Topic 12: DC Circuit Analysis

Advanced Placement Physics 1 & 2

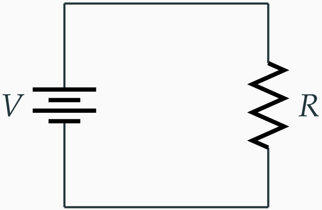
Dr. Timothy Leung

August 2020

Olympiads School, Toronto, ON, Canada

Circuits

Basic DC Circuit



A basic circuits consists of:

- A voltage source: battery, generator or capacitor
- Connecting wires
- A load: resistors, motor, LEDs

Current

Current

Electric current is the amount of *positive* chargess Q that flows through a wire in during a time interval Δt:

$$\boxed{I = \frac{Q}{\Delta t}}$$

It is also defined as the product of **charge density** J flowing past a surface A:

$$\boxed{I = JA = nvqA}$$

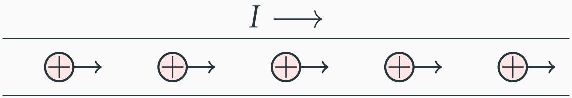
where

- v is the average **drift velocity** of the charge carriers
- n is the number of charge carriers per unit volume
- q is the charge of each particle

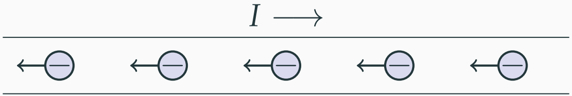
Basic algebra will show that these equations are consistent with one another.

Conventional Current vs. Electron Flow

We have *assumed* that the charge carriers are positively charged, which means that the current flows from high electric potential to low potential (i.e. from cathode to anode).



In a conducting wire, instead of positive charges flowing in one direction, we have, in fact, electrons (negative charges) flowing in the opposite direction, called the **electron current**:



For circuit analysis, we use the conventional current for simplicity.

Electric Field Inside the Wire

Inside the wire, there is a weak electric field, which exerts a force on the free electrons as they move in the wire

- As the charges move through a conductor, they lose potential energy
- Unlike a falling object which converts gravitational potential energy to kinetic energy, in a circuit, the energy is
  - converted into radiative heat or light through the resistance of the wire, or
  - converted into kinetic energy of a motor shaft

Resistors

Resistance of a Conductor

The resistance of a conductor is proportional to the resistivity ρ and its length L, and inversely proportional to the cross-sectional area A:

$$\boxed{R = \rho \frac{L}{A}}$$

Quantity	Symbol	SI Unit
Resistance	R	Ω
Resistivity	ρ	$\Omega\,\mathrm{m}$
Length of conductor	L	m
Cross-sectional area	A	m^2

The resistivity ρ is determined by the material.

Resistivity

The resistivity of a material is the ratio between strength of the electric field E inside the material and the charge density J:

$$\boxed{\rho = \frac{E}{J}}$$

- Conductor: the electrons are free to move, therefore the electric field tend to be very small, and the resistivity is low.
- Dielectric (non conducting material): electrons cannot move easily–they can only polarize themselves–the electric field are generally strong, and the resistivity is higher.

Resistance of a Conductor

$$\boxed{R = \rho \frac{L}{A}}$$

Gauge	Diameter (mm)	R/L ($10^{-3}\,\Omega/\mathrm{m}$)
0	9.35	0.31
10	2.59	2.20
14	1.63	8.54
18	1.02	21.90
22	0.64	51.70

Material	Resistivity ρ ($\Omega\,\mathrm{m}$)
silver	1.6×10^{-8}
copper	1.7×10^{-8}
aluminum	2.7×10^{-8}
tungsten	5.6×10^{-8}
Nichrome	100×10^{-8}
carbon	3500×10^{-8}
germanium	0.46
glass	10^{10} to 10^{14}

Ohm’s Law

Ohm's Law

The electric potential difference V across a resistor equals the product of the current I through the load and the resistance R:

$$\boxed{V = IR}$$

Quantity	Symbol	SI Unit
Potential difference	V	V
Current	I	A
Resistance	R	Ω

A resistor is considered "ohmic" if it obeys Ohm's law. Note that Ohm's law is not a fundamental law in physics.

Power Dissipated by a Resistor

Power is the rate at which work W is done, and from electrostatics, the change in electric potential energy ΔE_q (i.e. the work done!) is proportional to the amount of charge q and the voltage V. This gives a very simple expression for power through a resistor:

$$P = \frac{dW}{dt} = \frac{d(qV)}{dt} = \left(\frac{dq}{dt}\right) V \rightarrow \boxed{P = IV}$$

Quantity	Symbol	SI Unit
Power through a resistor	P	W
Current through a resistor	I	A
Voltage across the resistor	V	V

Other Equations for Power

When we combine Ohm's Law ($V = IR$) with power equation, we get two additional expressions for power through a resistor:

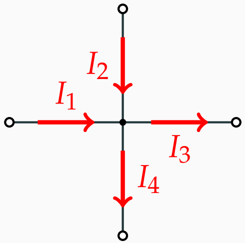
$$\boxed{P = \frac{V^2}{R}} \qquad \boxed{P = I^2 R}$$

Quantity	Symbol	SI Unit
Power	P	W
Voltage	V	V
Resistance	R	Ω
Current	I	A

Kirchkoff’s Laws

Kirchkoff's Current Law

The electric current that flows into any junction in an electric circuit must be equal to the current which flows out.



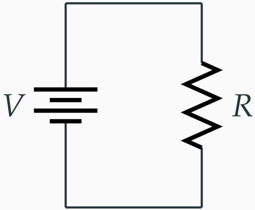
e.g. if there are 4 paths to the junction at the center, with I_1 and I_2 going into the junction, and I_3 and I_4 coming out, then the current law says that

$$I_1 + I_2 - I_3 - I_4 = 0$$

Basically, it means that there cannot be any accumulation of charges anywhere in the circuit. The law is a consequence of conservation of energy.

Kirchkoff's Voltage Law

The voltage changes around any closed loop in the circuit must sum to zero, no matter what path you take through an electric circuit.



Assume that the current flows clockwise and we draw a clockwise loop, we get

$$V - V_R = 0 \quad \rightarrow \quad V - IR = 0$$

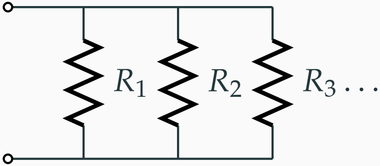
If I incorrectly guess that I flows counterclockwise, I will still have a similar expression

$$-V_R - V = 0 \quad \rightarrow \quad -V - IR = 0$$

When solving for I, we get a negative number, indicating that my guess was in the wrong direction.

Resistors in Circuits

Resistors in Parallel



From the current law, we know that the total current is the current through all the resistors, which we can rewrite in terms of voltage and resistance using Ohm's law:

$$I = I_1 + I_2 + I_3 \cdots = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} \cdots$$

But since we also know that $V_1 = V_2 = V_3 = \cdots = V$ from the voltage law, we can re-write as

$$I = \frac{V}{R_{\text{eq}}} = V \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \cdots \right)$$

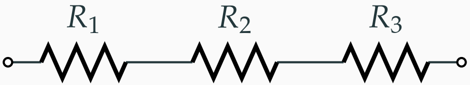
Equivalent Resistance of Resistors in Parallel

Through applying Ohm's Law and Kirchkoff's laws, we find the equivalent resistance of a parallel circuit: The inverse of the equivalent resistance for resistors connected in parallel is the sum of the inverses of the individual resistances.

$$\boxed{\frac{1}{R_p} = \sum_i^N \frac{1}{R_i}}$$

Quantity	Symbol	SI Unit
Equivalent resistance in parallel	R_p	Ω
Resistance of individual loads	$R_{1,2,3,\cdots,N}$	Ω

Resistors in Series



The analysis for resistors in series is similar (but easier). From the current law, the current through each resistor is the same:

$$I_1 = I_2 = I_3 = \cdots = I$$

And the total voltage drop across all resistor is therefore:

$$V = V_1 + V_2 + V_3 + \cdots = I(R_1 + R_2 + R_3 + \cdots)$$

Equivalent Resistance of Resistors in Series

Again, through applying Ohm's Law and Kirchkoff's laws, we find that when resistors are connected in series: the equivalent resistance of loads is the sum of the resistances of the individual loads.

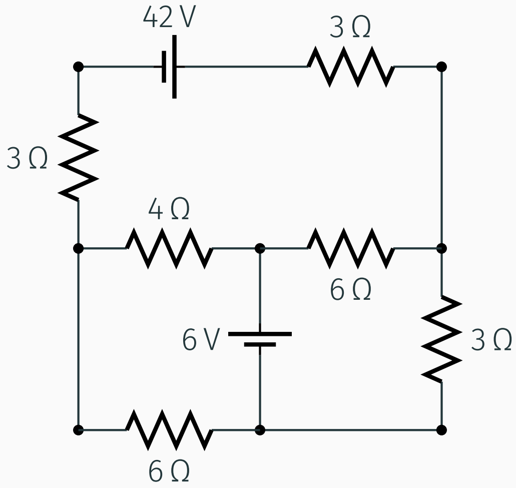
$$\boxed{R_s = \sum_{i=1}^{N} R_i}$$

Quantity	Symbol	SI Unit
Equivalent resistance in series	R_s	Ω
Resistance of individual loads	$R_{1,2,3,\cdots,N}$	Ω

Tips for Solving "Simple" Circuit Problems

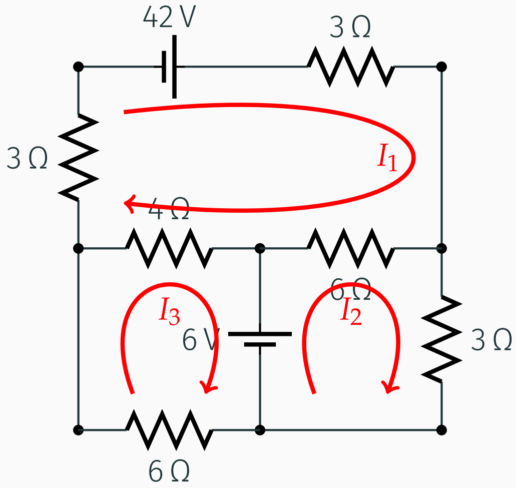
1. Identify groups of resistors that are in parallel or in series, and find their equivalent resistance.
2. Gradually reduce the entire circuit to one voltage source and one resistor.
3. Using Ohm's law, find the current out of the battery.
4. Using Kirchkoff's laws, find the current through each of the resistors.

A Very Difficult Example



- To solve this problem, we define a few " current loops" around the circuit: one on top, one on bottom left, and one on bottom right.

A Very Difficult Example



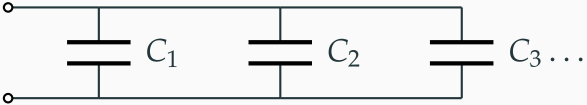
- To solve this problem, we define a few “current loops” around the circuit: one on top, one on bottom left, and one on bottom right.
- Apply the voltage law in the loops. For example, in the lower left:

$$4(I_1 - I_3) - 6 - 6I_3 = 0$$

- Solve the system of equations to find the current. If the current that you worked out is negative, it means that you have the direction wrong.

Capacitors in Circuit

Capacitors in Parallel



From the voltage law, we know that the voltage across all the capacitors are the same, i.e. $V_1 = V_2 = V_3 - \cdots = V$. We can express the total charge Q_{tot} stored across all the capacitors in terms of capacitance and this common voltage V:

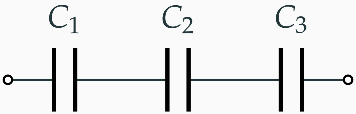
$$Q_{\text{tot}} = Q_1 + Q_2 + Q_3 + \cdots = C_1 V + C_2 V + C_3 V + \cdots$$

Factoring out V from each term gives us the equivalent capacitance:

$$\boxed{C_p = \sum_i C_i}$$

Capacitors in Series

Likewise, we can do a similar analysis to capacitors connected in series.



The total voltage across these capacitors are the sum of the voltages across each of them, i.e. $V_{\text{tot}} = V_1 + V_2 + V_3 + \cdots$

We recognize that the charge stored on all the capacitors must be the same! We can then write the total voltage in terms of capacitance and charge:

$$V_{\text{tot}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \cdots$$

Capacitors in Series

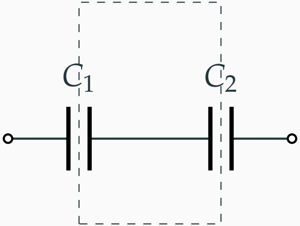
The inverse of the equivalent capacitance for N capacitors connected in series is the sum of the inverses of the individual capacitance.

$$\boxed{\frac{1}{C_s} = \sum_i \frac{1}{C_i}}$$

Make sure we don't confuse ourselves with resistors.

How Do We Know That Charges Are The Same?

It's simple to show that the charges across all the capacitors are the same

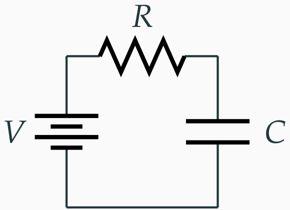


The capacitor plates and the wire connecting them are really one piece of conductor. There is nowhere for the charges to leave the conductor, therefore when charges are accumulating on C_1, C_2 must also have the same charge because of conservation of charges.

R-C Circuits

Circuits with Resistors and Capacitors

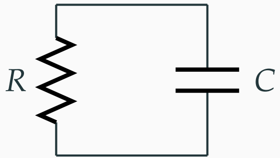
Now that we have seen how resistors and capacitors behave in a circuit, we can look into combining them in to an “R-C circuit”.



The simplest form is a resistor and capacitor connected in series, and then connect to a voltage source. Because of the nature of capacitors, the current through the circuit will not be steady as were the case with only resistors.

Discharging a Capacitor

We start the analysis with something even simpler. There is no voltage source, and the capacitor is already charged to $V_c = Q_{\text{tot}}/C$. What happens when the current begin to flow?



As current starts to flow, the charge on the capacitor decrease. Over time the current decreases, until the capacitor is fully discharged, and current stops flowing.

Apply the voltage law for the circuit, and substitute the definition of current $I = dQ/dt$ and the voltage across a capacitor $V_c = Q/C$:

$$V_c - IR = 0 \quad \rightarrow \quad -\frac{Q}{C} - R\frac{dQ}{dt} = 0$$

Discharging a Capacitor

Separating the Q terms on the left side of the equation, and leaving everything else on the right side, we get:

$$\frac{dQ}{Q} = \frac{-dt}{RC}$$

which we can now integrate and "exponentiate":

$$\int \frac{dQ}{Q} = \int \frac{-dt}{RC} \;\rightarrow\; \ln Q = \frac{-t}{RC} + K \;\rightarrow\; Q = e^{K} e^{-t/RC}$$

The constant of integration K is the initial charge on the capacitor Q_{tot}:

$$e^{K} = Q_{\mathrm{tot}}$$

Discharging a Capacitor

The expression of charge across the capacitor is time-dependent:

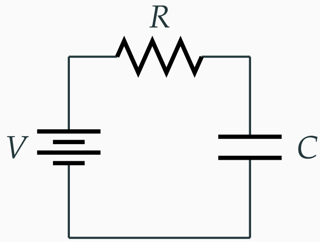
$$\boxed{Q(t) = Q_0 e^{-t/\tau}}$$

where $Q_0 = Q_{\text{tot}}$ is the initial charge on the capacitor, and $\tau = RC$ is called the **time constant**. Taking the time derivative of $Q(t)$ gives us the current through the circuit:

$$\boxed{I(t) = I_0 e^{-t/\tau}}$$

where the initially current at $t = 0$ is given by $I_0 = Q_{\text{tot}}/\tau = Q_{\text{tot}}/RC$.

Charging a Capacitor

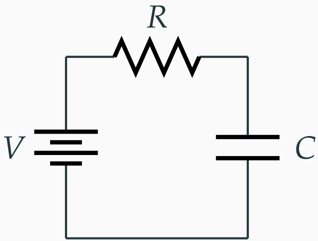


When charging the capacitor, when we apply the voltage law, and do some calculus, we find that the charge across the capacitor is:

$$\boxed{Q(t) = Q_{\text{tot}}(1 - e^{-t/RC})}$$

where the time constant $\tau = RC$ is the same as the discharging case

Charging a Capacitor



The current through the capacitor is:

$$\boxed{I_c(t) = I_0 e^{-t/\tau}}$$

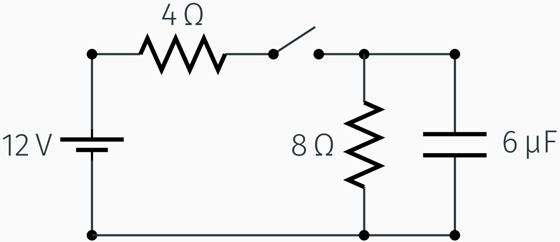
where the initial current $I_0 = Q_{\text{tot}}/\tau = V/R$. This makes sense because $V_C = 0$ at $t = 0$, and all of the energy must be dissipated through the resistor.

- At $t = \infty$, current through the capcitor is $I_c = 0$

Two Small Notes

1. When a capacitor is uncharged, there is no voltage across the plate, it acts like a short circuit.
2. When a capacitor is charged, there is a voltage across it, but no current flows *through* it. Effectively it acts like an open circuit.

A Slightly More Difficult Problem



Example: The capacitor in the circuit is initially uncharged. Find the current through the battery

1. Immediately after the switch is closed
2. A long time after the switch is closed