Class 12: Orbital Mechanics

Advanced Placement Physics C

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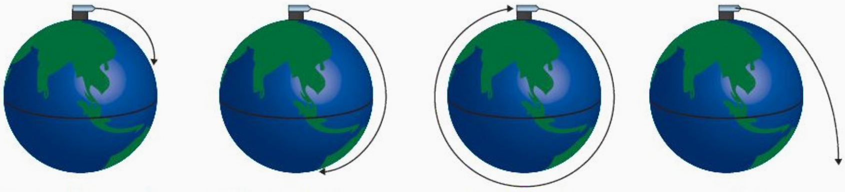
Olympiads School

Orbits

Newton’s Thought Experiment

In *Treatise of the System of the World*, the third book in *Principia*, Newton presented this thought experiment:



- How fast does the cannonball have to travel before it goes around Earth without falling? (i.e. goes into orbit)
- How fast does the cannonball have to travel before it never comes back?

Relating Gravitational and Centripetal Force

Assuming a small mass m in circular orbit around a much larger mass M. The required centripetal force is supplied by the gravitational force:

$$F_g = F_c \quad \longrightarrow \quad \frac{GMm}{r^2} = \frac{mv^2}{r}$$

Solving for v, we obtain the **orbital velocity** v_{orbit}, which does not depend on the mass of the small object in orbit:

$$\boxed{v_{\text{orbit}} = \sqrt{\frac{GM}{r}}}$$

This equation is only applicable for perfectly circular orbits.

Escape Velocity

An object can leave the surface of Earth at any speed. But when all the kinetic energy of that object is converted to gravitational potential energy, it will return back to the surface of the earth. There is, however, a *minimum* velocity at which the object *would not* fall back to Earth.

Escape Velocity

The calculation for escape is a simple exercise in conservation of energy, since gravity is a conservative force, i.e.:

$$K + U_g = K' + U_g'$$

- Initial gravitational potential energy at the surface is:

$$U_g = -\frac{GMm}{r_i}$$

- The final gravitational potential energy is at the other side of the universe ($r_f = \infty$), where $U_g' = 0$. At this point, the object has *escaped* the gravitational pull of the planet/star
- The minimum kinetic energy at $r = \infty$ is $K' = 0$

Escape Velocity from Circular Orbits

Set K to equal to $-U_g$:

$$\frac{1}{2}mv_i^2 = \frac{GMm}{r_i}$$

We can then solve for the initial speed $v_i = v_{\text{esc}}$ (**escape speed** or **escape velocity**):

$$\boxed{v_{\text{esc}} = \sqrt{\frac{2GM}{r_i}}}$$

where r_i is the initial distance from the center of the planet/star. There is a simple relationship between orbital speed and escape speed:

$$v_{\text{esc}} = \sqrt{2}v_{\text{orbit}}$$

Example Problem

Example: Determine the escape velocity and energy for a 1.60×10^4 kg rocket leaving the surface of Earth.

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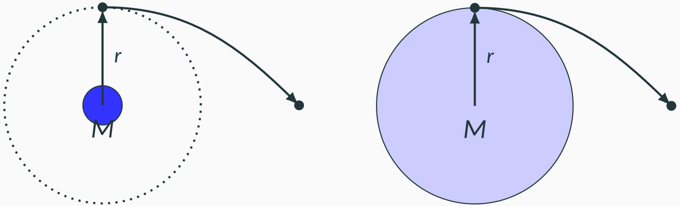
Note: The equation for the escape speed is based on the object have a *constant* mass, which is *not* the case for a rocket going into space.

What if I'm not escaping from the surface?

Both objects have the same escape velocity:

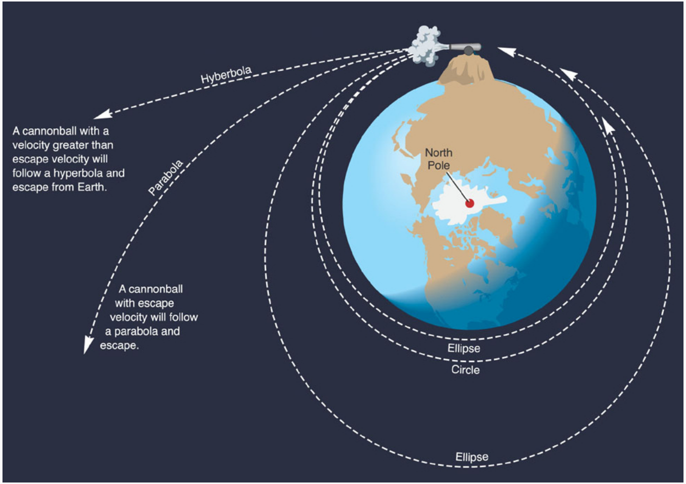


The difference is that the object in orbit (left) already has orbital speed v_{orbit}, so escaping from that orbit requires only an additional speed of

$$\Delta v = v_{\text{esc}} - v_{\text{orbit}} = (\sqrt{2} - 1) v_{\text{orbit}}$$

- What if $v_{\text{orbit}} < v < v_{\text{esc}}$?
- What if $v < v_{\text{orbit}}$?

Non-Circular Orbits



Orbital Energies

We can obtain the **orbital kinetic energy** in a perfectly circular orbit by using the orbital speed in our expression of kinetic energy:

$$K_{\text{orbit}} = \frac{1}{2}mv_{\text{orbit}}^2 = \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2 = \boxed{\frac{GMm}{2r}}$$

We already have an expression for **gravitational potential energy**:

$$U_g = -\frac{GMm}{r} = -2K_{\text{orbit}}$$

The **total orbital energy** is the sum of K and U_g:

$$E_T = K_{\text{orbit}} + U_g = -\frac{GMm}{2r} = -K_{\text{orbit}}$$

Orbital Mechanics

Orbital Mechanics

We turn our attention to applying the law of universal gravitation to the orbital motion of planets and stars in our solar system.

Properties of Gravitational Force

Two properties of gravity are crucial to understanding of orbital mechanics:

1. Gravity is a *conservative force*, in that
   - The total mechanical energy of objects under gravity is constant
   - Work done by gravity converts gravitational potential energy U_g into kinetic energy K; work against gravity converts K into U_g
2. Gravity is a *central force*, in that
   - Gravitational force $\vec{F}_g$ is always in the $-\hat{r}$ direction, i.e. $\vec{F} \times \vec{r} = \vec{0}$
   - Therefore gravity doesn't generate any torque
   - And therefore angular momentum $\vec{L}$ is constant

These two properties are true regardless of the shape of the orbit, and even for objects that are not in orbit at all!

Kepler’s Laws of Planetary Motion

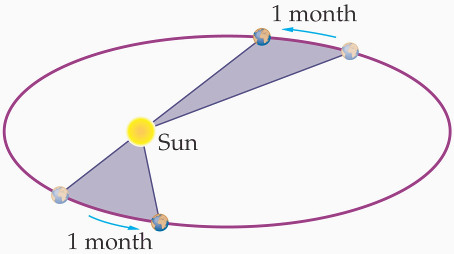
Johannes Kepler (1571–1630) formulated the **laws of planetary motion** between 1609 to 1619, by interpreting planetary motion data from his teacher, Tycho Brahe. It is an improvement over the heliocentric theory of Nicolaus Copernicus. Expressed in modern language:

1. **Law of ellipses:** The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. **Law of equal areas:** A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time
3. **Law of periods:** The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

(For anyone who is interested, there is a handout with the proofs of Kepler’s laws using Newton’s laws of motion.)

Kepler's Second Law: Law of Equal Areas

Law of Equal Areas: A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time



The second law of planetary motion is the easiest to proof, by applying the conservation of angular momentum $\vec{L} = m(\vec{r} \times \vec{v})$ (gravity is a central force).

Kepler's Second Law: Law of Equal Areas

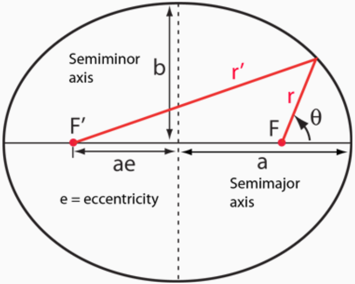
The rate of change of the area ($\mathrm{d}A/\mathrm{d}t$) swept out by a planet (called the **areal velocity**) is given by:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{L}{2m} = \text{constant}$$

The rate a planet sweeps out the area in orbit is its angular momentum around the sun divided by twice its mass.

Kepler’s First Law: Law of Ellipses

Proofing Kepler’s first law requires some understanding the ellipse. If the law is true, then orbital motion must agree with the equations of an ellipse.



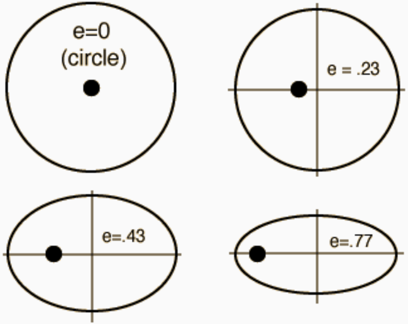
- $r' + r = 2a$
- The area of the ellipse is $A = \pi ab$
- The relationship between r and θ given by:

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \quad \text{where} \quad 0 \leq e < 1$$

- when $e = 0$ it’s a circle: $a = b = r$
- When $e = 1$ it’s no longer an ellipse

Kepler’s First Law: Law of Ellipses

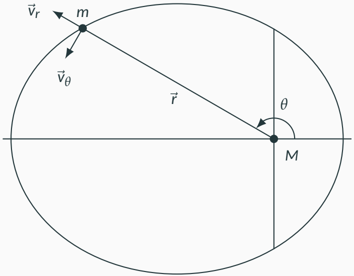
Most planets in the solar system have very small eccentricity, so their orbits are fairly close to being circular, but comets are much more eccentric



Object	e
Mercury	0.206
Venus	0.0068
Earth	0.0167
Mars	0.0934
Jupiter	0.0485
Saturn	0.0556
Uranus	0.0472
Neptune	0.0086
Pluto	0.25
Halley’s Comet	0.9671
Comet Hale-Bopp	0.9951
Comet Ikeya-Seki	0.9999

Kepler's First Law: Law of Ellipses

As m orbits around M, there are two velocity components: **radial velocity** $\vec{v}_r$ and **angular velocity** $\vec{v}_\theta$.



- $\vec{v}_\theta$ means a centripetal acceleration toward M
- Changes in $\vec{v}_r$ (i.e. accceleration in the radial direction) also means a force along $\hat{r}$
- Both components of acceleration are due entirely to gravitational force toward M
- Applying second law of motion gives a complicated (at least for students new to the concept) ordinary differential equation.

A full description for solving the differential equation is presented in the accompanied handout for anyone interested.

Kepler's First Law: Law of Ellipses

The solution to the ODE is the expression for $r(\theta)$, with eccentricity e determined by a constant B based on initial condition (how the planet is formed):

$$r = \left[\frac{L^2}{GMm^2}\right] \frac{1}{1 + e\cos\theta} \quad \text{where} \quad e = \frac{BL^2}{GMm^2}$$

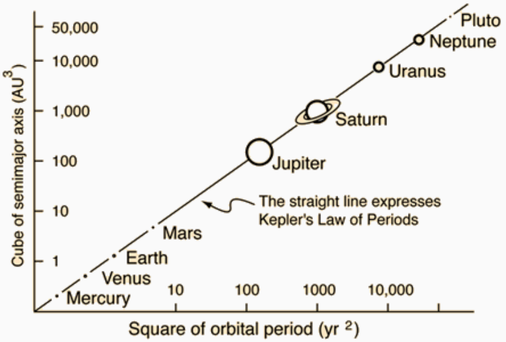
The semi-major axis is the average value between the minimum and maximum values of r:

$$a = \frac{1}{2}(r_{\text{min}} + r_{\text{max}}) = \left[\frac{L^2}{GMm^2}\right] \frac{1}{1 - e^2}$$

We can rearrange the terms to see that this is the equation for an ellipse.

Kepler's Third Law: The Law of Periods

Law of Periods: The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.



Kepler’s Third Law: The Law of Periods

The area swept by the planet through one orbital period is the areal velocity (constant!) integrated by time, from $t = 0$ to $t = T$:

$$A = \int \mathrm{d}A = \int_0^T \frac{\mathrm{d}A}{\mathrm{d}t}\mathrm{d}t = \frac{L}{2m}\int_0^T \mathrm{d}t = \frac{L}{2m}T$$

But this area is an ellipse, given by the equation based on a (semi-major axis), $b = a\sqrt{1-e^2}$ (semi-minor axis):

$$A = \pi ab = \pi a^2\sqrt{1-e^2}$$

Equating two equations above and squaring both sides give this expression:

$$T^2 = \frac{m^2}{L^2}4\pi^2 a^4(1-e^2)$$

Kepler's Third Law: The Law of Periods

But we also (from proving the first law) have:

$$a(1 - e^2) = \frac{L^2}{GMm^2}$$

Substituting this expression into the equation for the period, and after some simple algebra, we end up with this expression:

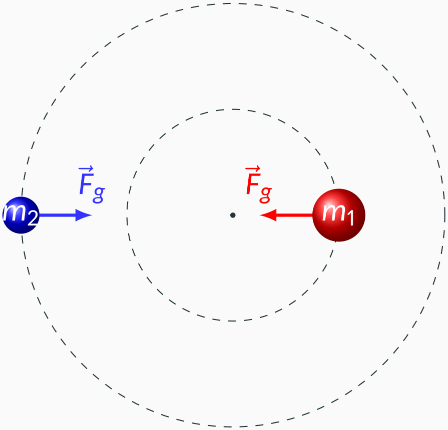
$$\boxed{T^2 = \left[\frac{4\pi^2}{GM}\right] a^3}$$

Reality of Orbital Motion

As always, nothing is as simple as is first seems

- Most AP Problems will be circular instead of elliptical, but you must know the nature of gravitational force (conservative, central)
- The analysis on the slides shown assumes a small mass m orbiting around a large mass M. In reality:
  - Just as planets experience a gravitational force by the Sun, the Sun experiences a gravitational force from the planets
  - The smaller mass m does not actually orbit about the center of M, but rather, the *center of mass between M and m*
  - Especially important when the two objects orbiting each other has similar masses (e.g. a binary star system)

Binary System



In a binary star system, two stars orbit around their center of mass. Both have the same period, and the gravitational force provides the centripetal force, but this time, the distance to the center of motion is empty space.