsymbol{v}}$$

Application: aerodynamics

- When an object moves through air, the applied force must overcome air resistance (drag force), which is proportional with v^2
- Therefore "aerodynamic power" must scale with v^3 (i.e. doubling your speed requires $2^3 = 8$ times more power)
- Important when aerodynamic forces dominate

Efficiency

Efficiency is the ratio of useful energy or work output to the total energy or work input

$$\boxed{\eta = \frac{E_o}{E_i} \times 100\,\%} \qquad \boxed{\eta = \frac{W_o}{W_i} \times 100\,\%}$$

Quantity	Symbol	SI Unit
Useful output energy	E_o	J
Input energy	E_i	J
Useful output work	W_o	J
Input work	W_i	J
Efficiency	η	no units

Efficiency is always $0 \leq \eta < 100\,\%$