Class 14: Electrostatics Part 1 (Point Charges)

Advanced Placement Physics C

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Olympiads School

Electrostatic Force

Review: The Charges Are

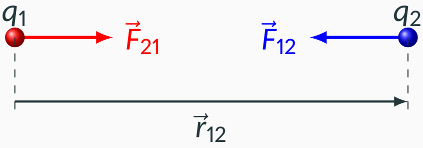
We should already know a bit about charge particles:

- A **proton** carries a **positive** charge
- An **electron** carries a **negative** charge
- A *net charge* of an object means an excess of protons or electrons
- Similar charges are repel; opposite charges attract

We start with electrostatics:

- Charges that are not moving relative to one another

Coulomb's Law for Electrostatic Force



The **electrostatic force** (or **coulomb force**) is a mutually repulsive/attractive force between all charged objects. The force that charge q_1 exerts on q_2 is given by **Coulomb's law**:

$$\boxed{\vec{F}_{12} = \frac{kq_1q_2}{|\vec{r}_{12}|^2}\hat{r}_{12}}$$

Coulomb's Law for Electrostatic Force

$$\boxed{\vec{F}_{12} = \frac{kq_1q_2}{\left|\vec{r}_{12}\right|^2}\hat{r}_{12}}$$

Quantity	Symbol	SI Unit
Electrostatic force	$\vec{F}_{12}$	N
Coulomb's constant	k	$\mathrm{N} \cdot \mathrm{m}^2 / \mathrm{C}^2$
Point charges 1 and 2	q_1, q_2	C
Distance between point charges	$\left|\vec{r}_{12}\right|$	m
Unit vector of direction between point charges	$\hat{r}_{12}$	

Coulomb's Constant

$$\boxed{\vec{F}_{12} = \frac{kq_1q_2}{|\vec{r}_{12}|^2}\hat{r}_{12}}$$

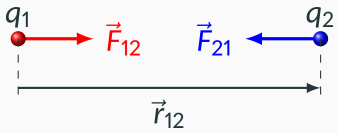
The constant k in the Coulomb's law is called the **coulomb's constant**, defined as:

$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \, \mathrm{N \cdot m^2/C^2}$$

where ϵ_0 is a fundamental constant called the **permittivity of free space**, or **vacuum permittivity**. It measures a vacuum's ability to resist the formation of an electric field:

$$\epsilon_0 = 8.85 \times 10^{-12} \, \mathrm{C^2/N \cdot m^2}$$

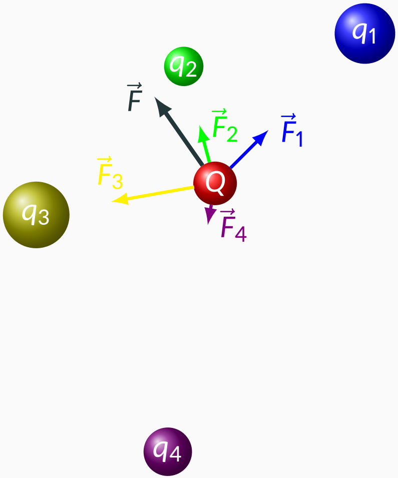
Coulomb's Law for Electrostatic Force



- Third law of motion: If q_1 exerts an electrostatic force $\vec{F}_{12}$ on q_2, then q_2 likewise exerts a force of $\vec{F}_{21} = -\vec{F}_{12}$ on q_1. The two forces are equal in magnitude and opposite in direction.
- q_1 and q_2 are assumed to be *point charges* that do not occupy any space
- The scalar form is often used as well, since the direction of F_q can easily be found:

$$\boxed{F_q = \frac{kq_1q_2}{r^2}}$$

More Than One Charge



For a charge Q that is subjected to the influence of multiple discrete point charges q_i, the total electrostatic force that Q experiences is the vector sum of all the forces $\vec{F}_i$:

$$\boxed{\vec{F} = \sum_i \vec{F}_i = kQ\left(\sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{r}_i\right)}$$

Continuous Distribution of Charges

As $N \to \infty$, the summation becomes an integral, and can now be used to describe the force from charges with *spatial extend* i.e. charges that take up physical space (e.g. a continuous distribution of charges):

$$\boxed{\vec{F} = \int \mathrm{d}\vec{F} = kQ \int \frac{\mathrm{d}q}{r^2}\hat{r}}$$

Infinitesimal Charge dq

The calculation for the infinitesimal charge dq is similar to the calculation for the infinitesimal mass dm earlier in the course (See Class 5: Center of Mass)

- Linear charge density (for 1D problems)

$$\gamma = \frac{\mathrm{d}q}{\mathrm{d}L} \quad \rightarrow \quad \mathrm{d}q = \gamma \mathrm{d}L$$

- Surface charge density (for 2D problems)

$$\sigma = \frac{\mathrm{d}q}{\mathrm{d}A} \quad \rightarrow \quad \mathrm{d}q = \sigma \mathrm{d}A$$

- Charge density (for 3D problems)

$$\rho = \frac{\mathrm{d}q}{\mathrm{d}V} \quad \rightarrow \quad \mathrm{d}q = \rho \mathrm{d}V$$

Electric Field

Electric Field

The expression for **electric field** is obtained by repeating the same procedure as with gravitational field, by grouping the variables in Coulomb's law:

$$F_q = \underbrace{\left[\frac{kq_1}{|\vec{r}_{12}|^2} \hat{r} \right]}_{\vec{E}} q_2$$

The electric field $\vec{E}$ created by q_1 is a vector function (called a **vector field**) that shows how it influences other charged particles around it.

Electric Field Near a Point Charge

The electric field a distance r away from a point charge q is given by:

$$\boxed{\vec{E}(q,\vec{r}) = \frac{kq}{|\vec{r}|^2}\hat{r}}$$

Quantity	**Symbol**	**SI Unit**
Electric field intensity	$\vec{E}$	N / C
Coulomb's constant	k	$\mathrm{N \cdot m^2 / C^2}$
Source charge	q	C
Distance from source charge	$\lvert\vec{r}\rvert$	m
Outward unit vector from point source	$\hat{r}$	

The direction of $\vec{E}$ is radially outward from a positive point charge and radially inward toward a negative charge.

More Than One Charge

When multiple point charges are present, the total electric field at any position $\vec{r}$ is the vector sum of all the fields $\vec{E}_i$:

$$\boxed{\vec{E} = \sum_i \vec{E}_i = k \left(\sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{r}_i \right)}$$

More Than One Charge

As $N \to \infty$, the summation becomes an integral, and can now be used to describe the electric field generated by charges with *spatial extend*:

$$\boxed{\vec{E} = \int \mathrm{d}\vec{E} = k \int \frac{\mathrm{d}q}{r^2}\hat{r}}$$

This integral may be difficult to compute if the geometry of is complicated, but in general, in AP Physics C, there are usually symmetry that can be exploited.

Think Electric Field

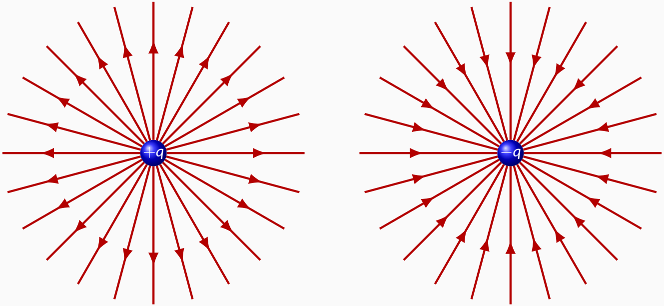
$\vec{E}$ itself *doesn't do anything* until another charge interacts with it. And when there is a charge q, the electrostatic force $\vec{F}_q$ that the charge experiences is proportional to q and $\vec{E}$, regardless of how the electric field is generated:

$$\boxed{\vec{F}_q = q\vec{E}}$$

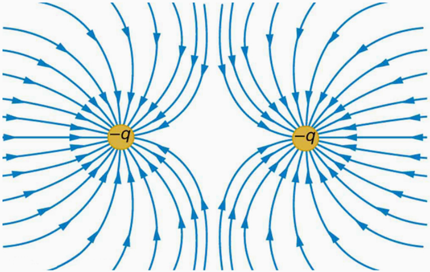
A positive charge in the electric field experiences an electrostatic force $\vec{F}$ in the same direction as $\vec{E}$.

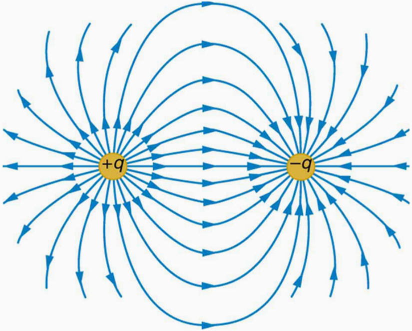
Electric Field Lines

Electric field lines can be used to visualize the direction of the electric field.



Electric Field from Multiple Charges

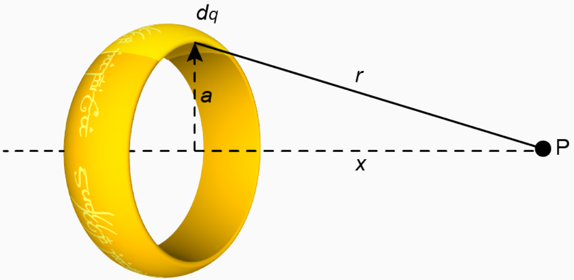




- Electric field lines must begin and/or end at a charge
- Field lines do not cross
- Direction of the electric field is tangent to the field lines

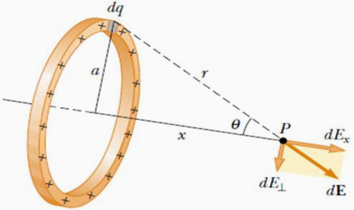
Lord of the Ring Charge

Suppose you have been given *The One Ring To Rule Them All*, and you found out that it is charged! What is its electric field at point *P* along its axis?



Note that calculating the electric field away from the axis is very difficult.

Electric Field Along Axis of a Ring Charge



- We can separate the electric field $d\vec{E}$ (generated by charge dq) into axial (dE_x) and radial ($dE_\perp$) components
- Based on symmetry, $dE_\perp$ doesn't contribute to anything; but dE_x is pretty easy to find:

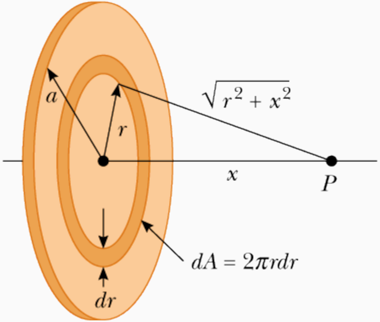
$$dE_x = \frac{k\,dq}{r^2}\cos\theta = \frac{k\,dq}{r^2}\frac{x}{r} = \frac{kx\,dq}{(x^2+a^2)^{3/2}}$$

Integrating this over all charges dq, we have:

$$E_x = \frac{kx}{(x^2+a^2)^{3/2}}\int dq = \boxed{\frac{kQx}{(x^2+a^2)^{3/2}}}$$

Electric Field Along Axis of a Uniformly Charged Disk

Let's extend what we know to a disk of radius a and charge density σ



We start with the solution from the ring problem, and replace Q with $dq = 2\pi\sigma r dr$:

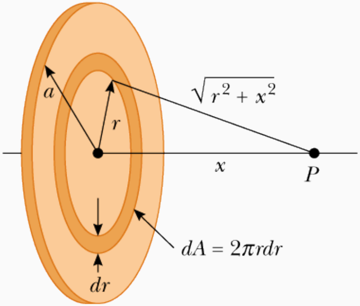
$$dE_x = \frac{2\pi k r \sigma x}{(x^2 + r^2)^{3/2}} dr$$

Integrating over the entire disk:

$$E_x = \pi k x \sigma \int_0^a \frac{2r}{(x^2 + r^2)^{3/2}} dr$$

This is not an easy integral!

Eclectic Field Along Axis of a Uniformly Charged Disk



Luckily for us, the integral is in the form of $\int u^n du$, with $u = x^2 + r^2$ and $n = \frac{-3}{2}$. You can find the integral in any math textbook:

$$E_x = 2\pi k\sigma \left(1 - \frac{x}{\sqrt{x^2 + a^2}}\right)$$

Electric Potential Energy

Electric Potential Energy

The electrostsatic force is a conservative force, therefore the work done by F_q is related to the **electric potential energy** U_q:

$$W = \int \vec{F}_q \cdot \mathrm{d}\vec{r} = kq_1q_2 \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r^2} = -\frac{kq_1q_2}{r}\bigg|_{r_1}^{r_2} = -\Delta U_q$$

where

$$\boxed{U_q = \frac{kq_1q_2}{r}}$$

- U_q can be $(+)$ or $(-)$, because charges can be either $(+)$ or $(-)$
- Positive work done by F_q decreases U_q, while
- Negative work done by F_q increases U_q
- W depends on r_1 and r_2 but not *how* the charge moves from $r_1 \to r_2$

How it Differs from Gravitational Potential Energy

Two positive charges:

$$U_q > 0$$

Two negative charges:

$$U_q > 0$$

One positive and one negative charge:

$$U_q < 0$$

- $U_q > 0$ means positive work is done to bring two charges together from $r = \infty$ to r (both charges of the same sign)
- $U_q < 0$ means negative work (the charges are opposite signs)
- For gravitational potential U_g is always < 0

Relating U_q to $\vec{F}_q$

From the fundamental theorem calculus, we can relate electrostatic force ($\vec{F}_q$) to electric potential energy (U_q) by the gradient operator:

$$\Delta U_q = -\int \vec{F}_q \cdot \mathrm{d}\vec{r} \quad \rightarrow \quad \vec{F}_q(r) = -\nabla U_q = -\frac{\partial U_q}{\partial r}\hat{r}$$

Electrostatic force $\vec{F}_q$ always points from high to low potential energy (steepest descent direction)

Electric Potential

Electric Potential: Using Gravity as Example

An object at a specific location inside a gravitational field has a gravitational potential energy proportional to its mass, i.e.

$$U_g = V_g m$$

This “constant” V_g is called the **gravitational potential**, which is the *gravitational potential energy per unit mass*. In the trivial case with a uniform gravitational field:

$$V_g = \frac{U_g}{m} = gh$$

This also applies to the general case of the gravitational potential energy:

$$V_g = \frac{U_g}{m} = -\frac{Gm}{r}$$

Electric Potential

This is also true for moving a charged particle q against an electric electric field created by q_s, and the “constant” is called the **electric potential**. The unit for electric potential is a *volt* which is *one joule per coulomb*, i.e. $1\,\mathrm{V} = 1\,\mathrm{J/C}$

$$\boxed{V = \frac{U_q}{q}}$$

The electric potential from a source point charge q_s is therefore:

$$\boxed{V = \frac{kq_s}{r}}$$

Electric Potential from Multiple Charges

When there are multiple point charges present, the electric potential is given by the summation:

$$\boxed{V = k\sum_{i=1}^{N} \frac{q_i}{r_i}}$$

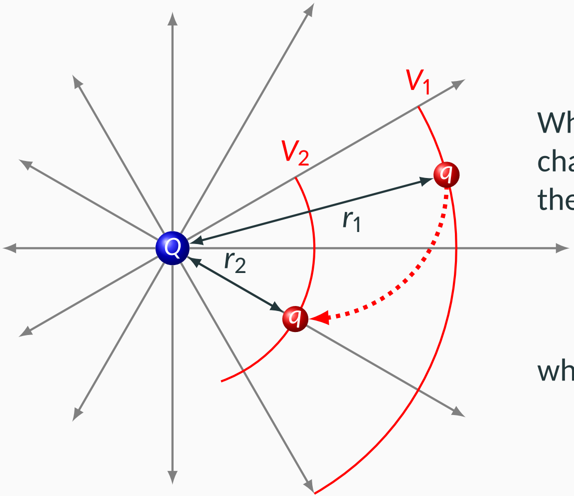
As $N \to \infty$ the summation becomes an integral:

$$\boxed{V = k\int \frac{\mathrm{d}q}{r}}$$

where r is the distance to the infinitesimal charge dq

Electric Potential Difference

Potential Difference

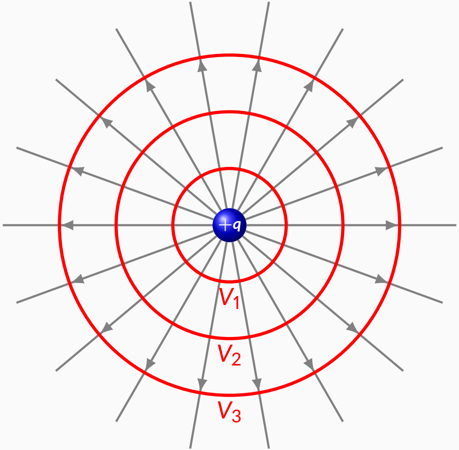


When a charge is moved from r_1 to r_2, the change in electric potential energy is related to the change in electric potential by:

$$\Delta U_q = U_2 - U_1 = q\Delta V$$

where ΔV is called the **potential difference**

Electric Potential

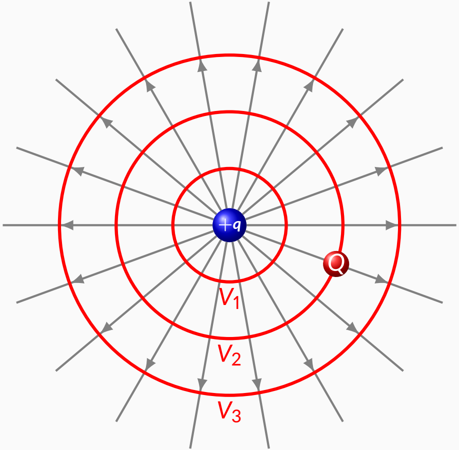


For a point charge q, every point at a distance r will have the same electric potential $V(r)$.

- The red lines have the same electric potential; they are called **equipotential lines**, or **equipotential contours**
- Equipotential lines are perpendicular to the electric field lines
- Electric field lines always points from higher V toward lower V, i.e.

$$V_1 > V_2 > V_3$$

Electric Potential



A charge Q that is placed inside this electric field will now have an electric potential energy of:

$$U_q = QV = Q\left[\frac{kq}{r}\right]$$

in agreement with equation for electric potential energy

Potential Difference (Voltage)

The change in electric potential is called the **electric potential difference** or **voltage**:

$$\boxed{\Delta V = \frac{\Delta U_q}{q}} \quad \text{and} \quad \boxed{dV = \frac{dU_q}{q}}$$

Here, we can relate ΔV to an equation that we knew from Grade 11 Physics and AP Physics 1/2, which related to the energy dissipated in a resistor in a circuit ΔU to the voltage drop ΔV:

$$\boxed{\Delta U_q = q\Delta V}$$

Electric potential difference also has the unit *volts* (V)

Relating V to $\vec{E}$

In the same way that the fundamental theorem of calculus relates the electrostatic force ($\vec{F}_q$) and electric potential energy (U_q) by the gradient operator, electric field ($\vec{E}$) and electric potential (V) are also related the same way:

$$\Delta V_q = -\int \vec{E}_q \cdot \mathrm{d}\vec{r} \quad \rightarrow \quad \vec{E}(r) = -\nabla V_q = -\frac{\partial V}{\partial r}\hat{r}$$

- Electrostatic field $\vec{E}$ always points from high to low electric potential
- Electric field is also called “potential gradient”

Getting Those Names Right

Remember that these three scalar quantities, as opposed to electrostatic force $\vec{F}_q$ and electric field $\vec{E}$ which are vectors

- Electric potential energy:

$$U_q = \frac{kq_1q_2}{r}$$

- Electric potential:

$$V = \frac{U_q}{q}$$

- Electric potential difference (voltage):

$$\Delta V = \frac{\Delta U_q}{q}$$