# Topic 6: Circular Motion

## Advanced Placement Physics C

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Olympiads School

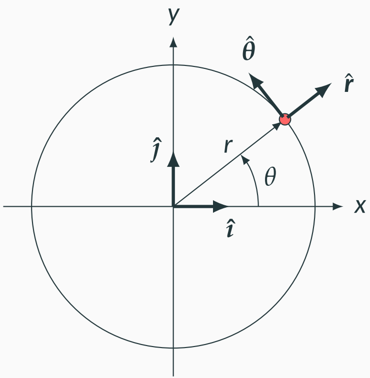
# Review of Circular Motion

In a **circular motion**, an object of mass $m$ moves in a circular path about a fixed center. In Grade 12 Physics, you should have studied *uniform* circular motion, where:

- the object's speed (magnitude of velocity) is constant
- the object's **centripetal acceleration** is toward the center
- the object's acceleration is caused by a **centripetal force**

# Polar Coordinates

# Polar Coordinate System in 2D




In the Cartesian coordinate system, an object's position is described by its $x$ and $y$ coordinates:

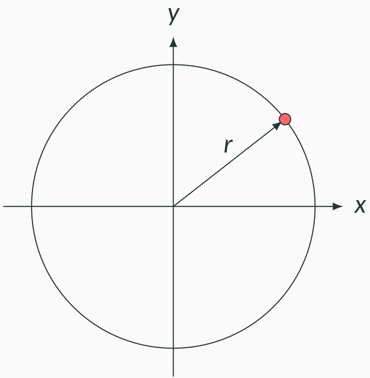
$$\boldsymbol{x}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}$$

For circular motion or general rotational motion, the **polar coordinate system** is preferred. The position of an object is described by:

$$\boldsymbol{r}(t) = r(t)\hat{\boldsymbol{r}} + \theta(t)\hat{\boldsymbol{\theta}}$$

- $r$ is distance from the origin
- $\theta$ is the standard angle, measured counter clockwise from the $x$ axis in *radians*

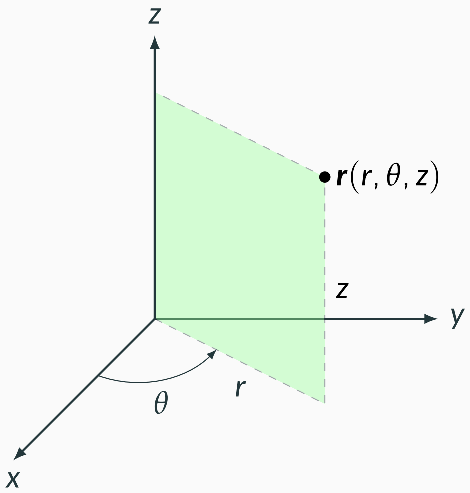
# Polar Coordinate System in 2D




- Like the Cartesian system, the polar coordinate system is also right-handed
- Both basic vectors $\hat{\boldsymbol{r}}$ (radial direction) and $\hat{\boldsymbol{\theta}}$ (angular direction) rotate as the object moves
- Simpler way is to think of position as just two parameters, which is *exactly* how position vectors are expressed in Grade 11/12 Physics: magnitude ($r$) and direction ($\theta$)!
- Cartesian and polar coordinates are related by:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

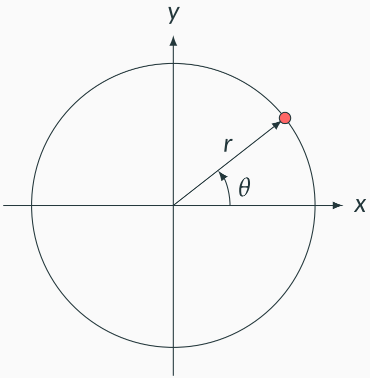
# Cylindrical Coordinates in 3D




One way to extend the coordinates coordinate system into 3D is the **cylindrical coordinate system**. Note that the discussions for this topic focuses on *xy* plane. Since the *z*-axis is linearly independent of the *xy* plane, motion along that direction is independent.

# Rigid-Body Circular Motion

# Angular Position and Angular Velocity




For a constant $r$, the **angular position** $\theta$ determines an object's position as a function of time:

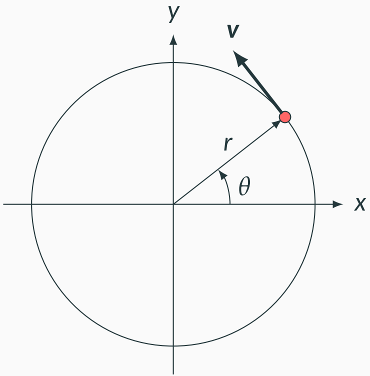
$$\boxed{\theta = \theta(t)}$$

**Angular velocity** $\omega$ (or **angular frequency**) is its time derivative:

$$\boxed{\omega(t) = \frac{d\theta(t)}{dt} = \dot{\theta}}$$

$\theta$ is measured in radians, and $\omega$ in rad/s

# Velocity and Angular Velocity

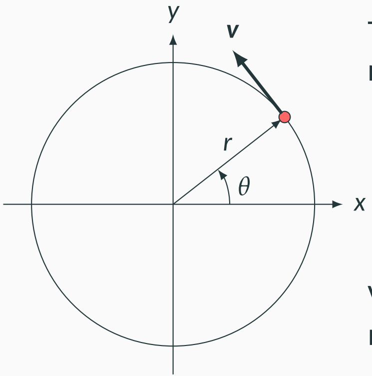



The velocity of the object in circular motion is related to the angular velocity (or angular frequency) by:

$$v = r\omega$$

- The direction of $\boldsymbol{v}$ is tangent to circle, along $\hat{\boldsymbol{\theta}}$, and therefore $\perp$ to $\hat{\boldsymbol{r}}$
- If $\omega > 0$, the motion is counter-clockwise
- If $\omega < 0$, the motion is clockwise

# Velocity and Angular Velocity



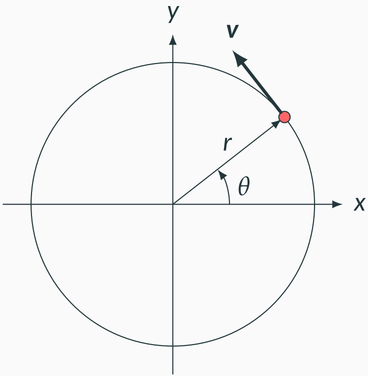

The velocity of the object in circular motion is more properly related to the angular velocity using this vector cross product:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$

- $\omega$: out of the page if motion is counter-clockwise
- $\omega$: into the page if motion is clockwise

Visualizing $\omega$ takes practice, but this vector notation is mathematically rigorious and consistent

# Period & Frequency




For constant angular velocity $\omega$ (uniform circular motion), the motion is periodic. Its **frequency** and **period** are given by:

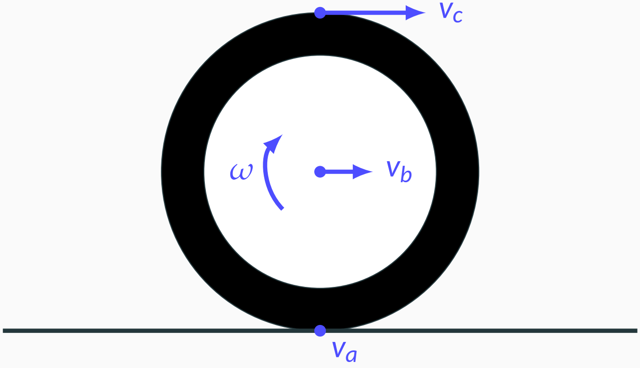
$$f = \frac{\omega}{2\pi} \quad T = \frac{2\pi}{\omega} \quad f = \frac{1}{T}$$

$T$ is in **seconds** (s) and $f$ is in **hertz** (Hz)

# Rotating Object Without Slipping

A tire with radius $r$ rolls along the road with an angular velocity $\omega$ *without slipping*. (This is a very common case for analysis.) What is its velocity $v$

a. at the contact between the ground and the tire?
b. at the center?
c. at the top of the tire?



# Angular Acceleration

The time derivative of $\omega$ is **angular acceleration**, which has a unit of rad/s$^2$:

$$\alpha = \dot{\omega} = \ddot{\theta}$$

Similar to the relationship between velocity and angular velocity, **tangential acceleration** $a_t$ is related to angular acceleration $\alpha$ by the radius $r$:

$$a_t(t) = \dot{v} = r\dot{\omega} = r\alpha$$

For *uniform* circular motion, $\omega$ is constant, and therefore $a_t = 0$

# With Calculus

Relationship between angular position and angular velocity:

$$\omega(t) = \frac{d\theta}{dt} \qquad \theta(t) = \int \omega(t)dt + \theta_0$$

Relationship between angular velocity and angular acceleration:

$$\alpha(t) = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \qquad \omega(t) = \int \alpha(t)dt + \omega_0$$

The relationships are the same as in rectilinear motion.

# Kinematics in the Angular Direction

For constant $\alpha$, the kinematic equations are just like in rectilinear motion:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\theta = \theta_0 + \frac{\omega_0 + \omega}{2} t$$
$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

If $\alpha$ is *not* constant, integration will be required.

# A Simple Example

**Example 1:** An object moves in a circle with angular acceleration $3.0\,\mathrm{rad/s^2}$. The radius is $2.0\,\mathrm{m}$ and it starts from rest. How long does it take for this object to finish a circle?

# Centripetal Acceleration & Centripetal Force

There is also a component of acceleration toward the center of the motion, called the **centripetal acceleration** $a_c$:

$$\boxed{\boldsymbol{a}_c = -\frac{v^2}{r}\hat{\boldsymbol{r}} = -(\omega^2 r)\hat{\boldsymbol{r}}}$$

(The negative sign is because $\hat{\boldsymbol{r}}$ is radially outward from the center.) The force that causes the centripetal acceleration is called the **centripetal force**:

$$\boxed{\boldsymbol{F}_c = m\boldsymbol{a}_c = -\frac{mv^2}{r}\hat{\boldsymbol{r}}}$$

# Centripetal Acceleration for Uniform Circular Motion

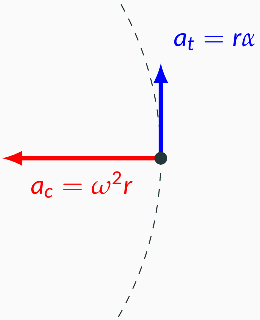
In uniform circular motion ($\alpha = 0$) problems where the period or frequency are known, the speed of the object is:

$$v = \omega r = 2\pi r f = \frac{2\pi r}{T}$$

Centripetal acceleration can therefore be expressed based on $T$ or $f$:

$$\boldsymbol{a}_c = -\frac{v^2}{r}\hat{\boldsymbol{r}} \quad \rightarrow \quad \boxed{\boldsymbol{a}_c = -\frac{4\pi^2 r}{T^2}\hat{\boldsymbol{r}} = -4\pi^2 r f^2 \hat{\boldsymbol{r}}}$$

# Acceleration: The General Case




In general circular motion, there are two components of acceleration:

- **Centripetal acceleration** $a_c$ depends on radius of curvature $r$ and instantaneous speed $v$. The direction of the acceleration is toward the center of the circle.
- **Tangential acceleration** $a_t$ depends on radius $r$ and angular acceleration $\alpha$. The direction of the acceleration is tangent to the circle

Most of the cases in AP Physics are uniform circular motion.

# How to Solve Circular Motion Problems

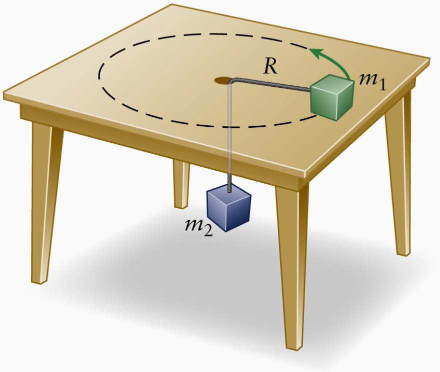
The condition for circular motion is the second law of motion:

$$\boldsymbol{F}_c = \sum \boldsymbol{F} = m\boldsymbol{a}_c$$

The forces that generate the centripetal force comes from the free-body diagram. It may include:

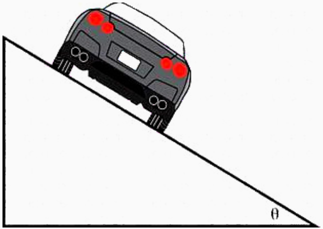
- Gravity
- Friction
- Normal force
- Tension
- Etc.

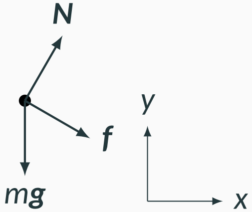
# Example: Horizontal Motion




**Example 2:** In the figure on the left, a mass $m_1 = 3.0\,\mathrm{kg}$ is rolling around a frictionless table with radius $R = 1.0\,\mathrm{m}$. with a speed of $2.0\,\mathrm{m/s}$. What is the mass of the weight $m_2$?

# Banked Curves on Highways and Racetracks




No motion in the *y* direction, i.e. no net force:

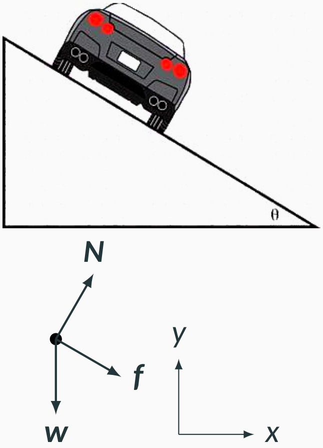
$$\sum F_y = N\cos\theta - f\sin\theta - w = 0$$

Net force in the *x* direction is the centripetal force:

$$\sum F_x = N\sin\theta + f\cos\theta = \frac{mv^2}{r}$$

Friction force ***f*** may be static or kinetic.

# Banked Curves on Highways and Racetracks

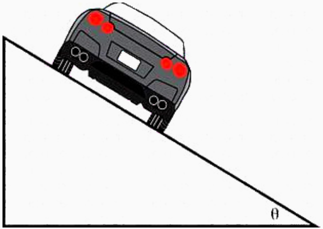


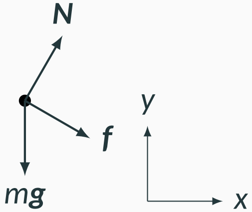

For analysis, use the simplified equation for friction $f = \mu N$ (i.e. assume either kinetic friction or maximum static friction), and weight $w = mg$, the equations on the previous slides can be arranged as:

$$N\left(\cos\theta - \mu\sin\theta\right) = mg$$

$$N\left(\sin\theta + \mu\cos\theta\right) = \frac{mv^2}{r}$$

# Banked Curves on Highways and Racetracks




Dividing the two equations removes both the normal force and mass terms:

$$\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg}$$

The *maximum* velocity $v_{max}$ can be expressed as:

$$\boxed{v_{max} = \sqrt{rg\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}}}$$

Note that $v_{max}$ does not depend on mass.

# Banked Curves on Highways and Racetracks

In the limit of $\mu = 0$ (frictionless case), the equation reduces to:

$$\boxed{v_{\max} = \sqrt{rg\tan\theta}}$$

And in the limit of a flat roadway with no banking ($\theta = 0$, $\sin\theta = 0$ and $\cos\theta = 1$), the equation reduces to:

$$\boxed{v_{\max} = \sqrt{\mu rg}}$$

# Vertical Circles

# Vertical Circles

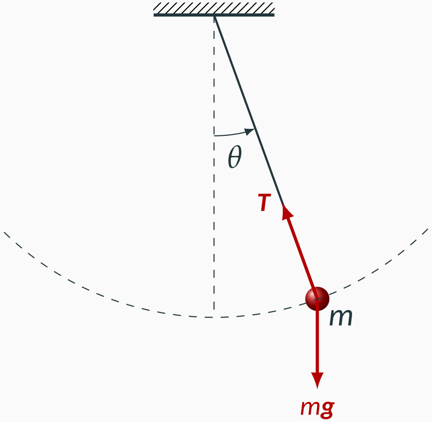
Circular motion with a horizontal path is straightforward. However, for vertical motion:

- Generally difficult to solve by dynamics and kinematics
- Instead, use conservation of energy to solve for $\boldsymbol{v}$
- Then use the equation for centripetal force to find other forces

**Remember:** If it is impossible to get the required centripetal force, then it could not continue the circular motion

# What About a Pendulum?

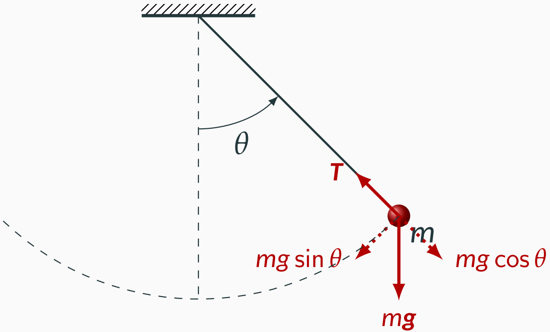
A simple pendulum is also like a vertical circular motion problem.




- There are two forces act on the pendulum: weight $\boldsymbol{w} = m\boldsymbol{g}$, and tension $\boldsymbol{T}$
- Speed of the pendulum at any height is found using conservation of energy
  - Tension $\boldsymbol{T}$ is always $\perp$ to motion, therefore it doesn't do any work
  - Work is done by gravity (a conservative force) alone
- Tangential and centripetal accelerations are based on the net force along the angular and radial directions

# Simple Pendulum

At the top of the swing, velocity $v$ is zero, therefore:




Centripetal acceleration is also zero:

$$a_c = \frac{v^2}{r} = 0$$

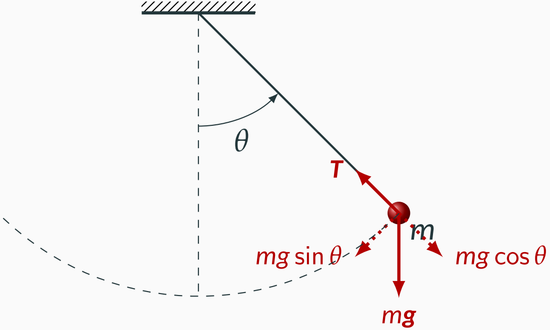
and therefore the net force along the radial direction $\hat{r}$ is zero. The tension force $T$ can be calculated:

$$T = mg\cos\theta$$

At the highest point when $\theta$ is largest, tension is the lowest.

# Simple Pendulum

In the radial direction $\hat{\theta}$, there is still a net force of $mg\sin\theta$, therefore:



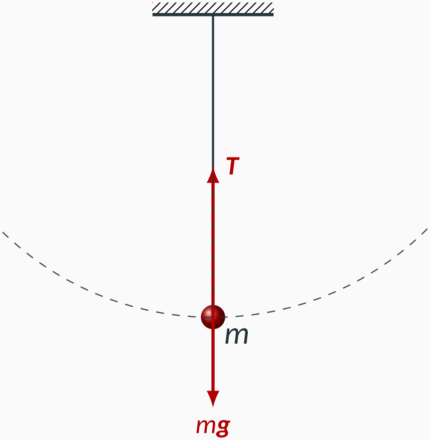

There is a tengential acceleration along $\hat{\theta}$, with a magnitude of:

$$a_t = g\sin\theta$$

This is the same acceleration as an object sliding down a frictionless ramp at an angle of $\theta$.

# Simple Pendulum

At the bottom of the swing, the velocity is at its maximum value,




- Maximum centripetal acceleration:

$$a_c = \frac{v^2}{r}$$

- No tangential acceleration:

$$a_t = 0$$

- At the lowest point, tension is the highest:

$$T = w + F_c = m\left(g + \frac{v^2}{r}\right)$$

# Example Problem

**Example 4:** You are playing with a yo-yo with a mass of 225 g. The full length of the string is 1.2 m. You decide to see how slowly you can swing it in a vertical circle while keeping the string fully extended, even when the yo-yo is at the top of its swing.

A. Calculate the minimum speed at which you can swing the yo-yo while keeping it on a circular path.

B. Find the tension in the string when the yo-yo is at the side and at the bottom of its swing.

# Example Problem

**Example 5:** A cord is tied to a pail of water, and the pail is swung in a vertical circle of 1.0 m. What must be the minimum velocity of the pail be at its highest point so that no water spills out?

(a) 3.1 m/s

(b) 5.6 m/s

(c) 20.7 m/s

(d) 100.5 m/s

# Example: Roller Coaster

**Example 6:** A roller coaster car is on a track that forms a circular loop, of radius $R$, in the vertical plane. If the car is to maintain contact with the track at the top of the loop (generally considered to be a good thing), what is the minimum speed that the car must have at the bottom of the loop. Ignore air resistance and rolling friction.

(a) $\sqrt{2gR}$

(b) $\sqrt{3gR}$

(c) $\sqrt{4gR}$

(d) $\sqrt{5gR}$

# Example

**Example 7:** A stone of mass $m$ is attached to a light strong string and whirled in a *vertical* circle of radius $r$. At the exact bottom of the path, the tension of the string is three times the weight of the stone. The stone's speed at that point is given by:

(a) $2\sqrt{gR}$

(b) $\sqrt{2gR}$

(c) $\sqrt{3gR}$

(d) $4gR$