Topic 8: Simple Harmonic Motion

Advanced Placement Physics 1

Dr. Timothy Leung

February 17, 2021

Olympiads School

Review: Hooke's Law

Review: Hooke's Law

Hooke's law relates the force $\boldsymbol{F}_s$ (**spring force**) exerted by a compressed or stretched spring onto another object to the stiffness of the spring k (**spring constant**, **Hooke's constant** or **force constant**) and spring displacement $\boldsymbol{x}$:

$$\boxed{\boldsymbol{F}_s = -k\boldsymbol{x}}$$

- The unit for k is **newton per meter** (N/m)
- $\boldsymbol{F}_s$ is a **restoring force**: its direction is always opposite to the displacement to always move it back towards $\boldsymbol{x} = 0$.
- The magnitude of $\boldsymbol{F}_s$ is non constant: the acceleration of objects connected to it changes with time

Elastic Potential Energy

Applying Hooke's law to the definition of work ($W = F\Delta x$), and after a bit of simple calculus, we find that a compressed/stretched spring stores **elastic potential energy**:

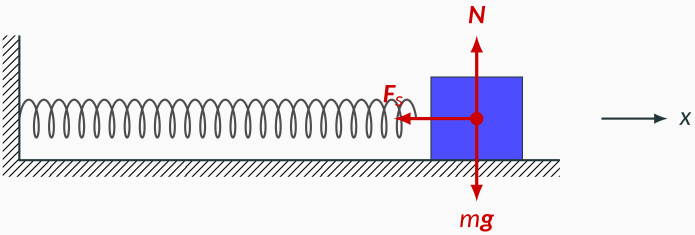
$$\boxed{U_e = \frac{1}{2}kx^2}$$

The spring force is a conservative force.

Spring-Mass Systems

Mass on a Spring

Consider the forces acting on a mass connected horizontally to a spring



mg and $\boldsymbol{N}$ cancel out, so net force is due only to spring force $\boldsymbol{F}_s = -k\boldsymbol{x}$ along the x-axis. This is true both when the spring is in compression or extension. (The spring is in extension in the diagram.)

Mass on a Spring

Applying second law of motion in the x-direction results in an equation containing both displacement and acceleration:

$$\sum F = F_s = ma \quad \rightarrow \quad -\frac{k}{m}x = a$$

We know that:

- Velocity is how quickly position changes in time, and
- Acceleration is how quickly velocity changes in time

So the left & right hand side of this equation are actually related. In calculus, we call this a *second-order ordinary differential equation with constant coefficients*.

Mass on a Spring

Solving this equation is something every calculus student have to learn. The general form of the solution is the cosine (or sine) function:

$$x(t) = A\cos(\omega t - \phi)$$

where ω is the **angular frequency**, A is the amplitude of the oscillation and ϕ is a **phase shift** (or **phase constant**) that depends on the initial condition

cos is preferred over sin because $\cos(0) = 1$, consistent with the fact that oscillations usually begin at maximum amplitude A at $t = 0$ and we can set $\phi = 0$. Mathematically, the two functions only differ in $\phi = \pi/2$

Mass on a Spring

Starting with the general form and applying some differential calculus, we obtain the velocity and acceleration of the mass as functions of time:

$$
\begin{aligned}
x(t) &= A\cos(\omega t - \phi) \\
v(t) &= -A\omega\sin(\omega t - \phi) \\
a(t) &= -A\omega^2\cos(\omega t - \phi) = -\omega^2 x
\end{aligned}
$$

Acceleration can be expressed as function of time or a function of position.

Angular Frequency

Comparing these two expressions

$$a(t) = -\omega^2 x(t) \qquad -kx(t) = ma(t)$$

It is obvious that the angular frequency ω must be related to the spring constant and mass. The angular frequency for the (undamped) simple harmonic oscillator is called the **natural frequency**:

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

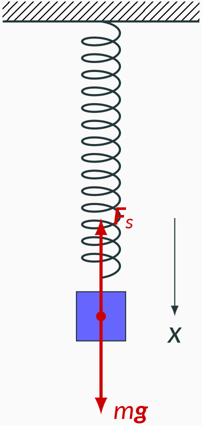
Frequency and Period of Mass on a Spring

The period T and frequency f of the simple harmonic motion are given by:

$$\boxed{f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}} \qquad \boxed{T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}}$$

Angular frequency ω, frequency f and period T are independent of amplitude A

Vertical Spring-Mass System

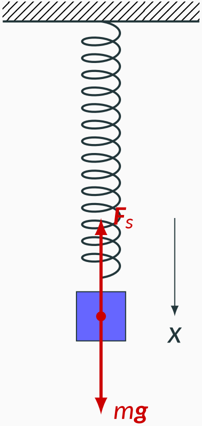


For a vertical spring-mass system, we must consider the weight mg of the mass as well, but since weight is constant, the only change is the addition of a constant B in the expression of $x(t)$:

$$
\begin{aligned}
x(t) &= A\cos(\omega t - \phi) + B \\
v(t) &= -A\omega \sin(\omega t - \phi) \\
a(t) &= -A\omega^2 \cos(\omega t - \phi)
\end{aligned}
$$

There is no change in the expressions for $v(t)$ and $a(t)$.

Vertical Spring-Mass System



The constant B is the amount that the equilibrium position is stretched downwards, and is the stretching of the spring due to its weight:

$$B = \frac{mg}{k}$$

Angular frequency (natural frequency) is the same as the horizontal case:

$$\omega = \sqrt{\frac{k}{m}}$$

Conservation of Energy in a Spring-Mass System

In spring-mass systems, assumingif no frictional losses, then the only forces doing work are the spring force (horizontal and vertical) and gravity (vertical). Both forces are *conservative*, therefore total mechanical energy is conserved:

$$K + U_e + U_g = K' + U'_e + U'_g$$

For the horizontal spring-mass system, the total energy of the simple harmonic oscillator is:

$$\boxed{E_T = \frac{1}{2}kA^2}$$

Simple Example

Example 2: A mass suspended from a spring is oscillating up and down. Consider the following two statements:

1. At some point during the oscillation, the mass has zero velocity but it is accelerating
2. At some point during the oscillation, the mass has zero velocity and zero acceleration.

(a) Both occur at some time during the oscillation

(b) Neither occurs during the oscillation

(c) Only (1) occurs

(d) Only (2) occurs

Another Example

Example 3: An object of mass 5 kg hangs from a spring and oscillates with a period of 0.5 s. By how much will the equilibrium length of the spring be shortened when the object is removed.

(a) 0.75 cm

(b) 1.50 cm

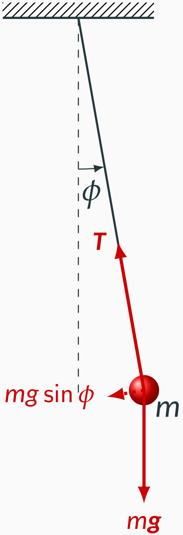
(c) 3.13 cm

(d) 6.21 cm

Simple Pendulum

What About a Simple Pendulum?

- Pendulums also exhibit oscillatory motion
- A **simple pendulum** is where the mass is concentrated at the end point
- There are two forces acting on the mass: weight mg and tension T
- When the mass is deflected by ϕ, the component of gravity in the tangential direction is $F_t = -mg \sin \phi$
- No need to worry about the radial direction because it does not have to do with the restoring force



The Simple Pendulum

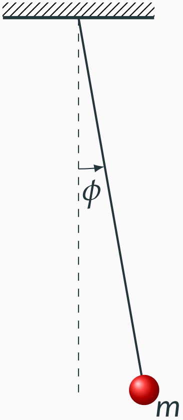
Substitute F_t into the second law of motion, and cancelling mass term, we get:

$$F_t = ma_t \quad \rightarrow \quad -g \sin\phi = \ell\alpha$$

where the acceleration α is the angular acceleration α, and tangential acceleration is $a_t = \ell\alpha$. Solving this ODE is difficult because of the $\sin\phi$ term. However, using the series expansion of the sine function

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \cdots$$

shows that for "small angles" (usually <10°), $\sin\phi \approx \phi$

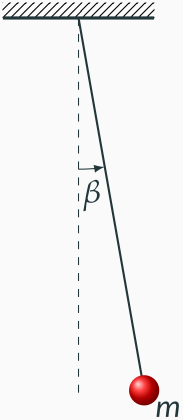


The Simple Pendulum

For small angles of ϕ, the equation that needs to be solved is essentially the same as the spring-mass system:

$$-g\phi = \ell\alpha$$

Instead of position x and acceleration a, we have *angular* position ϕ and *angular* acceleration α of the pendulum. This expression is similar to the spring-mass case.



Solution for Deflection

The solution for $\phi(t)$ is similar to the spring-mass system:

$$\boxed{\phi(t) = \Phi\cos(\omega t - \beta)}$$

where amplitude Φ is the maximum deflection of the pendulum (less than 10°), and angular frequency of the oscillation ω is given by:

$$\boxed{\omega = \sqrt{\frac{g}{L}}}$$

A Pendulum Example

Example: A bucket full of water is attached to a rope and allowed to swing back and forth as a pendulum from a fixed support. The bucket has a hole in its bottom that allows water to leak out. How does the period of motion change with the loss of water?

(a) The period does not change.
(b) The period continuously decreases.
(c) The period continuously increases.
(d) The period increases to some maximum and then decreases again.

Think About g

Example: A little girl is playing with a toy pendulum while riding in an elevator. Being an astute and educated young lass, she notes that the period of the pendulum is 0.5 s. Suddenly the cables supporting the elevator break and all of the brakes and safety features fail simultaneously. The elevator plunges into free fall. The young girl is astonished to discover that the pendulum has:

(a) continued oscillating with a period of 0.5 s.

(b) stopped oscillating entirely.

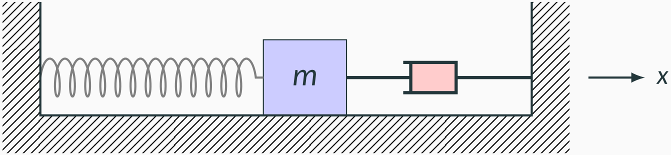
(c) decreased its rate of oscillation to have a longer period.

(d) increased its rate of oscillation to have a lesser period.

Damped and Forced Oscillations

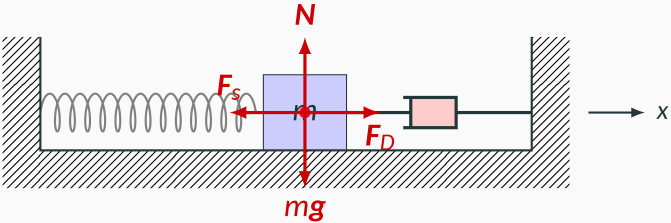
It’s Never Perfect

In reality, there are friction, or drag, or other forces present in any spring-mass system that takes away energy, represented by a shock absorber:



It's Never Perfect

In reality, there are friction, or drag, or other forces present in any spring-mass system that takes away energy, represented by a shock absorber:



The damping force is typically proportional to velocity, in the opposite direction:

$$\boldsymbol{F}_D = -b\boldsymbol{v}$$

Second law of motion:

$$-kx - bv = ma$$

Damped Oscillator

The solution this equation is a standard (albeit more difficult) calculus problem. The motion of the mass has both an exponential decay and a sinusoidal term:

$$x(t) = A_0 e^{-\frac{b}{2m}t} \cos(\omega' t + \phi)$$

where A_0 is the initial amplitude, and angular frequency ω' is now shifted from the natural frequency:

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Angular frequency ω' of the damped oscillator differs from the undamped case ω_0 depending on the damping factor b.

Critical Damping

Critical damping occurs when the angular frequency ω' term is zero:

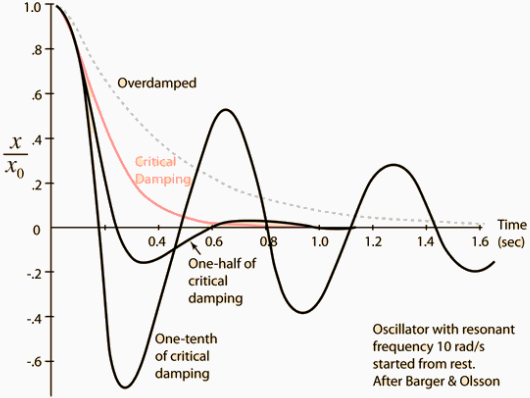
$$\omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = 0$$

which occurs when the damping constant is:

$$b_c = 2m\omega_0$$

- A critically damped system returns to its equilibrium position in the shortest time with *no* oscillation
- When $b > b_c$, the system is **over-damped**
- Critical or near-critical damping is desired in many engineering designs (e.g. shock absorbers on car suspensions)

Comparing Damped System



The motion of the damped oscillator is not strictly periodic.

Energy in a Damped System

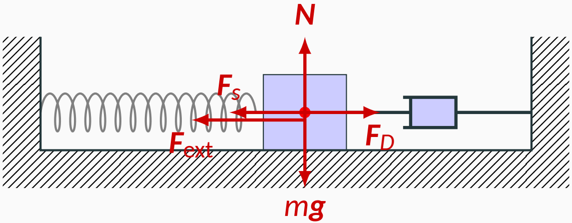
The non-conservative damping force dissipates energy from the oscillator. Therefore the total amount of energy decreases exponentially with time:

$$E(t) = E_0 e^{-\frac{b}{m}t}$$

Driven Oscillation

Forced Harmonic Motion

To keep a damped system going, energy must be added into the system.



Assume that the system is subjected to an external force that is sinusoidal with time, with a driving frequency ω:

$$F_{\text{ext}} = F_0 \cos(\omega t)$$

Forced Harmonic Motion

Accounting for all the forces in the x direction, and applying the second law of motion:

$$-kx - bv + F_0 \cos(\omega t) = ma$$

This is a difficult problem (even for calculus students), but its solution is well known. It has:

- A **transient solution** identical to the damped oscillator
  - Obtained by setting the external force term to zero
  - Depends on the initial condition
  - Solution becomes negligible over time because of exponential decay
- A **steady-state solution** which does not depend on the initial condition

Forced Harmonic Motion

Solving for the steady-state solution is a difficult calculus exercise, but the solution is a harmonic motion at the driving frequency ω of the external force:

$$\boxed{x(t) = A\cos(\omega t - \phi)}$$

The equation is in the same form as the SHM case, but the amplitude A and phase shift ϕ are now given by:

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} \qquad \phi = \tan^{-1}\left[\frac{b\omega}{m(\omega_0^2 - \omega^2)}\right]$$

Resonance

Resonance is caused by in-phase excitation at natural frequency. This means that:

- The frequency of the driving force is same as the natural frequency of the oscillator

$$\omega = \omega_0 = \sqrt{\frac{k}{m}}$$

- The driving force follows the motion of the oscillator.

Resonance

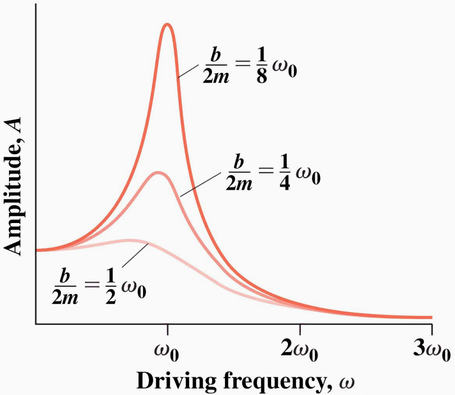
Looking at the expression for amplitude of the oscillation:

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}$$

Amplitude is at maximum when the frequency of the driving force ω is equal to the natural frequency ω_0, with a maximum value of:

$$A_{\text{max}} = \frac{F_0}{b\omega}$$

Resonance



Plotting amplitude A as a function of driving frequency ω shows that:

- Resonance response is highest when $\omega = \omega_0$, which we know already
- The smaller the damping constant b, the higher and narrower the peak is

Resonance

$$\boxed{\tan\phi = \frac{b\omega}{m(\omega_0^2 - \omega^2)}}$$

When $\omega = \omega_0$ is substituted into the phase shift expression, the right-hand side becomes undefined. From this, we obtain a phase shift of $\phi = \pi/2$. Using the expression for $v(t)$, and substituting $\phi = \pi/2$:

$$v(t) = -A\omega\sin(\omega t - \frac{\pi}{2}) = A\omega\cos(\omega t)$$

Resonance

At resonance, the object is always moving in the same direction as the driving force:

$$v(t) = A\omega \cos(\omega t)$$
$$F_{\text{ext}}(t) = F_0 \cos(\omega t)$$