Projectile Motion

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1 General Projectile Motion

A **projectile** is an object that is launched with an initial velocity of $\mathbf{v}_0$ at an angle θ above the horizontal. It travels along a parabolic trajectory and accelerates only due to gravity, as shown in Figure 1.

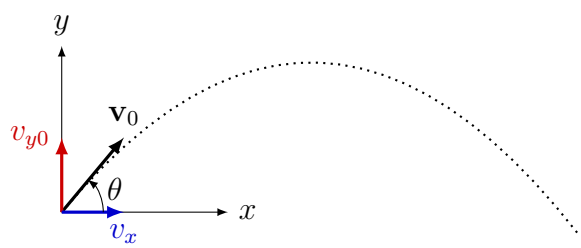

Figure 1: The parameters defining the motion of a projectile.

In general, when solving a projectile motion project, the x-axis is defined as the *horizontal* direction, with the (+) direction pointing forward, while the y-axis is the *vertical* direction, with the (+) direction pointing upwards. For simplicity, the origin of the coordinate system is usually the position where the projectile is launched. This method is consistent with the standard right-handed Cartesian coordinate system.

The initial velocity $\mathbf{v}_0$ can be resolved into its $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components, along the x and y axes:

$$\mathbf{v}_0 = v_x\hat{\boldsymbol{\imath}} + v_{y0}\hat{\boldsymbol{\jmath}} = v_0\cos\theta\hat{\boldsymbol{\imath}} + v_0\sin\theta\hat{\boldsymbol{\jmath}} \tag{1}$$

There is no acceleration (i.e. $a_x = 0$) along $\hat{\boldsymbol{\imath}}$ direction, therefore horizontal velocity v_x is constant, and the kinematic equations are reduce to a single equation:

$$x = v_x t = [v_0\cos\theta]\,t \tag{2}$$

where x is the horizontal position at time t, $v_0 = |\mathbf{v}_0|$ is the magnitude of the initial velocity, $v_x = v_0\cos\theta$ is its horizontal component.

There is a constant acceleration due to gravity alone along the $\hat{\boldsymbol{\jmath}}$ direction, i.e. $a_y = -g$. (Acceleration is *negative* due to the fact that the positive y-axis points upwards.) The kinematic equations

along the vertical direction are therefore:

$$y = [v_0 \sin\theta]\, t - \frac{1}{2} g t^2 \tag{3}$$

$$v_y = [v_0 \sin\theta] - gt \tag{4}$$

$$v_y^2 = [v_0 \sin\theta]^2 - 2gy \tag{5}$$

For most projectile motion problems, Equations 2 and 3 are the most important ones, and also most likely to be used for problem solving.

Because $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are perpendicular[1], horizontal and vertical motions are completely independent of each other. However, there are variables that are shared in both directions, namely:

- Time t
- Launch angle θ, measured above the horizontal[2]
- Initial speed v_0

When solving any projectile motion problems, there will be two unknowns that need to be solved (although you may not be explicitly told what one of them is), requiring two equations (the x and y kinematic equations). In some rare cases, if an object lands on an incline, there will be a third equation describing the relationship between x and y of the incline.

2 Symmetric Trajectory

A **symmetric trajectory** is a special case of projectile motion where an object is launched at an angle of θ (between 0° and 90°) above the horizontal[3] and then lands at the same height. Examples may include hitting a golf ball toward the hole, or shooting a bullet toward a horizontal target[4]. The equations for symmetric trajectory are *not* included in the AP Exam equation sheet; if you need these equations during the exams, you will need to derive them yourself. Thankfully, the derivation is quite striaghtforward.

Total time of flight $t_{\max}$: We apply the kinematic equation first in the y direction. When the object lands at the same height, the final velocity is the same in magnitude and opposite in direction as the initial velocity, i.e. $v_y = -v_{y0} = -v_0 \sin\theta$:

$$v_y = v_{y0} + a_y t$$

$$-v_0 \sin\theta = v_0 \sin\theta - g t_{\max}$$

[1] i.e. they are linearly independent

[2] It should be obvious that If the projectile is launched below horizontal, then $\theta < 0$

[3] This may be obvious, but angles *below* the horizontal will never have a symmetric trajectory.

[4] Shooting a bullet toward a horizontal target always require an upward angle because of gravity.

Solving for $t_{\max}$ we have:

$$\boxed{t_{\max} = \frac{2v_0 \sin\theta}{g}} \tag{6}$$

Not surprisingly, a projectile will stay in the air the longest when it is launched at $\theta = \frac{\pi}{2}$, or 90°. (As a fun exercise, for a known initial speed v_0, you can plot t vs. $\sin\theta$ to find the acceleration due to gravity g!)

Maximum height H: Apply the kinematic equation in the y-direction. Recognizing that at maximum height $H = y - y_0$, the vertical component of velocity is zero $v_y = 0$:

$$\begin{aligned} v_y^2 &= v_{y0}^2 + 2a_y(y - y_0) \\ 0 &= (v_0 \sin\theta)^2 - 2gH \end{aligned}$$

Solving for H, we get the maximum height equation:

$$\boxed{H = \frac{v_0^2 \sin^2\theta}{2g}} \tag{7}$$

The maximum height also (not surprisingly) has a maximum value at $\theta = \frac{\pi}{2}$.

Range R: We substitute the expression for $t_{\max}$ from Equation 6 into the t term, then apply the kinematic equation in the x-direction to compute $R = x - x_0$ for any given launch angle and initial speed:

$$R = (x - x_0) = v_x t = v_0 \cos\theta \left(\frac{2v_0 \sin\theta}{g}\right) \tag{8}$$

Using the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$, we simplify Equation 8 to:

$$\boxed{R = \frac{v_0^2 \sin(2\theta)}{g}} \tag{9}$$

It is obvious that for any given initial speed v_0, the maximum range $R_{\max}$ occurs at an angle where $\sin(2\theta) = 1$ (i.e. $\theta = \pi/4$, or 45°), with a value of

$$\boxed{R_{\max} = \frac{v_0^2}{g}} \tag{10}$$

Also, for a known initial speed v_0 and range R we can compute the launch angle θ:

$$\theta_1 = \frac{1}{2}\sin^{-1}\left(\frac{gR}{v_0^2}\right)$$

This angle is labelled θ_1 because it is *not* the only angle that can reach this range. Recall that for any angle $0 < \phi < \pi/2$, there is also another angle ϕ_2 where $\sin(\phi)$ is the same:

$$\sin\phi = \sin(\pi - \phi)$$

Which means that for any θ_1, there is also another angle θ_2 where $2\theta_2 = \pi - 2\theta_1$, or quite simply:

$$\theta_2 = \frac{\pi}{2} - \theta_1$$