Topic 7: Rotational Motion of a Rigid Body

Advanced Placement Physics C

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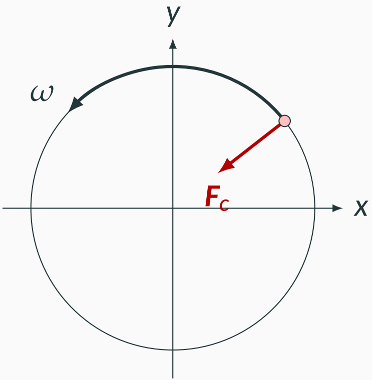
Olympiads School

Introduction

Uniform Circular Motion

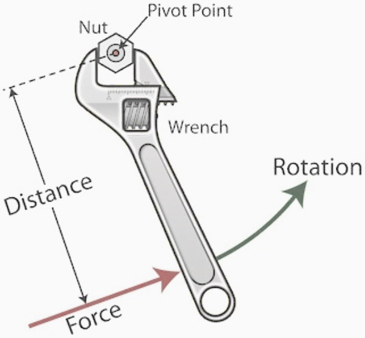
Consider the uniform circular motion of an object with (constant) angular velocity ω [1]



- Centripetal force F_c is always perpendicular to the motion of the object
- F_c not do any mechanical work
- Therefore, angular velocity ω remains constant
- **The "rotational state" of the object does not change**
- Rotation of an object is not determined by merely what forces are acting it

[1] FYI: if the rotation is counterclockwise, the direction of ω is *out of the page*; if rotation is clockwise, ω is *into the page*.

Turning A Wrench



Similarly, when tightening/loosening a bolt by turning a wrench.

- When the nut turns, its “rotational state” changes
- The applied force has to be directed at a distance away from the bolt
- How easy to turn the nut depends on *both* the distance and the force

Torque

Recall the second law of motion for objects with constant mass:

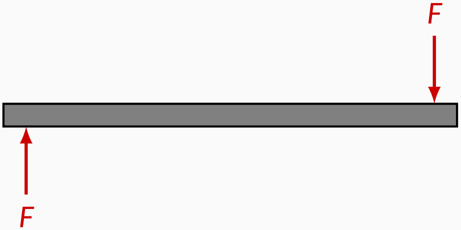
$$\boldsymbol{F}_{\text{net}} = m\boldsymbol{a}$$

Is it also true for *rotational* motion? If a net force $\boldsymbol{F}_{\text{net}}$ causes the center of mass of an object to begin to accelerate, what causes a mass to rotate?

Torque

Torque

I have a rod on a table, and with my fingers, I push the two ends of the rod with equal force F. *What happens?*

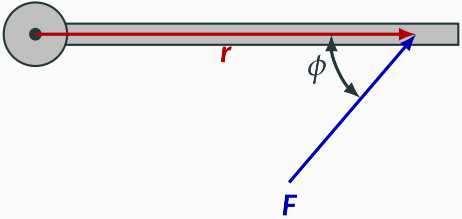


$\boldsymbol{F}_{\text{net}} = \boldsymbol{0}$, therefore $\boldsymbol{a} = \boldsymbol{0}$ at the center of mass. But it is also obvious that the rod would *rotate* instead.

What is Torque?

Torque (or **moment**) is the tendency for a force to change the rotational motion of a body. The unit for torque is a **newton meter** (N· m).

- A force $\boldsymbol{F_a}$ acting at a point some distance $\boldsymbol{r}$ (called the **moment arm**) from a **fulcrum** (or **pivot**) at an angle ϕ between $\boldsymbol{F_a}$ and $\boldsymbol{r}$
- e.g. the force to twist a screw
- In the example below, a force $\boldsymbol{F}$ is applied $\boldsymbol{r}$ away from the pivot at an angle ϕ. This generates a torque around the pivot.



Torque

We can express torque $\boldsymbol{\tau}$ in terms of the force $\boldsymbol{F}$, the **moment arm** $\boldsymbol{r}$ using the cross-product:

$$\boxed{\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}_a}$$

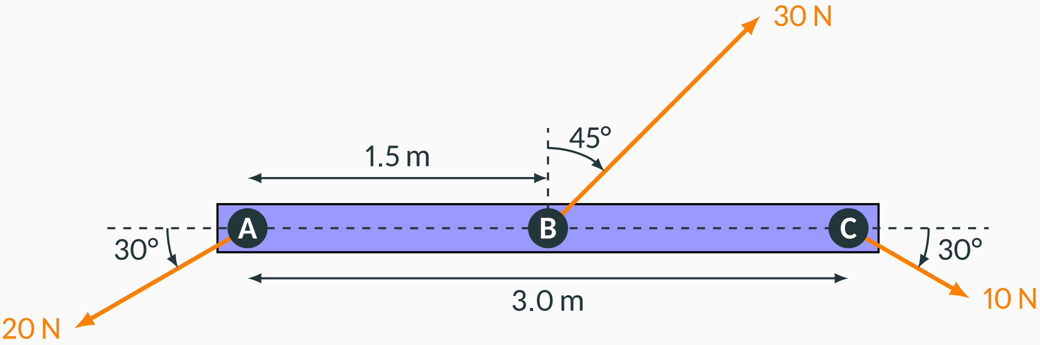
The magnitude of the torque can be calculated in scalar form using the angle ϕ between $\boldsymbol{F}$ and $\boldsymbol{r}$:

$$\boxed{\tau = Fr\sin\phi}$$

Quantity	**Symbol**	**SI Unit**
Torque	$\boldsymbol{\tau}$	N$\cdot$ m
Applied force	$\boldsymbol{F}_a$	N
Moment arm (from fulcrum to force)	$\boldsymbol{r}$	m
Angle between force and moment arm	ϕ	(no units)

Example Problem

Example: Find the net torque on point C.

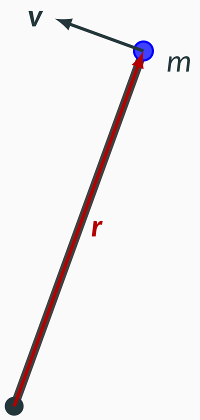


Angular Momentum

Angular Momentum

Consider a mass m connected to a massless beam rotates with speed v at a distance r from the center (shown on the right). It has an **angular momentum** ($\boldsymbol{L}$), defined as:

$$\boxed{\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m(\boldsymbol{r} \times \boldsymbol{v})}$$



Expanding the term with $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$, the expression for angular momentum can now be expressed in quantities related to rotations:

$$\boxed{\boldsymbol{L} = m(\boldsymbol{r} \times \boldsymbol{v}) = m(\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})) = mr^2\boldsymbol{\omega}}$$

Or in scalar form:

$$\boxed{L = rmv = mr^2\omega}$$

The unit for angular momentum is a **kilogram meter squared per second** ($\mathrm{N \cdot m^2/s}$).

Moment of Inertia

Look again at the definition of angular momentum:

$$\boldsymbol{L} = \underbrace{mr^2}_{I}\omega$$

The quantity $I = mr^2$ is called the **moment of inertia** with a unit of **kilogram meter squared** ($\mathrm{kg} \cdot \mathrm{m}^2$), and

$$\boxed{\boldsymbol{L} = I\omega}$$

Momentum of inertia can be considered to be an object's “rotational mass”

Moment of Inertia

For a *single particle* of m rotating at a distance r from the pivot:

$$\boxed{I = mr^2}$$

For a *collection of particles*, each of mass m_i at distance r_i from the pivot:

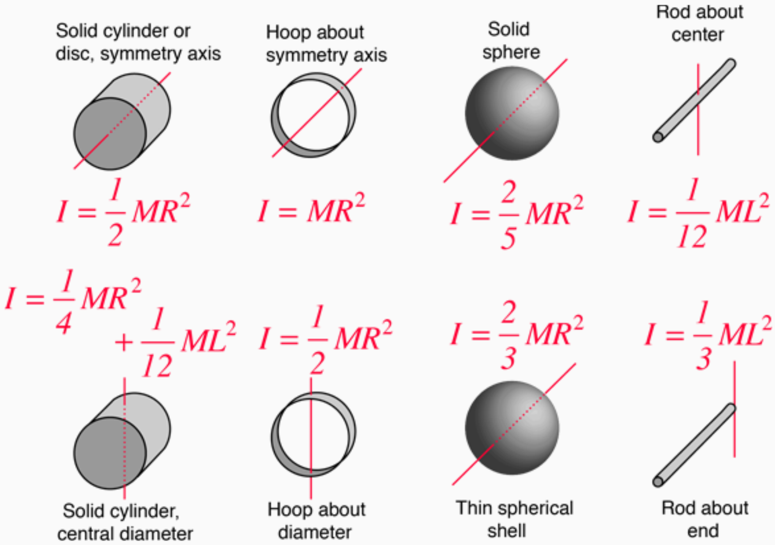
$$\boxed{I = \sum m_i r_i^2}$$

For a *continuous distribution of mass* about a pivot, integral calculus is need to calculate the momentum of inertia[2]

$$\boxed{I = \int r^2 dm}$$

[2] don't worry, no one will ask you to do this in this course!

Moment of Inertia



Angular Momentum and Moment of Inertia

Linear and angular momentum have very similar expressions

$$\boxed{\boldsymbol{p} = m\boldsymbol{v}} \qquad \boxed{\boldsymbol{L} = I\boldsymbol{\omega}}$$

Just as $\boldsymbol{p}$ describes the overall *translational* state of a physical system, $\boldsymbol{L}$ describes its overall *rotational* state

Laws of Motion

Equilibrium: First Law of Motion

An object is in **translational equilibrium** is when the net force acting on it is zero:

$$\boldsymbol{F}_{\text{net}} = \boldsymbol{0}$$

This does *not* mean that the object has no translational motion; it just means that the object's overall *transtational state* is not changing, i.e. the translational momentum $\boldsymbol{p}$ is constant. For constant mass, this means $\boldsymbol{a} = \boldsymbol{0}$.

Equilibrium: First Law of Motion

Likewise, an object is in **rotational equilibrium** when the net torque acting on it is zero:

$$\boldsymbol{\tau}_{\text{net}} = \mathbf{0}$$

This does *not* mean that the object has no rotational motion; it just means that the object's overall *rotational state* is not changing, i.e. $\boldsymbol{\alpha} = \mathbf{0}$, or that the angular momentum $\boldsymbol{L}$ is constant.

Second Law of Motion for Rotational Motion

The net torque is the time rate of change of angular momentum:

$$\boldsymbol{\tau}_{\text{net}} = \mathbf{r} \times \mathbf{F}_{\text{net}} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt} \quad \longrightarrow \quad \boxed{\boldsymbol{\tau}_{\text{net}} = \frac{d\mathbf{L}}{dt}}$$

- If the net torque on a system is zero, then the rate of change of angular momentum is zero, and we say that the angular momentum is conserved.
- e.g. When an ice skater starts to spin and draws his arms inward. Since angular momentum is conserved, a decrease in r means an increase in ω.

Second Law of Motion

For translational motion, the general form of the first and second laws of motion states that the net force is rate of change of the object's momentum:

$$\boldsymbol{F}_{\text{net}} = \frac{d\boldsymbol{p}}{dt}$$

For objects with constant mass, the second law reduces to the more familiar form:

$$\boldsymbol{F} = m\boldsymbol{a}$$

Second Law of Motion for Rotational Motion

Likewise, the second law of motion for rotational motion has a similar form, but with torque $\boldsymbol{\tau}$ replacing force $\boldsymbol{F}$, and angular momentum $\boldsymbol{L}$ replacing linear momentum $\boldsymbol{p}$:

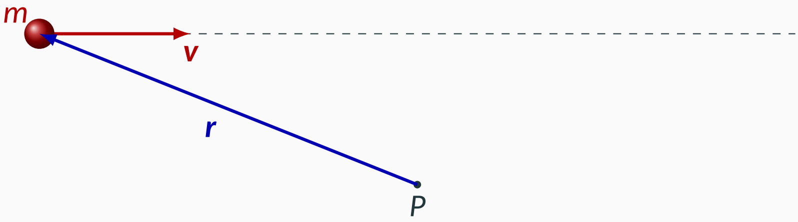
$$\boxed{\boldsymbol{\tau}_{\text{net}} = \frac{d\boldsymbol{L}}{dt}}$$

For objects with constant momentum of inertia I, the second law of motion reduces to:

$$\boldsymbol{\tau}_{\text{net}} = I\boldsymbol{\alpha}$$

But there is no rotational motion, is there?

Even when there is no apparent rotational motion, it does not necessarily mean that angular momentum is zero! In this case, mass m travels along a straight path at constant velocity (uniform motion), but the angular momentum around point P is not zero:



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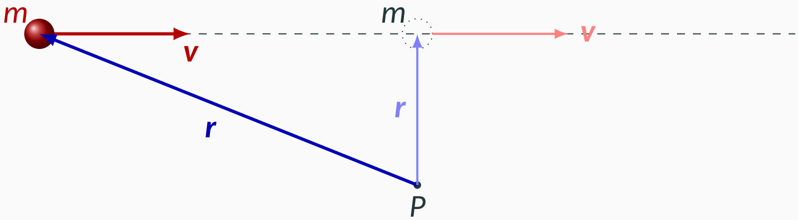

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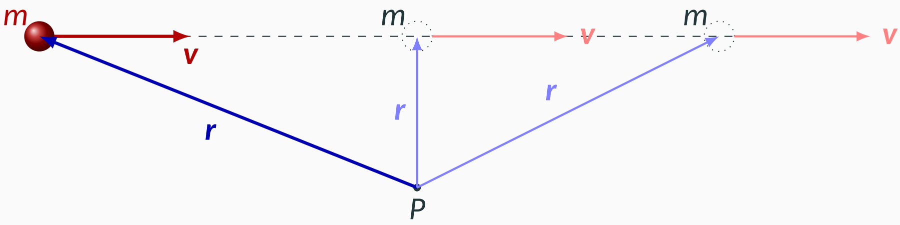

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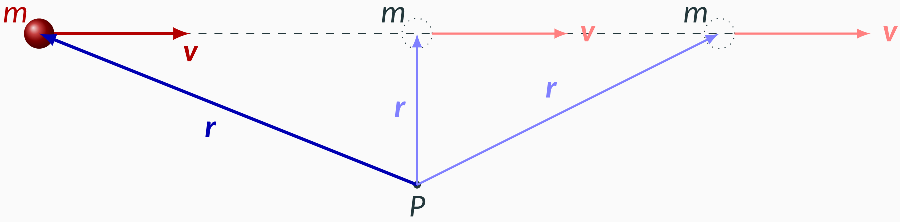

Since there is no force and no torque acting on the object, both the linear momentum ($\boldsymbol{p} = m\boldsymbol{v}$) and angular momentum ($\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{v}$) are constant.

Example Problem

Example: A skater extends her arms (both arms!), holding a 2.0 kg mass in each hand. She is rotating about a vertical axis at a given rate. She brings her arms inward toward her body in such a way that the distance of each mass from the axis changes from 1.0 mto 0.50 m. Her rate of rotation (neglecting her own mass) will?

Example Problem

Example: A 1.0 kg mass swings in a vertical circle after having been released from a horizontal position with zero initial velocity. The mass is attached to a massless rigid rod of length 1.5 m. What is the angular momentum of the mass, when it is in its lowest position?

Solving Rotational Problems

When solving for rotational problems like the ones described in the previous sections:

- Draw a free-body diagram to account for all forces
- The direction of friction force is not always obvious
- The magnitude of any static friction force cannot be assumed to be at maximum.
- If the object is to change its rotational state, there must be a net torque causing it.

Solving Rotational Problems

Once the free-body diagram is complete

- Breaks down the *forces* into $\hat{\imath}$, $\hat{\jmath}$ and $\hat{\boldsymbol{k}}$ components
- We have now three equations for translation, but it is likely that only *one* direction will have forces:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z$$

- And three equations for rotation, and torque is only applied in one direction (likely $\hat{\boldsymbol{k}}$):

$$\sum \tau_x = I_x\alpha_x \qquad \sum \tau_y = I_y\alpha_y \qquad \sum \tau_z = I_z\alpha_z$$

Solving Rotational Problems

For rotational motion dynamics equation:

1. Relate the force(s) that causes rotational motion to the net torque

$$\tau = Fr$$

2. Substitute the expression for momentum of inertia (which has both mass and radius terms in it) into the equation for rotational motion
3. Relate angular acceleration to linear acceleration, if applicable:

$$\alpha = \frac{a}{R}$$

Now there are two equations with force and acceleration terms. See handout

Work & Energy in Rotational Motion

Mechanical Work

For translational motion, mechanical work is defined as

$$W = \int_{x_1}^{x_2} \boldsymbol{F} \cdot d\boldsymbol{x}$$

For rotational motion, mechanical work is defined similarly as:

$$\boxed{W = \int_{\theta_1}^{\theta_2} \boldsymbol{\tau} \cdot d\boldsymbol{\theta}}$$

The work-energy theorem still applies to rotational motion, i.e.;

$$W = \Delta K$$

Rotational Kinetic Energy

To find the kinetic energy of a rotating system of particles (discrete number of particles, or continuous mass distribution), we sum the kinetic energy of the individual particles:

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

It's no surprise that in both case, rotational kinetic energy is given by:

$$\boxed{K = \frac{1}{2} I\omega^2}$$

Kinetic Energy of a Rotating System

The total kinetic energy of a rotating system is the sum of its translational and rotational kinetic energies at its center of mass:

$$\boxed{K = \frac{1}{2}mv_{\text{CM}}^2 + \frac{1}{2}I_{\text{CM}}\omega^2}$$

In this case, I_{CM} is calculated at the center of mass. For simple problems, we only need to compute rotational kinetic energy at the pivot:

$$\boxed{K = \frac{1}{2}I_{\text{P}}\omega^2}$$

In this case, the I_{P} is calculated at the pivot. **IMPORTANT:** $I_{\text{CM}} \neq I_{\text{P}}$