Topic 18: Special Relativity, Part 3

AP and IBHL Physics

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Olympiads School

Relativistic Momentum

Relativistic Momentum

In Grade 12 Physics, you were taught that momentum is mass times velocity. And in Grade 11 Physics, you were taught that velocity is displacement over time. *These definitions have not changed.*

$$\mathbf{p} = m\frac{d\mathbf{x}}{dt}$$

But now that you know $d\mathbf{x}$ and dt are relativistic quantities that depend on motion, we can find a new expression for "relativistic momentum":

$$\mathbf{p} = m\frac{d\mathbf{x}}{dt} = \frac{m d\mathbf{x}}{\sqrt{1-\left(\frac{v}{c}\right)^2}\, dt} = \boxed{\frac{m\mathbf{v}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}} = \gamma m\mathbf{v}$$

Relative Mass

Relativistic Mass

From the relativistic momentum expression, we see the relativistic aspect to mass as well. The **apparent mass** (or **relativistic mass**) m' as measured by a moving observer is related to its **rest mass** (or **intrinsic mass** or **invariant mass**) m by the Lorentz factor:

$$\boxed{m' = \frac{m}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma m}$$

The intrinsic mass of a moving object does not change, but a moving observer will observe that it behaves as if it is more massive. As $v \to c$, $m' \to \infty$.

Relativistic Energy

Work and Energy

Einstein published a fourth paper in *Annalen der Physik* on November 21, 1905 (received Sept. 27) titled "Does the Inertia of a Body Depend Upon Its Energy Content?" (In German: Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?)

- Einstein deduced the most famous of equations: $E = mc^2$

Work and Energy

In Grade 12 Physics, you were taught that force is the rate of change of momentum with respect to time. *This definition has not changed.*

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

and that work is the integral of the dot product between force and displacement vectors. *This definition has not changed either.*

$$W = \int \mathbf{F} \cdot d\mathbf{x} = \int \frac{d\mathbf{p}}{dt} \cdot \mathbf{dx}$$

Since we now have a relativistic expression for momentum, we substitute that new expression into the expression for force, and then integrate.

Work and Energy

For 1D motion (for simplicity), we can rearrange the terms in the integral:

$$W = \int F dx = \int \frac{dp}{dt} dx = \int v dp$$

Assuming that both v and p are continuous in time, we can apply the chain rule to find the infinitesimal change in momentum (dp) with respect to γ and v:

$$p = \gamma m v \quad \rightarrow \quad dp = \gamma dv + v d\gamma$$

Substituting that back into the integral, we have:

$$W = \int v dp = \int m v (\gamma dv + v d\gamma) = \int m \left(\gamma v dv + v^2 d\gamma\right)$$

Work and Energy

One of the integral is with respect to γ, so we express v and dv in terms of γ using its definition:

$$v^2 = c^2 \left[1 - \left(\frac{1}{\gamma}\right)^2\right] \quad \rightarrow \quad dv = \frac{c^2}{\gamma^3 v} d\gamma$$

Work and Energy

Putting everything together, we have

$$W = \int m(\gamma v dv + v^2 d\gamma) = \int m\left[\frac{c^2}{\gamma^2} + c^2\left(1 - \frac{1}{\gamma^2}\right)\right] d\gamma$$

This is a surprisingly simple integral:

$$W = \int_1^\gamma mc^2 d\gamma$$

The limit of the integral is from 1 because at $v = 0$, $\gamma = 1$

Work and Kinetic Energy

The integral gives us this expression:

$$W = \gamma mc^2 - mc^2$$

We know from the work-kinetic energy theorem that the work W done is equal to the change in kinetic energy K, therefore

$$\boxed{K = m'c^2 - mc^2}$$

Variable	Symbol	SI Unit
Kinetic energy of an object	K	J
Apparent mass (measured in moving frame)	m'	kg
Rest mass (measured in stationary frame)	m	kg
Speed of light	c	m / s

Relativistic Energy

$$\boxed{K = m'c^2 - mc^2}$$

The minimum amount of energy that any object has, regardless of it's motion (or lack of) is its **rest energy**:

$$E_0 = mc^2$$

The **total energy** of an object has is

$$E_T = m'c^2 = \gamma mc^2$$

The difference between total energy and rest energy is the kinetic energy:

$$K = E_T - E_0$$

Relativistic Energy

$$\boxed{E = mc^2}$$

Mass-energy equivalence:

- Whenever there is a change of energy, there is also a change of mass
- "Conservation of mass" and "conservation of energy" must be combined into a single concept of **conservation of mass-energy**
- Mass-energy equivalence doesn't merely mean that mass can be converted into energy, and vice versa (although this is true), but rather, one can be converted into the other *because they are fundamentally the same thing*

Energy-Momentum Relation

The **energy-momentum relation** relates an object's rest (intrinsic) mass m, total energy E, and momentum p:

$$\boxed{E^2 = p^2c^2 + m^2c^4}$$

Quantity	Symbol	SI Unit
Total energy	E	J
Momentum	p	$\mathrm{kg \cdot m/s}$
Rest mass	m	kg
Speed of light	c	$\mathrm{m/s}$

Energy-Momentum Relation

This equation is derived by squaring the expression for relativistic momentum:

$$p = \gamma m v = \frac{mv}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \quad \rightarrow \quad p^2 = \gamma^2 m^2 v^2 = \frac{m^2 v^2}{1 - \left(\frac{v}{c}\right)^2}$$

Energy-Momentum Relation

Solving for v^2 and substituting it back into the Lorentz factor, we obtain an alternative form for γ in terms of momentum and mass:

$$\gamma = \sqrt{1 + \left(\frac{p}{mc}\right)^2}$$

Inserting this form of the Lorentz factor into the energy equation, we have

$$E = mc^2 \sqrt{1 + \left(\frac{p}{mc}\right)^2}$$

Which is the same equation as in the last slide.

Energy-Momentum Relation

In the **stationary frame of reference**, (rest frame, center-of-momentum frame) the momentum is zero, so the equation simplifies to

$$\boxed{E = mc^2}$$

where m is the rest mass of the object.

If the object is **massless**, as is the case for a **photon**, then the equation reduces to

$$\boxed{E = pc}$$

Kinetic Energy–Classical vs. Relativistic

Relativistic:

$$K = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - mc^2$$

Newtonian:

$$K = \frac{1}{2}mv^2$$

But are they really that different?

- If space and time are indeed relative quantities, then the relativistic equation for K must apply to all velocities
- But we know that when $v \ll c$, the Newtonian expression works perfectly
- i.e. The Newtonian expression for K must be a very good approximation for the relativistic expression for K for $v \ll c$

Binomial Series Expansion

The **binomial series** is the Maclaurin series for the function $f(x) = (1+x)^{\alpha}$, given by:

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$$

In the case of relativistic kinetic energy, we use:

$$x = -\left(\frac{v}{c}\right)^2 \quad \text{and} \quad \alpha = -\frac{1}{2}$$

Binomial Series Expansion

Substituting these terms into the equation:

$$
\begin{aligned}
K &= mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right) - mc^2 \\
&\approx \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \cdots
\end{aligned}
$$

For $v \ll c$, we can ignore the high-order terms. The leading term reduces to the Newtonian expression

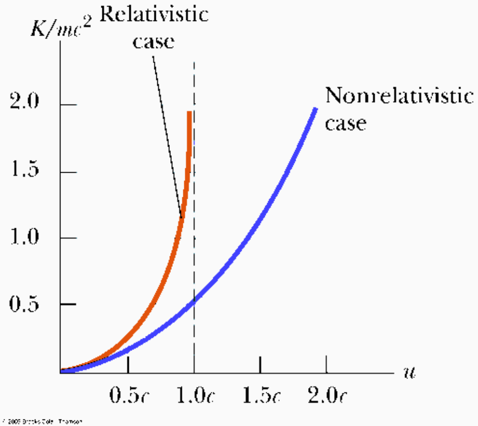
Comparing Classical and Relativistic Energy

In classical mechanics:

$$K = \frac{1}{2}mv^2$$

In relativistic mechanics:

$$K = \gamma mc^2 - mc^2$$



The classical expression is accurate for speeds up to $v \approx 0.3c$.