Projectile Motion
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1 Projectile Motion

For projectile motion, there is no acceleration in the horizontal ($\hat{\imath}$) direction, i.e. $a_x = 0$. Therefore the kinematic equations is reduced to a single equation

$$x - x_0 = v_x t \tag{1}$$

It is customary to define the $\hat{\imath}$ direction to be the forward horizontal direction.

The only acceleration is due to gravity in the vertical ($\hat{\jmath}$) direction. In the standard Cartesian coordinate system, $\hat{\jmath}$ is direction is therefore usually *up*, and the acceleration due to gravity is

$$a_y = -g \tag{2}$$

The kinematic equation are reduced to:

$$v_y = v_{y0} - gt \tag{3}$$

$$y = y_0 + v_{y0}t - \frac{1}{2}gt^2 \tag{4}$$

$$v_y^2 = v_{y0}^2 - 2g(y - y_o) \tag{5}$$

In general, only Eq. 4 is needed to solve projectile motion problems.

2 Symmetric Trajectory

A **symmetric trajectory** is a special case of projectile motion where an object is launched at an angle of θ (between 0° and 90°) above the horizontal[1] with an initial velocity $\mathbf{v}_0$, and then lands at the same height, as shown below in Fig. 1. Examples may include hitting a golf ball toward the hole, or shooting a bullet toward a horizontal target[2]. The equations for symmetric trajectory are *not* included in the AP Exam equation sheet; if you need these equations during the exams, you will need to derive them yourself. Thankfully, the derivation is quite striaght forward. To derive the equations, we use the x-axis for the horizontal direction and y-axis for the vertical.

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[1]This may be obvious, but angles *below* the horizontal will never have a symmetric trajectory.

[2]Shooting a bullet toward a horizontal target always require an upward angle because of gravity.
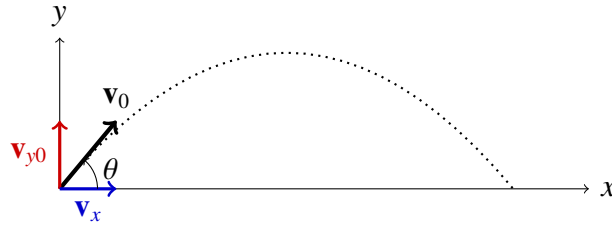
Figure 1: Symmetric project trajectory

The initial velocity $\mathbf{v}_0$ can be resolved into its $\hat{\imath}$ and $\hat{\jmath}$ components, also shown in Fig. 1:

$$\mathbf{v}_0 = v_x\hat{\imath} + v_{y0}\hat{\jmath} = v_0\cos\theta\hat{\imath} + v_0\sin\theta\hat{\jmath} \tag{6}$$

The x component of velocity $\mathbf{v}_x$ remains constant during the motion, as there are no forces acting in the x direction (as long as air resistance can be ignored), and therefore no acceleration. In the y direction, acceleration is due to gravity alone, $a_y = -g$.

Maximum height H: Apply the kinematic equation in the y-direction. Recognizing that at maximum height $H = y - y_0$, the vertical component of velocity is zero $v_y = 0$:

$$v_y^2 = v_{y0}^2 + 2a_y(y - y_0)$$
$$0 = (v_0\sin\theta)^2 - 2gH$$

Solving for H, we get the maximum height equation:

$$\boxed{H = \frac{v_0^2\sin^2\theta}{2g}} \tag{7}$$

Total time of flight $t_{\max}$: We apply the kinematic equation in the y direction. When the object lands at the same height, the final velocity is the same in magnitude and opposite in direction as the initial velocity, i.e. $v_{y2} = -v_{y1} = -v_0\sin\theta$:

$$v_y = v_{y0} + a_y t$$
$$-v_0\sin\theta = v_0\sin\theta - gt_{\max}$$

Solving for $t_{\max}$ we have:

$$\boxed{t_{\max} = \frac{2v_0\sin\theta}{g}} \tag{8}$$

Range R: We substitute the expression for $t_{\max}$ from Eq. 8 into the t term, then apply the kinematic equation in the x-direction to compute $R = x - x_0$ for any given launch angle and initial speed:

$$x = x_0 + v_x t$$
$$R = v_0\cos\theta\left(\frac{2v_0\sin\theta}{g}\right)$$

Using the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$, we simplify the equation to:

$$R = \frac{v_0^2 \sin(2\theta)}{g} \tag{9}$$

It is obvious that for any given initial speed v_0, the maximum range R_{max} occurs at an angle where $\sin(2\theta) = 1$ (i.e. $\theta = \pi/4$), with a value of

$$R_{max} = \frac{v_0^2}{g} \tag{10}$$

Also, for a known initial speed v_0 and range R we can compute the launch angle θ:

$$\theta_1 = \frac{1}{2}\sin^{-1}\left(\frac{gR}{v_0^2}\right)$$

This angle is labelled θ_1 because it is *not* the only angle that can reach this range. Recall that for any angle $0 < \phi < \pi/2$, there is also another angle ϕ_2 where $\sin(\phi)$ value is the same:

$$\sin\phi = \sin(\pi - \phi)$$

Which means that for any θ_1, there is also another angle θ_2 where $2\theta_2 = \pi - 2\theta_1$, or quite simply:

$$\theta_2 = \frac{\pi}{2} - \theta_1$$