Class 8: Rotational Motion of a Rigid Body, Part 2

Advanced Placement Physics C

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Olympiads School

Introduction

Curvilinear vs. Rectilinear Motion

Kinematic quantities for rectilinear (translational) vs. curvilinear (circular) motion are related:

$$\begin{aligned} \vec{r} &\quad\rightarrow\quad \theta \\ \vec{v} &\quad\rightarrow\quad \omega \\ \vec{a} &\quad\rightarrow\quad \alpha \end{aligned}$$

Dynamics:

$$\begin{aligned} m &\quad\rightarrow\quad I \\ \vec{F} &\quad\rightarrow\quad \vec{\tau} \\ \vec{p} = m\vec{v} &\quad\rightarrow\quad \vec{L} = I\vec{\omega} \end{aligned}$$

Laws of Motion

The laws of motion are also related between translational and rotational motion:

$$\vec{F}_{\text{net}} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \quad \rightarrow \quad \vec{\tau}_{\text{net}} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$$

$$\vec{F}_{\text{net}} = m\vec{a} \quad \rightarrow \quad \vec{\tau}_{\text{net}} = I\vec{\alpha}$$

Solving Rotational Problems

When solving for rotational problems like the ones described in the previous sections:

- Draw a free-body diagram to account for all forces
- The direction of friction force is not always obvious
- The magnitude of any static friction force cannot be assumed to be at maximum.
- If the object is to change its rotational state, there must be a net torque causing it.

Solving Rotational Problems

Once the free-body diagram is complete, the forces should break down into their *forces* into $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ components. If the axes are defined properly, only one direction should have acceleration (usually $\hat{\imath}$), i.e.:

$$\sum F_x = ma \qquad \sum F_y = 0 \qquad \sum F_z = 0$$

There are also three equations for rotation, and torque is only applied in one direction (likely $\hat{k}$):

$$\sum \tau_x = 0 \qquad \sum \tau_y = 0 \qquad \sum \tau_z = I_z\alpha$$

Solving Rotational Problems

For rotational motion dynamics equation:

1. Relate the force(s) that causes rotational motion to the net torque

$$\tau_{\text{net}} = \sum_i F_i r_i$$

2. Substitute the expression for momentum of inertia (which has both mass and radius terms in it) into the equation for rotational motion
3. Relate angular acceleration to linear acceleration, if applicable:

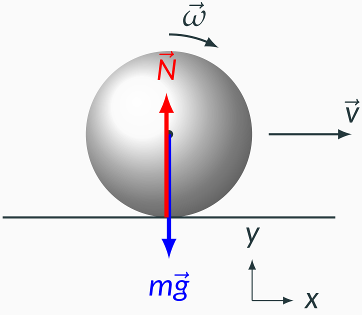
$$\alpha = \frac{a}{R}$$

Now there are two equations with force and acceleration terms.

Pure Rolling Problems

Pure Rolling Problems

In a **pure rolling** problem, a smooth solid sphere[1] rolls along a smooth surface without slipping



- Assumptions:
  - Both the sphere and the surface are both perfectly rigid (they do not deform)
  - The sphere and the surface are both perfectly smooth without defects even at the microscopic level
- There are only two forces acting on the sphere:
  - Gravitational force $m\vec{g}$
  - Normal force $\vec{N}$
- There is no friction

[1] any object that can roll with do!

Pure Rolling Problems

The free-body diagram is simple enough that we can see that:

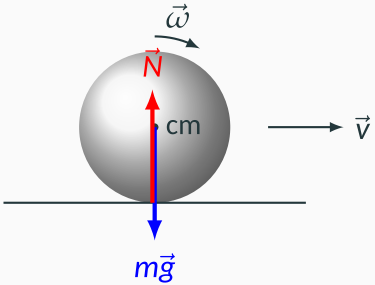
- There is no net force, therefore the translational state ($\vec{v}$) of the sphere is constant

$$\sum \vec{F} = \vec{0} \qquad \vec{v} = \text{constant}$$

- Neither gravity or normal force generate a torque about the center of mass, therefore there is no net torque, and the rotational state $\vec{\omega}$ is constant:

$$\sum \vec{\tau} = \vec{0} \qquad \vec{\omega} = \text{constant}$$

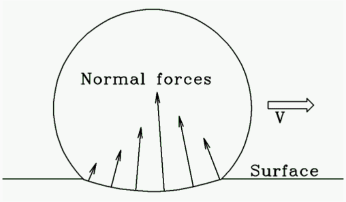
- In *theory*, this sphere will roll along with angular speed ω and speed $v = \omega R$ forever



Reality: Rolling Resistance

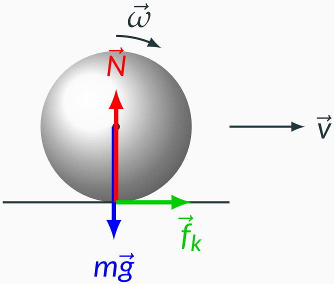
In reality, the rolling sphere will slow down and eventually come to a stop, because *nothing is perfectly rigid*: both the sphere and the surface deform when they make contact

- Example: a car's tires flatten when they make contact with the ground
- The normal force is larger in magnitude on the front side than on the other
- N exerts both a horizontal force to slow down the sphere, as well as a torque to slow down its rotation
- The normal force does not point toward the CM because of the deformation.



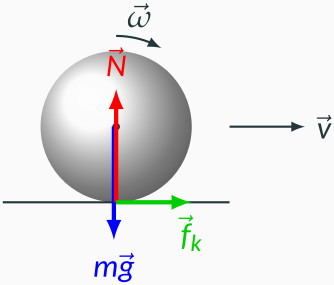
Rolling with Slipping on Rough Surface

What if the rolling sphere is slipping against the surface?



- Slippage at the point of contact between the sphere and the flat surface
- There is *kinetic* friction $f_k = \mu_k N$ in the $+\hat{\imath}$ direction (toward the right). The friction force generates:
  - A net force F_{net} in the $+\hat{\imath}$ direction, toward the right
  - A net torque τ_{net} in the $+\hat{k}$ direction, i.e. counter clockwise

Rolling with Slipping on Rough Surface



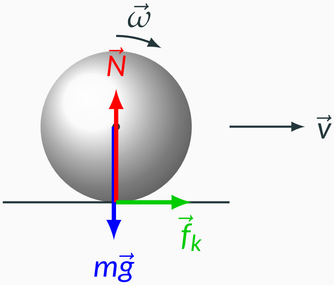
- Kinetic friction f_k causes the center of mass of the sphere to accelerate toward the right

$$F_{\text{net}} = f_k = ma_{\text{cm}}$$

- Since f_k is constant, acceleration is also constant as well, as long as the sphere slips.
- e.g.: A car with its tires spinning on ice still has a small forward acceleration
- The acceleration of the center of mass is:

$$a_{\text{cm}} = \mu_k g$$

Rolling with Slipping on Rough Surface



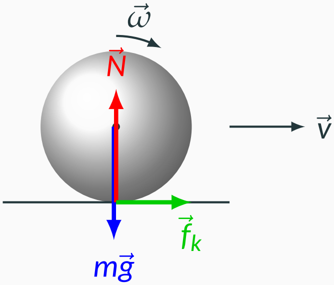
- The constant net torque in the $+\hat{k}$ (counter clockwise) direction generates a constant angular acceleration (because f_k is constant)

$$\tau_{\text{net}} = f_k R = I\alpha$$

- The angular acceleration causes the angular speed ω to decrease over time
- The angular acceleration in the $+\hat{k}$ direction:

$$\alpha = \frac{f_k R}{I} = \frac{\mu_k m g R}{\frac{2}{5} m R^2} = \frac{5\mu_k g}{2R}$$

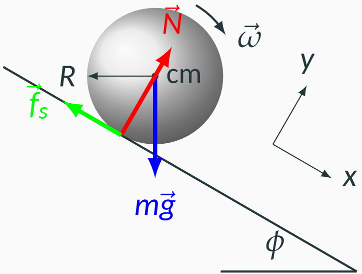
Rolling with Slipping on Rough Surface



- Unlike the no-slip case where where angular acceleration is related to linear acceleration by the radius, i.e. $a = \alpha R$, in this case, there is **no relationship between α and a.**
- There is also no relationship between the velocity and the center of mass and the angular velocity
- The speed of the sphere v increases, while the angular speed ω decreases, until...
- When $v = \omega r$, the sphere stops slipping, and the problem returns to the no-slip case

Rolling on an Inclined Surface

For a rigid and smooth sphere of radius R rolling down a ramp of angle ϕ without slippage down a ramp of angle θ.



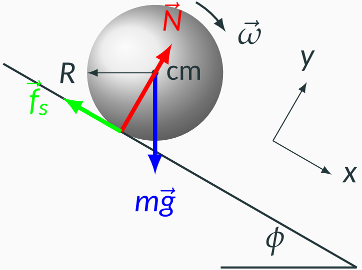
Three forces act on the sphere as it rolls down the ramp

- The weight (mg) of the sphere acts at the CM
- The normal force (N) acts at the point of contact
- The static friction (f_s) act at the point of contact

Only static friction generates a torque about the CM in the clockwise direction

- If f_s is not present, there would have been nothing that causes the sphere to rotate.

Rolling on an Inclined Surface

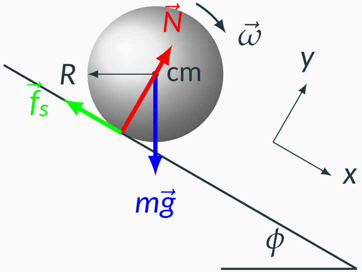


To solve this problem, there are three dynamics equations:

$$\sum F_x = mg\sin\theta - f_s = ma$$
$$\sum F_y = N - mg\cos\theta = 0$$
$$\sum \tau = Rf_s = I\alpha$$

At this point, static friction f_s is *not* known. The coefficient of static friction (μ_s) only tells us the *maximum* static friction, not the *actual* friction. (We will instead use it to check if the answer makes sense.)

Rolling on an Inclined Surface



For non-slip case, angular and translational acceleration are related using relative motion:

$$a = \alpha R$$

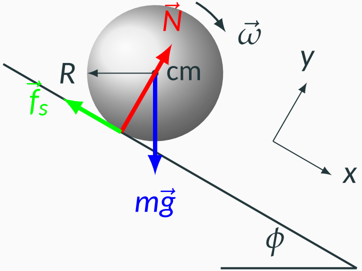
Solving for the static friction:

$$f_s = \frac{I\alpha}{R} = \frac{2}{5}mR^2 \cdot \frac{a}{R} \cdot \frac{1}{R} = \frac{2}{5}ma$$

It is substituted into the force equation in the $\hat{\imath}$ direction to solve for the acceleration of the CM down the ramp:

$$mg\sin\theta - \frac{2}{5}ma = ma$$

Rolling on an Inclined Surface



The acceleration of the center of mass is therefore:

$$a = \frac{5}{7} g \sin \theta$$

Compare this to an object *sliding* without friction down the same ramp, which is higher than the pure rolling case.

$$a = g \sin \theta$$

If the sphere starts from rest, the speed of the sphere when it reaches the bottom of the ramp, a distance d away, would be:

$$v = \sqrt{2ad} = \sqrt{\frac{10}{7} g d \sin \theta}$$

Work & Energy in Rotational Motion

Mechanical Work

For translational motion, mechanical work is defined as

$$W_t = \int_{x_1}^{x_2} \vec{F} \cdot d\vec{x}$$

For rotational motion, mechanical work is defined similarly as:

$$W = \int_{x_1}^{x_2} F dx = \int_{\theta_1}^{\theta_2} F(r d\theta) \quad \rightarrow \quad \boxed{W_r = \int_{\theta_1}^{\theta_2} \tau d\theta}$$

The work-energy theorem still applies to rotational motion, i.e.;

$$W_r = \Delta K_r$$

Rotational Kinetic Energy

To find the kinetic energy of a rotating system of particles (discrete number of particles, or continuous mass distribution), we sum the kinetic energies of the individual particles:

$$K_r = \sum_i \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

It's no surprise that rotational kinetic energy is given by:

$$\boxed{K_r = \frac{1}{2} I \omega^2}$$

Kinetic Energy of a Rotating System

The total kinetic energy of a rotating system is the sum of its translational and rotational kinetic energies at its center of mass:

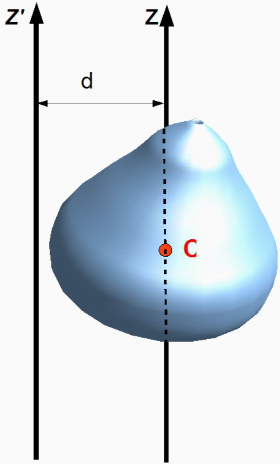
$$\boxed{K = K_t + K_r = \frac{1}{2}mv_{\mathrm{cm}}^2 + \frac{1}{2}I_{\mathrm{cm}}\omega^2}$$

In this case, I_{cm} is calculated at the center of mass. For simple problems, we only need to compute rotational kinetic energy at the pivot:

$$\boxed{K = \frac{1}{2}I_{\mathrm{P}}\omega^2}$$

In this case, the I_{P} is calculated at the pivot. **IMPORTANT:** $I_{\mathrm{cm}} \neq I_{\mathrm{P}}$

Parallel Axis Theorem



The **parallel axis theorem** relates the moment of inertia of an object along two different but parallel axis by:

$$\boxed{I = I_{\text{cm}} + md^2}$$