# Class 12: Electrostatics Part 1 (Point Charges)

Advanced Placement Physics C

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Olympiads School

# Electrostatic Force

# Review: The Charges Are

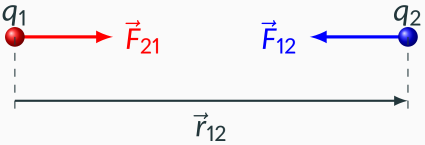
We should already know a bit about charge particles:

- A **proton** carries a **positive** charge
- An **electron** carries a **negative** charge
- A *net charge* of an object means an excess of protons or electrons
- Similar charges are repel; opposite charges attract

We start with electrostatics:

- Charges that are not moving relative to one another

# Coulomb's Law for Electrostatic Force




The **electrostatic force** (or **coulomb force**) is a mutually repulsive/attractive force between all charged objects. The force that charge $q_1$ exerts on $q_2$ is given by **Coulomb's law**:

$$\boxed{\vec{F}_{12} = \frac{kq_1q_2}{|\vec{r}_{12}|^2}\hat{r}_{12}}$$

# Coulomb's Law for Electrostatic Force

$$\boxed{\vec{F}_{12} = \frac{kq_1q_2}{\left|\vec{r}_{12}\right|^2}\hat{r}_{12}}$$

| Quantity | Symbol | SI Unit |
|---|---|---|
| Electrostatic force | $\vec{F}_{12}$ | N |
| Coulomb's constant | $k$ | $\mathrm{N} \cdot \mathrm{m}^2 / \mathrm{C}^2$ |
| Point charges 1 and 2 | $q_1, q_2$ | C |
| Distance between point charges | $\left\|\vec{r}_{12}\right\|$ | m |
| Unit vector of direction between point charges | $\hat{r}_{12}$ | |

# Coulomb's Constant

$$\boxed{\vec{F}_{12} = \frac{kq_1q_2}{|\vec{r}_{12}|^2}\hat{r}_{12}}$$

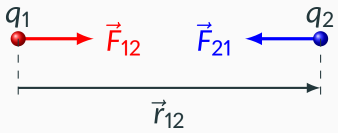
The constant $k$ in the Coulomb's law is called the **coulomb's constant**, defined as:

$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \, \mathrm{N \cdot m^2 / C^2}$$

where $\epsilon_0$ is a fundamental constant called the **permittivity of free space**, or **vacuum permittivity**. It measures a vacuum's ability to resist the formation of an electric field:

$$\epsilon_0 = 8.85 \times 10^{-12} \, \mathrm{C^2 / N \cdot m^2}$$

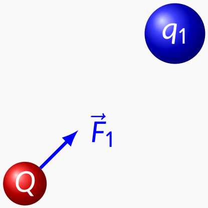
# Coulomb's Law for Electrostatic Force




- Third law of motion: If $q_1$ exerts an electrostatic force $\vec{F}_{12}$ on $q_2$, then $q_2$ likewise exerts a force of $\vec{F}_{21} = -\vec{F}_{12}$ on $q_1$. The two forces are equal in magnitude and opposite in direction.
- $q_1$ and $q_2$ are assumed to be *point charges* that do not occupy any space
- The scalar form is often used as well, since the direction of $F_q$ can easily be found:

$$\boxed{F_q = \frac{kq_1q_2}{r^2}}$$

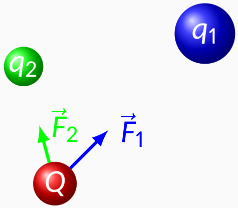
# More Than One Charge




For a charge Q that is subjected to the influence of multiple discrete point charges $q_i$, the total electrostatic force that Q experiences is the vector sum of all the forces $\vec{F}_i$:

$$\boxed{\vec{F} = \sum_i \vec{F}_i = kQ\left(\sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{r}_i\right)}$$

# More Than One Charge




For a charge $Q$ that is subjected to the influence of multiple discrete point charges $q_i$, the total electrostatic force that $Q$ experiences is the vector sum of all the forces $\vec{F}_i$:

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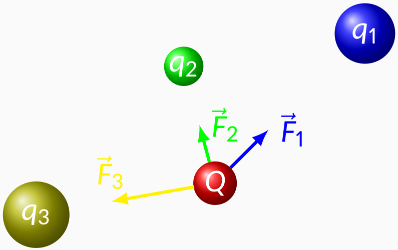


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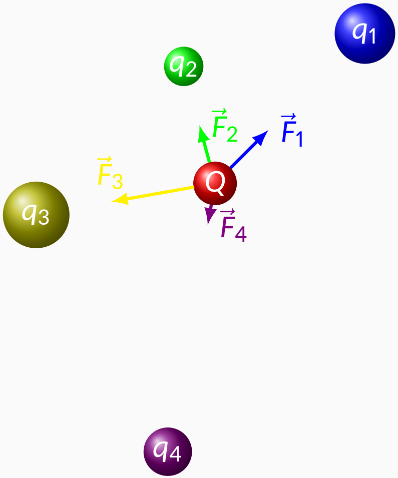


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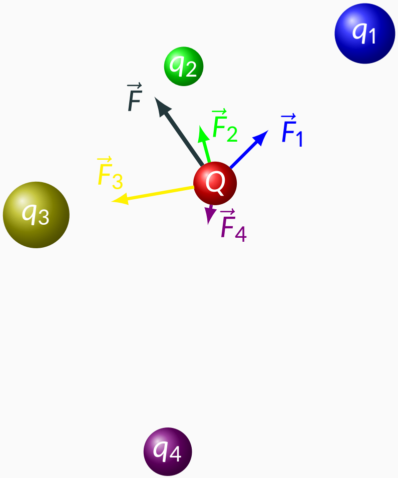



For a charge $Q$ that is subjected to the influence of multiple discrete point charges $q_i$, the total electrostatic force that $Q$ experiences is the vector sum of all the forces $\vec{F}_i$:

$$\boxed{\vec{F} = \sum_i \vec{F}_i = kQ\left(\sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{r}_i\right)}$$

# Continuous Distribution of Charges

As $N \to \infty$, the summation becomes an integral, and can now be used to describe the force from charges with *spatial extend* i.e. charges that take up physical space (e.g. a continuous distribution of charges):

$$\boxed{\vec{F} = \int \mathrm{d}\vec{F} = kQ \int \frac{\mathrm{d}q}{r^2}\hat{r}}$$

# Infinitesimal Charge d$q$

The calculation for the infinitesimal charge d$q$ is similar to the calculation for the infinitesimal mass d$m$ earlier in the course (See Class 5: Center of Mass)

- Linear charge density (for 1D problems)

$$\gamma = \frac{\mathrm{d}q}{\mathrm{d}L} \quad \rightarrow \quad \mathrm{d}q = \gamma \mathrm{d}L$$

- Surface charge density (for 2D problems)

$$\sigma = \frac{\mathrm{d}q}{\mathrm{d}A} \quad \rightarrow \quad \mathrm{d}q = \sigma \mathrm{d}A$$

- Charge density (for 3D problems)

$$\rho = \frac{\mathrm{d}q}{\mathrm{d}V} \quad \rightarrow \quad \mathrm{d}q = \rho \mathrm{d}V$$

# Electric Field

# Electric Field

The expression for **electric field** is obtained by repeating the same procedure as with gravitational field, by grouping the variables in Coulomb's law:

$$F_q = \underbrace{\left[ \frac{kq_1}{|\vec{r}_{12}|^2} \hat{r} \right]}_{\vec{E}} q_2$$

The electric field $\vec{E}$ created by $q_1$ is a vector function (called a **vector field**) that shows how it influences other charged particles around it.

# Electric Field Near a Point Charge

The electric field a distance $r$ away from a point charge $q$ is given by:

$$\boxed{\vec{E}(q,\vec{r}) = \frac{kq}{|\vec{r}|^2}\hat{r}}$$

| Quantity | Symbol | SI Unit |
|---|---|---|
| Electric field intensity | $\vec{E}$ | N / C |
| Coulomb's constant | $k$ | $\mathrm{N} \cdot \mathrm{m}^2 / \mathrm{C}^2$ |
| Source charge | $q$ | C |
| Distance from source charge | $\lvert\vec{r}\rvert$ | m |
| Outward unit vector from point source | $\hat{r}$ | |

The direction of $\vec{E}$ is radially outward from a positive point charge and radially inward toward a negative charge.

# More Than One Charge

When multiple point charges are present, the total electric field at any position $\vec{r}$ is the vector sum of all the fields $\vec{E}_i$:

$$\boxed{\vec{E} = \sum_i \vec{E}_i = k\left(\sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{r}_i\right)}$$

# More Than One Charge

As $N \to \infty$, the summation becomes an integral, and can now be used to describe the electric field generated by charges with *spatial extend*:

$$\boxed{\vec{E} = \int \mathrm{d}\vec{E} = k \int \frac{\mathrm{d}q}{r^2}\hat{r}}$$

This integral may be difficult to compute if the geometry of is complicated, but in general, in AP Physics C, there are usually symmetry that can be exploited.

# Think Electric Field

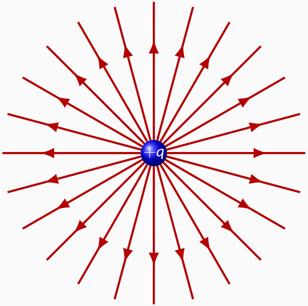
$\vec{E}$ itself *doesn't do anything* until another charge interacts with it. And when there is a charge $q$, the electrostatic force $\vec{F}_q$ that the charge experiences is proportional to $q$ and $\vec{E}$, regardless of how the electric field is generated:

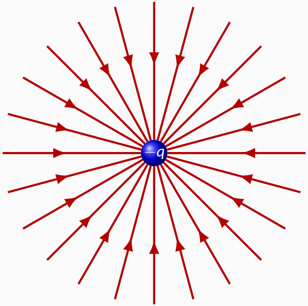
$$\boxed{\vec{F}_q = q\vec{E}}$$

A positive charge in the electric field experiences an electrostatic force $\vec{F}$ in the same direction as $\vec{E}$.

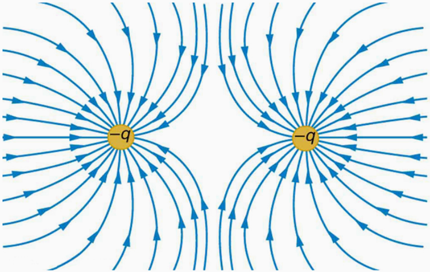
# Electric Field Lines

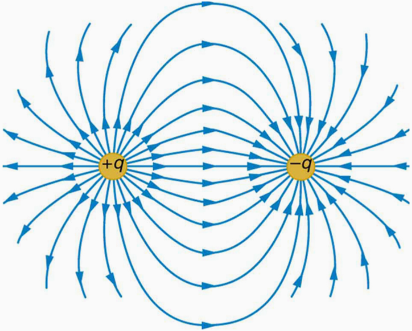
**Electric field lines** can be used to visualize the direction of the electric field.



# Electric Field from Multiple Charges

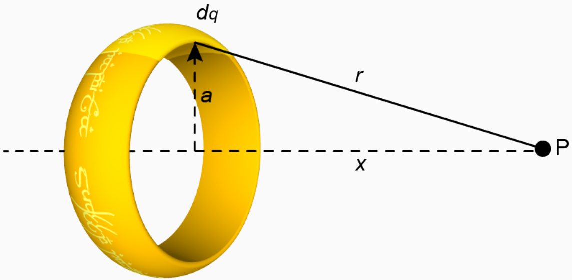



- Electric field lines must begin and/or end at a charge
- Field lines do not cross
- Direction of the electric field is tangent to the field lines

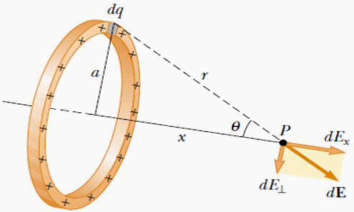
# Lord of the Ring Charge

Suppose you have been given *The One Ring To Rule Them All*, and you found out that it is charged! What is its electric field at point $P$ along its axis?




Note that calculating the electric field away from the axis is very difficult.

# Electric Field Along Axis of a Ring Charge




- We can separate the electric field $d\vec{E}$ (generated by charge $dq$) into axial ($dE_x$) and radial ($dE_\perp$) components
- Based on symmetry, $dE_\perp$ doesn't contribute to anything; but $dE_x$ is pretty easy to find:

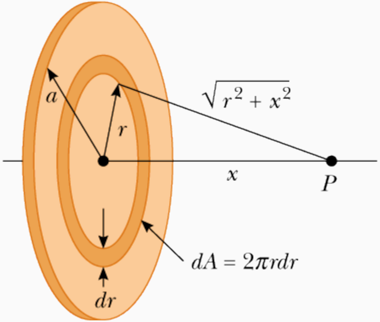
$$dE_x = \frac{kdq}{r^2}\cos\theta = \frac{kdq}{r^2}\frac{x}{r} = \frac{kxdq}{(x^2+a^2)^{3/2}}$$

Integrating this over all charges $dq$, we have:

$$E_x = \frac{kx}{(x^2+a^2)^{3/2}}\int dq = \boxed{\frac{kQx}{(x^2+a^2)^{3/2}}}$$

# Electric Field Along Axis of a Uniformly Charged Disk

Let's extend what we know to a disk of radius $a$ and charge density $\sigma$




We start with the solution from the ring problem, and replace Q with $\mathrm{d}q = 2\pi\sigma r\mathrm{d}r$:

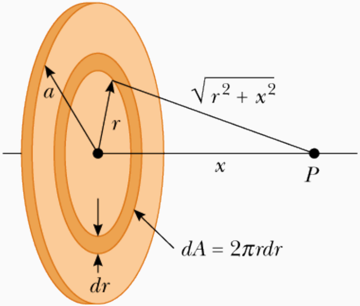
$$\mathrm{d}E_x = \frac{2\pi kr\sigma x}{(x^2 + r^2)^{3/2}}\mathrm{d}r$$

Integrating over the entire disk:

$$E_x = \pi kx\sigma \int_0^a \frac{2r}{(x^2 + r^2)^{3/2}}\mathrm{d}r$$

This is not an easy integral!

# Eclectic Field Along Axis of a Uniformly Charged Disk




Luckily for us, the integral is in the form of $\int u^n du$, with $u = x^2 + r^2$ and $n = \frac{-3}{2}$. You can find the integral in any math textbook:

$$E_x = 2\pi k\sigma \left(1 - \frac{x}{\sqrt{x^2 + a^2}}\right)$$

# Electric Potential Energy

# Electric Potential Energy

The electrostsatic force is a conservative force, therefore the work done by $F_q$ is related to the **electric potential energy** $U_q$:

$$W = \int \vec{F}_q \cdot \mathrm{d}\vec{r} = kq_1q_2 \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r^2} = -\frac{kq_1q_2}{r}\bigg|_{r_1}^{r_2} = -\Delta U_q$$

where

$$\boxed{U_q = \frac{kq_1q_2}{r}}$$

- $U_q$ can be $(+)$ or $(-)$, because charges can be either $(+)$ or $(-)$
- Positive work done by $F_q$ decreases $U_q$, while
- Negative work done by $F_q$ increases $U_q$
- $W$ depends on $r_1$ and $r_2$ but not *how* the charge moves from $r_1 \to r_2$

# How it Differs from Gravitational Potential Energy

Two positive charges:

$$U_q > 0$$

Two negative charges:

$$U_q > 0$$

One positive and one negative charge:

$$U_q < 0$$

- $U_q > 0$ means positive work is done to bring two charges together from $r = \infty$ to $r$ (both charges of the same sign)
- $U_q < 0$ means negative work (the charges are opposite signs)
- For gravitational potential $U_g$ is always $< 0$

# Relating $U_q$ to $\vec{F}_q$

From the fundamental theorem calculus, we can relate electrostatic force ($\vec{F}_q$) to electric potential energy ($U_q$) by the gradient operator:

$$\Delta U_q = -\int \vec{F}_q \cdot \mathrm{d}\vec{r} \quad \rightarrow \quad \vec{F}_q(r) = -\nabla U_q = -\frac{\partial U_q}{\partial r}\hat{r}$$

Electrostatic force $\vec{F}_q$ always points from high to low potential energy (steepest descent direction)

# Electric Potential

# Electric Potential: Using Gravity as Example

An object at a specific location inside a gravitational field has a gravitational potential energy proportional to its mass, i.e.

$$U_g = V_g m$$

This "constant" $V_g$ is called the **gravitational potential**, which is the *gravitational potential energy per unit mass*. In the trivial case with a uniform gravitational field:

$$V_g = \frac{U_g}{m} = gh$$

This also applies to the general case of the gravitational potential energy:

$$V_g = \frac{U_g}{m} = -\frac{Gm}{r}$$

# Electric Potential

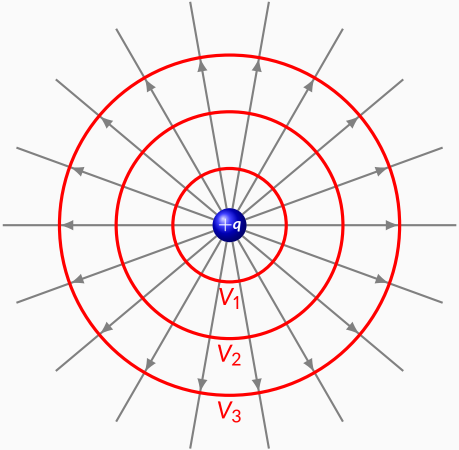
This is also true for moving a charged particle $q$ against an electric electric field created by $q_s$, and the "constant" is called the **electric potential**. The unit for electric potential is a *volt* which is *one joule per coulomb*, i.e. $1\,\mathrm{V} = 1\,\mathrm{J/C}$

$$\boxed{V = \frac{U_q}{q}}$$

The electric potential from a source point charge $q_s$ is therefore:

$$\boxed{V = \frac{kq_s}{r}}$$

# Electric Potential

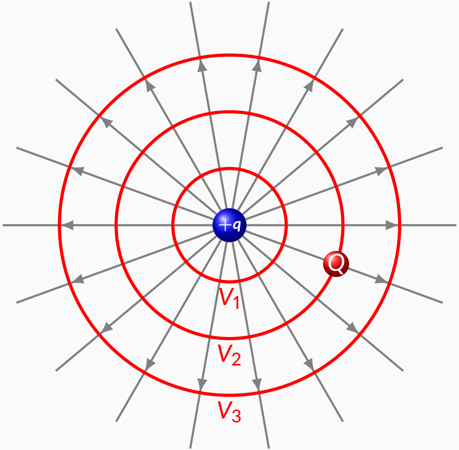



For a point charge $q$, every point at a distance $r$ will have the same electric potential $V(r)$.

- The red lines have the same electric potential; they are called **equipotential lines**, or **equipotential contours**
- Equipotential lines are perpendicular to the electric field lines
- Electric field lines always points from higher $V$ toward lower $V$, i.e.

$$V_1 > V_2 > V_3$$

# Electric Potential




A charge Q that is placed inside this electric field will now have an electric potential energy of:

$$U_q = QV = Q\left[\frac{kq}{r}\right]$$

in agreement with equation for electric potential energy

# Electric Potential from Multiple Charges

When there are multiple points charges present, the electric potential is given by the summation:

$$\boxed{V = \underbrace{\frac{1}{4\pi\epsilon_0}}_{k} \sum_{i=1}^{N} \frac{q_i}{r_i}}$$

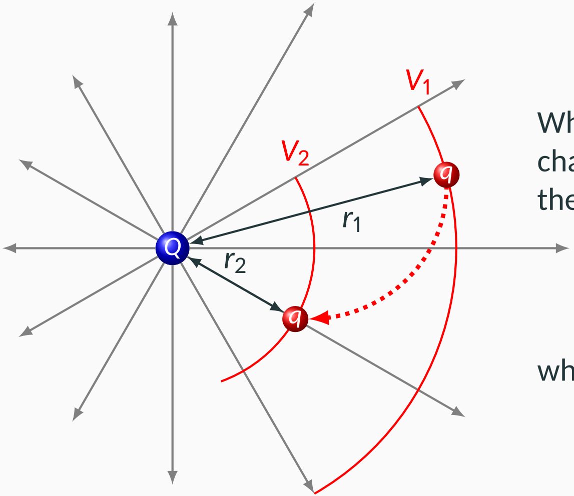
As $N \to \infty$ the summation becomes an integral:

$$\boxed{V = \frac{1}{4\pi\epsilon_0} \int \frac{\mathrm{d}q}{r}}$$

where $r$ is the distance to the infinitesimal charge $\mathrm{d}q$

# Electric Potential Difference

# Potential Difference




When a charge is moved from $r_1$ to $r_2$, the change in electric potential energy is related to the change in electric potential by:

$$\Delta U_q = U_2 - U_1 = q\Delta V$$

where $\Delta V$ is called the **potential difference**

# Potential Difference (Voltage)

The change in electric potential is called the **electric potential difference** or **voltage**:

$$\boxed{\Delta V = \frac{\Delta U_q}{q}} \quad \text{and} \quad \boxed{dV = \frac{dU_q}{q}}$$

Here, we can relate $\Delta V$ to an equation that we knew from Grade 11 Physics and AP Physics 1/2, which related to the energy dissipated in a resistor in a circuit $\Delta U$ to the voltage drop $\Delta V$:

$$\boxed{\Delta U_q = q\Delta V}$$

Electric potential difference also has the unit *volts* (V)

# Relating $V$ to $\vec{E}$

In the same way that the fundamental theorem of calculus relates the electrostatic force ($\vec{F}_q$) and electric potential energy ($U_q$) by the gradient operator, electric field ($\vec{E}$) and electric potential ($V$) are also related the same way:

$$\Delta V_q = -\int \vec{E}_q \cdot \mathrm{d}\vec{r} \quad \rightarrow \quad \vec{E}(r) = -\nabla V_q = -\frac{\partial V}{\partial r}\hat{r}$$

- Electrostatic field $\vec{E}$ always points from high to low electric potential
- Electric field is also called "potential gradient"

# Getting Those Names Right

Remember that these three scalar quantities, as opposed to electrostatic force $\vec{F}_q$ and electric field $\vec{E}$ which are vectors

- Electric potential energy:

$$U_q = \frac{kq_1q_2}{r}$$

- Electric potential:

$$V = \frac{U_q}{q}$$

- Electric potential difference (voltage):

$$\Delta V = \frac{\Delta U_q}{q}$$