Topic 5: Circular Motion

Advanced Placement Physics 1

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Olympiads School

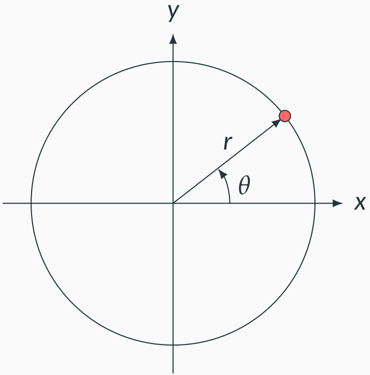
Review of Circular Motion

In **circular motion**, an object of mass m moves in a circular path about a fixed center. In Grade 12 Physics, you were introduced to *uniform* circular motion, where:

- the object's speed (magnitude of velocity) is constant
- the object's **centripetal acceleration** is toward the center
- the object's acceleration is caused by a **centripetal force**

Rigid-Body Circular Motion

Angular Position and Angular Velocity



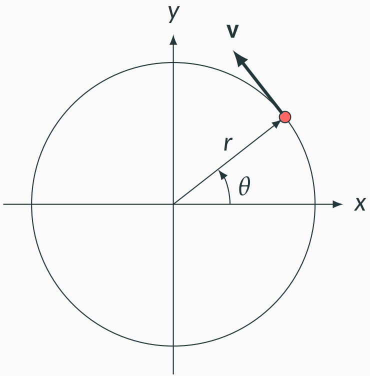
For circular motion with constant radius *r*, the **angular position** $\theta(t)$ fully describes an object's position. It is generally measured in radians (rad):

$$\boxed{\theta = \theta(t)}$$

Average angular velocity $\overline{\omega}$ (or **angular frequency**) is the change in angular position over a finite time interval. It is measured in rad/s.

$$\boxed{\overline{\omega} = \frac{\Delta\theta}{\Delta t}}$$

Velocity and Angular Velocity

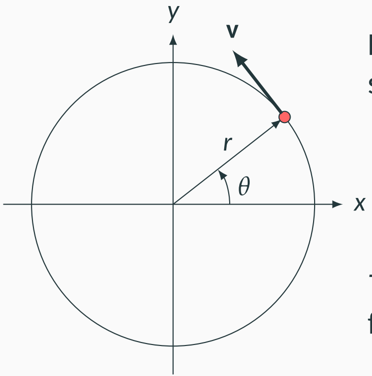


The actual velocity of an object in circular motion is related to the angular velocity by:

$$v(t) = r\omega(t)$$

- The direction of **v** is tangent to circle
- If $\omega > 0$, the motion is counter-clockwise
- If $\omega < 0$, the motion is clockwise

Period & Frequency



For a constant ω (uniform circular motion), the motion is strictly periodic; its **frequency** f and **period** T given by:

$$\boxed{f = \frac{\omega}{2\pi}} \qquad \boxed{T = \frac{2\pi}{\omega}} \qquad \boxed{f = \frac{1}{T}}$$

T is measured in seconds (s) and f in hertz (Hz). Period and frequency are reciprocals of each other.

Angular Acceleration

The change in anguler velocity $\Delta\omega$ over a finite time interval Δt is **average angular acceleration** $\overline{\alpha}$, with a unit of rad/s^2:

$$\boxed{\overline{\alpha} = \frac{\Delta\omega}{\Delta t}}$$

Similar to the relationship between velocity and angular velocity, **average tangential acceleration** $\overline{a}_t$ is related to angular acceleration $\overline{\alpha}$ by the radius r:

$$\boxed{\overline{a}_t = \frac{\Delta v}{\Delta t} = \frac{r\Delta\omega}{\Delta t} = r\overline{\alpha}}$$

For *uniform* circular motion (constant ω), $\alpha = 0$ and $a_t = 0$

Kinematics in the Angular Direction

For constant angular acceleration α, the kinematic equations are the same in rectilinear motion, but with θ replaces x, ω replaces v, and α replaces a:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \theta_0 + \alpha t$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Calculus (or other methods based on integral calculus) is required for non-constant α.

A Simple Example

Example 1: An object moves in a circle with angular acceleration $3.0\,\mathrm{rad/s^2}$. The radius is $2.0\,\mathrm{m}$ and it starts from rest. How long does it take for this object to finish a circle?

Centripetal Acceleration & Centripetal Force

There is also a component of acceleration toward the center of the rotation, called the **centripetal acceleration** a_c:

$$\boxed{a_c = \frac{v^2}{r} = \omega^2 r}$$

The force that causes the centripetal acceleration is called the **centripetal force**, also toward the center of rotation:

$$\boxed{F_c = m a_c = \frac{m v^2}{r}}$$

Centripetal Acceleration for Uniform Circular Motion

In uniform circular motion ($\alpha = 0$, constant ω) problems where the period or frequency are known, the speed of the object is:

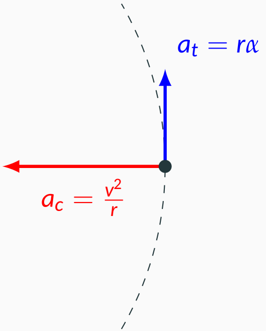
$$v = \frac{2\pi r}{T} = 2\pi r f$$

Centripetal acceleration can therefore be expressed based on T or f:

$$a_c = \frac{v^2}{r^2} \quad \rightarrow \quad \boxed{a_c = \frac{4\pi^2 r}{T^2} = 4\pi^2 r f^2}$$

Acceleration: The General Case

In general circular motion, there are two components of acceleration:



Centripetal acceleration a_c

- Depends on radius of curvature r and instantaneous speed v.
- The direction of a_c is toward the center of the circle.

Tangential acceleration a_t

- Depends on radius r and angular acceleration α.
- The direction of the acceleration is tangent to the circle, which is the same as the velocity vector $\mathbf{v}$.

How to Solve Circular Motion Problems

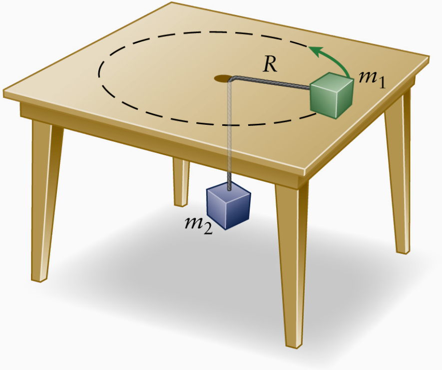
The condition for circular motion is the second law of motion:

$$\mathbf{F}_c = \sum \mathbf{F} = m\mathbf{a}_c$$

The forces that generate the centripetal force comes from the free-body diagram. It may include:

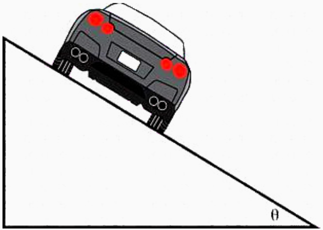
- Gravity
- Friction
- Normal force
- Tension
- Etc.

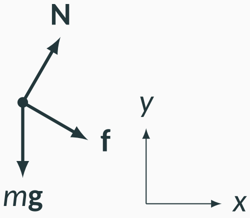
Example: Horizontal Motion



Example 2: In the figure on the left, a mass $m_1 = 3.0\,\text{kg}$ is rolling around a frictionless table with radius $R = 1.0\,\text{m}$. with a speed of $2.0\,\text{m}/\text{s}$. What is the mass of the weight m_2?

Banked Curves on Highways and Racetracks





No motion in the y direction, therefore $\sum F_y = 0$:

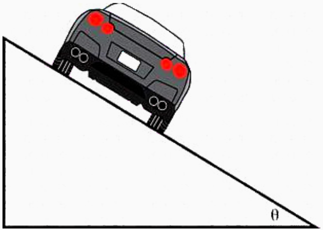
$$N\cos\theta - f\sin\theta - w = 0$$

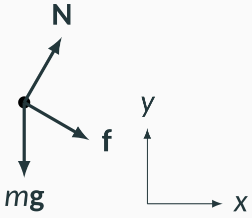
Net force in the x direction is the centripetal force, i.e. $\sum F_x = ma_c$

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r}$$

Friction force **f** may be static or kinetic, depending on the situation.

Banked Curves on Highways and Racetracks



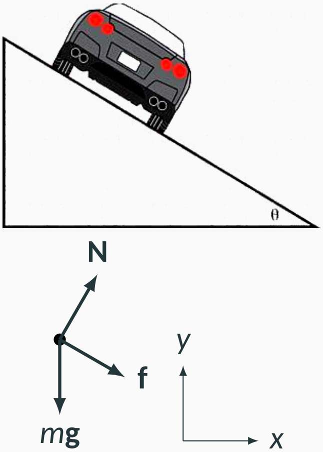


For analysis, use the simplified equation for friction $f = \mu N$ (i.e. assume either kinetic friction or maximum static friction), and weight $m\mathbf{g}$, the equations on the previous slides can be arranged as:

$$N\left(\cos\theta - \mu\sin\theta\right) = mg$$

$$N\left(\sin\theta + \mu\cos\theta\right) = \frac{mv^2}{r}$$

Banked Curves on Highways and Racetracks



Dividing the two equations removes both the normal force and mass terms:

$$\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg}$$

The *maximum* velocity $v_{\max}$ can be expressed as:

$$\boxed{v_{\max} = \sqrt{rg\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}}}$$

Note that $v_{\max}$ does not depend on mass.

Banked Curves on Highways and Racetracks

In the limit of $\mu = 0$ (frictionless case), the equation reduces to:

$$\boxed{v_{\text{max}} = \sqrt{rg\tan\theta}}$$

And in the limit of a flat roadway with no banking ($\theta = 0$, $\sin\theta = 0$ and $\cos\theta = 1$), the equation reduces to:

$$\boxed{v_{\text{max}} = \sqrt{\mu rg}}$$

Vertical Circles

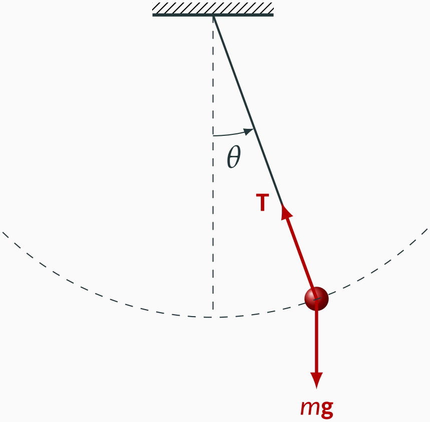
Vertical Circles

Circular motion with a horizontal path is straightforward. However, for vertical motion:

- Generally difficult to solve by dynamics and kinematics
- Instead, use conservation of energy to solve for speed v
- Then use the equation for centripetal force to find other forces

What About a Pendulum?

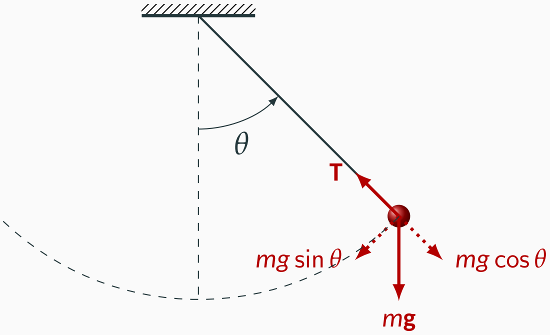
A simple pendulum is also like a vertical circular motion problem.



- There are two forces act on the pendulum: weight $m\mathbf{g}$, and tension $\mathbf{T}$
- Speed of the pendulum at any height is found using conservation of energy
  - $\mathbf{T}$ is $\perp$ to motion, therefore it does not do work
  - Work is done by gravity (conservative!) alone
- Tangential and centripetal accelerations are based on the net force along the angular and radial directions

Simple Pendulum

At the top of the swing, velocity v is zero, therefore:



Centripetal acceleration is also zero:

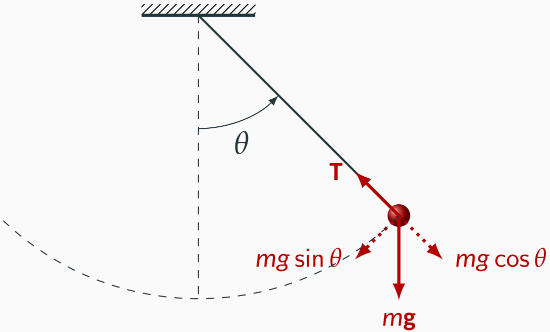
$$a_c = \frac{v^2}{r} = 0$$

and therefore the net force along the radial direction is zero. The tension force T can be calculated:

$$T = mg\cos\theta$$

At the highest point when θ is largest, tension is the lowest.

Simple Pendulum



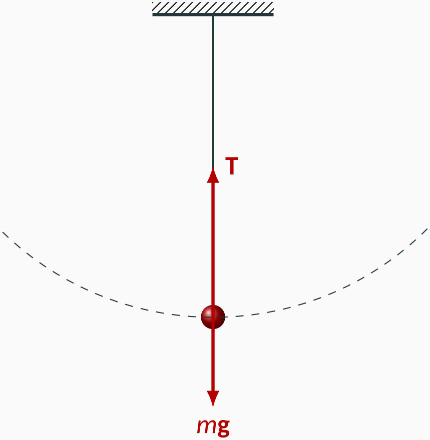
In the tangential direction , there is a net force of $mg\sin\theta$, therefore, a tangential acceleration along that direction, with a magnitude of:

$$a_t = g\sin\theta$$

This is the same acceleration as an object sliding down a frictionless ramp at an angle of θ.

Simple Pendulum

At the bottom of the swing, the velocity is at its maximum value,



- Maximum centripetal acceleration:

$$a_c = \frac{v^2}{r}$$

- No tangential acceleration:

$$a_t = 0$$

- At the lowest point, tension is the highest:

$$T = w + F_c = m\left(g + \frac{v^2}{r}\right)$$

Example Problem

Example 4: You are playing with a yo-yo with a mass M. The length of the string is R. You decide to see how slowly you can swing it in a vertical circle while keeping the string fully extended, even when it is at the top of its swing.

a. Calculate the minimum speed at which you can swing the yo-yo while keeping it on a circular path.
b. Find the tension in the string when the yo-yo is at the side and at the bottom of its swing.

Example: Roller Coaster

Example 5: A roller coaster car is on a track that forms a circular loop, of radius R, in the vertical plane. If the car is to maintain contact with the track at the top of the loop (generally considered to be a good thing), what is the minimum speed that the car must have at the bottom of the loop. Ignore air resistance and rolling friction.

A. $\sqrt{2gR}$
B. $\sqrt{3gR}$
C. $\sqrt{4gR}$
D. $\sqrt{5gR}$