Topic 5: Center of Mass

Advanced Placement Physics

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Olympiads School

Files to Download

- `PhysAPC-05-CM-print.pdf`—The “print version” of this topic.
- `PhysAPC-05-Homework.pdf`—Homework problems for Topics 4 & 5.

Please download/print the PDF file for the class slides before each class. There is no point copying notes that are already on the slides. Instead, focus on things that aren’t necessarily on the slides. If you wish to print the slides, we recommend printing 4 slides per page.

Center of Mass

Finding an object's center of mass is important, because

- The laws of motion are formulated by treating an objects as point masses (for real-life objects, we let the forces apply to the center of mass)
- Objects can have *rotational* motion in addition to *translational* motion as well (we will examine that a bit more next week)

Start with a Definition

The **center of mass** ("CM") is the *weighted average of the masses in a system*. The "system" may be:

- A collection of individual particles (use summation to compute CM)
- A continuous distribution of mass with constant density (use integration to compute CM); in this case, CM is also the geometric center of the object, called the **centroid**
- A continuous distribution of mass with varying density (use integral to compute CM)
- If the masses are inside of a gravitational field, then the CM is also its **center of gravity** ("CG")

Simple Example

We start with a very simple example: there are two equal masses along the x-axis. What is the center of mass of the system?

The answer is really simple: it's at the half way point between the two masses!

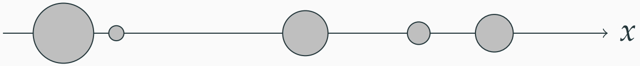
Things Aren't Always That Example

- What if one of the masses are increased to $2m$?
- This is still not a terribly difficult problem; you can still *guess* the right answer without knowing the equation for center of mass.

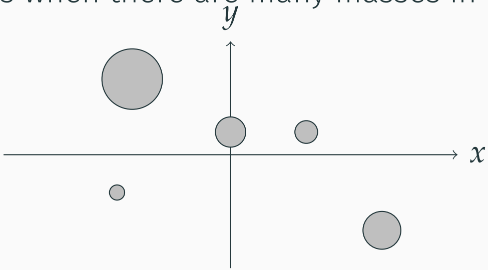
- The answer is still simple. The CM is no longer at the half way point between the two masses, but now $\frac{1}{3}$ the total distance from the larger masses.

Complicating Things Further

If we increase the number of point masses along the x-axis, our problem can become much more complicated (although still not devastatingly so)



Difficulties really arises when there are many masses in the system in 2D or 3D:



An Equation Helps

The center of mass is defined as:

$$\boxed{\mathbf{x}_{\mathrm{CM}} = \frac{\sum \mathbf{x}_i m_i}{\sum m_i}}$$

Quantity	Symbol	SI Unit
Position of center of mass (vector)	$\mathbf{x}_{\mathrm{CM}}$	m
Position of point mass i (vector)	$\mathbf{x}_i$	m
Point mass i	m_i	kg
Total mass	$\sum m_i$	kg

Breaking Down Into Components

$$\boxed{\mathbf{x}_{\mathrm{CM}} = \frac{\sum \mathbf{x}_i m_i}{\sum m_i}}$$

Position vectors have x, y and z components: $\mathbf{x} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ which we can deal with each component individually, i.e.:

$$x_{\mathrm{CM}} = \frac{\sum x_i m_i}{\sum m_i} \qquad y_{\mathrm{CM}} = \frac{\sum y_i m_i}{\sum m_i} \qquad z_{\mathrm{CM}} = \frac{\sum z_i m_i}{\sum m_i}$$

An Example

Example 1: Consider the following masses and their coordinates which make up a "discrete mass" rigid body"

$$
\begin{aligned}
m_1 &= 5.0\,\mathrm{kg} & \qquad \mathbf{x}_1 &= 3\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{k}} \\
m_2 &= 10.0\,\mathrm{kg} & \qquad \mathbf{x}_2 &= -4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 7\hat{\boldsymbol{k}} \\
m_3 &= 1.0\,\mathrm{kg} & \qquad \mathbf{x}_3 &= 10\hat{\boldsymbol{\imath}} - 17\hat{\boldsymbol{\jmath}} + 10\hat{\boldsymbol{k}}
\end{aligned}
$$

What are the coordinates for the center of mass of this system?

Continuous Mass Distribution

In general, objects are not a discrete collection of point masses, but a continuous distribution of mass. Therefore, we take the limit of when the number of masses approaches ∞:

$$\mathbf{x}_{\mathrm{CM}} = \lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n}\mathbf{x}_i m_i}{\sum_{i=1}^{n} m_i}\right)$$

This gives us an integral form of our equation:

$$\boxed{\mathbf{x}_{\mathrm{CM}} = \frac{\int \mathbf{x}\, dm}{\int dm}}$$

Densities

- Linear density (for 1D problems)

$$\gamma = \frac{m}{L}$$

- Surface area density (for 2D problems)

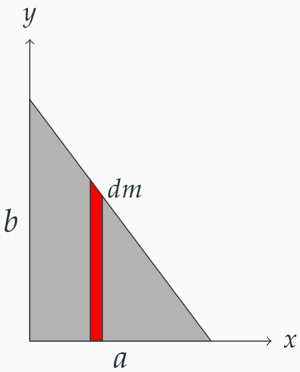
$$\sigma = \frac{m}{A}$$

- Volume density (for 3D problems)

$$\rho = \frac{m}{V}$$

An Example with Integrals

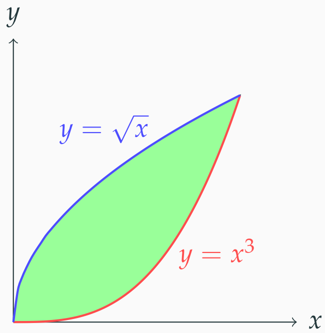
Example 2: A triangular plate is placed in a Cartesian coordinate system with two of its edges along the x and y-axis. The length of the edges along the axes are a and b respectively. Assuming that the surface area density σ is uniform, determine the coordinate of its center of mass.



A Difficult Example to Try at Home

Not typically an AP problem, this example shows how we can use integral to find the center of mass for something very complicated.

Example 3: Find the x-coordinate of the center of mass in the shape bound by the two functions shown on the right.

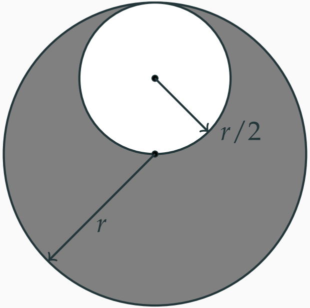


Symmetry

- Any plane of symmetry, mirror line, axis of rotation, point of inversion *must* contain the center of mass.
- Caveat: only works if the density distribution is also symmetric
- Again: if density is uniform, CM is also geometric center (centroid)

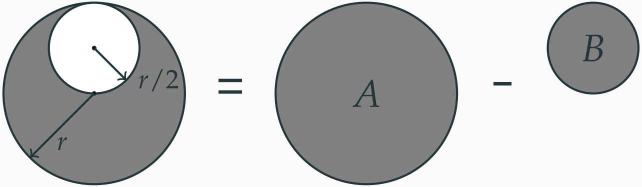
“Negative Mass”

- Where there is a “hole” in the geometry, treat it as having negative mass density $-\sigma$ in that region.
- Negative masses don't exist, so this is really just a trick.
- **Example:** What is the center of mass of this shape?



Negative Mass Example

- This is how we would think of it:



- Let the origin of the coordinate system to located at the center of A
- Based on symmetry: $x_{\mathrm{CM}} = 0$; only have to find y-coordinate.
- Sum our weighted average:

$$y_{\mathrm{CM}} = \frac{\sum y_i m_i}{\sum m_i} = \frac{m_A(0) + m_B(r/2)}{m_A + m_B} = \frac{-\sigma\pi\left(r/2\right)^2\left(r/2\right)}{\sigma\pi r^2 - \sigma\pi\left(r/2\right)^2} = \frac{-r}{6}$$

Velocity, Acceleration and Momentum

Take time derivative of the equation for $\mathbf{x}_{\mathrm{CM}}$ to get the velocity of the CM:

$$\mathbf{v}_{\mathrm{CM}} = \frac{d\mathbf{x}_{\mathrm{CM}}}{dt} = \frac{1}{m}\frac{d}{dt}\left(\int \mathbf{x}dm\right) = \frac{1}{m}\int \frac{d\mathbf{x}}{dt}dm = \frac{\int \mathbf{v}dm}{m}$$

The integral in the numerator is the sum of the momentum of all the masses in the system ($\mathbf{p}_{\mathrm{net}}$) which means that we have

$$\mathbf{p}_{\mathrm{net}} = m\mathbf{v}_{\mathrm{CM}}$$

Taking the derivative of $\mathbf{p}_{\mathrm{net}}$ relates force and acceleration at the CM as well:

$$\mathbf{F}_{\mathrm{net}} = \frac{d\mathbf{p}_{\mathrm{net}}}{dt} = m\frac{d\mathbf{v}_{\mathrm{CM}}}{dt} = m\mathbf{a}_{\mathrm{CM}}$$