One-Dimensional Motion Graphs for Constant Acceleration

Dr. Timothy Leung

When analyzing one-dimensional motion, particularly experimental results, we often use **motion graphs** to graphically express how motion quantities (position, velocity, acceleration) evolves in time, or how they are related to other motion quantities.

1 Motion Quantities as Functions of Time

The basic motion graphs most familiar to physics students in grades 11 and 12 are the graphs that express motion quantities as functions of time:

- position vs. time (or displacement vs. time)
- velocity vs. time
- acceleration vs. time

Since velocity is the time derivative of position ($v = \dot{x}$), and acceleration is the time derivative of velocity ($a = \dot{v} = \ddot{x}$), the relationship between the graphs are straightforward: the velocity graph is the slope of the position graph, and the acceleration graph is the slope of the velocity graph.

1.1 Areas Under a Graph

The area under the acceleration vs. time graph is the change in velocity Δv and the area under the velocity vs. time graph to find displacement Δx based on the integral relationship:

$$\Delta v = \int a(t)dt$$

$$\Delta x = \int v(t)dt$$

However, unless the functions $a(t)$ and $v(t)$ are known, or if the graphs are simple, it is almost impossible to directly integrate. Instead, numerical integration methods such as trapazoid rule or Simpson's rule are used to approximate the area.

1.2 Uniform Motion and Uniform Acceleration

For uniform motion, i.e. constant velocity, the basic motion graphs are shown in Fig. 1. All the graphs are linear. Computing the slopes (derivatives) and areas (integrals) are straightforward exercises.

For uniform acceleration, the position vs. time graph is a parabola, while velocity and acceleration graphs are linear (Fig. 2). Computing the area under the velocity and acceleration graphs are still

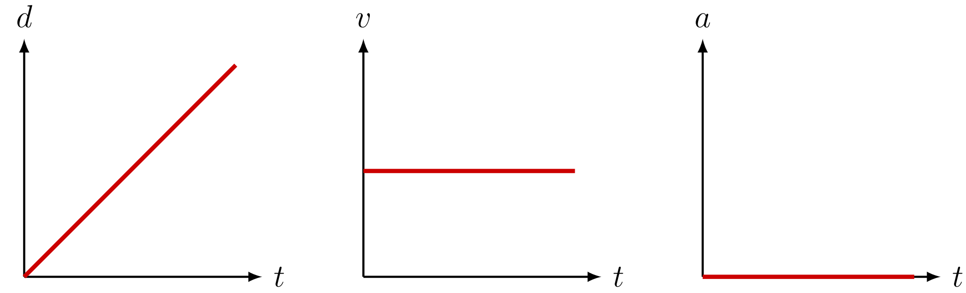

Figure 1: Position, velocity and acceleration are all linear functions of time for uniform motion.

straightforward, as is finding the slope of the velocity graph, however, finding the acceleration using only the position graph is much more difficult.

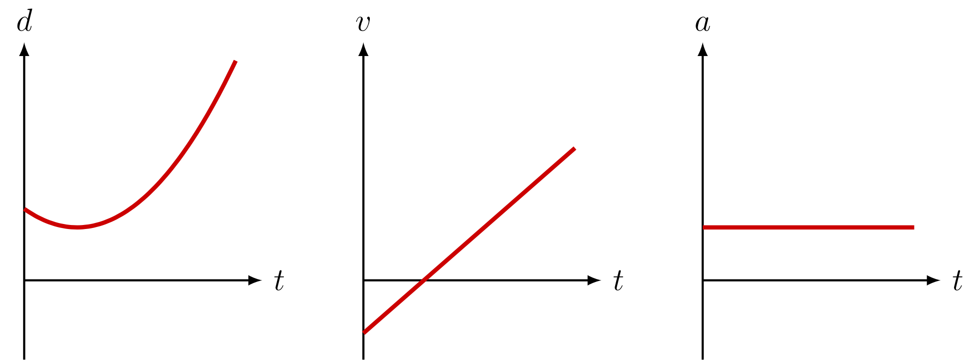

Figure 2: Displacement, velocity and acceleration as functions of time for uniformly accelerated motion.

2 Position vs. Time Squared

In many cases, the position vs. time graph is often obtained by plotting experimental values of x and t. For example, photo gates can be used to record the position of an object at regular time intervals. However, even if we know that the acceleration is uniform, like the position vs. time graph from Fig. 2, it is still difficult to determine the magnitude of acceleration[1] from the graph. One way to get around this problem is to do a curve fitting. However, curve fitting a parabola is time consuming. A much faster method is to plot the motion quantities as a linear function instead.

For example, if initial velocity is zero, ($v_0 = 0$), then instead of plotting position x directly against

[1] Of course, the *sign* of the acceleration is easy to see, by looking at the concavity of the graph.

time t, we re-interpret the kinematic equation as a linear function in the form of $y = mx + b$:

$$\underset{y}{x} = \underset{m}{\left(\tfrac{1}{2}a\right)}\underset{x}{t^2} + \cancel{v_0 t} + \underset{b}{x_0} \tag{1}$$

and plot position x against the *square* of time, t^2. If acceleration is indeed uniform, then the graph would be linear. That constant acceleration is twice slope, i.e.:

$$m = \frac{1}{2}a \quad \rightarrow \quad a = 2m$$

The x-intercept of both graphs are still the initial position x_0 (Fig. 3). The x vs. t and the x vs. t^2 graphs both describe the same motion.

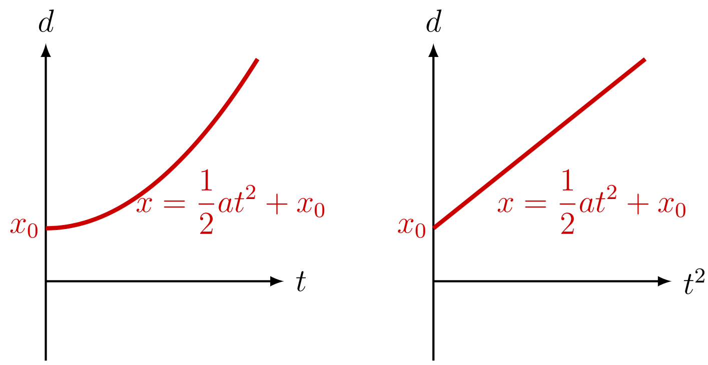

Figure 3: Position plotted against time t and time squared t^2 for uniformly accelerated motion.

3 Velocity Squared vs. Displacement

Likewise, some experimental equation is able to capture velocity information as an object moves in *space*. In other words, velocity and position information is available experimentally but not time. Instead of plotting velocity vs. time, position vs. time, we can plot velocity as a function of *position*. Again, even if acceleration is uniform, finding a from this graph is not straightforward. However, another option is to, like in the previous section, re-interpret the kinematic equation as a linear function:

$$\underset{y}{v^2} = \underset{b}{v_0^2} + \underset{m}{(2a)}\underset{x}{(x - x_0)} \tag{2}$$

and plot the *square* of velocity v^2 as a function of displacement $(x - x_0)$. If acceleration is uniform (constant a), the graph would be linear. The acceleration is half the slope m of the graph: $a = \frac{m}{2}$ and the y-intercept is the square of the initial velocity (v_0^2).

4 Use of Linear Graphs in Other Applications

There are many types of graphs that we plot as linear graphs even though the relationship between variables is not linear. In the AP Physics C exams, there is often one free-response question where "experimental" data is provided for a known algebraic relationship, and you as asked to find a constant by plotting a linear graph. Some example are shown here.

4.1 Orbital Mechanics

The relationship between the period T and orbital radius R of planets around the same star (Kepler's third law of planetary motion) is most often shown by plotting T^2 vs. R^3 rather than plotting T vs. R directly. The slope of the graph can be used to find the mass of the star M at the center:

$$\underset{\substack{\uparrow \\ y}}{T^2} = \underset{\substack{\uparrow \\ m}}{\frac{4\pi^2}{GM}} \underset{\substack{\uparrow \\ x}}{R^3} \tag{3}$$

4.2 Simple Pendulum

In the simple harmonic motion of a simple pendulum, the relationship between the frequency f and the length of the pendulum ℓ is given by[2]

$$f = \left[\frac{1}{2\pi\sqrt{g}}\right]\sqrt{\ell} \tag{4}$$

In this case, the slope of the graph of f vs. $\sqrt{\ell}$ can be used to experimentally determine the acceleration due to gravity g.

[2] Although *usually* we would write it as

$$f = \frac{1}{2\pi}\sqrt{\frac{\ell}{g}}$$

4.3 Refraction of Light

In the law of refraction[3], the incident angle θ_1 and the refracted angle θ_2 when light is refracted from a vaccuum into a material with refractive index n is given by:

$$\underset{\overset{\uparrow}{y}}{\sin\theta_1} = \underset{\overset{\uparrow}{m}}{n}\underset{\overset{\uparrow}{x}}{\sin\theta_2} \tag{5}$$

By plotting $\sin\theta_1$ vs. $\sin\theta_2$, we see that the slope of the graph is the refractive index.

[3] You may know it as Snell's law.