Topic 6: Circular Motion

Advanced Placement Physics C

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Olympiads School

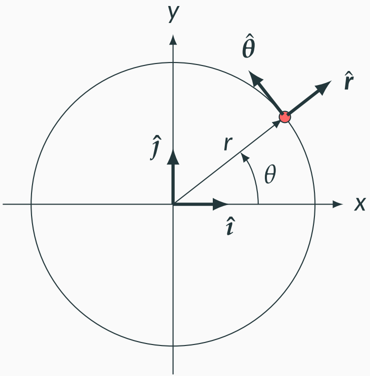
Review of Circular Motion

In a **circular motion**, an object of mass m moves in a circular path about a fixed center. In Grade 12 Physics, you should have studied *uniform* circular motion, where:

- the object's speed (magnitude of velocity) is constant
- the object's **centripetal acceleration** is toward the center
- the object's acceleration is caused by a **centripetal force**

Polar Coordinates

Polar Coordinate System in 2D



In the Cartesian coordinate system, an object's position is described by its x and y coordinates:

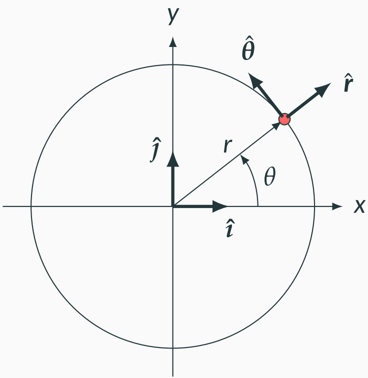
$$\boldsymbol{x}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}$$

For circular motion or general rotational motion, the **polar coordinate system** is preferred. The position of an object is described by:

$$\boldsymbol{r}(t) = r(t)\hat{\boldsymbol{r}} + \theta(t)\hat{\boldsymbol{\theta}}$$

- r is distance from the origin
- θ is the standard angle, measured counter clockwise from the x axis in *radians*

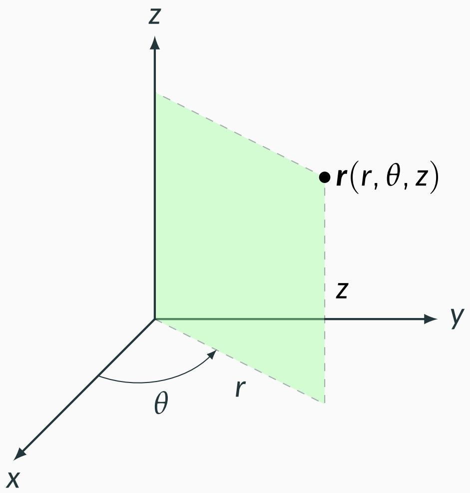
Polar Coordinate System in 2D



- Like the Cartesian system, the polar coordinate system is also right-handed
- Both basic vectors $\hat{\boldsymbol{r}}$ (radial direction) and $\hat{\boldsymbol{\theta}}$ (angular direction) rotate as the object moves
- Simpler way is to think of position as just two parameters, which is *exactly* how position vectors are expressed in Grade 11/12 Physics: magnitude (r) and direction (θ)!
- Cartesian and polar coordinates are related by:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

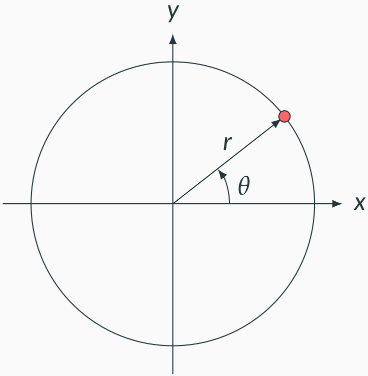
Cylindrical Coordinates in 3D



One way to extend the coordinates coordinate system into 3D is the **cylindrical coordinate system**. Note that the discussions for this topic focuses on *xy* plane. Since the *z*-axis is linearly independent of the *xy* plane, motion along that direction is independent.

Rigid-Body Circular Motion

Angular Position and Angular Velocity



For a constant r, the **angular position** θ determines an object's position as a function of time:

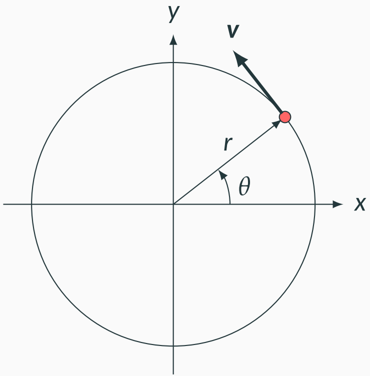
$$\boxed{\theta = \theta(t)}$$

Angular velocity ω (or **angular frequency**) is its time derivative:

$$\boxed{\omega(t) = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}}$$

θ is measured in radians, and ω in rad/s

Velocity and Angular Velocity

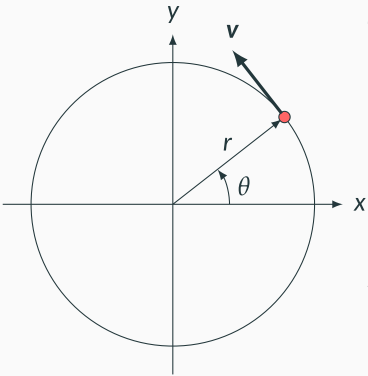


The velocity of the object in circular motion is related to the angular velocity (or angular frequency) by:

$$v = r\omega$$

- The direction of $\boldsymbol{v}$ is tangent to circle, along $\hat{\boldsymbol{\theta}}$, and therefore $\perp$ to $\hat{\boldsymbol{r}}$
- If $\omega > 0$, the motion is counter-clockwise
- If $\omega < 0$, the motion is clockwise

Velocity and Angular Velocity



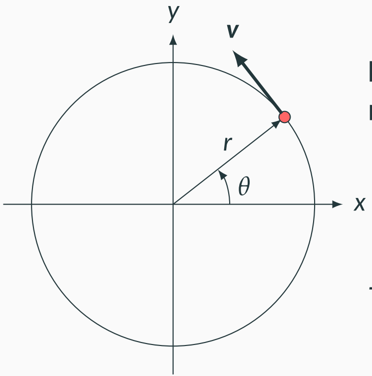
The velocity of the object in circular motion is more properly related to the angular velocity using this vector cross product:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$

- $\boldsymbol{\omega}$: out of the page if motion is counter-clockwise
- $\boldsymbol{\omega}$: into the page if motion is clockwise

Visualizing $\boldsymbol{\omega}$ takes practice, but this vector notation is mathematically rigorious and consistent

Period & Frequency



For constant angular velocity ω (uniform circular motion), the motion is periodic. Its **frequency** and **period** are given by:

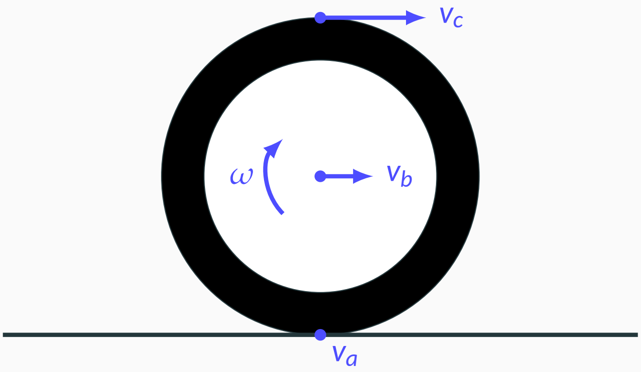
$$f = \frac{\omega}{2\pi} \quad T = \frac{2\pi}{\omega} \quad f = \frac{1}{T}$$

T is in **seconds** (s) and f is in **hertz** (Hz)

Rotating Object Without Slipping

A tire with radius r rolls along the road with an angular velocity ω *without slipping*. (This is a very common case for analysis.) What is its velocity v

a. at the contact between the ground and the tire?
b. at the center?
c. at the top of the tire?



Angular Acceleration

The time derivative of ω is **angular acceleration**, which has a unit of rad/s^2:

$$\alpha = \dot{\omega} = \ddot{\theta}$$

Similar to the relationship between velocity and angular velocity, **tangential acceleration** a_t is related to angular acceleration α by the radius r:

$$a_t(t) = \dot{v} = r\dot{\omega} = r\alpha$$

For *uniform* circular motion, ω is constant, and therefore $a_t = 0$

With Calculus

Relationship between angular position and angular velocity:

$$\omega(t) = \frac{\mathrm{d}\theta}{\mathrm{d}t} \qquad \theta(t) = \int \omega(t)\mathrm{d}t + \theta_0$$

Relationship between angular velocity and angular acceleration:

$$\alpha(t) = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \qquad \omega(t) = \int \alpha(t)\mathrm{d}t + \omega_0$$

The relationships are the same as in rectilinear motion.

Kinematics in the Angular Direction

For constant α, the kinematic equations are just like in rectilinear motion:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \theta_0 + \frac{\omega_0 + \omega}{2} t$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

If α is *not* constant, integration will be required.

A Simple Example

Example 1: An object moves in a circle with angular acceleration $3.0\,\mathrm{rad/s^2}$. The radius is $2.0\,\mathrm{m}$ and it starts from rest. How long does it take for this object to finish a circle?

Centripetal Acceleration & Centripetal Force

There is also a component of acceleration toward the center of the motion, called the **centripetal acceleration** a_c:

$$\boxed{\boldsymbol{a}_c = -\frac{v^2}{r}\hat{\boldsymbol{r}} = -(\omega^2 r)\hat{\boldsymbol{r}}}$$

(The negative sign is because $\hat{\boldsymbol{r}}$ is radially outward from the center.) The force that causes the centripetal acceleration is called the **centripetal force**:

$$\boxed{\boldsymbol{F}_c = m\boldsymbol{a}_c = -\frac{mv^2}{r}\hat{\boldsymbol{r}}}$$

Centripetal Acceleration for Uniform Circular Motion

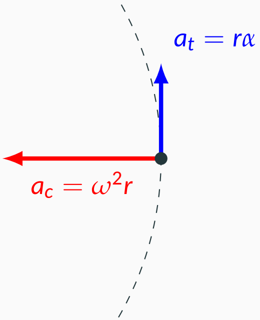
In uniform circular motion ($\alpha = 0$) problems where the period or frequency are known, the speed of the object is:

$$v = \omega r = 2\pi r f = \frac{2\pi r}{T}$$

Centripetal acceleration can therefore be expressed based on T or f:

$$\boldsymbol{a}_c = -\frac{v^2}{r}\hat{\boldsymbol{r}} \quad \rightarrow \quad \boxed{\boldsymbol{a}_c = -\frac{4\pi^2 r}{T^2}\hat{\boldsymbol{r}} = -4\pi^2 r f^2 \hat{\boldsymbol{r}}}$$

Acceleration: The General Case



In general circular motion, there are two components of acceleration:

- **Centripetal acceleration** a_c depends on radius of curvature r and instantaneous speed v. The direction of the acceleration is toward the center of the circle.
- **Tangential acceleration** a_t depends on radius r and angular acceleration α. The direction of the acceleration is tangent to the circle

Most of the cases in AP Physics are uniform circular motion.

How to Solve Circular Motion Problems

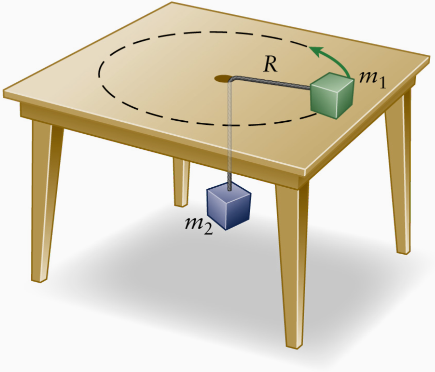
The condition for circular motion is the second law of motion:

$$\boldsymbol{F}_c = \sum \boldsymbol{F} = m\boldsymbol{a}_c$$

The forces that generate the centripetal force comes from the free-body diagram. It may include:

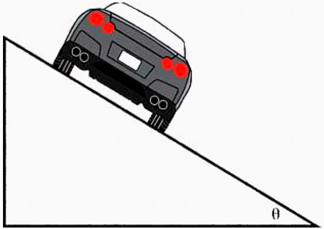
- Gravity
- Friction
- Normal force
- Tension
- Etc.

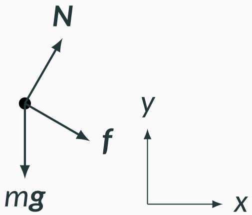
Example: Horizontal Motion



Example 2: In the figure on the left, a mass $m_1 = 3.0\,\mathrm{kg}$ is rolling around a frictionless table with radius $R = 1.0\,\mathrm{m}$. with a speed of $2.0\,\mathrm{m/s}$. What is the mass of the weight m_2?

Banked Curves on Highways and Racetracks





No motion in the *y* direction, i.e. no net force:

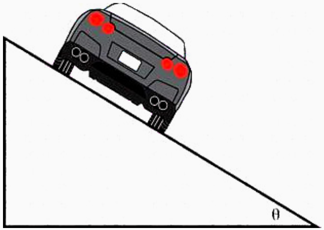
$$\sum F_y = N\cos\theta - f\sin\theta - w = 0$$

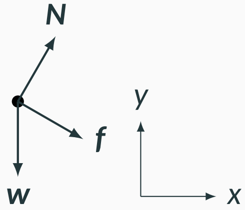
Net force in the *x* direction is the centripetal force:

$$\sum F_x = N\sin\theta + f\cos\theta = \frac{mv^2}{r}$$

Friction force ***f*** may be static or kinetic.

Banked Curves on Highways and Racetracks



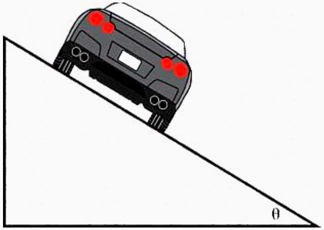


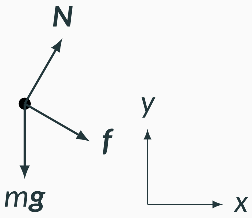
For analysis, use the simplified equation for friction $f = \mu N$ (i.e. assume either kinetic friction or maximum static friction), and weight $w = mg$, the equations on the previous slides can be arranged as:

$$N\left(\cos\theta - \mu\sin\theta\right) = mg$$

$$N\left(\sin\theta + \mu\cos\theta\right) = \frac{mv^2}{r}$$

Banked Curves on Highways and Racetracks





Dividing the two equations removes both the normal force and mass terms:

$$\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg}$$

The *maximum* velocity $v_{\max}$ can be expressed as:

$$\boxed{v_{\max} = \sqrt{rg\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}}}$$

Note that $v_{\max}$ does not depend on mass.

Banked Curves on Highways and Racetracks

In the limit of $\mu = 0$ (frictionless case), the equation reduces to:

$$\boxed{v_{\text{max}} = \sqrt{rg \tan \theta}}$$

And in the limit of a flat roadway with no banking ($\theta = 0$, $\sin \theta = 0$ and $\cos \theta = 1$), the equation reduces to:

$$\boxed{v_{\text{max}} = \sqrt{\mu rg}}$$

Vertical Circles

Vertical Circles

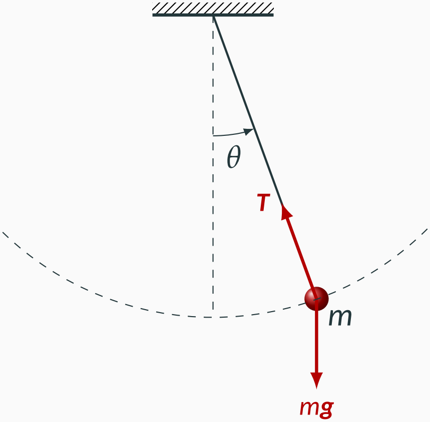
Circular motion with a horizontal path is straightforward. However, for vertical motion:

- Generally difficult to solve by dynamics and kinematics
- Instead, use conservation of energy to solve for $\boldsymbol{v}$
- Then use the equation for centripetal force to find other forces

Remember: If it is impossible to get the required centripetal force, then it could not continue the circular motion

What About a Pendulum?

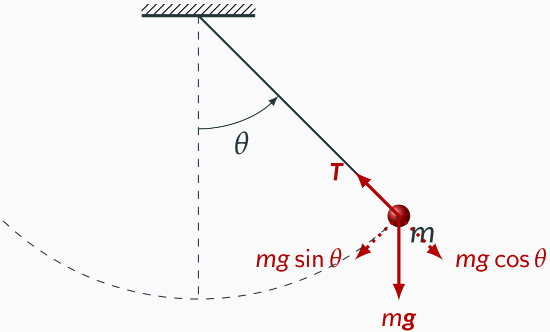
A simple pendulum is also like a vertical circular motion problem.



- There are two forces act on the pendulum: weight $\boldsymbol{w} = m\boldsymbol{g}$, and tension $\boldsymbol{T}$
- Speed of the pendulum at any height is found using conservation of energy
    - Tension $\boldsymbol{T}$ is always $\perp$ to motion, therefore it doesn't do any work
    - Work is done by gravity (a conservative force) alone
- Tangential and centripetal accelerations are based on the net force along the angular and radial directions

Simple Pendulum

At the top of the swing, velocity v is zero, therefore:



Centripetal acceleration is also zero:

$$a_c = \frac{v^2}{r} = 0$$

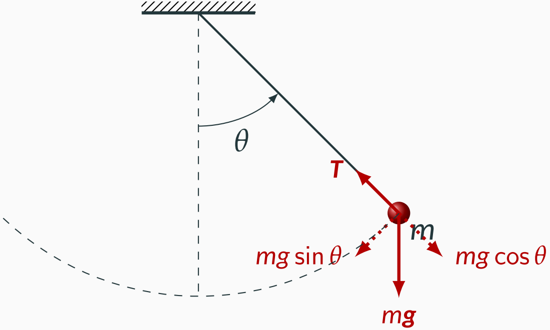
and therefore the net force along the radial direction $\hat{r}$ is zero. The tension force T can be calculated:

$$T = mg\cos\theta$$

At the highest point when θ is largest, tension is the lowest.

Simple Pendulum

In the radial direction $\hat{\theta}$, there is still a net force of $mg \sin\theta$, therefore:



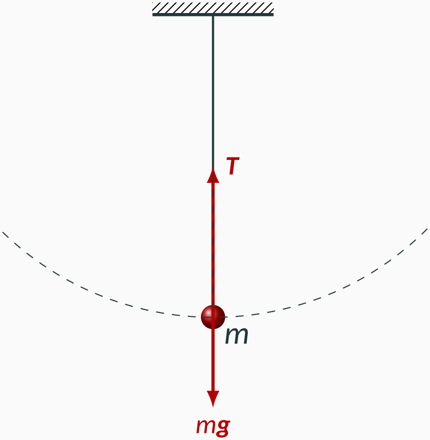
There is a tengential acceleration along $\hat{\theta}$, with a magnitude of:

$$a_t = g \sin\theta$$

This is the same acceleration as an object sliding down a frictionless ramp at an angle of θ.

Simple Pendulum

At the bottom of the swing, the velocity is at its maximum value,



- Maximum centripetal acceleration:

$$a_c = \frac{v^2}{r}$$

- No tangential acceleration:

$$a_t = 0$$

- At the lowest point, tension is the highest:

$$T = w + F_c = m\left(g + \frac{v^2}{r}\right)$$

Example Problem

Example 4: You are playing with a yo-yo with a mass of M. TThe full length of the string is R. You decide to see how slowly you can swing it in a vertical circle while keeping the string fully extended, even when the yo-yo is at the top of its swing.

A. Calculate the minimum speed at which you can swing the yo-yo while keeping it on a circular path.
B. Find the tension in the string when the yo-yo is at the side and at the bottom of its swing.

Example Problem

Example 5: A cord is tied to a pail of water, and the pail is swung in a vertical circle of 1.0 m. What must be the minimum velocity of the pail be at its highest point so that no water spills out?

(a) 3.1 m/s
(b) 5.6 m/s
(c) 20.7 m/s
(d) 100.5 m/s

Example: Roller Coaster

Example 6: A roller coaster car is on a track that forms a circular loop, of radius R, in the vertical plane. If the car is to maintain contact with the track at the top of the loop (generally considered to be a good thing), what is the minimum speed that the car must have at the bottom of the loop. Ignore air resistance and rolling friction.

(a) $\sqrt{2gR}$

(b) $\sqrt{3gR}$

(c) $\sqrt{4gR}$

(d) $\sqrt{5gR}$

Example

Example 7: A stone of mass m is attached to a light strong string and whirled in a *vertical* circle of radius r. At the exact bottom of the path, the tension of the string is three times the weight of the stone. The stone's speed at that point is given by:

(a) $2\sqrt{gR}$

(b) $\sqrt{2gR}$

(c) $\sqrt{3gR}$

(d) $4gR$