Topic 12: DC Circuit Analysis

Advanced Placement Physics 1 & 2

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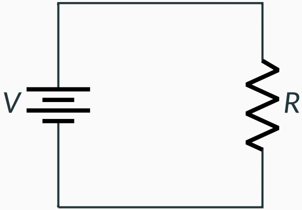
Olympiads School

Circuits

Basic DC Circuit



A basic circuits consists of:

- A voltage source: battery, generator or capacitor
- Connecting wires
- A load: resistors, motor, LEDs

Current

Current

The **electric current** I through conductor is the amount of charge carriers Q that passes through a point during a time interval t:

$$\boxed{I = \frac{Q}{t}}$$

Current Through the Conductor

Expanding the expression:

$$I = \frac{Q}{t} = \left(\frac{Q}{V}\right)\frac{V}{t} = [ne]\,[Av_d]$$

where

- Q/V is the amount of charge carriers *per volume*, which is just the **charge carrier density** (number of charge carriers per volume) n times the **elementary charge** e
- V/t is the rate the volume of charges moves through the conductor, give by the cross-section area of the conductor A times the **drift velocity** v_d of the charge carrier

Current Through the Conductor

Combining the terms:

$$\boxed{I = Qt = neAv_d}$$

Quantity	Symbol	SI Unit
Current	I	A
Charge carrier density	n	$1/\mathrm{m}^3$
Elementary charge	e	C
Cross-section area of the conductor	A	m^2
Drift velocity of the charge carriers	v_d	m/s

The calculation for the charge carrier density n requires some additional thoughts.

Charge Carrier Density

Calculating the charge carrier density in a *metal* conductor involves some physical information about the metal:

1. Divide the metal's density ρ by its molar mass M to find the *number of moles of atoms per unit volume*
2. Multiply by Avogadro's number $N_A = 6.0221 \times 10^{23}$ /mol to find *number of atoms per unit volume*
3. Multiply by the number of free electrons per atom k for that particular metal

Charge Carrier Density

Collecting all the terms from the last slide, we have:

$$\boxed{n = \frac{\rho k N_A}{M}}$$

Quantity	Symbol	SI Unit
Charge carrier density	n	$1/\mathrm{m}^3$
Density of material	ρ	kg/m^3
Free electrons per atom	k	
Avogadro's number	N_A	$1/\mathrm{mol}$
Molar mass	M	kg/mol

For copper, $M = 63.54 \times 10^{-3}\,\mathrm{kg}/\mathrm{mol}$, $\rho = 8.96 \times 10^{3}\,\mathrm{kg}/\mathrm{m}^3$, $k = 1$ and therefore $n = 8.5 \times 10^{28}\,/\mathrm{m}^3$.

Current

Another alternate description of the electric current is to express it in terms of the current density J, with a unit of of *ampère per meters squared* (A/m^2).

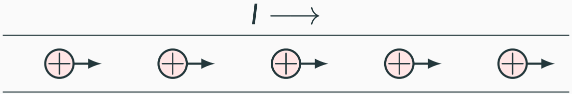
$$\boxed{I = JA}$$

It is obvious from the previous expression that the current density is the product of the charge carrier density, elementary charge, and the drift velocity:

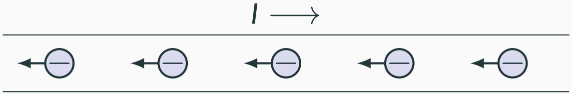
$$\boxed{J = nev_d}$$

Conventional Current vs. Electron Flow

We have *assumed* that the charge carriers are positively charged, which means that the current flows from high electric potential to low potential (i.e. from cathode to anode).



In a conducting wire, instead of positive charges flowing in one direction, we have, in fact, electrons (negative charges) flowing in the opposite direction, called the **electron current**:



For circuit analysis, we use the conventional current for simplicity.

Electric Field Inside the Wire

Inside the wire, there is a weak electric field, which exerts a force on the free electrons as they move in the wire

- As the charges move through a conductor, they lose potential energy
- Unlike a falling object which converts gravitational potential energy to kinetic energy, in a circuit, the energy is
    - converted into radiative heat or light through the resistance of the wire, or
    - converted into kinetic energy of a motor shaft

Resistors

Resistance of a Conductor

The resistance of a conductor is proportional to the resistivity ρ and its length L, and inversely proportional to the cross-sectional area A:

$$\boxed{R = \rho\frac{L}{A}}$$

Quantity	Symbol	SI Unit
Resistance	R	Ω
Resistivity	ρ	$\Omega\,\mathrm{m}$
Length of conductor	L	m
Cross-sectional area	A	m^2

The resistivity ρ is determined by the material.

Resistivity

The resistivity of a material is the ratio between strength of the electric field E inside the material and the current density J:

$$\boxed{\rho = \frac{E}{J}}$$

- Conductor: the electrons are free to move, therefore the electric field tend to be very small, and the resistivity is low.
- Insulators and dielectric: electrons cannot move easily (dielectric: they can only polarize themselves) the electric field are generally strong, and the resistivity is higher.

Resistance of a Conductor

$$\boxed{R = \rho \frac{L}{A}}$$

Gauge	Diameter (mm)	R/L ($10^{-3}\,\Omega/\text{m}$)
0	9.35	0.31
10	2.59	2.20
14	1.63	8.54
18	1.02	21.90
22	0.64	51.70

Material	Resistivity ρ ($\Omega\,\text{m}$)
silver	1.6×10^{-8}
copper	1.7×10^{-8}
aluminum	2.7×10^{-8}
tungsten	5.6×10^{-8}
Nichrome	100×10^{-8}
carbon	3500×10^{-8}
germanium	0.46
glass	10^{10} to 10^{14}

Ohm’s Law

Ohm's Law

The electric potential difference V across a resistor equals the product of the current I through the load and the resistance R:

$$\boxed{V = IR}$$

Quantity	Symbol	SI Unit
Potential difference	V	V
Current	I	A
Resistance	R	Ω

A resistor is considered "ohmic" if it obeys Ohm's law. Note that Ohm's law is not a fundamental law in physics.

Power Dissipated by a Resistor

Average power P is the rate at which work W is done, and from electrostatics, the change in electric potential energy ΔE_q (i.e. the work done!) is proportional to the amount of charge q and the voltage V. This gives a very simple expression for power through a resistor:

$$P = \frac{W}{\Delta t} = \frac{\Delta(qV)}{\Delta t} = \left(\frac{\Delta q}{\Delta t}\right) V \rightarrow \boxed{P = IV}$$

Quantity	**Symbol**	**SI Unit**
Power through a resistor	P	W
Current through a resistor	I	A
Voltage across the resistor	V	V

Other Equations for Power

When we combine Ohm's law ($V = IR$) with power equation, we get two additional expressions for power through a resistor:

$$\boxed{P = \frac{V^2}{R}} \qquad \boxed{P = I^2 R}$$

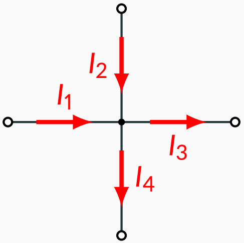
Quantity	Symbol	SI Unit
Power	P	W
Voltage	V	V
Resistance	R	Ω
Current	I	A

Not surprisingly, these equations will only apply to ohmic devices.

Kirchhoff’s Laws

Current Law

The electric current that flows into any junction in an electric circuit must be equal to the current which flows out.



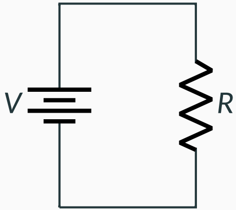
e.g. if there are 4 paths to the junction at the center, with I_1 and I_2 going into the junction, and I_3 and I_4 coming out, then the current law says that

$$I_1 + I_2 - I_3 - I_4 = 0$$

Basically, it means that there cannot be any accumulation of charges anywhere in the circuit. The law is a consequence of conservation of energy.

Voltage Law

The voltage changes around any closed loop in the circuit must sum to zero, no matter what path you take through an electric circuit.



Assume that the current flows clockwise and we draw a clockwise loop, we get

$$V - V_R = 0 \quad \rightarrow \quad V - IR = 0$$

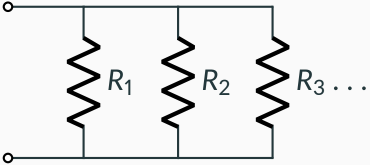
If I incorrectly guess that I flows counterclockwise, I will still have a similar expression

$$-V_R - V = 0 \quad \rightarrow \quad -V - IR = 0$$

When solving for I, we get a negative number, indicating that my guess was in the wrong direction.

Resistors in Circuits

Resistors in Parallel



From the current law, we know that the total current is the current through all the resistors, which we can rewrite in terms of voltage and resistance using Ohm's law:

$$I = I_1 + I_2 + I_3 \cdots = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} \cdots$$

But since we also know that $V_1 = V_2 = V_3 = \cdots = V$ from the voltage law, we can re-write as

$$I = \frac{V}{R_p} = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \cdots\right)$$

Equivalent Resistance in Parallel

Applying applying Ohm's law and Kirchhoff's laws gives the equivalent resistance of a parallel circuit: the inverse of the equivalent resistance for resistors connected in parallel is the sum of the inverses of the individual resistances.

$$\boxed{\frac{1}{R_p} = \sum_i^N \frac{1}{R_i}}$$

Quantity	Symbol	SI Unit
Equivalent resistance in parallel	R_p	Ω
Resistance of individual loads	$R_{1,2,3,\cdots,N}$	Ω

Resistors in Series

The analysis for resistors in series is similar (but easier). From the current law, the current through each resistor is the same:

$$I_1 = I_2 = I_3 = \cdots = I$$

And the total voltage drop across all resistor is therefore:

$$V = V_1 + V_2 + V_3 + \cdots = I(\underbrace{R_1 + R_2 + R_3 + \cdots}_{R_S})$$

Equivalent Resistance in Series

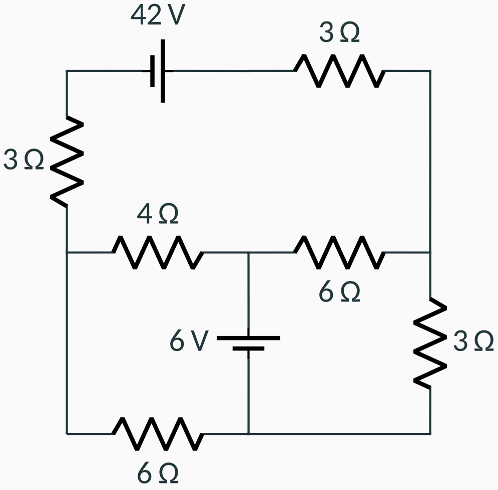
Again, through applying Ohm's law and Kirchhoff's laws, we find that when resistors are connected in series: the equivalent resistance of loads is the sum of the resistances of the individual loads.

$$\boxed{R_s = \sum_{i=1}^{N} R_i}$$

Quantity	Symbol	SI Unit
Equivalent resistance in series	R_s	Ω
Resistance of individual loads	$R_{1,2,3,\cdots,N}$	Ω

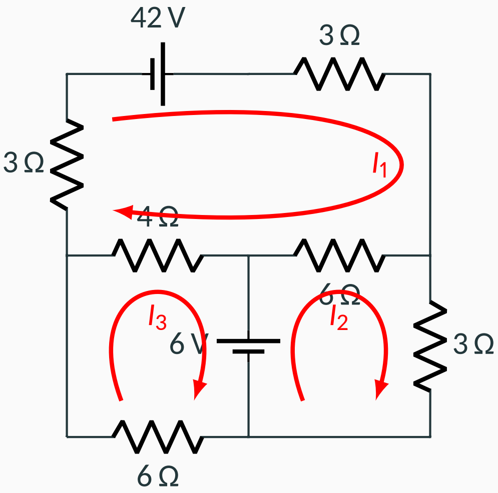
Tips for Solving “Simple” Circuit Problems

1. Identify groups of resistors that are in parallel or in series, and find their equivalent resistance.
2. Gradually reduce the entire circuit to one voltage source and one resistor.
3. Using Ohm’s law, find the current out of the battery.
4. Using Kirchhoff’s laws, find the current through each of the resistors.



- To solve this problem, we define a few “ current loops” around the circuit: one on top, one on bottom left, and one on bottom right.

Multi-Loop Circuits



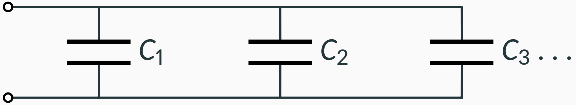
- To solve this problem, we define a few " current loops" around the circuit: one on top, one on bottom left, and one on bottom right.
- Apply the voltage law in the loops. For example, in the lower left:

$$4(I_1 - I_3) - 6 - 6I_3 = 0$$

- Solve the system of equations to find the current. If the current that you worked out is negative, it means that you have the direction wrong.

Capacitors in Circuit

Capacitors in Parallel



From the voltage law, we know that the voltage across all the capacitors are the same, i.e. $V_1 = V_2 = V_3 = \cdots = V$. We can express the total charge Q_{tot} stored across all the capacitors in terms of capacitance and this common voltage V:

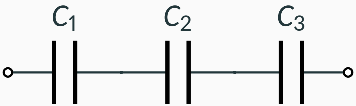
$$Q_{tot} = Q_1 + Q_2 + Q_3 + \cdots = C_1 V + C_2 V + C_3 V + \cdots$$

Factoring out V from each term gives us the equivalent capacitance:

$$\boxed{C_p = \sum_i C_i}$$

Capacitors in Series

Likewise, we can do a similar analysis to capacitors connected in series.



The total voltage across these capacitors are the sum of the voltages across each of them, i.e. $V_{tot} = V_1 + V_2 + V_3 + \cdots$

We recognize that the charge stored on all the capacitors must be the same! We can then write the total voltage in terms of capacitance and charge:

$$V_{tot} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \cdots$$

Capacitors in Series

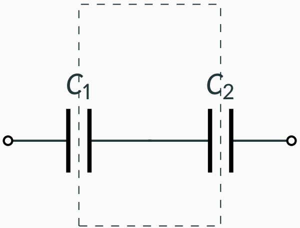
The inverse of the equivalent capacitance for N capacitors connected in series is the sum of the inverses of the individual capacitance.

$$\boxed{\frac{1}{C_s} = \sum_i \frac{1}{C_i}}$$

Make sure we don't confuse ourselves with resistors.

How Do We Know That Charges Are The Same?

It's simple to show that the charges across all the capacitors are the same

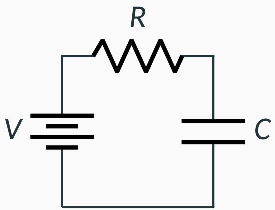


The capacitor plates and the wire connecting them are really one piece of conductor. There is nowhere for the charges to leave the conductor, therefore when charges are accumulating on C_1, C_2 must also have the same charge because of conservation of charges.

R-C Circuits

Circuits with Resistors and Capacitors

Now that we have seen how resistors and capacitors behave in a circuit, we can look into combining them in to an “R-C circuit”.



The simplest form is a resistor and capacitor connected in series, and then connect to a voltage source. Because of the nature of capacitors, the current through the circuit will not be steady as were the case with only resistors.

Discharging a Capacitor

The expression of charge across the capacitor is time-dependent:

$$\boxed{Q(t) = Q_0 e^{-t/\tau}}$$

where $Q_0 = Q_{tot}$ is the initial charge on the capacitor, and $\tau = RC$ is called the **time constant**. Using a bit of basic calculus gives the current through the circuit:

$$\boxed{I(t) = I_0 e^{-t/\tau}}$$

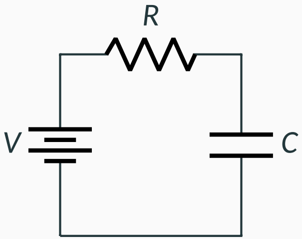
where the initially current is given by $I_0 = I(0) = \dfrac{Q_{tot}}{\tau}$.

Charging a Capacitor

When charging the capacitor, the charge across the capacitor is given by:

$$\boxed{Q(t) = Q_{tot}(1 - e^{-t/RC})}$$

the time constant $\tau = RC$ is the same as the discharging case



The current I through the circuit is the same as the discharging case:

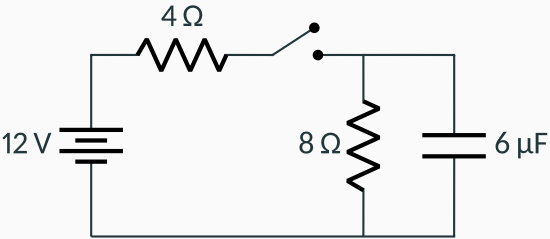
$$\boxed{I_c(t) = I_0 e^{-t/\tau}}$$

The initial current $I_0 = Q_{tot}/\tau = V/R$. This makes sense because $V_C = 0$ at $t = 0$, and all of the energy must be dissipated through the resistor. At $t = \infty$, current through the capacitor is $I_c = 0$

Two Small Notes

1. When a capacitor is uncharged, there is no voltage across the plate, it acts like a short circuit.
2. When a capacitor is charged, there is a voltage across it, but no current flows *through* it. Effectively it acts like an open circuit.

A Slightly More Difficult Problem



Example: The capacitor in the circuit is initially uncharged. Find the current through the battery

1. Immediately after the switch is closed
2. A long time after the switch is closed