Topic 5: Circular Motion

Advanced Placement Physics 1

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Olympiads School

Files to Download

Please download the following files from the school website if you have not already done so:

- `PhysAP1-05-circMotion.pdf`—The “print version” of the class slides for this topic.
- `PhysAP1-05-Homework.pdf`—Homework problems for this topic.

Please download/print the PDF file for the class slides before each class. There is no point copying notes that are already on the slides. Instead, focus on things that aren’t necessarily on the slides. If you wish to print the slides, we recommend printing 4 slides per page.

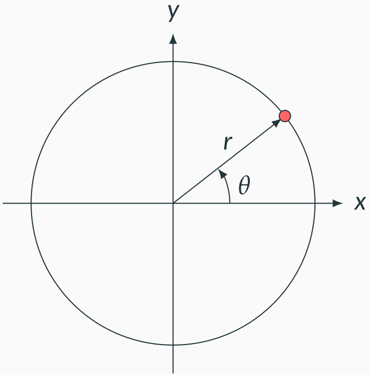
Review of Circular Motion

In **circular motion**, an object of mass m moves in a circular path about a fixed center. In Grade 12 Physics, you were introduced to *uniform* circular motion, where:

- the object's speed (magnitude of velocity) is constant
- the object's **centripetal acceleration** is toward the center
- the object's acceleration is caused by a **centripetal force**

Rigid-Body Circular Motion

Angular Position and Angular Velocity



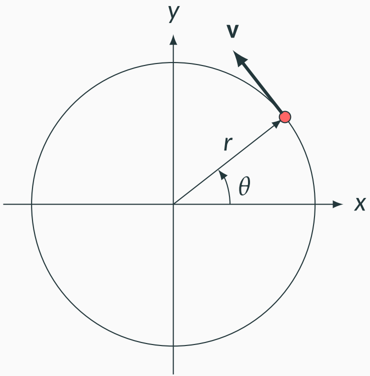
For circular motion with constant radius *r*, the **angular position** $\theta(t)$ fully describes an object's position. It is generally measured in radians (rad):

$$\boxed{\theta = \theta(t)}$$

Average angular velocity $\overline{\omega}$ (or **angular frequency**) is the change in angular position over a finite time interval. It is measured in rad/s.

$$\boxed{\overline{\omega} = \frac{\Delta\theta}{\Delta t}}$$

Velocity and Angular Velocity

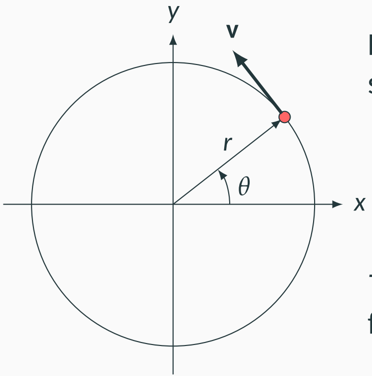


The actual velocity of an object in circular motion is related to the angular velocity by:

$$v(t) = r\omega(t)$$

- The direction of **v** is tangent to circle
- If $\omega > 0$, the motion is counter-clockwise
- If $\omega < 0$, the motion is clockwise

Period & Frequency



For a constant ω (uniform circular motion), the motion is strictly periodic; its **frequency** f and **period** T given by:

$$\boxed{f = \frac{\omega}{2\pi}} \quad \boxed{T = \frac{2\pi}{\omega}} \quad \boxed{f = \frac{1}{T}}$$

T is measured in seconds (s) and f in hertz (Hz). Period and frequency are reciprocals of each other.

Angular Acceleration

The change in anguler velocity $\Delta\omega$ over a finite time interval Δt is **average angular acceleration** $\overline{\alpha}$, with a unit of rad/s^2:

$$\boxed{\overline{\alpha} = \frac{\Delta\omega}{\Delta t}}$$

Similar to the relationship between velocity and angular velocity, **average tangential acceleration** $\overline{a}_t$ is related to angular acceleration $\overline{\alpha}$ by the radius r:

$$\boxed{\overline{a}_t = \frac{\Delta v}{\Delta t} = \frac{r\Delta\omega}{\Delta t} = r\overline{\alpha}}$$

For *uniform* circular motion (constant ω), $\alpha = 0$ and $a_t = 0$

Kinematics in the Angular Direction

For constant angular acceleration α, the kinematic equations are the same in rectilinear motion, but with θ replaces x, ω replaces v, and α replaces a:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \theta_0 + \alpha t$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

A Simple Example

Example 1: An object moves in a circle with angular acceleration $3.0\,\mathrm{rad/s^2}$. The radius is $2.0\,\mathrm{m}$ and it starts from rest. How long does it take for this object to finish a circle?

Centripetal Acceleration & Centripetal Force

There is also a component of acceleration toward the center of the rotation, called the **centripetal acceleration** a_c:

$$\boxed{a_c = \frac{v^2}{r} = \omega^2 r}$$

The force that causes the centripetal acceleration is called the **centripetal force**, also toward the center of rotation:

$$\boxed{F_c = ma_c = \frac{mv^2}{r}}$$

Centripetal Acceleration for Uniform Circular Motion

In uniform circular motion ($\alpha = 0$, constant ω) problems where the period or frequency are known, the speed of the object is:

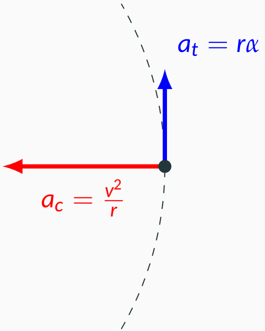
$$v = \frac{2\pi r}{T} = 2\pi r f$$

Centripetal acceleration can therefore be expressed based on T or f:

$$a_c = \frac{v^2}{r^2} \quad \rightarrow \quad \boxed{a_c = \frac{4\pi^2 r}{T^2} = 4\pi^2 r f^2}$$

Acceleration: The General Case

In general circular motion, there are two components of acceleration:



Centripetal acceleration a_c

- Depends on radius of curvature r and instantaneous speed v.
- The direction of a_c is toward the center of the circle.

Tangential acceleration a_t

- Depends on radius r and angular acceleration α.
- The direction of the acceleration is tangent to the circle, which is the same as the velocity vector $\mathbf{v}$.

How to Solve Circular Motion Problems

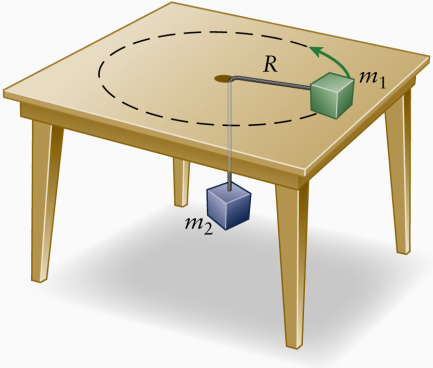
The condition for circular motion is the second law of motion:

$$\mathbf{F}_c = \sum \mathbf{F} = m\mathbf{a}_c$$

The forces that generate the centripetal force comes from the free-body diagram. It may include:

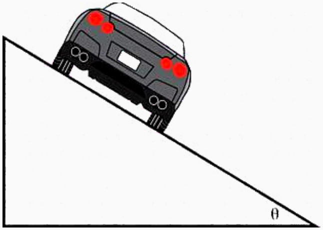
- Gravity
- Friction
- Normal force
- Tension
- Etc.

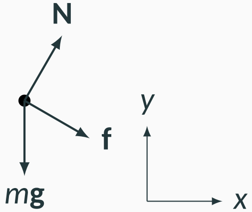
Example: Horizontal Motion



Example 2: In the figure on the left, a mass $m_1 = 3.0\,\text{kg}$ is rolling around a frictionless table with radius $R = 1.0\,\text{m}$. with a speed of $2.0\,\text{m/s}$. What is the mass of the weight m_2?

Banked Curves on Highways and Racetracks





No motion in the y direction, therefore $\sum F_y = 0$:

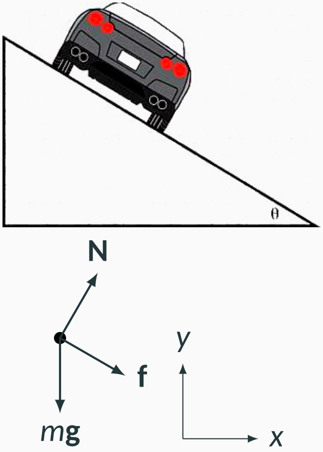
$$N\cos\theta - f\sin\theta - w = 0$$

Net force in the x direction is the centripetal force, i.e. $\sum F_x = ma_c$

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r}$$

Friction force **f** may be static or kinetic, depending on the situation.

Banked Curves on Highways and Racetracks

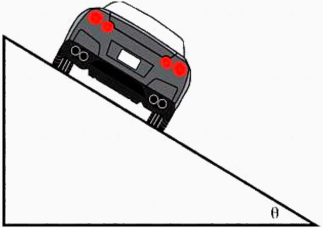


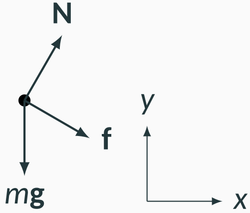
For analysis, use the simplified equation for friction $f = \mu N$ (i.e. assume either kinetic friction or maximum static friction), and weight $m\mathbf{g}$, the equations on the previous slides can be arranged as:

$$N\left(\cos\theta - \mu \sin\theta\right) = mg$$

$$N\left(\sin\theta + \mu \cos\theta\right) = \frac{mv^2}{r}$$

Banked Curves on Highways and Racetracks





Dividing the two equations removes both the normal force and mass terms:

$$\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg}$$

The *maximum* velocity v_{max} can be expressed as:

$$\boxed{v_{\text{max}} = \sqrt{rg\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}}}$$

Note that v_{max} does not depend on mass.

Banked Curves on Highways and Racetracks

In the limit of $\mu = 0$ (frictionless case), the equation reduces to:

$$\boxed{v_{\text{max}} = \sqrt{rg\tan\theta}}$$

And in the limit of a flat roadway with no banking ($\theta = 0$, $\sin\theta = 0$ and $\cos\theta = 1$), the equation reduces to:

$$\boxed{v_{\text{max}} = \sqrt{\mu rg}}$$

Vertical Circles

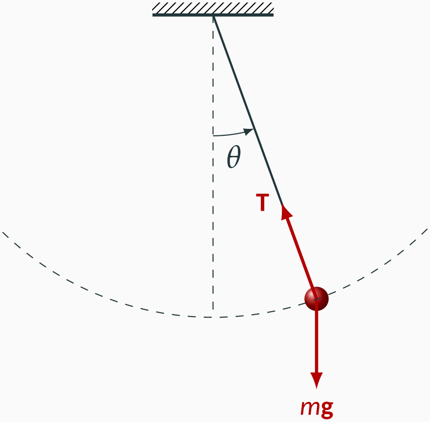
Vertical Circles

Circular motion with a horizontal path is straightforward. However, for vertical motion:

- Generally difficult to solve by dynamics and kinematics
- Instead, use conservation of energy to solve for speed v
- Then use the equation for centripetal force to find other forces

What About a Pendulum?

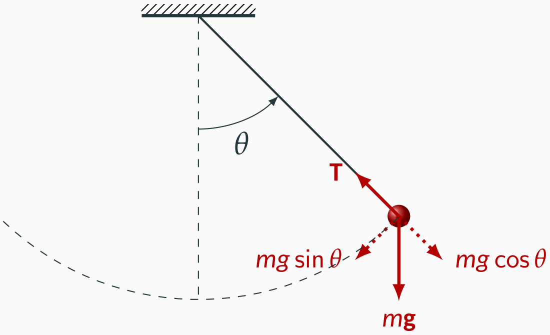
A simple pendulum is also like a vertical circular motion problem.



- There are two forces act on the pendulum: weight $m\mathbf{g}$, and tension $\mathbf{T}$
- Speed of the pendulum at any height is found using conservation of energy
  - $\mathbf{T}$ is $\perp$ to motion, therefore it does not do work
  - Work is done by gravity (conservative!) alone
- Tangential and centripetal accelerations are based on the net force along the angular and radial directions

Simple Pendulum

At the top of the swing, velocity v is zero, therefore:



Centripetal acceleration is also zero:

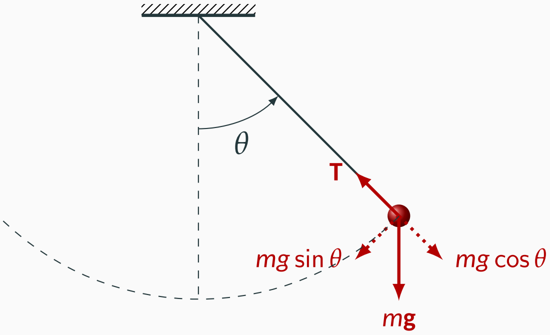
$$a_c = \frac{v^2}{r} = 0$$

and therefore the net force along the radial direction is zero. The tension force T can be calculated:

$$T = mg\cos\theta$$

At the highest point when θ is largest, tension is the lowest.

Simple Pendulum



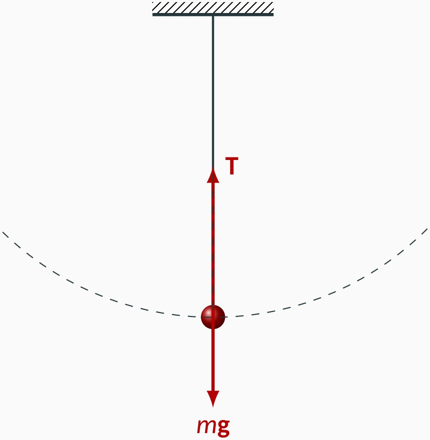
In the tangential direction , there is a net force of $mg \sin\theta$, therefore, a tangential acceleration along that direction, with a magnitude of:

$$a_t = g \sin\theta$$

This is the same acceleration as an object sliding down a frictionless ramp at an angle of θ.

Simple Pendulum

At the bottom of the swing, the velocity is at its maximum value,



- Maximum centripetal acceleration:

$$a_c = \frac{v^2}{r}$$

- No tangential acceleration:

$$a_t = 0$$

- At the lowest point, tension is the highest:

$$T = w + F_c = m\left(g + \frac{v^2}{r}\right)$$

Example Problem

Example 4: You are playing with a yo-yo with a mass M. The length of the string is R. You decide to see how slowly you can swing it in a vertical circle while keeping the string fully extended, even when it is at the top of its swing.

a. Calculate the minimum speed at which you can swing the yo-yo while keeping it on a circular path.
b. Find the tension in the string when the yo-yo is at the side and at the bottom of its swing.

Example: Roller Coaster

Example 5: A roller coaster car is on a track that forms a circular loop, of radius R, in the vertical plane. If the car is to maintain contact with the track at the top of the loop (generally considered to be a good thing), what is the minimum speed that the car must have at the bottom of the loop. Ignore air resistance and rolling friction.

A. $\sqrt{2gR}$

B. $\sqrt{3gR}$

C. $\sqrt{4gR}$

D. $\sqrt{5gR}$