Class 3: Work and Energy

Advanced Placement Physics C

Dr. Timothy Leung

Fall 2021

Olympiads School

Work and Energy

We start with some definition at are (unfortunately) not very useful:

- **Energy** is the ability to do work.
- **Work** is the mechanism in which energy is transformed.

Luckily, we can also use equations to define these concepts.

Work

Work

Mechanical work $\mathrm{d}W$ is done when a force $\vec{F}$ displaces an object by $\mathrm{d}\vec{x}$. If a varying force is applied to move an object from $\vec{x}_0$ to $\vec{x}_1$ along a path, then the total work done by the force is defined by the integral:

$$\boxed{W = \int_{x_0}^{x_1} \vec{F}(\vec{x}) \cdot \mathrm{d}\vec{x}}$$

- Work is a scalar quantity
- No work done if the force is perpendicular to displacement, when $\vec{F} \cdot \mathrm{d}\vec{x} = 0$ (i.e. the force did not cause the displacement)
- No work done if no displacement ($\mathrm{d}\vec{x} = \vec{0}$)
- Work can be positive or negative depending on the dot product
- When there are multiple forces acting on an object, we can compute the work done by each *each* force

In One Dimension

For motion confined to one dimension (which is common for AP Physics C), we can ignore the dot product:

$$W = \int_{x_0}^{x_1} F(x) \mathrm{d}x$$

Direction still matters for F and x, in that there is still a positive and negative direction.

Work by Constant Force

For a constant force, if the object moves along straight path, the integral simplifies to just the dot product of the two vectors:

$$\boxed{W = \vec{F} \cdot \Delta\vec{x}}$$

In scalar form that is more familiar in Grades 11/12 Physics:

$$\boxed{W = |\vec{F}||\Delta\vec{x}| \cos\theta}$$

where θ is the angle between the force and displacement vectors

Definition of Work

Work done by a force

- The work done by *one specific force*
- Example: A boy pushes a cart forward. The "work done by the boy" is the work done by the applied force.

Work done on an object

- There may be more than one force acting on an object
- The *sum* of all the work done on the object by each force
- The work done by the net force
- Also called the **net work** W_{net}

Kinetic Energy & Work-Energy Theorem

Kinetic Energy

When a net force on an object (with constant mass) accelerates it, the resulting amount of work done on the object (net work W_{net}) is given by:

$$W_{\text{net}} = \int_{x_0}^{x_1} \vec{F}_{\text{net}} \cdot \mathrm{d}\vec{x} = \int m\vec{a} \cdot \mathrm{d}\vec{x} = m \int \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \cdot \mathrm{d}\vec{x}$$

Both $\vec{v}$ and $\vec{x}$ are continuous functions in time, we can switch the order of differentiation.

$$= m \int \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \cdot \mathrm{d}\vec{v} = m \int \vec{v} \cdot \mathrm{d}\vec{v} = m \int_{v_0}^{v_1} v \mathrm{d}v$$

where $v_0 = v(x_0)$ and $v_1 = v(x_1)$.

Kinetic Energy

This integral, when integrated from v_0 to v_1, becomes:

$$= m \int_{v_0}^{v_1} v dv = \frac{1}{2} m v^2 \Big|_{v_0}^{v_1} = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_0^2 = \Delta K$$

where K is defined as the **translational kinetic energy**:

$$\boxed{K = \frac{1}{2} m v^2}$$

Later in the course we will discuss *rotational* kinetic energy.

Work-Energy Theorem

The *definition* of kinetic energy came from this integration, in that work equals to the change in *something*, and we define that as kinetic energy. This is the **work-energy theorem**:

$$\boxed{W_{\text{net}} = \Delta K}$$

- ΔK can be positive or negative depending on the dot product
- When multiple forces acting on an object; each force can add or remove kinetic energy from an object
- Therefore we use the “net” amount of work done in the above equation
- It does not matter *what* the net force is composed of

Example

Example 1: A force $\vec{F} = 4.0x\hat{\imath}$ (in newtons) acts on an object of mass 2.0 kg as it moves from $x = 1.0$ to $x = 5.0$ m. Given that the object is at rest at $x = 1$,

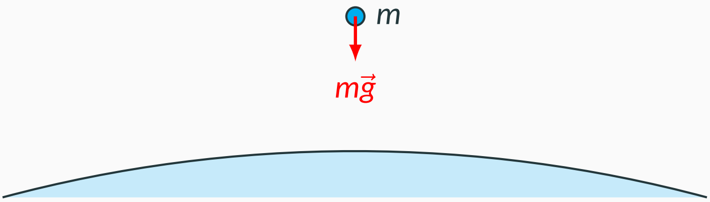
(a) Calculate the net work

(b) What is the final speed of the object?

Potential Energy

Gravitational Force & Gravitational Potential Energy

Consider an object that is free-falling under the force of gravity over a distance of Δx:

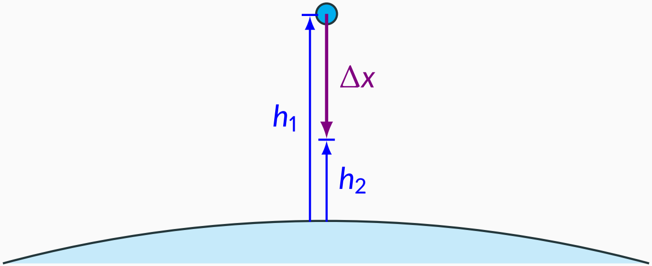


- Assuming that $\Delta\vec{x}$ is small, $\vec{g}$ can be considered to be constant
- The work done by the gravity (W_g) is *positive*, and therefore, there is an increase in kinetic energy. The object speeds up.

$$W_g = mg\Delta x = \Delta K > 0$$

Gravitational Potential Energy

The work done by gravity can also be expressed in terms of the change in height. Using ground as the reference level (i.e. $h = 0$):



$$\begin{aligned} W_g &= mg(h_1 - h_2) \\ &= -mg(h_2 - h_1) = -(mgh_2 - mgh_1) = -\Delta U_g \end{aligned}$$

Gravitational Potential Energy

Defining the **gravitational potential energy** U_g as:

$$\boxed{U_g = mgh}$$

The the work done by gravity can be related to this potential energy by:

$$\boxed{W_g = -\Delta U_g}$$

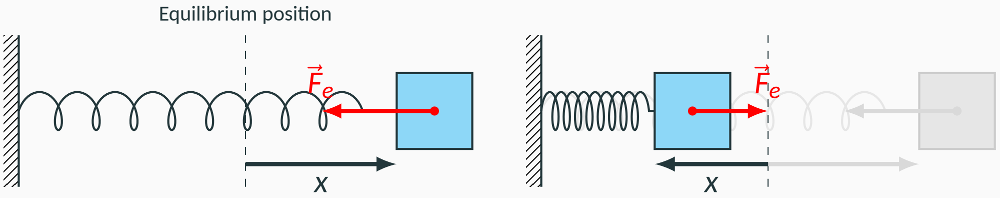
- *Positive* work decreases gravitational potential energy, while
- *Negative* work increases gravitational potential energy
- W_g depends on the end points h_1 and h_2, but not *how* it went from $h_1 \to h_2$

Spring Force & Elastic Potential Energy

The spring force $\vec{F}_e$ is the force that a compressed/stretched spring exerts on the object connected to it. An *ideal* spring obeys Hooke's law:

$$\boxed{\vec{F}_e = -k\vec{x}}$$

The spring force acts in the opposite direction to the spring's displacement, and is proportional to the amount of compression/stretching.



Elastic Potential Energy

The work done by the spring force W_e as it pushes any masses that are connected to a compressed/stretched spring is therefore:

$$W_e = \int_{x_0}^{x_1} \vec{F}_e \cdot d\vec{x} = -k \int_{x_0}^{x_1} x dx = -\frac{1}{2}kx^2 \Big|_{x_0}^{x_1} = -\Delta U_e$$

where U_e is the **elastic potential energy**, defined as:

$$\boxed{U_e = \frac{1}{2}kx^2}$$

Elastic Potential Energy

The the work done by the spring force can be related to the elastic potential energy by:

$$\boxed{W_e = -\Delta U_e}$$

- *Positive* work by the spring decreases spring potential energy, while
- *Negative* work by the spring increases spring potential energy
- W_e depends on the end points x_0 and x_1, but not *how* it went from x_0 to x_1

Conservative Forces

These forces are called **conservative forces**

- Gravitational force $\vec{F}_g$
- Spring force $\vec{F}_e$
- Electrostatic force $\vec{F}_q$
- Magnetic force $\vec{F}_m$
- Nuclear forces

Because they shared these common properties:

- The work done by these forces relate to a change of a potential energy
    - Positive work decreases this related potential energy
    - Negative work increases this related potential energy
- The work done by a conservative force is *path independent* (depends only on end points)

Conservative Forces

By the fundamental theorem of calculus, any conservative forces $\vec{F}$ must be the negative gradient of the potential energies:

$$\vec{F} = -\nabla U = -\frac{\partial U}{\partial x}\hat{\imath} - \frac{\partial U}{\partial y}\hat{\jmath} - \frac{\partial U}{\partial z}\hat{k}$$

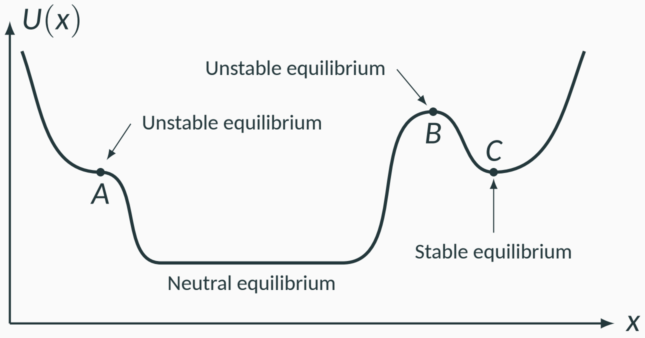
In one-dimension:

$$\boxed{F = -\frac{\mathrm{d}U}{\mathrm{d}x}}$$

The direction of a conservative force *always* decreases the potential energy. (Pay attention to the negative sign. Students often forget it.)

Energy Diagrams

Plot of potential energy (U) vs. position (x) for a conservative force



Conservation of Mechanical Energy

Positive work done by conservative forces on an object does two things:

1. Decrease its potential energy, while
2. Increase its kinetic energy by the same amount

Mathematically, this shows that mechanical energy must *always* be conserved when there are only conservative forces:

$$W_c = -\Delta U = \Delta K \quad \longrightarrow \quad \Delta K + \Delta U = 0$$

That's why those forces are called conservative forces, and they form the basis for conservation of energy.

Non-Conservative Forces

Examples of Non-Conservative Force

The majority of forces are **non-conservative**. The common forces discussed in the previous class (and also in Grade 11/12 Physics) are generally non-conservative:

- Applied force
- Tension force
- Normal force
- Friction
- Drag (fluid resistance)

The work-energy theorem still applies for non-conservative forces

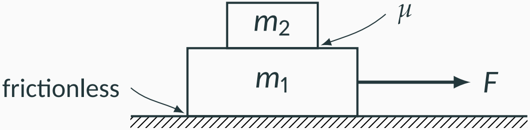
Work by Non-Conservative Forces

The work done by non-conservative forces differs from conservative forces in that:

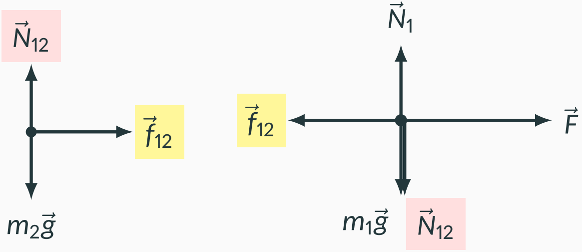
- There is **no related potential energies**: the work done by a non-conservative force transform energy from one form of kinetic energy to another
- The work is **path dependent**

Work by Friction, an Illustration

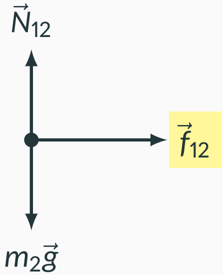
Work done by friction—and other non-conservative forces—is illustrated below. Two blocks (m_1 and m_2), stacked vertically, move to the right by external force.



The FBDs of the blocks are shown below. (The forces highlighted in the same color are action-reaction pairs.)



Work Done By Friction

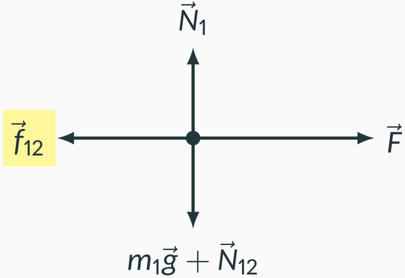


On the top block m_2, when it moves to the right

- Friction between the blocks $\vec{f}_{12}$ is the only force doing work
- The work done by $\vec{f}_{12}$ is positive
- Mass m_2 gains kinetic energy

On the bottom block m_1, when it moves to the right

- Applied force $\vec{F}$ does positive work on m_1, while
- Friction between the blocks $\vec{f}_{12}$ does negative work
- Therefore $\vec{f}_{12}$ decreases the kinetic energy



Work by Friction

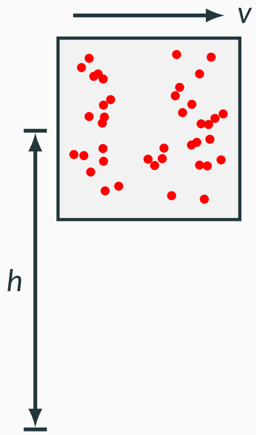
The work done by friction is

- positive on one object
- negative on another

Therefore, work by non-conservative forces transforms energy from the kinetic energy of one object into the kinetic energy of another object.

Internal Energy

Internal Energy



Consider a container of gas of mass M moving at speed v at a height h above Earth. It has a *bulk kinetic energy* of

$$K = \frac{1}{2}Mv^2$$

and a gravitational potential energy of

$$U_g = Mgh$$

But the random motion of the air molecules also contribute to additional energy, called the **internal energy** E_{int}.

Internal Energy

Internal energy of a system of molecules is the sum of all their kinetic and potential energies at the microscopic level:

$$\boxed{E_{\text{int}} = K_{\text{micro}} + U_{\text{micro}}}$$

It is a function of the molecules' **absolute temperature**, measured in *kelvin*. For example,

- For an ideal gas: $E_{\text{int}} = \frac{3}{2}NkT$
- For an diatomic gas: $E_{\text{int}} = \frac{3}{2}NkT$
- Solid: $E_{\text{int}} = 3NkT$

Conservation of Energy

Law of Conservation of Energy

The **law of conservation of energy**, which is based on the work-energy theorem, states that *the change in the total energy of a system is equal to the external work done to it.*

$$\boxed{\Delta K + \Delta U + \Delta E_{\text{int}} = W_{\text{ext}}}$$

In an isolated system that does not interact with the outside (and therefore no external work can be done), conservation of energy reduces to

$$\boxed{\Delta K + \Delta U + \Delta E_{\text{int}} = 0}$$

Law of Conservation of Energy

In almost all of the problem encountered in AP Physics C, there will be no change in the internal energy of the system, and conservation of energy reduces to:

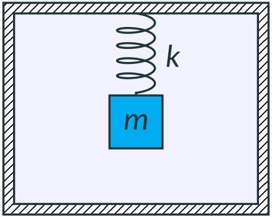
$$\boxed{\Delta K + \Delta U = W_{\text{ext}}} \quad \rightarrow \quad \boxed{U_1 + K_1 + W_{\text{ext}} = U_2 + K_2}$$

External work W_{ext} is

- **Positive** if work is done to the system
- **Negative** if work is done by by the system to the surrounding

Isolated Systems and the Conservation of Energy

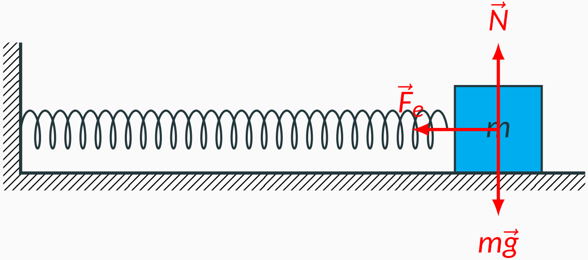
An **isolated system** is a system of objects that does not interact with the surrounding. Think of an isolated system as a bunch of objects inside an insulated box.



Since the system is isolated from the surrounding environment, the environment can't do any work on it. Likewise, the energy inside the system cannot escape either.

Example: Horizontal Spring-Mass System

Assuming that there are no friction, drag or other damping forces present, a horizontal spring-mass system is a closed system:

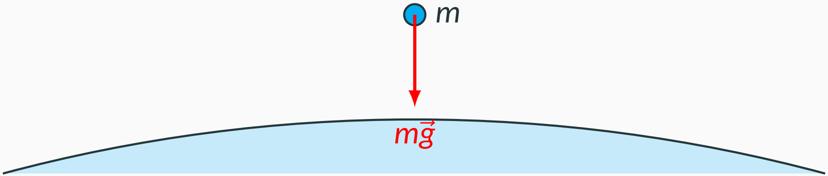


The sum of the kinetic energy of the mass (K) and the elastic potential energy stored in the spring (U_e) is constant

$$K + U_e = \text{constant}$$

Example: Gravity

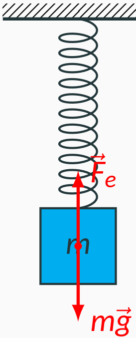
Assuming that there are no friction and drag, a free-falling object forms a closed system with Earth:



The sum of the kinetic energy of the mass (K) and the gravitational potential energy stored between the tow masses (U_g) is constant

$$K + U_g = \text{constant}$$

Example: Vertical Spring-Mass System



Assuming that there are no friction, drag or other damping forces in the spring, the vertical spring-mass system (consists of the mass, the spring and Earth) is a closed system.

$$K + U_g + U_e = \text{constant}$$

The sum of the kinetic energy of the mass (K), the gravitational potential energy stored between the mass and Earth (U_g), and the elastic potential energy stored in the spring (U_e) is constant.

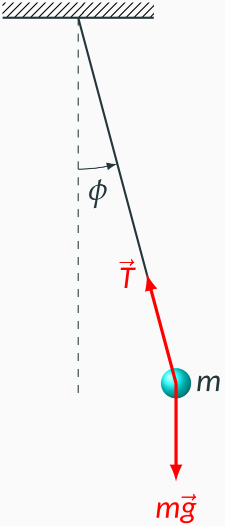
Simple Pendulum System

Assuming that there are no friction, drag or other damping forces in the spring, the simple pendulum system (consists of the mass and Earth) is a closed system.

- Gravity ($m\vec{g}$), which is conservative, is the only force that does work
- Tension ($\vec{T}$), which is non-conservative, does not do work on the pendulum because it is always perpendicular to the motion of the pendulum bob

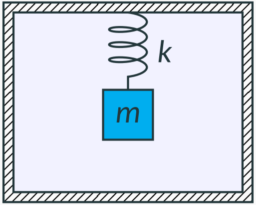
The sum of the kinetic energy of the mass (K), the gravitational potential energy stored between the mass and Earth (U_g) is constant:

$$K + U_g = \text{constant}$$



Isolated System with Changing Internal Energy

Energy is always conserved as long as your system is defined properly. In this case, the system consists of a mass, a spring, Earth and all the air molecules inside the box:

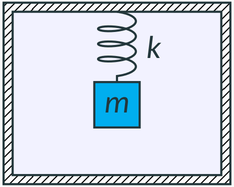


The energies of this system include

- Kinetic energy of the mass (K)
- Gravitational potential energy (U_g) between the mass and Earth
- Elastic potential energy (U_e) stored in the spring
- Internal energy (E_{int}) of the air molecules

Isolated System with Changing Internal Energy

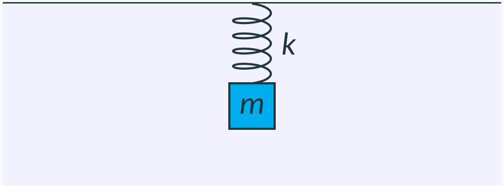
As the mass vibrates, friction and drag with air slows it down, converting the kinetic energy of the mass into the internal energy of the air. Total energy is conserved even as the mass stops moving



$$K + E_{\text{int}} + U_g + U_e = \text{constant}$$

Isolated System vs. Open System

Accounting for the change in the internal energy of the air molecules is not always practical, especially when the air molecules are not confined to a box.



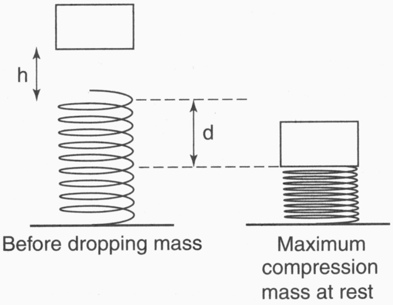
The solution:

- Take the air molecule out of the *system*
- No longer an isolated system
- Treat the negative work done by kinetic friction and drag as *external work* between initial (1) and final (2) states

$$K_1 + U_{g1} + U_{e1} + W_f = K_2 + U_{g2} + U_{e2}$$

Example

Example: A mass m is dropped from a height of h above the equilibrium position of a spring. Set up the equation that determines the spring's compression d when the object is instantaneously at rest.



Power & Efficiency

Power

Power is the *rate* at which work is done, i.e. the rate at which energy is being transformed:

$$\boxed{P(t) = \frac{\mathrm{d}W}{\mathrm{d}t}} \qquad \boxed{\overline{P} = \frac{W}{\Delta t}}$$

Quantity	Symbol	SI Unit
Instantaneous and average power	P, $\overline{P}$	W
Work done	W	J
Time interval	Δt	s

In engineering, power is often more critical than the actual amount of work done.

Power

If a constant force is used to push an object at a constant velocity, the power produced by the force is:

$$P = \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{x}}{dt} = \vec{F} \cdot \frac{d\vec{x}}{dt} \quad \rightarrow \quad \boxed{P = \vec{F} \cdot \vec{v}}$$

Application: aerodynamics

- When an object moves through air, the applied force must overcome air resistance (drag force), which is proportional with v^2
- Therefore "aerodynamic power" must scale with v^3 (i.e. doubling your speed requires $2^3 = 8$ times more power)
- Important when aerodynamic forces dominate

Efficiency

Efficiency is the ratio of useful energy or work output to the total energy or work input

$$\boxed{\eta = \frac{E_o}{E_i} \times 100\,\%} \qquad \boxed{\eta = \frac{W_o}{W_i} \times 100\,\%}$$

Quantity	Symbol	SI Unit
Useful output energy	E_o	J
Input energy	E_i	J
Useful output work	W_o	J
Input work	W_i	J
Efficiency	η	no units

Efficiency is always $0 \leq \eta < 100\,\%$