Topic 5a: Gravity

AP and IBHL Physics

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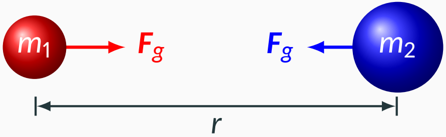
Summer 2021

Olympiads School

Gravitational Force

Law of Universal Gravitation

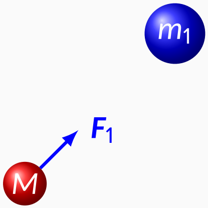


In classical mechanics, **gravity** is a mutually attractive force between all massive objects; the magnitude F_g is determined by the law of universal gravitation:

$$\boxed{F_g = G\frac{m_1 m_2}{r^2}}$$

where $G = 6.674 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$ is the **universal gravitation constant**, and r is the distance between the centers of the masses. m_1 and m_2 are assumed to be *point masses* that do not occupy any space.

More Than One Mass



For a mass M that is subjected to the influence of multiple discrete point masses m_i, the total gravitational force that M experiences is the vector sum of all the forces $\boldsymbol{F}_i$:

$$\boxed{\boldsymbol{F} = \sum_i \boldsymbol{F}_i = GM\left(\sum_{i=1}^{N} \frac{m_i}{r_i^2}\hat{\boldsymbol{r}}_i\right)}$$

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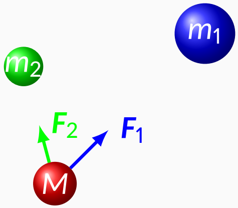

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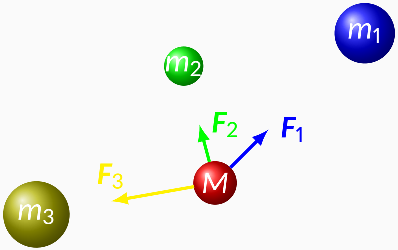

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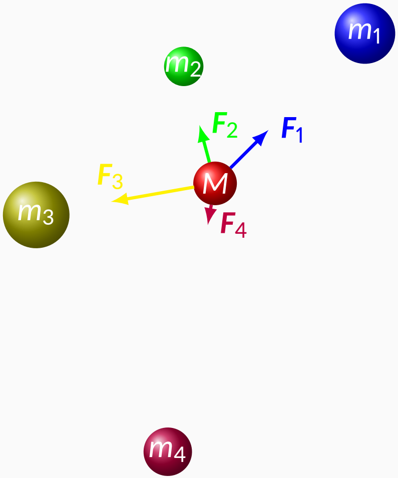

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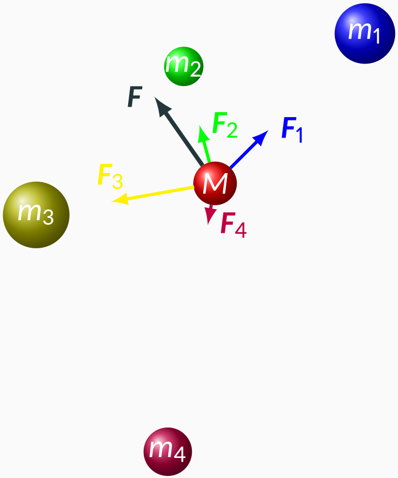

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Gravitational Field

Gravitational Field

We generally describe the gravitational force $\boldsymbol{F}_g$ (or weight $\boldsymbol{w}$) as:

$$\boxed{\boldsymbol{w} = \boldsymbol{F}_g = m\boldsymbol{g}}$$

To find $\boldsymbol{g}$, we group the variables in the law of universal gravitation:

$$\boldsymbol{F}_g = \underbrace{\left[-\frac{Gm_1}{|\boldsymbol{r}|^2}\hat{\boldsymbol{r}}\right]}_{=\boldsymbol{g}} m = m\boldsymbol{g}$$

The vector field function $\boldsymbol{g}$ is known as the **acceleration due to gravity** in kinematics, and **gravitational field** in field theory.

Gravitational Field

On/near the surface of Earth, we can use Earth's mass and radius

$$m_1 = 5.972 \times 10^{24}\,\mathrm{kg}$$
$$r = 6.371 \times 10^{6}\,\mathrm{m}$$

to compute the commonly known value of

$$g \approx 9.81\,\mathrm{m/s^2}$$
$$g \approx 9.81\,\mathrm{N/kg}$$

Gravitational Field

The **gravitational field** $\boldsymbol{g}$ generated by point mass m shows how it influences the gravitational forces on other masses:

$$\boxed{\boldsymbol{g}(m, \boldsymbol{r}) = -\frac{Gm}{|\boldsymbol{r}|^2}\hat{\boldsymbol{r}}}$$

Quantity	Symbol	SI Unit
Gravitational field	$\boldsymbol{g}$	N / kg
Universal gravitational constant	G	$\mathrm{Nm}^2 / \mathrm{kg}^2$
Source mass	m	kg
Distance from source mass	$\lvert\boldsymbol{r}\rvert$	m
Outward radial unit vector from source	$\hat{\boldsymbol{r}}$	N/A

The *direction* of the gravitational field is toward m (that's why the negative sign)

More Than One Mass

When there are multiple point masses present, the total gravitational field at any position $\boldsymbol{r}$ is the vector sum of all the fields $\boldsymbol{g}_i$:

$$\boxed{\boldsymbol{g} = \sum_i \boldsymbol{g}_i = G\left(\sum_i \frac{m_i}{r_i^2}\hat{\boldsymbol{r}}_i\right)}$$

Relating Gravitational Field & Gravitational Force

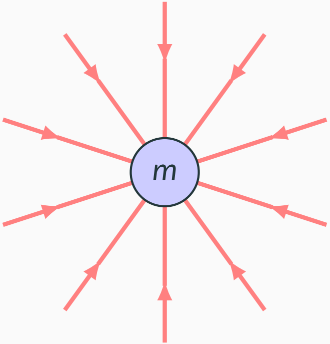
$\boldsymbol{g}$ acts on any mass m that enters the field. Then, m experiences a gravitational force $\boldsymbol{F}_g$, regardless of how $\boldsymbol{g}$ is created:

$$\boxed{\boldsymbol{F}_g = m\boldsymbol{g}}$$

Quantity	Symbol	SI Unit
Gravitational force on a mass	$\boldsymbol{F}_g$	N
Mass inside the gravitational field	m	kg
Gravitational field	$\boldsymbol{g}$	N/kg

Note: A point mass is not affected by the gravitational field that itself generates.

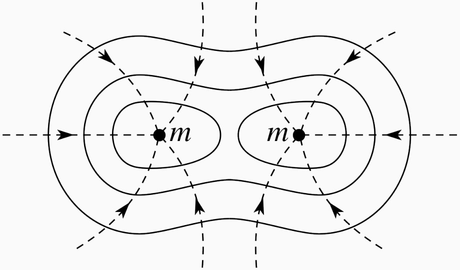
Gravitational Field Lines



- The direction of $\boldsymbol{g}$ is toward the center of the object that created it
- Field lines do not tell the intensity (i.e. magnitude) of $\boldsymbol{g}$, only the direction

Gravitational Field Lines

When there are multiple masses, the total gravitational field (dotted line) is the vector sum of all the individual fields.



The solid lines are called **equipotential lines**, where the potential energy is constant. Equipotential lines are perpendicular to gravitational field lines.

Gravitational Potential Energy

Gravitational Potential Energy

The expression for **gravitational potential energy** can be obtained from the law of universal gravitation using some basic calculus:

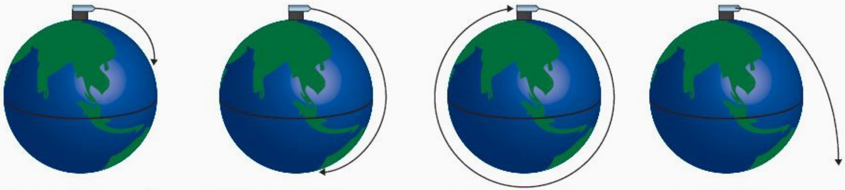
$$\boxed{U_g = -G\frac{m_1 m_2}{r}}$$

- U_g is the work required to move two objects from r to ∞
- $U_g = 0$ at $r = \infty$ and *decrease* as r decreases
- The fundamental theorem of calculus shows that the direction of gravitational force $\boldsymbol{F}_g$ always points from high to low potential energy,
    - A free-falling object is always decreasing in U_g, while
    - Objects traveling $\perp$ to $\boldsymbol{F}_g$ has constant U_g

Orbital Motion

Newton's Thought Experiment

In *Treatise of the System of the World*, the third book in *Principia*, Newton presented this thought experiment:



- How fast does the cannonball have to travel before it goes around Earth without falling? (i.e. goes into orbit)
- How fast does the cannonball have to travel before it never comes back?

Relating Gravitational and Centripetal Force

Assuming a small mass m in circular orbit around a much larger mass M. The required centripetal force is supplied by the gravitational force:

$$F_g = F_c \quad \longrightarrow \quad \frac{GMm}{r^2} = \frac{mv^2}{r}$$

Solving for v, we obtain the **orbital velocity** v_{orb} which does not depend on the mass of the small object in orbit:

$$\boxed{v_{\text{orb}} = \sqrt{\frac{GM}{r}}}$$

This equation is only applicable for perfectly circular orbits. (The orbital velocity equation is not part of the equation sheet, but it may be very easy to derive.)

Escape Velocity

An object can leave the surface of Earth at any speed. But when all the kinetic energy of that object is converted to gravitational potential energy, it will return back to the surface of the earth. There is, however, a *minimum* velocity at which the object *would not* fall back to Earth.

Escape Velocity

The calculation for escape velocity is an exercise in conservation of energy, since gravity is a conservative force, i.e.:

$$K + U_g = K' + U'_g$$

- Initial gravitational potential energy at the surface is:

$$U_g = -\frac{GMm}{r_i}$$

- Final gravitational potential energy is at the other side of the universe ($r = \infty$), where $U'_g = 0$. At this point, the object has *escaped* the gravitational pull of the planet/star
- The minimum kinetic energy at $r = \infty$ is $K' = 0$

Escape Velocity from Circular Orbits

Set K to equal to $-U_g$:

$$\frac{1}{2}mv_i^2 = \frac{GMm}{r_i}$$

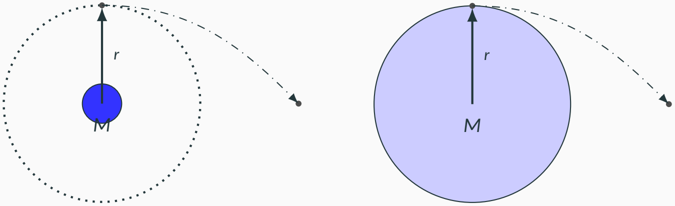
We can then solve for the initial speed $v_i = v_{\text{esc}}$ (called the **escape velocity**):

$$\boxed{v_{\text{esc}} = \sqrt{\frac{2GM}{r_i}}}$$

where r_i is the initial distance from the center of the planet/star, and it does not have to be on the surface.

What if I'm not escaping from the surface?

Both objects have the same escape velocity:

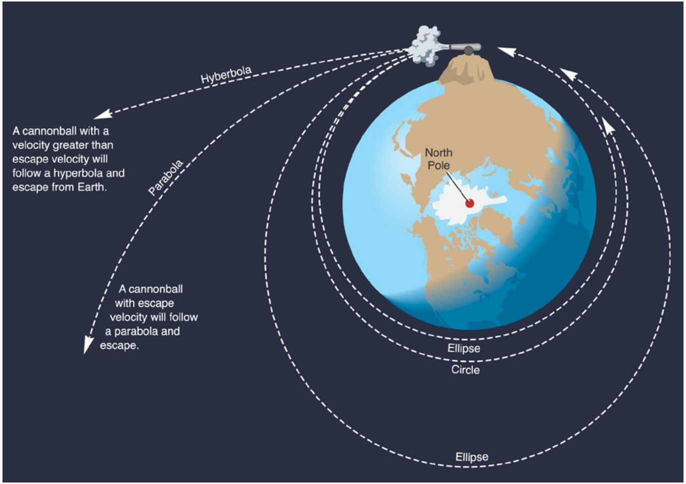


The difference is that the object in orbit (left) already has orbital speed v_{orb}, so escaping from that orbit requires only an additional speed of

$$\Delta v = v_{\text{esc}} - v_{\text{orbit}} = (\sqrt{2} - 1) v_{\text{orbit}}$$

- What if $v_{\text{orbit}} < v < v_{\text{esc}}$?
- What if $v < v_{\text{orbit}}$?

Non-Circular Orbits



Orbital Energies

We can obtain the **orbital kinetic energy** in a perfectly circular orbit by using the orbital speed in our expression of kinetic energy:

$$K_{\text{orb}} = \frac{1}{2}mv_{\text{orb}}^2 = \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2 = \boxed{\frac{GMm}{2r}}$$

We already have an expression for gravitational potential energy:

$$U_g = -\frac{GMm}{r} = -2K_{\text{orb}}$$

Orbital Energies

The **total orbital energy** is the sum of K and U_g:

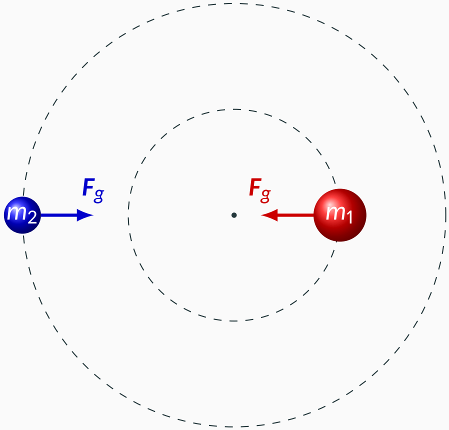
$$E_T = K_{\text{orb}} + U_g = -\frac{GMm}{2r} = -K_{\text{orb}}$$

Note the relationship:

$$K = -\frac{1}{2}U_g = -E_T$$

You are unlikely to encounter this in an AP exam, but this relationship, when applied to *electrostatic* force, was crucial in developing the first model for the hydrogen atom

Binary System



In a binary star system, two stars orbit around their center of mass. Both have the same period, and the gravitational force provides the centripetal force, but this time, the distance to the center of motion is empty space.