Class 22: Maxwell's Equations

Advanced Placement Physics C

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Olympiads School

Making Ampère's Law Better

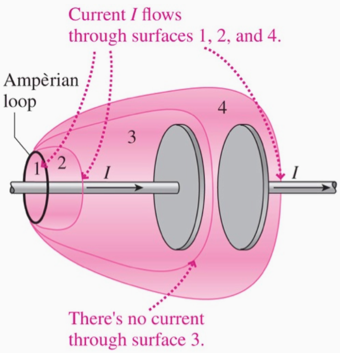
Ampère's law, as we know it, only applies to *steady* currents I_c:

$$\boxed{\oint_C \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_c}$$

However,

- Current are usually not steady in RC, RL, LC or RLC circuits
- Applying Ampère's law at a charging/discharging capacitor gives an ambiguous answer

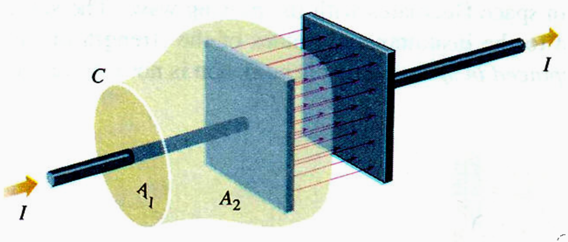
Modifying Ampère’s Law for Unsteady Current



Four surfaces bounded by the same circular Amperian loop (think blowing a soap bubble). Surfaces 1, 2 and 4 have currents penetrating through them, but surface 3 does not.

Modifying Ampère's Law

This might give a better view of what the "soap bubble" looks like



There is no current through the surface A_2 (same as surface 3 in the last slide), but there is definitely a changing *electric flux*

Maxwell's Modification to Ampère's Law

James Clerk Maxwell, in 1860, proposed a modification to Ampère's Law to make it work with unsteady current as well

$$\boxed{\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 \left(I + \varepsilon_0 \frac{\mathrm{d}\Phi_E}{\mathrm{d}t} \right)}$$

Maxwell called the correction term $\varepsilon_0 \dfrac{\mathrm{d}\Phi_E}{\mathrm{d}t}$ **displacement current**.

- The word "displacement" has historical roots, but no physical meaning
- However, "current" means that the effect of changing the electric flux is indistinguishable from real currents in producing magnetic field

Maxwell’s Equations

- Maxwell recognized the relationship between electricity and magnetism in **Gauss’s law**, **Faraday’s law** and **Ampère’s law**
- Combined them into a unified set of equations, now known as **Maxwell’s equations** for electrodynamics

Maxwell's Equations in Integral Form

Maxwell's equations can be expressed in its integral form, which is how we have studied the equations in the first place:

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0} \qquad \text{(Gauss, for } \vec{E}\text{)}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \qquad \text{(Gauss, for } \vec{B}\text{)}$$

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} \qquad \text{(Faraday)}$$

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \qquad \text{(Ampère-Maxwell)}$$

Maxwell’s Equations in Vacuum

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = 0$$

$$\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0$$

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 \epsilon_0 \frac{\mathrm{d}\Phi_E}{\mathrm{d}t}$$

In vacuum, we can remove all references to matter in the equation, and Maxwell’s equations simplifies.

- The equations show “symmetry”
- Magnetic and electric fields are on equal footing
- In a vacuum where charges are currents are absent, the only source of either field is a change in the other field

Maxwell's Equations in Differential Form

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

- Maxwell's equations are usually expressed in *differential* form, which is obtained using vector calculus. Follow [this link] if you want to see how it's done.
- The differential form shows how the *time derivatives* of $\vec{E}$ and $\vec{B}$ are related to the *spatial derivatives* of the other field
- The last two equations (Faraday's and Ampère's laws) together represent two set of second order partial differential equations (one for each field), the solution of which represents a traveling wave

Electromagnetic (EM) Wave

Maxwell's equations show that an "electromagnetic wave" must exist. In a simple case where electric and magnetic fields only vary in x and time t only, i.e. $\vec{E} = E(x, t)$ and $\vec{B} = B(x, t)$, Faraday's and Ampère's laws reduce to:

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \qquad \frac{\partial B}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

(A negative sign appears on the right-hand side because we have ignored some of the vector operations.) Taking the spatial derivative of E with respect to x on both side of Faraday's law, and switch the order of differentiation, we get:

$$\frac{\partial}{\partial x}\left(\frac{\partial E}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial B}{\partial t}\right) \quad \rightarrow \quad \frac{\partial^2 E}{\partial x^2} = -\frac{\partial}{\partial t}\left(\frac{\partial B}{\partial x}\right)$$

Electromagnetic (EM) Wave

But we already have an expression for $\partial B / \partial x$ from Ampère's law:

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial}{\partial t}\left(\frac{\partial B}{\partial x}\right) = -\frac{\partial}{\partial t}\left(-\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

Rearranging the terms on the right hand side, we get

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

This is the standard form of the **wave equation** (a second-order partial differential equation):

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}$$

Electromagnetic (EM) Wave

- "Second-order" means that the equation deals with second derivatives, in this case, in x and in t.
- "Partial" means the equation involves partial derivatives (i.e. when a function has more than one variables, and you only differentiate against one variable)
- We can also repeat the exercise by first differentiating Ampère's law to get

$$\frac{\partial^2 B}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}$$

Electromagnetic (EM) Wave

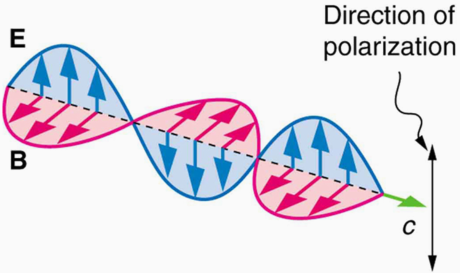
The wave equation shows that disturbances in electric and magnetic fields propagate as an electromagnetic wave (“EM wave”) with a universal speed generally referred to as the **speed of light**.

$$v = c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299\,792\,458\,\mathrm{m/s}$$

The simplified 1D example cannot (because we have ignored the cross-product) that $\vec{E}$ and $\vec{B}$ are actually perpendicular to each other

Electromagnetic (EM) Wave

A EM wave is considered to be **polarized** if both $\vec{E}$ (and therefore) $\vec{B}$ of the wave are confined to a single plane. The direction of the polarization is the direction of $\vec{E}$.



“Failure” of Maxwell’s Equation

A peculiar feature of Maxwell’s equation:

- When applying *Galilean transformation* (our classical equation for *relative motion*) to Maxwell’s equations, they seem to “fail”
- Gauss’s law for magnetism break down: magnetic field lines appear to have beginnings/ends
- So does that mean that in *some* inertial frames of reference, Maxwell’s equations are valid, but in others, they are not?
- Physicists theorized that, perhaps, there is/are actually *preferred* inertial frame(s) of references
- This violate the long-standing *principle of relativity*, which says that *the laws of physics are equal in all inertial frames of reference*

Making The Equations Work Again

Maxwell's equations didn't "fail"; it was our understanding of space and time that needed to change

- Albert Einstein believed in the principle of relativity, and rejected the concept of a preferred frame of reference
- In Maxwell's equations, the speed of an electromagnetic wave (speed of light) is independent of the frame of reference
- In order to make the equations to work again, Einstein revisited the most basic concepts involved in our understanding of physics: space and time

Einstein and Special Relativity

Einstein's Postulates of Special Relativity:

1. **Principle of relavity:** All laws of physics must apply equally in all inertial frames of reference.
2. **Principle of invariant speed of light:** As measured in any inertial frame of reference, light always propagated in a vacuum with a definite velocity c_0 that is independent of the state of motion of the emitting body.

Published in 1905 in the article *On the Electrodynamics of Moving Bodies* when Einstein was 26 years old working as a patent clerk in Switzerland