Topic 8: Universal Gravitation & Orbital Mechanics

Advanced Placement Physics

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January, 2020

Olympiads School
Toronto, Ontario, Canada

Intro

Files to Download

If you have not done so already, please download the following files.

- `PhysAP-08-Gravity-print.pdf`—The presentation slides for this topic.
- `PhysAP-08-Homework.pdf`—Homework questions for this topic.

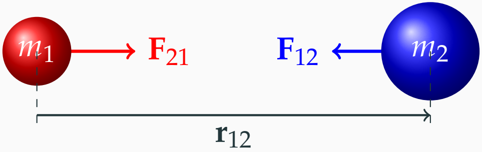
Gravitational Force

Universal Gravitation

This topics we will discuss in this class are covered in AP Physics 1 and C exams.

- Gravitational force ($\mathbf{F}_g$)
- Gravitational field ($\mathbf{g}$)
- Gravitational potential energy (U_g)
- Kepler's laws of planetary motion

Law of Universal Gravitation

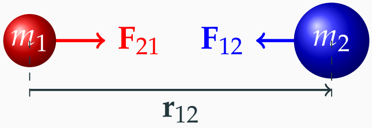


In classical mechanics, **gravity** is the mutual attractive force between all massive objects. The force m_1 exerts on m_2 is given by the law of universal gravitation:

$$\boxed{\mathbf{F}_{12} = -G\frac{m_1 m_2}{|\mathbf{r}_{12}|^2}\hat{\mathbf{r}}_{12}}$$

where $G = 6.674 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$ is the **gravitation constant**, $r = |\mathbf{r}_{12}|$ is the distance between the centers of the masses, and $\hat{\mathbf{r}}_{12} = \mathbf{r}_{12}/|\mathbf{r}_{12}|$ is the unit vector pointing in the direction from m_1 to m_2.

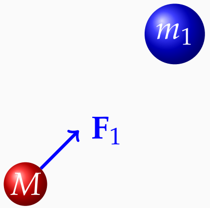
Law of Universal Gravitation



- If m_1 exerts a gravitational force $\mathbf{F}_{12}$ on m_2, then m_2 likewise also exerts a force of $\mathbf{F}_{21} = -\mathbf{F}_{12}$ on m_1. The two forces are equal in magnitude and opposite in direction (third law of motion).
- m_1 and m_2 are assumed to be *point masses* that do not occupy any space
- The (more familiar) scalar form is often used as well:

$$\boxed{F = G\frac{m_1 m_2}{r^2}}$$

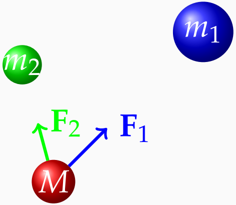
More Than One Mass



For a mass that is subjected to the influence of multiple discrete point masses m_i, the total gravitational force that M experiences is the vector sum of all the forces $\mathbf{F}_i$:

$$\boxed{\mathbf{F} = \sum_i \mathbf{F}_i = GM\left(\sum_{i=1}^{N} \frac{m_i}{r_i^2}\hat{\mathbf{r}}_i\right)}$$

More Than One Mass



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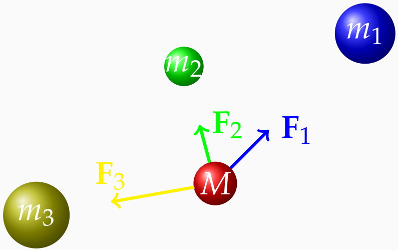

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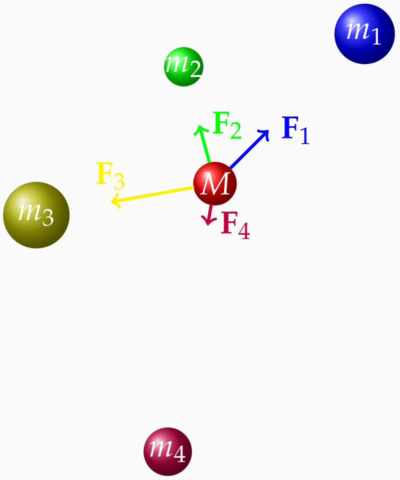

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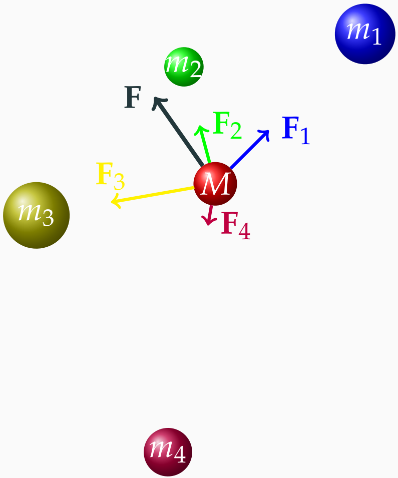

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Continuous Distribution of Mass

At the limit $N \to \infty$, the summation becomes an integral, and can now be used to describe the gravitational force from objects with *spatial extend* i.e. masses that take up space (e.g. a continuous distribution of mass):

$$\boxed{\mathbf{F} = \int d\mathbf{F} = GM \int \frac{dm}{r^2}\hat{\mathbf{r}}}$$

Objects that are symmetrically spherical (e.g. planets are stars in our solar system) can be treated as point masses, and integration can be avoided. However, this is not necessarily the case for some celestial objects.

Gravitational Field

Gravitational Field

We generally describe the gravitational force (weight) as:

$$\boxed{\mathbf{F}_g = m\mathbf{g}}$$

To find $\mathbf{g}$, we group the variables in the law of universal gravitation:

$$\mathbf{F}_g = \underbrace{\left[-\frac{Gm_1}{|\mathbf{r}|^2}\hat{\mathbf{r}}\right]}_{=\mathbf{g}} m = m\mathbf{g}$$

The vector field function $\mathbf{g}$ is known as the **acceleration due to gravity** in kinematics, and **gravitational field** in field theory.

Gravitational Field

On/near the surface of Earth, we can use

$$m_1 = m_\oplus = 5.972 \times 10^{24}\,\mathrm{kg}$$
$$r = r_\oplus = 6.371 \times 10^{6}\,\mathrm{m}$$

to compute the commonly known value of

$$g \approx 9.81\,\mathrm{m/s^2}$$
$$g \approx 9.81\,\mathrm{N/kg}$$

both units are equivalent

Gravitational Field

The **gravitational field** $\mathbf{g}$ generated by a point mass m shows how it influences the gravitational forces on other masses:

$$\boxed{g(m,\mathbf{r}) = -\frac{Gm}{|\mathbf{r}|^2}\hat{\mathbf{r}}}$$

Quantity	Symbol	SI Unit
Gravitational field	$\mathbf{g}$	N/kg
Universal gravitational constant	G	Nm^2/kg^2
Source mass of	m	kg
Distance from source mass	$\|\mathbf{r}\|$	m
Outward radial unit vector from source mass	$\hat{\mathbf{r}}$	N/A

The *direction* of the gravitational field is toward m.

More Than One Mass

When there are multiple point masses present, the total gravitational field at any position $\mathbf{r}$ is the vector sum of all the forces $\mathbf{F}_i$:

$$\boxed{\mathbf{g} = \sum_i \mathbf{g}_i = G\left(\sum_i \frac{m_i}{r_i^2}\hat{\mathbf{r}}_i\right)}$$

At the limit $N \to \infty$, the summation becomes an integral, and can now be used to describe the gravitational field generated by objects with *spatial extend*:

$$\boxed{\mathbf{g} = \int d\mathbf{g} = G\int \frac{dm}{r^2}\hat{\mathbf{r}}}$$

This integral may be difficult to compute, if the geometry of the object is complicated.

Relating Gravitational Field & Gravitational Force

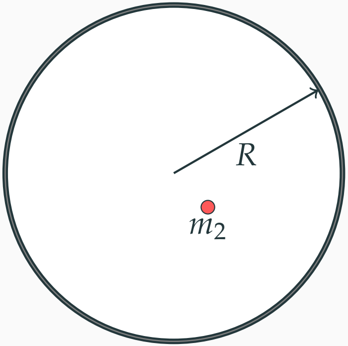
$\mathbf{g}$ itself doesn't *do* anything until there is another mass m inside the field. Then, m experiences a gravitational force $\mathbf{F}_g$ proportional to m and $\mathbf{g}$, regardless of how the field is created:

$$\boxed{\mathbf{F}_g = m\mathbf{g}}$$

Quantity	Symbol	SI Unit
Gravitational force on a mass	$\mathbf{F}_g$	N
Mass inside the gravitational field	m	kg
Gravitational field	$\mathbf{g}$	N/kg

Note: A point mass is not affected by the gravitational field that itself generates.

What If You Are Inside Another Mass?

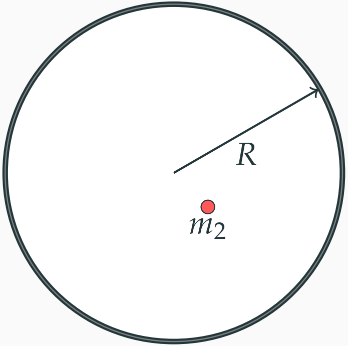


Newton used the **shell theorem** to show that if a mass m_2 is *inside* a spherical shell of mass m_1, the gravitational force that it experiences is *zero*.

$$\boxed{\mathbf{F}_g = \begin{cases} \mathbf{0} & \text{if } r < R \\ -Gm_1m_2/r^2\hat{\mathbf{r}} & \text{otherwise} \end{cases}}$$

It also means that gravitational field is also *zero*

What If You Are Inside Another Mass?

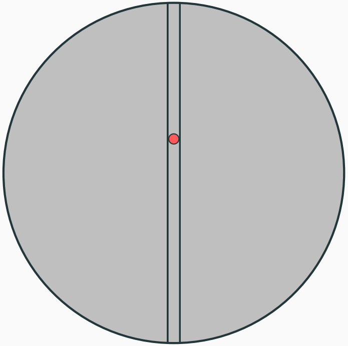


That $\mathbf{g}_{\text{inside}} = \mathbf{0}$ can be calculated by:

- Integrating the fields created by infinitesimal mass elements dm at any point inside the shell, or
- Using **Gauss's law** for gravity, similar to finding the electric field inside a charged conducting sphere:

$$\oint \mathbf{g} \cdot d\mathbf{A} = -4\pi G M_{\text{encl}}$$

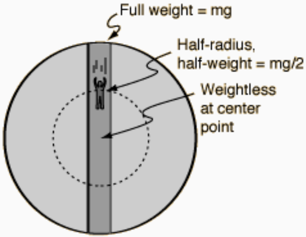
What If You Are Inside Another Mass?



Suppose you could drill a hole through the Earth and then jump into it. How long would it take you to emerge on the other side of the Earth?

To calculate this, we need to know how the gravitational force changes as you fall through Earth.

Falling Toward the Center of Earth



As you fall through Earth, we can separate the part of Earth that is "above" you, and the part that is "below" you

- The part that is "above" you is like the spherical shell, and does not contribute to the gravitational field, and therefore does not exert any force
- The part that is "below" you gets smaller as you fall toward the center

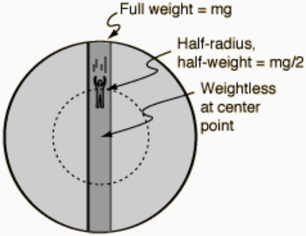
Falling Toward the Center of Earth

Assuming that Earth's density is uniform, and neglecting air resistance and other factors, the value of g as the person falls through Earth ($r < R$) is given by finding how much mass is still "below" the person, $M(r)$:

$$g(r) = \frac{GM(r)}{r^2} \quad M(r) = \frac{4}{3}\rho\pi r^3 \quad \rho = \frac{3M_\oplus}{4\pi r_\oplus^3}$$

where $M_\oplus$ is the mass of Earth, $r_\oplus$ is the radius of Earth, ρ is the (constant) density, and r is the distance from Earth's center. Then $M(r)$ is the amount of mass "below" the person as he/she falls toward the center.

Falling Toward the Center of Earth



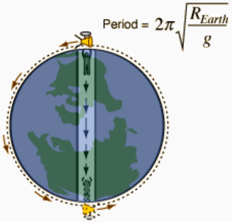
The gravitational field strength inside this hypothetical Earth is a linear function of distance r from the center:

$$g(r) = \frac{GM_{\oplus} r}{r_{\oplus}^3} = \left[\frac{g_0}{r_{\oplus}}\right] r$$

where $g_0 = 9.81\,\mathrm{N/kg}$ is the field strength at the surface. At the center $(r = 0)$, $g = 0$. The gravitational force is:

$$F_g(r) = -\underbrace{\left[\frac{m g_0}{r_{\oplus}}\right]}_{k} r$$

Falling Toward the Center of Earth



A satellite at the Earth's radius would have the same period as one falling through the Earth.

The gravitational force has the same form as Hooke's law: it is proportional to displacement from the center, but in the opposite direction:

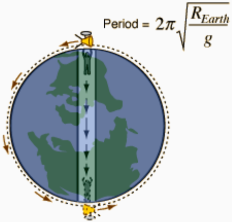
$$F_g(r) = -kr$$

The motion is a simple harmonic motion. The traveller will oscillate through Earth with a period of:

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{r_\oplus}{g_0}}$$

For Earth, $T = 5068\,\mathrm{s}$. The traveller would pop up on the opposite side every 42 min.

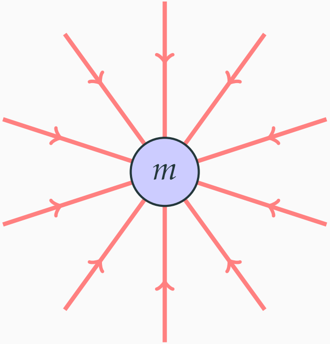
Falling Toward the Center of Earth



A satellite at the Earth's radius would have the same period as one falling through the Earth.

Since simple harmonic motion is a projection of a uniform circular motion, if a satellite is in a circular orbit just above the surface, and passes overhead just above the traveller as he/she popped up out of the hole. The period of such an orbit would be the same as oscillating traveller.

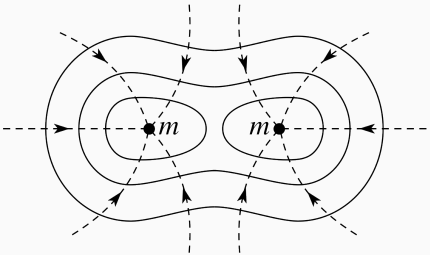
Gravitational Field Lines



- The direction of $\mathbf{g}$ is toward the center of the object that created it
- Field lines do not tell the intensity (i.e. magnitude) of $\mathbf{g}$, only the direction

Gravitational Field Lines

When there are multiple masses, the total gravitational field (dotted line) is the vector sum of all the individual fields.



The solid lines are called **equipotential lines**, where the potential energy is constant. Equipotential lines are perpendicular to gravitational field lines.

Gravitational Potential Energy

Gravitational Potential Energy

Gravitational potential energy is found by integrating the work equation and using the law of universal gravitation:

$$W = \int \mathbf{F}_g \cdot d\mathbf{r} = -\int_{r_1}^{r_2} \frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \cdot d\mathbf{r}$$

$$= -\int_{r_1}^{r_2} \frac{Gm_1m_2}{r^2} dr = \frac{Gm_1m_2}{r}\bigg|_{r_1}^{r_2} = -\Delta U_g$$

where

$$\boxed{U_g = -G\frac{m_1m_2}{r}}$$

- U_g is the work required to move two objects from r to ∞
- $U_g = 0$ at $r = \infty$ and *decrease* as r decreases

Relating Gravitational Potential Energy to Force

The fundamental theorem of calculus shows that gravitational force ($\mathbf{F}_g$) is the negative gradient of the gravitational potential energy (U_g):

$$\mathbf{F}_g = -\nabla U_g = -\frac{\partial U_g}{\partial r}\hat{\mathbf{r}}$$

The direction of $\mathbf{F}_g$ always points from high to low potential energy

- A free-falling object is always decreasing in U_g
- "Steepest descent": the direction of $\mathbf{F}$ is the shortest path to decrease U_g
- Objects travelling perpendicular to $\mathbf{F}_g$ has constant U_g

Relating U_g, $\mathbf{F}_g$ and $\mathbf{g}$

Knowing that $\mathbf{F}_g$ and $\mathbf{g}$ only differ by a constant (mass m), we can also relate gravitational field to potential energy by the gradient operator:

$$\mathbf{g} = -\nabla V_g = -\frac{\partial V_g}{\partial r}\hat{\mathbf{r}} \quad \text{where} \quad V_g = \frac{U_g}{m}$$

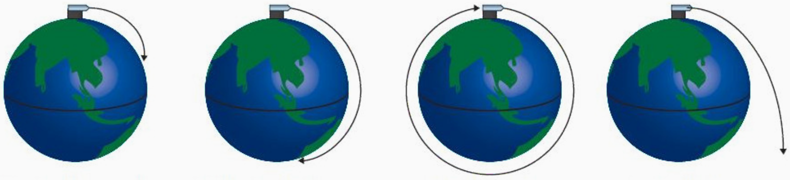
We already know that the direction of $\mathbf{g}$ is the same as $\mathbf{F}_g$, i.e.

- The direction of $\mathbf{g}$ is the shortest path to decrease U_g
- Objects travelling perpendicular to $\mathbf{g}$ has constant U_g
- The quantity V_g is called the **gravitational potential** but it is not often used

Orbits

Newton's Thought Experiment

In *Treatise of the System of the World*, the third book in *Principia*, Newton presented this thought experiment:



- How fast does the cannonball have to travel before it goes around Earth without falling? (i.e. goes into orbit)
- How fast does the cannonball have to travel before it never comes back?

Relating Gravitational and Centripetal Force

Assuming a small mass m in circular orbit around a much larger mass M. The required centripetal force is supplied by the gravitational force:

$$F_g = F_c \quad \longrightarrow \quad \frac{GMm}{r^2} = \frac{mv^2}{r}$$

Solving for v, we get the **orbital speed** v_{orbit} (aka **orbital velocity**), which does not depend on the mass of the small object in orbit:

$$\boxed{v_{\text{orbit}} = \sqrt{\frac{GM}{r}}}$$

This equation is only applicable for perfectly circular orbits.

Escape Speed

An object can leave the surface of Earth at any speed. But when all the kinetic energy of that object is converted to gravitational potential energy, it will return back to the surface of the earth. There is, however, a *minimum* velocity at which the object *would not* fall back to Earth.

Escape Speed

- The gravitational potential energy of an object with mass m on the surface of a planet (with mass M and radius r) is

$$U_g = -\frac{GMm}{r}$$

- The most amount of work that can be done against gravity is to bring it to the other side of the universe $r = \infty$, where $U_g = 0$.
- The work against gravity converts kinetic energy into gravitational potential energy.
- If you start with *more* kinetic energy than required to do all the work, then the object has *escaped* the gravitational pull of the planet.

Escape Speed from Circular Orbits

Set K to equal to $-U_g$:

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

We can then solve for **escape speed** $v = v_{\text{esc}}$ (or **escape velocity**):

$$\boxed{v_{\text{esc}} = \sqrt{\frac{2GM}{r}}}$$

There is a simple relationship between orbital speed and escape speed:

$$\boxed{v_{\text{esc}} = \sqrt{2}v_{\text{orbit}}}$$

Example Problem

Example: Determine the escape velocity and energy for a 1.60×10^4 kg rocket leaving the surface of Earth.

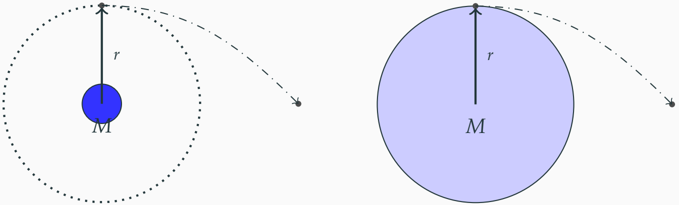
Example Problem

Example: Determine the escape velocity and energy for a 1.60×10^4 kg rocket leaving the surface of Earth.

Note: The equation for the escape speed is based on the object have a *constant* mass, which is *not* the case for a rocket going into space.

What if I'm not escaping from the surface?

Both objects have the same escape velocity:

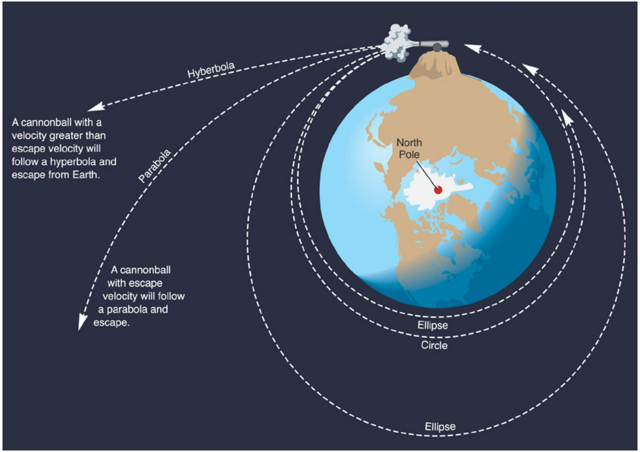


The difference is that the object in orbit (left) already has orbital speed v_{orbit}, so escaping from that orbit requires only an additional speed of

$$\Delta v = v_{\text{esc}} - v_{\text{orbit}} = (\sqrt{2} - 1)v_{\text{orbit}}$$

- What if $v_{\text{orbit}} < v < v_{\text{esc}}$?
- What if $v < v_{\text{orbit}}$?

Non-Circular Orbits



Orbital Energies

We can obtain the **orbital kinetic energy** in a perfectly circular orbit by using the orbital speed in our expression of kinetic energy:

$$K_{\text{orbit}} = \frac{1}{2} m v_{\text{orbit}}^2 = \frac{1}{2} m \left(\sqrt{\frac{GM}{r}} \right)^2 = \boxed{\frac{GMm}{2r}}$$

We already have an expression for **gravitational potential energy**:

$$U_g = -\frac{GMm}{r} = -2K_{\text{orbit}}$$

The **total orbital energy** is the sum of K and U_g:

$$E_T = K_{\text{orbit}} + U_g = -\frac{GMm}{2r} = -K_{\text{orbit}}$$

Orbital Mechanics

Orbital Mechanics

We turn our attention to applying the law of universal gravitation to the orbital motion of planets and stars in our solar system.

Properties of Gravitational Force

Two properties of gravity are crucial to understanding of orbital mechanics:

1. Gravity is a *conservative force*, in that
    - The total mechanical energy of objects under gravity is constant
    - Work done by gravity converts gravitational potential energy U_g into kinetic energy K; work against gravity converts K into U_g
2. Gravity is a *central force*, in that
    - Gravitational force $\mathbf{F}_g$ is always in the $-\hat{\mathbf{r}}$ direction, i.e. $\mathbf{F} \times \mathbf{r} = \mathbf{0}$
    - Therefore gravity doesn't generate any torque
    - And therefore angular momentum $\mathbf{L}$ is constant

These two properties are true regardless of the shape of the orbit, and even for objects that are not in orbit at all!

Kepler's Laws of Planetary Motion

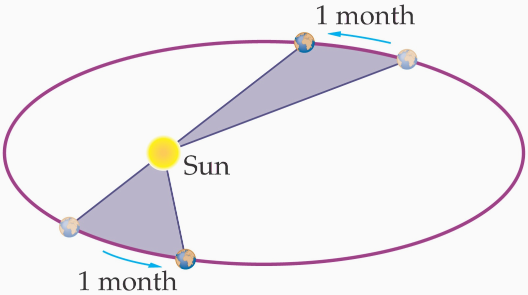
Johannes Kepler (1571–1630) formulated the **laws of planetary motion** between 1609 to 1619, by interpreting planetary motion data from his teacher, Tycho Brahe. It is an improvement over the heliocentric theory of Nicolaus Copernicus. Expressed in modern language:

1. **Law of ellipses:** The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. **Law of equal areas:** A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time
3. **Law of periods:** The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

Kepler's Second Law: Law of Equal Areas

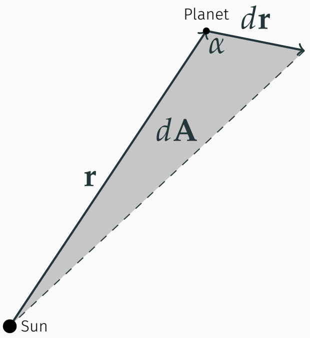
The second law of planetary motion is the easiest to proof using concepts in rotational motion. It states that:

A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time



Kepler's Second Law: Law of Equal Areas

A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time



The infinitesimal area $d\mathbf{A}$ swept out by an object (such as a planet) as it moves in orbit by an infinitesimal amount $d\mathbf{r}$ is given by:

$$dA = \frac{1}{2} r dr \sin\alpha \;\rightarrow\; d\mathbf{A} = \frac{1}{2}\mathbf{r} \times d\mathbf{r}$$

Its time derivative is called the **areal velocity**:

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2}\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \frac{1}{2}\mathbf{r} \times \mathbf{v}$$

Kepler's Second Law: Law of Equal Areas

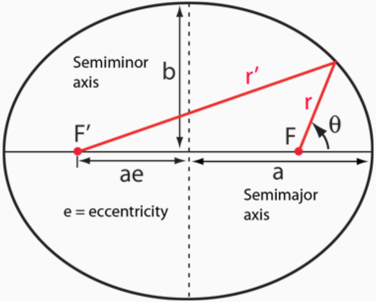
We can express $\mathbf{r} \times \mathbf{v}$ in terms of angular momentum, $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$. But in motion under any central force (such as gravity), angular momentum is a constant, and therefore:

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2}(\mathbf{r} \times \mathbf{v}) = \frac{\mathbf{L}}{2m} = \text{constant}$$

as predicted by Kepler's second law. The rate a planet sweeps out the area in orbit is its angular momentum around the sun divided by twice its mass.

Kepler's First Law: Law of Ellipses

To proof Kepler's first law, we have to first understand the ellipse, at least a little bit. If Kepler's first law is true, then orbital motion must agree with the equations of an ellipse.



$$r' + r = 2a$$

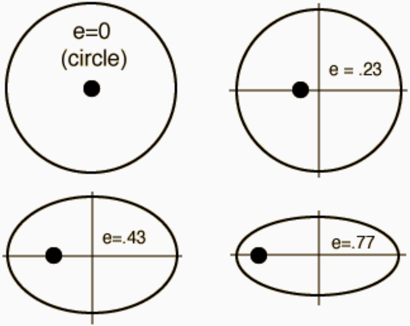
- The area of the ellipse is $A = \pi ab$
- For an ellipse, the relationship between r and θ given by:

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \quad \text{where} \quad 0 \le e < 1$$

- when $e = 0$ it's a circle: $a = b = r$
- When $e = 1$ it's no longer an ellipse

Kepler's First Law: Law of Ellipses

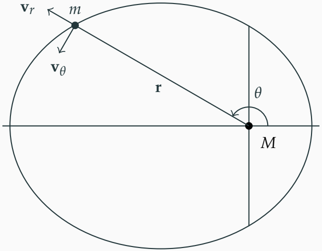
Most of the planets have very small eccentricity, so their orbits are fairly close to being circular, but comets are much more eccentric



Object	e
Mercury	0.206
Venus	0.0068
Earth	0.0167
Mars	0.0934
Jupiter	0.0485
Saturn	0.0556
Uranus	0.0472
Neptune	0.0086
Pluto	0.25
Halley's Comet	0.9671
Comet Hale-Bopp	0.9951
Comet Ikeya-Seki	0.9999

Kepler's First Law: Law of Ellipses

As m orbits around M, there are two velocity components



- **Angular velocity $\mathbf{v}_\theta$.** The presence of $\mathbf{v}_\theta$ means a centripetal acceleration toward M:

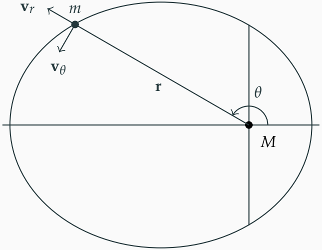
$$a = -r\omega^2\hat{\mathbf{r}}$$

- **Radial velocity $\mathbf{v}_r$.** If this velocity changes with time (i.e. elliptical orbits), then there is an acceleration, also in the radial direction:

$$a = \frac{dr}{dt}\hat{\mathbf{r}}$$

Both components of acceleration are due entirely to gravitational force.

Kepler's First Law: Law of Ellipses

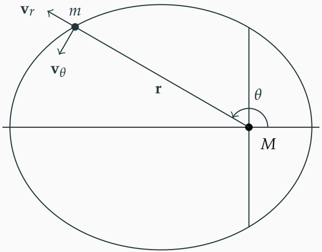


Applying Newton's second law of motion gives the differential equation:

$$\frac{d^2r}{dt^2} - r\omega^2 = -\frac{GM}{r^2}$$

- The $(+)$ direction is radially outward from M.
- In the circular motion case, where $d^2r/dt^2 = 0$, we are left with only the centripetal force.

Kepler's First Law: Law of Ellipses



Substituting angular momentum $L = mr^2\omega$ (which is a constant), the differential equation becomes:

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + \frac{L^2}{mr^3}$$

This differential equation is difficult to solve, and the equation for an ellipse (that we've shown a few slides ago) depends on θ but not on t.

Kepler's First Law: Law of Ellipses

To make things simpler, We define a new variable (this is *not* obvious):

$$u = \frac{1}{r}$$

Using the fact that angular momentum L is constant to relate derivatives in time t to derivatives in angle θ:

$$L = m{\color{red}r^2}\frac{d\theta}{dt} = \frac{m}{{\color{red}u^2}}\frac{d\theta}{dt} \quad \rightarrow \quad \frac{d}{d\theta} = \frac{Lu^2}{m}\frac{d}{dt}$$

Kepler's First Law: Law of Ellipses

With this change, $\ddot{r}$ can be expressed in terms of u and θ. First, we find the first derivative $\dot{r}$ in terms of θ:

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{L}{m}\frac{du}{d\theta}$$

Then, we find the second derivative:

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = -\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2}$$

Kepler's First Law: Law of Ellipses

The original differential equation, now written in terms of u and θ, becomes a second-order ordinary differential equation with constant coefficients and a constant forcing function (this form of differential equation was first encountered in the harmonic motion problem):

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + \frac{L^2}{mr^3} \quad \rightarrow \quad \frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2}$$

The solution to the differential equation is:

$$u(\theta) = \frac{GMm^2}{L^2} + B\cos\theta$$

Kepler's First Law: Law of Ellipses

Solving for r, and factoring terms appropriately,

$$r = \frac{1}{\frac{GMm^2}{L^2}(1 + e\cos\theta)} = \left(\frac{L^2}{GMm^2}\right)\frac{1}{1 + e\cos\theta}$$

where e is a constant:

$$e = \frac{BL^2}{GMm^2}$$

The *aphelion* and *perihelion* (points of furthest and closest distance to the Sun) are given by:

$$r_{\mathrm{max}} = \left(\frac{L^2}{GMm^2}\right)\frac{1}{1 - e} \quad r_{\mathrm{min}} = \left(\frac{L^2}{GMm^2}\right)\frac{1}{1 + e}$$

Kepler's First Law: Law of Ellipses

The semi-major axis is therefore

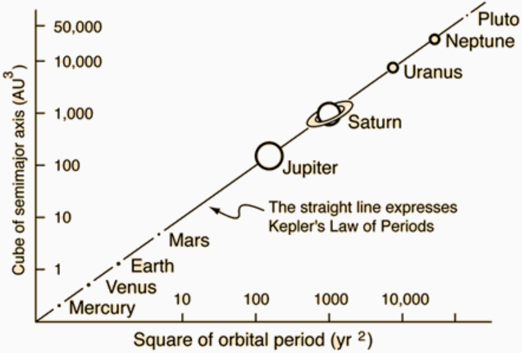
$$a = \frac{1}{2}(r_{\min} + r_{\max}) = \left(\frac{L^2}{GMm^2}\right)\frac{1}{1-e^2}$$

Or more importantly, we can relate the eccentricity e and the semi-major axis a to the numerator in the r expression from the last slide:

$$a(1-e^2) = \frac{L^2}{GMm^2} = r(1 + e\cos\theta)$$

Kepler’s Third Law: The Law of Periods

Law of Periods: The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.



Kepler's Third Law: The Law of Periods

The area swept by the planet through one orbital period is the areal velocity (constant!) integrated by time, from $t = 0$ to $t = T$:

$$A = \int dA = \int_0^T \frac{dA}{dt} dt = \frac{L}{2m} \int_0^T dt = \frac{L}{2m} T$$

From Kepler's first law, this area is an ellipse, given by the equation based on a (the semi-major axis), $b = a\sqrt{1 - e^2}$ (the semi-minor axis):

$$A = \pi ab = \pi a^2 \sqrt{1 - e^2}$$

Equating two equations above and squaring both sides give this expression:

$$T^2 = \frac{m^2}{L^2} 4\pi^2 a^4 (1 - e^2)$$

Kepler's Third Law: The Law of Periods

But we also (from proving the first law) have:

$$a(1-e^2) = \frac{L^2}{GMm^2}$$

Substituting this expression into the equation for the period, and after some simple algebra, we end up with this expression:

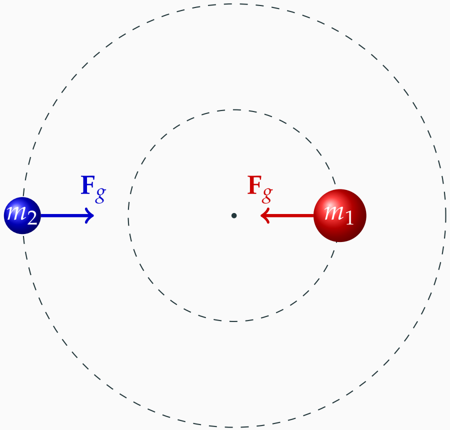
$$\boxed{T^2 = \left[\frac{4\pi^2}{GM}\right] a^3}$$

Reality of Orbital Motion

As always, nothing is as simple as is first seems

- Centripetal motion is based on rotation around a fixed point, but this is *not* the case for orbital mechanics!
- Just as planets experiences a gravitational force by the Sun, the Sun also experiences a gravitational force from the planets
- The smaller mass m does not actually orbit about the center of the larger mass M, but rather, the center of mass between M and m.
- This problem is especially important when the two objects orbiting each other has similar masses (e.g. a binary star system)

Binary System



In a binary star system, two stars orbit around their center of mass. Both have the same period, and the gravitational force provides the centripetal force, but this time, the distance to the center of motion is empty space.