# Class 5: Center of Mass

## Advanced Placement Physics C

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Olympiads School

# Center of Mass

Finding an object's center of mass is important, because

- The laws of motion are formulated by treating an objects as point masses (for real-life objects, we let the forces apply to the center of mass)
- Objects can have *rotational* motion in addition to *translational* motion as well (we will examine that a bit more in a very-important topic later)

# Start with a Definition

The **center of mass** (“CM”) is the *weighted average of the masses in a system.* The “system” may be:

- A collection of individual particles
- A continuous distribution of mass with constant density. In this case, CM is also the geometric center (**centroid**) of the object
- A continuous distribution of mass with varying density
- If the masses are inside of a gravitational field, then the CM is also its **center of gravity** (“CG”)

# Simple Example

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Answer: the half way point between the two masses!

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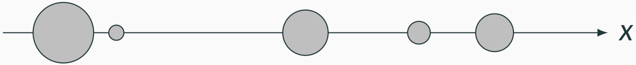
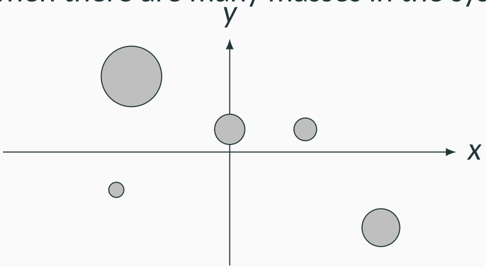
The answer is still simple. The CM is no longer half way between the two masses, but now $\frac{1}{3}$ the total distance from the larger masses.

# Complicating Things Further

If we increase the number of point masses along the $x$-axis, our problem can become much more complicated (although still not devastatingly so)




Difficulties really arises when there are many masses in the system in 2D or 3D:



# An Equation Helps

The center of mass is defined for discrete number of masses as:

$$\boxed{\mathbf{x}_{CM} = \frac{\sum \mathbf{x}_i m_i}{\sum m_i}}$$

| Quantity | Symbol | SI Unit |
|---|---|---|
| Position of center of mass (vector) | $\mathbf{x}_{CM}$ | m |
| Position of point mass *i* (vector) | $\mathbf{x}_i$ | m |
| Point mass *i* | $m_i$ | kg |

In components:

$$x_{CM} = \frac{\sum x_i m_i}{\sum m_i} \qquad y_{CM} = \frac{\sum y_i m_i}{\sum m_i} \qquad z_{CM} = \frac{\sum z_i m_i}{\sum m_i}$$

# An Example

**Example 1:** Consider the following masses and their coordinates which make up a "discrete mass" rigid body"

$$
\begin{aligned}
m_1 &= 5.0\,\mathrm{kg} & \qquad \boldsymbol{x}_1 &= 3\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{k}} \\
m_2 &= 10.0\,\mathrm{kg} & \qquad \boldsymbol{x}_2 &= -4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 7\hat{\boldsymbol{k}} \\
m_3 &= 1.0\,\mathrm{kg} & \qquad \boldsymbol{x}_3 &= 10\hat{\boldsymbol{\imath}} - 17\hat{\boldsymbol{\jmath}} + 10\hat{\boldsymbol{k}}
\end{aligned}
$$

What are the coordinates for the center of mass of this system?

# Continuous Mass Distribution

In general, objects are not a discrete collection of point masses, but a continuous distribution of mass. Therefore, we take the limit of when the number of masses approaches $\infty$:

$$\boldsymbol{x}_{CM} = \lim_{n\to\infty} \left( \frac{\sum_{i=1}^{n} \boldsymbol{x}_i m_i}{\sum_{i=1}^{n} m_i} \right)$$

This gives us an integral form of our equation:

$$\boxed{\boldsymbol{x}_{CM} = \frac{\int \boldsymbol{x} \mathrm{d}m}{\int \mathrm{d}m}}$$

# Densities

- Linear density (for 1D problems)

$$\gamma = \frac{\mathrm{d}m}{\mathrm{d}L} \quad \rightarrow \quad \mathrm{d}m = \gamma \mathrm{d}L$$

- Surface area density (for 2D problems)

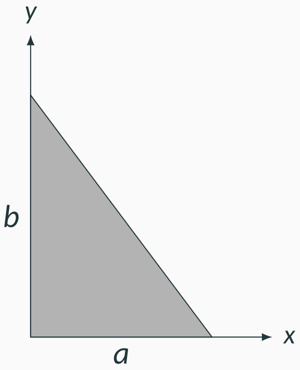
$$\sigma = \frac{\mathrm{d}m}{\mathrm{d}A} \quad \rightarrow \quad \mathrm{d}m = \sigma \mathrm{d}A$$

- Volume density (for 3D problems)

$$\rho = \frac{\mathrm{d}m}{\mathrm{d}V} \quad \rightarrow \quad \mathrm{d}m = \rho \mathrm{d}V$$

# An Example with Integrals

**Example 2:** A triangular plate is placed in a Cartesian coordinate system with two of its edges along the $x$ and $y$-axis. The length of the edges along the axes are $a$ and $b$ respectively. Assuming that the surface area density $\sigma$ is uniform, determine the coordinate of its center of mass.



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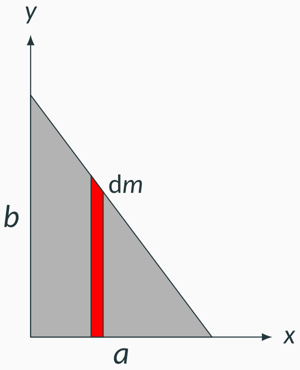
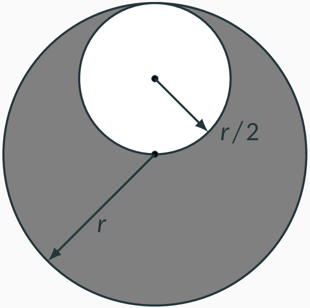


# Symmetry

- Any plane of symmetry, mirror line, axis of rotation, point of inversion *must* contain the center of mass.
- Caveat: only works if the density distribution is also symmetric
- Again: if density is uniform, CM is also geometric center (centroid)

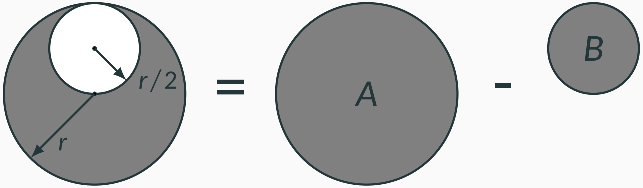
# “Negative Mass”

- Where there is a “hole” in the geometry, treat it as having negative mass density $-\sigma$ in that region.
- Negative masses don’t exist, so this is really just a trick.
- **Example:** What is the center of mass of this shape?



# Negative Mass Example

- This is how we would think of it:




- Let the origin of the coordinate system to located at the center of $A$
- Based on symmetry: $x_{CM} = 0$; only have to find $y$-coordinate.
- Sum our weighted average:

$$y_{CM} = \frac{\sum y_i m_i}{\sum m_i} = \frac{m_A(0) + m_B(r/2)}{m_A + m_B} = \frac{-\sigma\pi\left(r/2\right)^2\left(r/2\right)}{\sigma\pi r^2 - \sigma\pi\left(r/2\right)^2}$$

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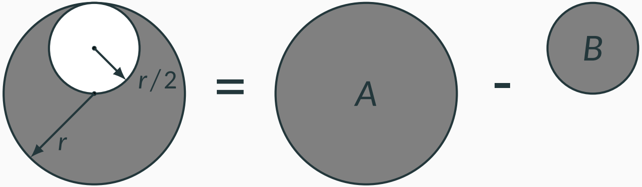


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# Velocity, Acceleration and Momentum

Take time derivative of the equation for $\boldsymbol{x}_{\mathrm{CM}}$ to get the velocity at the CM:

$$\boldsymbol{v}_{\mathrm{CM}} = \frac{\mathrm{d}\boldsymbol{x}_{\mathrm{CM}}}{\mathrm{d}t} = \frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \boldsymbol{x}\mathrm{d}m\right) = \frac{1}{m}\int \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\mathrm{d}m = \frac{\int \mathbf{v}\mathrm{d}m}{m}$$

The integral in the numerator is the sum of the momentum of all the masses in the system ($\boldsymbol{p}_{\mathrm{net}}$) which means that we have

$$\boldsymbol{p}_{\mathrm{net}} = m\boldsymbol{v}_{\mathrm{CM}}$$

Taking the derivative of $\boldsymbol{p}_{\mathrm{net}}$ relates force and acceleration at the CM as well:

$$\boldsymbol{F}_{\mathrm{net}} = \frac{\mathrm{d}\boldsymbol{p}_{\mathrm{net}}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{v}_{\mathrm{CM}}}{\mathrm{d}t} = m\boldsymbol{a}_{\mathrm{CM}}$$