Class 20: Circuit Analysis, Part 1

AP Physics C

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Olympiads School

Electric Current

Current

The **electric current** is defined as the rate at which **charges** Q pass through a point in a circuit. In differential form:

$$\boxed{I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t}} = \frac{Q}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = (ne)(Av_d)$$

- Q/V is the amount of charge carriers *per volume*, which is just the **charge carrier density** (number of charge carriers per volume) n times the **elementary charge** e
- $\mathrm{d}V/\mathrm{d}t$ is the rate the volume of charges moves through the conductor, give by the wire's cross-section area A times the **drift velocity** v_d of the charge carrier

Charge Carrier Density

Calculating the charge carrier density in a *metal* conductor involves some physical information about the metal:

1. Divide the metal's density ρ by its molar mass M to find the *number of moles of atoms per unit volume*
2. Multiply by Avogadro's number $N_A = 6.0221 \times 10^{23}\,/\,\mathrm{mol}$ to find *number of atoms per unit volume*
3. Multiply by the number of free electrons per atom k for that particular metal

Charge Carrier Density

Collecting all the terms from the last slide, we have:

$$\boxed{n = \frac{\rho k N_A}{M}}$$

Quantity	Symbol	SI Unit
Charge carrier density	n	$1/\mathrm{m}^3$
Density of material	ρ	kg/m^3
Free electrons per atom	k	
Avogadro's number	N_A	$1/\mathrm{mol}$
Molar mass	M	kg/mol

For copper, $M = 63.54 \times 10^{-3}\,\mathrm{kg}/\mathrm{mol}$, $\rho = 8.96 \times 10^3\,\mathrm{kg}/\mathrm{m}^3$, $k = 1$ and therefore $n = 8.5 \times 10^{28}\,/\mathrm{m}^3$. The drift velocity is in the order of $v \approx 1\,\mathrm{mm}/\mathrm{s}$.

Current

Another alternate description of the electric current is to express it in terms of the **current density** J, with a unit of of *ampère per meters squared* $(\mathrm{A}/\mathrm{m}^2)$.[1]

$$\boxed{I(t) = J(t)A}$$

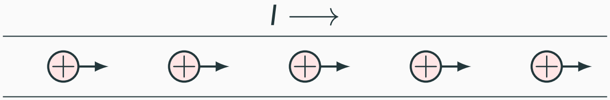
It is obvious from the previous expression that the current density is the product of the charge carrier density, elementary charge, and the drift velocity:

$$\boxed{J = nev_d}$$

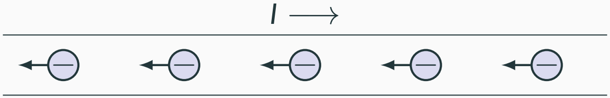
[1] We had previously encountered this quantity when studying Ampère's law.

Electric Current: Conventional vs. Electron Flow

The flow of electric current assumes the flow of *positive* charges. We call this the **conventional current**:



In a conducting wire, however, negatively charged electrons flow in the opposite direction. We call this the **electron current**:



Resistors

Resistivity and Electric Field

The resistivity of a material is proportional to the electric field and current density:

$$\boxed{\vec{E} = \rho \vec{J}} \quad \text{or} \quad \boxed{\rho = \left| \frac{E}{J} \right|}$$

Quantity	Symbol	SI Unit
Electric field	$\vec{E}$	N/C
Current density	$\vec{J}$	A/m^2
Resistivity	ρ	$\Omega \cdot m$

- In a conductor, the electrons are free to move, and the electric field tend to be weak, and the resistivity is low.
- In an insulator, electrons cannot move easily, therefore the electric field are generally strong, and the resistivity is high.

Resistance of a Conductor

The resistance of a conductor is proportional to the resistivity ρ and its length L, and inversely proportional to the cross-sectional area A:

$$\boxed{R = \int \mathrm{d}R = \rho \int_0^L \frac{\mathrm{d}x}{A(x)}}$$

Quantity	Symbol	SI Unit
Resistance	R	Ω
Resistivity	ρ	$\Omega \cdot \mathrm{m}$
Length of conductor	L	m
Cross-sectional area	$A(x)$	m^2

Resistance of a Conductor

$$\boxed{R = \rho \frac{L}{A}}$$

Gauge	Diameter (mm)	R/L ($10^{-3}\,\Omega/\mathrm{m}$)
0	9.35	0.31
10	2.59	2.20
14	1.63	8.54
18	1.02	21.90
22	0.64	51.70

Material	Resistivity ρ ($\Omega \cdot \mathrm{m}$)
silver	1.6×10^{-8}
copper	1.7×10^{-8}
aluminum	2.7×10^{-8}
tungsten	5.6×10^{-8}
Nichrome	100×10^{-8}
carbon	3500×10^{-8}
germanium	0.46
glass	10^{10} to 10^{14}

Ohm’s Law

Ohm's Law

The electric potential difference V across a "load" (resistor) equals the product of the current I through the load and the resistance R of the load.

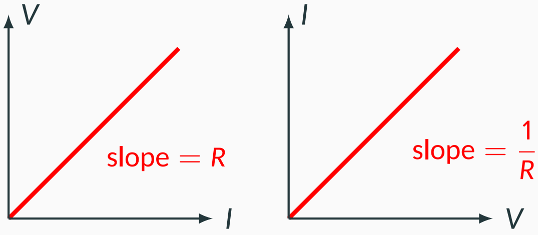
$$\boxed{V = IR}$$

Quantity	**Symbol**	**SI Unit**
Potential difference	V	V
Current	I	A
Resistance	R	Ω

A resistor is considered "ohmic" if it obeys Ohm's law. Note that Ohm's law is not a fundamental law in physics.

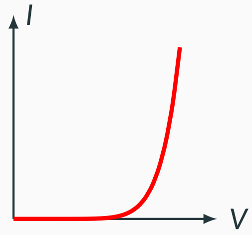
Ohmic Devices

In an ohmic device such as a resistor, the relationship between voltage across the device and the current through the device is linear, i.e.:



Non-Ohmic Devices

Loads that are **non-ohmic** do not obey Ohm's law: the relationship between voltage and current is *not* linear.



For example, diodes (e.g. your TV's LED screen) are non-ohmic.

Power Dissipated by a Resistor

Power is the rate at which work W is done, and from electrostatics, the change in electric potential energy ΔE_q is proportional to the amount of charge q and the voltage V. This gives a very simple expression for power through a resistor:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}E_q}{\mathrm{d}t} = \frac{\mathrm{d}(qV)}{\mathrm{d}t} = \left(\frac{\mathrm{d}q}{\mathrm{d}t}\right) V \rightarrow \boxed{P = IV}$$

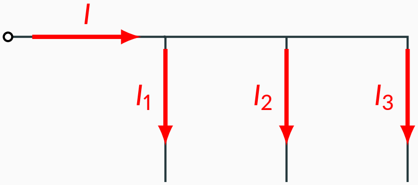
Combining Ohm's law with the above equation gives two additional expressions for power through a resistor:

$$\boxed{P = \frac{V^2}{R}} \qquad \boxed{P = I^2 R}$$

Kirchhoff’s Laws

Kirchhoff's Current Law

The electric current that flows into any junction in an electric circuit must be equal to the current which flows out.



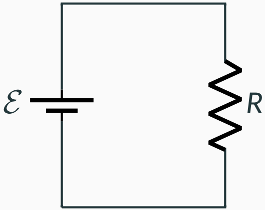
In the example on the left, with I going into the junction, and I_1, I_2 and I_4 coming out, the current law says that

$$I = I_1 + I_2 + I_3$$

Basically, it means that there cannot be any accumulation of charges anywhere in the circuit. The law is a consequence of conservation of charge.

Kirchhoff's Voltage Law

The voltage changes around any closed loop in the circuit must sum to zero, no matter what path you take through an electric circuit.

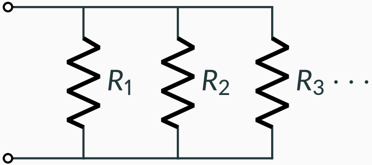


Assume that the current flows clockwise and we draw a clockwise loop, we get

$$\mathcal{E} - V_R = 0 \quad \rightarrow \quad \mathcal{E} - IR = 0$$

Resistors in Circuits

Resistors in Parallel



The total current is the current through all the resistors, which can be rewritten in terms of voltage and resistance using Ohm's law:

$$I = I_1 + I_2 + I_3 \cdots = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} \cdots$$

Since $V_1 = V_2 = V_3 = \cdots = V$ from the voltage law, we can re-write as

$$I = \frac{V}{R_p} = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \cdots\right)$$

Equivalent Resistance of Resistors in Parallel

The reciprocal of the equivalent resistance for resistors connected in parallel is the sum of the inverses of the individual resistances.

$$\boxed{\frac{1}{R_p} = \sum_{i}^{N} \frac{1}{R_i}}$$

Quantity	Symbol	SI Unit
Equivalent resistance in parallel	R_p	Ω
Resistance of individual loads	R_i	Ω

Resistors in Series

The analysis for resistors in series is similar (but easier). From the current law, the current through each resistor is the same:

$$I_1 = I_2 = I_3 = \cdots = I$$

And the total voltage drop across all resistor is therefore:

$$V = V_1 + V_2 + V_3 + \cdots = I\big(R_1 + R_2 + R_3 + \cdots\big)$$

Equivalent Resistance: Resistors in Series

The equivalent resistance of loads is the sum of the resistances of the individual loads.

$$\boxed{R_s = \sum_{i=1}^{N} R_i}$$

Quantity	Symbol	SI Unit
Equivalent resistance in series	R_s	Ω
Resistance of individual loads	R_i	Ω

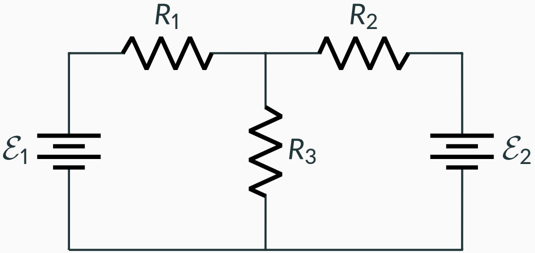
Tips for Solving “Simple” Circuit Problems

1. Identify groups of resistors that are in parallel or in series, and find their equivalent resistance.
2. Gradually reduce the entire circuit to one voltage source and one resistor.
3. Using Ohm’s law, find the current out of the battery.
4. Using Kirchhoff’s laws, find the current through each of the resistors.

Multi-loop Circuit

Circuits Aren't Always Simple

Some of these problems require you to solve a system of linear equations. The following is a simple example with two voltage sources:



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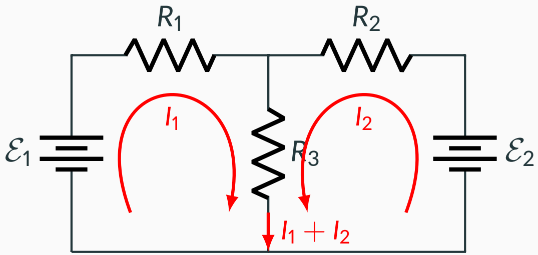
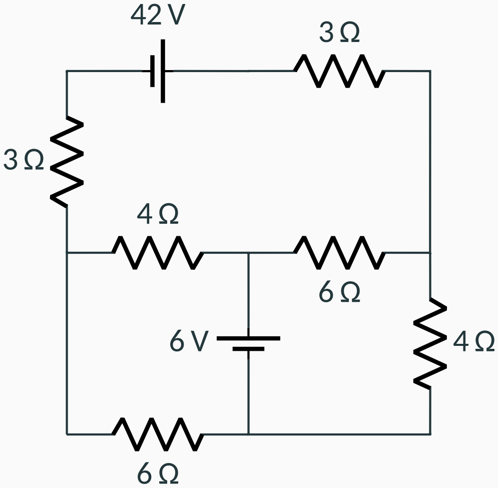


In this case, we have to draw two loops of current.

As Difficult As It Gets



- To solve this problem, we define a few "loops" around the circuit: one on top, one on bottom left, and one on bottom right.

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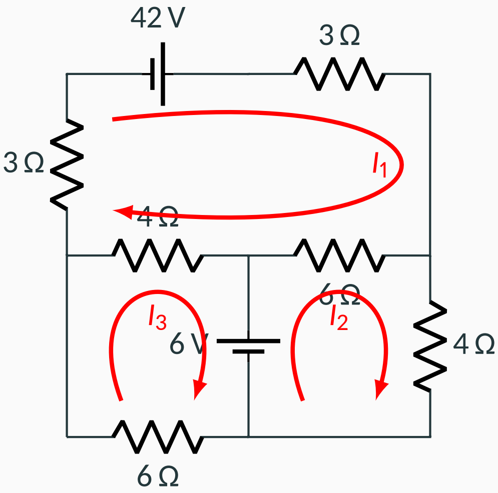

- To solve this problem, we define a few "loops" around the circuit: one on top, one on bottom left, and one on bottom right.
- Apply the voltage law in the loops. For example, in the lower left:

$$4(I_1 - I_3) - 6 - 6I_3 = 0$$

- Solve the linear system to find the current. If the current that you worked out is negative, it means that you have the direction wrong.