Topic 18: Special Relativity

AP and IBHL Physics

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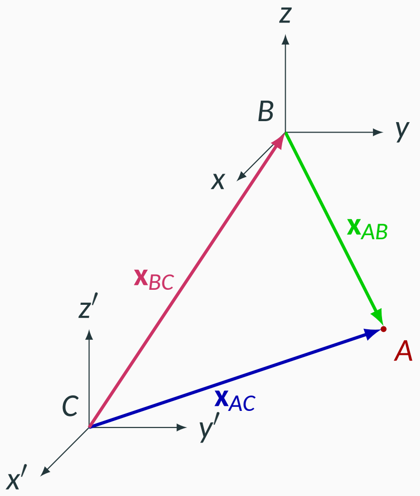
Olympiads School

Reference Frame

Relative Motion in Classical Physics



In the first topic (kinematics) we studied the definitions **relative position**:

$$\boxed{\mathbf{x}_{AC} = \mathbf{x}_{AB} + \mathbf{x}_{BC}}$$

relative velocity:

$$\boxed{\mathbf{v}_{AC} = \mathbf{v}_{AB} + \mathbf{v}_{BC}}$$

and **relative acceleration**:

$$\boxed{\mathbf{a}_{AC} = \mathbf{a}_{AB} + \mathbf{a}_{BC}}$$

Relative Motion

All motion quantities must be measured relative to a frame of reference

- A frame of reference is the coordinate system from which all physical measurements are made.
- Because all motions are relative, there is no absolute motion/rest

Frame of Reference

Think of a **frame of reference** (or just “frame”) as a hypothetical mobile “laboratory” an observer uses to make measurements (e.g. mass, lengths, time). At a minimum, it includes:

- Some rulers (i.e. coordinate system) to measure positions and lengths
- A clock to measure the passage of time
- A scale to compare forces
- A balance to measure masses

Frame of Reference

- We assume that the hypothetical laboratory is *perfect*—all the hypothetical "instruments" have zero errors
- What matters is the *motion* (at rest, uniform motion, acceleration etc) of your laboratory, and how it affects the measurement that you make
- "From the point of view of..."

Inertial Frame of Reference

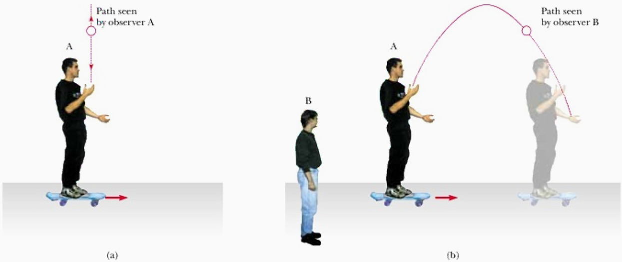
An **inertial frame of reference** (or a **rest frame**) is one that is moving in uniform motion

- In an inertial frame, Newton's first and second laws of motion are valid
- Since all uniform motion are treated the same way, you may consider any inertial frames of reference to be *at rest*

The Principle of Relativity
All laws of motion must apply equally in all inertial frames of reference.

Inertial Frame of Reference

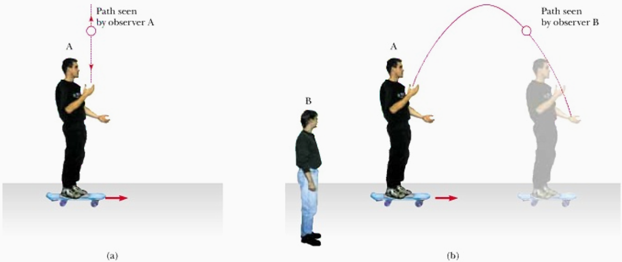
Observer A moves uniformly with the skateboard, while Observer B stands on the side of the road. So, when A tosses a ball upward:



- A sees only vertical motion, while
- B sees the ball traveling in a parabolic curve, although
- A & B observe different motion, but they agree on the *equations* that govern the motion

Inertial Frame of Reference

Observer A sees the same motion (only vertical motion) regardless of whether he is moving uniformly w.r.t. B or not (as long as *neither* are accelerating)



- Valid for A to conclude that he is at rest, but that B and the rest of the world are moving
- Likewise, it is also valid for B to think that he is at rest, but it is only A and his skateboard that are moving

Newtonian (Classical) Relativity

When studying kinematics and dynamics, we made some untested assumptions that seemed obvious: space and time are *absolute*

- 1 m is 1 m no matter where you are, or how you are moving
- 1 s is 1 s no matter where you are, or how you are moving
- Measurements of space and time do not depend on motion

If space and time are absolute, then *all* velocities are relative to the observer

- Measured velocities depend on the motion of the observer
- Galilean velocity addition rule:

$$\boxed{\mathbf{v}_{AC} = \mathbf{v}_{AB} + \mathbf{v}_{BC}}$$

Maxwell's Equations

Maxwell's equations on electrodynamics in a vacuum (studied previously):

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Disturbances in **E** and **B** travel as an *electromagnetic wave* with a speed *c*:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 299\,792\,458\ \mathrm{m/s}$$

Peculiar features of Maxwell’s equation

- Does not mention the *medium* in which EM waves travels
- When applying *Galilean transformation* (from which we obtain the velocity addition rule) to Maxwell’s equations, asymmetry is introduced
- Gauss’s law for magnetism break down: magnetic field lines appear to have beginnings/ends
- In *some* inertial frames of reference, Maxwell’s equations are simple and elegant, but transform the equation into another inertial frame, the equations are ugly and complex!
- Physicists at the time began to theorize that (perhaps) there is an actual *preferred* inertial frame of references
- This seems to violate the *principle of relativity*

The Illusive Ether

Maxwell's hypothesis: the speed of light c_0 is relative to a hypothetical subtance called **luminiferous aether** (or just **ether**) that permeates the universe. Ether must have some fantastic properties:

- All space is filled with ether
- Massless
- Zero viscosity
- Non-dispersive
- Incompressible
- Continuous at a very small (sub-atomic) scale

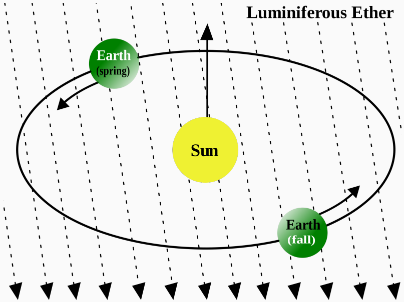
It was thought that the preferred inertial reference frame is that of the ether

Spoiler Alert

Spoiler alert: Ether doesn't exist.

The Michelson-Morley Experiment

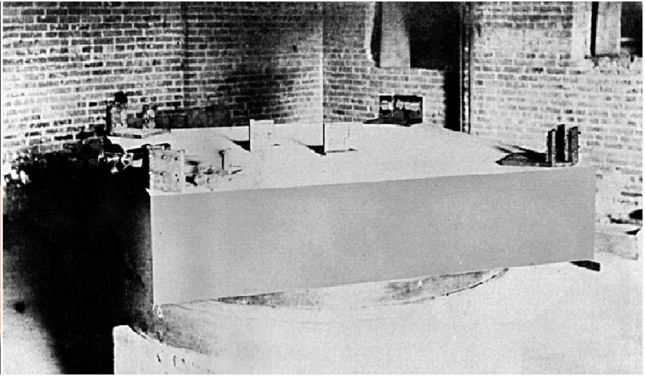
If ether exists, then at different times of the year, the Earth will have a different relative velocity with respect to it:



And it causes light to speed up or slow down. By measuring and comparing the speed of light at various times of the year, we should be able to determine to flow of ether relative to Earth.

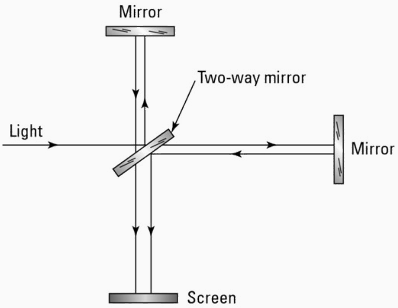
Michelson-Morley Experiment

American phycisists Albert Michelson[1] and Edward Morley designed an ingenious but very difficult experiment to detect ether using an **interferometer** designed by Michelson



[1] Nobel Prize in Physics, 1907

The Michelson Interferometer



- A beam of light is split into two using a two-way (half-silvered) mirror
- The two beams are reflected off mirrors and finally arriving at the screen where interference patterns are observed
- The two paths are the same length, so if the *speed* of the light changes, we should see an interference pattern
- **Except none were ever found!** The interference patterns that could be observed were well within experimental errors, and far below expected values

What To Do with “Null Result”

The Michelson-Morley experiment failed to detect the flow of ether, even after many refinements. What does this mean?

- Majority view
    - **The experiment was flawed!** It is actually a reasonable guess, since the experiment is known to be a difficult one, errors can be introduced
    - Keep improving the experiment (or design a better experiment) and Earth’s motion relative to ether will eventually be found
- Minority view:
    - **The ether hypothesis is wrong!**
    - The experiment showed it for what it is: ether either cannot be detected or it doesn’t exist
- A few physicists: The must be **another explanation** that saves both experiment and theory

Hendrik Lorentz

Dutch physicist Hendrik Antoon Lorentz[2] was one of the first to consider the findings of Michelson-Morley experiment to be significant

- Lorentz's hypothesis: objects traveling in the direction of ether must contract in length, nullifying the experimental results:

$$\boxed{\beta = \sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

- But *No known physical phenomenon* causes an object to contract

[2] 1853–1928, Nobel Prize in Physics, 1902

Strange Behavior in Absolute Space Time

French mathematician Henri Poincaré also hypothesized that ether affects the flow of time the direction of motion. His equation is similar to the hypothesis by Lorentz and contains the same factor:

$$\boxed{t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

But *no known physical phenomenon* can alter the flow of time either!

Strange Behavior in Absolute Space Time

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$$\boxed{t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

But *no known physical phenomenon* can alter the flow of time either!

Both Poincaré and Lorentz depended their hypothesis on

- Absolute time and space
- Existence of ether

Making Maxwell’s Equations Work

Albert Einstein in 1905

- In 1905, at the age of 26, Albert Einstein was working as a patent clerk in Bern, Switzerland while completing his Ph.D.
  - Believed in the principle of relativity, and therefore
  - Rejected the concept of a “preferred” frame of reference
- The failure of the Michelson-Morley experiment to find the flow of ether proves that it does not exist
- In order to make Maxwell’s equations to work again, Einstein revisited two most fundamental concepts in physics: *space* and *time*

Special Relativity

Published in *Annalen der Physik* on September 26, 1905 in the article *On the Electrodynamics of Moving Bodies*

- Submitted on June 30, 1905 and passed for publication by a referee
- Einstein's third paper (of four) that year
- Mentions only five other scientists by name: Issac Newton, James Clerk Maxwell, Heinrich Hertz, Christian Doppler and Hendrik Lorentz, but does not contain references to any publications
- Ignored by most physicists at first, until Max Planck took interests
- Called "special relativity" because it describes a "special case" without effects of gravity

Postulates of Special Relativity

The Principle of Relativity: *All* laws of physics must apply equally in *all* inertial frames of reference.

- Reaffirms the principle in which physics is based on
- Extends the principle to include electrodynamics

The Principle of Invariant Light Speed: As measured in any inertial frame of reference, light always propagates in a vaccum with a definite velocity c_0, independent of the state of motion of the emitting body.

- Reaffirms the results from Michelson-Morley experiment
- Disproves the existence of ether

The two postulates are unremarkable by themselves, but when combined, the consequences are profound

What’s so Special About Special Relativity?

Classical (Newtonian) relativity:

- Space and time are absolute (invariant), therefore
- The speed of light must be relative to the observer

Einstein’s special relativity:

- The speed of light is absolute (invariant), therefore
- Space and time must be relative to the observer

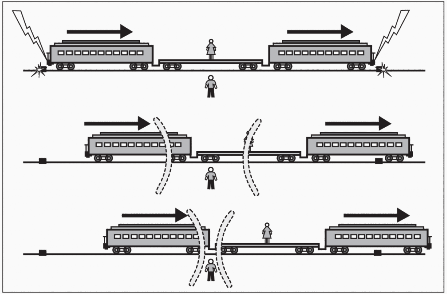
We must modify our traditional concepts:

- Measurement of *space*
- Measurement of *time*
- Concept of *simultaneity*

Simultaneity

The Relativity of Simultaneity

This *thought experiment* is similar to the one that Einstein presented. Suppose lightning bolt strikes the two ends of a high-speed moving train. Does it happen simultaneously?



- Two *independent* events: lightning striking the front, and lightning striking the back of the train
- The man on the ground sees the lightning bolt striking at the same time
- The woman on the moving train sees the lightning bolt on the front first

Relativity of Simultaneity

From the man's perspective:

- He is stationary, but the train is moving
- When the lightnings strike, he is at an equal distance from the front and the back of the train
- Flashes from the two lightning bolts arrive at his eyes at the same time
- Since the speed of light is a constant regardless of motion

Therefore, his conclusions are:

- The two lightnings must have happened at the same time
- The woman in the train made the wrong observation: she only *thinks* that the lightning struck the front first because she is moving toward the light from the front

Relativity of Simultaneity

From the woman's perspective:

- *She* is stationary, but the man and the rest of the world are moving
- When the lightnings strike, she is at an equal distance from the two ends of the train
- The flash from the front arrive first, then the back
- Since the speed of light is a constant regardless of motion

Therefore, her conclusions are:

- Lightnings must have struck the front first
- The man on the road made the wrong observation: he only *thinks* that the lightning struck at the same time because he's moving toward the light from the back

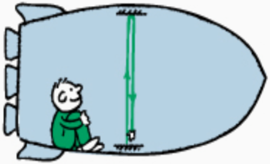
Relativity of Simultaneity

- The two observers disagree on the result, but
  - Neither person is wrong
  - Neither person is misinformed
- Both observers are valid *inertial* frames of reference, and therefore both can consider themselves at rest
- This means that *simultaneity depends on your motion*

Relativity of Simultaneity: Events that are simultaneous in one inertial frame of reference are not simultaneous in another.

Time Dilation

Relativity of Time: Time Dilation

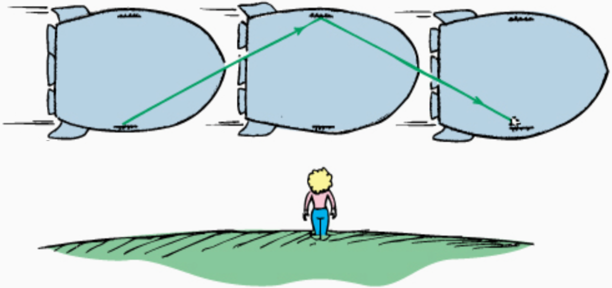


I'm on a spaceship travelling in deep space, and I shine a light from A to B. The distance between A and B is:

$$|AB| = c\Delta t_0$$

I know the speed of light c, and I know how long it took for the light pulse to reach B. (The reason I used Δt_0 will be obvious later.)

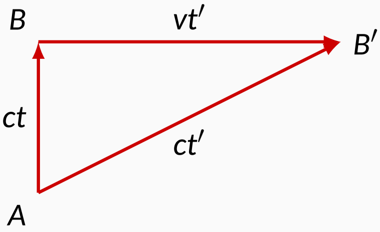
Relativity of Time: Time Dilation



You are on a small planet watching my spaceship go past you at speed v. You would see that same beam of light travel from A to B' instead.

Relativity of Time: Time Dilation

We can relate the time interval observed by me on the spaceship (t) and your time interval on the small planet (t') using Pythagorean theorem:



$$
\begin{aligned}
(ct')^2 &= (vt')^2 + (ct)^2 \\
\left(c^2 - v^2\right) t'^2 &= c^2 t^2 \\
\left(1 - \frac{v^2}{c^2}\right) t'^2 &= t^2 \\
t' &= \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}
\end{aligned}
$$

Relativity of time: the passage of time as measured by two observers in two different inertial references are different

Relativity of Time: Time Dilation

The passage of time as measured by two observers in two different inertial frames of reference are related by:

$$\boxed{t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

Variable	Symbol	SI Unit
Proper time (ordinary time)	t	s
Dilated time (expanded time)	t'	s
Speed	v	m/s
Speed of light	c	m/s

- **Proper time** is measured by an observer *at rest* relative to the events
- **Dilated time** is measured by a *moving* observer in another inertial frame

Example Problem

Example 1a: Kim is riding a rocket that speeds past an asteroid at $v = 0.600c$. If Kim sees 10.0 s pass on her watch, how long would that time interval be as seen by Jim, an observer on the asteroid?

$$t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{10.0}{\sqrt{1 - 0.600^2}} = 16.7\,\mathrm{s}$$

Example Problem

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$$t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{10.0}{\sqrt{1 - 0.600^2}} = 16.7\,\mathrm{s}$$

- Jim observes that in the time it took Kim's clock to run 10.0 s, his watch has already gone 16.7 s, therefore
- Jim concludes that Kim's watch must be running slow

Relativity of Time: A moving clock appears to run slow.

Example Problem

Example 1b: Kim is riding a rocket that speeds past an asteroid at $0.600c$. If Jim, an observer in the *asteroid*, sees 10.0 s pass on his watch, how long would that time interval be as seen by Kim?

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$$t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{10.0}{\sqrt{1 - 0.600^2}} = 16.7\,\mathrm{s}$$

- This problem is exactly the same as the last one!
- Kim observes that in the time it took Jim's clock to run 10.0 s, her watch has already gone 16.7 s, therefore
- Kim concludes that Jim's watch must be running slow

How can that be?

How can the observer in the asteroid sees time in the rocket runs slowly, while the observer in the rocket *also* sees time in the asteroid runs slowly?

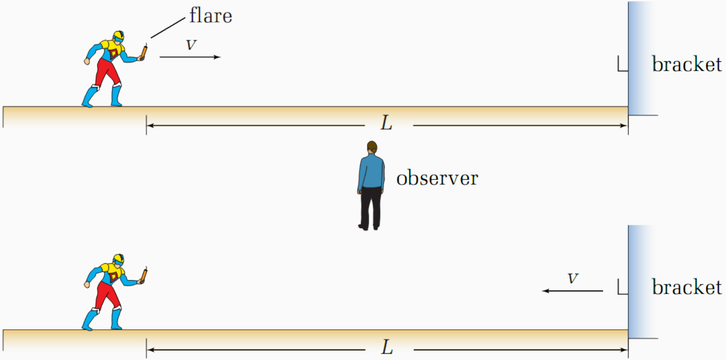
Answer: the relativty of simultaneity. The clocks on the asteroid and on the rocket are *not* synchronized.

- In example 1a, when Kim (on the rocket) starts measuring a 10.0 s time interval, in order for Jim to compare that interval to *his* watch, he has to start and end at the same time (simultaeously!) as Kim.
- But simultaneity is only relative. In Kim's reference frame, Jim never got the timing right!
- This problem reverses itself when Kim tries to synchronize her watch to Jim's 10.0 s interval.

Length Contraction

Relativity of Space

Captain Quick is a comic book hero who can run at nearly the speed of light. In his hand, he is carrying a bomb set to explode in 1.5 μs. The bomb must be placed into its bracket before this happens. The distance (L) between the flare and the bracket is 402 m.



Relativity of Space

Suppose Captain Quick runs at $2.00 \times 10^8\,\mathrm{m/s}$, according to classical mechanics, he will not make it in time:

$$t = \frac{L}{v} = \frac{402\,\mathrm{m}}{2.00 \times 10^8\,\mathrm{m/s}} = 2.01 \times 10^{-6}\,\mathrm{s} = 2.01\,\mu\mathrm{s}$$

But according to relativistic mechanics, he makes it just in time...

Relativity of Space

To a stationary observer, the time on the flare is slowed:

$$t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{1.5 \times 10^{-6}}{\sqrt{1 - \left(\frac{2.00}{3.00}\right)^2}} = 2.01 \times 10^{-6}\,\mathrm{s}$$

The stationary observer sees a passage of time of $t' = 2.01\,\mu\mathrm{s}$, but Captain Quick, who is in the same reference frame as the flare, experiences a passage of time of $t = 1.50\,\mu\mathrm{s}$, precisely the time for the flare to explode.

Relativity of Space

If Captain Quick sees only $t = 1.50\,\mu\text{s}$, then how far did he travel?

- Both Captain Quick and the observer on the side of the road agree that he is traveling at $v = 2.00 \times 10^8\,\text{m/s}$
- The only possibility is that *the distance actually got shorter* in Captain Quick's frame of reference, by this amount:

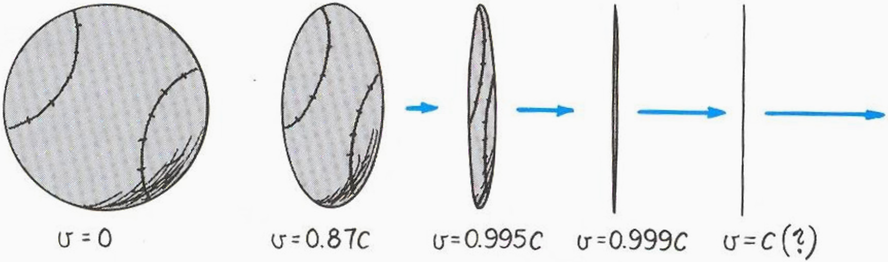
$$\boxed{L' = L\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

For this example:

$$L' = L\sqrt{1 - \left(\frac{v}{c}\right)^2} = 402\sqrt{1 - \left(\frac{2.00}{3.00}\right)^2} = 300\,\text{m}$$

Length Contraction

Length contraction only occurs in the direction of motion



Example Problem

Example 2: A spacecraft passes Earth at a speed of $2.00 \times 10^8\ \mathrm{m/s}$. If observers on Earth measure the length of the spacecraft to be $554\ \mathrm{m}$, how long would it be according to its passengers?

Lorentz Factor

Lorentz Factor

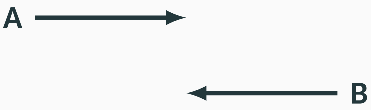
The **Lorentz factor** γ is a short-hand for writing length contraction, time dilation and relativistic mass:

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

Then time dilation and length contraction can be written simply as:

$$\boxed{t' = \gamma t} \quad \boxed{L' = \frac{L}{\gamma}}$$

Summary



If observers A and B are moving at constant velocity relative to one another (doesn't matter if they're moving toward, or away from each other)

- They cannot agree whether any events happens at the same time or not
- Each sees the other's clock running slow
- Each sees the other "contracted" in length along the direction of motion

Lorentz Transformation

Time dilation and length contraction only tell part of the story. To account for the loss of simultaneity from one inertial frame to another, we need to use the **Lorentz transformation:**

$$
\begin{aligned}
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z \\
t' &= \gamma\left(t - \frac{vx}{c^2}\right)
\end{aligned}
$$

The Lorentz transformation “solves” many paradoxes (e.g. the twin paradox) from the time-dilation and length-contraction equations, but aren’t really there.

Lorentz Transformation

For slow speeds $v \ll c$, Lorentz transformation reduces to the Galilean transformation from classical mechanics, from which the velocity addition rule is formulated:

$$
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t'
\end{aligned}
$$

Relative Velocity

Relative Velocity

Unlike in classical mechanics, velocities (speeds) do not simply add. We have to account for time dilation and length contraction, which are included in the Lorentz transformation

Einstein velocity addition rule:

$$\boxed{\mathbf{v}_{AC} = \frac{\mathbf{v}_{AB} + \mathbf{v}_{BC}}{1 + \frac{\mathbf{v}_{AB} \cdot \mathbf{v}_{BC}}{c^2}}}$$

If $v_{AB} \ll c$ and $v_{BC} \ll c$, we recover Galilean velocity addition rule

Relativistic Momentum

Relativistic Momentum

In Grade 12 Physics, you were taught that momentum is mass times velocity. And in Grade 11 Physics, you were taught that velocity is displacement over time. *These definitions have not changed.*

$$\mathbf{p} = m\frac{d\mathbf{x}}{dt}$$

But now that you know $d\mathbf{x}$ and dt are relativistic quantities that depend on motion, we can find a new expression for "relativistic momentum":

$$\mathbf{p} = m\frac{d\mathbf{x}}{dt} = \frac{m d\mathbf{x}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}\, dt} = \boxed{\frac{m\mathbf{v}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}} = \gamma m\mathbf{v}$$

Relative Mass

Relativistic Mass

From the relativistic momentum expression, we see the relativistic aspect to mass as well. The **apparent mass** (or **relativistic mass**) m' as measured by a moving observer is related to its **rest mass** (or **intrinsic mass** or **invariant mass**) m by the Lorentz factor:

$$\boxed{m' = \frac{m}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma m}$$

The intrinsic mass of a moving object does not change, but a moving observer will observe that it behaves as if it is more massive. As $v \to c$, $m' \to \infty$.

Energy

Work and Energy

Einstein published a fourth paper in *Annalen der Physik* on November 21, 1905 (received Sept. 27) titled “Does the Inertia of a Body Depend Upon Its Energy Content?” (In German: Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?)

- Einstein deduced the most famous of equations: $E = mc^2$

Work and Energy

In Grade 12 Physics, you were taught that force is the rate of change of momentum with respect to time. *This definition has not changed.*

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

and that work is the integral of the dot product between force and displacement vectors. *This definition has not changed either.*

$$W = \int \mathbf{F} \cdot d\mathbf{x} = \int \frac{d\mathbf{p}}{dt} \cdot \mathbf{dx}$$

Since we now have a relativistic expression for momentum, we substitute that new expression into the expression for force, and then integrate.

Work and Energy

For 1D motion (for simplicity), we can rearrange the terms in the integral:

$$W = \int F dx = \int \frac{dp}{dt} dx = \int v dp$$

Assuming that both v and p are continuous in time, we can apply the chain rule to find the infinitesimal change in momentum (dp) with respect to γ and v:

$$p = \gamma m v \quad \rightarrow \quad dp = \gamma dv + v d\gamma$$

Substituting that back into the integral, we have:

$$W = \int v dp = \int m v (\gamma dv + v d\gamma) = \int m \left(\gamma v dv + v^2 d\gamma \right)$$

Work and Energy

One of the integral is with respect to γ, so we express v and dv in terms of γ using its definition:

$$v^2 = c^2 \left[1 - \left(\frac{1}{\gamma} \right)^2 \right] \quad \rightarrow \quad dv = \frac{c^2}{\gamma^3 v} d\gamma$$

Work and Energy

Putting everything together, we have

$$W = \int m(\gamma v dv + v^2 d\gamma) = \int m \left[\frac{c^2}{\gamma^2} + c^2\left(1 - \frac{1}{\gamma^2}\right)\right] d\gamma$$

This is a surprisingly simple integral:

$$W = \int_1^{\gamma} mc^2 d\gamma$$

The limit of the integral is from 1 because at $v = 0$, $\gamma = 1$

Work and Kinetic Energy

The integral gives us this expression:

$$W = \gamma mc^2 - mc^2$$

We know from the work-kinetic energy theorem that the work W done is equal to the change in kinetic energy K, therefore

$$\boxed{K = m'c^2 - mc^2}$$

Variable	Symbol	SI Unit
Kinetic energy of an object	K	J
Apparent mass (measured in moving frame)	m'	kg
Rest mass (measured in stationary frame)	m	kg
Speed of light	c	m/s

Relativistic Energy

$$\boxed{K = m'c^2 - mc^2}$$

The minimum amount of energy that any object has, regardless of it's motion (or lack of) is its **rest energy**:

$$E_0 = mc^2$$

The **total energy** of an object has is

$$E_T = m'c^2 = \gamma mc^2$$

The difference between total energy and rest energy is the kinetic energy:

$$K = E_T - E_0$$

Relativistic Energy

$$\boxed{E = mc^2}$$

Mass-energy equivalence:

- Whenever there is a change of energy, there is also a change of mass
- "Conservation of mass" and "conservation of energy" must be combined into a single concept of **conservation of mass-energy**
- Mass-energy equivalence doesn't merely mean that mass can be converted into energy, and vice versa (although this is true), but rather, one can be converted into the other *because they are fundamentally the same thing*

Energy-Momentum Relation

The **energy-momentum relation** relates an object's rest (intrinsic) mass m, total energy E, and momentum p:

$$\boxed{E^2 = p^2c^2 + m^2c^4}$$

Quantity	Symbol	SI Unit
Total energy	E	J
Momentum	p	kg m / s
Rest mass	m	kg
Speed of light	c	m / s

Energy-Momentum Relation

This equation is derived by squaring the expression for relativistic momentum:

$$p = \gamma m v = \frac{mv}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \quad \rightarrow \quad p^2 = \gamma^2 m^2 v^2 = \frac{m^2 v^2}{1 - \left(\frac{v}{c}\right)^2}$$

Energy-Momentum Relation

Solving for v^2 and substituting it back into the Lorentz factor, we obtain an alternative form for γ in terms of momentum and mass:

$$\gamma = \sqrt{1 + \left(\frac{p}{mc}\right)^2}$$

Inserting this form of the Lorentz factor into the energy equation, we have

$$E = mc^2\sqrt{1 + \left(\frac{p}{mc}\right)^2}$$

Which is the same equation as in the last slide.

Energy-Momentum Relation

In the **stationary frame of reference**, (rest frame, center-of-momentum frame) the momentum is zero, so the equation simplifies to

$$\boxed{E = mc^2}$$

where m is the rest mass of the object.

If the object is **massless**, as is the case for a **photon**, then the equation reduces to

$$\boxed{E = pc}$$

Kinetic Energy–Classical vs. Relativistic

Relativistic:

$$K = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - mc^2$$

Newtonian:

$$K = \frac{1}{2}mv^2$$

But are they really that different?

- If space and time are indeed relative quantities, then the relativistic equation for K must apply to all velocities
- But we know that when $v \ll c$, the Newtonian expression works perfectly
- i.e. The Newtonian expression for K must be a very good approximation for the relativistic expression for K for $v \ll c$

Binomial Series Expansion

The **binomial series** is the Maclaurin series for the function $f(x) = (1 + x)^\alpha$, given by:

$$(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \cdots$$

In the case of relativistic kinetic energy, we use:

$$x = -\left(\frac{v}{c}\right)^2 \quad \text{and} \quad \alpha = -\frac{1}{2}$$

Binomial Series Expansion

Substituting these terms into the equation:

$$K = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right) - mc^2$$
$$\approx \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \cdots$$

For $v \ll c$, we can ignore the high-order terms. The leading term reduces to the Newtonian expression

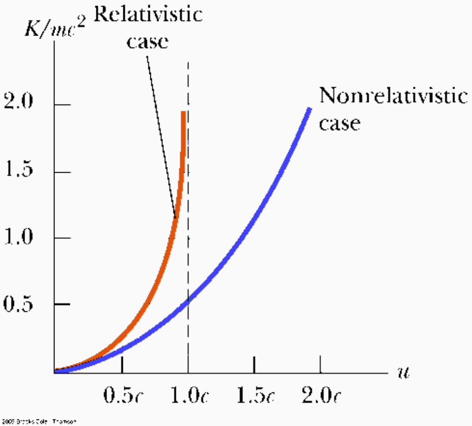
Comparing Classical and Relativistic Energy

In classical mechanics:

$$K = \frac{1}{2}mv^2$$

In relativistic mechanics:

$$K = \gamma mc^2 - mc^2$$



The classical expression is accurate for speeds up to $v \approx 0.3c$.