Solutions Manual

For

Special Relativity
A Heuristic Approach

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List of Symbols, Phrases, and Acronyms

$\hat{a}$ unit vector in the direction of $\vec{a}$

anti-particle a particle whose mass and spin are exactly the same as its corresponding particle, but the sign of all its "charges" are opposite. If a particle is represented by the letter p, then it is customary to denote its anti-particle by $\bar{p}$. If a particle is represented by the letter q^- (or q^+), then it is customary to denote its anti-particle by q^+ (or q^-).

as arcsecond; an arcsecond is an angle 1/3600 of a degree.

baryon a hadron whose spin is an odd multiple of $\hbar/2$. Baryons are composed of three quarks. Examples of baryons are protons and neutrons.

$\vec{\boldsymbol{\beta}}$ fractional velocity of one observer relative to another, $\vec{\beta} = \vec{v}/c$

boson a particle whose spin is an integer multiple of $\hbar$. All gauge particles are bosons as are all mesons, as well as the Higgs particle.

causally connected referring to two two events. If an observer or a light signal can be present at two events, those events are said to be causally connected.

causally disconnected referring to two two events. If an observer or a light signal cannot be present at two events, those events are said to be causally disconnected.

CBR Cosmic Background Radiation

CM center of mass

CS coordinate system

$\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$ unit vectors along the three Cartesian axes.

EM electromagnetic or electromagnetism, one of the four fundamental forces of nature.

equilibrium temperature temperature of the universe at which matter and radiation densities are equal

eV electron volt, unit of energy equal to 1.6×10^{-19} J

fermion a particle whose spin is an odd multiple of $\hbar/2$. Fermions obey Pauli's exclusion principle: no two identical fermions can occupy a single quantum state. Electrons, protons, and neutrons are fermions, so are all leptons and quarks, as well as all baryons.

γ the Lorentz factor, $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-(v/c)^2}$

gauge bosons According to the modern theory of forces, fundamental particles interact via the exchange of gauge bosons. Excluding gravity, whose microscopic behavior is not well understood, there are 12 gauge bosons whose exchange explains all the interactions: Z^0, $W^{\pm}$ and γ (photon) are responsible for electroweak interaction, while 8 gluons are responsible for strong interaction.

gluons the particles responsible for strong interactions: two or more quarks participate in strong interaction by exchanging gluons. There are four gluons, which with their antiparticles comprise the eight gluons whose exchange binds quarks together.

GTR general theory of relativity; the relativistic theory of gravity.

Gyr gigayear, equal to 10^9 years

hadron a particle capable of participating in strong nuclear interactions. Examples of hadrons are protons, neutrons and pions. All hadrons are made up of quarks and/or anti-quarks.

half life the time interval in which one half of the initial decaying particles survive.

LAV Law of Addition of Velocities

lepton a particle that participates only in electromagnetic and weak nuclear interactions, but not in strong nuclear interactions. Leptons are elementary particles in the sense that they are not made up of anything more elementary. There are three electrically charged leptons: electron, muon, and tauon. Each charged lepton has its own neutrino. So, altogether there are six leptons.

LHC Large Hadron Collider

light cone (at an event E) The set of all events that are causally connected to E.

light hour the *distance* that light travels in one hour, $\approx 1.08 \times 10^{12}$ m

light minute the *distance* that light travels in one minute, $\approx 1.8 \times 10^{10}$ m

light second the *distance* that light travels in one second, $\approx 3 \times 10^8$ m

lightlike referring to two events, when $c\Delta t = \Delta x$ or $(\Delta s)^2 = 0$.

luminally connected referring to two two events. If a light signal can be present at two events, those events are said to be luminally connected.

ly light year; one light year is 9.467×10^{15} m.

mean time the time interval in which $1/e$ of the initial decaying particles survive.

meson a hadron whose spin is an integer multiple of $\hbar$. Mesons are composed of one quark and one anti-quark. Examples of mesons are pions.

MeV million electron volt, unit of energy equal to 1.6×10^{-13} J

μm micrometer $= 10^{-6}$ m

Minkowskian distance also called "spacetime distance,"

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$$

is an expression involving the coordinates of two events which is independent of the coordinates used to describe those events.

MM clock sometimes called "light clock" is described on page 13.

Mpc Megaparsec

muon an elementary particle belonging to the group of particles named "leptons," to which electron belongs as well. Muon is called a "fat electron" because it behaves very much like an electron except that it is heavier.

neutrino a neutral lepton with very small mass. Neutrinos participate only in weak nuclear force. That's why they are very weakly interacting.

ns nanosecond or 10^{-9} s

Parsec A distance of about 3.26 light years. One parsec corresponds to the distance at which the mean radius of the Earth's orbit subtends an angle of one second of arc.

positron the anti-particle of the electron

quarks elementary particles which make up all hadrons. There are six quarks: up, down, strange, charm, bottom, top. Quarks participate in all interactions, in particular, the strong interaction.

RF reference frame

spacelike referring to two events, when $c\Delta t < \Delta x$ or $(\Delta s)^2 < 0$.

spacetime distance see Minkowskian distance

STR special theory of relativity

tauon an elementary particle belonging to the group of particles named "leptons," to which electron belongs as well. It is the heaviest lepton discovered so far.

timelike referring to two events, when $c\Delta t > \Delta x$ or $(\Delta s)^2 > 0$.

CHAPTER 1

Qualitative Relativity

Problems With Solutions

1.1. A rod of length L emits light from all of its points simultaneously (in its rest frame) when a remote switch is turned on. Its center is on the x-axis and is moving on the axis in a plane parallel to a very large photographic plate and infinitesimally close to it. When it reaches the middle of the plate, the switch is turned on.

(a) Compare the length $L_{||}$ of the image on the photographic plate with L when the rod is along the x-axis: $L_{||} > L$, $L_{||} = L$, or $L_{||} < L$? Give a reason for your answer.

(b) Compare the length $L_\perp$ of the image on the photographic plate with L when the rod is perpendicular to the x-axis: $L_\perp > L$, $L_\perp = L$, or $L_\perp < L$? Give a reason for your answer.

Solution:

(a) Length parallel to the direction of motion shrinks regardless of who sees the events of light emission simultaneously. See the discussion in Section 1.3 for the reason (as well as how to capture the length of a moving object).

(b) Length perpendicular to the direction of motion does not change.

Note the importance of the fact that the distance between the rod and the photographic plate is zero. ■

1.2. A rod is placed along the x-axis with its center at the origin. A pinhole camera C_1 is located on the z-axis and takes a picture of the stationary rod. Now the rod starts moving along the x-axis parallel to itself from $-\infty$. Camera C_1 is removed and another pinhole camera C_2 replaces it on the z-axis. As soon as the center of the rod reaches the origin (call it $t = 0$), C_2 takes a picture.

(a) Is the pinhole of C_2 collecting the light rays from the two ends of the rod that were emitted at $t = 0$?

(b) Is the pinhole collecting the light rays from the two ends of the rod that were emitted simultaneously, but not at $t = 0$?

(c) If the answer to (b) is no, which end emitted its light first, the trailing end or the leading end?

(d) Is it possible for the image of the rod in C_2 to be *longer* than its image in C_1? Hint: Consider the location of each end as it emits the light ray captured by C_2.

Solution:

(a) No. It takes time for the light to reach the camera once it leaves its source.

(b) No.

(c) The trailing end is farther away from the camera, so it must emit the light sooner than the leading end.

(d) The trailing end emits its light, the rod moves a little, then the leading end emits its light. So, the distance between the source of the light from the trailing end and that of the leading end is indeed larger than the length of the rod.

Note that the image in camera C_2, which is longer than the image in C_1, has nothing to do with the actual length of the rod! ■

1.3. A rod is placed along the y-axis with its center at the origin. A pinhole camera C_1 is located on the z-axis and takes a picture of the stationary rod. Now the rod starts moving along the x-axis parallel to itself from $-\infty$. Camera C_1 is removed and another pinhole camera C_2 replaces it on the z-axis. As soon as the center of the rod reaches the origin (call it $t = 0$), C_2 takes a picture.

(a) Is the pinhole of C_2 collecting the light rays from the two ends of the rod that were emitted at $t = 0$?

(b) Is the pinhole collecting the light rays from the two ends of the rod that were emitted simultaneously, but not at $t = 0$?

(c) If the answer to (b) is no, which end emitted its light first, the top or the bottom?

(d) Is it possible for the image of the rod in C_2 to be longer than its image in C_1? Hint: Consider the locations of the ends as they emit their light rays captured by C_2, the distance between those locations and the pinhole, and the angle they subtend at the pinhole.

Solution:

(a) No. It takes time for the light to reach the camera once it leaves its source.

(b) Yes. The perpendicular distance does not change. So, the top and bottom of the rod are equidistant from the pinhole, and to reach it at $t = 0$, the rays must have been emitted at the same time in the past.

(c) The answer to (b) is yes!

(d) No. Since the locations of the sources of the rays are farther from C_2 (they have negative x-coordinates) than the locations in C_1, they must have a *smaller* image.

Chapter 7 discusses this in gory mathematical detail! ■

1.4. A circular ring emits light from all of its points simultaneously (in its rest frame) when a remote switch is turned on. It is moving in a plane parallel to a photographic plate and infinitesimally close to it. When it reaches the plate, the switch is turned on. What is the shape of the image of the photograph? Hint: See Problem 1.1.

Solution: The diameter along the direction of motion shrinks; the diameter perpendicular to the direction of motion stays the same. So, the shape is an ellipse flattened in the direction of motion. ■

1.5. A conveyor belt moving at relativistic speed carries cookie dough. A circular stamp cuts out cookies as the dough rushes by beneath it. What is the shape of these cookies? Are they flattened in the direction of the belt, stretched in that direction, or circular?

Solution: The answer is identical to that of the previous problem. So, the cookies are flattened in the direction of the belt. ■

1.6. A conveyor belt moving at relativistic speed carries cookie dough. A laser gun one meter above the belt emits a beam in the shape of the surface of a circular cone that cuts the dough perpendicularly. Are these cookies flattened in the direction of the belt, stretched in that direction, or circular? Hint: Concentrate on the two ends of the diameter of the beam along the dough, and note that their light beams arrive simultaneously at the stationary bed on which the dough is moving. Now consider how the two events appear in the RF of the moving dough and what implication it has on the length of the diameter. The discussion surrounding Figure 1.3 may be helpful.

Solution: The image of the laser beam on the stationary bed is circular and the two events of the arrival of the beams from the two ends of the horizontal diameter occur simultaneously. Consider two experiments. In the first experiment, there is no dough and the laser imprints a circle on the stationary bed. In the second experiment, an observer riding with the dough records the coincidence of the location of the event in front of her with the front end of the imprint *before* the coincidence of the event in the back. She concludes that for her, the distance between the two events is *larger* than the two marks on the stationary bed. So, the image is an ellipse *elongated* along the direction of the belt. Thus, the cookies are stretched in the direction of the belt. ■

1.7. A circular ring is centered at the origin in the xy-plane. A pinhole camera C_1 is located on the z-axis and takes a picture of the stationary ring. Now the ring starts moving along the x-axis from $-\infty$. Camera C_1 is removed and another pinhole camera C_2 replaces it on the z-axis. As soon as the center of the rod reaches the origin (at $t = 0$), C_2 takes a picture.

(a) Is the pinhole of C_2 collecting the light rays from the two ends of the horizontal diameter (along the x-axis) of the ring that were emitted at $t = 0$? Hint: Look at Problem 1.2.

(b) Is the pinhole of C_2 collecting the light rays from the two ends of the horizontal diameter of the ring that were emitted simultaneously, but not at $t = 0$?

(c) If the answer to (b) is no, which end emitted its light first, the trailing end or the leading end?

(d) Is it possible for the image of the horizontal diameter in C_2 to be *longer* than its image in C_1?

(e) Is the image of the vertical diameter (along the y-axis) in C_2 equal to, longer than, or shorter than its image in C_1? Hint: Look at Problem 1.3.

(f) Can you guess what the shape of the image of the ring is in C_2?

Solution:

(a) No.

(b) No.

(c) The trailing end.

(d) Yes, it always is.

(e) It is shorter than its image in C_1.

(f) It is an ellipse elongated along the direction of motion.

See Example 2.2.6 for a quantitative analysis of this problem. ■

CHAPTER 2

Relativity of Time and Space

Problems With Solutions

2.1. Take the most rigid rod you can find, and hit one end of it with the hammer. The rod *as a whole* starts to move because it is rigid.[1] Actually not! It takes time for the information that one end of the rod was hit to reach the other end, because of Note 2.1.7. Now go to the rest frame of the rod which is now moving relative to the hammer. Assume that the hammer hits the rod in such a way as to cause (the front end of) it to stop. But the other end knows nothing about the hammer yet. So, it keeps moving! What does this say about the concept of "rigidity" in relativity?

Solution: Rigidity is not a well defined concept in relativity. The other end of the rod gets compressed because of its motion. ■

2.2. In this problem you'll learn more about superluminal transverse speeds.

(a) Show that the angle that maximizes Equation (2.6) is given by $\cos\theta = \beta$.

(b) Substitute this in (2.6) to obtain $(v_{tr})_{\max} = c\beta\gamma$.

(c) Show that $(v_{tr})_{\max}$ is larger than c for any $\beta > 1/\sqrt{2}$.

(d) What speed makes $(v_{tr})_{\max}$ ten times faster than light? What is the angle corresponding to this speed?

Solution:

(a) Set the derivative of v_{tr}

$$v'_{tr}(\theta) = \frac{c\beta(\cos\theta - \beta)}{(\beta\cos\theta - 1)^2}$$

equal to zero to get $\cos\theta = \beta$. The second derivative test shows that $v''_{tr}(\theta) < 0$ when $\cos\theta = \beta$. Therefore, v_{tr} is indeed maximum at $\cos\theta = \beta$.

[1] If you hit one end of a slinky, the other end does not move, at least not immediately.

(b)

$$(v_{tr})_{\max} = c\beta\frac{\sin\theta}{1-\beta\cos\theta} = c\beta\frac{\sqrt{1-\beta^2}}{1-\beta^2} = \frac{c\beta}{\sqrt{1-\beta^2}} = c\beta\gamma.$$

(c)

$$c\beta\gamma > c \iff \beta^2\gamma^2 > 1 \iff \beta^2 > 1-\beta^2 \iff \beta^2 > 1/2.$$

(d)

$$\beta\gamma = 10 \iff \beta^2\gamma^2 = 100 \iff \beta^2 = 100 - 100\beta^2 \iff \beta^2 = 100/101,$$

or if $\beta = 0.995$. The angle corresponding to this β is $\theta = \cos^{-1} 0.995 = 0.0997$ or $\theta = 5.71°$.

■

2.3. Consider an MM clock moving horizontally with speed β relative to observer O. Denote its length in motion by L and at rest by L_0. Let Δt_1 be the time it takes light to go from the emitter to the mirror according to O. Let Δt_2 be the time it takes light to go from the mirror to the emitter according to O.

(a) Show that

$$c\Delta t_1 = L + \beta c\Delta t_1, \quad c\Delta t_2 = L - \beta c\Delta t_2.$$

(b) Show that a "tick" according to O is

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{2L/c}{1-\beta^2}.$$

(c) Now use the time dilation formula with $\Delta\tau = 2L_0/c$ to derive the length contraction formula.

Solution:

(a) By the time the light that is emitted at the emitter reaches the mirror, the MM clock has moved. So, the distance that the light covers is L plus the distance that the MM clock moves in the same time interval. So, $c\Delta t_1 = L + \beta c\Delta t_1$, and

$$c\Delta t_1(1-\beta) = L \iff \Delta t_1 = \frac{L}{c(1-\beta)}.$$

On reflection, the light and the MM clock move in opposite directions. Therefore, $c\Delta t_2 = L - \beta c\Delta t_2$, and

$$c\Delta t_2(1+\beta) = L \iff \Delta t_2 = \frac{L}{c(1+\beta)}.$$

(b) A tick according to O is therefore,

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{L}{c(1-\beta)} + \frac{L}{c(1+\beta)} = \frac{2L/c}{1-\beta^2}.$$

But $\Delta t = \gamma\Delta\tau = \gamma(2L_0/c)$. Thus,

$$\gamma(2L_0/c) = \frac{2L/c}{1-\beta^2} = \gamma^2(2L/c) \iff L = L_0/\gamma = L_0\sqrt{1-\beta^2}.$$

■

2.4. The spaceship Enterprise goes to a planet in a star system far away with a speed of $0.9c$, spends 6 months on the planet, and comes back with a speed of $0.95c$. The entire trip takes 5 years for the crew.

(a) How far is the planet according to Earth observers?

(b) How long did it take the crew to get to the planet?

(c) How long did the entire trip take for the Earth observers?

Solution:

(a) On the outbound part, the distance for the crew is $L_0\sqrt{1-0.9^2}$ and the time it takes them to get there is $L_0\sqrt{1-0.9^2}/(0.9c)$, where L_0 is the distance according to Earth observers. Similarly, the time it takes them to come back is $L_0\sqrt{1-0.95^2}/(0.95c)$. So, the entire round trip time is

$$\frac{L_0\sqrt{1-0.9^2}}{0.9c}+\frac{L_0\sqrt{1-0.95^2}}{0.95c}=4.5 \text{ years}.$$

This gives $L_0 = 5.54$ light years.

(b)

$$\Delta\tau_{\text{outbound}}=\frac{L_0\sqrt{1-0.9^2}}{0.9c}=0.484 \text{ years}.$$

(c) The distance according to Earth is 5.54 light years. So, for outbound trip

$$\Delta t_{\text{outbound}}=\frac{5.54 \text{ light years}}{0.9c}=6.15 \text{ years}.$$

Similarly,

$$\Delta t_{\text{inbound}}=\frac{5.54 \text{ light years}}{0.95c}=5.83 \text{ years}.$$

Therefore the entire trip takes $6.15+0.5+5.83=12.48$ years.

■

2.5. A rocket ship leaves the Earth at a speed of $0.8c$. When a clock on the rocket says 1 hour has elapsed, the rocket ship sends a light signal back to Earth.

(a) According to Earth clocks, when was the signal sent?

(b) According to Earth clocks, how long after the rocket left did the signal arrive back on Earth?

(c) According to the rocket clock, how long after the rocket left did the signal arrive back on Earth?

Solution:

(a) The rocket is measuring the proper time of 1 hour. So, for Earth

$$\Delta t=\gamma\Delta\tau=\frac{1 \text{ hour}}{\sqrt{1-0.8^2}}=1.67 \text{ hours}.$$

(b) The distance of the rocket from Earth when it sends the signal is

$$1.67 \text{ hours} \times 0.8c = 1.33 \text{ light hours.}$$

So, it took 1.33 hours for the signal to arrive at Earth after it was sent. Therefore, between the rocket leaving and the signal arriving, it took $1.33 + 1.67 = 3$ hours according to Earth. Note that *this is proper time for Earth.*

(c) The rocket measures coordinate time:

$$\Delta t_{\text{rocket}} = \frac{3 \text{ hour}}{\sqrt{1-0.8^2}} = 5 \text{ hours.}$$

It is a good exercise to find the answer to (c) by calculating in the rocket frame. Note that the Earth moves at $0.8c$ away from the rocket. So, when the rocket sends the signal, the Earth is at a distance of 0.8 light hour from rocket. Now the rocket sends a signal that chases the Earth. When does the light catch up with Earth according to rocket? ■

2.6. A bicycle wheel of rest radius R is rotating in such a way that the rim has a linear speed of $0.866c$. What is the circumference of the rim? What is the length of the spokes in motion? But spokes are perpendicular to the direction of motion! Discuss whether in relativity anything can be considered "incompressible" or "rigid." See also Problem 2.1.

Solution: For the same reason as Problem 2.1, the spokes are not "rigid." ■

2.7. The spaceship Viking goes to a planet in a star system 30 light years away from Earth with a speed of $0.99c$, spends 1 year on the planet, and then returns home. The entire trip takes 10 years for the crew.

(a) How far is the planet according to crew?

(b) How long does it take the crew to get to the planet?

(c) How long does it take the crew to return to Earth?

(d) What is the speed of the crew on return? Warning! The distance for the crew is *not* the same as the distance on their way to the planet.

(e) How far is the Earth from the planet according to crew on their return?

(f) How long did the entire trip take for the Earth observers?

Solution:

(a) $L = 30\sqrt{1-.99^2} = 4.23$ light years.

(b) The distance is 4.23 light years, and they are going at $0.99c$, so it takes them $4.23/0.99$, or 4.27 years to get there.

(c) Since the entire trip is 10 years, the return time is $10 - 4.27 - 1 = 4.73$ years.

(d) Let β_2 be the speed of return. Then the distance is $30\sqrt{1-\beta_2^2}$ light years. And the time, in terms of speed, is $30\sqrt{1-\beta_2^2}/c\beta_2$. So, we have to solve the equation

$$\frac{30\sqrt{1-\beta_2^2}\text{ light years}}{c\beta_2} = 4.73\text{ years.}$$

The answer comes out to be $\beta_2 = 0.988$.

(e) $L = 30\sqrt{1-0.988^2} = 4.67$ light years.

(f)
$$\frac{30\text{ light years}}{0.99c} + 1 + \frac{30\text{ light years}}{0.988c} = 61.67\text{ years.}$$

■

2.8. The spaceship Diracus goes to a planet in a star system with a speed whose Lorentz factor is γ, spends 1 year on the planet, and then returns home with a speed whose Lorentz factor is 4γ. The captain of the spaceship is 29 years old and has just had a newborn son. The entire trip takes 11 years for the crew. The odometer of the spaceship shows that the "milage" for the round trip is a quarter of the Earth-planet distance as measured by Earth observers.

(a) What is the outbound speed? The inbound speed?

(b) What is the Earth-planet distance according to the Earth observers?

(c) What is the Earth-planet distance according to the crew on their way to the planet? On their way back?

(d) How long does it take the crew to go to the planet? To return?

(e) Who is older, the son or the father when the ship lands on Earth? By how many years?

Solution:

(a) Let L_0 be the distance according to Earth. Then

$$\frac{L_0}{\gamma} + \frac{L_0}{4\gamma} = \frac{L_0}{4}.$$

This gives $\gamma = 5$ and

$$\beta_{\text{out}} = \frac{\sqrt{\gamma^2-1}}{\gamma} = \frac{\sqrt{24}}{5} = 0.98,$$

and

$$\beta_{\text{in}} = \frac{\sqrt{(4\gamma)^2-1}}{4\gamma} = \frac{\sqrt{399}}{20} = 0.9987,$$

(b) The round trip time is 10 years. So,

$$\frac{L_{\text{out}}}{\beta_{\text{out}}} + \frac{L_{\text{in}}}{\beta_{\text{in}}} = \frac{L_0/\gamma}{\beta_{\text{out}}} + \frac{L_0/(4\gamma)}{\beta_{\text{in}}} = 10$$

or

$$\frac{L_0}{5\times 0.98} + \frac{L_0}{20\times 0.9987} = 10.$$

This gives $L_0 = 39.35$ light years.

(c) $L_{\text{out}} = L_0/\gamma = 7.87$ light years; $L_{\text{in}} = L_0/(4\gamma) = 1.97$ light years.

(d) $\Delta\tau_{\text{out}} = L_{\text{out}}/c\beta_{\text{out}} = 8.03$ years; $\Delta\tau_{\text{in}} = L_{\text{in}}/c\beta_{\text{in}} = 1.97$ years; and these two answers are consistent with the roundtrip time being 10 years.

(e) $\Delta t_{\text{out}} = L_0/c\beta_{\text{out}} = 40.15$ years; $\Delta t_{\text{in}} = L_0/c\beta_{\text{in}} = 39.4$. So, the son is $40.15 + 1 + 39.4 = 80.55$ years old, while the father is just $29 + 11 = 40$ years old.

■

2.9. A rod of rest length L_0 moves with speed v along the positive x'-direction of observer O'. The rod makes an angle θ_0 with respect to the x-axis of its rest frame.

(a) Find the length of the rod as measured by O'.

(b) Find the angle θ the rod makes with the x'-axis as measured by O'.

Solution: The projections along the axes in the rest frame are

$$\Delta x = L_0 \cos\theta_0, \quad \Delta y = L_0 \sin\theta_0.$$

In the O' frame, we have

$$\Delta x' = \Delta x\sqrt{1-(v/c)^2} = L_0\sqrt{1-(v/c)^2}\cos\theta_0, \quad \Delta y' = L_0 \sin\theta_0$$

(a) The length in O' is

$$\begin{aligned} L &= \sqrt{(\Delta x')^2 + (\Delta y')^2} = \sqrt{L_0^2[1-(v/c)^2]\cos^2\theta_0 + L_0^2\sin^2\theta_0} \\ &= L_0\sqrt{1-\beta^2\cos^2\theta_0}, \quad \beta \equiv v/c. \end{aligned}$$

Note that the answer is consistent with the special cases $\theta_0 = 0$ and $\theta_0 = \pi/2$.

(b)

$$\cos\theta = \frac{\Delta x'}{L} = \frac{\sqrt{1-\beta^2}\cos\theta_0}{\sqrt{1-\beta^2\cos^2\theta_0}} = \frac{\cos\theta_0}{\gamma\sqrt{1-\beta^2\cos^2\theta_0}}$$

or

$$\tan\theta = \frac{\Delta y'}{\Delta x'} = \frac{\sin\theta_0}{\sqrt{1-(v/c)^2}\cos\theta_0} = \gamma\tan\theta_0.$$

■

2.10. A flasher produces a flash of light every second when at rest. It is moving away from you at $0.9c$.

(a) How frequently does it flash according to you?

(b) By how much does the distance between you and the flasher increase between consecutive flashes?

(c) How long after the emission of a given flash does it reach you?

(d) How often do you receive the flashes?

Solution: The flasher keeps proper time.

(a)

$$\Delta t = \frac{1 \text{ s}}{\sqrt{1-0.9^2}} = 2.294 \text{ s}.$$

(b)

$$\Delta x = 0.9c \times 2.294 \text{ s} = 2.065 \text{ light second}.$$

(c) From its emission, it takes the flash 2.065 seconds to reach you.

(d) The time intervals between flashes is the sum of the time interval between emissions and the time it takes the flashes to reach you: $2.294 + 2.065 = 4.35$ seconds.

Note that this is related to Doppler effect: Think of a flash as a wavefront. ■

2.11. Charged pions are produced in many collisions in accelerators. They decay in their rest frame according to

$$N(t) = N_0 e^{-t/T},$$

where $T = 2.6 \times 10^{-8}$ s is their mean life. A burst of charged pions is produced at the target of an accelerator and it is observed that only one percent of them decay at a distance of 1 m from the target. What is the Lorentz factor for pions and how fast are they moving?

Solution: Note that t in the decay formula is proper time. So, we have to calculate things in the rest frame of the pions. The the distance in the rest frame is L_0/γ. Therefore, $t = L_0/(\gamma c\beta)$, and

$$0.99 = e^{-t/T} = e^{-L_0/(\gamma c\beta T)} \iff \ln(0.99) = -\frac{L_0}{\gamma c\beta T} \iff \gamma\beta = \sqrt{\gamma^2 - 1} = -\frac{L_0}{\ln(0.99)cT}.$$

This gives $\gamma = 12.795$ and $\beta = 0.997$. ■

2.12. Derive Equations (2.8), (2.9), and (2.10).

Solution: Since $x_c < 0$, the negative sign in the previous equation must be chosen. The expression under square root sign can be written as

$$\beta^2\gamma^2(\beta^2 L^2 + b^2 + L^2/\gamma^2) = \beta^2\gamma^2\left[b^2 + L^2\left(\beta^2 + 1 - \beta^2\right)\right] = \beta^2\gamma^2(b^2 + L^2)$$

This yields (2.8). For (2.9), we have

$$x_A = x_c - L/\gamma = -\beta^2\gamma L - \beta\gamma\sqrt{b^2 + L^2} - L/\gamma = -\gamma\left(\beta^2 L + \frac{L}{\gamma^2} + \beta\sqrt{b^2 + L^2}\right).$$

Equation (2.9) now follows immediately from the definition of γ in terms of β. Finally,

$$\begin{aligned}
c^2 t_A^2 &= \gamma^2\left(L + \beta\sqrt{b^2 + L^2}\right)^2 + b^2 = \gamma^2 L^2 + \gamma^2\beta^2(b^2 + L^2) + 2\gamma^2\beta L\sqrt{b^2 + L^2} + b^2 \\
&= \gamma^2 L^2 + (\gamma^2 - 1)b^2 + \gamma^2\beta^2 L^2 + 2\gamma^2\beta L\sqrt{b^2 + L^2} + b^2 \\
&= \gamma^2(L^2 + b^2) + \gamma^2\beta^2 L^2 + 2\gamma^2\beta L\sqrt{b^2 + L^2} \\
&= \gamma^2\left[(L^2 + b^2) + \beta^2 L^2 + 2\beta L\sqrt{b^2 + L^2}\right] = \gamma^2\left(\beta L + \sqrt{b^2 + L^2}\right)^2.
\end{aligned}$$

When taking the square root, the negative sign is chosen because the light from A was emitted in the past. ■

2.13. Derive Equations (2.11) and (2.12).

Solution: Just as in the case of A, write

$$\frac{|x'_c|}{v} = \frac{\sqrt{(x'_c + L/\gamma)^2 + b^2}}{c} \iff |x'_c| = \beta\sqrt{(x'_c + L/\gamma)^2 + b^2}.$$

Squaring both sides gives

$$x'^2_c = \beta^2(x'^2_c + L^2/\gamma^2 + 2x'_c L/\gamma + b^2) \iff \frac{x'^2_c}{\gamma^2} - \frac{2\beta^2 L}{\gamma}x'_c - \beta^2(b^2 + L^2/\gamma^2) = 0.$$

Solve this quadratic equation for x_c to obtain

$$x'_c = \beta^2\gamma L \pm \sqrt{\beta^4\gamma^2 L^2 + \beta^2\gamma^2(b^2 + L^2/\gamma^2)}.$$

The negative sign must be chosen because $x'_c < 0$. The rest of the solution is identical to the previous problem. In fact, all the answers are obtained from that problem by changing the sign of L. ■

2.14. Derive Equations (2.14) and (2.15).

Solution: Square both sides of $|x_c| = \beta\sqrt{(b+L)^2 + x_c^2}$ to get

$$x_c^2 = \beta^2[(b+L)^2 + x_c^2] = \beta^2 x_c^2 + \beta^2(b+L)^2 \iff (1-\beta^2)x_c^2 = \frac{x_c^2}{\gamma^2} = \beta^2(b+L)^2.$$

Take the square root and chose the negative sign to get the answer. For x'_c, just change L to $-L$. ■

2.15. Derive Equation (2.16).

Solution: For an arbitrary point, you square both sides of $|x| = \beta\sqrt{(b-z)^2 + x^2}$. Then

$$x^2 = \beta^2[(b-z)^2 + x^2] = \beta^2 x^2 + \beta^2(b-z)^2 \iff \frac{x^2}{\gamma^2} = \beta^2(b-z)^2.$$

Take the square root to get $x = \pm\gamma\beta|b-z|$. Choosing the negative sign and noting that $|b-z| = b-z$, you obtain the answer. ■

2.16. Find t_A and t_B, the times at which A and B emit their light rays when the rod is oriented along the z-axis as in Example 2.2.4.

Solution: A and B have coordinates $(x_c, 0, -L)$ and $(x'_c, 0, L)$, respectively. So, their distances, $|ct_A|$ and $|ct_B|$ from the camera are given by

$$c^2t_A^2 = x_c^2 + (b+L)^2, \quad c^2t_B^2 = x'^2_c + (b-L)^2.$$

I'll find the first one, leaving the second for you. From (2.14), we have

$$c^2t_A^2 = \gamma^2\beta^2(b+L)^2 + (b+L)^2 = (\gamma^2\beta^2 + 1)(b+L)^2 = \gamma^2(b+L)^2.$$

Therefore, $ct_A = -\gamma(b+L)$. Similarly, $ct_B = -\gamma(b-L)$. Note that $b > L$. ■

2.17. Derive Equation (2.17).

Solution: From $|x| = \beta\sqrt{x^2+y^2+b^2}$, you get

$$x^2 = \beta^2(x^2+y^2+b^2) \iff \frac{x^2}{\gamma^2} = \beta^2(y^2+b^2) \iff \frac{x^2}{\gamma^2\beta^2} - y^2 = b^2,$$

and the final form follows immediately. ■

2.18. In Example 2.2.5, find t_A and t_B, the times at which A and B emit their light rays when the rod is oriented along the y-axis.

Solution: Let t_P be the time that the light from an arbitrary point P of the rod with coordinates $(x, y, 0)$ was emitted. Then using the results of the example, you have

$$c^2 t_P^2 = \sqrt{x^2+y^2+b^2} = \frac{x^2}{\beta^2} = \gamma^2(y^2+b^2).$$

Therefore, $ct_P = -\gamma\sqrt{y^2+b^2}$. Thus, $ct_A = ct_B = -\gamma\sqrt{L^2+b^2}$. ■

2.19. Derive Equations (2.20) and (2.21).

Solution: From $x_c^2 = \beta^2\left[x^2+a^2-\gamma^2(x-x_c)^2+b^2\right]$, you get

$$x_c^2 = \beta^2\left(x^2+a^2-\gamma^2x^2-\gamma^2x_c^2+2x\gamma^2x_c+b^2\right)$$

or

$$x_c^2(1+\beta^2\gamma^2) - 2x\beta^2\gamma^2x_c - \beta^2\left(a^2+b^2-\beta^2\gamma^2x^2\right) = 0$$

or

$$\gamma^2x_c^2 - 2x\beta^2\gamma^2x_c - \beta^2\left(a^2+b^2-\beta^2\gamma^2x^2\right) = 0.$$

The solution is

$$\begin{aligned} x_c &= \frac{x\beta^2\gamma^2 \pm \sqrt{x^2\beta^4\gamma^4 + \beta^2\gamma^2(a^2+b^2-\beta^2\gamma^2x^2)}}{\gamma^2} \\ &= \frac{x\beta^2\gamma^2 \pm \gamma\beta\sqrt{x^2\beta^2\gamma^2+a^2+b^2-\beta^2\gamma^2x^2}}{\gamma^2} = \frac{x\beta^2\gamma^2 \pm \gamma\beta\sqrt{a^2+b^2}}{\gamma^2}. \end{aligned}$$

Choosing the negative sign gives (2.20). Then

$$x - x_c = x - \beta^2x + \frac{\beta}{\gamma}\sqrt{a^2+b^2} = \frac{x+\beta\gamma\sqrt{a^2+b^2}}{\gamma^2}.$$

Substituting this in the equation of the ellipse (2.18) yields (2.21). ■

CHAPTER 3

Lorentz Transformation

Problems With Solutions

3.1. A line in the xz-plane has slope m and intercept b.

(a) Show that Equation (3.5) maps this line onto a line in the $x'z'$-plane.

(b) What are the slope and the intercept of the line in the $x'z'$-plane?

(c) Show that the transformation

$$x' = a_0 + a_1 x + a_2 z, \quad z' = b_0 + b_1 x^2 + b_2 z$$

transforms a straight line in the xz-plane into a parabola in the $x'z'$-plane.

Solution: The equation of the line is $z = mx + b$.

(a) Substitute for z in (3.5) to get

$$x' = a_0 + a_1 x + a_2 m x + a_2 b, \quad z' = b_0 + b_1 x + b_2 m x + b_2 b.$$

Find x from first equation

$$x = \frac{x' - a_0 - a_2 b}{a_1 + a_2 m}$$

and substitute it in the second

$$z' = b_0 + b_2 b + (b_1 + b_2 m)\frac{x' - a_0 - a_2 b}{a_1 + a_2 m},$$

or

$$z' = \frac{b_1 + b_2 m}{a_1 + a_2 m} x' + b_0 + b_2 b - \frac{(b_1 + b_2 m)(a_0 + a_2 b)}{a_1 + a_2 m}.$$

(b) The slope m' and the intercept b' are

$$m' = \frac{b_1 + b_2 m}{a_1 + a_2 m}, \quad b' = b_0 + b_2 b - \frac{(b_1 + b_2 m)(a_0 + a_2 b)}{a_1 + a_2 m}.$$

(c) Substitute $z = mx + b$ in the equations as before

$$x' = a_0 + (a_1 + a_2 m)x + a_2 b, \quad z' = b_0 + b_1 x^2 + b_2 mx + b_2 b.$$

Find x from the first and plug it in the second

$$z' = b_0 + b_1 \left(\frac{x' - a_0 - a_2 b}{a_1 + a_2 m} \right)^2 + b_2 m \frac{x' - a_0 - a_2 b}{a_1 + a_2 m} + b_2 b.$$

Expanding and collecting terms, you get an expression of the form $z' = Ax'^2 + Bx' + C$, which is the equation of a parabola.

■

3.2. Start with Equation (3.6) and provide all the missing steps that lead to Equation (3.8).

Solution: Most of the missing steps are actually explained in the textbook. ■

3.3. Derive Equation (3.10).

Solution: With $e = \tan\alpha$, we get $\sqrt{1+e^2} = \sec\alpha$ and

$$\frac{1}{\sqrt{1+e^2}} = \cos\alpha, \quad \frac{e}{\sqrt{1+e^2}} = \frac{\tan\alpha}{\sec\alpha} = \sin\alpha.$$

■

3.4. Use Figure 3.5 to prove the coordinate transformation

$$\begin{aligned} x' &= a + x\cos\alpha - y\sin\alpha \\ y' &= b + x\sin\alpha + y\cos\alpha. \end{aligned}$$

Solution: Refer to Figure 3.1 of the manual. I'll do the first equation. The second one is very similar.

$$\begin{aligned} x' &= \overline{O'A'} = \overline{O'A} + \overline{AA'} = a + \overline{OD} - \overline{CD} \\ &= a + x\cos\alpha - \overline{QR} = a + x\cos\alpha - y\sin\alpha. \end{aligned}$$

■

3.5. In Euclidean space, the locus of points equidistant from the origin of a plane is a circle. What is the locus of points equidistant (in the spacetime distance sense) from the origin of a spacetime plane?

Solution: Instead of $\sqrt{x^2+y^2} = R$, we now have $\sqrt{(ct)^2 - x^2} = R$, or $(ct)^2 - x^2 = R^2$, which is a hyperbola. ■

3.6. Verify that LT preserves the spacetime distance (3.13). In other words, if E_1 and E_2 are two events with coordinates (x_1, ct_1) and (x_2, ct_2) in O and (x'_1, ct'_1) and (x'_2, ct'_2) in O', where the primed coordinates are obtained from the unprimed coordinates by an LT, then

$$c^2(t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2.$$

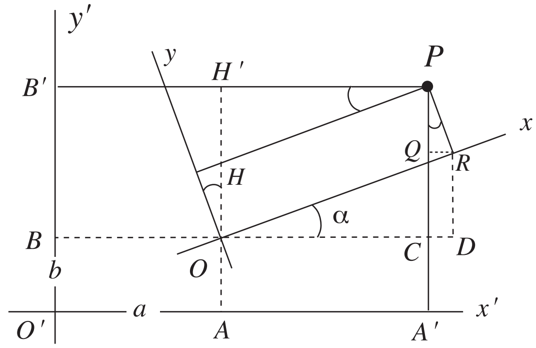


Figure 3.1: The same point P has different pairs of coordinates in different coordinate systems. Note that $a = \overline{O'A}$ and $b = \overline{O'B}$.

Solution: To save writing use notations like $\Delta t_{21} = t_2 - t_1$, etc. Then

$$c\Delta t'_{21} = \gamma(c\Delta t_{21} + \beta\Delta x_{21}), \quad \Delta x'_{21} = \gamma(\Delta x_{21} + \beta c\Delta t_{21})$$

and

$$\begin{aligned} c^2\Delta t'^2_{21} &= \gamma^2(c^2\Delta t^2_{21} + \beta^2\Delta x^2_{21} + 2c\beta\Delta t_{21}\Delta x_{21}) \\ \Delta x'^2_{21} &= \gamma^2(\Delta x^2_{21} + \beta^2 c^2\Delta t^2_{21} + 2c\beta\Delta t_{21}\Delta x_{21}). \end{aligned}$$

Subtracting the second from the first gives

$$c^2\Delta t'^2_{21} - \Delta x'^2_{21} = \gamma^2(c^2\Delta t^2_{21} - \Delta x^2_{21}) - \gamma^2\beta^2(c^2\Delta t^2_{21} - \Delta x^2_{21}).$$

Now invoke the very useful identity $\gamma^2\beta^2 = \gamma^2 - 1$. ■

3.7. Starting with (3.19), show that the origin of O' has the coordinates

$$\begin{aligned} x_0 &= \gamma(-x'_0 + \beta ct'_0) \\ ct_0 &= \gamma(\beta x'_0 - ct'_0) \end{aligned}$$

relative to O. Show also that the inverse of Equation (3.19) can be written as

$$\begin{aligned} x &= x_0 + \gamma(x' - \beta ct') \\ ct &= ct_0 + \gamma(\beta x' + ct'). \end{aligned}$$

Solution: Set the left-hand side of (3.19) equal to zero and solve for x and ct, which are now labeled x_0 and ct_0:

$$\begin{aligned} 0 &= x'_0 + \gamma(x_0 + \beta ct_0) \\ 0 &= ct'_0 + \gamma(\beta x_0 + ct_0). \end{aligned}$$

Multiply the second equation by β and subtract it from the first:

$$0 = x'_0 - \beta ct'_0 + \gamma(1-\beta^2)x_0 \iff 0 = x'_0 - \beta ct'_0 + \frac{x_0}{\gamma}.$$

The first equation now follows trivially. The second equation is obtained similarly.

To obtain the inverse of Equation (3.19), multiply the second equation by β and subtract it from the first:

$$x' - \beta ct' = x'_0 - \beta ct'_0 + \gamma(1-\beta^2)x \iff \gamma(x' - \beta ct') = \underbrace{\gamma(x'_0 - \beta ct'_0)}_{=-x_0} + x.$$

This yields the first inverse equation. The second inverse equation is derived similarly. ■

3.8. A ruler of length L is at rest in O with its left end at the origin. O moves from left to right with speed β relative to O' along the length of the ruler. The two origins coincide at time zero for both, at which time a photon is emitted toward the other end of the ruler. What are the coordinates in O' of the event at which the photon reaches the other end?

Solution: The event has coordinates $\big(c(L/c), L\big) = (L, L)$ in O. Therefore, in O', it has coordinates

$$ct' = \gamma(L + \beta L) = \sqrt{\frac{1+\beta}{1-\beta}}\,L$$
$$x' = \gamma(L + \beta L) = \sqrt{\frac{1+\beta}{1-\beta}}\,L,$$

verifying that light travels with the same speed in both RFs.

There is another way of obtaining the same answer not using Lorentz transformation. The length of the rod in O' is L/γ. As the light catches up with the other end of the rod in time t', the rod moves a distance of $\beta ct'$. Therefore,

$$ct' = \frac{L}{\gamma} + \beta ct' \iff (1-\beta)ct' = \frac{L}{\gamma} \iff ct' = \frac{\sqrt{1-\beta^2}L}{1-\beta} = \sqrt{\frac{1+\beta}{1-\beta}}\,L.$$

The location of the event is where the leading end of the rod is when the light catches up with it. The leading end was at L/γ at $t' = 0$ and it moved a distance of $\beta ct'$, so

$$x' = \frac{L}{\gamma} + \beta ct' = ct' = \sqrt{\frac{1+\beta}{1-\beta}}\,L.$$

I hope you appreciate the power of Lorentz transformation. Once you identify the coordinates of an event, Lorentz transformation takes care of the rest! ■

3.9. Start with the inverse LT and derive the time dilation (corresponding to $\Delta x = 0$) and length contraction (corresponding to $\Delta t' = 0$) formulas.

Solution: With

$$\Delta x = \gamma(\Delta x' - \beta c\Delta t'), \quad c\Delta t = \gamma(c\Delta t' - \beta\Delta x'), \tag{3.1}$$

$\Delta t' = 0$ immediately gives the length contraction formula:

$$\Delta x = \gamma\Delta x' \iff \Delta x' = \frac{\Delta x}{\gamma} = \sqrt{1-\beta^2}\Delta x,$$

remembering that Δx is moving in O'.

For time dilation, we have to set $\Delta x = 0$ and note that then Δt is the proper time $\Delta\tau$. Then, the first equation in (3.1) gives $\Delta x' = \beta c \Delta t'$, and the second equation yields

$$c\Delta\tau = \gamma(c\Delta t' - c\beta^2\Delta t') = \frac{c\Delta t'}{\gamma} \iff \Delta t' = \gamma\Delta\tau.$$

Alternatively, you can set $\Delta x' = 0$ and note that in that case $\Delta t'$ is the proper time. Then the second equation in (3.1) yields the answer immediately. ■

3.10. Two rockets A and B of rest length L_0 travel towards each other along their x-axes with relative speed β.[1] According to A, when the tail of B passes the nose of A, a missile is fired from the tail of A towards B. It will clearly miss due to length contraction of B as seen by A. But B will see the length of A contracted. So, when the nose of A coincides with the tail of B, the tail of A is in the middle of B's rocket and it will hit B. Who is right? Consider five events: nose of A passes nose of B (call this the origin of spacetime for both), tail of B passes nose of A, missile is fired, nose of B passes tail of A, and tail of B passes tail of A. Find the coordinates of these five events in both frames to see who is right.

Solution: Label the events E_1 through E_5. Assume that A has its nose at $x = 0$ and its tail at $x = L_0$, and that B is moving in its positive x direction. Then, considering that the length of B is L_0/γ for A, these events have the following coordinates according to A:

$$\begin{aligned} &E_1 : (0,0), \quad E_2 : \left(0, \frac{L_0/\gamma}{\beta}\right), \quad E_3 : \left(L_0, \frac{L_0/\gamma}{\beta}\right), \\ &E_4 : \left(L_0, \frac{L_0}{\beta},\right), \quad E_5 : \left(L_0, \frac{L_0}{\beta} + \frac{L_0/\gamma}{\beta}\right). \end{aligned}$$

A moves in the negative x direction of B, and if the two noses meet at $(0,0)$, then the tail of B must be at $x = -L_0$ according to B. Obviously, E_1 has the same coordinates in B as in A. The coordinates of E_2 in B are

$$\begin{aligned} x_2 &= \gamma\left(0 - \beta\frac{L_0}{\gamma\beta}\right) = -L_0 \\ ct_2 &= \gamma\left(\frac{L_0}{\gamma\beta} - 0\right) = \frac{L_0}{\beta}. \end{aligned}$$

Of course, we could have guessed this without any equations! But, it is a good idea to use Lorentz transformation to check our calculations. The coordinates of E_3 in B are

$$\begin{aligned} x_3 &= \gamma\left(L_0 - \beta\frac{L_0}{\gamma\beta}\right) = (\gamma - 1)L_0 \\ ct_3 &= \gamma\left(\frac{L_0}{\gamma\beta} - \beta L_0\right) = \frac{L_0}{\beta} - \gamma\beta L_0. \end{aligned}$$

Thus, $t_3 < t_2$, meaning that the missile is fired before the nose of A passes the tail of B. The location of the firing of the missile is $x_3 > 0$, which is outside the range of B (remember

[1] Assume that their path is slightly shifted in the y-direction so they do not collide.

that the nose of B is at the origin and its tail at $-L_0$). So, according to B, the missile misses the rocket. The coordinates of E_4 in B are

$$x_4 = \gamma\left(L_0 - \beta\frac{L_0}{\beta}\right) = 0$$
$$ct_4 = \gamma\left(\frac{L_0}{\beta} - \beta L_0\right) = \frac{\gamma L_0}{\beta} - \gamma\beta L_0 = \frac{L_0}{\gamma\beta},$$

and finally the coordinates of E_5 in B are

$$x_5 = \gamma\left[L_0 - \beta\left(\frac{L_0}{\beta} + \frac{L_0}{\gamma\beta}\right)\right] = -L_0$$
$$ct_5 = \gamma\left(\frac{L_0}{\beta} + \frac{L_0}{\gamma\beta} - \beta L_0\right) = \gamma\left(\frac{L_0}{\gamma^2\beta} + \frac{L_0}{\gamma\beta}\right) = \frac{L_0}{\gamma\beta} + \frac{L_0}{\beta}$$

Note that the location of E_5 is at the tail of either rocket, and the time of occurrence is the same for both. This makes sense by the symmetry of the problem. Note also that event E_4 for B is the analogue of E_2 for A. In both events, the tail of the moving rocket passes the nose of the stationary one. Since the length of the moving rocket is L_0/γ and its speed is β, it takes $L_0/\beta\gamma$ for the length of the moving rocket to pass the nose of the stationary one.

Alternative solution: We can avoid using length contraction, as I have done above. Designate the coordinates of A with primes. Assuming that A has its nose at $x' = 0$ and its tail at $x' = L_0$, B must have its nose at $x = 0$ and its tail at $x = -L_0$. Note that when the primed coordinates are on the left-hand side of a Lorentz transformation, β is positive, and when they are on the right-hand side, β is negative. Now assign coordinates to the five events in the two RFs, entering as much info as you have for each event. You should be able to get the following set of coordinates:

	E_1	E_2	E_3	E_4	E_5
For A	$(0,0)$	$(0,t_2')$	(L_0,t_2')	$(L_0,L_0/\beta)$	(L_0,t_5')
For B	$(0,0)$	$(-L_0,L_0/\beta)$	(x_3,t_3)	$(0,t_4)$	$(-L_0,t_5)$

I let you do the Lorentz transforming of these fives events and find the coordinates of all of them in both RFs. ■

3.11. This problem is the continuation of Example 3.4.4. It is important for you to remember that LT applies only to *events*. The coincidence of Sonya's clock with C_1 is an event. But what is the event corresponding to the reading of C_2? Note 3.4.3 gives the answer: There is an event E_0 at the location of C_2 with coordinates $(L,0)$ in Sam's RF.

(a) What are the coordinates of E_0 in Sonya's RF? Note that although C_2 is at the 12 o'clock position for Sam, it is not for Sonya!

(b) How long does it take her to reach C_2? Warning: The length Sonya has to travel to reach C_2 *is not* L/γ, because when Sonya is at C_1, C_2 is not at $x = L/\gamma$ for her!

(c) What time does Sonya's clock show when she reaches C_2? This should be $L/(c\beta\gamma)$ as in Example 3.4.4.

(d) What are the coordinates (in her RF) of the event at which Sonya reaches C_2?

(e) Using LT, show that the time coordinate of the same event in Sam's RF is $L/(c\beta)$.

Solution:

(a)
$$x_0 = \gamma(L - 0) = \gamma L, \quad ct_0 = \gamma(0 - \beta L) = -\gamma\beta L.$$

(b) The event is at x_0 and she is going with speed β toward it. So,
$$\Delta t = \frac{|x_0|}{c\beta} = \frac{\gamma L}{c\beta}.$$

(c) The time of Sonya's arrival at C_2 is
$$ct_2 = ct_0 + c\Delta t = -\gamma\beta L + \frac{\gamma L}{\beta} = \frac{\gamma L}{\beta}(1-\beta^2) = \frac{L}{\gamma\beta}.$$

(d) Sonya is at her origin; and at event E_2, she is also at C_2. So, $x_2 = 0$. Therefore, by (c), $(x_2, ct_2) = (0, L/\gamma\beta)$.

(e)
$$x_2' = \gamma(0 + \beta L/\gamma\beta) = L, \quad ct_2' = \gamma(L/\gamma\beta + 0) = L/\beta.$$
This agrees with what I found directly in the example.

■

3.12. To gain more insight into the clock comparisons of Example 3.4.4 and Problem 3.11, consider the synchronization process of the two clocks C_1 and C_2 in Sam's RF as seen by Sonya. To synchronize the clocks, let two light signals be sent simultaneously to the two clocks from the midpoint between them. Call this event E.

(a) Show that E has coordinates $(L/2, -L/2)$ in Sam's RF.

(b) Show that E has coordinates
$$x = \sqrt{\frac{1+\beta}{1-\beta}}\frac{L}{2}, \quad ct = -\sqrt{\frac{1+\beta}{1-\beta}}\frac{L}{2}$$
in Sonya's RF.

(c) How long does it take this signal to arrive at the (common) origin? What is the time at which it arrives at the origin? Is that what you expect?

(d) What is the location of C_2 relative to Sonya? What is the distance between E and C_2 as measured by Sonya? Show that it takes $\gamma(1+\beta)L/2c$ for the signal to cover this distance.

(e) From (b) and (d), show that the time of arrival of the signal at C_2 is $-\gamma\beta L/c$ according to Sonya.

(f) From (d), show that it takes Sonya $\gamma L/(c\beta)$ to reach C_2. Therefore, her time of arrival at C_2 is $L/(c\gamma\beta)$ *after* the arrival of the light signal at C_2, as in Example 3.4.4.

(g) Using (f) and LT show that the time shown on C_2 is $L/(c\beta)$.

Solution:

(a) The event E is equidistant from the clocks; so it has to have $x' = L/2$. In order for the light signals to arrive at the two clocks at $t' = 0$, they must have been emitted at $-L/2c$.

(b) You have to use the inverse Lorentz transformation with $\beta \to -\beta$

$$x = \gamma\left[\frac{L}{2} - \beta\left(-\frac{L}{2}\right)\right] = \gamma(1+\beta)\frac{L}{2} = \sqrt{\frac{1+\beta}{1-\beta}}\frac{L}{2}$$

$$ct = \gamma\left(-\frac{L}{2} - \beta\frac{L}{2}\right) = -\gamma(1+\beta)\frac{L}{2} = -\sqrt{\frac{1+\beta}{1-\beta}}\frac{L}{2}.$$

(c) The distance is x; so it takes $|x|/c$ for the light to arrive at the origins. Since it was emitted at t, it arrives at the origin at $(|x|/c) + t = 0$, as expected because the origin of time is the same for Sam and Sonya: if the signal arrives at $t' = 0$ according to Sam, it arrives at $t = 0$ according to Sonya.

(d) Recall that C_2 has coordinates $(L, 0)$ in Sam's RF. So, its location is $x_2 = \gamma(L - 0)$ in Sonya's RF. The distance between E and C_2 is

$$|x - x_2| = \left|\gamma(1+\beta)\frac{L}{2} - \gamma L\right| = \gamma(1-\beta)\frac{L}{2},$$

and light covers it in $\gamma(1-\beta)L/2c$.

(e) The time of arrival is

$$t + \frac{|x - x_2|}{c} = -\gamma(1+\beta)\frac{L}{2c} + \gamma(1-\beta)\frac{L}{2c} = -\gamma\beta\frac{L}{c}.$$

(f) The arrival of signal at C_2 sets that clock. In other words, according to Sonya's clock, the time at C_2 is $-\gamma\beta L/c$ when the clock at the origin reads 0. So, when Sonya arrives at C_2 after $x_2/c\beta = \gamma L/c\beta$, her clock reads

$$-\frac{\gamma\beta L}{c} + \frac{\gamma L}{c\beta} = \frac{\gamma L}{c\beta}(1 - \beta^2) = \frac{L}{c\beta\gamma}.$$

Therefore the event of her arrival at C_2 has coordinates $(0, L/\beta\gamma)$ in her RF.

(g) Lorentz transform the event of Sonya's arrival to Sam's RF:

$$x'_{\text{arrive}} = \gamma\left(0 + \beta\frac{L}{\beta\gamma}\right) = L$$

$$ct'_{\text{arrive}} = \gamma\left(\frac{L}{\beta\gamma} + 0\right) = \frac{L}{\beta}.$$

The second equation is the time of arrival of Sonya according to Sam, and therefore what C_2 actually reads.

$\blacksquare$

3.13. Obtain Equation (2.16) using Lorentz transformation.

Solution: In the rest frame of the rod, where the rod is assumed to lie along the z-axis, the emission event has coordinates $(0, -|b-z|)$. Lorentz transforming this, you get

$$x' = \gamma(0 - \beta|b-z|) = -\gamma\beta|b-z| = -\gamma\beta(b-z')$$

because $b > z$ and $z = z'$. $\blacksquare$

3.14. Generalize the discussion of Example 3.4.5 to any clock length L and any speed β by showing that

$$c\Delta t'_{21} = \gamma L(1+\beta), \quad \text{and} \quad c\Delta t'_{32} = \gamma L(1-\beta).$$

Solution: In the rest frame of the MM clock, events E_1, E_2, and E_3 have coordinates $(0,0)$, (L,L), and $(0,2L)$, respectively. If the clock moves relative to O' in the positive x direction with speed β, then

$$c\Delta t'_{21} \equiv c(t'_2 - t'_1) = \gamma(L + \beta L) = \gamma L(1+\beta)$$

and

$$c\Delta t'_{32} \equiv c(t'_3 - t'_3) = \gamma[(2L - L) + \beta(0 - L)] = \gamma L(1-\beta).$$

Therefore,

$$c\Delta t' = c\Delta t'_{21} + c\Delta t'_{32} = 2\gamma L,$$

and

$$\Delta t' = \gamma \frac{2L}{c} = \gamma\Delta\tau.$$

$\blacksquare$

3.15. Reorient the rod of Example 3.4.6 so that now it lies along the y-axis. Let t be the event at which the light from an arbitrary point of the rod is emitted in such a way that it reaches P_0 of Figure 3.4 at $t = 0$.

(a) Find ct in terms of y and b.

(b) Find the x'-coordinate of the event according to O', with respect to which the rod is moving.

(c) Show that the locus of source points whose light rays are collected simultaneously at the pinhole is a hyperbola in the $x'y'$-plane as in Example 2.2.5.

(d) Do you expect the rays from the two ends of the rod to have been emitted simultaneously according to O? According to O'?

(e) Find t_A, t_B, t'_A, and t'_B, the times of occurrence of the emission of the light rays from the two ends of the rod according to the two observers.

Solution:

(a) In the rest frame of the rod the location of the event is $(0, y, 0)$. So, its distance from the pinhole is $\sqrt{y^2 + b^2}$, and $ct = -\sqrt{y^2 + b^2}$.

(b) The spacetime coordinates of the event is $(0, -\sqrt{y^2+b^2})$. So, in O', we have

$$x' = \gamma(0 - \beta\sqrt{y^2+b^2}) = -\gamma\beta\sqrt{y'^2+b^2} \iff \frac{x'}{\gamma\beta} = -\sqrt{y'^2+b^2}$$

because $y' = y$.

(c) Squaring both sides and dividing by b^2 yields

$$\frac{x'^2}{\gamma^2\beta^2 b^2} - \frac{y'^2}{b^2} = 1.$$

(d) The two ends of the rod are equidistant from the pinhole in O. So, they must have been emitted at the same time in O. Since the rod is perpendicular to the direction of motion, simultaneity is not affected.

(e) $ct_A = ct_B = -\sqrt{L^2+b^2}$. Therefore,

$$ct'_A = \gamma(-\sqrt{L^2+b^2} + 0) = -\gamma\sqrt{L^2+b^2}$$
$$ct'_B = \gamma(-\sqrt{L^2+b^2} + 0) = -\gamma\sqrt{L^2+b^2} = ct'_A$$

■

3.16. The Earth and Alpha Centauri are 4.3 light years apart. Ignore their relative motion. Events A and B occur at $t = 0$ on Earth and at 1 year on Alpha Centauri, respectively.

(a) What is the time difference between the events according to an observer moving at $\beta = 0.98$ from Earth to Alpha Centauri?

(b) What is the time difference between the events according to an observer moving at $\beta = 0.98$ from Alpha Centauri to Earth?

(c) What is the speed of a spacecraft that makes the trip from Alpha Centauri to Earth in 2.5 years according to the spacecraft clocks?

(d) What is the trip time in the Earth RF?

Solution:

(a) Let Earth be the origin O. Then A has coordinates $(0, 0)$ and B has coordinates $(4.3, 1)$ in units of light years. The observer O', is moving in the positive x direction. Therefore, the Earth is moving in the negative direction, and

$$ct'_{BA} = \gamma[c(t_B - t_A) - \beta(x_B - x_A)] = 5(1 - 4.3 \times 0.98) = -16.15 \text{ light years.}$$

Thus, $t'_{BA} = -16.15$ years. The event on Alpha Centauri occurs *before* that on earth. This switching of the order of events should not surprise you because the two events are not causally connected: You can't send a probe (even a laser beam) that can be present at the two events, because such a probe must cover a distance of 4.3 light years in one year.

(b) Just change the sign of β:

$$ct'_{BA} = \gamma[c(t_B - t_A) + \beta(x_B - x_A)] = 5(1 + 4.3 \times 0.98) = 26.2 \text{ light years},$$

and $t'_{BA} = 26.2$ years. The event on Alpha Centauri occurs *after* that on earth.

(c) The distance for the spacecraft is $4.3/\gamma$; so the time is $4.3/(c\gamma\beta)$. Now you can find the speed:

$$2.5 \text{ years} = \frac{4.3 \text{ light years}}{c\gamma\beta} = \frac{4.3 \text{ years}}{\gamma\beta} \iff 2.5\gamma\beta = 4.3 \iff 2.5\sqrt{\gamma^2 - 1} = 4.3,$$

or $\gamma = 1.99$. Therefore, $\beta = \gamma\beta/\gamma = (4.3/2.5)/1.99 = 0.865$.

(d) The trip time is 4.3 light years$/(0.865c) = 4.97$ years.

■

3.17. All muons in a group move towards Earth with the same speed. After covering a distance of 2911 m, half of them survive. Recall that for any decay process, $N(t) = N_0 e^{-t/\tau}$, where τ is the mean life of the decaying particles as measured in their rest frame, and for muons $\tau = 2.2\,\mu$s.

(a) What is the speed of the muons?

(b) What is the mean life of the muons in the Earth frame?

(c) According to the muons, how far did they travel?

(d) A spaceship is launched from Earth with a speed of $0.95c$. What is the mean life of the muons in the frame of the spaceship?

Solution:

(a) In the rest frame of the muons, the distance is L_0/γ. So, the time they cover that distance is $L_0/(c\beta\gamma)$. Therefore,

$$0.5 = \exp\left(-\frac{L_0}{c\beta\gamma\tau}\right) \iff \ln(0.5) = -\frac{L_0}{c\beta\gamma\tau}$$

or

$$\beta\gamma = \sqrt{\gamma^2 - 1} = -\frac{L_0}{\ln(0.5)c\tau} = 6.36 \iff \gamma = 6.44,$$

and $\beta = \beta\gamma/\gamma = 6.36/6.44 = 0.988$.

(b) τ is the proper time. So, in the Earth frame

$$t_{\text{Earth}} = \gamma\tau = 6.44 \times 2.2\ \mu s = 14.17\ \mu s.$$

(c) $2911/6.44 = 452$ m.

(d) The speed of the muons relative to the spaceship is

$$\beta' = \frac{\beta + \beta_{\text{ship}}}{1 + \beta\beta_{\text{ship}}} = \frac{1.938}{1.9386} = 0.99969$$

and

$$\gamma' = \frac{1}{\sqrt{1 - \beta'^2}} = 40.2.$$

Therefore,

$$t_{\text{ship}} = \gamma'\tau = 40.2 \times 2.2\ \mu s = 88.4\ \mu s.$$

You can also find the answer to (d) by using Lorentz transformation. In Earth's RF, $\Delta x = 2911$ m and $\Delta t = L_0/c\beta = 9.82\ \mu$s. The spaceship is moving in the negative x direction according to Earth. Therefore, Earth is moving in the positive direction of spaceship, and thus,

$$c\Delta t_{\text{ship}} = \gamma(c\Delta t + \beta\Delta x) = 3.2(3 \times 10^8 \times 9.82 \times 10^{-6} + 0.95 \times 2911) = 18276.6 \text{ m}$$

or $\Delta t_{\text{ship}} = 61\ \mu$s. This is a fraction of the mean life, because in their rest frame, the muons cover this distance in $452/(0.988 \times 3 \times 10^8) = 1.525\ \mu$s, which is 0.693 of the mean time. So, the muon mean life in the spaceship frame is $61/0.693 = 88\ \mu$s, as before. ■

3.18. A ruler moves past another ruler of rest length L and parallel to it with speed β to the right. Two clocks are attached to the ends of the moving ruler and synchronized in the RF of that ruler. When the left end of the moving ruler reaches the left end of the stationary ruler, the left clock reads zero. When the right end of the moving ruler reaches the right end of the stationary ruler, the right clock reads Δt.

(a) Find the rest length of the moving ruler in terms of β, L, and Δt.

(b) Relative to the stationary ruler, how long after the left coincidence does the right coincidence occur?

Solution: Let $(0, 0)$ be the coincidence of the two left ends in both frames. Let E be the coincidence of the two right ends. Then in the moving frame, E has coordinates $(L_0, c\Delta t)$, with L_0 to be determined.

(a) In the moving frame, the stationary ruler has length L/γ. Therefore, the distance between its right end and the right end of the moving frame is $L/\gamma - L_0$, assuming that $\Delta t > 0$. Therefore,

$$\frac{L}{\gamma} - L_0 = \beta c\Delta t \iff L_0 = \frac{L}{\gamma} - \beta c\Delta t$$

(b) The moving ruler O is moving in the positive x direction of the stationary ruler O'. Thus, Lorentz transforming the coordinates of E, we get

$$c\Delta t' = \gamma(c\Delta t + \beta L_0) = \gamma\left[c\Delta t + \beta\left(\frac{L}{\gamma} - \beta c\Delta t\right)\right] = \frac{c\Delta t}{\gamma} + \beta L.$$

It is also instructive to find the location of the event in O', although it is obviously L:

$$x' = \gamma(L_0 + \beta c\Delta t) = \gamma\left(\frac{L}{\gamma} - \beta c\Delta t + \beta c\Delta t\right) = L.$$

We could find (a) by calculating in the stationary ruler's frame O'. The presence of a clocks at an end of the moving ruler constitutes an event. One has coordinates $(0,0)$ in both frames, the other E_0, has coordinates $(L_0, 0)$ in the moving frame. Therefore, in the stationary frame E_0 has coordinates

$$x'_0 = \gamma(L_0 + 0) = \gamma L_0, \quad ct'_0 = \gamma(0 + \beta L_0) = \gamma\beta L_0.$$

The coincidence of the right end occurs at $\Delta t' = \gamma(\Delta t + \beta L_0)$. Thus in the time interval $\Delta t' - t'_0$ the moving ruler covers a distance of $L - x'_0$. Thus,

$$c\beta(\Delta t' - t'_0) = L - \gamma L_0 \iff \beta(\gamma c\Delta t + \gamma\beta L_0 - \gamma\beta L_0) = L - \gamma L_0,$$

or $\gamma L_0 = L - \gamma\beta c\Delta t$, which is what we obtained before. ■

3.19. Sonya (observer O) is holding a long pole of length L in the middle as she runs with speed β towards a barn. Sam (observer O'), standing in the barn, sees that the pole fits exactly between the entrance and exit doors of the barn. On the other hand, Sonya sees her pole—which is at rest relative to her—longer than the barn and concludes that there is no way that the pole can fit in the barn. Who is right? To find out, let E_1 be the event when the front end of the pole exits the barn and E_2 the event when the rear end of the pole enters the barn. Write the general Lorentz transformation for these events, and answer the following questions:

(a) What are x_1 and x_2?

(b) How does t'_1 compare to t'_2?

(c) How does t_1 compare to t_2? Conclude that Sam and Sonya are both right!

(d) What is $x'_1 - x'_2$? Is that what you expect?

Solution: Let $(0,0)$ be the coordinates of E_1 in both frames.

(a) For both observers $x_1 = x'_1 = 0$. For Sonya, E_2 is on the negative x-axis and has $x_2 = -L$.

(b) For Sam the two ends coincide simultaneously. Therefore, $t'_1 = t'_2 = 0$.

(c) You don't have to use the length contraction formula! Write the Lorentz transformation for E_2:

$$\begin{aligned} x'_2 &= \gamma(x_2 + \beta ct_2) = \gamma(-L + \beta ct_2) \\ ct'_2 &= \gamma(ct_2 + \beta x_2) = \gamma[ct_2 + \beta(-L)]. \end{aligned}$$

With $t'_2 = 0$, the second equation gives $ct_2 = \beta L$. So, E_2 occurs after E_1 for Sonya, indicating that the pole is longer than the length of the barn.

(d) Substituting βL for ct_2 in the first equation yields

$$x'_2 = \gamma(-L + \beta^2 L) = -\frac{L}{\gamma}.$$

Hence, $x'_1 - x'_2 = L/\gamma$, which is the length contraction formula as expected.

■

3.20. Sam sees a firecracker explode at $x = 1.5$ km (event E_1), then after $5\,\mu$s, another firecracker explode at $x = 100$ m (event E_2).

(a) What is the velocity relative to Sam of Sonya for whom the events occur at the same place?

(b) Which event occurs first according to Sonya and what is the time interval between the explosions for her?

Solution: In order for the two events to occur at the same place for Sonya, she has to be present at both events.

(a) Therefore, she has to cover the distance of 1400 m in $5\,\mu$s. Thus, her speed should be

$$v = \frac{1400 \text{ m}}{5 \times 10^{-6} \text{ s}} = 2.8 \times 10^8 \text{ m/s} \iff \beta = \frac{2.8 \times 10^8}{3 \times 10^8} = 0.933$$

(b) Sonya has to move from the earlier event to the later event. So, she has to move in the negative x direction of Sam. So, Sam is moving in the positive x direction of Sonya. In Sam's RF, the events E_1 and E_2 have coordinates $(1.5 \text{ km}, 0)$ and $(100 \text{ m}, 5\,\mu\text{s})$, respectively. Obviously, E_1 occurs first, and

$$c\Delta t_{21} = c(t_2 - t_1) = \gamma[3 \times 10^8 \times 5 \times 10^{-6} + \beta(-1400)] \text{ m},$$

or $c\Delta t_{21} = 539.8$ m and $\Delta t_{21} = 1.8\,\mu$s. Note that Sonya measures proper time. So, Δt_{21} is indeed $5\ \mu\text{s}/\gamma$.

■

3.21. Sonya is in the middle of a train car of length L moving relative to Sam with speed β along Sam's positive x'-direction. Firecrackers A in the back and B in the front of the car explode simultaneously according to Sam at the same time that Sonya passes him. Let $t = t' = 0$ be the time that Sam and Sonya pass each other, and assume that both observers are at the origins of their coordinate systems. Without using length contraction or time dilation formulas, find the coordinates of all the following events for both observers: explosion of A, explosion of B, reception of light from A by Sam, reception of light from B by Sam, reception of light from A by Sonya, reception of light from B by Sonya.

Solution: Label the events E_1 through E_6. Let Sonya be observer O and Sam O'. Plug in all the info for each event. So, in Sonya's RF, the events E_1 through E_6 have the following coordinates, respectively:

$$(-L/2, ct_1), \quad (L/2, ct_2), \quad (x_3, ct_3), \quad (x_4, ct_4), \quad (0, ct_1 + L/2), \quad (0, ct_2 + L/2),$$

and in Sam's RF, they have the following coordinates:

$$(x'_1, 0), \quad (x'_2, 0), \quad (0, |x'_1|), \quad (0, |x'_2|), \quad (x'_5, ct'_5), \quad (x'_6, ct'_6).$$

Now write the Lorentz transformation for each event and solve for the unknowns. For E_1, we have

$$x'_1 = \gamma(-L/2 + \beta ct_1), \quad 0 = \gamma(ct_1 - \beta L/2).$$

These two equations give $ct_1 = \beta L/2$ and $x'_1 = -L/(2\gamma)$. For E_2, we have

$$x'_2 = \gamma(L/2 + \beta ct_2), \quad 0 = \gamma(ct_2 + \beta L/2).$$

These two equations give $ct_2 = -\beta L/2$ and $x'_2 = L/(2\gamma)$. Note that $x'_2 - x'_1 = L/\gamma$, indicating the contraction of the length of the train car for Sam.

The coordinates of E_3 and E_4 are the same in Sam's RF. They are both $\big(0, L/(2\gamma)\big)$. So, they must also be the same in Sonya's RF:

$$\begin{aligned} x_3 = x_4 &= \gamma\big(0 - \beta L/(2\gamma)\big) = -\beta L/2 \\ ct_3 = ct_4 &= \gamma\big(L/(2\gamma) - 0\big) = L/2. \end{aligned}$$

It is important to note that the equality of t_3 and t_4 *does not mean* that Sonya receives the signals simultaneously! It means that Sam receives the two signals simultaneously at $L/2c$ *according to Sonya*. Sonya can reason as follows to get this answer: "I know that at my zero time I passed Sam and at the same time *according to him*, the two firecrackers, which I know are at $\pm L/2$ exploded. Therefore, Sam must have gotten the signals, *according to me* at $L/2c$ after I passed him.

With ct_1 and ct_2 determined, the coordinates of E_5 and E_6 can be written in O. E_5 has coordinate $\big(0, (1+\beta)L/2\big)$ and E_6 has coordinate $\big(0, (1-\beta)L/2\big)$. Lorentz transforming to O', we get

$$\begin{aligned} x'_5 &= \gamma\big(0 + \beta(1+\beta)L/2\big) = \gamma\beta(1+\beta)L/2 \\ ct'_5 &= \gamma\big((1+\beta)L/2 + 0\big) = \gamma(1+\beta)L/2 \end{aligned}$$

and

$$\begin{aligned} x'_6 &= \gamma\big(0 + \beta(1-\beta)L/2\big) = \gamma\beta(1-\beta)L/2 \\ ct'_6 &= \gamma\big((1-\beta)L/2 + 0\big) = \gamma(1-\beta)L/2. \end{aligned}$$

■

3.22. Two spaceships of rest length L_0 are approaching the Earth from opposite directions at velocities $\pm 0.8c$. How long does one of them appear to the other?

Solution: One spaceship moves relative to the other with speed

$$\beta' = \frac{0.8 + 0.8}{1 + 0.8^2} = 0.9756,$$

and the length contraction formula gives

$$L = L_0\sqrt{1 - \beta'^2} = 0.2195 L_0.$$

■

3.23. Spaceship A is twice as long as spaceship B when they are at rest. Spaceship B is moving at three quarters the speed of light relative to Earth. As A overtakes B, an Earth observer notices that they both have the same length.

(a) How fast is A moving?

(b) What is the length of A relative to B?

(c) What is the length of B relative to A?

Solution:

(a) Let L_0 be the rest length of B. Then

$$2L_0\sqrt{1-\beta_A^2} = L_0\sqrt{1-0.75^2} \iff 4(1-\beta_A^2) = 0.4375 \iff \beta_A = 0.9437.$$

(b) The relative speed of the two spaceships is

$$\beta_{AB} = \frac{0.9437-0.75}{1-0.9437\times 0.75} = 0.663.$$

Therefore,

$$L_{AB} = 2L_0\sqrt{1-0.663^2} = 1.5L_0.$$

(c)

$$L_{BA} = L_0\sqrt{1-0.663^2} = 0.75L_0.$$

So, A says that the length of B is 37.5% of his rest length and B say that the length of A is 1.5 times her rest length. ■

3.24. Sam and Pat move in the same direction at $0.8c$ and $0.6c$ relative to Earth, respectively.

(a) How fast should Sonya move relative to Earth so that she observes Sam and Pat approaching her at the same speed?

(b) What is the speed of Sam (or Pat) relative to Sonya?

Solution:

(a) Sonya must move away from Sam and toward Pat. Let her speed relative to Earth be β. Then

$$\frac{0.8-\beta}{1-0.8\beta} = \frac{0.6+\beta}{1+0.6\beta} \iff \beta^2 - 5.2\beta + 1 = 0,$$

with solutions

$$\beta = 2.6 \pm \sqrt{2.6^2-1} \iff \beta = 2.6 - \sqrt{5.76} = 0.2,$$

because the positive choice gives $\beta > 1$.

(b) Her speed relative to Sam is

$$\beta_{ss} = \frac{0.8-0.2}{1-0.8\times 0.2} = 0.714$$

and relative to Pat it is

$$\beta_{sp} = \frac{0.6+0.2}{1+0.6\times 0.2} = 0.714.$$

■

3.25. Sonya (observer O) is approaching Sam (observer O') with speed β from the negative values of his x'-axis. Sonya has a source of light of wavelength λ sending signals to Sam. Consider two events: E_1, the source sends a wave crest, and E_2, the source sends the next wave crest. So, $t_2 - t_1$ is the period of the wave according to Sonya.

(a) Write the Lorentz transformation for these two events assuming that Sonya (at her origin) is holding the source of light.

(b) What is the period according to Sam? Hint: It is *not* $t'_2 - t'_1$!

(c) Show that λ', the wavelength as measured by Sam is related to λ via the following formula:

$$\lambda' = \sqrt{\frac{1-\beta}{1+\beta}}\,\lambda.$$

Solution: Let $(0,0)$ be the origin of both coordinate systems. So, the two events have negative time coordinates. Let E_1 and E_2 have coordinates $(0, ct_1)$ and $(0, ct_2)$, with $t_1 < t_2 < 0$.

(a) According to Sam, E_1 and E_2 have coordinates

$$\begin{aligned} x'_1 &= \gamma(0 + \beta ct_1) = \gamma\beta ct_1 \\ ct'_1 &= \gamma(ct_1 + 0) = \gamma ct_1 \end{aligned}$$

and

$$\begin{aligned} x'_2 &= \gamma(0 + \beta ct_2) = \gamma\beta ct_2 \\ ct'_1 &= \gamma(ct_2 + 0) = \gamma ct_2. \end{aligned}$$

(b) Sam receives the first wave front $|x'_1|/c$ after it was emitted. Thus

$$ct_1^{\text{rec}} = ct'_1 + |x'_1| = \gamma ct_1 + \gamma\beta c|t_1| = \gamma ct_1 - \gamma\beta ct_1 = (1-\beta)\gamma ct_1,$$

because $t_1 < 0$. Similarly

$$ct_2^{\text{rec}} = ct'_2 + |x'_2| = \gamma ct_2 + \gamma\beta c|t_2| = \gamma ct_2 - \gamma\beta ct_2 = (1-\beta)\gamma ct_2.$$

So, the difference in the time of reception of the two wave fronts for Sam is

$$ct_2^{\text{rec}} - ct_1^{\text{rec}} = (1-\beta)\gamma ct_2 - (1-\beta)\gamma ct_1 = \gamma(1-\beta)c(t_2 - t_1) = \gamma(1-\beta)cT.$$

Now, $t_2^{\text{rec}} - t_1^{\text{rec}}$ is the period T' according to Sam, which can also be written as

$$T' = \frac{1-\beta}{\sqrt{1-\beta^2}}\,T = \sqrt{\frac{1-\beta}{1+\beta}}\,T.$$

$\lambda' = cT'$ and $\lambda = cT$ should now give the relation between the wavelengths.

■

3.26. Sonya (observer O) is moving away from Sam (observer O') with speed β along his positive x'-axis. At time t', Sam sends a light signal to Sonya (event E), which gets reflected from a reflector at Sonya's origin (event $E_{\rm ref}$), and received by a detector at Sam's origin (event $E_{\rm det}$).

(a) Find (x, t), the coordinates of E according to Sonya in terms of t'.

(b) Find $t_{\rm ref}$, the time of $E_{\rm ref}$ according to Sonya in terms of t'.

(c) Find $(x'_{\rm ref}, t'_{\rm ref})$, the coordinates of $E_{\rm ref}$ according to Sam in terms of t'.

(d) Show that

$$t'_{\rm det} = \frac{1+\beta}{1-\beta}\, t'$$

Solution: Event E has coordinates $(0, t')$ in Sam's RF, and Sam is moving in the negative x direction of Sonya.

(a)

$$x = \gamma(0 - \beta ct') = -\gamma\beta ct', \quad t = \gamma(ct' - 0) = \gamma ct'$$

(b) The reflection occurs immediately after the signal hits the reflector. Thus, $t_{\rm ref} = t = \gamma ct'$. Therefore, $E_{\rm ref}$ has coordinates $(0, \gamma ct')$ according to Sonya.

(c)

$$\begin{aligned} x'_{\rm ref} &= \gamma(0 + \beta ct_{\rm ref}) = \gamma^2 \beta ct' \\ ct'_{\rm ref} &= \gamma(ct_{\rm ref} + 0) = \gamma^2 ct' \end{aligned}$$

(d)

$$ct'_{\rm det} = ct'_{\rm ref} + |x'_{\rm ref}| = \gamma^2 ct' + \gamma^2 \beta ct' = \frac{1+\beta}{1-\beta^2} ct' = \frac{1+\beta}{1-\beta} ct'$$

■

3.27. Speeder O is in a spacecraft moving away from a policeman (observer O') with speed β. The policeman sends an EM signal of wavelength λ to O and receives the reflected wave of wavelength $\lambda_{\rm ref}$.

(a) Use the result of Problem 3.26 for the emission two successive wave crests to show that

$$\lambda_{\rm ref} = \frac{1+\beta}{1-\beta}\, \lambda.$$

(b) The speed limit in Metropolis is half the speed of light. Will the speeder get a ticket if the policeman sends a violet signal of 400 nm and receives an infrared signal of 1.3 μm?

Solution:

(a) The result is immediate.

(b) With $\lambda = 400$ nm and $\lambda_{\text{ref}} = 1300$ nm, we get

$$1300 = \frac{1+\beta}{1-\beta}400 \iff \frac{1+\beta}{1-\beta} = \frac{13}{4} \iff \beta = \frac{9}{17} > 0.5.$$

■

3.28. Sam is on a train of rest length L_0 moving at speed β relative to Sonya standing on the ground. Consider the time interval between the front of the train coinciding with Sonya and the back of the train coinciding with her.

(a) Find this time interval in Sonya's frame by calculating in her frame.

(b) Find this time interval in Sonya's frame by calculating in Sam's frame.

(c) Find the time interval in Sam's frame by calculating in Sonya's frame.

(d) Find the time interval in Sam's frame by calculating in his frame.

Solution: Let $\Delta t'$ be the time interval according to Sonya and Δt according to Sam.

(a) The train's length is L_0/γ according to Sonya and moves at $c\beta$. So, she calculates $c\Delta t'$ to be $(L_0/\gamma)/\beta$.

(b) For Sam the time interval between coincidences is $c\Delta t = L_0/\beta$. He concludes that since Sonya is measuring proper time, for her the time interval is $c\Delta t' = (L_0/\beta)/\gamma$.

(c) Sonya know that she is calculating proper time. So, she concludes that Sam measures the time interval to be $c\Delta t = \gamma c\Delta t'$ or $[(L_0/\gamma)/\beta]\gamma = L_0/\beta$.

(d) Sam sees Sonya move from the front to back with speed $c\beta$. So, he measures the time to be $c\Delta t = L_0/\beta$.

It is instructive to use Lorentz transformation to find the answers. The two events according to Sonya are $(0, 0)$ and $(0, \Delta t')$ and according to Sam, they are $(0, 0)$ and $(-L_0, \Delta t)$. Write the Lorentz transformation for the second event:

$$0 = \Delta x' = \gamma(-L_0 + \beta c\Delta t), \quad c\Delta t' = \gamma(c\Delta t - \beta L_0).$$

The first equation gives $c\Delta t = L_0/\beta$ and plugging this in the second equation yields

$$c\Delta t' = \gamma\left(\frac{L_0}{\beta} - \beta L_0\right) = \frac{\gamma L_0}{\beta}(1-\beta^2) = \frac{L_0}{\gamma\beta}.$$

This should show you once again the power of Lorentz transformation. Once you specify your events, you don't have to second guess. Lorentz transformation will take care of the rest! ■

3.29. Sam and Sonya sit in two identical rockets of length L. Sam approaches Sonya from behind with relative speed β along the length of their rockets. Assume that at time zero for both, the nose of Sam's rocket coincides with the tail of Sonya's. Now consider three evens occurring in succession: Event A is when the tail of Sam's rocket coincides with the tail of Sonya's; B is when the nose of Sam's rocket coincides with the nose of Sonya's; C is when the tail of Sam's rocket coincides with the nose of Sonya's. Using only Lorentz transformation and inverse Lorentz transformation (no length contraction or time dilation) find the coordinates of these three events in Sam's and Sonya's RFs paying attention to signs.

Solution: Let Sam be O and Sonya O'. Since the coincidence of the nose of Sam's rocket occurs at the tail of Sonya's at $t = t' = 0$, it is convenient to have Sam's space origin at the nose and Sonya's space origin at the tail. Write all the coordinates in the two frames inserting all the info given. Then in O, events A, B, and C have coordinates

$$(-L, ct_A), \quad (0, ct_B), \quad (-L, ct_C)$$

and in O' they have coordinates

$$(0, ct'_A), \quad (L, ct'_B), \quad (L, ct'_C).$$

Now Lorentz transform the events. For A, we get

$$0 = x'_A = \gamma(-L + \beta ct_A), \quad ct'_A = \gamma(ct_A - \beta L).$$

The first equation gives $ct_A = L/\beta$, and pugging this in the second equation yields

$$ct'_A = \gamma\left(\frac{L}{\beta} - \beta L\right) = \frac{\gamma L}{\beta}(1 - \beta^2) = \frac{L}{\gamma\beta}.$$

For B, we get

$$L = \gamma(0 + \beta ct_B), \quad ct'_B = \gamma(ct_B - 0).$$

Finding ct_B from first and substituting in the second, we get $ct_B = L/(\gamma\beta)$ and $ct'_B = L/\beta$. Finally, for C, we get

$$L = \gamma(-L + \beta ct_C), \quad ct'_C = \gamma(ct_C - \beta L).$$

These equations yield

$$ct_C = \frac{(1+\gamma)L}{\gamma\beta}, \quad ct'_C = \gamma\left[\frac{(1+\gamma)L}{\gamma\beta} - \beta L\right].$$

The second equation can be simplified:

$$ct'_C = \frac{(1+\gamma)L}{\beta} - \gamma\beta L = \frac{(1+\gamma)L - \gamma\beta^2 L}{\beta} = \frac{L + \gamma L(1-\beta^2)}{\beta} = \frac{(1+\gamma)L}{\gamma\beta}.$$

The equality $t'_C = t_C$ should come as no surprise because of the symmetry of the problem: They both see nose-tail to tail-nose coincidences during the entire motion (one in reverse order). You should convince yourself that the results make sense based on length contraction and time dilation. ■

3.30. Sam is on a train of rest length L_0 moving at speed β relative to Sonya standing on the ground. He fires a gun from the back of the train to the front. The speed of the bullet relative to the train is β_b.

(a) How much time does the bullet spend in the air before hitting the front of the train according to Sonya?

(b) What is the distance that the bullet travels according to Sonya?

Solution: I'll do the problem in two ways. First I use length contraction and the law of addition of velocities. Then I use Lorentz transformation to show you its power!

(a) According to Sonya, the length of the train is L_0/γ and the speed of the bullet is $\beta_b' = (\beta_b + \beta)/(1 + \beta_b\beta)$. The bullet is catching up with the front of the train which is moving away from it with speed β, all according to Sonya. So,

$$\frac{L_0}{\gamma} + c\beta\Delta t = c\beta_b'\Delta t \iff c\Delta t = \frac{L_0}{\gamma(\beta_b' - \beta)}.$$

But

$$\beta_b' - \beta = \frac{\beta_b + \beta}{1 + \beta_b\beta} - \beta = \frac{\beta_b + \beta - \beta - \beta^2\beta_b}{1 + \beta_b\beta} = \frac{\beta_b}{\gamma^2(1 + \beta_b\beta)}.$$

Therefore,

$$c\Delta t = \frac{L_0}{\gamma(\beta_b' - \beta)} = \frac{L_0\gamma(1 + \beta_b\beta)}{\beta_b}.$$

(b)

$$\Delta x = c\beta_b'\Delta t = c\left(\frac{\beta_b + \beta}{1 + \beta_b\beta}\right)\left(\frac{L_0\gamma(1 + \beta_b\beta)}{\beta_b}\right) = \frac{L_0\gamma(\beta_b + \beta)}{\beta_b}.$$

Now let's do the problem using Lorentz transformation. In Sam's RF, the event of the bullet hitting the front of the train has coordinates $(L_0, L_0/\beta_b)$. Lorentz transforming these, we get

$$\Delta x = \gamma(L_0 + \beta L_0/\beta_b), \quad c\Delta t = \gamma(L_0/\beta_b + \beta L_0).$$

And that's it! ■

3.31. Sam and Sonya are newborn twins. Sam is placed on a rocket that moves at speed $0.99c$ to a star system 25 light years away. At the moment of his departure a light signal is sent to the star system and gets reflected.

(a) How old is Sam when he receives the reflected signal? How old is Sonya?

(b) How old is Sonya when *she* receives the reflected signal? How old is Sam?

(c) How old is Sam when he reaches the star system?

(d) How old is Sonya when Sam reaches the star system?

Hint: See Example 3.4.9.

Solution: The hint makes this just a plug and chug problem! But you should do it anyway to get a feel for the numbers. ■

3.32. Multiply both sides of the inequality $\beta_b < 1$ by the positive quantity $1 - \beta$ and show that $\beta + \beta_b < 1 + \beta\beta_b$. From this conclude that the relativistic law of addition of velocities never yields a speed larger than the speed of light.

Solution:

$$\beta_b(1 - \beta) < 1 - \beta \iff \beta_b - \beta_b\beta < 1 - \beta \iff \beta_b + \beta < 1 + \beta_b\beta.$$

■

3.33. A super-ball has the property that when it hits a wall with a given speed *relative to the wall*, it bounces back with the same speed in the opposite direction. What do you measure the speed of the bounced ball to be if you throw it at speed β_s towards a wall which is moving towards you at speed β_w?

Solution: The speed of the super-ball relative to the wall is

$$\beta_{sw} = \frac{\beta_s + \beta_w}{1 + \beta_s \beta_w}.$$

Therefore, it bounces back *relative to the wall* with speed β_{sw}. To find the speed β_s' of the bounced super-ball relative to you, you have to *add* the speeds again:

$$\beta_s' = \frac{\beta_{sw} + \beta_w}{1 + \beta_{sw}\beta_w} = \frac{\beta_s + \beta_w + \beta_w(1 + \beta_s\beta_w)}{1 + \beta_s\beta_w + (\beta_s + \beta_w)\beta_w} = \frac{\beta_s + 2\beta_w + \beta_s\beta_w^2}{1 + 2\beta_s\beta_w + \beta_w^2}.$$

It is interesting to note that $\beta_s' \to 1$ when $\beta_s \to 1$, regardless of the speed of the wall! This is, of course, consistent with the second postulate of relativity. ■

3.34. In an intergalactic race, team A is moving at speed $0.8c$ relative to the finish line. They notice that a faster team B passes them at $0.9c$. Team B observes another team C to pass them at $0.95c$. What are the speeds of teams B and C relative to the finish line?

Solution:

$$\begin{aligned}\beta_B &= \frac{0.8 + 0.9}{1 + 0.8 \times 0.9} = \frac{1.7}{1.72} = 0.9884 \\ \beta_C &= \frac{0.9884 + 0.95}{1 + 0.9884 \times 0.95} = \frac{1.9384}{1.939} = 0.9997.\end{aligned}$$

■

3.35. Sam sees Sonya and Pat flying in opposite directions with a speed $\beta = 0.995$. What is the speed of Pat relative to Sonya?

Solution:

$$\beta_{ps} = \frac{0.995 + 0.995}{1 + 0.995 \times 0.995} = 0.999987.$$

■

3.36. Two spaceships approach each other with relative speed $0.9c$. What are the velocities of the spaceships relative to Earth assuming that they move with the same speed relative to Earth?

Solution: Let β_{AB} be the speed of spaceship A relative to spaceship B. Let β_{AE} be the speed of A relative to Earth and β_{EB} the speed of Earth relative to B. Then the law of addition of velocities can be written as

$$\beta_{AB} = \frac{\beta_{AE} + \beta_{EB}}{1 + \beta_{AE}\beta_{EB}}.$$

Note the order of subscripts. Assume that A is moving in the positive direction relative to Earth. Then B is moving in the negative direction. This means that $\beta_{BE} < 0$. Therefore, $\beta_{EB} = -\beta_{BE} > 0$. Furthermore, by assumption, $\beta_{AE} = \beta_{EB}$. So, the above equation becomes

$$0.9 = \frac{2\beta_{AE}}{1 + \beta_{AE}^2} \iff 0.9\beta_{AE}^2 - 2\beta_{AE} + 0.9 = 0 \iff \beta_{AE} = \frac{1 \pm \sqrt{1 - 0.9^2}}{0.9}$$

or $\beta_{AE} = 0.627$. ■

3.37. Observer O moves in the positive x'-direction of observer O' with speed β. Observer O' moves in the positive x''-direction of observer O'' with speed β'. Use the relativistic law of addition of velocities (3.26) to obtain γ'' in terms of β, β', γ, and γ'.

Solution: Substitute

$$\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'}$$

in the definition of γ'':

$$\begin{aligned}
\gamma'' &= \frac{1}{\sqrt{1 - \left(\frac{\beta+\beta'}{1+\beta\beta'}\right)^2}} = \frac{1 + \beta\beta'}{\sqrt{(1+\beta\beta')^2 - (\beta+\beta')^2}} \\
&= \frac{1+\beta\beta'}{\sqrt{(1-\beta^2)(1-\beta'^2)}} = \gamma\gamma'(1+\beta\beta').
\end{aligned}$$

■

3.38. All motions are in the positive x-direction. Observer O_1 moves relative to observer O with speed β_1. Observer O_2 moves relative to O_1 with speed β_2. Let $P_2^+ \equiv (1+\beta_1)(1+\beta_2)$ and $P_2^- \equiv (1-\beta_1)(1-\beta_2)$.

(a) Show that O_2 moves relative to O with speed β_2' given by

$$\beta_2' = \frac{P_2^+ - P_2^-}{P_2^+ + P_2^-}.$$

(b) Now introduce another observer O_3 moving relative to O_2 with speed β_3, and let

$$\begin{aligned}
P_3^+ &= (1+\beta_1)(1+\beta_2)(1+\beta_3) \\
P_3^- &= (1-\beta_1)(1-\beta_2)(1-\beta_3).
\end{aligned}$$

Use the relativistic law of addition of velocities for β_2' and β_3 in the form given in (a) to prove that O_3 moves relative to O with speed β_3' given by

$$\beta_3' = \frac{P_3^+ - P_3^-}{P_3^+ + P_3^-}.$$

(c) Can you predict what happens if you introduce a fourth observer?

(d) If you are familiar with mathematical induction, prove the formula for N observers.

Solution:

(a)

$$\begin{aligned}
\frac{P_2^+ - P_2^-}{P_2^+ + P_2^-} &= \frac{(1+\beta_1)(1+\beta_2) - (1-\beta_1)(1-\beta_2)}{(1+\beta_1)(1+\beta_2) + (1-\beta_1)(1-\beta_2)} \\
&= \frac{2(\beta_1+\beta_2)}{2(1+\beta_1\beta_2)} = \beta_2'.
\end{aligned}$$

(b)

$$\beta_3' = \frac{\beta_2' + \beta_3}{1 + \beta_2'\beta_3} = \frac{\dfrac{P_2^+ - P_2^-}{P_2^+ + P_2^-} + \beta_3}{1 + \dfrac{P_2^+ - P_2^-}{P_2^+ + P_2^-}\beta_3} = \frac{P_2^+ - P_2^- + (P_2^+ + P_2^-)\beta_3}{P_2^+ + P_2^- + (P_2^+ - P_2^-)\beta_3}$$

$$= \frac{P_2^+(1+\beta_3) - P_2^-(1-\beta_3)}{P_2^+(1+\beta_3) + P_2^-(1-\beta_3)} = \frac{P_3^+ - P_3^-}{P_3^+ + P_3^-}$$

(c)

$$\beta_4' = \frac{\beta_3' + \beta_4}{1 + \beta_3'\beta_4} = \frac{\dfrac{P_3^+ - P_3^-}{P_3^+ + P_3^-} + \beta_4}{1 + \dfrac{P_3^+ - P_3^-}{P_3^+ + P_3^-}\beta_4} = \frac{P_3^+ - P_3^- + (P_3^+ + P_3^-)\beta_4}{P_3^+ + P_3^- + (P_3^+ - P_3^-)\beta_4}$$

$$= \frac{P_3^+(1+\beta_4) - P_3^-(1-\beta_4)}{P_3^+(1+\beta_4) + P_3^-(1-\beta_4)} = \frac{P_4^+ - P_4^-}{P_4^+ + P_4^-}$$

(d) In proof by mathematical induction, you show that the equality holds for some particular value, usually for a low value like $N = 1$ or $N = 2$. Then *assume* that it holds for $N - 1$ and *prove* that it holds for N. We have seen that the equation holds for $N = 2$. Now assume that the equality holds for $N - 1$, i.e., assume that the following is true:

$$\beta_{N-1}' = \frac{P_{N-1}^+ - P_{N-1}^-}{P_{N-1}^+ + P_{N-1}^-}.$$

Now we have to *prove* that

$$\beta_N' = \frac{P_N^+ - P_N^-}{P_N^+ + P_N^-}$$

is true. To do so, we use the law of addition of velocities:

$$\beta_N' = \frac{\beta_{N-1}' + \beta_N}{1 + \beta_{N-1}'\beta_N} = \frac{\dfrac{P_{N-1}^+ - P_{N-1}^-}{P_{N-1}^+ + P_{N-1}^-} + \beta_N}{1 + \dfrac{P_{N-1}^+ - P_{N-1}^-}{P_{N-1}^+ + P_{N-1}^-}\beta_N} = \frac{P_{N-1}^+ - P_{N-1}^- + (P_{N-1}^+ + P_{N-1}^-)\beta_N}{P_{N-1}^+ + P_{N-1}^- + (P_{N-1}^+ - P_{N-1}^-)\beta_N}$$

$$= \frac{P_{N-1}^+(1+\beta_N) - P_{N-1}^-(1-\beta_N)}{P_{N-1}^+(1+\beta_N) + P_{N-1}^-(1-\beta_N)} = \frac{P_N^+ - P_N^-}{P_N^+ + P_N^-}$$

■

3.39. Recall that **hyperbolic** functions are defined by

$$\cosh\phi = \frac{e^{\phi} + e^{-\phi}}{2}, \quad \sinh\phi = \frac{e^{\phi} - e^{-\phi}}{2}.$$

(a) Using these definitions, show the following properties

$$
\begin{aligned}
\cosh^2\phi - \sinh^2\phi &= 1 \\
\sinh(\phi_1+\phi_2) &= \sinh\phi_1\cosh\phi_2 + \sinh\phi_2\cosh\phi_1 \\
\cosh(\phi_1+\phi_2) &= \cosh\phi_1\cosh\phi_2 + \sinh\phi_1\sinh\phi_2 \\
\tanh(\phi_1+\phi_2) &= \frac{\tanh\phi_1+\tanh\phi_2}{1+\tanh\phi_1\tanh\phi_2}
\end{aligned}
$$

(b) Now define the **rapidity** ϕ by $\tanh\phi = \beta$ and show that

$$\cosh\phi = \gamma, \quad \sinh\phi = \beta\gamma, \quad e^{\phi} = \left(\frac{1+\beta}{1-\beta}\right)^{1/2}.$$

Solution:

(a) From

$$\cosh^2\phi = \frac{e^{2\phi}+e^{-2\phi}+2}{4}, \quad \sinh^2\phi = \frac{e^2\phi + e^{-2\phi}-2}{2}$$

the first equality follows immediately. For the second equality, note that

$$e^{\phi} = \cosh\phi + \sinh\phi, \quad e^{-\phi} = \cosh\phi - \sinh\phi$$

Therefore,

$$
\begin{aligned}
\sinh(\phi_1+\phi_2) &= \frac{e^{\phi_1+\phi_2} - e^{-\phi_1-\phi_2}}{2} = \frac{e^{\phi_1}e^{\phi_2} - e^{-\phi_1}e^{-\phi_2}}{2} \\
&= \frac{(\cosh\phi_1+\sinh\phi_1)(\cosh\phi_2+\sinh\phi_2)}{2} \\
&\quad - \frac{(\cosh\phi_1-\sinh\phi_1)(\cosh\phi_2-\sinh\phi_2)}{2}.
\end{aligned}
$$

The identity now follows immediately. The identity involving $\cosh(\phi_1+\phi_2)$ can be derived similarly. For $\tanh(\phi_1+\phi_2)$, divide both sides of the $\sinh(\phi_1+\phi_2)$ and $\cosh(\phi_1+\phi_2)$ identities to get $\tanh(\phi_1+\phi_2)$ on the left. Then divide both numerator and the denominator on the right by $\cosh\phi_1\cosh\phi_2$.

(b) Divide both sides of $\cosh^2\phi - \sinh^2\phi = 1$ by $\cosh^2\phi$ to get

$$1-\tanh^2\phi = \frac{1}{\cosh^2\phi} \iff 1-\beta^2 = \frac{1}{\cosh^2\phi} \iff \cosh\phi = \gamma.$$

Then,

$$\beta = \tanh\phi = \frac{\sinh\phi}{\cosh\phi} = \frac{\sinh\phi}{\gamma} \iff \sinh\phi = \beta\gamma.$$

Finally,

$$e^{\phi} = \cosh\phi + \sinh\phi = \gamma + \beta\gamma = \frac{1+\beta}{\sqrt{1-\beta^2}} = \left(\frac{1+\beta}{1-\beta}\right)^{1/2}.$$

$\blacksquare$

3.40. Show that in terms of rapidity of Problem 3.39, the LT can be written as

$$\begin{aligned} x' &= x\cosh\phi + ct\sinh\phi \\ ct' &= x\sinh\phi + ct\cosh\phi. \end{aligned}$$

Solution: With $\gamma = \cosh\phi$ and $\beta\gamma = \sinh\phi$, the result is immediate. ■

3.41. Redo Problem 3.37 using LTs in terms of rapidities. How is the rapidity ϕ'' of O'' relative to O related to the rapidity ϕ' of O'' relative to O' and the rapidity ϕ of O' relative to O? The identities in Problem 3.39 may be useful.

Solution: The law of addition of velocities can be written a

$$\tanh\phi'' = \frac{\tanh\phi' + \tanh\phi}{1 + \tanh\phi'\tanh\phi} = \tanh(\phi' + \phi),$$

i.e., rapidities add. ■

3.42. Derive the following relations among the coordinates of the RFs O and O':

$$x' + ct' = e^{\phi}(x + ct), \quad x' - ct' = e^{-\phi}(x - ct),$$

where ϕ is the rapidity as defined in Problem 3.39. Now define two new coordinates (ξ, η) as

$$\xi = x + ct, \quad \eta = x - ct,$$

and show that they are perpendicular to each other (in the Euclidean sense). Show that under a LT, ξ and η do not change direction, but their calibration changes. By how much?

Solution: Add the two sides of the Lorentz transformation written in terms of rapidity. On the left-hand side you get $x' + ct'$ and on the right-hand side

$$x\underbrace{(\cosh\phi + \sinh\phi)}_{=e^{\phi}} + ct\underbrace{(\sinh\phi + \cosh\phi)}_{=e^{\phi}} = e^{\phi}(x + ct).$$

If you subtract, you get

$$x' - ct' = x(\cosh\phi - \sinh\phi) + ct(\cosh\phi - \sinh\phi) = e^{-\phi}(x - ct).$$

The coordinate ξ is defined by $\eta = 0$ or $x = ct$, which is the equation of the world line of light moving in the positive x-direction. The coordinate η is defined by $\xi = 0$ or $x = -ct$, which is the equation of the world line of light moving in the negative x-direction. And these two world lines make a $90°$ with one another.

Under a Lorentz transformation, $\xi \to \xi' = e^{\phi}\xi$, so its scale changes by a factor of e^{ϕ}, and $\eta \to \eta' = e^{-\phi}\eta$, so its scale changes by a factor of $e^{-\phi}$. ■

3.43. The speed limit in Metropolis is $0.8c$. A driver is moving at $0.92c$ (with respect to the ground) as he passes a policeman without noticing him. The policeman, accelerating to $0.996c$ (also with respect to the ground) instantaneously, catches up with the driver 10 minutes later according to his own clock. The origins of the RFs of both observers are when and where they pass each other.

(a) What are the coordinates (in the policeman's RF) of the event at which the policeman starts chasing the driver?

(b) What are the coordinates of that event in the driver's RF?

(c) How long after the driver passes the policeman does the policeman catch up with him according to the driver's clock?

Solution:

(a) The coordinates are $(0, 10 \text{ light minutes})$.

(b) In the driver's RF, with $\beta = -0.92$, $\gamma = 2.55$,

$$\begin{aligned} x &= 2.55(0 - 0.92 \times 10) = 23.47 \text{ light minutes} \\ ct &= 2.55(\times 10 - 0) = 25.5 \text{ light minutes.} \end{aligned}$$

(c) When the policeman starts chasing the driver, he will be moving in the driver's positive direction with speed

$$\beta = \frac{0.996 - 0.92}{1 - 0.996 \times 0.92} = 0.908.$$

According to the driver, the policeman is 23.47 light minutes away and approaching him at $0.908c$. So, it takes him

$$\frac{23.47 \text{ light minutes}}{0.908c} = 25.85 \text{ minutes}$$

to catch up with the driver after he starts chasing him. So, $25.5 + 25.85 = 51.35$ minutes after the driver passes the policeman, the policeman catches up with him.

■

3.44. The acceleration of a "bullet" in O' is $a'_b = dv'_b/dt'$ and in O, moving relative to O' with constant speed β in the positive direction of the x'-axis is $a_b = dv_b/dt$. Use (infinitesimal) LT and Equation (3.26) to show that

$$a'_b = \frac{a_b}{\gamma^3(1 + \beta\beta_b)^3}$$

Solution: The differential of Equation (3.26) is

$$d\beta'_b = \frac{(1 + \beta\beta_b)d\beta_b - (\beta_b + \beta)\beta d\beta_b}{(1 + \beta\beta_b)^2} = \frac{d\beta_b}{\gamma^2(1 + \beta\beta_b)^2},$$

and

$$cdt' = \gamma(cdt + \beta dx) = \gamma cdt(1 + \beta\beta_b) \iff dt' = \gamma dt(1 + \beta\beta_b).$$

Therefore,

$$a'_b = \frac{cd\beta'_b}{dt'} = \frac{\dfrac{cd\beta_b}{\gamma^2(1 + \beta\beta_b)^2}}{\gamma dt(1 + \beta\beta_b)} = \frac{a_b}{\gamma^3(1 + \beta\beta_b)^3}.$$

■

3.45. In Problem 3.44, let $\beta'_b = \beta$, that is let O be moving with the bullet. Furthermore, assume that the acceleration a_b is constant. Call it a.

(a) Show that (ignoring the prime on t)

$$adt = \frac{cd\beta}{(1-\beta^2)^{3/2}}.$$

(b) Integrate the equation above and show that if $\beta = 0$ at $t = 0$, then

$$\beta = \frac{at}{\sqrt{c^2 + a^2t^2}}.$$

How long do you have to wait for the bullet to reach a speed of $0.99c$ if the acceleration is 5 m/s^2? If you wait long enough, can you accelerate the bullet to a speed larger than c?

(c) Ignoring the prime on x as well and noting that $c\beta = v = dx/dt$, find x as a function of time assuming that $x = 0$ at $t = 0$. What kind of a curve do you get on the xt-plane?

(d) How long does it take the "bullet" to reach Alpha Centauri 4 light years away if the acceleration of the bullet is 5 m/s^2? What is its speed when it reaches there?

Solution: If $\beta'_b = \beta$, the $\beta_b = 0$, and

$$a'_b = \frac{a_b}{\gamma^3} \iff cd\beta/dt = \frac{a}{\gamma^3}.$$

(a) Therefore,

$$adt = \gamma^3 cd\beta \iff adt = \frac{cd\beta}{(1-\beta^2)^{3/2}}.$$

(b) Integrate both sides and choose the constant of integration so that $\beta = 0$ at $t = 0$. Then

$$at = \frac{c\beta}{\sqrt{1-\beta^2}} \iff a^2t^2 = \frac{c^2\beta^2}{1-\beta^2} \iff \beta = \frac{at}{\sqrt{c^2+a^2t^2}}.$$

In units of c, the acceleration is

$$\alpha \equiv a/c = \frac{5\ \text{m/s}^2}{3\times 10^8\ \text{m/s}} = 1.67\times 10^{-8}\ \text{s}^{-1} = 0.526\ \text{yr}^{-1}.$$

So, using the first equality in the preceding equation, we get

$$0.526\ \text{yr}^{-1} = \frac{0.99}{\sqrt{1-0.99^2}} = 7.02 \iff t = \frac{7.02}{0.526\ \text{yr}^{-1}} = 13.35\ \text{yr}.$$

Also it is clear that $\beta \to 1$ as $t \to \infty$; so you cannot accelerate to the speed of light in a finite time.

(c) From (b)

$$\frac{dx}{dt} = \frac{cat}{\sqrt{c^2+a^2t^2}} \iff dx = \frac{catdt}{\sqrt{c^2+a^2t^2}} = \frac{c\alpha tdt}{\sqrt{1+\alpha^2t^2}}.$$

Integration and the condition that $x = 0$ at $t = 0$ yield

$$x = \frac{c}{\alpha}(\sqrt{1+(\alpha t)^2} - 1).$$

You can easily show that

$$\left(\frac{\alpha x}{c} + 1\right)^2 - (\alpha t)^2 = 1,$$

which is a hyperbola.

(d) Note that

$$\frac{\alpha x}{c} = \frac{\alpha(4 \text{ light years})}{c} = 0.526 \text{ yr}^{-1}(4 \text{ years}) = 2.1.$$

So, the last equation yields

$$3.1^2 - (\alpha t)^2 = 1, \iff \alpha t = 2.94 \iff t = \frac{2.94}{0.526 \text{ yr}^{-1}} = 5.59 \text{ yr}.$$

From (b), we have

$$\beta = \frac{at}{\sqrt{c^2 + a^2 t^2}} = \frac{\alpha t}{\sqrt{1 + (\alpha t)^2}} = \frac{2.94}{\sqrt{1 + 2.94^2}} = 0.9467.$$

■

CHAPTER 4

Spacetime Geometry

Problems With Solutions

4.1. When axes are perpendicular to each other, it does not matter whether you draw lines that are parallel or perpendicular to the axes to find the coordinates of a point. Why can't you draw *perpendicular* lines when axes are not at right angles to each other? Hint: If a point lies on an axis, what do you expect its "other" coordinate to be?

Solution: When you drop a perpendicular line from a point of one axis to another axis it crosses the latter. That cannot happen because points on one axis should have zero coordinates corresponding to the other axis. ■

4.2. Consider a non-perpendicular coordinate system with axes x and y. Take any two points P_1 and P_2 with coordinates (x_1, y_1) and (x_2, y_2), respectively. Show that if

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

then the axes must be perpendicular.

Solution: Figure 4.1 of the manual shows the two points and their coordinates in a non-perpendicular coordinate system. The law of cosines gives

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\theta.$$

Therefore,

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \iff (x_2 - x_1)(y_2 - y_1)\cos\theta = 0 \iff \cos\theta = 0,$$

because $x_2 \neq x_1$ and $y_2 \neq y_1$. ■

4.3. Draw the x and ct axes with acute angles. Draw a wavy line through the origin to represent the world line of a light signal. Show that if the speed of this light signal is to be c, then its world line has to make equal angles with both axes. That is, it has to be the bisector of the angle between the x and ct axes.

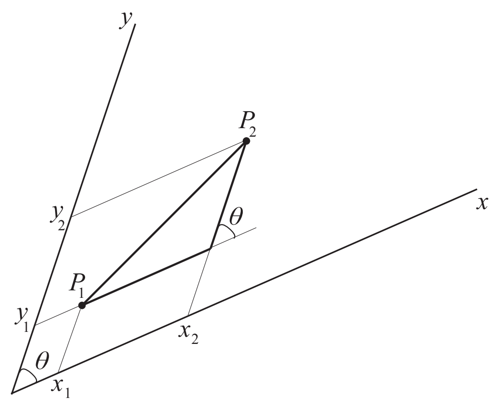


Figure 4.1: Two points and their coordinates in a non-perpendicular set of axes.

Solution: I'll use a dashed line instead of a wavy line. The event E in Figure 4.2 of the manual, lying on the worldline of a light signal, covers a distance of x in time t. Therefore, $x = ct$ and the two triangles shown must be isosceles. Therefore all the angles shown must be equal. ■

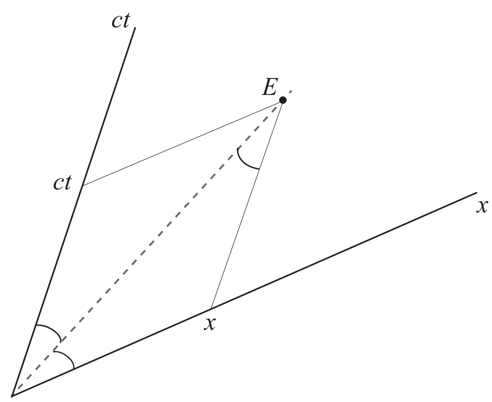


Figure 4.2: The event E covers a distance of x in time t.

4.4. A train of rest length L_0 travels at speed β in the positive x'-direction of Sam. As the front of the train passes Sam at $t' = 0$, a light signal is sent from the front of the train to the rear.

(a) Draw a spacetime diagram showing the worldlines of the front and rear of the train and the photon in Sam's RF.

(b) When does the rear of the train pass Sam?

(c) When does the signal reach the rear of the train according to Sam?

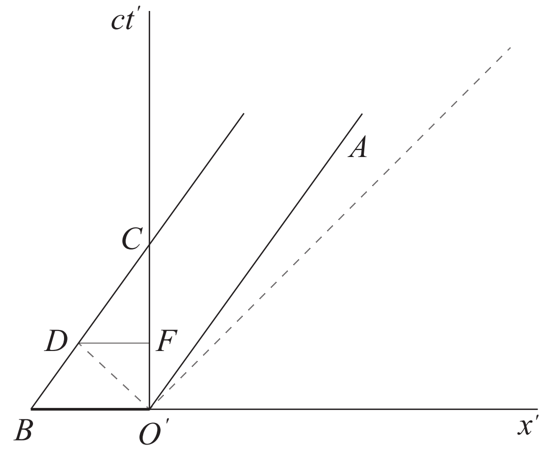


Figure 4.3: The worldlines of the front and back of the train are the lines $O'A$ and BC, respectively.

Solution:

(a) The worldlines of the front and back of the train are, respectively, the lines $O'A$ and BC of Figure 4.3 of the manual. The train has a length of L_0/γ according to Sam. So, in the figure, the line segment $\overline{O'B}$ is that length.

(b) By Rule 2 of Note 4.2.6, $\beta = \overline{BO'}/\overline{O'C}$ or $\beta = (L_0/\gamma)/ct'_2$, where t'_2 is when the back of the train reaches Sam. Therefore, $ct'_2 = L_0/(\beta\gamma)$.

(c) By Rule 2 of Note 4.2.6, $\beta = \overline{DF}/\overline{FC}$. Therefore, $\overline{FC} = \overline{DF}/\beta$. But $\overline{DF} = \overline{O'F}$ and, by (b) $ct'_2 = \overline{O'F} + \overline{FC}$. Hence,

$$\overline{O'F} = ct'_2 - \overline{FC} = \frac{L_0}{\beta\gamma} - \frac{\overline{O'F}}{\beta} \iff \frac{\overline{O'F}}{\beta}(1+\beta) = \frac{L_0}{\beta\gamma} \iff \overline{O'F} = \frac{L_0}{(1+\beta)\gamma},$$

and $\overline{O'F} \equiv ct'_1$ is the time (times c) that the signal reaches the back of the train according to Sam.

It is instructive to check our answers by using Lorentz transformation. So, place Sonya in the train and call her observer O. Let the front of the train be $x = 0$, and assume that at $t = 0 = t'$, the front of the train coincides with Sam. Then according to Sonya, the light reaches the back of the train at $(-L_0, L_0)$ and the back of the train reaches Sam at $(-L_0, L_0/\beta)$. Therefore, the first event has coordinates

$$\begin{aligned} x'_1 &= \gamma(-L_0 + \beta L_0) = -\gamma(1-\beta)L_0 \\ ct'_1 &= \gamma(L_0 - \beta L_0) = \gamma(1-\beta)L_0 = \frac{\gamma(1-\beta)(1+\beta)L_0}{1+\beta} = \frac{L_0}{\gamma(1+\beta)}, \end{aligned}$$

in Sam's RF, as before. Note that although we didn't calculate x'_1 before, the spacetime diagram clearly shows that $x'_1 = -\overline{DF} = -\overline{O'F} = -ct'_1$.

The second event has coordinates

$$\begin{aligned} x'_2 &= \gamma(-L_0 + \beta L_0/\beta) = 0 \\ ct'_2 &= \gamma(L_0/\beta - \beta L_0) = \frac{\gamma}{\beta}(1-\beta^2)L_0 = \frac{L_0}{\gamma\beta}, \end{aligned}$$

which is identical to what we obtained earlier. ■

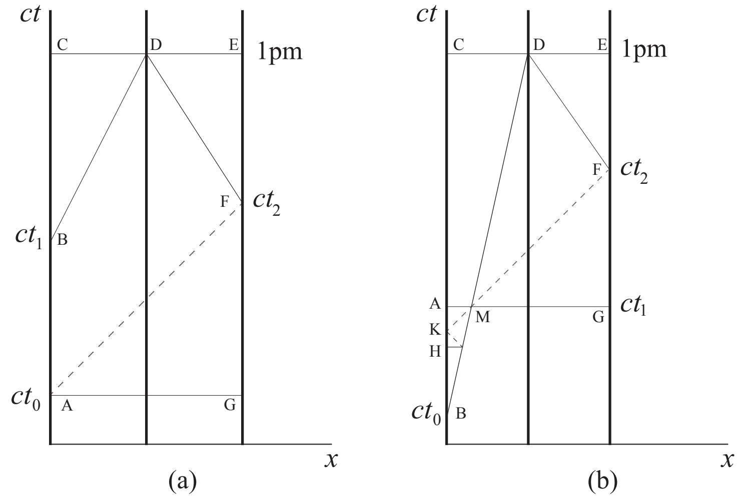


Figure 4.4: The spacetime diagram of Sonya-Sam meeting at 1pm. The worldline of Sonya's planet is the ct axis. (a) Sonya's speed is $0.5c$. (b) Sonya's speed is $0.2c$.

4.5. Sonya and Sam live on two different planets one light hour apart. A space station is located half way between the two planets. The planets and the space station are all stationary relative to each other. Sonya wants to arrange a meeting with Sam at 1:00 PM on the space station. She decides to send him a light signal so that as soon as he receives it he starts to move toward the space station to get there at exactly 1:00 PM. Sonya's spaceship moves at half the speed of light while Sam's moves at $0.75c$. All times are according to the common reference frame of the planets and the space station.

(a) At what time should Sonya send the signal?

(b) At what time should she leave her planet?

(c) At what time should Sam leave his planet?

Solution: Figure 4.4(a) of the manual shows the relevant spacetime diagram. Just to let you know, I drew the lines backward: I started at D and drew Sam and Sonya's worldlines. Then from F, I drew a light signal toward the ct axis. I'll have to do (b) and (c) before I can do (a)!

(b) By Rule 2 of Note 4.2.6, $\beta = \overline{CD}/\overline{CB}$, or $\overline{CB} = 0.5$ light hour$/0.5c = 1$ hour. So, Sonya has to start her trip at 12:00 pm.

(c) Similarly, $\beta = \overline{DE}/\overline{EF}$ gives $\overline{EF} = 0.5$ light hour$/0.75c$ or $\overline{EF} = 40$ minutes. So, Sam has to leave at 12:20 pm.

(a) From the equality $\overline{GF} = \overline{AG} = 1$ light hour, we conclude that t_0 is one hour before t_2. So, Sonya has to send the signal at 11:20 am.

■

4.6. Same as the previous problem except that Sonya's spaceship moves at $0.2c$.

(a) At what time should Sonya leave her planet?

(b) At what time should she send the signal?

(c) As she tries to contact Sam, Sonya realizes that she can't contact him directly. She decides to send a radio signal to her planet telling them to contact Sam instead. What is the latest time that Sonya can contact her planet so that Sam receives the message in time to be able to make it to the meeting?

Solution: Figure 4.4(b) of the manual shows the relevant spacetime diagram.

(a) By Rule 2 of Note 4.2.6, $\beta = \overline{CD}/\overline{CB}$, or $\overline{CB} = 0.5$ light hour$/0.2c = 2.5$ hours. So, Sonya has to start her trip at 10:30 am. Similarly, $\beta = \overline{DE}/\overline{EF}$ gives $\overline{EF} = 0.5$ light hour$/0.75c$ or $\overline{EF} = 40$ minutes. So, Sam has to leave at 12:20 pm.

(b) K is one hour below F (why?). So, K occurs at 11:20 am. Thus $\overline{BK}$ is 50 minutes. Again, by rule 2, $\overline{AM} = 0.2\overline{AB}$. Therefore, $\overline{AK} = 0.2\overline{AB}$. Thus,

$$\overline{BK} + \overline{AK} = \overline{AB} = \overline{AK}/0.2 = 5\overline{AK} \iff \overline{BK} = 4\overline{AK},$$

or $\overline{AK} = 12.5$ minutes. Therefore, A occurs 62.5 minutes after Sonya's take-off, or at 11:32.5 am. Since Sonya is measuring proper time, she has to send the signal

$$\frac{62.5 \text{ minutes}}{\gamma} = \sqrt{1 - 0.2^2} \times 62.5 \text{ minutes} = 61.24 \text{ minutes}$$

after her take-off by her own watch.

(c) In order for Sam to receive the signal at F, the planet people have to send it at K. And in order for them to receive Sonya's signal at K, she has to send hers at H. With a reasoning along the same line as in (b), you can show that $\overline{BK} = 6\overline{HK}$, so that $\overline{HK} = 50/6 = 8.67$ minutes, and $\overline{BH} = 5\overline{HK} = 250/6 = 41.67$ minutes. So, Sonya has to send the stress signal 41.67 minutes after take-off according to her planet's time. According to her own time, she has to send the signal

$$\frac{41.67 \text{ minutes}}{\gamma} = \sqrt{1 - 0.2^2} \times 41.67 \text{ minutes} = 40.82 \text{ minutes}$$

after take-off.

■

4.7. In Example 4.2.4, I showed that when two events are causally disconnected, you can always find an observer for whom the order of occurrence of the events is switched. I used algebraic method to prove the statement. Geometry make the argument much easier! Let O' have perpendicular axes. Pick two events E_1 and E_2 that are causally disconnected. Now choose a set of axes passing through the origin of O' in such as way that the order of occurrence of E_1 and E_2 is switched. Hint: Draw parallel lines from the two events in such a way that the earlier event cuts ct' axis at a *later* time. From this decide what the new x-axis should be. Make sure that the angle between the x- and x'-axes is not larger than $45°$.

Solution: The idea is first to find the x'-axis. If the two events are causally disconnected, then the line passing through them must make an angle less that $45°$ with the x-axis. Now consider candidates for the x'-axis as lines passing through the origin and making various

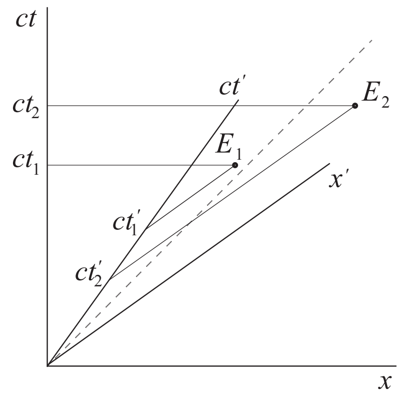


Figure 4.5: To find the x'-axis, draw a line from the origin that makes a larger angle with the x-axis than the line E_1E_2 does.

angles with the x-axis. As you increase the angle from zero (corresponding to the x-axis itself), and draw two lines from E_1 and E_2 parallel to the current candidate, the spacing between those two lines decreases, meaning that the corresponding $ct'_2 - ct'_1$ decreases. As you continue increasing the angle, you reach a line that is parallel to $\overline{E_1E_2}$. For this line, $ct'_2 = ct'_1$. It should now be clear that any line that makes an angle with the x-axis larger than the angle of $\overline{E_1E_2}$ (but less than 45°) is a good candidate for the x'-axis. Once the x'-axis is determined, the ct'-axis can be drawn. Figure 4.5 of the manual shows one of the infinitely many coordinate systems in which the order of occurrence of the two events E_1 and E_2 has been switched. ■

4.8. Figure 4.26 shows five firecrackers separated by 4 light seconds from one another in Sam's reference frame O. F_1 and F_3 occur simultaneously 4 seconds into the future; F_2, F_4, and F_5 occur 16 seconds, 8 seconds, and 20/3 seconds into the future, respectively. Sonya (reference frame O') moves at 1/3 the speed of light relative to Sam in the positive x-direction in such a way that at $t = t' = 0$ the origins of the two RFs coincide.

(a) With dots, show the events corresponding to the explosion of the firecrackers.

(b) At what times does Sam receive the light signals from the firecrackers?

(c) Draw Sonya's spacetime axes, x' and ct'.

(d) In what order do the firecrackers explode according Sonya?

(e) In what order do the light signals from the firecrackers reach Sonya?

(f) At what times according to Sam do the light signals from the firecrackers reach Sonya?

Solution:

(a) Figure 4.6 of the manual shows the coordinates of the events corresponding to the explosion of the firecrackers.

(b) Sam receives the light signals from F_1, 8 s into the future, from F_2 and F_4, 24 s into the future, from F_3, 16 s into the future, and from F_5, 26.667 s into the future.

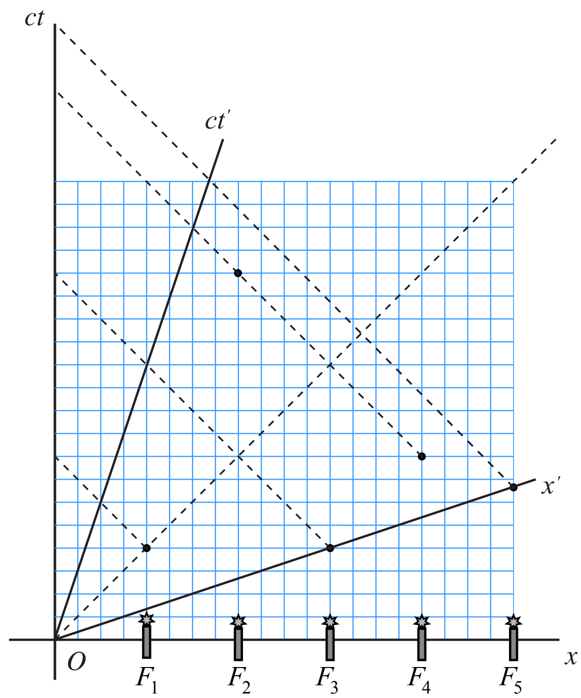


Figure 4.6: The distance between consecutive firecrackers is 4 light second. The unit on the vertical axis is also light second. The dashed lines are light worldlines.

(c) Sonya's spacetime axes, x' and ct', are as shown in the figure.

(d) According to Sonya, F_3 and F_5 explode simultaneously first, then F_1 and F_4 also simultaneously, and finally F_2.

(e) F_1, then F_3, then F_2 and F_4 simultaneously, and finally, F_5.

(f) F_1, 6 s into the future; F_3, 12 s into the future; F_2 and F_4, 18 s into the future; and finally F_5, 20 s into the future.

■

4.9. Sonya, who is in reference frame O, sees a flash of red light at $x = 1500$ m, and after 5 μs, a flash of green light at $x = 300$ m. Use spacetime diagrams for the problem.

(a) What should Sam's speed be relative to Sonya so that he sees the two events at the same point in his RF?

(b) Which event occurs first according to Sam and what is the time interval between the two flashes?

Solution: Figure 4.7 of the manual shows the details of the problem. In order for Sam's RF to be present at both events, his world line (his time axis) must be parallel to the line connecting E_1 and E_2. That's how ct'-axis is constructed. The x'-axis is then drawn as usual.

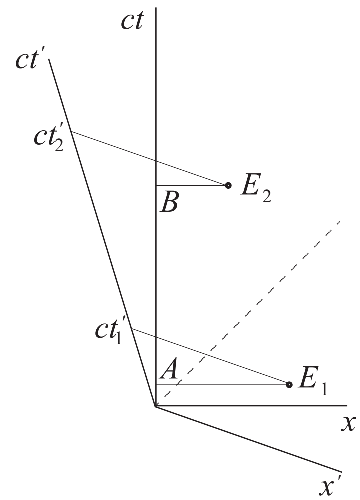


Figure 4.7: The spacetime diagram for the red and green flashes according to Sonya (observer O) and Sam (observer O').

(a) Sam's speed relative to Sonya is the slope of the ct'-axis relative to the ct-axis, which is the slope of $\overline{E_1E_2}$ relative to the ct-axis:

$$v = \frac{300 \text{ m} - 1500 \text{ m}}{5 \ \mu\text{s}} = -2.4 \times 10^8 \text{ m/s} \iff \beta = -0.8$$

(b) The red flash occurs first according to Sam. To find the time interval between flashes, draw lines parallel to the x-axis from E_1 and E_2, and let them cut the ct-axis at A and B. It is then clear that $\overline{AB} = ct_2 - ct_1$. By Rule 4 of Note 4.2.7,

$$ct_2 - ct_1 = \overline{AB} = \gamma\overline{E_1E_2} = \gamma(ct_2' - ct_1'),$$

or

$$t_2' - t_1' = \frac{t_2 - t_1}{\gamma} = \sqrt{1 - 0.8^2} \times 5 \ \mu\text{s} = 3 \ \mu\text{s}.$$

The answer to (b) could also be obtained by noting that Sam measures proper time. ■

4.10. In a galactic rocket race, Sam and Sonya drive two rockets in opposite directions towards their respective finish lines as shown in Figure 4.27 (the units are not the same on the two axes). Sam moves to the right and Sonya to the left. The drivers start their motion at $t = 0$ according to all three RFs. When they reach their finish lines—placed at 20 light minutes on either side of the referee—each driver sends a light signal to the referee relaying their arrival. The referee receives the signals from Sam and Sonya exactly 45 minutes after their departure, indicating a tie.

(a) Draw the two events of the arrival of the rockets at the finish lines on the referee's RF.

(b) What is the speed of each rocket?

(c) Draw the spacetime axes of both rockets.

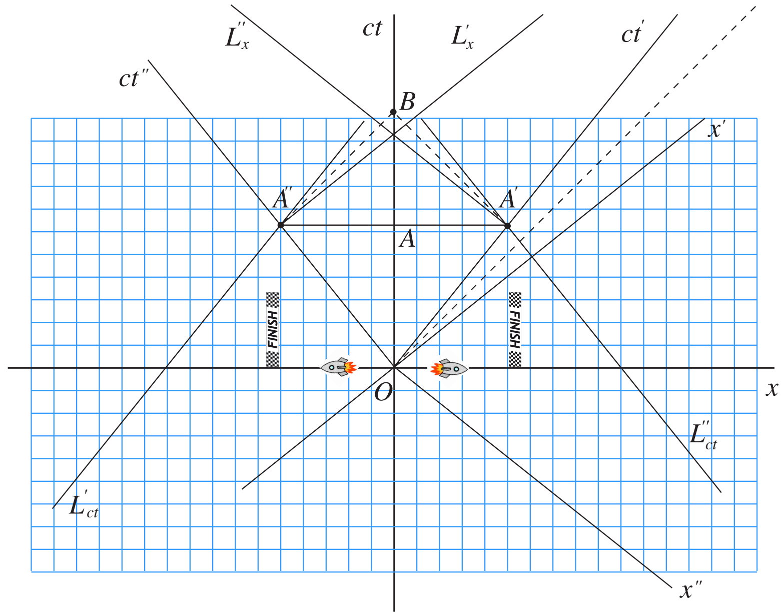


Figure 4.8: The diagram shows all the events. The units are the same for both axes. The prime indicates Sam and the double-prime indicates Sonya.

(d) Graphically show the coordinates of the two arrival events on the axes of all three observers.

(e) What is the order of arrivals according to Sam? Does he think he is the winner?

(f) What is the order of arrivals according to Sonya? Does she think she is the winner?

Solution: In Figure 4.8 of the manual the units are the same on the two axes.

(a) The two events labeled A' and A'' are the events of the arrival of Sam's and Sonya's rockets at the finish lines, respectively.

(b) The speeds are the same. For Sam, it is the slope of ct' relative to ct:

$$\beta = \frac{\overline{AA'}}{\overline{OA}} = \frac{\overline{AA'}}{\overline{OB} - \overline{AB}} = \frac{20}{45 - 20} = 0.8$$

(c) To draw the time axis of each rocket, connect O to its arrival event.

(d) The coordinates of the arrival events are already drawn in O: their x's are the locations of their finish lines, and their common ct is $\overline{OA}$. To find the coordinates of Sam's arrival in Sonya's RF, draw lines from A' parallel to Sonya's axes. In the figure, the line L_x'' is parallel to x''-axis; if its intersection with ct'' is a point C'', then $\overline{OC''}$ is the time coordinate of A' in Sonya's RF. Similarly, the line L_{ct}'' is parallel to ct''-axis; if its intersection with x'' is a point D'', then $\overline{OD''}$ is the x'' coordinate of A' in Sonya's RF. Note that the x'' coordinate is negative, as should be evident. To find the coordinates of Sonya's arrival in Sam's RF, draw lines from A'' parallel to Sam's axes.

(e) Obviously, $\overline{OC''} > \overline{OA''}$. So, Sam thinks that he reached the finish line earlier than Sonya, and therefore that he is the winner.

(f) Same as (e) for Sonya.

It is a good idea to put some quantitative flesh on all the qualitative discussion skeleton I have made so far. First note that $\overline{OA}$ is the projection on ct-axis of the interval $\overline{OA'}$ on ct'-axis. Therefore, by Rule 4 of Note 4.2.7,

$$\overline{OA} = \gamma\overline{OA'} \iff \overline{OA'} = \frac{\overline{OA}}{\gamma} = \sqrt{1-0.8^2} \times 25 \text{ light minutes} = 15 \text{ light minutes}.$$

Next, I want to find the time of Sam's arrival according to Sonya. For this, I need their relative speed:

$$\beta_{O'O''} = \frac{0.8+0.8}{1+0.8\times 0.8} = 0.9756 \iff \gamma_{O'O''} = \frac{1}{\sqrt{1-0.9756^2}} = 4.56.$$

Since I know $\overline{OA'}$, I can invoke Rule 4 of Note 4.2.7:

$$\overline{OC''} = \gamma_{O'O''}\overline{OA'} = 4.56 \times 15 \text{ light minutes} = 68.33 \text{ light minutes}.$$

Finally, I want to find the location of A' in O''. That's where Rule 2 of Note 4.2.6 comes in handy:

$$\beta_{O'O''} = \frac{\overline{OD''}}{\overline{OC''}} \iff 0.9756 = \frac{\overline{OD''}}{68.33 \text{ light minutes}} \iff \overline{OD''} = 66.67 \text{ light minutes},$$

and $x''_{A'} = -66.67$ light minutes. ■

4.11. Sonya, who is in reference frame O, throws a ball with speed β_b at time t_0 (event E_0) in her positive x-direction, which reaches a point (event E) Δx from the origin at time $t_0 + \Delta t$. Sam, in RF O', with respect to whom Sonya is moving with speed β in the positive x'-direction, looks at the same two events. Use a spacetime diagram in which Sam's axes are perpendicular to derive the relativistic law of addition of velocities.

Solution: The relevant diagram is depicted in Figure 4.9 of the manual. We need to find the ratio of $\overline{AB}$ to $\overline{MF}$. I'll be using Rules 2 and 4 without mentioning them each time. Let's start with $\overline{AB}$:

$$\overline{AB} = \overline{OB} - \overline{OA} = \overline{OB} - \beta\overline{OM} = \overline{OB} - \beta\gamma ct_0.$$

Now, I calculate $\overline{OB}$:

$$\overline{OB} = \overline{OH} + \underbrace{\overline{HB}}_{=\overline{DC}} = \gamma\overline{OG} + \beta\overline{OD} = \gamma\Delta x + \beta\gamma\overline{OC} = \gamma\Delta x + \beta\gamma(ct_0 + c\Delta t).$$

Therefore,

$$\overline{AB} = \gamma\Delta x + \beta\gamma c\Delta t = \gamma(\beta_b + \beta)c\Delta t$$

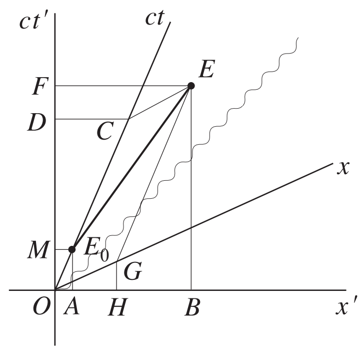


Figure 4.9: The spacetime diagram for the derivation of the relativistic law of addition of velocities.

Now, let's calculate $\overline{MF}$:

$$\begin{aligned}\overline{MF} &= \overline{OF} - \overline{OM} = \overline{OF} - \gamma ct_0 = \overline{OD} + \underbrace{\overline{DF}}_{=\overline{GH}} - \gamma ct_0 \\ &= \gamma\overline{OC} + \beta\overline{OH} - \gamma ct_0 = \gamma(ct_0 + c\Delta t) + \beta\gamma\Delta x - \gamma ct_0.\end{aligned}$$

Hence,

$$\overline{MF} = \gamma(c\Delta t + \beta\Delta x) = \gamma(c\Delta t + \beta c\beta_b\Delta t) = \gamma(1+\beta\beta_b)c\Delta t.$$

Taking the ratio $\overline{AB}/\overline{MF}$ yields the relativistic law of addition of velocities. ■

4.12. Alpha Centauri is about 4 light years away from Earth and does not move significantly relative to it, so they are both in the same RF. Assume Earth is at the origin of this RF. Event E_1 occurs at $t = 0$ on Earth. Event E_2 occurs on Alpha Centauri 3 years later. Use spacetime diagrams.

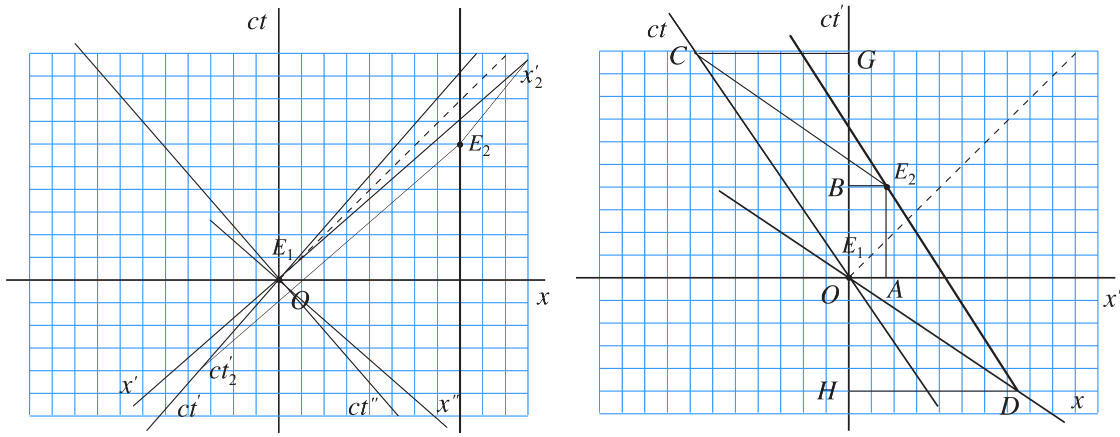
(a) What is the time difference between the two events according to an observer moving from Earth to Alpha Centauri at $0.9c$? Which event occurs first?

(b) What is the time difference between the two events according to an observer moving from Alpha Centauri to Earth at $0.9c$? Which event occurs first?

What happened to the invariance of "earlier" and "later"?

Solution: Figure 4.10 of the manual shows the spacetime diagrams. In Figure 4.10(a) the Earth RF is drawn with axes perpendicular. The Alpha Centauri worldline is the vertical line. The RF moving toward Alpha Centauri is primed, and the RF coming toward Earth is double-primed. To avoid cluttering the figure too much, only the lines constructing the coordinates of E_2 in the primed coordinate system are shown. It is seen that in this coordinate system, E_2 occurs earlier.

(a) In Figure 4.10(b), the Earth and the primed RF are isolated. Furthermore, to ease calculation, the primed RF is drawn orthogonal. You can derive the coordinates of E_2 in the primed RF as was done in the re-derivation of the Lorentz transformation in Section 4.4 (see also the solution to the previous problem). What you'll be doing is essentially re-deriving the Lorentz transformation. So, I'll leave the derivation as an exercise. I'll just use the Lorentz transformation to calculate the time difference.



(a) (b)

Figure 4.10: The Earth RF is unprimed. The RF moving toward Alpha Centauri is primed, and the RF coming toward Earth is double-primed. (a) All three RFs are shown. (b) Only Earth and the RF moving toward Alpha Centauri are shown. Earth RF has slanted axes.

The Earth is moving in the negative direction of the primed RF. So, β appears with a negative sign in the Lorentz transformation.

$$c\Delta t' = \gamma(c\Delta t - \beta\Delta x) = \frac{1}{\sqrt{1-0.9^2}}(3 - 0.9 \times 4) = -1.38 \text{ light year}.$$

Thus, $\Delta t' = -1.38$ years. Therefore, E_2 occurs *before* E_1. This is okay, because the two events are causally disconnected.

(b) The Earth is moving in the positive direction of the double-primed RF. So, β appears with a positive sign in the Lorentz transformation.

$$c\Delta t'' = \frac{1}{\sqrt{1-0.9^2}}(3 + 0.9 \times 4) = +15.14 \text{ light year},$$

and $\Delta t'' = 15.14$ years.

■

4.13. Rework Example 3.5.3 using geometric method in which the speeder's RF has orthogonal axes. Instead of numbers, use β_1 for the speed of the speeder and $\beta_2 > \beta_1$ for the policeman's speed, both relative to the ground. Let T be the time according to the speeder that the policeman catches up with the speeder.

Solution: Figure 4.11 of the manual shows the relevant spacetime diagram. Using Rule 2, you get

$$\beta_1 = \frac{\overline{E_1A}}{\overline{OA}}, \quad \beta_{ps} = \frac{\overline{E_1A}}{\overline{AE_2}} \iff \beta_1\overline{OA} = \beta_{ps}\overline{AE_2},$$

where β_{ps} is the speed of the policeman relative to the speeder. Relativistic LAV gives

$$\beta_{ps} = \frac{\beta_{pg} + \beta_{gs}}{1 + \beta_{pg}\beta_{gs}} = \frac{\beta_{pg} - \beta_{sg}}{1 - \beta_{pg}\beta_{sg}}.$$

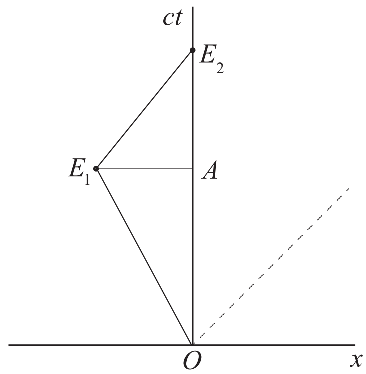


Figure 4.11: Event E_1 is when the policeman starts chasing the speeder. Event E_2 is when the policeman catches up with the speeder.

Note the order of subscripts in the first equation. That's how the LAV should be remembered! Concentrating only on the numerator, it says that the speed of the police relative to the speeder is equal to the speed of the police relative to the ground plus the speed of the ground relative to the speeder (divided by the appropriate denominator). Therefore,

$$\beta_{ps} = \frac{\beta_2 - \beta_1}{1 - \beta_2\beta_1},$$

and

$$\beta_1 \overline{OA} = \frac{\beta_2 - \beta_1}{1 - \beta_2\beta_1}\overline{AE_2} \iff \overline{OA} = \frac{\beta_2 - \beta_1}{\beta_1(1 - \beta_2\beta_1)}\overline{AE_2}.$$

We also have $cT = \overline{OA} + \overline{AE_2}$. Hence,

$$cT = \left[\frac{\beta_2 - \beta_1}{\beta_1(1 - \beta_2\beta_1)} + 1\right]\overline{AE_2} = \frac{\beta_2(1 - \beta_1^2)}{\beta_1(1 - \beta_2\beta_1)}\overline{AE_2}$$

or

$$\overline{AE_2} = \frac{\gamma_1^2\beta_1(1 - \beta_2\beta_1)cT}{\beta_2}, \quad \overline{OA} = \frac{\gamma_1^2(\beta_2 - \beta_1)cT}{\beta_2}.$$

The policeman is at a distance

$$\overline{E_1A} = \beta_1\overline{OA} = \frac{\gamma_1^2\beta_1(\beta_2 - \beta_1)cT}{\beta_2}$$

from the speeder when he starts the chase, making his x-coordinate relative to the speeder

$$x_1 = -\frac{\gamma_1^2\beta_1(\beta_2 - \beta_1)cT}{\beta_2}.$$

The time that the policeman starts chasing the speeder, according to the policeman is $\overline{OE_1}$. Using Rule 4, we get

$$\overline{OA} = \gamma_1\overline{OE_1} \iff ct_1' = \overline{OE_1} = \frac{\overline{OA}}{\gamma_1} = \frac{\gamma_1(\beta_2 - \beta_1)cT}{\beta_2}.$$

Similarly, using $\gamma_{ps} = \gamma_1\gamma_2(1-\beta_1\beta_2)$, we obtain

$$\overline{AE_2} = \gamma_{ps}\overline{E_1E_2} \iff \overline{E_1E_2} = \frac{\overline{AE_2}}{\gamma_1\gamma_2(1-\beta_1\beta_2)} = \frac{\beta_1\gamma_1}{\beta_2\gamma_2}cT.$$

As a check, it's a good idea to plug in the numbers of Example 3.5.3 and make sure you get the results of that example. ■

4.14. This is a variation of Example 3.5.3 for which you are to use spacetime diagrams. Let the speeder's RF have orthogonal axes. The speeder passes the policeman with a speed of $0.99c$. Two minutes later (policeman's time), the policeman starts chasing the speeder with a speed of $0.995c$.

(a) How long does it take the policeman to catch up with the speeder according to the speeder's watch?

(b) How long does it take the policeman to catch up with the speeder according to the policeman's watch?

To answer these questions you have to find all the following on the speeder's spacetime coordinates: the time when the policeman starts the chase, the distance between the two when the chase starts, the time it takes for the policeman to catch up with the speeder after the start of the chase.

Solution: We can use Figure 4.11 of the manual for this problem as well. I calculated $\overline{OE_1}$ in the previous problem.

(a) By Rules 4 and 2,

$$\overline{OA} = \gamma_1\overline{OE_1}, \quad \overline{AE_1} = \beta_1\overline{OA} = \beta_1\gamma_1\overline{OE_1}.$$

Also by Rule 2,

$$\overline{AE_1} = \beta_{ps}\overline{AE_2} \iff \overline{AE_2} = \frac{\overline{AE_1}}{\beta_{ps}} = \frac{\beta_1\gamma_1\overline{OE_1}}{\beta_{ps}} = \frac{\beta_1\gamma_1(1-\beta_1\beta_2)\overline{OE_1}}{\beta_2-\beta_1},$$

where β_{ps} is the speed of the policeman relative to the driver in the second leg of his trip, and in the last equality, I used the relativistic LAV. So, the time according to the speeder is

$$ct = \overline{OE_2} = \overline{OA} + \overline{AE_2} = \gamma_1\overline{OE_1} + \frac{\beta_1\gamma_1(1-\beta_1\beta_2)\overline{OE_1}}{\beta_2-\beta_1} = \frac{\beta_2\overline{OE_1}}{\gamma_1(\beta_2-\beta_1)}.$$

Plugging in the numbers, you get

$$ct = \frac{0.995 \times 2 \text{ minutes}}{2.294(0.995-0.99)} = 173.5 \text{ minutes}$$

(b) To find the time according to the policeman, we need $\overline{E_1E_2}$:

$$\overline{AE_2} = \gamma_{ps}\overline{E_1E_2} \iff \overline{E_1E_2} = \frac{\overline{AE_2}}{\gamma_{ps}} = \frac{\overline{AE_2}}{\gamma_1\gamma_2(1-\beta_1\beta_2)} = \frac{\beta_1\overline{OE_1}}{\gamma_2(\beta_2-\beta_1)}.$$

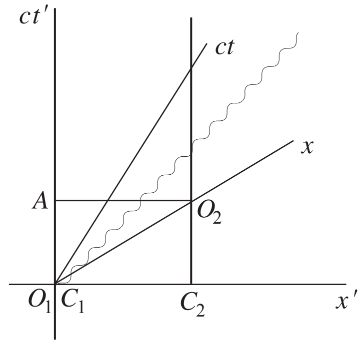


Figure 4.12: Clocks C_1 and C_2 and the observers O_1 and O_2.

Thus,

$$ct' = \overline{OE_1} + \overline{E_1E_2} = \left[1 + \frac{\beta_1}{\gamma_2(\beta_2 - \beta_1)}\right]\overline{OE_1}.$$

Plugging in the numbers, you get

$$ct' = \left[1 + \frac{0.99}{10.01(0.995 - 0.99)}\right] 2 \text{ minutes} = 41.55 \text{ minutes}$$

■

4.15. Two clocks C_1 and C_2 separated by a fixed distance L' are placed on the x'-axis of an observer O' and synchronized according to O'.[1] Reference frame O is moving relative to O' in the positive direction of x' with speed β. Observer O_1 is at the origin of O and observer O_2 is placed strategically along the x-axis in such a way that they can read the two moving clocks *at the same time*. O_1 records the reading as 12:00 (the zero time). Draw a spacetime diagram with O' axes perpendicular, and show the world lines of the two clocks in the O' reference frame.

(a) What is the location of O_2 in O?

(b) What time does O_2 record? Hint: It is the time coordinate in O' of the intersection of the x-axis with the worldline of C_2.

Solution: The two clocks C_1 and C_2 and the observers O_1 and O_2 are shown in Figure 4.12 of the manual.

(a) By Rule 4, $\overline{C_1C_2} = \gamma\overline{O_1O_2}$. Therefore, with $x_1 = 0$, we get $x_2 = L'/\gamma$.

(b) The time is $ct' = \overline{O_2C_2} = \beta L'$ (using Rule 2).

These are the results we obtained in Example 3.4.10. ■

4.16. Two photons are moving in O at a fixed distance L apart. Show that in O' with respect to which O moves with fractional speed β in the *negative* direction, the two photons are separated by

$$L' = \sqrt{\frac{1+\beta}{1-\beta}}\, L.$$

[1] This is the same problem done in Example 3.4.10.

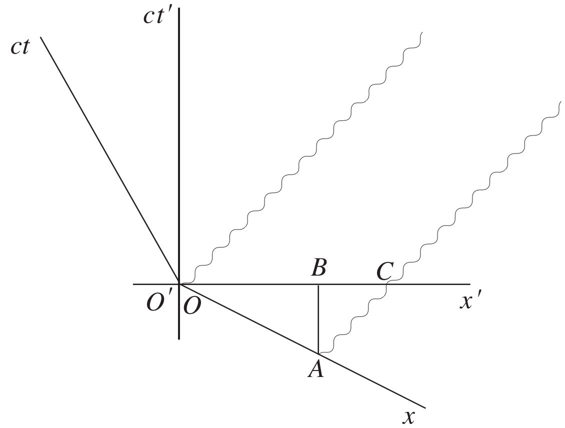


Figure 4.13: The two photons are a distance of $L = \overline{OA}$ apart in O.

Solution: The separation of the two photons in O' is $\overline{O'C}$ of Figure 4.13 of the manual. Using Rules 2 and 4, we have

$$L' = \overline{O'B} + \overline{BC} = \gamma\overline{OA} + \overline{AB} = \gamma\overline{OA} + \beta\overline{O'B}$$
$$= \gamma\overline{OA} + \beta\gamma\overline{OA} = \gamma(\beta+1)L = \sqrt{\frac{1+\beta}{1-\beta}}\,L.$$

■

4.17. Sam and Sonya are twins. Sam is put on a spaceship O moving with relative speed β in the positive x' direction of Sonya's reference frame O'. At time T_0', O' sends a light signal toward O. Draw a spacetime diagram showing the relevant axes and the light signal. Use the rules of spacetime geometry.

(a) Show that when Sam receives the light signal, the spaceship is at a distance of $\beta T_0'/(1-\beta)$ from Sonya.

(b) Show that Sam receives the light signal when he is $\sqrt{(1+\beta)/(1-\beta)}\,T_0'$ older than when he left his sister.

Solution: Figure 4.14 of the manual shows the two RFs of Sonya (O') and Sam (O). The problem is to find $\overline{OE_1}$ and $\overline{AE_1}$ in terms of $cT_0' = \overline{O'E_0}$. Rule 4 gives

$$\overline{O'A} = \overline{O'E_0} + \overline{E_0A} = \overline{O'E_0} + \overline{AE_1} = \overline{O'E_0} + \beta\overline{O'A}.$$

Therefore,

$$(1-\beta)\overline{O'A} = \overline{O'E_0} \iff \overline{O'A} = \frac{cT_0'}{1-\beta}.$$

(a) By Rule 2,

$$\overline{OB} = \overline{AE_1} = \beta\overline{O'A} = \frac{\beta cT_0'}{1-\beta}.$$

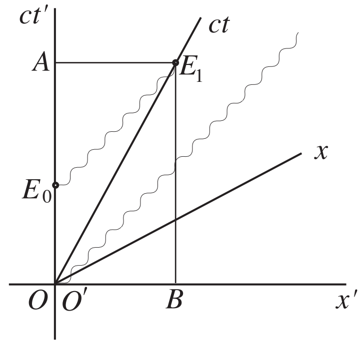


Figure 4.14: Spacetime diagram for Sonya sending a light signal to her twin brother. Event E_0 takes place at T'_0.

(b) By Rule 4, $\overline{O'A} = \gamma \overline{OE_1}$. Thus,

$$ct_1 = \overline{OE_1} = \frac{\overline{O'A}}{\gamma} = \frac{cT'_0}{\gamma(1-\beta)},$$

and

$$t_1 = \frac{\beta T'_0}{\gamma(1-\beta)} = \sqrt{\frac{1+\beta}{1-\beta}}\, T'_0$$

■

4.18. Tomorrow is "only one day away;" it is probably not too much to ask the theory of relativity to help us get there. Utilizing the experience you gained in the case of Bruno's death, find observer O who is only 24.5 light hours away.[2]

(a) What speed should O have? In which direction?

(b) How far away is "tomorrow" taking place from O?

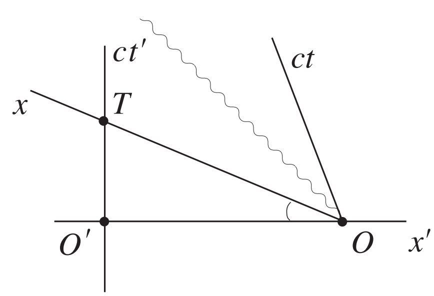


Figure 4.15: Spacetime diagram for "traveling" to the future. T stands for tomorrow. OT must be the x-axis, because T has to happen NOW for O.

Solution: Figure 4.15 of the manual shows the spacetime diagram for the problem.

[2]A light hour is the distance that light travels in one hour. For comparison, Saturn is about 1.25 light hours away from Sun.

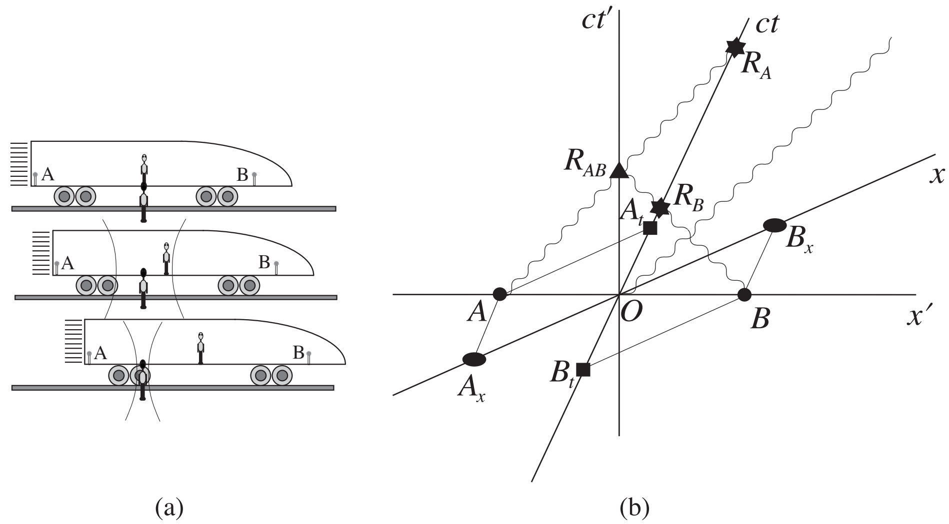


Figure 4.16: (a) To Sam, the explosion of the two firecrackers occur at the same time that Sonya passes him by. (b) The spacetime geometry of the events as seen by Sam (O') and Sonya (O).

(a) By Rule 2,

$$\beta = \frac{\overline{O'T}}{\overline{O'O}} = \frac{24}{24.5} = 0.9796.$$

It is clear that the speed should be in the negative x' direction.

(b) By Rule 4, $\overline{O'O} = \gamma \overline{OT}$. Therefore,

$$\overline{OT} = \frac{\overline{O'O}}{\gamma} = \sqrt{1-\beta^2}\,\overline{O'O} = 0.2 \times 24.5 \text{ light hours} = 4.9 \text{ light hours}.$$

■

4.19. Sonya (observer O) moves at $0.99c$ relative to Sam (observer O') as shown in Figure 4.9. Assume that the length of the train is 50 m. Find the coordinates of all the points marked as triangle, square, circle, oval, and stars in Figure 4.9(b). Hint: Go through Example 4.3.1 for warmup.

Solution: I've reproduced the figure in Figure 4.16 of the manual, enlarged part (b), and added some labels in part (b) to facilitate the solution. I'll just use β for the speed and $2L$ for the length of the train. From Rule 4, we get $\overline{OB_x} = \gamma\overline{OB} = \gamma L$. From Rule 2, we get $\overline{OB_t} = \overline{BB_x} = \beta\overline{OB_x} = \beta\gamma L$ (the time coordinate of B in O is $-\overline{OB_t}$). Similarly, $\overline{OA_x} = \gamma L$ (the space coordinate of A in O is $-\overline{OA_x}$) and $\overline{OA_t} = \overline{AA_x} = \beta\gamma L$.

The triangle BB_tR_B is an isosceles triangle (show this by drawing a line from B_t parallel to the light world line passing through O). Therefore, $\overline{B_tR_B} = \overline{BB_t} = \overline{OB_x}$, and

$$\overline{OR_B} = \overline{B_tR_B} - \overline{OB_t} = \overline{OB_x} - \overline{OB_t} = \gamma L - \beta\gamma L = (1-\beta)\gamma L.$$

Similarly, the triangle AA_tR_A is an isosceles triangle. Therefore, $\overline{A_tR_A} = \overline{AA_t} = \overline{OA_x}$, and

$$\overline{OR_A} = \overline{A_tR_A} + \overline{OA_t} = \overline{OA_x} + \overline{OA_t} = \gamma L + \beta\gamma L = (1+\beta)\gamma L.$$

Finally, it is clear that $\overline{OR_{AB}} = \overline{OA} = \overline{OB} = L$. ■

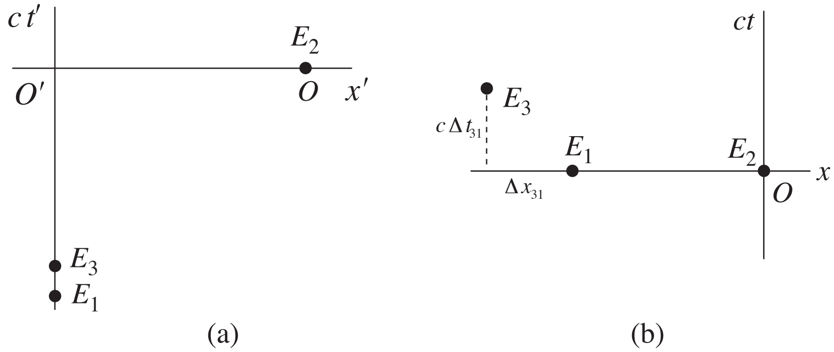


Figure 4.17: (a) The events E_1, E_2, and E_3 in the Earth RF. Note that the time of the origin (and thus the time of all the events on the x'-axis) is NOW which is the year 2163. (b) The same three events as seen by the crew of Diracus II.

4.20. Use the result of Problem 3.7 to find the coordinates of E_1, E_2, and E_3 of Figure 4.18 in both coordinate systems O and O'.

Solution: I'll reproduce the figure for convenience. The coordinates of the origin of O relative to O' are just the coordinates of E_2 of Figure 4.17 of the manual relative to O'. These are $(x'_0, ct'_0) \equiv (x'_2, 0)$. Now, we calculate x_0 and ct_0 of Problem 3.7:

$$x_0 = \gamma(-x'_0 + \beta ct'_0) = -\gamma x'_2, \quad ct_0 = \gamma(\beta x'_0 - ct'_0) = \gamma\beta x'_2.$$

Thus, the inverse Lorentz transformation that includes the coordinates of the origin is

$$\begin{aligned} x =& -\gamma x'_2 + \gamma(x' - \beta ct') \\ ct =& \gamma\beta x'_2 + \gamma(-\beta x' + ct'). \end{aligned}$$

As a check, note that

$$x_2 = -\gamma x'_2 + \gamma(x'_2 - \beta ct'_2) = 0, \quad ct_2 = \gamma\beta x'_2 + \gamma(-\beta x'_2 + ct'_2) = 0.$$

The coordinates of E_1 and E_3 in O' are $(0, -\beta x'_2)$ and $(0, ct'_3)$, respectively. Therefore,

$$\begin{aligned} x_1 &= -\gamma x'_2 + \gamma[0 - \beta(-\beta x'_2)] = -\gamma x'_2(1 - \beta^2) = -\frac{x'_2}{\gamma} \\ ct_1 &= \gamma\beta x'_2 + \gamma(-0 - \beta x'_2) = 0, \end{aligned}$$

and

$$\begin{aligned} x_3 =& -\gamma x'_2 + \gamma(0 - \beta ct'_3) = -\gamma(x'_2 + \beta ct'_3) \\ ct_3 =& \gamma\beta x'_2 + \gamma(-0 + ct'_3) = \gamma(\beta x'_2 + ct'_3). \end{aligned}$$

You can verify all the results of Section 4.4.1 pertaining to the Kennedy assassination. ■

4.21. Provide the missing steps leading to the final result of Equation (4.12).

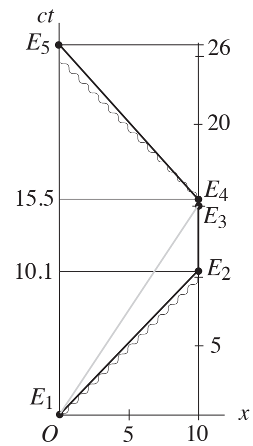


Figure 4.18: Sonya moves on the heavy black worldline. Sam moves on the grey worldline first and then joins Sonya to return home. Pat remains on Earth.

Solution: Consider the equation

$$\beta_{\text{probe}} = \beta + \frac{T_1 + T_2}{\beta\gamma^2 T_2},$$

which is already derived in the text. Manipulate the second term:

$$\frac{T_1 + T_2}{\beta\gamma^2 T_2} = \frac{T_1}{\beta\gamma^2 T_2} + \frac{1}{\beta\gamma^2} = \frac{T_1}{\beta\gamma^2 T_2} + \frac{1-\beta^2}{\beta} = \frac{T_1}{\beta\gamma^2 T_2} + \frac{1}{\beta} - \beta.$$

Substituting this in the previous equation yields the final result. ■

4.22. Sam, Sonya, and Pat are newly born triplets. Sam and Sonya are put on two different spaceships that travel to a planet of a star system 10 ly away. Sonya lands on the planet 10.1 years later as seen by observer O, Pat. She waits 4.9 years until Sam, who is traveling slower, lands on the same planet (see Figure 4.18 of the manual). After six months they both return home on the same spaceship and land on Earth 26 years after their departure according to Earth calendar. All times and distances of the figure are given according to the Earth observers, and all units shown are in light years, and for easier reading most of the calibration of the ct-axis is made on the worldline parallel to it.

(a) What is the speed of Sonya's spaceship on her journey to the planet?

(b) What is the speed of Sam's spaceship on his journey to the planet?

(c) How old is Sonya when she meets Sam? How old is Sam?

(d) How old is Sonya when she lands back on Earth? How old is Sam? How old is Pat?

Solution:

(a) Distance is 10 light years, and time is 10.1 years. Therefore, $\beta_1 = \frac{10}{10.1} = 0.99$.

(b) $\beta_2 = \frac{10}{15} = 0.67$.

(c) Find the spacetime distance for each. For Sonya, there are two pieces to calculate:

$$\Delta s_{12} = \sqrt{10.1^2 - 10^2} = 1.418 \text{ light years} \iff \Delta\tau_{12} = 1.418 \text{ years},$$

and $\Delta\tau_{23} = 4.9$ years. Therefore, she is 6.318 years old when she meets Sam. For Sam, there is only one piece:

$$\Delta s_{13} = \sqrt{15^2 - 10^2} = 11.18 \text{ light years} \iff \Delta\tau_{13} = 11.18 \text{ years}.$$

Therefore, he is 11.18 years old when he meets Sonya.

(d) We have to add $0.5 + \Delta s_{45}$ to each, where

$$\Delta s_{45} = \sqrt{(26 - 15.5)^2 - 10^2} = 3.2 \text{ light years} \iff \Delta\tau_{45} = 3.2 \text{ years}.$$

Hence Sonya is $6.318 + 0.5 + 3.2 = 10.02$ years old; Sam is $11.18 + 0.5 + 3.2 = 14.88$ years old; and Pat is 26 years old.

■

4.23. Verify that Equation (4.14) is the left half of a hyperbola with center at $\big[\beta_0(\kappa^2 + 1)T/2, T/2\big]$, semi-major axis $a = \beta_0\kappa\sqrt{\kappa^2+1}\,T/2$ and semi-minor axis $b = \kappa T/2$.

Solution: Rewrite (4.14) as

$$x_s = \frac{c\beta_0(\kappa^2+1)T}{2} - \frac{c\beta_0\sqrt{\kappa^2+1}}{2}\sqrt{\kappa^2T^2 + (2t-T)^2},$$

or

$$x_s - \frac{c\beta_0(\kappa^2+1)T}{2} = -\frac{c\beta_0}{2}\sqrt{(\kappa^2+1)[\kappa^2T^2 + (2t-T)^2]}$$

and note that $x_s \to -\infty$ as $t \to \infty$, indicating that the graph is in the left half-plane. Now square both sides and write the result as follows:

$$\left[x_s - \frac{c\beta_0(\kappa^2+1)T}{2}\right]^2 = \frac{c^2\beta_0^2}{4}(\kappa^2+1)\kappa^2T^2 + c^2\beta_0^2(\kappa^2+1)(t - T/2)^2.$$

Now transfer the t term to the left and divide both sides by $c^2\beta_0^2(\kappa^2+1)\kappa^2T^2/4$ to obtain

$$\frac{\big[x_s - c\beta_0(\kappa^2+1)T/2\big]^2}{(c\beta_0\sqrt{\kappa^2+1}\kappa T/2)^2} - \frac{(t - T/2)^2}{(\kappa T/2)^2} = 1.$$

This is a hyperbola with the said characteristics. ■

4.24. Show that the absolute value of the speed in (4.15) is always less than c for the duration of all the three trips.

Solution:

$$\left|\frac{dx_s}{dt}\right| < c \iff \beta_0^2(\kappa^2+1)(2t-T)^2 < \kappa^2T^2 + (2t-T)^2.$$

The last inequality is equivalent to

$$\beta_0^2\kappa^2(2t-T)^2 < \kappa^2T^2 + (1-\beta_0^2)(2t-T)^2.$$

The second term on the right-hand side is positive, so the minimum of the right hand side is κ^2T^2. Since $0 < t < T$, the maximum of the left hand side is $\beta_0^2\kappa^2T^2$. Since the minimum of the right hand side is larger than the maximum of the left hand side, the right-hand side is *always* larger than the left hand side. ■

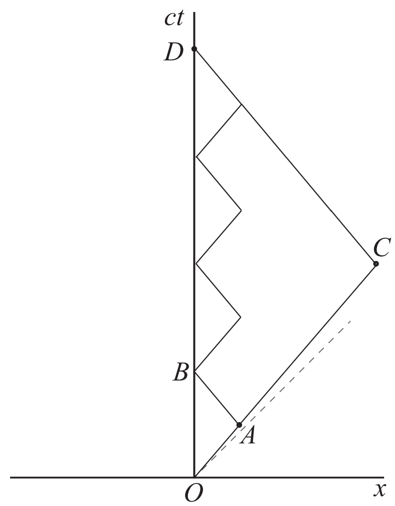


Figure 4.19: Each side of the big triangle is 4 times the corresponding side of each small triangle.

4.25. In Example 4.5.4, assume that Pat makes four identical round trips between $t = 0$ and $t = T$ with the same speed β_0.

(a) How far does he have to go before turning around in each trip?

(b) How much does he age in each trip?

(c) Compare his total age with his age in the example. Can you give a simple geometric reason for this? Hint: The geometric reason is identical to ordinary Euclidean geometry.

Solution: Refer to Figure 4.19 of the manual.

(a) Each leg of each trip now lasts $T/8$. Therefore, the outbound distance travelled is $c\beta_0 T/8$.

(b) Equation (4.16) of Example 4.5.4 shows that Pat's age is $\sqrt{1-\beta_0^2}$ times the time of landing. If the time of landing is $T/4$, then he ages $\sqrt{1-\beta_0^2}\,T/4$ during each trip.

(c) Hence, during the entire 4 trips, he ages $\sqrt{1-\beta_0^2}\,T$, as before. This can be explained geometrically as shown in Figure 4.19 of the manual. It is obvious that $\overline{OC} = 4\overline{OA}$ and $\overline{DC} = 4\overline{AB}$. Therefore, $\overline{OD} = 4\overline{OB}$.

■

4.26. In Example 4.5.4, assume that Pat travels with the initial speed of $0.999c$ to a destination 40 light years away.

(a) How old is Sonya when Pat returns? How old is Pat?

(b) Sam goes to a destination 30 light years away with the same initial speed as Pat. What is κ?

(c) How old is Sam when he returns? You'll have to numerically integrate (4.16) to find the answer.

Solution: It is shown in Example 4.5.4 that $x_{p,\max} = c\beta_0 T/2$.

(a) Therefore,

$$40 \text{ light years} = \frac{0.999cT}{2} \iff T = \frac{80 \text{ light years}}{0.999c} = 80.08 \text{ years},$$

and this is Sonya's age. Example 4.5.4 derived Pat's age as $\sqrt{1-\beta_0^2}\,T$. Thus,

$$T_{\text{Pat}} = \sqrt{1-0.999^2}\,80.08 \text{ years} = 3.58 \text{ years}.$$

(b) It is shown in Example 4.5.4 that

$$x_{s,\max} = \frac{c\beta_0 T}{2}\left(\kappa^2+1-\kappa\sqrt{\kappa^2+1}\right) = x_{p,\max}\left(\kappa^2+1-\kappa\sqrt{\kappa^2+1}\right).$$

Therefore,

$$30 \text{ light years} = 40 \text{ light years}\left(\kappa^2+1-\kappa\sqrt{\kappa^2+1}\right)$$

or

$$0.75 = \kappa^2+1-\kappa\sqrt{\kappa^2+1} \iff \kappa\sqrt{\kappa^2+1} = \kappa^2+0.25.$$

Squaring both sides and simplifying leads to $\kappa^2 = 0.125$, or $\kappa = 0.3536$.

(c) Sam's age is given by Equation (4.16), or—in years—by

$$T_{\text{Sam}} = \tau_s(0.999, \sqrt{0.125}) = \int_0^{80.08}\sqrt{1-\frac{0.999^2(1.125)(2t-80.08)^2}{0.125\times 80.08^2+(2t-80.08)^2}}\,dt.$$

Numerical integration yields $T_{\text{Sam}} = 49.69$ years.

■

4.27. Sonya gets on a spaceship that travels on a parabolic path given by

$$x(t) = c\kappa\left(t-\frac{t^2}{T}\right),$$

in Sam's coordinate system O, where κ is a positive constant. Note that $x(0) = x(T) = 0$. This means that Sonya leaves Sam at $t = 0$ and meets him again at $t = T$.

(a) What are Sonya's velocities relative to Sam at the times of her departure and return? What restriction does this put on κ?

(b) Show that during the entire Sonya's trip, her speed is always less than the speed of light.

(c) When and where does Sonya stop momentarily and come back?

(d) Verify that it take Sonya

$$T\left(\frac{\sin^{-1}\kappa+\kappa\sqrt{1-\kappa^2}}{2\kappa}\right)$$

to make her round trip. Show that as $\kappa \to 0$, this travel time reduces to Sam's time, as it should.

(e) Show that no matter how fast Sonya starts to travel, the ratio of her increase in age to Sam's can't be smaller than $\pi/4$.

Solution: Sonya's speed as a function of t is given by

$$\frac{dx}{dt} = c\kappa\left(1 - \frac{2t}{T}\right).$$

(a) Her speeds at $t = 0$ and $t = T$ are

$$\left.\frac{dx}{dt}\right|_{t=0} = c\kappa, \quad \left.\frac{dx}{dt}\right|_{t=T} = c\kappa\left(1 - \frac{2T}{T}\right) = -c\kappa.$$

Therefore, κ is both the departure and arrival speeds and $0 < \kappa < 1$.

(b)

$$\left|\frac{dx}{dt}\right| < c \iff \kappa\left|1 - \frac{2t}{T}\right| < 1.$$

$|1 - 2t/T|$ is a decreasing function of t for $0 < t < T/2$ with a maximum initial value of 1 and an increasing function of t for $T/2 < t < T$ with a maximum final value of 1. Thus, $|1 - 2t/T| < 1$ for all t, and therefore, the speed is less than 1 during the entire trip.

(c) The speed is zero when $t = T/2$, for which $x = c\kappa T/4$.

(d)

$$s = c\tau = \int_0^T \sqrt{(cdt)^2 - (dx)^2} = c\int_0^T \sqrt{1 - \left(\frac{dx}{cdt}\right)^2}\, dt$$

or

$$\tau = \int_0^T \sqrt{1 - \kappa^2\left(1 - \frac{2t}{T}\right)^2}\, dt = T\left(\frac{\sin^{-1}\kappa + \kappa\sqrt{1-\kappa^2}}{2\kappa}\right)$$

As $\kappa \to 0$, each term in the numerator goes to κ, so that the ratio becomes T.

(e) As $\kappa \to 1$, the first term in the numerator of the right-hand side of the last equation tends to $\pi/2$ and the second term to zero, while the denominator goes to 2. So, the entire right-hand side goes to $(\pi/4)T$.

■

4.28. Sonya gets on a spaceship that travels on a path given parametrically by

$$x(\theta) = cT\kappa(\sqrt{2}\cos\theta - 1), \quad ct(\theta) = cT(\sqrt{2}\sin\theta + 1), \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

in Sam's coordinate system O, where κ is a positive constant less than one. Note that $x(-\pi/4) = ct(-\pi/4) = 0$ and $x(\pi/4) = 0, ct(\pi/4) = 2cT$. This means that Sonya leaves Sam at $t = 0$ and meets him again at $t = 2T$.

(a) Show that during the entire Sonya's trip, her speed is always less than the speed of light.

(b) What are Sonya's velocities relative to Sam at the times of her departure and return?

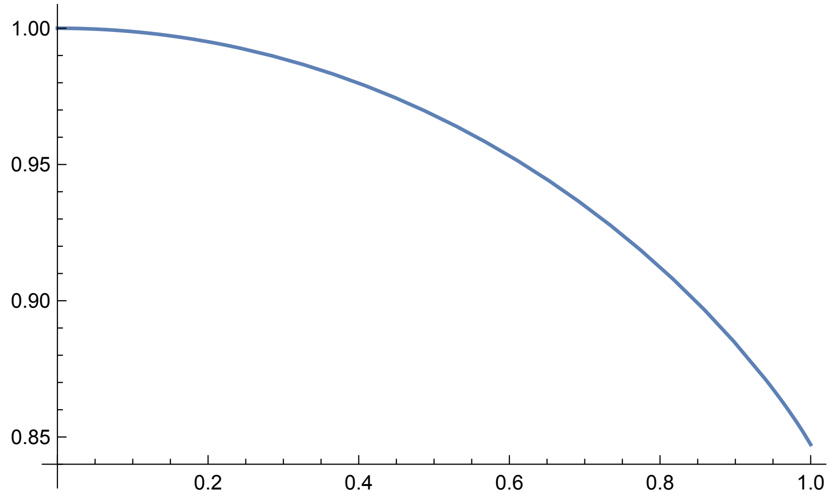


Figure 4.20: Plot of the ratio of Sam's round-trip time to Sonya's.

(c) When and where does Sonya stop momentarily and come back?

(d) Verify that it take Sonya

$$\sqrt{2}T \int_{-\pi/4}^{\pi/4} \sqrt{1-(\kappa^2+1)\sin^2\theta}\, d\theta$$

to make her round trip. Show that as $\kappa \to 0$, this travel time reduces to Sam's time, as it should.

(e) Numerically integrate the integral in (d) for various κ and plot the ratio of Sam's round-trip time to Sonya's. What is the limit of this ratio as $\kappa \to 1$?

Solution:

(a) Take the differential of the two equations and divide dx by cdt:

$$\beta = \frac{dx}{cdt} = \frac{-cT\kappa\sqrt{2}\sin\theta d\theta}{cT\sqrt{2}\cos\theta d\theta} = -\kappa\tan\theta$$

and $|\beta| = \kappa|\tan\theta|$. Now, $|\tan\theta| \le 1$ for the allowed range of θ, and $\kappa < 1$, therefore, $|\beta| < 1$.

(b) Departure corresponds to $\theta = -\pi/4$. Thus, from (b), $\beta_0 = \kappa$. On return, $\theta = \pi/4$ and the speed is $-\beta_0$.

(c) Within the allowed range of θ, the speed is zero at $\theta = 0$, for which

$$x(0) = cT\kappa(\sqrt{2}-1), \quad ct(0) = cT.$$

(d) Integrating $ds = cd\tau$, we get

$$c\tau = \int_{-\pi/4}^{\pi/4} \sqrt{(cdt)^2 - (dx)^2} = \int_{-\pi/4}^{\pi/4} \sqrt{(cT\sqrt{2}\cos\theta d\theta)^2 - (-cT\kappa\sqrt{2}\sin\theta d\theta)^2}$$
$$= \sqrt{2}cT \int_{-\pi/4}^{\pi/4} \sqrt{\cos^2\theta - \kappa^2 \sin^2\theta}\, d\theta = \sqrt{2}cT \int_{-\pi/4}^{\pi/4} \sqrt{1 - (\kappa^2+1)\sin^2\theta}\, d\theta.$$

As $\kappa \to 0$, this travel time reduces

$$\tau = \sqrt{2}T \int_{-\pi/4}^{\pi/4} \sqrt{1 - \sin^2\theta}\, d\theta = \sqrt{2}T \int_{-\pi/4}^{\pi/4} \cos\theta\, d\theta = 2T.$$

(e) Let $R(\kappa)$ denote the ratio of Sam's time to Sonya's:

$$R(\kappa) = \sqrt{2} \int_{-\pi/4}^{\pi/4} \sqrt{1 - (\kappa^2+1)\sin^2\theta}\, d\theta.$$

Figure 4.20 of the manual shows the plot of this ratio. It is a decreasing function of κ and its minimum value is $R(1) = 0.847213$.

■

CHAPTER 5

Spacetime Momentum

Problems With Solutions

5.1. Show that $u_{bt} = \gamma_b$, and that $u_{bt}^2 - u_{bx}^2 = 1$. Show that for any other observer O', $u_{bt}'^2 - u_{bx}'^2 = 1$ as well.

Solution: First find u_{bt}:

$$\begin{aligned} u_{bt} &= \frac{c\Delta t_b}{c\Delta\tau_b} = \frac{c\Delta t_b}{\Delta s_b} = \frac{c\Delta t_b}{\sqrt{(c\Delta t_b)^2 - (\Delta x_b)^2}} \\ &= \frac{c\Delta t_b}{c\Delta t_b\sqrt{1 - (\Delta x_b/c\Delta t_b)^2}} = \frac{1}{\sqrt{1-\beta_b^2}} = \gamma_b. \end{aligned}$$

Now note that

$$u_{bt}^2 - u_{bx}^2 = \frac{(c\Delta t_b)^2}{(\Delta s_b)^2} - \frac{(\Delta x_b)^2}{(\Delta s_b)^2} = \frac{(c\Delta t_b)^2 - (\Delta x_b)^2}{(\Delta s_b)^2} = 1.$$

Furthermore,

$$u_{bt}'^2 - u_{bx}'^2 = \frac{(c\Delta t_b')^2}{(\Delta s_b)^2} - \frac{(\Delta x_b')^2}{(\Delta s_b)^2} = \frac{(c\Delta t_b')^2 - (\Delta x_b')^2}{(\Delta s_b)^2} = 1,$$

because $(\Delta s_b)^2 = (c\Delta t_b)^2 - (\Delta x_b)^2 = (c\Delta t_b')^2 - (\Delta x_b')^2$. ■

5.2. Derive the relativistic law of addition of velocities from the second equation in (5.4).

Solution: The second equation in (5.4) can be written as

$$\gamma_b' = \gamma(\beta\gamma_b\beta_b + \gamma_b) = \gamma\gamma_b(\beta\beta_b + 1),$$

or

$$\frac{1}{1-\beta_b'^2} = \frac{(\beta\beta_b+1)^2}{(1-\beta^2)(1-\beta_b^2)} \iff 1 - \beta_b'^2 = \frac{(1-\beta^2)(1-\beta_b^2)}{(\beta\beta_b+1)^2},$$

or

$$\beta_b'^2 = 1 - \frac{(1-\beta^2)(1-\beta_b^2)}{(\beta\beta_b+1)^2} = \frac{(\beta\beta_b+1)^2 - (1-\beta^2)(1-\beta_b^2)}{(\beta\beta_b+1)^2},$$

or

$$\beta_b'^2 = \frac{2\beta\beta_b + \beta^2 + \beta_b^2}{(\beta\beta_b+1)^2} = \frac{(\beta+\beta_b)^2}{(\beta\beta_b+1)^2}.$$

■

5.3. I defined spacetime velocity by differentiating x and ct with respect to $s = c\tau$. Now define **spacetime acceleration** a_b by differentiating spacetime velocity with respect to s:

$$a_{bt} = \frac{du_{bt}}{ds}, \qquad a_{bx} = \frac{du_{bx}}{ds}.$$

(a) Show that

$$a_{bt}u_{bt} - a_{bx}u_{bx} = 0. \tag{5.1}$$

[Hint: Use Equation (5.2).] You'll see later that it is possible to define a "dot product" in two-dimensional relativity by *subtracting* the product of the two components of the vectors. More specifically, if $\mathbf{A} = (A_x, A_t)$ and $\mathbf{B} = (B_x, B_t)$ are two vectors, then the "dot product" is defined as

$$\mathbf{A} \cdot \mathbf{B} = A_t B_t - A_x B_x \tag{5.2}$$

Equation (5.15) therefore, says that spacetime acceleration is orthogonal to spacetime velocity.

(b) Verify that spacetime acceleration transforms via LTs. That is, if O measures the components of the spacetime acceleration of a particle to be (a_{bx}, a_{bt}) and O' measures them to be (a'_{bx}, a'_{bt}), then

$$a'_{bx} = \gamma(a_{bx} + \beta a_{bt}), \qquad a'_{bt} = \gamma(\beta a_{bx} + a_{bt}),$$

where β is the speed of O relative to O'.

Solution:

(a) Differentiate Equation (5.2) with respect to s and note that the right-hand side vanishes.

(b) Take the differential of Equation (5.4):

$$\begin{aligned} du'_{bx} &= \gamma(du_{bx} + \beta du_{bt}) \\ du'_{bt} &= \gamma(\beta du_{bx} + du_{bt}), \end{aligned}$$

and divide both sides by ds.

■

5.4. Let $y = f(x)$ describe a curve in the xy-plane on which a particle moves.

(a) Verify that for the velocity vector of the particle to be perpendicular to the position vector of that particle, $f(x)$ must satisfy the following differential equation

$$f(x)\frac{df}{dx} + x = 0.$$

(b) Solve this simple equation and show that the solution is

$$f(x) = \pm\sqrt{C - x^2},$$

where C is the constant of integration. This is a circle of radius $\sqrt{C}$.

(c) Now let $x = g(t)$ describe the world line of a particle in the spacetime plane. Refer to Equation (5.16) for the definition of the dot product in spacetime plane, and show that for the spacetime velocity vector of the particle to be perpendicular to the "position" vector (x, ct) of that particle, $g(t)$ must satisfy the following differential equation

$$g(t)\frac{dg}{dt} - c^2 t = 0.$$

(d) Solve this simple equation and show that the solution is

$$g(t) = \pm\sqrt{C + c^2t^2},$$

where C is the constant of integration. What kind of a curve is this worldline?

Solution:

(a) The position vector of the particle is $\langle x, f(x)\rangle$ and its velocity vector is

$$\langle \dot{x}, \dot{y}\rangle = \langle \dot{x}, \dot{x}df/dx\rangle,$$

where dot represents differentiation with respect to t. If the position vector is to be perpendicular to the velocity vector, we must have

$$\langle x, f(x)\rangle \cdot \langle \dot{x}, \dot{x}df/dx\rangle = 0 \iff x\dot{x} + \dot{x}f(x)df/dx = 0$$

or

$$f(x)\frac{df}{dx} + x = 0.$$

(b) Rewrite the equation as $fdf + xdx = 0$, and integrate it to get

$$\tfrac{1}{2}f^2 + \tfrac{1}{2}x^2 = \text{constant} \iff f^2 + x^2 = C \iff f(x) = \pm\sqrt{C - x^2}.$$

(c) The spacetime "position" vector of the particle is $\langle g(t), ct\rangle$ and its spacetime velocity vector is $\langle dg/ds, cdt/ds\rangle$. For the spacetime velocity vector of the particle to be perpendicular to its spacetime velocity vector, we must have

$$\langle g(t), ct\rangle \cdot \langle dg/ds, cdt/ds\rangle = 0 \iff g(dg/ds) - c^2t(dt/ds) = 0.$$

(d) Multiply the previous equation by ds and integrate to get

$$\tfrac{1}{2}g^2 - \tfrac{1}{2}c^2t^2 = \text{constant} \iff g^2 - c^2t^2 = C \iff g(t) = \pm\sqrt{C + c^2t^2},$$

which is a hyperbola.

■

5.5. Use Equations (5.7) and (5.8) to derive the relativistic law of addition of velocities.

Solution: Multiply the first equation in (5.8) by c, divide it by the second equation, and use (5.7):

$$\beta_b' = \frac{p_b' c}{E_b'} = \frac{\gamma(p_b c + \beta E_b)}{\gamma(\beta p_b c + E_b)} = \frac{p_b c/E_b + \beta}{\beta p_b c/E_b + 1} = \frac{\beta_b + \beta}{\beta \beta_b + 1}.$$

■

5.6. Derive Equation (5.10) directly from the definition of energy and momentum in Equations (5.6) and (5.5). Now use (5.8) to show that Equation (5.10) holds in all RFs.

Solution: From (5.5) and (5.6), we obtain

$$E_b^2 - p_b^2 c^2 = (\gamma_b m c^2)^2 - (mc\gamma_b \beta_b)^2 c^2 = m^2 c^4 \gamma_b^2 (1 - \beta_b^2) = m^2 c^4.$$

From (5.8), we get

$$\begin{aligned} E_b'^2 - p_b'^2 c^2 &= \gamma^2 (\beta p_b c + E_b)^2 - \gamma^2 (p_b c + \beta E_b)^2 \\ &= \gamma^2 (\beta^2 p_b^2 c^2 + E_b^2 + 2\beta p_b c E_b) - \gamma^2 (p_b^2 c^2 + \beta^2 E_b^2 + 2\beta p_b c E_b) \\ &= \gamma^2 [E_b^2 (1 - \beta^2) - p_b^2 c^2 (1 - \beta^2)] = E_b^2 - p_b^2 c^2 \end{aligned}$$

■

5.7. Using the definition of relativistic momentum and the relativistic law of addition of velocities, show that if relativistic momentum is conserved in all RFs, then relativistic energy must also be conserved.

Solution: Let two particles of masses m_1 and m_2 and speeds $c\beta_1$ and $c\beta_2$ collide to produce two other particles of masses m_3 and m_4 and speeds $c\beta_3$ and $c\beta_4$. Then, in some RF O, the conservation of relativistic momentum gives

$$m_1 \gamma_1 c \beta_1 + m_2 \gamma_2 c \beta_2 = m_3 \gamma_3 c \beta_3 + m_4 \gamma_4 c \beta_4.$$

Now let O' be another RF with respect to which O moves with speed β in the positive x direction. Let i be one of the numbers 1, 2, 3, or 4. Then, in O', the speeds and the Lorentz factors of the particles are

$$\beta_i' = \frac{\beta_i + \beta}{1 + \beta_i \beta}, \quad \gamma_i' = \gamma \gamma_i (1 + \beta_i \beta), \iff \beta_i' \gamma_i' = \gamma \gamma_i (\beta_i + \beta), \quad i = 1, 2, 3, 4.$$

I have already derived the middle equality, but it is a good exercise for you to derive it again. In O', the conservation of momentum is

$$m_1 \gamma_1' \beta_1' + m_2 \gamma_2' \beta_2' = m_3 \gamma_3' \beta_3' + m_4 \gamma_4' \beta_4',$$

or

$$m_1 \gamma \gamma_1 (\beta_1 + \beta) + m_2 \gamma \gamma_2 (\beta_2 + \beta) = m_3 \gamma \gamma_3 (\beta_3 + \beta) + m_4 \gamma \gamma_4 (\beta_4 + \beta),$$

or

$$m_1 \gamma_1 \beta_1 + m_1 \gamma_1 \beta + m_2 \gamma_2 \beta_2 + m_2 \gamma_2 \beta = m_3 \gamma_3 \beta_3 + m_3 \gamma_3 \beta + m_4 \gamma_4 \beta_4 + m_4 \gamma_4 \beta.$$

Conservation of momentum in O eliminates the sum of the first and third terms on each side, leaving

$$m_1 \gamma_1 \beta + m_2 \gamma_2 \beta = m_3 \gamma_3 \beta + m_4 \gamma_4 \beta.$$

Canceling β and multiplying by c^2 yields the conservation of energy. ■

5.8. A particle of mass m moving at speed v collides with another particle of the same mass at rest. They stick together and move with speed V. What is V in terms of v? What is the mass of the final combined particle?

Solution: Let M be the mass of the final particle. Then, conservation of momentum and energy yield the following two equations:

$$m\gamma_v v = M\gamma_V V, \quad mc^2\gamma_v + mc^2 = Mc^2\gamma_V.$$

The second equation gives $M\gamma_V = m(1+\gamma_v)$. Substituting this in the first equation yields

$$m\gamma_v v = m(1+\gamma_v)V \iff V = \frac{\gamma_v v}{1+\gamma_v} = \frac{v}{1+\sqrt{1-(v/c)^2}}$$

and

$$\gamma_V = \frac{1}{\sqrt{1-\left(\dfrac{\gamma_v v}{1+\gamma_v}\right)^2}} = \frac{1+\gamma_v}{\sqrt{(1+\gamma_v)^2-\gamma_v^2 v^2}} = \sqrt{\frac{1+\gamma_v}{2}}.$$

Now we can calculate M:

$$M = \frac{m(1+\gamma_v)}{\gamma_V} = m\sqrt{2(1+\gamma_v)}.$$

Note that when $\gamma_v \to 1$, i.e., in the classical limit, $M = 2m$ confirming the conservation of mass in classical collisions. ■

5.9. Electrons in projection TV sets are accelerated through a potential difference of 50 kV.

(a) Find the speed of the electrons using the *relativistic* form of KE (relativistic energy minus rest energy) assuming that the electrons start from rest.

(b) Find the speed of the electrons using the *classical* form of KE.

(c) Does the difference in speed have to be taken into account when designing the TV set?

Solution: By the definition of the electron volt, the potential energy of the electrons is 50 keV.

(a) If electrons start from rest, their KE must equal their potential energy:

$$m_e c^2\gamma_e - m_e c^2 = 50 \text{ keV} \iff 511(\gamma_e - 1) \text{ keV} = 50 \text{ keV} \iff \gamma_e = 1.0978,$$

because $m_e c^2 = 0.511$ MeV. This gives $\beta_e = 0.413$.

(b) From

$$KE_e = \tfrac{1}{2}m_e v_e^2 = \tfrac{1}{2}m_e c^2(v_e/c)^2 = \tfrac{1}{2}m_e c^2\beta_e^2 \iff 50 = \tfrac{1}{2}\times 511\beta_e^2,$$

we get $\beta_e = 0.442$.

(c) The difference between classical and relativistic speeds is only 7%, which is within the tolerance of the operation of a TV set.

■

5.10. Two identical particles of mass m approaching each other with velocity β collide and as a result of their collision a particle of mass M and a photon are produced.

(a) Can M remain stationary as in Example 5.3.3?

(b) If the answer to (a) is no, then what are the energies of M and the photon?

(c) What are the momenta of M and the photon?

Solution:

(a) No. The initial momentum is zero. Photon cannot be stationary, so M must have some momentum to cancel the momentum of the photon.

(b) Conservation of momentum and energy give two equations:

$$2mc^2\gamma = E + e, \quad 0 = P + p \iff |p| = |P| \iff e = |P|c = \sqrt{E^2 - M^2c^4},$$

where upper case letters correspond to M and lower case letters to photon. Substitute the last equation in the first to get

$$2mc^2\gamma = E + \sqrt{E^2 - M^2c^4} \iff (2mc^2\gamma - E)^2 = E^2 - M^2c^4,$$

which in combination with $2mc^2\gamma = E + e$ yields

$$E = mc^2\gamma + \frac{M^2c^2}{4m\gamma} \quad \text{and} \quad e = mc^2\gamma - \frac{M^2c^2}{4m\gamma}.$$

(c) The magnitude of the two momenta are equal:

$$|p| = e/c = mc\gamma - \frac{M^2c}{4m\gamma} = |P|.$$

■

5.11. Two particles of masses m_1 and m_2 are moving in opposite directions with the same relativistic momentum p.

(a) Find their speeds v_1 and v_2, in terms of m_1, m_2, and p.

(b) Find their energies E_1 and E_2, in terms of m_1, m_2, and p.

(c) The two particles collide and form a particle of mass M at rest. Find the mass of the final particle.

(d) Verify that the extra mass (times c^2) is equal to the total initial kinetic energy of the two particles.

Solution: Equation (5.7) gives speed in terms of momentum and energy.

(a) Therefore,

$$v_1 = \frac{pc^2}{E_1} = \frac{pc^2}{\sqrt{p^2c^2 + m_1^2c^4}} = \frac{pc}{\sqrt{p^2 + m_1^2c^2}}, \quad v_2 = \frac{pc}{\sqrt{p^2 + m_2^2c^2}}.$$

(b)

$$E_1 = \sqrt{p^2c^2 + m_1^2c^4}, \quad E_2 = \sqrt{p^2c^2 + m_2^2c^4}.$$

(c) Conservation of energy gives $E_1 + E_2 = Mc^2$. Therefore,

$$M = \frac{E_1 + E_2}{c^2} = \frac{\sqrt{p^2 + m_1^2c^2} + \sqrt{p^2 + m_2^2c^2}}{c}$$

(d)

$$\begin{aligned}(M - m_1 - m_2)c^2 &= \left(\frac{E_1 + E_2}{c^2} - m_1 - m_2\right)c^2 \\ &= E_1 - m_1c^2 + E_2 - m_2c^2 = KE_1 + KE_2.\end{aligned}$$

■

5.12. A particle of mass M at rest decays into two particles of masses m_1 and m_2.

(a) What is the sum of the momenta of the two particles?

(b) Find their energies E_1 and E_2, in terms of M, m_1, and m_2.

(c) Verify that the common momentum of the particles can be expressed as

$$p = \frac{\sqrt{(M^2 - m_1^2)^2 + (M^2 - m_2^2)^2 - M^4 - 2m_1^2m_2^2}}{2M}\, c.$$

Note the symmetry of the expression in m_1 and m_2.

(d) Show that if the masses are equal, then $E_1 = E_2$ and that these are independent of the common mass of the particles.

(e) Without going through the previous parts, show that if a particle of mass M at rest decays into two identical particles, the energies of the two particles are equal and this energy is the same regardless of their mass. In particular, whether M decays into two massive or massless identical particles, the particles carry the same amount of energy. What is that energy?

Solution:

(a) Initial momentum is zero. Therefore, the sum of the final momenta is also zero. Let p denote the *magnitude* of the common momentum of the two particles.

(b) Conservation of energy is $Mc^2 = E_1 + E_2$, or

$$Mc^2 = E_1 + \sqrt{p^2c^2 + m_2^2c^4} = E_1 + \sqrt{E_1^2 - m_1^2c^4 + m_2^2c^4}$$

or

$$(Mc^2 - E_1)^2 = E_1^2 - m_1^2c^4 + m_2^2c^4,$$

yielding

$$E_1 = \frac{Mc^2}{2} + \frac{(m_1^2 - m_2^2)c^2}{2M}.$$

Similarly,

$$E_2 = \frac{Mc^2}{2} + \frac{(m_2^2 - m_1^2)c^2}{2M}.$$

(c) From (b), we have

$$p^2c^2 = E_1^2 - m_1^2c^4 = \left[\frac{M^2c^2 + (m_1^2 - m_2^2)c^2}{2M}\right]^2 - m_1^2c^4,$$

or

$$\begin{aligned}\frac{p^2}{c^2} &= \frac{M^4 + (m_1^2 - m_2^2)^2 + 2M^2(m_1^2 - m_2^2)}{4M^2} - m_1^2 \\ &= \frac{M^4 + m_1^4 + m_2^4 - 2m_1^2m_2^2 - 2M^2(m_1^2 + m_2^2)}{4M^2} \\ &= \frac{(M^2 - m_1^2)^2 + (M^2 - m_2^2)^2 - M^4 - 2m_1^2m_2^2}{4M^2}.\end{aligned}$$

(d) If $m_1 = m_2$, then (b) gives

$$E_1 = \frac{Mc^2}{2} = E_2.$$

(e) Identity of the particles plus the fact that the initial momentum is zero implies that the particles have the same mass and momentum, therefore the same energy. Conservation of energy now gives $2E = Mc^2$, or $E = \frac{1}{2}Mc^2$.

■

5.13. The neutral pion π^0 has a mass of 2.41×10^{-28} kg. It decays into two photons 98.8% of the times.

(a) What is the energy of each photon in the rest frame of the pion?

(b) What is the wavelength of each photon? Remember that $\lambda = hc/E$ where h is the Planck constant.

(c) Assume that the pion is moving at $0.9c$ in the lab frame in the same direction as one of the photons. What are the energies and wavelengths of each photon in the lab?

Solution:

(a) From symmetry (or look at the solution to Problem 5.12), the energies of the two photons should be equal. Therefore,

$$e = \frac{m_{\pi^0}c^2}{2} = \frac{2.41 \times 10^{-28} \times (3 \times 10^8)^2}{2} = 1.08 \times 10^{-11} \text{ J}.$$

(b)

$$\lambda = \frac{hc}{e} = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{1.08 \times 10^{-11}} = 1.84 \times 10^{-14} \text{ m}.$$

This is in the short-wave gamma ray region of the EM spectrum.

(c) Assume that the pion is moving in the positive x direction in the lab. Then (5.8) gives

$$e_1' = \gamma(\beta p_1 c + e) = \gamma(\beta e + e) = \sqrt{\frac{1+\beta}{1-\beta}}\, e = 4.7 \times 10^{-11} \text{ J}$$

$$e_2' = \gamma(\beta p_2 c + e) = \gamma(-\beta e + e) = \sqrt{\frac{1-\beta}{1+\beta}}\, e = 2.48 \times 10^{-12} \text{ J},$$

and

$$\lambda_1' = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{4.7 \times 10^{-11}} = 4.23 \times 10^{-15} \text{ m}$$
$$\lambda_2' = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{2.48 \times 10^{-12}} = 8.08 \times 10^{-14} \text{ m}.$$

■

5.14. The negative pion π^- has a mass of 2.49×10^{-28} kg. It decays into a muon and a neutrino 99.988% of the times. Muon has a mass of 1.89×10^{-28} kg and neutrino is essentially massless.

(a) What are the energies of the muon and the neutrino in the pion's rest frame?

(b) What is the momentum of the neutrino?

(c) What is the speed of the muon?

(d) Assume that the pion is moving at $0.9c$ in the lab frame in the same direction as the neutrino. What are the energies and momenta of the muon and the neutrino in the lab?

Solution: The formulas are given in the solution of Problem 5.12.

(a) The unit that is more conveniently used for the masses (times c^2) is MeV. In that unit $m_{\pi^-}c^2 = 139.57$ MeV and $m_\mu c^2 = 105.66$ MeV, and

$$E_\mu = \frac{m_{\pi^-}c^2}{2} + \frac{m_\mu^2 c^2}{2m_{\pi^-}} = \frac{139.57}{2} + \frac{105.66^2}{279.14} = 109.78 \text{ MeV}$$
$$E_\nu = \frac{m_{\pi^-}c^2}{2} - \frac{m_\mu^2 c^2}{2m_{\pi^-}} = \frac{139.57}{2} - \frac{105.66^2}{279.14} = 29.79 \text{ MeV}.$$

(b) Since neutrino is assumed massless, $E_\nu = p_\nu c$, and $p_\nu = 29.79$ MeV/c.

(c) Muon has the same momentum as the neutrino. Thus, by (5.7),

$$\beta_\mu = \frac{p_\nu c}{E_\mu} = \frac{E_\nu}{E_\mu} = \frac{29.79}{109.78} = 0.27.$$

(d) Assume that the pion is moving in the positive x direction in the lab. Then (5.8) gives

$$E_\mu' = \gamma(\beta p_\mu c + E_\mu) = 2.29(-0.9 \times 29.79 + 109.78) = 190 \text{ MeV}$$
$$E_\nu' = \gamma(\beta p_\nu c + E_\nu) = 2.29(0.9 \times 29.79 + 29.79) = 129.6 \text{ MeV}.$$

and

$$p_\mu' c = \sqrt{E_\mu'^2 - m_\mu^2 c^4} = \sqrt{190^2 - 105.66^2} = 157.9 \text{ MeV} \iff p_\mu' = 157.9 \text{ MeV}/c$$
$$p_\nu' c = E_\nu' = 129.6 \text{ MeV} \iff p_\nu' = 129.6 \text{ MeV}/c.$$

■

5.15. This problem continues the discussion of Example 5.3.4. Two 0.5-kg pieces of clay, each moving at the rate of 2000 m/s toward the other, collide.

(a) What is the total KE distributed among the molecules of the end piece?

(b) Each kilogram of clay has about 4 moles, or about 2.4×10^{24} molecules. What is the average KE that each molecule receives?

(c) Use $KE_{\text{avg}} = \frac{3}{2}k_B T$, where $k_B = 1.38 \times 10^{-23}$ J/K is the Boltzmann constant, to find the temperature of the final piece of clay.

Solution: The problem is entirely classical.

(a)

$$KE_{\text{tot}} = 2KE = mv^2 = 0.5 \times 2000^2 = 2 \times 10^6 \text{ J}.$$

(b)

$$KE_{\text{avg}} = \frac{2 \times 10^6}{2.4 \times 10^{24}} = 8.33 \times 10^{-19} \text{ J}.$$

(c)

$$T = \frac{2KE_{\text{avg}}}{3k_B} = \frac{2 \times 8.33 \times 10^{-19}}{3 \times 1.38 \times 10^{-23}} = 40258\ K.$$

■

CHAPTER 6

Relativity in Four Dimensions

Problems With Solutions

6.1. Multiply the matrices in Equation (6.5) to obtain the three equations of (6.6). Solve these equations to find all matrix elements in terms of a_{11}.

Solution: Multiply the two matrices on the right:

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ -a_{21} & -a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Continue the multiplication:

$$\begin{pmatrix} a_{11}^2 - a_{21}^2 & a_{11}a_{12} - a_{21}a_{22} \\ a_{11}a_{12} - a_{21}a_{22} & a_{12}^2 - a_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now note that two matrices are equal if and only if all their corresponding elements are equal. This leads to (6.6). The rest is given in the text. ■

6.2. Show that if $\vec{\beta}$ is along the x-axis, then (6.14) reduces to (6.13) and (6.16) to (6.12).

Solution: With $\beta_x = \beta$ and $\beta_y = 0 = \beta_z$, the 4×4 matrix in (6.14) becomes obviously equal to the matrix in (6.13) except for the element in the second row and second column. But that element is equal to:

$$1 + \hat{\beta}_x^2(\gamma - 1) = 1 + \hat{\beta}^2(\gamma - 1) = \gamma$$

because $\hat{\beta}^2 = 1$.

With $\vec{\beta} = \langle \beta, 0, 0 \rangle$ and $\vec{r} = \langle x, y, z \rangle$, we get $\vec{\beta} \cdot \vec{r} = \beta x$. Plugging this in the first equation of (6.16) gives the first equation in (6.12). Write the second equation of (6.16) as a column vector:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \gamma\beta ct \\ 0 \\ 0 \end{pmatrix} + (\gamma - 1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x,$$

where I used $\hat{\beta} \cdot \vec{r} = x$. Equating the first component on either side yields

$$x' = x + \gamma\beta ct + (\gamma - 1)x = \gamma(\beta ct + x)$$

and the equality of the second and third components gives $y' = y$ and $z' = z$. ■

6.3. Derive Equation (6.17).

Solution: Write $\vec{r} = \vec{r}_{||} + \vec{r}_{\perp}$ where $\vec{r}_{||} \cdot \vec{\beta} = |\vec{r}_{||}||\vec{\beta}|$ and $\vec{r}_{\perp} \cdot \vec{\beta} = 0$. Then the first equation of (6.17) follows immediately. Writing $\vec{r}\,' = \vec{r}_{||}\,' + \vec{r}_{\perp}\,'$ as well, the second equation of (6.16) can be expressed as

$$\vec{r}_{||}\,' + \vec{r}_{\perp}\,' = \vec{r}_{||} + \vec{r}_{\perp} + \gamma\vec{\beta}ct + (\gamma - 1)\underbrace{\hat{\beta}\,|\vec{r}_{||}|}_{=\vec{r}_{||}} = \gamma(\vec{r}_{||} + \vec{\beta}ct) + \vec{r}_{\perp}.$$

Equating the parallel and perpendicular components of both sides yield the remaining equations of (6.17). ■

6.4. Derive Equation (6.18).

Solution: The differentials of (6.16) for a "ball" are

$$\begin{aligned} cdt' &= \gamma(cdt + \vec{\beta} \cdot d\vec{r}_b) \\ d\vec{r}\,'_b &= d\vec{r}_b + \gamma\vec{\beta}cdt + (\gamma - 1)\hat{\beta}\hat{\beta} \cdot d\vec{r}_b. \end{aligned}$$

Divide the second equation by the first to get

$$\begin{aligned} \vec{\beta}\,'_b &= \frac{d\vec{r}_b + \gamma\vec{\beta}cdt + (\gamma - 1)\hat{\beta}\hat{\beta} \cdot d\vec{r}_b}{\gamma(cdt + \vec{\beta} \cdot d\vec{r}_b)} \\ &= \frac{d\vec{r}_b/cdt + \gamma\vec{\beta} + (\gamma - 1)\hat{\beta}\hat{\beta} \cdot (d\vec{r}_b/cdt)}{\gamma[1 + \vec{\beta} \cdot (d\vec{r}_b/cdt)]} \\ &= \frac{\vec{\beta}_b + \gamma\vec{\beta} + (\gamma - 1)\hat{\beta}\hat{\beta} \cdot \vec{\beta}_b}{\gamma(1 + \vec{\beta} \cdot \vec{\beta}_b)}. \end{aligned}$$

■

6.5. Observer O moves relative to observer O' with velocity $\vec{\beta}_1$. Observer O' moves relative to observer O'' with velocity $\vec{\beta}_2$. Therefore, observer O moves relative to observer O'' with some velocity $\vec{\beta}$. Using matrices—as given in Equations (6.14) and (6.15)—find the general relativistic law of addition of velocities. Hint: See Example 3.5.1 and note that calculating the two elements in the first column of the block form of the product $\Lambda_1\Lambda_2$ is sufficient to yield the answer.

Solution: If $(ct, \vec{r})$ is the 4-vector in O, then $\Lambda_1(ct, \vec{r})$ is $(ct'\vec{r}\,')$, the 4-vector in O' and $\Lambda_2\Lambda_1(ct, \vec{r})$ is $(ct'', \vec{r}\,'')$, the 4-vector in O''. Therefore, if Λ_1 is parametrized by $\vec{\beta}_1$ and Λ_2 is parametrized by $\vec{\beta}_2$, then $\Lambda = \Lambda_2\Lambda_1$ is parametrized by $\vec{\beta}$. This means that

$$\begin{pmatrix} \gamma & \gamma\widetilde{\vec{\beta}} \\ \gamma\vec{\beta} & \overleftrightarrow{\Lambda} \end{pmatrix} = \begin{pmatrix} \gamma_2 & \gamma_2\widetilde{\vec{\beta}}_2 \\ \gamma_2\vec{\beta}_2 & \overleftrightarrow{\Lambda}_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_1\widetilde{\vec{\beta}}_1 \\ \gamma_1\vec{\beta}_1 & \overleftrightarrow{\Lambda}_1 \end{pmatrix} = \begin{pmatrix} \gamma_2\gamma_1 + \gamma_2\gamma_1\widetilde{\vec{\beta}}_2\vec{\beta}_1 & \gamma_2\gamma_1\widetilde{\vec{\beta}}_1 + \gamma_2\widetilde{\vec{\beta}}_2\overleftrightarrow{\Lambda}_1 \\ \gamma_2\gamma_1\vec{\beta}_2 + \gamma_1\overleftrightarrow{\Lambda}_2\vec{\beta}_1 & \gamma_2\gamma_1\vec{\beta}_2\widetilde{\vec{\beta}}_1 + \overleftrightarrow{\Lambda}_2\overleftrightarrow{\Lambda}_1 \end{pmatrix}. \tag{6.1}$$

Note that $\vec{\beta}_1$ and $\vec{\beta}_2$ are column vectors and $\widetilde{\vec{\beta}}_1$ and $\widetilde{\vec{\beta}}_2$ row vectors. So,

$$\widetilde{\vec{\beta}}_2\vec{\beta}_1 = \begin{pmatrix}\beta_{2x} & \beta_{2y} & \beta_{2z}\end{pmatrix}\begin{pmatrix}\beta_{1x}\\ \beta_{1y}\\ \beta_{1z}\end{pmatrix} = \beta_{2x}\beta_{1x} + \beta_{2y}\beta_{1y} + \beta_{2z}\beta_{1z} = \vec{\beta}_2 \cdot \vec{\beta}_1.$$

Therefore, equating the element in the first row and first column on the left and the right, we get

$$\gamma = \gamma_1\gamma_2(1 + \vec{\beta}_1 \cdot \vec{\beta}_2). \tag{6.2}$$

I'm not going to find all the elements of the product. All I need for the purposes of this problem is the element in the second row and first column of the product. First I calculate the matrix product of the second term:

$$\begin{aligned}\overleftrightarrow{\Lambda}_2\,\vec{\beta}_1 &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\beta_{1x}\\ \beta_{1y}\\ \beta_{1z}\end{pmatrix} + (\gamma_2 - 1)\begin{pmatrix}\hat{\beta}_{2x}^2 & \hat{\beta}_{2x}\hat{\beta}_{2y} & \hat{\beta}_{2x}\hat{\beta}_{2z}\\ \hat{\beta}_{2x}\hat{\beta}_{2y} & \hat{\beta}_{2y}^2 & \hat{\beta}_{2y}\hat{\beta}_{2z}\\ \hat{\beta}_{2x}\hat{\beta}_{2z} & \hat{\beta}_{2y}\hat{\beta}_{2z} & \hat{\beta}_{2z}^2\end{pmatrix}\begin{pmatrix}\beta_{1x}\\ \beta_{1y}\\ \beta_{1z}\end{pmatrix}\\ &= \vec{\beta}_1 + (\gamma_2 - 1)\begin{pmatrix}\hat{\beta}_{2x}\hat{\beta}_2 \cdot \vec{\beta}_1\\ \hat{\beta}_{2y}\hat{\beta}_2 \cdot \vec{\beta}_1\\ \hat{\beta}_{2z}\hat{\beta}_2 \cdot \vec{\beta}_1\end{pmatrix} = \vec{\beta}_1 + (\gamma_2 - 1)\begin{pmatrix}\hat{\beta}_{2x}\\ \hat{\beta}_{2y}\\ \hat{\beta}_{2z}\end{pmatrix}\hat{\beta}_2 \cdot \vec{\beta}_1,\end{aligned}$$

or

$$\overleftrightarrow{\Lambda}_2\,\vec{\beta}_1 = \vec{\beta}_1 + (\gamma_2 - 1)\hat{\beta}_2\hat{\beta}_2 \cdot \vec{\beta}_1.$$

So, the second-row-first-column element of the matrix on the right-hand side of (6.1) is

$$\gamma_2\gamma_1\vec{\beta}_2 + \overleftrightarrow{\Lambda}_2\,\vec{\beta}_1 = \gamma_2\gamma_1\vec{\beta}_2 + \gamma_1\left[\vec{\beta}_1 + (\gamma_2 - 1)\hat{\beta}_2\hat{\beta}_2 \cdot \vec{\beta}_1\right],$$

and, using (6.2), the corresponding element on the left is

$$\gamma\vec{\beta} = \gamma_1\gamma_2(1 + \vec{\beta}_1 \cdot \vec{\beta}_2)\vec{\beta}.$$

Equating the last two equations gives

$$\begin{aligned}\vec{\beta} &= \frac{\gamma_2\gamma_1\vec{\beta}_2 + \gamma_1\left[\vec{\beta}_1 + (\gamma_2 - 1)\hat{\beta}_2\hat{\beta}_2 \cdot \vec{\beta}_1\right]}{\gamma_1\gamma_2(1 + \vec{\beta}_1 \cdot \vec{\beta}_2)}\\ &= \frac{\vec{\beta}_1 + \gamma_2\vec{\beta}_2 + (\gamma_2 - 1)\hat{\beta}_2\hat{\beta}_2 \cdot \vec{\beta}_1}{\gamma_2(1 + \vec{\beta}_1 \cdot \vec{\beta}_2)},\end{aligned}$$

which is the relativistic LAV (6.18). ■

6.6. In Fizeau's experiment (see Example 3.5.2), light moves *perpendicular* to the direction of motion of the transparent medium.

(a) Show that

$$v' = \sqrt{\left(\frac{c}{n}\right)^2 + v^2\left(1 - \frac{1}{n^2}\right)}.$$

(b) Show that for $v << c$, a complete drag—in which the classical LAV holds—yields

$$v' = \frac{c}{n} + \frac{nv^2}{2c}.$$

(c) Show that relativistically, for low velocity, the drag is not complete but is smaller by $v^2/(2nc)$.

Solution: In Equation (6.18) of the text, let $\vec{\beta}_b = (c/n)\hat{\mathbf{e}}_y$, and $\vec{\beta} = \beta\hat{\mathbf{e}}_x$.

(a) Then

$$\frac{\vec{v}\,'}{c} = \frac{(1/n)\hat{\mathbf{e}}_y + \gamma\beta\hat{\mathbf{e}}_x}{\gamma} = \frac{1}{n\gamma}\hat{\mathbf{e}}_y + \beta\hat{\mathbf{e}}_x$$

and

$$\frac{|\vec{v}\,'|^2}{c^2} = \frac{1}{n^2\gamma^2} + \beta^2 = \frac{1}{n^2}(1-\beta^2) + \beta^2 = \frac{1}{n^2} + \beta^2\left(1 - \frac{1}{n^2}\right)$$

or

$$|\vec{v}\,'| = \sqrt{\left(\frac{c}{n}\right)^2 + v^2\left(1 - \frac{1}{n^2}\right)}.$$

(b) From the classical LAV, we get

$$\vec{v}\,' = \frac{c}{n}\hat{\mathbf{e}}_y + v\hat{\mathbf{e}}_x \iff |\vec{v}\,'|^2 = \frac{c^2}{n^2} + v^2 \iff |\vec{v}\,'| = \frac{c}{n}\sqrt{1 + \frac{n^2v^2}{c^2}}.$$

Approximating the square root by

$$\sqrt{1 + \frac{n^2v^2}{c^2}} \approx 1 + \frac{n^2v^2}{2c^2}$$

because $v << c$, we get

$$|\vec{v}\,'| = \frac{c}{n} + \frac{nv^2}{2c}.$$

(c) Write the relativistic result in (a) as

$$|\vec{v}\,'| = \frac{c}{n}\sqrt{1 + \frac{n^2v^2}{c^2}\left(1 - \frac{1}{n^2}\right)}.$$

For $v << c$, this can be approximated as

$$|\vec{v}\,'| = \frac{c}{n}\sqrt{1 + \frac{n^2v^2}{c^2}\left(1 - \frac{1}{n^2}\right)} \approx \frac{c}{n}\left[1 + \frac{n^2v^2}{2c^2}\left(1 - \frac{1}{n^2}\right)\right].$$

or

$$|\vec{v}\,'| \approx \frac{c}{n} + \frac{nv^2}{2c}\left(1 - \frac{1}{n^2}\right) = \frac{c}{n} + \frac{nv^2}{2c} - \frac{v^2}{2nc}.$$

Comparing this with the result in (b), we see that the drag is smaller by $v^2/(2nc)$.

■

6.7. Derive the velocity (6.23) directly from (6.18).

Solution: In (6.18), let $\vec{\beta}_b = \alpha\hat{\mathbf{e}}_y$ and $\vec{\beta} = \beta\hat{\mathbf{e}}_x$. Equation (6.23) follows immediately. ■

6.8. Use (6.25) and the trigonometric identity

$$\tan(\tfrac{1}{2}\varphi') = \frac{\sin\varphi'}{1+\cos\varphi'}$$

to show that Equation (6.26) holds.

Solution:

$$\tan(\tfrac{1}{2}\varphi') = \frac{\sin\varphi'}{1+\cos\varphi'} = \frac{\dfrac{\sin\varphi}{\gamma\left(1+|\vec{\beta}|\cos\varphi\right)}}{1+\dfrac{|\vec{\beta}|+\cos\varphi}{1+|\vec{\beta}|\cos\varphi}} = \frac{\sin\varphi}{\gamma(1+|\vec{\beta}|\cos\varphi+|\vec{\beta}|+\cos\varphi)}$$

$$= \frac{\sin\varphi}{\gamma(1+|\vec{\beta}|)(1+\cos\varphi)} = \frac{\sqrt{1-|\vec{\beta}|^2}}{1+|\vec{\beta}|}\frac{\sin\varphi}{1+\cos\varphi} = \sqrt{\frac{1-|\vec{\beta}|}{1+|\vec{\beta}|}}\tan(\tfrac{1}{2}\varphi).$$

■

6.9. The headlights of a car send light only in the forward *hemisphere* as seen in the rest frame of the car. Show that the maximum angle as seen by a ground observer is $\tan^{-1}[1/(\gamma\beta)]$, or $\cos^{-1}\beta$.

Solution: The maximum angle in the rest frame is $\pi/2$. Therefore, by (6.25), $\cos\varphi' = |\vec{\beta}|$, $\sin\varphi' = 1/\gamma$, and $\tan\varphi' = \sin\varphi'/\cos\varphi' = 1/(|\vec{\beta}|\gamma)$. ■

6.10. Use Figure 6.1(d) and the law of sines to obtain

$$\frac{c}{\cos\theta'} = \frac{c\beta}{\sin(\theta'-\theta)}.$$

Show that this can also be written as

$$\tan\theta' = \frac{\beta+\sin\theta}{\cos\theta}$$

which is the non-relativistic version of (6.28).

Solution: Figure 6.1 of the manual labels the appropriate angles. With those labels, the law of sines immediately leads to

$$\frac{c}{\cos\theta'} = \frac{c\beta}{\sin(\theta'-\theta)}.$$

Rewrite the equation above as

$$\frac{1}{\cos\theta'} = \frac{\beta}{\sin\theta'\cos\theta-\cos\theta'\sin\theta} \iff \sin\theta'\cos\theta-\cos\theta'\sin\theta = \beta\cos\theta'.$$

Divide both sides by $\cos\theta'$ and solve for $\tan\theta'$ to get

$$\tan\theta' = \frac{\beta+\sin\theta}{\cos\theta}.$$

■

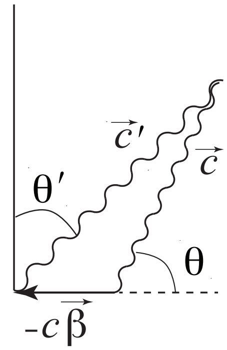


Figure 6.1: The non-relativistic LAV for aberration.

6.11. Derive Equation (6.29). Hint: Substitute $\theta' = \theta + \epsilon$ on the left-hand side of (6.28), expand both sides keeping only the first powers of ϵ and β; then equate the small terms on both sides.

Solution: The left-hand side is

$$\tan\theta' = \tan(\theta+\epsilon) = \frac{\tan\theta + \tan\epsilon}{1-\tan\theta\tan\epsilon},$$

which, to first order in ϵ gives

$$\tan\theta' \approx \frac{\tan\theta+\epsilon}{1-\epsilon\tan\theta} = (\tan\theta+\epsilon)(1-\epsilon\tan\theta)^{-1}.$$

Use binomial expansion to first order in ϵ to get

$$\tan\theta' \approx (\tan\theta+\epsilon)(1+\epsilon\tan\theta) \approx \tan\theta+\epsilon+\epsilon\tan^2\theta. = \tan\theta + \frac{\epsilon}{\cos^2\theta}.$$

With $\gamma \approx 1$, the right-hand side of (6.28) becomes $\tan\theta + |\vec{\beta}|/\cos\theta$. Thus equating the two sides, we get

$$\tan\theta + \frac{\epsilon}{\cos^2\theta} = \tan\theta + \frac{|\vec{\beta}|}{\cos\theta} \iff \epsilon = |\vec{\beta}|\cos\theta,$$

and $\theta' = \theta + \epsilon = \theta + |\vec{\beta}|\cos\theta$. ■

6.12. From a classical perspective, since satellite S_2 of Figure 6.3 is moving away from R, the light it emits moves slower than c while the light from S_1 moves faster. To simplify the classical derivation of (6.39) and (6.40), let $y = 0$. Let d_1 and d_2 be the distances of S_1 and S_2 from R, respectively. Since time is universal classically, the emission of signals occur at the same time according to O' as well as O. Use this to show that

$$\frac{d_1}{c+v} = \frac{d_2}{c-v}.$$

From this plus $d_1 + d_2 = L$ find d_1 and show that

$$x'_R = x'_1 + \frac{1}{2}L(1+\beta),$$

which is (6.40) for $\gamma \approx 1$ and $y = 0$. From this you can also find (6.39).

Solution: The light from S_1 moves at $c+v$ and covers the distance of d_1 in the same time interval that the light from S_2 moves at $c-v$ and covers the distance of d_2. Therefore,

$$\frac{d_1}{c+v} = \frac{d_2}{c-v} \iff d_2 = \frac{c-v}{c+v}\,d_1 = \frac{1-\beta}{1+\beta}\,d_1,$$

and $d_1 + d_2 = L$ yields

$$L = d_1 + \frac{1-\beta}{1+\beta}\,d_1 = \frac{2d_1}{1+\beta} \iff d_1 = \tfrac{1}{2}L(1+\beta).$$

The final answer is obtained once we note that $d_1 = x'_1 - x'_R$. ■

6.13. Show that

(a) Equation (6.41) reduces to the non-relativistic law when all velocities are small and to (3.26) when all velocities are in the same direction.

(b) Take the dot product of both sides of Equation (6.41) to find $|\vec{\alpha}'|^2 = \vec{\alpha}' \cdot \vec{\alpha}'$ in terms of unprimed quantities.

(c) From (b) calculate $\gamma_{\alpha'}$ and verify Equation (6.46).

(d) Show also that if $\vec{\alpha}$ is the velocity of light, so that $\vec{\alpha} \cdot \vec{\alpha} = 1$, then $\vec{\alpha}' \cdot \vec{\alpha}' = 1$ as well.

Solution:

(a) Keeping only first order terms in Equation (6.41), we get $\gamma \approx 1$ and $\vec{\beta} \cdot \vec{\alpha} \approx 0$, and the equation reduces to $\vec{\alpha}' = \vec{\alpha} + \vec{\beta}$, which is the classical LAV.

When all velocities are in the same direction, $\hat{\beta}\hat{\beta} \cdot \vec{\alpha} = \vec{\alpha}$, and $\vec{\beta} \cdot \vec{\alpha} = |\vec{\beta}||\vec{\alpha}|$. Then Equation (6.41) becomes

$$\vec{\alpha}' = \frac{\gamma(\vec{\alpha} + \vec{\beta})}{\gamma(1 + |\vec{\beta}||\vec{\alpha}|)} = \frac{\vec{\alpha} + \vec{\beta}}{1 + |\vec{\beta}||\vec{\alpha}|},$$

which is Equation (3.26) if you ignore the vector signs.

(b) The resulting denominator is simply $\gamma^2(1+\vec{\beta}\cdot\vec{\alpha})^2$. So, let's manipulate the numerator

$$\begin{aligned} Num &= [\vec{\alpha} + \gamma\vec{\beta} + (\gamma-1)\hat{\beta}\hat{\beta}\cdot\vec{\alpha}] \cdot [\vec{\alpha} + \gamma\vec{\beta} + (\gamma-1)\hat{\beta}\hat{\beta}\cdot\vec{\alpha}] \\ &= |\vec{\alpha}|^2 + \gamma^2|\vec{\beta}|^2 + (\gamma-1)^2(\hat{\beta}\cdot\vec{\alpha})^2 + 2\gamma\vec{\beta}\cdot\vec{\alpha} + 2(\gamma-1)(\hat{\beta}\cdot\vec{\alpha})^2 + 2\gamma(\gamma-1)\vec{\beta}\cdot\vec{\alpha}. \end{aligned}$$

The sum of the third and fifth terms can be simplified:

$$\begin{aligned} \text{sum 3rd+5th} &= (\gamma-1)[(\gamma-1)(\hat{\beta}\cdot\vec{\alpha})^2 + 2(\hat{\beta}\cdot\vec{\alpha})^2] \\ &= (\gamma^2-1)(\hat{\beta}\cdot\vec{\alpha})^2 = \gamma^2|\vec{\beta}|^2(\hat{\beta}\cdot\vec{\alpha})^2 = \gamma^2(\vec{\beta}\cdot\vec{\alpha})^2. \end{aligned}$$

So the numerator now becomes

$$\begin{aligned} Num &= |\vec{\alpha}|^2 + \gamma^2|\vec{\beta}|^2 + \gamma^2(\vec{\beta}\cdot\vec{\alpha})^2 + 2\gamma^2\vec{\beta}\cdot\vec{\alpha} \\ &= |\vec{\alpha}|^2 + \gamma^2 - 1 + \gamma^2(\vec{\beta}\cdot\vec{\alpha})^2 + 2\gamma^2\vec{\beta}\cdot\vec{\alpha} \\ &= |\vec{\alpha}|^2 - 1 + \gamma^2[1 + (\vec{\beta}\cdot\vec{\alpha})^2 + 2\vec{\beta}\cdot\vec{\alpha}], \end{aligned}$$

and dividing it by the denominator leads to

$$|\vec{\alpha}'|^2 = \frac{|\vec{\alpha}|^2 - 1}{\gamma^2(1+\vec{\beta}\cdot\vec{\alpha})^2} + 1 = -\frac{1}{\gamma_\alpha^2\gamma^2(1+\vec{\beta}\cdot\vec{\alpha})^2} + 1.$$

(c) The last equation in (b) leads to

$$\frac{1}{\gamma_{\alpha'}^2} = \frac{1}{\gamma_\alpha^2\gamma^2(1+\vec{\beta}\cdot\vec{\alpha})^2} \iff \gamma_{\alpha'} = \gamma_\alpha\gamma(1+\vec{\beta}\cdot\vec{\alpha}).$$

(d) This trivially follows from the first equality of the last equation in (b).

■

6.14. What are the coordinates of a particle's 4-velocity in its own rest frame? Use this and Equation (6.45) to obtain (6.42), the particle's 4-velocity in an arbitrary frame.

Solution: The coordinates of a particle's 4-velocity in its own rest frame is $(u_0, \vec{u}) = (1, 0, 0, 0)$. Substituting this in Equation (6.45) yields $u_0' = \gamma \equiv \gamma_\beta$ and $\vec{u}\,' = \gamma\vec{\beta} \equiv \gamma_\beta\vec{\beta}$. These are the components of (6.42), except for the fact that now $\vec{\beta}$ is the velocity of the particle. ■

6.15. Provide the details of the proof of the statement: *a particle is massless if and only if it moves at light speed.*

Solution: From Equation (6.52), we get

$$|\vec{v}| = c \iff |\vec{p}|/E = 1/c \iff E = |\vec{p}|c \iff m = 0.$$

These are all if-and-only-if implications. In the last implication, I used (6.50). ■

6.16. Apply (6.53) to a photon moving in the $\vec{\beta}$-direction and use $|\vec{p}| = E/c$ to show that

$$E' = \sqrt{\frac{1-\beta}{1+\beta}}\,E.$$

Now use $E = hc/\lambda$ to find a formula for the relativistic Doppler shift.

Solution: If $\vec{p}$ is parallel to $\vec{\beta}$, then $\vec{p}\cdot\vec{\beta} = |\vec{p}||\vec{\beta}|$. Therefore, (6.53) becomes

$$E' = \gamma(E - c|\vec{p}||\vec{\beta}|) = \gamma(E - E|\vec{\beta}|) = \sqrt{\frac{1-\beta}{1+\beta}}\,E,$$

using the notation $\beta \equiv |\vec{\beta}|$. Writing the last equation in terms of wavelengths, we get

$$\frac{hc}{\lambda'} = \sqrt{\frac{1-\beta}{1+\beta}}\,\frac{hc}{\lambda} \iff \lambda' = \sqrt{\frac{1+\beta}{1-\beta}}\,\lambda.$$

This is the formula for the relativistic Doppler shift. ■

6.17. Use (6.52)—which holds for any particle in any frame—and (6.54) to find the general relativistic law of addition of velocities (6.41).

Solution: Write (6.52) as $c\vec{p}/E = \vec{\alpha}$. Then divide the left-hand side of the second equation in (6.54) by the first one, noting that $c = 1$:

$$\vec{\alpha}\,' \equiv \frac{\vec{p}\,'}{E'} = \frac{\vec{p} + \gamma\vec{\beta}E + (\gamma-1)\hat{\beta}\hat{\beta}\cdot\vec{p}}{\gamma(E + \vec{\beta}\cdot\vec{p})}.$$

Now divide the numerator and denominator by E:

$$\vec{\alpha}\,' = \frac{\vec{\alpha} + \gamma\vec{\beta} + (\gamma-1)\hat{\beta}\hat{\beta}\cdot\vec{\alpha}}{\gamma(1 + \vec{\beta}\cdot\vec{\alpha})}.$$

This is precisely (6.41). ■

6.18. Show that the 4-acceleration is orthogonal to the 4-velocity.

Solution: From Equation (6.43) we have $\mathbf{u}\bullet\mathbf{u} = 1$. Now differentiate this with respect to proper time and note that the right-hand side gives zero:

$$\mathbf{a}\bullet\mathbf{u} + \mathbf{u}\bullet\mathbf{a} = 2\mathbf{u}\bullet\mathbf{a} = 0.$$

■

6.19. Differentiate the space component of the 4-velocity with respect to τ to get the second equation in (6.56). Now show directly that $u_0 a_0 - \vec{u}\cdot\vec{a} = 0$.

Solution: Use dot to denote differentiation with respect to t: $\dot{A} \equiv dA/dt$.

$$\vec{a} = \frac{d\vec{u}}{d\tau} = \gamma_\alpha\frac{d\vec{u}}{dt} = \gamma_\alpha\frac{d}{dt}(\gamma_\alpha\vec{\alpha}) = \gamma_\alpha\left(\vec{\alpha}\dot{\gamma}_\alpha + \gamma_\alpha\dot{\vec{\alpha}}\right).$$

Now use $\dot{\gamma}_\alpha = \gamma_\alpha^3\vec{\alpha}\cdot\dot{\vec{\alpha}}$ obtained in the calculation of a_0 just before Equation (6.56) to obtain

$$\vec{a} = \gamma_\alpha\left[\vec{\alpha}\left(\gamma_\alpha^3\vec{\alpha}\cdot\dot{\vec{\alpha}}\right) + \gamma_\alpha\dot{\vec{\alpha}}\right] = \gamma_\alpha^2\left[\gamma_\alpha^2(\vec{\alpha}\cdot\dot{\vec{\alpha}})\vec{\alpha} + \dot{\vec{\alpha}}\right].$$

From (6.42), we have

$$\begin{aligned} u_0 a_0 - \vec{u}\cdot\vec{a} &= \gamma_\alpha(a_0 - \vec{\alpha}\cdot\vec{a}) = \gamma_\alpha\left[\gamma_\alpha^4\vec{\alpha}\cdot\dot{\vec{\alpha}} - \gamma_\alpha^4(\vec{\alpha}\cdot\dot{\vec{\alpha}})\vec{\alpha}\cdot\vec{\alpha} - \gamma_\alpha^2\vec{\alpha}\cdot\dot{\vec{\alpha}}\right] \\ &= \gamma_\alpha^3\vec{\alpha}\cdot\dot{\vec{\alpha}}\left[\gamma_\alpha^2 - \gamma_\alpha^2\vec{\alpha}\cdot\vec{\alpha} - 1\right] = 0 \end{aligned}$$

because $\gamma_\alpha^2\vec{\alpha}\cdot\vec{\alpha} = \gamma_\alpha^2 - 1$. ■

6.20. Derive Equation (6.58).

Solution: If you dot the first equation on the second line of (6.57) with $\vec{\alpha}$ you get

$$\vec{\alpha}\cdot\dot{\vec{u}} = \gamma_\alpha^3(\vec{\alpha}\cdot\dot{\vec{\alpha}})\vec{\alpha}\cdot\vec{\alpha} + \gamma_\alpha\vec{\alpha}\cdot\dot{\vec{\alpha}} = \gamma_\alpha(\vec{\alpha}\cdot\dot{\vec{\alpha}})\underbrace{\gamma_\alpha^2\vec{\alpha}\cdot\vec{\alpha}}_{=\gamma_\alpha^2-1} + \gamma_\alpha\vec{\alpha}\cdot\dot{\vec{\alpha}} = \gamma_\alpha^3\vec{\alpha}\cdot\dot{\vec{\alpha}} = \dot{u}_0.$$

■

6.21. Derive Equation (6.60).

Solution: Using (6.59) and (6.46), we can write

$$\dot{\vec{u}}' = \frac{\vec{a}'}{\gamma_{\alpha'}} = \frac{\vec{a} + \gamma\vec{\beta}a_0 + (\gamma - 1)\hat{\beta}\hat{\beta}\cdot\vec{a}}{\gamma\gamma_\alpha(1 + \vec{\beta}\cdot\vec{\alpha})}.$$

Now write $a_0 = \gamma_\alpha \dot{u}$ and $\vec{a} = \gamma_\alpha \dot{\vec{u}}$—coming from (6.57)—to rewrite the above as

$$\begin{aligned}\dot{\vec{u}}' &= \frac{\gamma_\alpha\dot{\vec{u}} + \gamma_\alpha\gamma\vec{\beta}\dot{u}_0 + \gamma_\alpha(\gamma - 1)\hat{\beta}\hat{\beta}\cdot\dot{\vec{u}}}{\gamma\gamma_\alpha(1 + \vec{\beta}\cdot\vec{\alpha})}\\ &= \frac{\dot{\vec{u}} + \gamma\vec{\beta}\vec{\alpha}\cdot\dot{\vec{u}} + (\gamma - 1)\hat{\beta}\hat{\beta}\cdot\dot{\vec{u}}}{\gamma(1 + \vec{\beta}\cdot\vec{\alpha})},\end{aligned}$$

where I used (6.58) for $\dot{u}_0$. ■

6.22. How long does it take a particle to attain a speed of $0.999c$, if its acceleration is 10 m/s^2? What is the answer based on Newtonian mechanics? How do the answers change if the ultimate speed of the particle is $0.99999c$?

Solution: The appropriate equation is that before (6.68), with $F/m = 10\text{m/s}^2$. For $\alpha = 0.999$, we get

$$\frac{0.999 \times 3 \times 10^8}{\sqrt{1 - 0.999^2}} = 10t \iff t = 6.7 \times 10^8 \text{ s} = 21.28 \text{ years}.$$

Note that I had to restore the factor of c to make units right on both sides. If you use Newtonian mechanics, you get

$$c\alpha = Ft/m \iff 0.999 \times 3 \times 10^8 = 10t \iff t = 2.997 \times 10^7 \text{ s} = 0.95143 \text{ year}.$$

For $0.99999c$, relativity gives

$$\frac{0.99999 \times 3 \times 10^8}{\sqrt{1 - 0.99999^2}} = 10t \iff t = 6.7 \times 10^{10} \text{ s} = 2130 \text{ years},$$

while classical mechanics gives

$$0.99999 \times 3 \times 10^8 = 10t \iff t = 2.99997 \times 10^7 \text{ s} = 0.95237 \text{ year}.$$

The difference between this and the previous classical time is only 8.25 hours! ■

6.23. Differentiate both sides of $E^2 = \vec{p}\cdot\vec{p} + m^2$ with respect to time and use $\vec{p} = E\vec{\alpha}$ to obtain (6.65).

Solution:

$$\frac{d}{dt}(E^2) = \frac{d}{dt}(\vec{p}\cdot\vec{p} + m^2) \iff 2E\frac{dE}{dt} = 2\vec{p}\cdot\left(\frac{d\vec{p}}{dt}\right) \iff \frac{dE}{dt} = (\vec{p}/E)\cdot\vec{F}.$$

This is the result we are after, because $\vec{p}/E = \vec{\alpha}$. ■

6.24. Write $\vec{F}$ as $\vec{F}_{||} + \vec{F}_\perp$ on both sides of (6.66) and derive (6.67).

Solution: Write (6.66) as

$$\vec{F}_{||}' + \vec{F}_\perp' = \frac{\vec{F}_{||} + \vec{F}_\perp + \gamma\vec{\beta}\,\vec{\alpha}\cdot(\vec{F}_{||} + \vec{F}_\perp) + (\gamma-1)\hat{\beta}\,\hat{\beta}\cdot(\vec{F}_{||} + \vec{F}_\perp)}{\gamma(1+\vec{\beta}\cdot\vec{\alpha})}.$$

Now note that $\hat{\beta}\cdot\vec{F}_\perp = 0$ and $\hat{\beta}\,\hat{\beta}\cdot\vec{F}_{||} = \vec{F}_{||}$. These give us a new equation:

$$\begin{aligned}\vec{F}_{||}' + \vec{F}_\perp' &= \frac{\vec{F}_\perp + \gamma\vec{\beta}\,\vec{\alpha}\cdot\vec{F}_{||} + \gamma\vec{\beta}\,\vec{\alpha}\cdot\vec{F}_\perp + \gamma\vec{F}_{||}}{\gamma(1+\vec{\beta}\cdot\vec{\alpha})}\\ &= \frac{\vec{F}_\perp + \gamma\vec{F}_{||}\vec{\beta}\cdot\vec{\alpha} + \gamma\vec{\beta}\,\vec{\alpha}\cdot\vec{F}_\perp + \gamma\vec{F}_{||}}{\gamma(1+\vec{\beta}\cdot\vec{\alpha})}.\end{aligned}$$

Notice the change in the second term of the numerator. We can now finally write this as

$$\vec{F}_{||}' + \vec{F}_\perp' = \vec{F}_{||} + \frac{\vec{F}_\perp + \gamma\vec{\beta}\,\vec{\alpha}\cdot\vec{F}_\perp}{\gamma(1+\vec{\beta}\cdot\vec{\alpha})} = \vec{F}_{||} + \frac{\vec{\beta}\,\vec{\alpha}\cdot\vec{F}_\perp}{1+\vec{\beta}\cdot\vec{\alpha}} + \frac{\vec{F}_\perp}{\gamma(1+\vec{\beta}\cdot\vec{\alpha})}.$$

The sum of the first two terms on the right-hand side is the parallel component, and the third term is the perpendicular component of the right-hand side. Setting the corresponding components equal on both sides yields (6.67). ∎

6.25. Confine yourself to one dimension.

(a) Use $E^2 = p^2 + m^2$ for a moving object of mass m to show that $dE/dt = F\alpha$, where $F = dp/dt$ is the force and α is the speed of the object.

(b) Lorentz transform E, p, t, and x to another reference frame and use the result obtained in (a) to show that *in one dimension* force is invariant, i.e., it does not change when you go from one RF to another.

Solution: Remember that $\alpha = p/E$.

(a) With that in mind, differentiating $E^2 = p^2 + m^2$ gives

$$2EdE/dt = 2pdp/dt \iff EdE/dt = pF \iff dE/dt = (p/E)F = \alpha F.$$

(b) Let F act on an object for dt to displace it by dx and change it momentum by dp and its energy by dE, all in reference frame O. In another RF O' with respect to which O moves in positive x direction, we have

$$dp' = \gamma(dp + \beta dE), \quad dt' = \gamma(dt + \beta dx).$$

Now divide the first by the second to get

$$F' = dp'/dt' = \frac{\gamma(dp+\beta dE)}{\gamma(dt+\beta dx)} = \frac{dp/dt + \beta dE/dt}{1+\beta dx/dt} = \frac{F+\beta\alpha F}{1+\beta\alpha} = F,$$

where I used the result in (a).

∎

6.26. Define the power P consumed by an object as $P \equiv dE/dt = \vec{\alpha} \cdot \vec{F}$, where E is the relativistic energy of the object and $\vec{F}$ and $\vec{\alpha}$ are, respectively, the force acting on the object (the source of the power) and the velocity of the object. If P and P' are powers in the reference frames O and O', respectively, then

$$P' = \frac{P + \vec{\beta} \cdot \vec{F}}{1 + \vec{\beta} \cdot \vec{\alpha}},$$

where $\vec{\beta}$ is the velocity of O relative to O'. Prove this in three different ways:

(a) by writing the Lorentz transformation of dE' and dt', and dividing the first by the second;

(b) by noting that $P' = \vec{\alpha}' \cdot \vec{F}'$ and using the relativistic law of addition of velocities and the transformation rule for the three-force $\vec{F}$;

(c) by Lorentz transforming the zeroth component of the 4-force.

Solution:

(a) In a reference frame O, let $\vec{F}$ act on an object for dt to displace it by $d\vec{r}$ and change its momentum by $d\vec{p}$ and its energy by dE. In another RF O' with respect to which O moves with velocity $\vec{\beta}$, we have

$$dE' = \gamma(dE + \vec{\beta} \cdot d\vec{p}), \quad dt' = \gamma(dt + \vec{\beta} \cdot d\vec{r}).$$

Divide the first by the second:

$$P' = \frac{dE'}{dt'} = \frac{\gamma(dE + \vec{\beta} \cdot d\vec{p})}{\gamma(dt + \vec{\beta} \cdot d\vec{r})} = \frac{dE/dt + \vec{\beta} \cdot (d\vec{p}/dt)}{1 + \vec{\beta} \cdot (d\vec{r}/dt)} = \frac{P + \vec{\beta} \cdot \vec{F}}{1 + \vec{\beta} \cdot \vec{\alpha}}.$$

(b) Use (6.41) and (6.66) in $P' = \vec{\alpha}' \cdot \vec{F}'$ to obtain

$$P' = \left[\frac{\vec{\alpha} + \gamma\vec{\beta} + (\gamma - 1)\hat{\beta}\hat{\beta} \cdot \vec{\alpha}}{\gamma(1 + \vec{\beta} \cdot \vec{\alpha})}\right] \cdot \left[\frac{\vec{F} + \gamma\vec{\beta}\, \vec{\alpha} \cdot \vec{F} + (\gamma - 1)\hat{\beta}\, \hat{\beta} \cdot \vec{F}}{\gamma(1 + \vec{\beta} \cdot \vec{\alpha})}\right].$$

The denominator of the right-hand side is simply $\gamma^2(1 + \vec{\beta} \cdot \vec{\alpha})^2$. So, let's calculate the numerator:

$$\begin{aligned} Num = {} & \vec{\alpha} \cdot \vec{F} + \gamma\vec{\alpha} \cdot \vec{\beta}\, \vec{\alpha} \cdot \vec{F} + 2(\gamma - 1)\hat{\beta} \cdot \vec{\alpha}\hat{\beta} \cdot \vec{F} + \gamma\vec{\beta} \cdot \vec{F} + \underbrace{\gamma^2\vec{\beta} \cdot \vec{\beta}}_{=\gamma^2 - 1}\, \vec{\alpha} \cdot \vec{F} \\ & + \gamma(\gamma - 1)\vec{\beta} \cdot \vec{F} + \gamma(\gamma - 1)\vec{\alpha} \cdot \vec{\beta}\vec{\alpha} \cdot \vec{F} + (\gamma - 1)^2\hat{\beta} \cdot \vec{\alpha}\hat{\beta} \cdot \vec{F}. \end{aligned}$$

The sum of the third and last term is

$$\begin{aligned} \text{3rd+last} &= 2(\gamma - 1)\hat{\beta} \cdot \vec{\alpha}\hat{\beta} \cdot \vec{F} + \gamma\vec{\beta} \cdot \vec{F} + (\gamma - 1)^2\hat{\beta} \cdot \vec{\alpha}\hat{\beta} \cdot \vec{F} \\ &= (\gamma - 1)(2 + \gamma - 1)\hat{\beta} \cdot \vec{\alpha}\hat{\beta} \cdot \vec{F} = (\gamma^2 - 1)\hat{\beta} \cdot \vec{\alpha}\hat{\beta} \cdot \vec{F} \\ &= \gamma^2\beta^2\hat{\beta} \cdot \vec{\alpha}\hat{\beta} \cdot \vec{F} = \gamma^2\vec{\beta} \cdot \vec{\alpha}\vec{\beta} \cdot \vec{F}. \end{aligned}$$

Substituting this in the previous expression and simplifying yields

$$\begin{aligned} Num &= \gamma^2\vec{\alpha} \cdot \vec{F} + \gamma^2\vec{\alpha} \cdot \vec{\beta}\, \vec{\beta} \cdot \vec{F} + \gamma^2\vec{\beta} \cdot \vec{F} + \gamma^2\vec{\alpha} \cdot \vec{\beta}\vec{\alpha} \cdot \vec{F} \\ &= \gamma^2(1 + \vec{\alpha} \cdot \vec{\beta})(\vec{\alpha} \cdot \vec{F} + \vec{\beta} \cdot \vec{F}) = \gamma^2(1 + \vec{\alpha} \cdot \vec{\beta})(P + \vec{\beta} \cdot \vec{F}). \end{aligned}$$

Dividing this by the denominator gives the result.

(c) By (6.64), $f_0' = \gamma_{\alpha'} P'$. Lorentz transforming this gives

$$\gamma_{\alpha'} P' = \gamma(f_0 + \beta \cdot \vec{f}) = \gamma[\gamma_\alpha P + \beta \cdot (\gamma_\alpha \vec{F})]$$

or

$$P' = \frac{\gamma\gamma_\alpha(P + \beta \cdot \vec{F})}{\gamma_{\alpha'}}.$$

Using Equation (6.46) gives the final answer.

■

6.27. Provide the missing steps of (6.71) and (6.72) to arrive at (6.73).

Solution: To go from the second line of (6.71) to the last line, just note that

$$1 + \frac{\gamma^2\beta^2}{\gamma+1} = 1 + \frac{\gamma^2 - 1}{\gamma+1} = 1 + \frac{(\gamma-1)(\gamma+1)}{\gamma+1} = \gamma.$$

The first line of (6.72) comes from the identity

$$\frac{\gamma}{\gamma+1} = \frac{\gamma(\gamma+1) - \gamma^2}{\gamma+1},$$

and the second line from $\vec{\beta} = \beta\hat{\beta}$ and therefore, $\vec{\beta} \times (\vec{\beta} \times \vec{F}) = \beta^2\hat{\beta} \times (\hat{\beta} \times \vec{F})$. ■

CHAPTER 7

Relativistic Photography

Problems With Solutions

7.1. Consider the cube of Figure 7.10(a) which moves with speed β along the positive x-axis relative to observer O'. Define the angle θ as $\sin\theta \equiv \beta$. It is argued[1] that—since $\overline{A_b A_f}$ is perpendicular to the direction of motion and thus does not change length—it takes light $2a/c$ to travel from A_b to A_f according to O'. During this time, the cube moves a distance of

$$d = (\beta c)(2a/c) = 2a\beta \equiv 2a\sin\theta,$$

exposing the trailing face of the cube to the camera. Furthermore, the top and bottom sides, $\overline{A_f D_f}$ and $\overline{B_f C_f}$, shrink from $2a$ to

$$s = 2a\sqrt{1-\beta^2} = 2a\sqrt{1-\sin^2\theta} = 2a\cos\theta.$$

As Figures 7.10(c) and 7.10(d) indicate, the combination of these two effects manifests itself as the appearance of a rotation on the photographic plate. To see what is wrong with this argument, do the following.

(a) Consider two events: emission of light from A_b and its arrival at A_f. Write the spacetime coordinates of these two events in O where the cube is at rest, assuming that the emission event takes place at time t_b.

(b) Lorentz transform these two events to O'.

(c) What is the time difference $t'_f - t'_b$ between the emission at A_b and arrival at A_f according to O'? How far does the cube move during this time according to O'? Do these results agree with the argument for rotation?

(d) You can derive the result of (c) without using Lorentz transformation. Simply note that according to O', the emission and arrival events occur at the two ends of the hypotenuse of a right triangle whose other two sides are of lengths $2a$ and $\beta(t'_f - t'_b)$.

[1] See, for example, Weisskopf, V. F., Phys. Today **13**(9), 24–27 (1960).

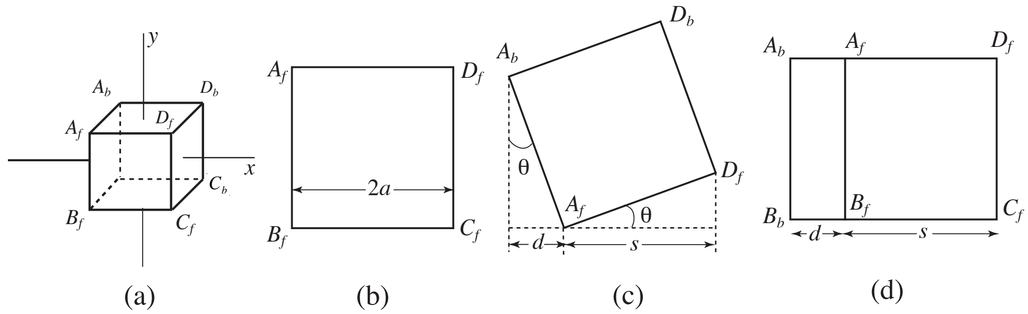


Figure 7.1: (a) The distant cube of side length $2a$ is centered at the origin of O, its rest frame. (b) The cube as it appears on the photographic plate of camera C in O. (c) The orientation of the top face of the cube when the cube is rotated counterclockwise by an angle $\theta = \sin^{-1}\beta$ about the y-axis. (d) The image of the rotated cube on the photographic plate of camera C in O.

Solution: The figure has been reproduced in Figure 7.1 of the manual.

(a) Let E_b denote the emission of light from A_b and E_f its arrival at A_f. Then, with $c = 1$, E_b has coordinates $(t_b, -a, a, -a)$ and E_f has coordinates $(t_b + 2a, -a, a, a)$.

(b) For E_b, we get

$$t'_b = \gamma(t_b - \beta a), \quad x'_b = \gamma(-a + \beta t_b),$$

and for E_f, we get

$$t'_f = \gamma(t_b + 2a - \beta a), \quad x'_f = \gamma[-a + \beta(t_b + 2a)].$$

(c) $t'_f - t'_b = 2a\gamma$ and $x'_f - x'_b = 2a\gamma\beta = \beta(t'_f - t'_b)$. These results do not agree with the argument for rotation!

(d) The hypotenuse of the right triangle is the path of light. So, it has length $t'_f - t'_b$. Hence,

$$(t'_f - t'_b)^2 = 4a^2 + \beta^2(t'_f - t'_b)^2 \iff (t'_f - t'_b)^2(1 - \beta^2) = 4a^2 \iff t'_f - t'_b = 2a\gamma.$$

This is precisely the kind of reasoning that gave us the time dilation formula (2.3) from Figure 2.2. The distance $x'_f - x'_b$ is just the speed times this time interval.

■

7.2. Use the space coordinates of (7.7) to calculate $|\mathbf{r}'_i - \mathbf{r}_0|$ and show that the result agrees with the right-hand side of t'_i.

Solution: With $\mathbf{r}'_i - \mathbf{r}_0 = \langle x'_i, y'_i, z'_i - b\rangle$, we have

$$\begin{aligned}
|\mathbf{r}'_i - \mathbf{r}_0|^2 &= x'^2_i + y'^2_i + (z'_i - b)^2 = \gamma^2(x_i - \beta|\mathbf{r}_i - \mathbf{r}_0|)^2 + y_i^2 + (z_i - b)^2 \\
&= \gamma^2 x_i^2 + \gamma^2\beta^2|\mathbf{r}_i - \mathbf{r}_0|^2 - 2\gamma^2\beta x_i|\mathbf{r}_i - \mathbf{r}_0| + \underbrace{y_i^2 + (z_i - b)^2}_{=|\mathbf{r}_i - \mathbf{r}_0|^2 - x_i^2} \\
&= (\gamma^2 - 1)x_i^2 + (\gamma^2 - 1)|\mathbf{r}_i - \mathbf{r}_0|^2 - 2\gamma^2\beta x_i|\mathbf{r}_i - \mathbf{r}_0| + |\mathbf{r}_i - \mathbf{r}_0|^2,
\end{aligned}$$

where I used the ever useful formula: $\gamma^2\beta^2 = \gamma^2 - 1$. Now use it backwards in the first term and cancel terms to obtain

$$\begin{aligned}|\mathbf{r}_i' - \mathbf{r}_0|^2 &= \gamma^2\beta^2 x_i^2 + \gamma^2|\mathbf{r}_i - \mathbf{r}_0|^2 - 2\gamma^2\beta x_i|\mathbf{r}_i \\ &= \gamma^2(-\beta x_i + |\mathbf{r}_i - \mathbf{r}_0|)^2,\end{aligned}$$

or $|\mathbf{r}_i' - \mathbf{r}_0| = \gamma(|\mathbf{r}_i - \mathbf{r}_0| - \beta x_i)$, where the sign is chosen to make the right-hand side positive. ■

7.3. Substitute Equations (7.7) and (7.8) in (7.9) and follow the steps similar to the ones that led to (7.5) and (7.6) to obtain (7.10).

Solution: The first equation in (7.7) provides the denominator of (7.9):

$$\begin{aligned}Den &= \gamma^2(|\mathbf{r}_1 - \mathbf{r}_0| - \beta x_1)(|\mathbf{r}_2 - \mathbf{r}_0| - \beta x_2) \\ &= \gamma^2|\mathbf{r}_1 - \mathbf{r}_0||\mathbf{r}_2 - \mathbf{r}_0|\left(1 - \frac{\beta x_1}{|\mathbf{r}_1 - \mathbf{r}_0|}\right)\left(1 - \frac{\beta x_2}{|\mathbf{r}_2 - \mathbf{r}_0|}\right).\end{aligned}$$

The numerator can be expressed as

$$\begin{aligned}Num &= \gamma^2(x_1 - \beta|\mathbf{r}_1 - \mathbf{r}_0|)(x_2 - \beta|\mathbf{r}_2 - \mathbf{r}_0|) + y_1y_2 + (z_1 - b)(z_2 - b) \\ &= \gamma^2(x_1 - \beta|\mathbf{r}_1 - \mathbf{r}_0|)(x_2 - \beta|\mathbf{r}_2 - \mathbf{r}_0|) + |\mathbf{r}_1 - \mathbf{r}_0||\mathbf{r}_2 - \mathbf{r}_0|\cos\alpha_{12} - x_1x_2 \\ &= (\gamma^2 - 1)x_1x_2 + \gamma^2\beta^2|\mathbf{r}_1 - \mathbf{r}_0||\mathbf{r}_2 - \mathbf{r}_0| \\ &\quad - \gamma^2 x_1\beta|\mathbf{r}_2 - \mathbf{r}_0| - \gamma^2 x_2\beta|\mathbf{r}_1 - \mathbf{r}_0| + |\mathbf{r}_1 - \mathbf{r}_0||\mathbf{r}_2 - \mathbf{r}_0|\cos\alpha_{12}.\end{aligned}$$

Invoking the identity $\gamma^2\beta^2 = \gamma^2 - 1$ in the first and the second terms and simplifying, you get

$$\begin{aligned}Num &= \gamma^2\beta^2 x_1x_2 + \gamma^2|\mathbf{r}_1 - \mathbf{r}_0||\mathbf{r}_2 - \mathbf{r}_0| - |\mathbf{r}_1 - \mathbf{r}_0||\mathbf{r}_2 - \mathbf{r}_0| \\ &\quad - \gamma^2 x_1\beta|\mathbf{r}_2 - \mathbf{r}_0| - \gamma^2 x_2\beta|\mathbf{r}_1 - \mathbf{r}_0| + |\mathbf{r}_1 - \mathbf{r}_0||\mathbf{r}_2 - \mathbf{r}_0|\cos\alpha_{12} \\ &= \gamma^2(|\mathbf{r}_1 - \mathbf{r}_0| - \beta x_1)(|\mathbf{r}_2 - \mathbf{r}_0| - \beta x_2) - |\mathbf{r}_1 - \mathbf{r}_0||\mathbf{r}_2 - \mathbf{r}_0|(1 - \cos\alpha_{12}).\end{aligned}$$

Dividing the first term by the first form of the denominator and the second term by the second form of the denominator gives

$$\cos\alpha_{12}' = 1 - \frac{1 - \cos\alpha_{12}}{\gamma^2\left(1 - \frac{\beta x_1}{|\mathbf{r}_1 - \mathbf{r}_0|}\right)\left(1 - \frac{\beta x_2}{|\mathbf{r}_2 - \mathbf{r}_0|}\right)}.$$

Now use the trigonometric identity $1 - \cos\theta = 2\sin^2(\theta/2)$ to obtain the final result. ■

7.4. Derive Equations (7.14) and (7.15).

Solution: It is actually more convenient to use the first equality in (7.12). If you write the second equation of (7.12) as

$$x_q'x' - b(z - b) = \frac{y_0\sqrt{x_q'^2 + b^2}}{v'} \iff x' = \frac{y_0\sqrt{x_q'^2 + b^2}}{x_q'v'} + \frac{b(z_0 - b)}{x_q'}$$

and substitute both in the first equation, you obtain

$$\begin{aligned} u' &= \frac{v'}{y_0\sqrt{x_q'^2+b^2}}[bx' + x_q'(z_0-b)] = \frac{b}{y_0\sqrt{x_q'^2+b^2}}v'x' + \frac{x_q'(z_0-b)}{y_0\sqrt{x_q'^2+b^2}}v' \\ &= \frac{b}{y_0\sqrt{x_q'^2+b^2}}v'\left[\frac{y_0\sqrt{x_q'^2+b^2}}{x_q'v'} + \frac{b(z_0-b)}{x_q'}\right] + \frac{x_q'(z_0-b)}{y_0\sqrt{x_q'^2+b^2}}v' \end{aligned}$$

or

$$x_q'u' = b + v'\frac{(z_0-b)(x_q'^2+b^2)}{y_0\sqrt{x_q'^2+b^2}} = b + v'\frac{(z_0-b)\sqrt{x_q'^2+b^2}}{y_0},$$

and

$$v' = \frac{y_0x_q'}{(z_0-b)\sqrt{x_q'^2+b^2}}u' - \frac{by_0}{(z_0-b)\sqrt{x_q'^2+b^2}}.$$

Equation (7.14) is obtained by noting that $x_q' = x_q/\gamma$.

Derivation of (7.15) is very straightforward, and I'll leave it for you to do. ■

7.5. Let $x_q = 0$ and derive (7.17) from (7.16).

Solution: With $x_q = 0$, (7.16) becomes

$$\begin{aligned} u' &= -\frac{\gamma\left[x_0 - \beta\sqrt{x_0^2+y^2+(z_0-b)^2}\right]}{z_0-b} \\ v' &= -\frac{y}{z_0-b}, \end{aligned}$$

or

$$\begin{aligned} u' + \frac{\gamma x_0}{z_0-b} &= \frac{\gamma\beta\sqrt{x_0^2+y^2+(z_0-b)^2}}{z_0-b} \\ v' &= -\frac{y}{z_0-b}, \end{aligned}$$

or

$$\begin{aligned} \left(u' + \frac{\gamma x_0}{z_0-b}\right)^2 &= \frac{\gamma^2\beta^2[x_0^2+y^2+(z_0-b)^2]}{(z_0-b)^2} \\ y^2 &= v'^2(z_0-b)^2. \end{aligned}$$

Substituting the second equation in the first yields

$$\left(u' + \frac{\gamma x_0}{z_0-b}\right)^2 = \gamma^2\beta^2v'^2 + \frac{\gamma^2\beta^2[x_0^2+(z_0-b)^2]}{(z_0-b)^2}.$$

Dividing both sides by the last term gives (7.17). ■

7.6. Derive Equation (7.19) from Equation (7.18).

Solution: Transfer γx_0 to the left of Equation (7.18) and square both sides. ■

7.7. Show that the line of Equation (7.20) makes an angle θ with the x'-axis given by $\cos\theta = \beta$.

Solution: The slope is $1/(\gamma\beta)$. Therefore, $\tan\theta = 1/(\gamma\beta)$, and

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{\gamma\beta}{\sqrt{\gamma^2\beta^2+1}} = \frac{\gamma\beta}{\sqrt{\gamma^2-1+1}} = \beta.$$

■

7.8. Derive Equation (7.23) from Equation (7.22).

Solution: Substitute for $b-z$ from the second equation of (7.22) in the first:

$$u' = \frac{\gamma\left[x_0 - \beta\sqrt{x_0^2+y_0^2+(y_0/v')^2}\right]}{y_0/v'}$$

or

$$y_0u' = \gamma\left[x_0v' - \beta\sqrt{v'^2(x_0^2+y_0^2)+y_0^2}\right]$$

or

$$y_0u' - \gamma x_0v' = -\gamma\beta\sqrt{v'^2(x_0^2+y_0^2)+y_0^2}.$$

Squaring both sides and simplifying yields (7.23). ■

7.9. Define a new pair of photographic plate coordinates by

$$u' = u'_{\text{new}}\cos\theta + v'_{\text{new}}\sin\theta, \quad v' = -u'_{\text{new}}\sin\theta + v'_{\text{new}}\cos\theta$$

with

$$\sin\theta = \frac{x_0}{\sqrt{\gamma^2y_0^2+x_0^2}}, \quad \cos\theta = \frac{\gamma y_0}{\sqrt{\gamma^2y_0^2+x_0^2}}.$$

Note that the new axes are obtained from the old by a clockwise rotation of angle θ.

(a) Substitute these values in Equation (7.23) and simplify to show that that equation reduces to

$$(x_0^2+y_0^2)u'^2_{\text{new}} - \beta^2\gamma^2y_0^2v'^2_{\text{new}} - \gamma^2\beta^2y_0^2 = 0,$$

which is the equation of a hyperbola.

(b) What are the coordinates of the center of this hyperbola in the new coordinate system?

(c) What are the equations of the axes of the hyperbola in the new coordinate system? In the old coordinate system?

Solution:

(a) Substitute the new pair of photographic plate coordinates in (7.23) and look at each resulting term separately:

$$\begin{aligned}
\text{1st term} &= y_0^2(u'^2_{\text{new}}\cos^2\theta + v'^2_{\text{new}}\sin^2\theta + 2u'_{\text{new}}v'_{\text{new}}\sin\theta\cos\theta)\\
\text{2nd term} &= \left(x_0^2-\gamma^2\beta^2y_0^2\right)(u'^2_{\text{new}}\sin^2\theta + v'^2_{\text{new}}\cos^2\theta - 2u'_{\text{new}}v'_{\text{new}}\sin\theta\cos\theta)\\
\text{3rd term} &= -2\gamma x_0y_0[-u'^2_{\text{new}}\sin\theta\cos\theta + u'_{\text{new}}v'_{\text{new}}(\cos^2\theta-\sin^2\theta) + v'^2_{\text{new}}\sin\theta\cos\theta].
\end{aligned}$$

First consider the new cross term, which can be written as

$$\text{cross term} = 2u'_{\text{new}}v'_{\text{new}}\left[\left(y_0^2 - x_0^2 + \gamma^2\beta^2 y_0^2\right)\sin\theta\cos\theta - \gamma x_0 y_0(\cos^2\theta - \sin^2\theta)\right].$$

Let $\mathcal{X}$ denote the expression inside the square bracket and use $\gamma^2\beta^2 = \gamma^2 - 1$ in its first term. Then, substituting

$$\sin\theta = \frac{x_0}{\sqrt{\gamma^2 y_0^2 + x_0^2}}, \quad \cos\theta = \frac{\gamma y_0}{\sqrt{\gamma^2 y_0^2 + x_0^2}}$$

in that expression yields

$$\mathcal{X} = \left(\gamma^2 y_0^2 - x_0^2\right)\frac{\gamma x_0 y_0}{\gamma^2 y_0^2 + x_0^2} - \gamma x_0 y_0 \left(\frac{\gamma^2 y_0^2 - x_0^2}{\gamma^2 y_0^2 + x_0^2}\right) = 0.$$

The coefficient that multiplies u'^2_{new} is

$$y_0^2\cos^2\theta + \left(x_0^2 - \gamma^2\beta^2 y_0^2\right)\sin^2\theta + 2\gamma x_0 y_0 \sin\theta\cos\theta$$

or

$$\frac{\gamma^2 y_0^4}{\gamma^2 y_0^2 + x_0^2} + \left(x_0^2 - \gamma^2\beta^2 y_0^2\right)\frac{x_0^2}{\gamma^2 y_0^2 + x_0^2} + \frac{2\gamma^2 x_0^2 y_0^2}{\gamma^2 y_0^2 + x_0^2}$$

or

$$\frac{\gamma^2 y_0^4 + x_0^4 + \gamma^2 x_0^2 y_0^2 \overbrace{(2-\beta^2)}^{=1+1/\gamma^2}}{\gamma^2 y_0^2 + x_0^2} = \frac{\gamma^2 y_0^4 + x_0^4 + \gamma^2 x_0^2 y_0^2 + x_0^2 y_0^2}{\gamma^2 y_0^2 + x_0^2} = x_0^2 + y_0^2.$$

Similarly, you can show that the coefficient of v'^2_{new} is $-\beta^2\gamma^2 y_0^2$.

(b) Transfer the constant term of the equation in part (a) of the problem to the right-hand side and divide both sides by that term to obtain

$$\frac{(x_0^2 + y_0^2)u'^2_{\text{new}}}{\gamma^2\beta^2 y_0^2} - v'^2_{\text{new}} = 1 \iff \frac{u'^2_{\text{new}}}{\gamma^2\beta^2 y_0^2/(x_0^2 + y_0^2)} - v'^2_{\text{new}} = 1.$$

This shows that the center is at the origin, the semi-major axis is $a = \gamma\beta y_0/\sqrt{x_0^2 + y_0^2}$, and the semi-minor axis is $b = 1$.

(c) The equations of the axes of the hyperbola in the new coordinate system are

$$v'_{\text{new}} = \pm\frac{\sqrt{x_0^2 + y_0^2}}{\gamma\beta y_0}u'_{\text{new}}.$$

Let α be the angle that the line with positive slope makes with the u'_{new} axis. Then

$$\tan\alpha = \frac{\sqrt{x_0^2 + y_0^2}}{\gamma\beta y_0} \iff \cos\alpha = \frac{1}{\sqrt{1 + \tan^2\alpha}} = \beta\cos\theta$$

To obtain the equations in the old coordinate system, substitute for the new in terms of the old:

$$u'\sin\theta + v'\cos\theta = \pm\tan\alpha(u'\cos\theta - v'\sin\theta).$$

or

$$\cos\alpha(u'\sin\theta + v'\cos\theta) = \pm\sin\alpha(u'\cos\theta - v'\sin\theta).$$

or

$$v'(\cos\alpha\cos\theta \mp \sin\alpha\sin\theta) = u'(\pm\sin\alpha\cos\theta - \cos\alpha\sin\theta).$$

or

$$\cos(\alpha \pm \theta)v' = \mp u'\sin(\alpha \pm \theta) \iff v' = \mp\tan(\alpha \pm \theta)u'.$$

Thus, the axes of the old hyperbola are rotated by θ compared to the new axes. This is what we should expect, because the new photographic plate coordinates are obtained from the old by the same rotation.

■

7.10. Derive Equations (7.25) and (7.26).

Solution: All the front-face corners have $z = a$ and their x and y coordinates are $\pm a$. Therefore, their distance from the pinhole is $\sqrt{2a^2 + (b-a)^2}$. With $c = 1$, and the fact that t is negative, we get the first equation in (7.25). The second equation follows the same way. The only difference is that now $z = -a$.

To get the equations in (7.26), note that the only coordinates that matter are x and t. Therefore, all the front corners have the same t-term in the Lorentz transformation. So, the left front corners have the same x of $-a$, and their Lorentz transformations are identical. Similarly for the rest of the Lorentz transformations. ■

7.11. Derive Equations (7.28) and (7.29).

Solution: With $x_q = 0$, (7.12) becomes

$$u' = -\frac{x'}{z-b}, \quad v' = -\frac{y}{z-b}.$$

So, to get (7.28) and (7.29) just substitute for x', y, and z. Since all the front corners have $z = a$, the denominator for all u's and v's is $b - a$. Similarly, the denominator for all u's and v's is $b + a$ for the back corners. To get (7.28) substitute the x's from (7.26), and to get (7.29), substitute for y. ■

7.12. Using Equation (7.28), show that $u'_{A_b} > u'_{A_f}$ and $v'_{A_b} < v'_{A_f}$ for all b and β. Hint: Show that each of the two terms on the right-hand side of u'_{A_b} is larger than the corresponding term on the right-hand side of u'_{A_f}.

Solution: Write u'_{A_b} and u'_{A_f} as

$$\begin{aligned} u'_{A_b} &= -\frac{\gamma a}{b+a} - \gamma\beta\sqrt{\frac{2a^2}{(b+a)^2} + 1} \\ u'_{A_f} &= -\frac{\gamma a}{b-a} - \gamma\beta\sqrt{\frac{2a^2}{(b-a)^2} + 1}. \end{aligned}$$

It is now clear that the *magnitude* of each term in u'_{A_b} is *smaller* than the magnitude of the corresponding term in u'_{A_f}. Therefore, $u'_{A_b} > u'_{A_f}$. The inequality $v'_{A_b} < v'_{A_f}$ is self-evident.

■

7.13. Recall that the aberration formula involves the angle with the direction of motion of a ray of light from a single point source. Therefore, if the aberration formula is to be interpreted as the angle between *two* light rays originating from an object (as is the case when image formation of the object is being considered), then one of those rays ought to be along the direction of motion. Show that when that is the case, then (7.5) reduces to the aberration formula (6.25).

Solution: Assume that the first photon is in the direction of motion. Then, $\cos\alpha'_{12} = \cos\varphi'$, $\cos\alpha_{12} = \cos\varphi$, $\hat{e}_1 = -\hat{\beta}$, and $\vec{\beta}\cdot\hat{e}_2 = -\beta\cos\varphi$, and (7.5) can be written as

$$\begin{aligned}\cos\varphi' &= 1 - \frac{1-\cos\varphi}{\gamma^2(1+\beta)(1+\beta\cos\varphi)} = 1 - \frac{(1-\beta)(1-\cos\varphi)}{1+\beta\cos\varphi}\\ &= \frac{1+\beta\cos\varphi-(1-\beta)(1-\cos\varphi)}{1+\beta\cos\varphi} = \frac{\beta+\cos\varphi}{1+\beta\cos\varphi}.\end{aligned}$$

■

7.14. Let's see what a cube looks like when it moves toward or away from the camera. Place the camera at the origin (i.e., let $b = 0$) and point it to the right face of the approaching cube as shown in Figure 7.3(b). Let the center of the cube have coordinates $(x_c, 0, 0)$ in O.

(a) Write down the coordinates of all the eight corners of the cube in O.

(b) From (a), (7.8), and (7.10) with $b = 0$, obtain

$$\begin{aligned}\sin(\alpha'_{D_fD_b}/2) &= \frac{a}{\gamma\big[\sqrt{(x_c+a)^2+2a^2}-\beta(x_c+a)\big]},\\ \sin(\alpha'_{A_fA_b}/2) &= \frac{a}{\gamma\big[\sqrt{(x_c-a)^2+2a^2}-\beta(x_c-a)\big]},\end{aligned} \tag{7.1}$$

the angles formed by sides $\overline{D_fD_b}$ and $\overline{A_fA_b}$ (and the other three sides of the leading and trailing faces) at the pinhole.

(c) For $x_c < 0$ and $\beta > 0$, show that $\alpha'_{D_fD_b} > \alpha'_{A_fA_b}$, with the other three angles satisfying the same kind of inequality. Thus, the trailing face is hidden behind the leading face when the cube is approaching the camera.

(d) For $x_c < 0$ and $\beta < 0$, show that $\alpha'_{D_fD_b} > \alpha'_{A_fA_b}$ if and only if $|x_c|/a > |\beta|\sqrt{2\gamma^2+1}$.

(e) Set $\beta = 0$ in (7.1) above to find $\sin(\alpha_{D_fD_b}/2)$. Then show that, for approach, $\alpha'_{D_fD_b} < \alpha_{D_fD_b}$, indicating that the picture in C' is smaller than in C.

(f) The case of recession is more complicated. The limiting case of Equation (7.11) may lead you to believe that the C' image is larger. For the moment assume that and see what condition makes the assumption true. For definiteness, let $\beta > 0$ and $x_c > 0$. Then, for $\alpha'_{A_fD_f} > \alpha_{A_fD_f}$ to hold, you should have

$$\gamma\big[\sqrt{(x_c+a)^2+2a^2}-\beta(x_c+a)\big] < \sqrt{(x_c+a)^2+2a^2}.$$

Show that this is equivalent to

$$[(x_c+a)^2 - X_-][(x_c+a)^2 - X_+] > 0, \quad X_- < 0$$

for certain X_+ and X_- that you have to find. Thus, for the inequality to hold, $(x_c+a)^2$ must be larger than X_+. Now show that

$$\frac{(x_c+a)^2}{a^2} > \gamma - 1. \tag{7.2}$$

When the cube is sufficiently far away, this condition holds, but not in general. Therefore, if the cube is close enough to the camera C' when the pictures are taken, the image in C can be larger.

(g) Show that (7.2) above holds also when $\beta < 0$ and $x_c < 0$.

(h) Plot the ratio

$$\frac{\sin(\alpha'_{D_f D_b}/2)}{\sin(\alpha_{D_f D_b}/2)}$$

for $x_c = +3a$ as a function of β for $-1 < \beta < 1$ to get a feel for the magnification of the image in C' relative to the image in C.

(i) Verify directly that, when $|x_c| \to \infty$, the ratio in (h) reduces to

$$\frac{\alpha'_{D_f D_b}}{\alpha_{D_f D_b}} \approx \sqrt{\frac{1+\beta|x_c|/x_c}{1-\beta|x_c|/x_c}},$$

as in Equation (7.11).

Solution:

(a) Referring to Figure 7.3, with the center of the cube at $(x_c, 0, 0)$, we get

$$\begin{array}{lll}
\text{Leading face} & D_f:\ (x_c+a, -a, a), & D_b:\ (x_c+a, a, a), \\
 & C_f:\ (x_c+a, -a, -a), & C_b:\ (x_c+a, a, -a) \\
\text{Trailing face} & A_f:\ (x_c-a, -a, a), & A_b:\ (x_c-a, a, a), \\
 & B_f:\ (x_c-a, -a, -a), & B_b:\ (x_c-a, -a, -a)
\end{array}$$

(b) Substituting from (a) in (7.8) with $b = 0$, we get

$$\cos\alpha_{D_f D_b} = \frac{x_{D_f}x_{D_b} + y_{D_f}y_{D_b} + z_{D_f}z_{D_b}}{|\mathbf{r}_{D_f}|\,|\mathbf{r}_{D_b}|} = \frac{(x_c+a)^2}{(x_c+a)^2 + 2a^2}$$

and

$$2\sin^2(\alpha_{D_f D_b}/2) = 1 - \cos\alpha_{D_f D_b} = 1 - \frac{(x_c+a)^2}{(x_c+a)^2+2a^2} = \frac{2a^2}{(x_c+a)^2+2a^2},$$

or

$$\sin(\alpha_{D_f D_b}/2) = \frac{a}{\sqrt{(x_c+a)^2+2a^2}}.$$

Substitute this in (7.10) with $b = 0$ to obtain

$$\sin(\alpha'_{D_f D_b}/2) = \frac{\frac{a}{\sqrt{(x_c+a)^2+2a^2}}}{\gamma\left(1 - \frac{\beta(x_c+a)}{\sqrt{(x_c+a)^2+2a^2}}\right)} = \frac{a}{\gamma\left(\sqrt{(x_c+a)^2+2a^2} - \beta(x_c+a)\right)}.$$

$\sin(\alpha'_{A_f A_b}/2)$ can be obtained in exactly the same way.

(c) For $x_c < 0$ and $\beta > 0$, (7.1) becomes

$$\sin(\alpha'_{D_f D_b}/2) = \frac{a}{\gamma\left[\sqrt{(|x_c|-a)^2+2a^2}+\beta(|x_c|-a)\right]},$$
$$\sin(\alpha'_{A_f A_b}/2) = \frac{a}{\gamma\left[\sqrt{(|x_c|+a)^2+2a^2}+\beta(|x_c|+a)\right]}.$$

The denominator of the first is smaller than the denominator of the second. Therefore,

$$\sin(\alpha'_{D_f D_b}/2) > \sin(\alpha'_{A_f A_b}/2) \iff \alpha'_{D_f D_b} > \alpha'_{A_f A_b}.$$

(d) For $x_c < 0$ and $\beta < 0$, (7.1) becomes

$$\sin(\alpha'_{D_f D_b}/2) = \frac{a}{\gamma\left[\sqrt{(|x_c|-a)^2+2a^2}-|\beta|(|x_c|-a)\right]},$$
$$\sin(\alpha'_{A_f A_b}/2) = \frac{a}{\gamma\left[\sqrt{(|x_c|+a)^2+2a^2}-|\beta|(|x_c|+a)\right]},$$

and $\alpha'_{D_f D_b} > \alpha'_{A_f A_b}$ if and only if

$$\sqrt{(|x_c|-a)^2+2a^2}-|\beta|(|x_c|-a) < \sqrt{(|x_c|+a)^2+2a^2}-|\beta|(|x_c|+a)$$

or

$$\sqrt{(|x_c|-a)^2+2a^2}+2|\beta|a < \sqrt{(|x_c|+a)^2+2a^2}.$$

Note how I arranged terms on either side so that both sides are always positive. So, I can square both sides and keep the direction of the inequality. Doing so gives

$$(|x_c|-a)^2+2a^2+4\beta^2a^2+4a|\beta|\sqrt{(|x_c|-a)^2+2a^2} < (|x_c|+a)^2+2a^2.$$

This can be simplified to

$$\beta^2 a+|\beta|\sqrt{(|x_c|-a)^2+2a^2} < |x_c|$$

or

$$|\beta|\sqrt{(|x_c|-a)^2+2a^2} < |x_c|-\beta^2 a.$$

Both sides are still positive because $|x_c| > a$. (Otherwise the camera will be inside the cube!) So, I square both sides and get

$$\beta^2(|x_c|-a)^2+2\beta^2a^2 < (|x_c|-\beta^2a)^2,$$

which can be simplified to

$$|x_c|^2 > \gamma^2\beta^2a^2(3-\beta^2) = \gamma^2\beta^2a^2(2+1/\gamma^2) = \beta^2a^2(2\gamma^2+1).$$

(e) With $\beta = 0$, we have

$$\sin(\alpha_{D_f D_b}/2) = \frac{a}{\sqrt{(x_c+a)^2+2a^2}}.$$

On approach, $x_c < 0$ and $\beta > 0$, therefore

$$\sin(\alpha_{D_f D_b}/2) = \frac{a}{\sqrt{(|x_c|-a)^2+2a^2}}.$$

and

$$\sin(\alpha'_{D_f D_b}/2) = \frac{a}{\gamma\left[\sqrt{(|x_c|-a)^2+2a^2}+\beta(|x_c|-a)\right]}.$$

It is clear that the denominator of $\sin(\alpha'_{D_f D_b}/2)$ is larger than that of $\sin(\alpha_{D_f D_b}/2)$. Hence, $\alpha'_{D_f D_b} < \alpha_{D_f D_b}$.

(f) Note that both sides of the inequality are positive. Therefore, I can square both sides

$$\gamma^2\left[\sqrt{(x_c+a)^2+2a^2}-\beta(x_c+a)\right]^2 < (x_c+a)^2+2a^2.$$

or

$$\gamma^2\left[(x_c+a)^2+2a^2+\beta^2(x_c+a)^2-2\beta(x_c+a)\sqrt{(x_c+a)^2+2a^2}\right] < (x_c+a)^2+2a^2.$$

Using $\gamma^2-1=\gamma^2\beta^2$, this inequality simplifies to

$$2\gamma^2\beta^2(x_c+a)^2+2\gamma^2\beta^2a^2 < 2\gamma^2\beta(x_c+a)\sqrt{(x_c+a)^2+2a^2},$$

or

$$\beta(x_c+a)^2+\beta a^2 < (x_c+a)\sqrt{(x_c+a)^2+2a^2}$$

Square both sides to get

$$\beta^2(x_c+a)^4+\beta^2a^4+2a^2\beta^2(x_c+a)^2 < (x_c+a)^4+2a^2(x_c+a)^2$$

or

$$(x_c+a)^4+2a^2(x_c+a)^2-\beta^2\gamma^2a^4 > 0.$$

Find the two roots of the left-hand side by setting it equal to zero and solving the quadratic equation:

$$(x_c+a)^2 = -a^2 \pm \sqrt{a^4+\beta^2\gamma^2a^4} = -a^2 \pm \gamma a^2.$$

Then the preceding inequality can be expressed as

$$[(x_c+a)^2+(\gamma+1)a^2][(x_c+a)^2-(\gamma-1)a^2] > 0.$$

The factor on the left is positive, so for the inequality to hold, we must have

$$(x_c+a)^2 > (\gamma-1)a^2.$$

(g) If $\beta<0$ and $x_c<0$, then the starting inequality in (f) becomes

$$\gamma\left[\sqrt{(|x_c|-a)^2+2a^2}-|\beta|(|x_c|-a)\right] < \sqrt{(|x_c|-a)^2+2a^2}.$$

This differs from the previous case only by the change in the sign of a. So, without going through any calculation, we can write the final inequality by changing a to $-a$:

$$(|x_c|-a)^2 > (\gamma-1)(-a)^2 \iff (-x_c-a)^2 > (\gamma-1)(-a)^2 \iff (x_c+a)^2 > (\gamma-1)a^2$$

which is the inequality in (f).

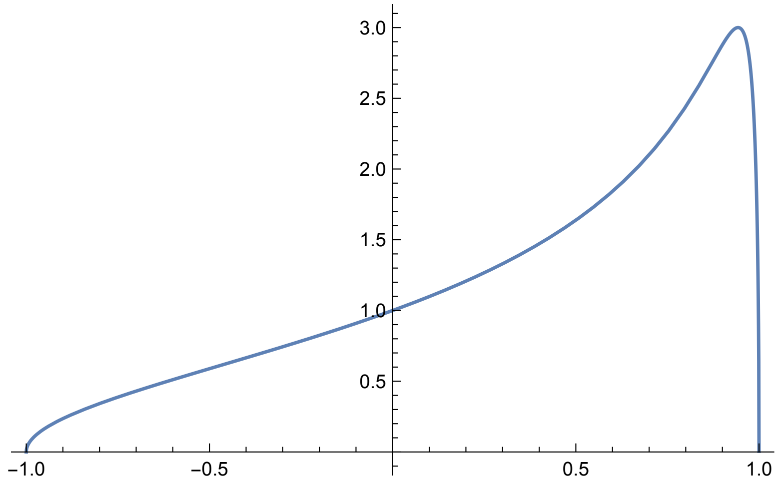


Figure 7.2: The plot of the ratio $\sin(\alpha'_{D_f D_b}/2)/\sin(\alpha_{D_f D_b}/2)$ when $x_c = 3$.

(h) From the solution to part (b), we have

$$\frac{\sin(\alpha'_{D_f D_b}/2)}{\sin(\alpha_{D_f D_b}/2)} = \frac{\sqrt{(x_c+a)^2+2a^2}}{\gamma\left(\sqrt{(x_c+a)^2+2a^2}-\beta(x_c+a)\right)}.$$

The plot of this ratio is shown in Figure 7.2 of the manual.

(i) When $|x_c| \to \infty$, we can ignore a in the equation above. Therefore, since angles get small, the ratio of sines becomes the ratio of the angles:

$$\begin{aligned}\frac{\alpha'_{D_f D_b}}{\alpha_{D_f D_b}} &\approx \frac{|x_c|}{\gamma\left(|x_c|-\beta x_c\right)} = \frac{1}{\gamma\left(1-\beta x_c/|x_c|\right)} \\ &= \frac{\sqrt{1-(\beta x_c/|x_c|)^2}}{(1-\beta x_c/|x_c|)} = \sqrt{\frac{1+\beta x_c/|x_c|}{1-\beta x_c/|x_c|}}.\end{aligned}$$

Note that $\beta^2 = (\beta x_c/|x_c|)^2$, because $x_c/|x_c| = \pm 1$.

■

7.15. Derive Equations (7.33) and (7.34).

Solution: With $\mathbf{r} = \langle x, y, a^2/b\rangle = \langle x, y, a\sin\alpha\rangle$, $\mathbf{r}_0 = \langle 0,0,b\rangle$, and referring to Figure 7.6 of the book, we obtain

$$\begin{aligned}|\mathbf{r}_0-\mathbf{r}|^2 &= x^2+y^2+(b-a\sin\alpha)^2 = R^2+b^2(1-\sin^2\alpha)^2 \\ &= a^2\cos^2\alpha + b^2\cos^4\alpha = b^2\sin^2\alpha\cos^2\alpha + b^2\cos^4\alpha = b^2\cos^2\alpha.\end{aligned}$$

The rest of (7.33) follows easily. For (7.34), use (6.25) and (7.33):

$$\cos\varphi' = \frac{|\vec{\beta}| + \cos\varphi}{1 + |\vec{\beta}|\cos\varphi} = \frac{|\vec{\beta}| - x/(b\cos\alpha)}{1 - |\vec{\beta}|x/(b\cos\alpha)} = \frac{|\vec{\beta}|b\cos\alpha - x}{b\cos\alpha - |\vec{\beta}|x}$$

$$\sin\varphi' = \frac{\sin\varphi}{\gamma\left(1 + |\vec{\beta}|\cos\varphi\right)} = \frac{\sqrt{b^2\cos^2\alpha - x^2}/(b\cos\alpha)}{\gamma\left(1 - |\vec{\beta}|x/(b\cos\alpha)\right)} = \frac{\sqrt{b^2\cos^2\alpha - x^2}}{\gamma\left(b\cos\alpha - |\vec{\beta}|x\right)}.$$

■

7.16. Show that if $\hat{\mathbf{v}}$ of Equation (7.35) is to make an angle φ' with the x-axis, then

$$\eta_z = \pm\frac{\eta_x b\cos^2\alpha}{\sqrt{y^2 + (b\cos^2\alpha)^2}}\tan\varphi'.$$

Solution: From

$$\cos\varphi' = \hat{\mathbf{v}}\cdot\hat{\mathbf{e}}_x = \frac{\eta_x b\cos^2\alpha}{\sqrt{\eta_x^2(b\cos^2\alpha)^2 + \eta_z^2[y^2 + (b\cos^2\alpha)^2]}},$$

we obtain

$$\cos^2\varphi' = \frac{\eta_x^2(b\cos^2\alpha)^2}{\eta_x^2(b\cos^2\alpha)^2 + \eta_z^2[y^2 + (b\cos^2\alpha)^2]},$$

or

$$\eta_x^2(b\cos^2\alpha)^2\cos^2\varphi' + \eta_z^2[y^2 + (b\cos^2\alpha)^2]\cos^2\varphi' = \eta_x^2(b\cos^2\alpha)^2$$

or

$$\eta_z^2[y^2 + (b\cos^2\alpha)^2]\cos^2\varphi' = \eta_x^2(b\cos^2\alpha)^2\sin^2\varphi'$$

or finally

$$\eta_z^2 = \frac{\eta_x^2(b\cos^2\alpha)^2\sin^2\varphi'}{[y^2 + (b\cos^2\alpha)^2]\cos^2\varphi'} = \frac{\eta_x^2(b\cos^2\alpha)^2}{y^2 + (b\cos^2\alpha)^2}\tan^2\varphi'.$$

■

7.17. With $\hat{\mathbf{e}}(\phi, \beta)$ defined as in Equation (7.40), show that

$$\hat{\mathbf{e}}(\phi,\beta)\cdot\hat{\mathbf{e}}(0,\beta) = 1 - \frac{\sin^2\alpha(1 - \cos\phi)}{\gamma^2(1 - |\vec{\beta}|\sin\alpha\cos\phi)(1 - |\vec{\beta}|\sin\alpha)}.$$

Now prove that the dot product has a maximum at $\phi = 0$ and a minimum at $\phi = \pi$.

Solution: With $\hat{\mathbf{e}}(\phi, \beta)$ defined as in Equation (7.40), we get

$$\hat{\mathbf{e}}(0,\beta) = \left\langle -\frac{|\vec{\beta}| - \sin\alpha}{1 - |\vec{\beta}|\sin\alpha}, 0, -\frac{\cos\alpha}{\gamma(1 - |\vec{\beta}|\sin\alpha)}\right\rangle,$$

and the dot product becomes

$$\hat{\mathbf{e}}(\phi,\beta)\cdot\hat{\mathbf{e}}(0,\beta) = \left(\frac{|\vec{\beta}| - \sin\alpha\cos\phi}{1 - |\vec{\beta}|\sin\alpha\cos\phi}\right)\left(\frac{|\vec{\beta}| - \sin\alpha}{1 - |\vec{\beta}|\sin\alpha}\right)$$
$$+ \frac{\cos^2\alpha}{\gamma^2(1 - |\vec{\beta}|\sin\alpha\cos\phi)(1 - |\vec{\beta}|\sin\alpha)}$$

or

$$\hat{\mathbf{e}}(\phi,\beta)\cdot\hat{\mathbf{e}}(0,\beta) = \frac{\gamma^2(|\vec{\beta}| - \sin\alpha\cos\phi)(|\vec{\beta}| - \sin\alpha) + \cos^2\alpha}{\gamma^2(1 - |\vec{\beta}|\sin\alpha\cos\phi)(1 - |\vec{\beta}|\sin\alpha)}.$$

Let's manipulate the numerator

$$\begin{aligned} Num &= \gamma^2\beta^2 - \gamma^2(|\vec{\beta}|\sin\alpha + |\vec{\beta}|\sin\alpha\cos\phi) \\ &\quad + [(\gamma^2 - 1)\sin^2\alpha\cos\phi + \sin^2\alpha\cos\phi] + \cos^2\alpha \\ &= \gamma^2 - 1 - \gamma^2(|\vec{\beta}|\sin\alpha + |\vec{\beta}|\sin\alpha\cos\phi) \\ &\quad + [\gamma^2\beta^2\sin^2\alpha\cos\phi + \sin^2\alpha\cos\phi] + \cos^2\alpha \\ &= \gamma^2(1 - |\vec{\beta}|\sin\alpha\cos\phi)(1 - |\vec{\beta}|\sin\alpha) - \sin^2\alpha(1 - \cos\phi). \end{aligned}$$

Dividing this by the denominator, gives the final result.

For the purposes of finding the extrema, we just have to differentiate the function

$$f(\phi) \equiv \frac{1 - \cos\phi}{1 - |\vec{\beta}|\sin\alpha\cos\phi},$$

whose derivative is

$$\begin{aligned} f'(\phi) &= \frac{\sin\phi(1 - |\vec{\beta}|\sin\alpha\cos\phi) - (1 - \cos\phi)|\vec{\beta}|\sin\alpha\sin\phi}{(1 - |\vec{\beta}|\sin\alpha\cos\phi)^2} \\ &= \frac{(1 - |\vec{\beta}|\sin\alpha)\sin\phi}{(1 - |\vec{\beta}|\sin\alpha\cos\phi)^2}. \end{aligned}$$

So, $f'(\phi) = 0$ if $\sin\phi = 0$ or if $\phi = 0, \pi$. ■

7.18. Provide all the missing steps leading to Equation (7.41).

Solution: Equation (7.40) gives

$$\begin{aligned} \hat{\mathbf{e}}(0,\beta) &= \left\langle -\frac{|\vec{\beta}| - \sin\alpha}{1 - |\vec{\beta}|\sin\alpha}, 0, -\frac{\cos\alpha}{\gamma(1 - |\vec{\beta}|\sin\alpha)}\right\rangle \\ \hat{\mathbf{e}}(\pi,\beta) &= \left\langle -\frac{|\vec{\beta}| + \sin\alpha}{1 + |\vec{\beta}|\sin\alpha}, 0, -\frac{\cos\alpha}{\gamma(1 + |\vec{\beta}|\sin\alpha)}\right\rangle. \end{aligned}$$

Therefore,

$$\hat{\mathbf{e}}_0 + \hat{\mathbf{e}}_\pi = \frac{2\cos\alpha}{1 - |\vec{\beta}|^2\sin^2\alpha}\left\langle -|\vec{\beta}|\cos\alpha, 0, -1/\gamma\right\rangle.$$

Although $\hat{\mathbf{e}}_{axis}$ is defined as $(\hat{\mathbf{e}}_0 + \hat{\mathbf{e}}_\pi)/|\hat{\mathbf{e}}_0 + \hat{\mathbf{e}}_\pi|$, the constant multiplying the angle brackets cancels in the ratio. So, you can define $\hat{\mathbf{e}}_{axis}$ as the ratio in (7.41). ■

7.19. Derive Equation (7.42).

Solution: From (7.40) and (7.41), we obtain

$$\hat{\mathbf{e}}(\phi,\beta)\cdot\hat{\mathbf{e}}_{axis} = \frac{1}{\sqrt{1 - |\vec{\beta}|^2\sin^2\alpha}}\left(\frac{(|\vec{\beta}| - \sin\alpha\cos\phi)|\vec{\beta}|\cos\alpha}{1 - |\vec{\beta}|\sin\alpha\cos\phi} + \frac{\cos\alpha}{\gamma^2(1 - |\vec{\beta}|\sin\alpha\cos\phi)}\right).$$

The expression in the big parentheses can be written as

$$\frac{\gamma^2(|\vec{\beta}| - \sin\alpha\cos\phi)|\vec{\beta}|\cos\alpha + \cos\alpha}{\gamma^2(1 - |\vec{\beta}|\sin\alpha\cos\phi)},$$

whose numerator is

$$\underbrace{\gamma^2\beta^2}_{=\gamma^2-1}\cos\alpha - \gamma^2|\vec{\beta}|\sin\alpha\cos\phi\cos\alpha + \cos\alpha$$

or

$$\gamma^2\cos\alpha(1 - |\vec{\beta}|\sin\alpha\cos\phi).$$

Dividing this by the denominator, you get $\cos\alpha$. Therefore,

$$\hat{\mathbf{e}}(\phi,\beta)\cdot\hat{\mathbf{e}}_{axis} = \frac{\cos\alpha}{\sqrt{1 - |\vec{\beta}|^2\sin^2\alpha}}.$$

■

7.20. Derive Equation (7.46).

Solution: Use determinant to find the cross product.

$$\begin{aligned}
\vec{\theta}\times\vec{\varphi} &= \det\begin{pmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ a\cos\theta\cos\varphi & a\cos\theta\sin\varphi & -a\sin\theta \\ -a\sin\theta\sin\varphi & a\sin\theta\cos\varphi & 0 \end{pmatrix} \\
&= a^2\sin^2\theta\cos\varphi\hat{\mathbf{e}}_x + a^2\sin^2\theta\sin\varphi\hat{\mathbf{e}}_y + a^2(\sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi)\hat{\mathbf{e}}_z \\
&= a^2\sin\theta(\sin\theta\cos\varphi\hat{\mathbf{e}}_x + \sin\theta\sin\varphi\hat{\mathbf{e}}_y + \cos\theta\hat{\mathbf{e}}_z)
\end{aligned}$$

■

7.21. Using (7.45) and (7.7), derive Equation (7.49).

Solution: With $\mathbf{r}$ as given in (7.45), (7.7) yields

$$\begin{aligned}
x' &= \gamma\Big(a\sin\theta\cos\varphi - \beta|\mathbf{r} - \mathbf{r}_0|\Big) \\
y' &= y = a\sin\theta\sin\varphi \\
z' &= z = a\cos\theta.
\end{aligned}$$

The only thing left to do is to calculate $|\mathbf{r} - \mathbf{r}_0|$:

$$|\mathbf{r} - \mathbf{r}_0|^2 = x^2 + y^2 + (z-b)^2 = \underbrace{x^2 + y^2 + z^2}_{=a^2} + b^2 - 2zb = a^2 + b^2 - 2ab\cos\theta.$$

or

$$|\mathbf{r} - \mathbf{r}_0| = \sqrt{a^2 + b^2 - 2ab\cos\theta}.$$

■

7.22. Derive Equations (7.50) and (7.51).

Solution: With

$$\vec{\theta}' \equiv \left\langle \frac{\partial x'}{\partial\theta}, \frac{\partial y'}{\partial\theta}, \frac{\partial z'}{\partial\theta} \right\rangle, \quad \vec{\varphi}' \equiv \left\langle \frac{\partial x'}{\partial\varphi}, \frac{\partial y'}{\partial\varphi}, \frac{\partial z'}{\partial\varphi} \right\rangle,$$

the only non-obvious component is $\vec{\theta}'_x$. So, let's calculate it:

$$\begin{aligned} \frac{\partial x'}{\partial\theta} &= \gamma \frac{\partial}{\partial\theta}\left(a\sin\theta\cos\varphi - \beta\sqrt{a^2+b^2-2ab\cos\theta}\right) \\ &= \gamma\left(a\cos\theta\cos\varphi - \beta\frac{ab\sin\theta}{\sqrt{a^2+b^2-2ab\cos\theta}}\right) \end{aligned}$$

For the cross product, use determinant:

$$\begin{aligned} \vec{\theta}' \times \vec{\varphi}' &= a^2 \det \begin{pmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \gamma\cos\theta\cos\varphi - \frac{\beta\gamma b\sin\theta}{\sqrt{a^2+b^2-2ab\cos\theta}} & \cos\theta\sin\varphi & -\sin\theta \\ -\gamma\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix} \\ &= a^2\left[\sin^2\theta\cos\varphi\hat{\mathbf{e}}_x + \gamma\sin^2\theta\sin\varphi\hat{\mathbf{e}}_y + \left(\gamma\sin\theta\cos\theta - \frac{\beta\gamma b\sin^2\theta\cos\varphi}{\sqrt{a^2+b^2-2ab\cos\theta}}\right)\hat{\mathbf{e}}_z\right] \\ &= a^2\sin\theta\left[\sin\theta\cos\varphi\hat{\mathbf{e}}_x + \gamma\sin\theta\sin\varphi\hat{\mathbf{e}}_y + \left(\gamma\cos\theta - \frac{\beta\gamma b\sin\theta\cos\varphi}{\sqrt{a^2+b^2-2ab\cos\theta}}\right)\hat{\mathbf{e}}_z\right]. \end{aligned}$$

■

7.23. Obtain Equation (7.52) by substituting (7.49) and (7.51) in (C.17).

Solution: In the present context, (C.17) can be written as

$$(\vec{\theta}' \times \vec{\varphi}')_x x' + (\vec{\theta}' \times \vec{\varphi}')_y y + (\vec{\theta}' \times \vec{\varphi}')_z (z-b) = 0.$$

Ignoring the factor $a^2\sin\theta$ in the expression for $\vec{\theta}' \times \vec{\varphi}'$, we have

$$\begin{aligned} 0 &= \gamma\sin\theta\cos\varphi\left(a\sin\theta\cos\varphi - \beta\sqrt{a^2+b^2-2ab\cos\theta}\right) + a\gamma\sin^2\theta\sin^2\varphi \\ &\quad + \left(\gamma\cos\theta - \frac{\beta\gamma b\sin\theta\cos\varphi}{\sqrt{a^2+b^2-2ab\cos\theta}}\right)(a\cos\theta - b) \\ &= a\sin^2\theta - \beta\sin\theta\cos\varphi\sqrt{a^2+b^2-2ab\cos\theta} \\ &\quad + a\cos^2\theta - b\cos\theta - \frac{\beta b\sin\theta\cos\varphi(a\cos\theta-b)}{\sqrt{a^2+b^2-2ab\cos\theta}} \end{aligned}$$

or

$$\begin{aligned} 0 &= a - b\cos\theta - \frac{\beta\sin\theta\cos\varphi[(a\cos\theta-b)b + (a^2+b^2-2ab\cos\theta)]}{\sqrt{a^2+b^2-2ab\cos\theta}} \\ &= (a - b\cos\theta)\left[1 - \frac{\beta a\sin\theta\cos\varphi}{\sqrt{a^2+b^2-2ab\cos\theta}}\right]. \end{aligned}$$

■

7.24. Show that (x', y', z') of Equation (7.53) satisfy

$$\sqrt{x'^2 + y'^2 + (z'-b)^2} = \frac{\sqrt{b^2-a^2}}{b}\gamma(b - \beta a\cos\varphi).$$

Solution:

$$\begin{aligned}|\mathbf{r}' - \mathbf{r}_0|^2 \equiv x'^2 + y'^2 + (z'-b)^2 &= \gamma^2 a^2 \left(1 - \frac{a^2}{b^2}\right)\cos^2\varphi + \gamma^2\beta^2(b^2 - a^2) \\ &- 2\gamma^2\beta a \frac{b^2 - a^2}{b}\cos\varphi + a^2\left(1 - \frac{a^2}{b^2}\right)\sin^2\varphi + \frac{(a^2 - b^2)^2}{b^2} \\ &= (\gamma^2 - 1)a^2\left(1 - \frac{a^2}{b^2}\right)\cos^2\varphi + (\gamma^2 - 1)(b^2 - a^2) \\ &- 2\gamma^2\beta a \frac{b^2 - a^2}{b}\cos\varphi + a^2\left(1 - \frac{a^2}{b^2}\right) + \frac{(a^2 - b^2)^2}{b^2},\end{aligned}$$

where I wrote $\sin^2\varphi$ as $1 - \cos^2\varphi$ and used $\gamma^2\beta^2 = \gamma^2 - 1$. Let's continue

$$\begin{aligned}|\mathbf{r}' - \mathbf{r}_0|^2 &= \gamma^2\beta^2 a^2 \left(\frac{b^2 - a^2}{b^2}\right)\cos^2\varphi + \gamma^2(b^2 - a^2) - b^2 + a^2 \\ &- 2\gamma^2\beta a \frac{b^2 - a^2}{b}\cos\varphi + a^2\left(1 - \frac{a^2}{b^2}\right) + \frac{(a^2 - b^2)^2}{b^2}.\end{aligned}$$

The sum of the last two terms is

$$a^2\left(1 - \frac{a^2}{b^2}\right) + \frac{(a^2 - b^2)^2}{b^2} = \frac{a^2 - b^2}{b^2}(a^2 - b^2 - a^2) = b^2 - a^2,$$

which cancels the last two terms on the first line of the previous equation. So, we now have

$$\begin{aligned}|\mathbf{r}' - \mathbf{r}_0|^2 &= \gamma^2\beta^2 a^2 \left(\frac{b^2 - a^2}{b^2}\right)\cos^2\varphi + \gamma^2(b^2 - a^2) - 2\gamma^2\beta a \frac{b^2 - a^2}{b}\cos\varphi \\ &= \frac{b^2 - a^2}{b^2}\gamma^2\left(\beta^2 a^2\cos^2\varphi + b^2 - 2\beta a b\cos\varphi\right) = \frac{b^2 - a^2}{b^2}\gamma^2\left(b - \beta a\cos\varphi\right)^2.\end{aligned}$$

■

7.25. Show that the line from $(0, 0, b)$ in the direction of $\hat{\mathbf{e}}_{axis}$ of Equation (7.41) cuts the x axis at $-|\vec{\beta}|\gamma\sqrt{b^2 - a^2}$.

Solution: The line in the direction of $\hat{\mathbf{e}}_{axis}$ is $t\hat{\mathbf{e}}_{axis}$, where $-\infty < t < \infty$. A vector parallel to this line with its tail at $\mathbf{r}_0 = \langle 0, 0, b\rangle$ cuts the x-axis for some $t = t_0$. So, we have to solve the equation

$$\mathbf{r}_0 + t_0\hat{\mathbf{e}}_{axis} = x\hat{\mathbf{e}}_x \iff \langle 0, 0, b\rangle + \left\langle -t_0|\vec{\beta}|\cos\alpha, 0, -t_0/\gamma\right\rangle = \langle x, 0, 0\rangle$$

or

$$\left\langle -t_0|\vec{\beta}|\cos\alpha, 0, b - t_0/\gamma\right\rangle = \langle x, 0, 0\rangle.$$

The equality of the last component gives $t_0 = \gamma b$. Plugging this in the equality of the first component yields

$$x = -b\gamma|\vec{\beta}|\cos\alpha = -b\gamma|\vec{\beta}|\sqrt{1 - a^2/b^2} = -\gamma|\vec{\beta}|\sqrt{b^2 - a^2}.$$

■

7.26. A point source produces a circular cone of light. It is placed at $(0,0,b)$ with its axis along the z-axis in reference frame O, moving with speed β in the positive direction of the x'-axis of O'. At $t = t' = 0$, when the two origins coincide, the source emits a pulse, producing a circular image of radius a in the xy-plane of O. What is the shape of the image in the $x'y'$-plane of O'? Hint: Look at the *events* of the arrival of light beams to the planes.

Solution: In the reference frame O, the event of a beam arriving at (x,y) has time coordinate $t = \sqrt{a^2+b^2}$, because the beam was emitted at $t = 0$. Lorentz transform to the RF O':

$$x' = \gamma(x + \beta\sqrt{a^2+b^2}), \quad y' = y \iff \frac{x'}{\gamma} - \beta\sqrt{a^2+b^2} = x, \quad y' = y$$

or

$$\left(\frac{x'}{\gamma} - \beta\sqrt{a^2+b^2}\right)^2 + y'^2 = a^2$$

or

$$\frac{\left(x' - \beta\gamma\sqrt{a^2+b^2}\right)^2}{\gamma^2 a^2} + \frac{y'^2}{a^2} = 1,$$

which is an ellipse centered at $(\beta\gamma\sqrt{a^2+b^2}, 0)$ with semi-major axis γa and semi-minor axis a. ■

7.27. Derive Equation (7.57).

Solution: The cross product is identical to that given in the solution of Problem 7.20. Let $\mathbf{r}_c \equiv \langle x_c, 0, 0\rangle$, $\mathbf{r}_0 \equiv \langle 0, 0, b\rangle$, and $\mathbf{r} \equiv \langle x, y, z\rangle$. With these notations, we can use the result of Problem 7.20 and find what we are looking for with minimal effort. We want to find the solution to

$$0 = (\vec{\theta} \times \vec{\varphi}) \cdot (\mathbf{r} + \mathbf{r}_c - \mathbf{r}_0) = (\vec{\theta} \times \vec{\varphi}) \cdot (\mathbf{r} - \mathbf{r}_0) + (\vec{\theta} \times \vec{\varphi}) \cdot \mathbf{r}_c.$$

The first term on the right has been calculated and is given as the left-hand side of the equation after Equation (7.46). The second term is

$$(\vec{\theta} \times \vec{\varphi})_x x_c = x_c a^2 \sin^2\theta \cos\varphi.$$

Thus, our equation is now

$$0 = a^2 \sin\theta[a\sin^2\theta + \cos\theta(a\cos\theta - b)] + x_c a^2 \sin^2\theta\cos\varphi = 0, \quad \theta \neq 0, \pi,$$

or

$$0 = a^2 \sin\theta[x_c \sin\theta\cos\varphi + a\sin^2\theta + a\cos^2\theta - b\cos\theta], \quad \theta \neq 0, \pi,$$

or

$$0 = x_c \sin\theta\cos\varphi + a - b\cos\theta = 0, \quad \theta \neq 0, \pi.$$

■

7.28. Starting with Equation (7.57), provide all the missing steps leading to (7.58).

Solution: First note that

$$\begin{aligned} x^2+y^2+(z-b)^2 &= x_c^2+a^2\sin^2\theta\cos^2\varphi+2x_c a\sin\theta\cos\varphi+a^2\sin^2\theta\sin^2\varphi \\ &+a^2\cos^2\theta+b^2-2ab\cos\theta = x_c^2+a^2+2x_c a\sin\theta\cos\varphi+b^2-2ab\cos\theta \\ &= x_c^2+a^2+b^2+2a\underbrace{(x_c\sin\theta\cos\varphi-b\cos\theta)}_{=-a \text{ by previous problem}} = x_c^2+b^2-a^2. \end{aligned}$$

The denominators of (7.12) are all the same. With $x_q = x_c$ and using the result obtained above, they can be written as

$$\begin{aligned} Den &= x_c\left[x_c+a\sin\theta\cos\varphi-\beta\sqrt{x_c^2+b^2-a^2}\right]-b(a\cos\theta-b) \\ &= x_c^2+ax_c\sin\theta\cos\varphi-\beta x_c\sqrt{x_c^2+b^2-a^2}-ab\cos\theta+b^2 \\ &= x_c^2+b^2-a^2-\beta x_c\sqrt{x_c^2+b^2-a^2} \end{aligned}$$

by the result obtained in the previous problem. ■

7.29. Define $\tan\alpha \equiv b/(x_c\cos\varphi)$ and show that (7.57) can be written as

$$\sin\theta\cos\alpha-\cos\theta\sin\alpha+\frac{a\cos\alpha}{x_c\cos\varphi}=0,$$

or

$$\sin(\theta-\alpha)=-\frac{a\cos\alpha}{x_c\cos\varphi}.$$

Now define

$$\sin\eta \equiv \frac{a\cos\alpha}{x_c\cos\varphi}$$

and show that $\theta=\alpha-\eta$. Using the trigonometric identity

$$\cos\alpha=\frac{1}{\sqrt{1+\tan^2\alpha}}$$

write η in terms of φ:

$$\sin\eta=\frac{a}{\sqrt{b^2+x_c^2\cos^2\varphi}}.$$

Since $b>a$, this shows that our definition of $\sin\eta$ is consistent with the fact that $|\sin\eta|<1$. With both α and η given in terms of φ, derive (7.59).

Solution: Divide (7.57) by $x_c\cos\varphi$ to write it as

$$\sin\theta-\underbrace{\left(\frac{b}{x_c\cos\varphi}\right)}_{=\tan\alpha}\cos\theta+\frac{a}{x_c\cos\varphi}=0$$

or

$$\sin\theta-\frac{\sin\alpha}{\cos\alpha}\cos\theta+\frac{a}{x_c\cos\varphi}=0$$

or

$$\sin\theta\cos\alpha-\cos\theta\sin\alpha+\frac{a\cos\alpha}{x_c\cos\varphi}=0.$$

Therefore,

$$\sin(\theta-\alpha) = -\frac{a\cos\alpha}{x_c\cos\varphi} = \sin(-\eta),$$

and $\theta-\alpha=-\eta$, or $\theta=\alpha-\eta$.

Now, I want to express everything in terms of the original parameters. For this, I need the sines and cosines of the newly defined angles.

$$\begin{aligned}\cos\alpha &= \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+(b/x_c\cos\varphi)^2}} = \frac{x_c\cos\varphi}{\sqrt{x_c^2\cos^2\varphi+b^2}}\\ \sin\alpha &= \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{b/x_c\cos\varphi}{\sqrt{1+(b/x_c\cos\varphi)^2}} = \frac{b}{\sqrt{x_c^2\cos^2\varphi+b^2}},\end{aligned}$$

$\sin\eta$ has already been calculated. So, I just need $\cos\eta$:

$$\cos\eta = \sqrt{1-\sin^2\eta} = \sqrt{1-\frac{a^2}{b^2+x_c^2\cos^2\varphi}} = \sqrt{\frac{b^2-a^2+x_c^2\cos^2\varphi}{b^2+x_c^2\cos^2\varphi}}.$$

Now we are ready to calculate $\sin\theta$ and $\cos\theta$:

$$\begin{aligned}\sin\theta &= \sin(\alpha-\eta) = \sin\alpha\cos\eta - \cos\alpha\sin\eta\\ &= \left(\frac{b}{\sqrt{x_c^2\cos^2\varphi+b^2}}\right)\left(\sqrt{\frac{b^2-a^2+x_c^2\cos^2\varphi}{b^2+x_c^2\cos^2\varphi}}\right)\\ &\quad -\left(\frac{x_c\cos\varphi}{\sqrt{x_c^2\cos^2\varphi+b^2}}\right)\left(\frac{a}{\sqrt{b^2+x_c^2\cos^2\varphi}}\right)\\ &= \frac{b\sqrt{b^2-a^2+x_c^2\cos^2\varphi}-ax_c\cos\varphi}{b^2+x_c^2\cos^2\varphi}.\end{aligned}$$

This is the first equation of (7.59). The second equation can be derived similarly. ■

7.30. Show that with $\beta=0$, Equations (7.58) and (7.59) give (7.60).

Solution: With $\beta=0$, (7.58) becomes

$$\begin{aligned}u' &= \frac{b\,(x_c+a\sin\theta\cos\varphi)+x_c(a\cos\theta-b)}{x_c^2+b^2-a^2}\\ v' &= \frac{a\sqrt{x_c^2+b^2}\sin\theta\sin\varphi}{x_c^2+b^2-a^2}.\end{aligned}$$

I calculate the first equation, leaving the easier second equation for you. Substitute for $\sin\theta$ and $\cos\theta$ from (7.59) in the first equation:

$$\begin{aligned}u &= \frac{bx_c + a\frac{b^2\sqrt{b^2-a^2+x_c^2\cos^2\varphi}-abx_c\cos\varphi}{b^2+x_c^2\cos^2\varphi}\cos\varphi}{x_c^2+b^2-a^2}\\ &\quad + \frac{a\frac{x_c^2\cos\varphi\sqrt{b^2-a^2+x_c^2\cos^2\varphi}+abx_c}{b^2+x_c^2\cos^2\varphi}-bx_c}{x_c^2+b^2-a^2},\end{aligned}$$

or

$$u = \frac{bx_c(b^2 + x_c^2\cos^2\varphi) + ab^2\sqrt{b^2 - a^2 + x_c^2\cos^2\varphi}\,\cos\varphi - a^2bx_c\cos^2\varphi}{(x_c^2 + b^2 - a^2)(b^2 + x_c^2\cos^2\varphi)} + \frac{ax_c^2\cos\varphi\sqrt{b^2 - a^2 + x_c^2\cos^2\varphi} + a^2bx_c - bx_c(b^2 + x_c^2\cos^2\varphi)}{(x_c^2 + b^2 - a^2)(b^2 + x_c^2\cos^2\varphi)},$$

or

$$u = \frac{a(b^2 + x_c^2)\cos\varphi\sqrt{b^2 - a^2 + x_c^2\cos^2\varphi} + a^2bx_c\sin^2\varphi}{(b^2 + x_c^2 - a^2)(b^2 + x_c^2\cos^2\varphi)}.$$

■

7.31. Show that with $\cos\phi$ defined as (7.61), Equation (7.62) follows and u and v of (7.60) could be expressed as in (7.63).

Solution: Verifying (7.63) once (7.61) and (7.62) are given is trivial. Verifying the definition of $\cos\phi$ by (7.61) is also very straightforward. The only challenge of the problem is to show that $\sin\phi$ as defined by (7.62) is indeed $\sqrt{1-\cos^2\phi}$. Instead of *deriving* (7.62) it is easier to *verify* it: square the numerator of (7.62) and add it to the square of the numerator of (7.61) and show that the sum is the square of the common denominator. I'll leave that for you. ■

7.32. Derive Equation (7.66) from Equations (7.59), (7.64), and (7.65).

Solution: Rather than go through the algebra, which happens to be very messy, I show you how to obtain the results using Mathematica. Figure 7.3 of the manual shows the outline of the procedure. Note how I first found the semi-major axis and center of the ellipse, and then used the first equation in (7.65) to find $\cos\vartheta$. ■

7.33. Substitute (7.56) in (7.7) and simplify to obtain (7.67).

Solution: The time of emission for the event at (x, y, z) is the negative of the distance from that event to the pinhole. Thus

$$\begin{aligned} t &= -\sqrt{x^2 + y^2 + (z-b)^2} = -\sqrt{(x_c + a\sin\theta\cos\varphi)^2 + a^2\sin^2\theta\sin^2\varphi + (a\cos\theta - b)^2} \\ &= -\sqrt{x_c^2 + a^2\sin^2\theta\cos^2\varphi + 2ax_c\sin\theta\cos\varphi + a^2\sin^2\theta\sin^2\varphi + a^2\cos^2\theta + b^2 - 2ab\cos\theta} \\ &= -\sqrt{x_c^2 + a^2 + 2ax_c\sin\theta\cos\varphi + b^2 - 2ab\cos\theta}. \end{aligned}$$

With this t, (7.67) is immediately obtained. ■

7.34. Derive Equation (7.68).

Solution: First find $\vec{\theta}'$ and $\vec{\varphi}'$. The first component of $\vec{\theta}'$ is

$$\frac{\partial x'}{\partial\theta} = \gamma\left(a\cos\theta\cos\varphi - \beta\frac{ax_c\cos\theta\cos\varphi + ab\sin\theta}{\sqrt{x_c^2 + a^2 + b^2 + 2a(x_c\sin\theta\cos\varphi - b\cos\theta)}}\right),$$

and the first component of $\vec{\varphi}'$ is

$$\frac{\partial x'}{\partial\varphi} = \gamma\left(-a\sin\theta\sin\varphi + \beta\frac{ax_c\sin\theta\sin\varphi}{\sqrt{x_c^2 + a^2 + b^2 + 2a(x_c\sin\theta\cos\varphi - b\cos\theta)}}\right).$$

The following 2 lines are Eq (7.59):

$$\texttt{sintheta}[\varphi\_, a\_, b\_, xc\_] := \frac{b\sqrt{b^2 - a^2 + xc^2\,\mathrm{Cos}[\varphi]^2} - a\,xc\,\mathrm{Cos}[\varphi]}{b^2 + xc^2\,\mathrm{Cos}[\varphi]^2}$$

$$\texttt{costheta}[\varphi\_, a\_, b\_, xc\_] := \frac{xc\,\mathrm{Cos}[\varphi]\sqrt{b^2 - a^2 + xc^2\,\mathrm{Cos}[\varphi]^2} + a\,b}{b^2 + xc^2\,\mathrm{Cos}[\varphi]^2}$$

The following 2 lines are Eq (7.64):

$$\texttt{utheta}[\varphi\_, a\_, b\_, xc\_] := \frac{xc + a\,\texttt{sintheta}[\varphi, a, b, xc]\,\mathrm{Cos}[\varphi]}{b - a\,\texttt{costheta}[\varphi, a, b, xc]}$$

$$\texttt{vtheta}[\varphi\_, a\_, b\_, xc\_] := \frac{a\,\texttt{sintheta}[\varphi, a, b, xc]\,\mathrm{Sin}[\varphi]}{b - a\,\texttt{costheta}[\varphi, a, b, xc]}$$

The following 2 lines determine the semi–major axis and the center of the ellipse:

$$\texttt{A}[a\_, b\_, xc\_] := \frac{\texttt{utheta}[0, a, b, xc] - \texttt{utheta}[\pi, a, b, xc]}{2}$$

$$\texttt{uc}[a\_, b\_, xc\_] := \frac{\texttt{utheta}[0, a, b, xc] + \texttt{utheta}[\pi, a, b, xc]}{2}$$

```
FullSimplify[A[a, b, xc]]
```

$$-\frac{a\sqrt{-a^2 + b^2 + xc^2}}{a^2 - b^2}$$

```
FullSimplify[uc[a, b, xc]]
```

$$\frac{b\,xc}{-a^2 + b^2}$$

The following line finds the cosine in (7.65):

$$\texttt{cosnew}[\varphi\_, a\_, b\_, xc\_] := \frac{\texttt{utheta}[\varphi, a, b, xc] - \texttt{uc}[a, b, xc]}{\texttt{A}[a, b, xc]}$$

To get the cosine as given in (7.65), you might FullSimplify cosnew, but Mathematica gives you a comlicated expression. However, when you simplify the ratio of cosnew to Cos[ϕ], you get the coefficient of Cos[ϕ] in (7.65).

$$\texttt{FullSimplify}\left[\frac{\texttt{cosnew}[\varphi, a, b, xc]}{\mathrm{Cos}[\varphi]}\right]$$

$$\frac{\sqrt{-a^2 + b^2 + xc^2}}{\sqrt{-a^2 + b^2 + xc^2\,\mathrm{Cos}[\varphi]^2}}$$

Figure 7.3: Outline of solution using Mathematica.

The other components are also trivially calculate. To save space I'll use $|t|$ for the square root in the calculation of the cross product using determinant.

$$\vec{\theta}' \times \vec{\varphi}' = \det \begin{pmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ a\gamma\cos\theta\cos\varphi - \gamma\beta\dfrac{ax_c\cos\theta\cos\varphi + ab\sin\theta}{|t|} & a\cos\theta\sin\varphi & -a\sin\theta \\ a\gamma\sin\theta\sin\varphi(\beta x_c/|t| - 1) & a\sin\theta\cos\varphi & 0 \end{pmatrix}$$

$$\begin{aligned} &= a\sin^2\theta\cos\varphi\,\hat{\mathbf{e}}_x - a^2\gamma\sin^2\theta\sin\varphi(\beta x_c/|t| - 1)\hat{\mathbf{e}}_y \\ &\quad + \Big[a^2\gamma\sin\theta\cos\theta\cos^2\varphi - \frac{\gamma\beta}{|t|}\Big(a^2x_c\sin\theta\cos\theta\cos^2\varphi \\ &\quad + a^2b\sin^2\theta\cos\varphi\Big) - (\beta x_c/|t| - 1)a^2\gamma\sin\theta\cos\theta\sin^2\varphi\Big]\hat{\mathbf{e}}_z. \end{aligned}$$

A little algebra in the z-component yields the desired result. ■

7.35. Substitute Equations (7.67) and (7.68) in Equation (C.17) to get one expression. Multiply out the parentheses in Equation (7.69) to get another expression. Now show that the two expressions are equal.

Solution: This is the messiest problem in the book if you were to do it by hand! It begs for a solution using computer algebra. So, I have shown my calculation in Mathematica in Figure 7.4 of the manual. ■

7.36. For the second parentheses of (7.69) to be zero, you should have

$$\beta^2(x_c + a\sin\theta\cos\varphi)^2 = x_c^2 + a^2 + b^2 + 2a(x_c\sin\theta\cos\varphi - b\cos\theta).$$

(a) Show that this gives a quadratic equation in $\cos\varphi$ whose solution is

$$\cos\varphi = \frac{x_c/\gamma^2 \pm \sqrt{x_c^2/\gamma^2 + \beta^2(a^2 + b^2 - 2ab\cos\theta)}}{\beta^2 a\sin\theta}.$$

(b) Verify that when x_c is positive (negative) and you choose the positive (negative) sign for the square root, $|\cos\varphi| > 1$ is trivially satisfied.

(c) For $x_c < 0$ and the positive sign for the square root, show that the inequality

$$\frac{-|x_c|/\gamma^2 + \sqrt{x_c^2/\gamma^2 + \beta^2[(b - a\cos\theta)^2 + a^2\sin^2\theta]}}{\beta^2 a\sin\theta} > 1$$

is equivalent to the inequality

$$(|x_c| - a\sin\theta)^2 + \gamma^2(b - a\cos\theta)^2 > 0,$$

which is obviously true. Therefore, $\cos\varphi > 1$.

```
γ[β_] := 1/Sqrt[1 - β^2]
```

The following 3 lines are Eq (7.67).

```
xPrime[φ_, θ_, a_, b_, xc_, β_] :=
  γ[β] (xc + a Sin[θ] Cos[φ] - β Sqrt[xc^2 + a^2 + b^2 + 2 a (xc Sin[θ] Cos[φ] - b Cos[θ])])

yPrime[φ_, θ_, a_] := a Sin[θ] Sin[φ]

zPrime[θ_, a_] := a Cos[θ]
```

The following 3 lines are Eq (7.68) (you have to multiply the x and y components by Sin θ to be consistent with the z component).

```
ThetaCrossPhiPrimeX[φ_, θ_] := Sin[θ] Cos[φ]

ThetaCrossPhiPrimeY[φ_, β_, xc_, a_, b_, θ_] :=
  γ[β] Sin[θ] Sin[φ] (1 - (β xc)/Sqrt[xc^2 + a^2 + b^2 + 2 a (xc Sin[θ] Cos[φ] - b Cos[θ])])

ThetaCrossPhiPrimeZ[φ_, β_, xc_, a_, b_, θ_] :=
  γ[β] (Cos[θ] - (β (xc Cos[θ] + b Sin[θ] Cos[φ]))/Sqrt[xc^2 + a^2 + b^2 + 2 a (xc Sin[θ] Cos[φ] - b Cos[θ])])
```

The following line defines the left-hand side of Eq (7.69):

```
Prod[φ_, β_, xc_, a_, b_, θ_] :=
 γ[β] (xc Sin[θ] Cos[φ] - b Cos[θ] + a)
  (1 - (β (xc + a Sin[θ] Cos[φ]))/Sqrt[xc^2 + a^2 + b^2 + 2 a (xc Sin[θ] Cos[φ] - b Cos[θ])])
```

The following line shows that the left-hand side of Eq (7.69) is the same as the product of the components of Eqs (7.68) and (7.67):

```
FullSimplify[ThetaCrossPhiPrimeX[φ, θ] xPrime[φ, θ, a, b, xc, β] +
  ThetaCrossPhiPrimeY[φ, β, xc, a, b, θ] yPrime[φ, θ, a] +
  ThetaCrossPhiPrimeZ[φ, β, xc, a, b, θ] (zPrime[θ, a] - b) -
  Prod[φ, β, xc, a, b, θ]]

0
```

Figure 7.4: Outline of solution of Problem 7.35 using Mathematica.

(d) For $x_c > 0$ and the negative sign for the square root $\cos\varphi$ is negative. Therefore, you have to prove the inequality

$$\frac{x_c/\gamma^2 - \sqrt{x_c^2/\gamma^2 + \beta^2[(b-a\cos\theta)^2 + a^2\sin^2\theta]}}{\beta^2 a\sin\theta} < -1.$$

Show that this leads to the same inequality as in (c), proving that $\cos\varphi < -1$.

Solution:

(a) Squaring the left-hand side and collecting terms, you get

$$\beta^2 a^2 \sin^2\theta\cos^2\varphi - \frac{2ax_c\sin\theta}{\gamma^2}\cos\varphi - (x_c^2/\gamma^2 + a^2 + b^2 - 2ab\cos\theta) = 0.$$

The quadratic rule gives the solution as

$$\begin{aligned}\cos\varphi &= \frac{ax_c\sin\theta/\gamma^2 \pm \sqrt{a^2x_c^2\sin^2\theta/\gamma^4 + \beta^2a^2\sin^2\theta(x_c^2/\gamma^2 + a^2 + b^2 - 2ab\cos\theta)}}{\beta^2a^2\sin^2\theta} \\ &= \frac{x_c/\gamma^2 \pm \sqrt{x_c^2/\gamma^4 + \beta^2(x_c^2/\gamma^2 + a^2 + b^2 - 2ab\cos\theta)}}{\beta^2 a\sin\theta}.\end{aligned}$$

Now note that

$$\frac{x_c^2}{\gamma^4} + \frac{\beta^2x_c^2}{\gamma^2} = \frac{x_c^2}{\gamma^2}\left(\frac{1}{\gamma^2} + \beta^2\right) = \frac{x_c^2}{\gamma^2}.$$

This gives

$$\cos\varphi = \frac{x_c/\gamma^2 \pm \sqrt{x_c^2/\gamma^2 + \beta^2(a^2 + b^2 - 2ab\cos\theta)}}{\beta^2 a\sin\theta},$$

which is what we are after, but I'll rewrite it as

$$\cos\varphi = \frac{x_c/\gamma^2 \pm \sqrt{x_c^2/\gamma^2 + \beta^2[(b-a\cos\theta)^2 + a^2\sin^2\theta]}}{\beta^2 a\sin\theta}$$

and note that every term under the radical sign is positive.

(b) It is now clear that if $x_c > 0$ and I choose the positive sign the numerator is larger than the denominator. In fact, if you keep just the last term under the radical sign the ratio of the numerator to the denominator will be

$$\frac{\beta a\sin\theta}{\beta^2 a\sin\theta} = \frac{1}{\beta} > 1.$$

Similarly, if $x_c < 0$ and I choose the negative sign the magnitude of the numerator is larger than the denominator.

(c) For $x_c < 0$ and the positive sign for the square root, the inequality

$$\frac{-|x_c|/\gamma^2 + \sqrt{x_c^2/\gamma^2 + \beta^2[(b-a\cos\theta)^2 + a^2\sin^2\theta]}}{\beta^2 a\sin\theta} > 1$$

can be written as

$$\sqrt{x_c^2/\gamma^2 + \beta^2[(b-a\cos\theta)^2 + a^2\sin^2\theta]} > \beta^2 a\sin\theta + \frac{|x_c|}{\gamma^2}.$$

Since both sides are positive, I can square the inequality:

$$\frac{x_c^2}{\gamma^2} + \beta^2[(b - a\cos\theta)^2 + a^2\sin^2\theta] > \left(\beta^2 a \sin\theta + \frac{|x_c|}{\gamma^2}\right)^2,$$

or

$$\frac{x_c^2}{\gamma^2} + \beta^2(b - a\cos\theta)^2 + \beta^2 a^2 \sin^2\theta > \beta^4 a^2 \sin^2\theta + \frac{x_c^2}{\gamma^4} + 2\frac{\beta^2 a \sin\theta |x_c|}{\gamma^2}.$$

Multiply both sides by γ^4 and collect terms using $\gamma^2(1-\beta^2) = 1$ and $\gamma^2 - 1 = -\beta^2\gamma^2$ to get

$$\gamma^2\beta^2 x_c^2 + \gamma^4\beta^2(b - a\cos\theta)^2 + \gamma^2\beta^2 a^2 \sin^2\theta > 2\gamma^2\beta^2 a \sin\theta |x_c|,$$

or

$$x_c^2 + \gamma^2(b - a\cos\theta)^2 + a^2\sin^2\theta - 2a\sin\theta|x_c| > 0,$$

which is the same as

$$(|x_c| - a\sin\theta)^2 + \gamma^2(b - a\cos\theta)^2 > 0.$$

(d) For $x_c > 0$ and the negative sign for the square root, multiply both sides of the inequality

$$\frac{x_c/\gamma^2 - \sqrt{x_c^2/\gamma^2 + \beta^2[(b - a\cos\theta)^2 + a^2\sin^2\theta]}}{\beta^2 a \sin\theta} < -1$$

by -1 and change the direction of the inequality

$$\frac{-x_c/\gamma^2 + \sqrt{x_c^2/\gamma^2 + \beta^2[(b - a\cos\theta)^2 + a^2\sin^2\theta]}}{\beta^2 a \sin\theta} > 1.$$

Noting that $x_c = |x_c|$ (because x_c is positive), this is identical to the inequality in (c).

■

7.37. Derive Equation (7.70) from Equation (7.58).

Solution: For $\varphi = 0$ and $\varphi = \pi$, with obvious notation, (7.59) gives

$$\sin\theta_0 = \frac{b\sqrt{b^2 - a^2 + x_c^2} - a x_c}{b^2 + x_c^2}, \qquad \cos\theta_0 = \frac{x_c\sqrt{b^2 - a^2 + x_c^2} + ab}{b^2 + x_c^2}$$

$$\sin\theta_\pi = \frac{b\sqrt{b^2 - a^2 + x_c^2} + a x_c}{b^2 + x_c^2}, \qquad \cos\theta_\pi = \frac{-x_c\sqrt{b^2 - a^2 + x_c^2} + ab}{b^2 + x_c^2},$$

and the first equation of (7.58) gives

$$u'_0 = \frac{b\gamma\left(x_c + a\sin\theta_0 - \beta\sqrt{x_c^2 + b^2 - a^2}\right) + x_c(a\cos\theta_0 - b)/\gamma}{x_c^2 + b^2 - a^2 - \beta x_c\sqrt{x_c^2 + b^2 - a^2}}$$

$$u'_\pi = \frac{b\gamma\left(x_c - a\sin\theta_\pi - \beta\sqrt{x_c^2 + b^2 - a^2}\right) + x_c(a\cos\theta_\pi - b)/\gamma}{x_c^2 + b^2 - a^2 - \beta x_c\sqrt{x_c^2 + b^2 - a^2}}.$$

Adding these two and dividing by 2 gives u_c':

$$u_c' = \frac{b\gamma\left[x_c + a(\sin\theta_0 - \sin\theta_\pi)/2 - \beta\sqrt{x_c^2 + b^2 - a^2}\right] + \dfrac{x_c}{\gamma}[a(\cos\theta_0 + \cos\theta_\pi)/2 - b]}{x_c^2 + b^2 - a^2 - \beta x_c\sqrt{x_c^2 + b^2 - a^2}},$$

or

$$\begin{aligned}
u_c' &= \frac{b\gamma^2\left[x_c - \dfrac{a^2 x_c}{x_c^2 + b^2} - \beta\sqrt{x_c^2 + b^2 - a^2}\right] + x_c\left[\dfrac{a^2 b}{x_c^2 + b^2} - b\right]}{\gamma\big(x_c^2 + b^2 - a^2 - \beta x_c\sqrt{x_c^2 + b^2 - a^2}\,\big)} \\
&= \frac{b\gamma^2\left[x_c^3 + x_c b^2 - a^2 x_c - \beta(x_c^2 + b^2)\sqrt{x_c^2 + b^2 - a^2}\right] + x_c b\left(a^2 - x_c^2 - b^2\right)}{\gamma(x_c^2 + b^2)\big(x_c^2 + b^2 - a^2 - \beta x_c\sqrt{x_c^2 + b^2 - a^2}\,\big)}
\end{aligned}$$

or

$$\begin{aligned}
u_c' &= \frac{b\left[\gamma^2\beta^2 x_c^3 + \gamma^2\beta^2 x_c b^2 - \gamma^2\beta^2 a^2 x_c - \gamma^2\beta(x_c^2 + b^2)\sqrt{x_c^2 + b^2 - a^2}\right]}{\gamma(x_c^2 + b^2)\sqrt{x_c^2 + b^2 - a^2}\big(\sqrt{x_c^2 + b^2 - a^2} - \beta x_c\big)} \\
&= \frac{b\gamma\left[\beta^2 x_c(x_c^2 + b^2 - a^2) - \beta(x_c^2 + b^2)\sqrt{x_c^2 + b^2 - a^2}\right]}{(x_c^2 + b^2)\sqrt{x_c^2 + b^2 - a^2}\big(\sqrt{x_c^2 + b^2 - a^2} - \beta x_c\big)} \\
&= \frac{b\gamma\left[\beta^2 x_c\sqrt{x_c^2 + b^2 - a^2} - \beta(x_c^2 + b^2)\right]}{(x_c^2 + b^2)\big(\sqrt{x_c^2 + b^2 - a^2} - \beta x_c\big)}.
\end{aligned}$$

I leave the calculation of the semi-major axis A for you. $\blacksquare$

CHAPTER 8

Relativistic Interactions

Problems With Solutions

8.1. Two identical particles of mass m approach each other along a straight line with speed $v = \beta c$ as measured in the lab frame. Show that the energy of one particle as measured in the rest frame of the other is

$$\frac{1+\beta^2}{1-\beta^2}mc^2.$$

Solution: There are two ways to do the problem. In the first, I use the relativistic LAV. Recall that the Lorentz factor associated with the combined velocity is given by

$$\gamma' = \gamma_1\gamma_2(1+\beta_1\beta_2) = \gamma^2(1+\beta^2) = \frac{1+\beta^2}{1-\beta^2}$$

since the two speeds are equal. Now note that the energy is just $E' = \gamma' mc^2$.

In the second way, I use Lorentz transformation of energy. In the lab, with $c = 1$, the energy and momentum of one of the particles are $E = m\gamma$ and $p = m\gamma\beta$. In the rest frame of one of the particles with respect to which the lab moves with speed β, the energy is

$$E' = \gamma(E + \beta p) = \gamma(m\gamma + \beta m\gamma\beta) = m\gamma^2(1+\beta^2).$$

■

8.2. Find an expression for the magnitude of the momentum (8.8), energy (8.9), and velocity (8.10) of the produced particle in the lab frame all in terms of only masses m_1, m_2, and M.

Solution: Since everything is given in terms of $\mathcal{E}_1$ and $\mathcal{E}_1$ is already given in terms of m_1, m_2, and M in (8.7), the problem reduces to substituting (8.7) in the relevant equations. From (8.8), we get

$$\begin{aligned}|\vec{P}| &= \sqrt{\mathcal{E}_1^2 - m_1^2} = \sqrt{\left(\frac{M^2 - m_1^2 - m_2^2}{2m_2}\right)^2 - m_1^2} \\ &= \frac{1}{2m_2}\sqrt{M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2 + m_2^2)},\end{aligned}$$

from (8.9), we get

$$\mathcal{E} = \mathcal{E}_1 + m_2 = \frac{M^2 - m_1^2 + m_2^2}{2m_2}$$

and from (8.10), we get

$$V = \frac{|\vec{P}|}{\mathcal{E}} = \frac{\sqrt{M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2 + m_2^2)}}{M^2 - m_1^2 + m_2^2}.$$

■

8.3. Insert (8.13) in (8.14) to derive (8.15). Another way of obtaining this result is to use Equation (8.3) with $E_{\text{cm}} = E_{1\text{cm}} + E_{2\text{cm}}$, and the fact that

$$E_{2\text{cm}}^2 = E_{1\text{cm}}^2 - m_1^2 + m_2^2,$$

which you should derive.

Solution: First note that from the first equation of (8.13), we have

$$\mathcal{E}_1 - \vec{\beta}_{\text{cm}} \cdot \vec{p}_1 = \mathcal{E}_1 - \frac{|\vec{p}_1|^2}{\mathcal{E}_1 + m_2} = \mathcal{E}_1 - \frac{\mathcal{E}_1^2 - m_1^2}{\mathcal{E}_1 + m_2} = \frac{\mathcal{E}_1 m_2 + m_1^2}{\mathcal{E}_1 + m_2}.$$

This, in combination with the second equation of (8.13), yields the final result.

In the CM, both initial particles have the same momentum. Thus

$$E_{2\text{cm}}^2 = |\vec{p}_2|^2 + m_2^2 = |\vec{p}_1|^2 + m_2^2 = E_{1\text{cm}}^2 - m_1^2 + m_2^2.$$

Now use this and $E_{\text{cm}} = E_{1\text{cm}} + E_{2\text{cm}}$ in Equation (8.3) to obtain

$$(E_{1\text{cm}} + E_{2\text{cm}})^2 = m_1^2 + m_2^2 + 2m_2\mathcal{E}_1,$$

or

$$2E_{1\text{cm}}^2 - m_1^2 + m_2^2 + 2E_{1\text{cm}}E_{2\text{cm}} = m_1^2 + m_2^2 + 2m_2\mathcal{E}_1,$$

or

$$E_{1\text{cm}}^2 - m_1^2 - m_2\mathcal{E}_1 = -E_{1\text{cm}}E_{2\text{cm}},$$

or

$$(E_{1\text{cm}}^2 - m_1^2 - m_2\mathcal{E}_1)^2 = E_{1\text{cm}}^2(E_{1\text{cm}}^2 - m_1^2 + m_2^2),$$

or

$$(m_1^2 + m_2\mathcal{E}_1)^2 - 2E_{1\text{cm}}^2(m_1^2 + m_2\mathcal{E}_1) = E_{1\text{cm}}^2(-m_1^2 + m_2^2).$$

A tiny amount of further algebra gives the desired result. ■

8.4. Suppose that in the elastic scattering of two identical relativistic particles in the lab frame, one of the final particles is produced at rest. Use (8.22) and the conservation of 4-momentum to show that the other final particle moves with the same velocity as the initial incident particle.

Solution: Assume that the third particle is produced at rest. Then $\vec{p}_3 = 0$ and the first equation in (8.22) gives $m\mathcal{E}_1 = m\mathcal{E}_4$ or $\mathcal{E}_1 = \mathcal{E}_4$. Since the total initial momentum is that of the first particle, the fourth particle carries that momentum. Therefore,

$$\vec{v}_1 = \frac{\vec{p}_1}{\mathcal{E}_1} = \frac{\vec{p}_4}{\mathcal{E}_4} = \vec{v}_4.$$

■

8.5. A photon of momentum p_γ hits a macroscopic object of mass M initially at rest, gets absorbed, and sets the object in motion. All motions are in one dimension. Write (8.18) for this process assuming that the mass of the macroscopic object does not change. What is the final momentum of the macroscopic object? What is wrong? How can you resolve the issue?

Solution: For this problem, with E_γ, E, and P denoting the initial energy of the photon and the final energy and momentum of the macroscopic object, Equation (8.18) becomes

$$E_\gamma + M = E, \quad p_\gamma = P \iff E_\gamma = P \iff P + M = E.$$

or

$$(P+M)^2 = E^2 = P^2 + M^2 \iff 2PM = 0 \iff P = 0,$$

which violates momentum conservation. This means that the mass of the macroscopic object *must* change. If M_f is its final mass, then the last equation yields

$$(P+M)^2 = E^2 = P^2 + M_f^2 \iff P = \frac{M_f^2 - M^2}{2M}.$$

■

8.6. A photon of momentum p_γ hits a perfectly reflecting (i.e., the photon does not lose any of its initial energy upon reflection) macroscopic object of mass M initially at rest. All motions are in one dimension. Write (8.18) for this process. Can you assume that the mass of the macroscopic object does not change? If not, write an expression connecting the final mass with M and p_γ.

Solution: With obvious notation, Equation (8.18) becomes

$$\begin{aligned} E_\gamma + M &= E_\gamma + E \\ \vec{p}_\gamma &= -\vec{p}_\gamma + \vec{P}. \end{aligned}$$

or, ignoring the vector signs,

$$M = E, \quad 2p_\gamma = P.$$

If you assume that the mass of the macroscopic object does not change, then the first equation implies that $P = 0$, which contradicts the second equation. So assume that the final mass is M_f. Then

$$M^2 = M_f^2 + P^2 = M_f^2 + 4p_\gamma^2 \iff M_f^2 = M^2 - 4p_\gamma^2.$$

So, the object must *lose* some of its mass. What this really means is that the final object is a different object than the initial one. ■

8.7. Derive (8.20) by transferring $\mathbf{p}_3$ to the left-hand side of (8.16) and squaring both sides.

Solution: Squaring both sides of $\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{p}_4$, yields

$$m_4^2 = m_1^2 + m_2^2 + m_3^2 + 2\mathbf{p}_1 \bullet \mathbf{p}_2 - 2\mathbf{p}_1 \bullet \mathbf{p}_3 - 2\mathbf{p}_2 \bullet \mathbf{p}_3.$$

Assuming that the second particle is at rest, we get

$$m_4^2 = m_1^2 + m_2^2 + m_3^2 + 2m_2\mathcal{E}_1 - 2(\mathcal{E}_1\mathcal{E}_3 - \vec{p}_1 \cdot \vec{p}_3) - 2m_2\mathcal{E}_3,$$

which is a rearranged version of (8.20). ■

8.8. A particle of mass m and energy E_1 collides with an identical stationary particle. The two particles scatter with momenta $\vec{p}_3$ and $\vec{p}_4$ making angles θ_3 and θ_4 with the direction of motion of the incident particle.

(a) Use (8.20) to show that

$$\cos\theta_3 = \frac{(E_1+m)(E_3-m)}{|\vec{p}_1||\vec{p}_3|}.$$

(b) Using the square of (a) and a similar result for θ_4 show that

$$\tan^2\theta_3 = \frac{2m(E_1-E_3)}{(E_1+m)(E_3-m)}$$

and

$$\tan^2\theta_4 = \frac{2m(E_1-E_4)}{(E_1+m)(E_4-m)} = \frac{2m(E_3-m)}{(E_1+m)(E_1-E_3)}$$

(c) Take the product of the last two results to show that

$$\tan\theta_3\tan\theta_4 = \frac{2m}{E_1+m} = \frac{2}{\gamma_1+1}$$

where γ_1 is the gamma factor for the incident particle.

Solution: With all the masses equal, (8.20) becomes

$$m^2 + mE_1 = E_3(E_1+m) - \vec{p}_1\cdot\vec{p}_3 = E_3(E_1+m) - |\vec{p}_1||\vec{p}_3|\cos\theta_3$$

or

(a)

$$|\vec{p}_1||\vec{p}_3|\cos\theta_3 = E_3(E_1+m) - m(m+E_1) = (E_3-m)(m+E_1).$$

Similarly,

$$|\vec{p}_1||\vec{p}_4|\cos\theta_4 = (E_4-m)(m+E_1).$$

(b)

$$\tan^2\theta_3 = \frac{1-\cos^2\theta_3}{\cos^2\theta_3} = \frac{|\vec{p}_1|^2|\vec{p}_3|^2 - (E_3-m)^2(m+E_1)^2}{(E_3-m)^2(m+E_1)^2}$$
$$= \frac{(E_1^2-m^2)(E_3^2-m^2) - (E_3-m)^2(m+E_1)^2}{(E_3-m)^2(m+E_1)^2}.$$

Write the first term of the numerator as

$$(E_1-m)(E_1+m)(E_3-m)(E_3+m),$$

factor out the common terms and simplify to get the final result. The final expression for $\tan^2\theta_4$ comes from the conservation of energy: $E_1+m = E_3+E_4$.

(c) Multiplying the two tangent terms in (b) immediately gives you the first expression on the right-hand side. The second expression follows from $E_1 = m\gamma_1$.

■

8.9. Show that in the ultra-relativistic case, Equation (8.20) leads to the angle between the final two particles being zero.

Solution: Actually, Equation (8.20) gives more than that. It shows that both particles move in the original direction of the projectile. Ultra-relativistic means that we can ignore masses compared with energies. Therefore, all momentum magnitudes are equal to the corresponding energies. With these in mind, Equation (8.20) becomes

$$m_2\mathcal{E}_1 \approx \mathcal{E}_3\mathcal{E}_1 - |\vec{p}_1|| \cdot \vec{p}_3|\cos\theta_3 \approx \mathcal{E}_3\mathcal{E}_1(1-\cos\theta_3)$$

or

$$\cos\theta_3 \approx 1 - \frac{m_2}{\mathcal{E}_3} \approx 1,$$

with a similar result for θ_4. ■

8.10. A particle of mass m and relativistic energy $4mc^2$ collides with another stationary particle of mass $2m$ and sticks to it. What is the mass of the resulting composite particle.

Solution: Set $c = 1$. Initially, the energy is $6m$, and the momentum is

$$p_{in} = \sqrt{16m^2 - m^2} = \sqrt{15}\, m.$$

Let M be the mass of the final particle. Then, conservation of energy and momentum gives

$$M^2 = E^2 - P^2 = 36m^2 - 15m^2 = 21m^2 \iff M = \sqrt{21}\, m$$

■

8.11. An electron of kinetic energy 1 GeV strikes a positron (antielectron) at rest and the two particles annihilate each other and produce two photons, one moving in the forward direction (the direction that electron had before collision) and the other in the backward direction. What are the energies of the two photons. The mass (times c^2) of electron and positron are the same and equal to 0.511 MeV.

Solution: Using Equation (8.20) with obvious notation, you can write

$$\begin{aligned} m(\mathcal{E}_e + m_e) &= \mathcal{E}_{\gamma i}(\mathcal{E}_e + m_e) - |\vec{p}_e||\vec{p}_{\gamma i}|\cos\theta_i \\ &= \mathcal{E}_{\gamma i}(\mathcal{E}_e + m_e) - |\vec{p}_e|\mathcal{E}_{\gamma i}\cos\theta_i, \quad i = 1, 2 \end{aligned}$$

or

$$\mathcal{E}_{\gamma i} = \frac{m_e(\mathcal{E}_e + m_e)}{\mathcal{E}_e + m_e - |\vec{p}_e|\cos\theta_i}, \quad i = 1, 2.$$

So, the two photons have energies,

$$\begin{aligned} \mathcal{E}_{\gamma 1} &= \frac{m_e(\mathcal{E}_e + m_e)}{\mathcal{E}_e + m_e - |\vec{p}_e|} = \frac{\mathcal{E}_e + m_e + |\vec{p}_e|}{2} \\ \mathcal{E}_{\gamma 2} &= \frac{m_e(\mathcal{E}_e + m_e)}{\mathcal{E}_e + m_e + |\vec{p}_e|} = \frac{\mathcal{E}_e + m_e - |\vec{p}_e|}{2}. \end{aligned}$$

(You should obtain the last equality on each line.) For $\mathcal{E}_e >> m_e$, as is the case here, we get $\mathcal{E}_{\gamma 1} \approx \mathcal{E}_e$ and $\mathcal{E}_{\gamma 2} \approx \frac{1}{2}m_e$. I let you plug in the numbers. ■

8.12. An electron of energy E and momentum $\vec{p}$ strikes a positron (antielectron) at rest. The two particles annihilate each other and produce two photons of energies $E_{\gamma 1}$ and $E_{\gamma 2}$.

(a) Use (8.20) to express the energy of each photon in terms of E, the mass of the electron (or positron) m_e, and the photon's scattering angle.

(b) Show that the photons have the same scattering angle if and only if they have the same energy.

(c) Prove that when the photons have the same scattering angles θ, then

$$\cos\theta = \sqrt{\frac{E - m_e}{E + m_e}} = \frac{|\vec{p}|}{2E_\gamma},$$

where E_γ is the energy of either photon.

(d) Show that for an ultra-relativistic incident electron, both photons move in the forward direction.

Solution:

(a) This is done in the previous problem. I reproduce the result here:

$$E_{\gamma i} = \frac{m_e(E + m_e)}{E + m_e - |\vec{p}_e| \cos\theta_i}, \quad i = 1, 2.$$

(b) Clearly, if $\theta_1 = \theta_2$, then $E_{\gamma 1} = E_{\gamma 2}$.

(c) Solve (b) for the common $\cos\theta$ in terms of the common E_γ:

$$\cos\theta = \frac{(E + m_e)(E_\gamma - m_e)}{|\vec{p}_e| E_\gamma}.$$

Conservation of energy gives $E + m_e = 2E_\gamma$, which also leads to $2(E_\gamma - m_e) = E - m_e$. Thus,

$$\cos\theta = \frac{2E_\gamma(E_\gamma - m_e)}{|\vec{p}_e| E_\gamma} = \frac{2(E_\gamma - m_e)}{|\vec{p}_e|} = \frac{E - m_e}{|\vec{p}_e|} = \sqrt{\frac{E - m_e}{E + m_e}}.$$

Multiplying the numerator and the denominator of the fraction under the radical sign by $E + m_e$ yields

$$\cos\theta = \sqrt{\frac{E^2 - m_e^2}{(E + m_e)^2}} = \frac{|\vec{p}_e|}{E + m_e} = \frac{|\vec{p}_e|}{2E_\gamma}.$$

(d) We can ignore m_e compared to E and $|\vec{p}_e|$. Then

$$\cos\theta = \frac{|\vec{p}_e|}{E + m_e} \approx \frac{|\vec{p}_e|}{E} \approx \frac{E}{E} = 1.$$

■

8.13. An electron of energy E, momentum $\vec{p}$, and mass m_e strikes a positron (antielectron) at rest. The two particles annihilate each other and produce two photons. Transfer the collision to the center of mass and do the following:

(a) Find the energy and momentum of each photon in the CM in terms of E, $\vec{p}$, and m_e.

(b) Transfer back to the rest frame of the positron and show the results in Problem 8.12.

Solution: The velocity and γ of the center of mass are given in (8.13):

$$\vec{\beta}_{\rm cm} = \frac{\vec{p}}{E+m_e}, \quad \gamma_{\rm cm} = \frac{E+m_e}{\sqrt{2m_e^2+2m_eE}} = \sqrt{\frac{E+m_e}{2m_e}}.$$

(a) The energy and momentum of the electron in the CM are obtained by Lorentz transforming to the CM using Equation (6.54):

$$\begin{aligned}
E_{\rm cm} &= \gamma_{\rm cm}(E - \vec{\beta}_{\rm cm}\cdot\vec{p}) = \sqrt{\frac{E+m_e}{2m_e}}\left(E - \frac{|\vec{p}|^2}{E+m_e}\right)\\
&= \sqrt{\frac{E+m_e}{2m_e}}\left(E - \frac{E^2-m_e^2}{E+m_e}\right) = \sqrt{\frac{m_e(E+m_e)}{2}}\\
\vec{p}_{\rm cm} &= \vec{p} - \gamma_{\rm cm}\vec{\beta}_{\rm cm}E + (\gamma_{\rm cm}-1)\hat{\beta}_{\rm cm}\hat{\beta}_{\rm cm}\cdot\vec{p}\\
&= \vec{p} - \sqrt{\frac{E+m_e}{2m_e}}\frac{\vec{p}}{E+m_e}E + \left(\sqrt{\frac{E+m_e}{2m_e}}-1\right)\underbrace{\hat{\beta}_{\rm cm}\hat{\beta}_{\rm cm}\cdot\vec{p}}_{=\vec{p}}\\
&= \sqrt{\frac{E+m_e}{2m_e}}\left(-\frac{\vec{p}E}{E+m_e}+\vec{p}\right) = \sqrt{\frac{m_e}{2(E+m_e)}}\,\vec{p}.
\end{aligned}$$

As a check, you should verify that $E_{\rm cm}^2 - \vec{p}_{\rm cm}\cdot\vec{p}_{\rm cm} = m_e^2$. Note that since the positron has no momentum in the lab, its energy and momentum in the CM are $E^+_{\rm cm} = \gamma_{\rm cm}m_e$ and $\vec{p}^{\,+}_{\rm cm} = -\gamma_{\rm cm}\vec{\beta}_{\rm cm}m_e$. As a further check, show that $E^+_{\rm cm} = E_{\rm cm}$ and $\vec{p}^{\,+}_{\rm cm} = -\vec{p}_{\rm cm}$. By energy conservation, the sum of the energies of the two photons should be $2E_{\rm cm}$. Since photons have equal and opposite momenta, their energies should be equal. So, $E_{\gamma\rm cm} = E_{\rm cm}$, and since they are massless, $|\vec{p}_{\gamma\rm cm}| = E_{\gamma\rm cm}$. The direction of the photon momenta cannot be determined.

(b) Now transform the energy and momentum of the photons back to the lab frame:

$$\begin{aligned}
E_\gamma &= \gamma_{\rm cm}(E_{\gamma\rm cm} + \vec{\beta}_{\rm cm}\cdot\vec{p}_{\gamma\rm cm})\\
\vec{p}_\gamma &= \vec{p}_{\gamma\rm cm} + \gamma_{\rm cm}\vec{\beta}_{\rm cm}E_{\gamma\rm cm} + (\gamma_{\rm cm}-1)\hat{\beta}_{\rm cm}\hat{\beta}_{\rm cm}\cdot\vec{p}_{\gamma\rm cm}.
\end{aligned}$$

To find the energy of each photon in the lab, we need $\vec{\beta}_{\rm cm}\cdot\vec{p}_{\gamma\rm cm}$. So, take the dot product of the second equation with $\vec{\beta}_{\rm cm}$:

$$\begin{aligned}
\vec{\beta}_{\rm cm}\cdot\vec{p}_\gamma &= \vec{\beta}_{\rm cm}\cdot\vec{p}_{\gamma\rm cm} + \gamma_{\rm cm}|\vec{\beta}_{\rm cm}|^2E_{\gamma\rm cm} + (\gamma_{\rm cm}-1)\vec{\beta}_{\rm cm}\cdot\vec{p}_{\gamma\rm cm}\\
&= \gamma_{\rm cm}|\vec{\beta}_{\rm cm}|^2E_{\gamma\rm cm} + \gamma_{\rm cm}\vec{\beta}_{\rm cm}\cdot\vec{p}_{\gamma\rm cm}.
\end{aligned}$$

Thus,

$$\vec{\beta}_{\rm cm}\cdot\vec{p}_{\gamma\rm cm} = \frac{\vec{\beta}_{\rm cm}\cdot\vec{p}_\gamma}{\gamma_{\rm cm}} - |\vec{\beta}_{\rm cm}|^2E_{\gamma\rm cm},$$

and

$$E_\gamma = \gamma_{\rm cm}\left(E_{\gamma\rm cm} + \frac{\vec{\beta}_{\rm cm}\cdot\vec{p}_\gamma}{\gamma_{\rm cm}} - |\vec{\beta}_{\rm cm}|^2E_{\gamma\rm cm}\right) = \vec{\beta}_{\rm cm}\cdot\vec{p}_\gamma + \frac{E_{\gamma\rm cm}}{\gamma_{\rm cm}}$$

or

$$E_\gamma = |\vec{\beta}_{\rm cm}||\vec{p}_\gamma|\cos\theta + \frac{E_{\rm cm}}{\gamma_{\rm cm}} = |\vec{\beta}_{\rm cm}|E_\gamma\cos\theta + \frac{E_{\rm cm}}{\gamma_{\rm cm}}$$

or

$$E_\gamma = \frac{E_{\rm cm}/\gamma_{\rm cm}}{1-|\vec{\beta}_{\rm cm}|\cos\theta} = \frac{\sqrt{\frac{m_e(E+m_e)}{2}}\sqrt{\frac{2m_e}{E+m_e}}}{1-\frac{|\vec{p}|}{E+m_e}\cos\theta} = \frac{m_e(E+m_e)}{E+m_e-|\vec{p}|\cos\theta}.$$

This is exactly what we had in Problem 8.12.

■

8.14. A particle of mass m and energy E collides with an identical particle at rest. The collision results in the formation of a single particle. Show that the mass and the speed of the formed particle are, respectively, $\sqrt{2m(E+m)}$ and $\sqrt{(E-m)/(E+m)}$.

Solution: The 4-momentum conservation gives $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{P}$. Squaring both sides gives

$$2m^2 + 2\mathbf{p}_1 \bullet \mathbf{p}_2 = M^2 \iff 2m^2 + 2mE = M^2.$$

Conservation of momentum and energy give

$$P = p = \sqrt{E^2 - m^2}, \quad \mathcal{E} = E + m.$$

The speed of the final particle is therefore

$$V = \frac{P}{\mathcal{E}} = \frac{\sqrt{E^2-m^2}}{E+m} = \sqrt{\frac{E-m}{E+m}}.$$

As a check, note that

$$P^2 + M^2 = E^2 - m^2 + 2m^2 + 2mE = (E+m)^2,$$

which is the square of the energy of the final particle. ■

8.15. A photon of energy E is absorbed by a stationary nucleus of mass M. The collision results in an excitation of the nucleus. Show that the mass and the speed of the excited nucleus are, respectively, $\sqrt{M(2E+M)}$ and $E/(E+M)$.

Solution: The 4-momentum conservation gives $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{P}$. Squaring both sides gives

$$0 + M^2 + 2\mathbf{p}_1 \bullet \mathbf{p}_2 = M^{*\,2} \iff M^2 + 2EM = M^{*\,2}.$$

Conservation of momentum and energy give

$$P = p = E, \quad \mathcal{E} = E + M.$$

The speed of the final particle is therefore

$$V = \frac{P}{\mathcal{E}} = \frac{E}{E+m}.$$

As a check, note that

$$P^2 + M^{*\,2} = E^2 + M^2 + 2ME = (E+M)^2,$$

which is the square of the energy of the excited nucleus. ■

8.16. Derive Equation (8.26).

Solution: From (8.25), we have

$$|\vec{p}_1| = \frac{M^2}{2E_2} - E_1 \iff E_1^2 - m_1^2 = \left(\frac{M^2}{E_2} - E_1\right)^2,$$

or

$$-m_1^2 = \frac{M^4}{4E_2^2} - \frac{M^2}{E_2}E_1 \iff E_1 = \frac{M^2}{4E_2} + \frac{m_1^2 E_2}{M^2}.$$

■

8.17. Show that (8.32) follows from the equation before it.

Solution: From $\mathcal{M} = Zm_p + Nm_n - BE$, the equation before (8.32) can be written as

$$\begin{aligned}\mathcal{E}_1 &= \frac{(\mathcal{M} + BE)^2 - m_1^2 - \mathcal{M}^2}{2\mathcal{M}} = \frac{BE^2 + 2(BE)\mathcal{M} - m_1^2}{2\mathcal{M}} \\ &= BE + \frac{BE^2 - m_1^2}{2\mathcal{M}}.\end{aligned}$$

■

8.18. Derive Equation (8.34) from (8.5).

Solution: For decays Equation (8.5) becomes $E_1 + E_2 = M$ and $\vec{p}_2 = -\vec{p}_1$. So, we have $E_2 = M - E_1$ and $|\vec{p}_2| = |\vec{p}_1|$. These two equations lead to

$$E_2^2 = (M - E_1)^2, \quad E_2^2 - m_2^2 = E_1^2 - m_1^2 \iff (M - E_1)^2 - m_2^2 = E_1^2 - m_1^2,$$

or

$$M^2 - 2ME_1 - m_2^2 = -m_1^2 \iff 2ME_1 = M^2 + m_1^2 - m_2^2.$$

This is the first equation of (8.34). The second equation follows similarly. ■

8.19. Obtain Equation (8.34) by transferring $\mathbf{p}_1$ or $\mathbf{p}_2$ to the left-hand side of $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and squaring both sides.

Solution: This problem demonstrates the advantage of using 4-vectors. Square both sides of $\mathbf{P} - \mathbf{p}_1 = \mathbf{p}_2$ to get

$$M^2 + m_1^2 - 2\mathbf{P} \bullet \mathbf{p}_1 = m_2^2.$$

The first equation of (8.34) follows immediately from $\mathbf{P} \bullet \mathbf{p}_1 = ME_1$. To get the second equation of (8.34), write the 4-momentum conservation as $\mathbf{P} - \mathbf{p}_2 = \mathbf{p}_1$. ■

8.20. The interior of a star like the Sun has a temperature of about 15-20 million Kelvin. Using the familiar statistical physics formula $\langle KE \rangle = \frac{3}{2}k_B T$, with k_B the Boltzmann constant, estimate the kinetic energy of a proton in the interior of the Sun. What is the speed β of this proton?

Solution: Use $T = 2 \times 10^7$ K and $m_p = 938.272$ MeV. Then

$$\langle KE \rangle = \tfrac{3}{2}(1.38 \times 10^{-23})(2 \times 10^7) = 4.14 \times 10^{-16} \text{ J},$$

which is about 2.6 keV or 0.0026 MeV. This is the KE of the proton. The total energy of the proton is thus 938.2746 MeV. The Lorentz factor for the proton is given by

$$\gamma - 1 = \frac{E}{M} - 1 = \frac{938.2746}{938.272} - 1 = 2.76 \times 10^{-6} \approx \tfrac{1}{2}\beta^2.$$

This gives a speed of $\beta = 0.00235$. ■

8.21. Derive Equation (8.45).

Solution: We want to have the identity

$$\beta^2 E^2 + (\beta\vec{\alpha} + \vec{\alpha}\beta) \cdot \vec{p}E + (\vec{\alpha} \cdot \vec{p})^2 - m^2 = E^2 - \vec{p} \cdot \vec{p} - m^2.$$

For the equality to hold, the coefficients must match on both sides. E^2 has 1 as coefficient on the right. So, $\beta^2 = 1$. There is no $p_x E$ term on the right. Therefore, its coefficient on the left must be zero. This gives $\alpha_x \beta + \beta\alpha_x = 0$; similarly for $p_y E$ and $p_z E$ terms. The terms that are of second order in momentum are $(\vec{\alpha} \cdot \vec{p})^2$ on the left and $-\vec{p} \cdot \vec{p}$ on the right. So, they too must equal. ■

8.22. Derive Equations (8.46) and (8.47).

Solution: Let's evaluate $(\vec{\alpha} \cdot \vec{p})^2$, being careful not to commute the components of $\vec{\alpha}$:

$$\begin{aligned}(\vec{\alpha} \cdot \vec{p})^2 &= (\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)\\ &= \alpha_x p_x(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z) + \alpha_y p_y(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)\\ &\quad + \alpha_z p_z(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z),\end{aligned}$$

or

$$\begin{aligned}(\vec{\alpha} \cdot \vec{p})^2 = \alpha_x^2 p_x^2 &+ \alpha_x\alpha_y p_x p_y + \alpha_x\alpha_z p_x p_z\\ &+ \alpha_y\alpha_x p_y p_x + \alpha_y^2 p_y^2 + \alpha_y\alpha_z p_y p_z\\ &+ \alpha_z\alpha_x p_z p_x + \alpha_z\alpha_y p_z p_y + \alpha_z^2 p_z^2.\end{aligned}$$

In the final product on the right-hand side I haven't commuted the p terms. I can! Because they are ordinary functions. Commuting the ps and collecting terms, I get

$$\begin{aligned}(\vec{\alpha} \cdot \vec{p})^2 = \alpha_x^2 p_x^2 &+ \alpha_y^2 p_y^2 + \alpha_z^2 p_z^2\\ &+ (\alpha_x\alpha_y + \alpha_y\alpha_x)p_x p_y\\ &+ (\alpha_x\alpha_z + \alpha_z\alpha_x)p_x p_z\\ &+ (\alpha_y\alpha_z + \alpha_z\alpha_y)p_y p_z.\end{aligned}$$

To get the identity $(\vec{\alpha} \cdot \vec{p})^2 = -\vec{p} \cdot \vec{p} = -p_x^2 - p_y^2 - p_z^2$, I must have

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = -1$$

$$\alpha_x\alpha_y + \alpha_y\alpha_x = \alpha_x\alpha_z + \alpha_z\alpha_x = \alpha_y\alpha_z + \alpha_z\alpha_y = 0.$$

■

8.23. A proton of mass $m_p = 938.3$ MeV collides with a stationary nucleus of mass M to produce antiprotons. The minimum number of particles at the end is two protons, one antiproton, and the nucleus.

(a) Show that the minimum amount of energy for this production is

$$\mathcal{E}_{\min} = 3m_p + \frac{4m_p^2}{M}.$$

(b) If pions of mass $m_\pi \approx 140$ MeV are also produced, this minimum energy increases. Show that if N pions are added to the outcome, then

$$\mathcal{E}_{\min} = 3m_p + Nm_\pi + \frac{(3m_p + Nm_\pi)^2 - m_p^2}{2M}.$$

As a check for your answer, make sure you get (a) as a special case.

(c) Find $\mathcal{E}_{\min}$ for $N = 12$ when the initial proton impinges on a copper nucleus of mass $M = 59150$ MeV.

Solution: Let me denote the numerator of (8.31) by $\mathcal{M}$ not to confuse it with the mass of the nucleus. Then

$$\mathcal{M}^2 = (M + 3m_p)^2 - M^2 - m_p^2 = 8m_p^2 + 6Mm_p.$$

(a) Therefore,

$$\mathcal{E}_{\min} = \frac{\mathcal{M}^2}{2M} = 3m_p + \frac{4m_p^2}{M}$$

(b) If N pions are added to the outcome, then

$$\begin{aligned}\mathcal{M}^2 &= (M + 3m_p + Nm_\pi)^2 - M^2 - m_p^2 \\ &= (3m_p + Nm_\pi)^2 + 2M(3m_p + Nm_\pi) - m_p^2,\end{aligned}$$

and

$$\mathcal{E}_{\min} = \frac{\mathcal{M}^2}{2M} = 3m_p + Nm_\pi + \frac{(3m_p + Nm_\pi)^2 - m_p^2}{2M}$$

(c)

$$\mathcal{E}_{\min} = 3 \times 938.3 + 12 \times 140 + \frac{(3 \times 938.3 + 12 \times 140)^2 - 938.3^2}{2 \times 59150},$$

or $\mathcal{E}_{\min} \approx 4.7$ GeV. ■

CHAPTER 9

Interstellar Travel

Problems With Solutions

9.1. Assume that $M >> dm_g$ and consider the ejection of dm_g from the rocket as a decay. Then use the energy and 3-momentum components of the 4-momentum conservation $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ to show that

$$d(M\gamma) + dm_g\gamma_g' = 0$$
$$d(M\gamma\vec{\beta}) + dm_g\gamma_g'\vec{\beta}_g' = 0.$$

Solution: Write the 4-momentum conservation as $\mathbf{P}_{\text{bef}} = \mathbf{p}_{\text{gas}} + \mathbf{P}_{\text{aft}}$ or $0 = \mathbf{p}_{\text{gas}} + d\mathbf{P}$, and note that

$$\mathbf{p}_{\text{gas}} = (dm_g\gamma_g', dm_g\gamma_g'\vec{\beta}_g') \quad \text{and} \quad d\mathbf{P} = \Big(d(M\gamma), d(M\gamma\vec{\beta})\Big).$$

■

9.2. Show that

$$d(M\gamma) = \gamma dM + M\beta\gamma^3 d\beta$$
$$d(M\gamma\beta) = \gamma\beta dM + M\gamma^3 d\beta.$$

Solution: Clearly, $d(M\gamma) = \gamma dM + Md\gamma$. But

$$d\gamma = d(1-\beta^2)^{-1/2} = \beta(1-\beta^2)^{-3/2}d\beta = \beta\gamma^3 d\beta$$

Similarly, $d(M\gamma\beta) = \gamma\beta dM + Md(\gamma\beta)$, with

$$d(\gamma\beta) = \gamma d\beta + \beta d\gamma = \gamma d\beta + \beta(\beta\gamma^3 d\beta) = \gamma(1+\beta^2\gamma^2)d\beta = \gamma^3 d\beta.$$

■

9.3. Derive Equation (9.3) from the equations preceding it.

Solution: Substitute the given expressions in (9.2) to obtain

$$(1 \mp \beta\beta_g)(\gamma\beta dM + M\gamma^3 d\beta) - (\beta \mp \beta_g)(\gamma dM + M\beta\gamma^3 d\beta) = 0,$$

or

$$\underbrace{[(1 \mp \beta\beta_g)\beta - (\beta \mp \beta_g)]}_{=\pm\beta_g(1-\beta^2)}\gamma dM + \underbrace{[(1 \mp \beta\beta_g) - (\beta^2 \mp \beta_g\beta)]}_{1-\beta^2} M\gamma^3 d\beta = 0,$$

or

$$\pm\frac{\beta_g}{\gamma}dM + M\gamma d\beta = 0 \iff \pm\beta_g dM + M\gamma^2 d\beta = 0.$$

■

9.4. Consider the acceleration part of the photon propulsion (9.6) with initial speed zero. Show that you get Equation (9.4) with $\beta_g = 1$.

Solution: The acceleration part of the photon propulsion (9.6) with initial speed zero reads:

$$\frac{M}{M_0} = \frac{1}{\gamma(1+\beta)} = \frac{\sqrt{1-\beta^2}}{1+\beta} = \frac{\sqrt{(1-\beta)(1+\beta)}}{1+\beta} = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}.$$

This is the accelerating version of Equation (9.4) with $\beta_g = 1$. ■

9.5. Suppose that a rocket accelerates from rest to β_0 upon launch and decelerates from β_0 to zero upon landing. If M_0 is the mass at launch and M_f is the mass at landing, show that (9.7) gives $M_f/M_0 = [\gamma_0(\beta_0 + 1)]^{-2}$, in agreement with (the first half of) Equation (9.5).

Solution: With $\beta_0 = 0$ and $\beta(t_1) = \beta_0$, the first equation of (9.7) gives

$$\frac{M_1}{M_0} = \frac{1}{\gamma_0(\beta_0+1)}.$$

With $\beta(t_f) = 0$, the second equation of (9.7) gives

$$\frac{M_f}{M_1} = \frac{1}{\gamma_0(\beta_0+1)}.$$

Therefore,

$$\frac{M_f}{M_0} = \left(\frac{M_f}{M_1}\right)\left(\frac{M_1}{M_0}\right) = \frac{1}{\gamma_0^2(\beta_0+1)^2}.$$

■

9.6. Show that both equations in (9.8) are consistent with $m(0) = 1$ for $\beta(0) = \beta_0$.

Solution: In (9.8), replace $\beta(t)$ with β_0 on the left and set $m(t) = 1$ on the right and show that the equality holds. For the first equation, we get

$$\beta_0 = \frac{\gamma_0^2(\beta_0+1)^2 - 1}{\gamma_0^2(\beta_0+1)^2 + 1},$$

or

$$\beta_0\gamma_0^2(\beta_0+1)^2 + \beta_0 = \gamma_0^2(\beta_0+1)^2 - 1,$$

or

$$1 + \beta_0 = \gamma_0^2(\beta_0+1)^2(1-\beta_0) \iff 1 = \gamma_0^2(\beta_0+1)(1-\beta_0),$$

which holds identically. The second equation is very similar. ■

9.7. The first equation in (9.8) yields $\beta = 1$ when $m(t)$ is completely depleted. Is that a violation of relativity or a confirmation of it? Think about it for a while before you look up the answer in Note 5.2.3.

Solution: It is a confirmation of the theory. When $m(t) = 0$, the rocket is massless, so by Note 5.2.3, it *has to* move at the speed of light. ■

9.8. Let $-k$ be the rate of depletion of the fuel of a rocket.

(a) Solve $dM_{\text{fuel}}/dt = -k$, when k is a positive constant subject to the condition that $M_{\text{fuel}}(0) = M_{\text{fuel}}^{\text{beg}}$. Here $M_{\text{fuel}}^{\text{beg}}$ is the amount of fuel at the beginning (which is *not* necessarily the take-off) of the part of motion under consideration.

(b) Assuming that the fuel mass at take-off is M_0, determine k from the condition that $M_{\text{fuel}}(T_0) = 0$, where T_0 is the time it takes for the fuel to be completely depleted.

(c) Show that if time is measured in units of T_0, then

$$M_{\text{tot}}(t) = M_f + M_{\text{fuel}}(t) = M_f + M_{\text{fuel}}^{\text{beg}} - M_0 t,$$

where M_f is the mass left over at the end of the trip.

Solution:

(a) The solution is $M_{\text{fuel}}(t) = -kt + C$, where C is the constant of integration. With $M_{\text{fuel}}(0) = M_{\text{fuel}}^{\text{beg}}$, we get $C = M_{\text{fuel}}^{\text{beg}}$. Therefore, the complete solution can be written as

$$M_{\text{fuel}}(t) = -kt + M_{\text{fuel}}^{\text{beg}}.$$

(b) We have $0 = -kT_0 + M_0$. Thus, $k = M_0/T_0$.

(c) With k given in (b), write $M_{\text{fuel}}(t)$ of (a) in terms of T_0:

$$M_{\text{fuel}}(t) = -M_0 t/T_0 + M_{\text{fuel}}^{\text{beg}}.$$

Then,

$$M_{\text{fuel}}(t) = M_f + M_{\text{fuel}}(t) = M_f + M_{\text{fuel}}^{\text{beg}} - M_0(t/T_0).$$

■

9.9. Derive Equation (9.11) and by taking the time derivative of (9.12) show that the latter is the integral of (9.11).

Solution: With $\beta_0 = 0$, the first equation of (9.8) becomes

$$\beta(t) = \frac{1 - m^2(t)}{1 + m^2(t)}.$$

On the other hand, at the beginning of the journey, $M_{\text{fuel}}^{\text{beg}} = M_0$. Thus, (9.10) becomes

$$m(t) = 1 - m_0 t, \quad m_0 = \frac{M_0}{M_f + M_0}, \quad t \leq 1.$$

Now take the derivative of (9.12):

$$\frac{dx}{dt} = -1 - \frac{2}{m_0}\frac{d}{dt}\left[\tan^{-1}(1-m_0t)\right] = -1 + \frac{2}{m_0}\left[\frac{m_0}{1+(1-m_0t)^2}\right]$$
$$= -1 + \frac{2}{1+(1-m_0t)^2} = \frac{1-(1-m_0t)^2}{1+(1-m_0t)^2}.$$

■

9.10. Show that the time required to accelerate the rocket from rest to β_0 is given by (9.13).

Solution: By (9.11), we have to solve the equation

$$\beta_0 = \frac{1-(1-m_0t_{\text{acc}})^2}{1+(1-m_0t_{\text{acc}})^2} \iff (1-m_0t_{\text{acc}})^2(1+\beta_0) = 1-\beta_0,$$

or

$$(1-m_0t_{\text{acc}})^2 = \frac{1-\beta_0}{1+\beta_0} = \frac{1-\beta_0^2}{(1+\beta_0)^2},$$

or

$$1-m_0t_{\text{acc}} = \frac{1}{\gamma_0(1+\beta_0)} \iff m_0t_{\text{acc}} = \frac{\gamma_0(1+\beta_0)-1}{\gamma_0(1+\beta_0)}.$$

■

9.11. Show that $x(t_{\text{acc}})$ is given by Equation (9.14).

Solution: Just substitute the result of the previous problem in (9.12). No algebra required! ■

9.12. Differentiate Equation (9.16) to get (9.15).

Solution: The derivative of (9.16) is

$$\frac{dx}{dt} = 1 + \frac{2}{m_0\gamma_0^2(1+\beta_0)^2}\frac{d}{dt}\left[\tan^{-1}\left\{\gamma_0(1+\beta_0)\left[1-m_0\gamma_0(1+\beta_0)t\right]\right\}\right]$$
$$= 1 - \frac{2}{m_0\gamma_0^2(1+\beta_0)^2}\left\{\frac{m_0\gamma_0^2(1+\beta_0)^2}{1+\gamma_0^2(1+\beta_0)^2[1-\gamma_0(1+\beta_0)m_0t]}\right\}$$
$$= 1 - \frac{2}{1+\gamma_0^2(1+\beta_0)^2[1-\gamma_0(1+\beta_0)m_0t]}.$$

This can be trivially shown to equal to (9.15). ■

9.13. Show that the time required to decelerate the rocket from β_0 to rest is given by (9.17) and the distance covered by (9.18).

Solution: Setting (9.15) equal to zero yields

$$\gamma_0^2(1+\beta_0)^2[1-m_0\gamma_0(1+\beta_0)t_{\text{dec}}]^2 - 1 = 0,$$

or

$$1-m_0\gamma_0(1+\beta_0)t_{\text{dec}} = \frac{1}{\gamma_0(1+\beta_0)},$$

or

$$m_0\gamma_0(1+\beta_0)t_{\text{dec}} = \frac{\gamma_0(1+\beta_0)-1}{\gamma_0(1+\beta_0)}.$$

This is essentially (9.17). ■

9.14. The maximum possible speed attainable is when no fuel is left upon landing. Show that this speed is given by Equation (9.19), and for such a speed, t_{acc} and t_{dec} are given by Equation (9.20).

Solution: Setting the sum of (9.13) and (9.17) equal to one yields

$$1 = \frac{\gamma_0(1+\beta_0)-1}{m_0\gamma_0(1+\beta_0)} + \frac{\gamma_0(1+\beta_0)-1}{m_0\gamma_0^2(1+\beta_0)^2} = \frac{\gamma_0^2(1+\beta_0)^2-1}{m_0\gamma_0^2(1+\beta_0)^2},$$

or

$$\gamma_0^2(1+\beta_0)^2(1-m_0) = 1 \iff \gamma_0(1+\beta_0) = \frac{1}{\sqrt{m_f}},$$

because

$$m_0 + m_f \equiv \frac{M_0}{M_f+M_0} + \frac{M_f}{M_f+M_0} = 1.$$

Rewrite the last equation as

$$\frac{\sqrt{1-\beta_0^2}}{1+\beta_0} = \sqrt{\frac{1-\beta_0}{1+\beta_0}} = \frac{1}{\sqrt{m_f}}.$$

Square both sides and solve for β_0 to get Equation (9.19).

Now use $\gamma_0(1+\beta_0) = 1/\sqrt{m_f}$—obtained above—in (9.13) to get

$$t_{\text{acc}} = \frac{(1/\sqrt{m_f})-1}{(1-m_f)(1/\sqrt{m_f})} = \frac{1-\sqrt{m_f}}{\underbrace{1-m_f}_{=(1-\sqrt{m_f})(1+\sqrt{m_f})}} = \frac{1}{1+\sqrt{m_f}}.$$

Equation (9.17) now gives

$$t_{\text{dec}} = \frac{t_{\text{acc}}}{1/\sqrt{m_f}} = \sqrt{m_f}\,t_{\text{acc}} = \frac{\sqrt{m_f}}{1+\sqrt{m_f}}.$$

■

9.15. Derive Equations (9.21) and (9.22). Plot each separately as well as the sum $x_{\text{acc}}+x_{\text{dec}}$ and verify that the sum is a decreasing function with a maximum of 0.57 light second occurring when $m_f = 0$.

Solution: Substitute $\gamma_0(1+\beta_0) = 1/\sqrt{m_f}$ and $m_0 + m_f = 1$ in (9.14):

$$\begin{aligned} x(t_{\text{acc}}) &= \frac{\pi}{2(1-m_f)} - \frac{(1/\sqrt{m_f})-1}{(1-m_f)/\sqrt{m_f}} - \frac{2}{1-m_f}\tan^{-1}\left(\sqrt{m_f}\right) \\ &= \frac{\pi/2 - 1 + \sqrt{m_f} - 2\tan^{-1}\left(\sqrt{m_f}\right)}{1-m_f}. \end{aligned}$$

This is (9.21). I let you derive (9.22). Figure 9.1 of the manual shows x_{acc}, x_{dec}, and $x_{\text{acc}} + x_{\text{dec}}$ as functions of m_f. ■

9.16. I discussed the case of maximum attainable speed when m_f is small. What about the case when $m_f \approx 1$? Write $m_f = 1 - m_0$ and assume that m_0 is very small.

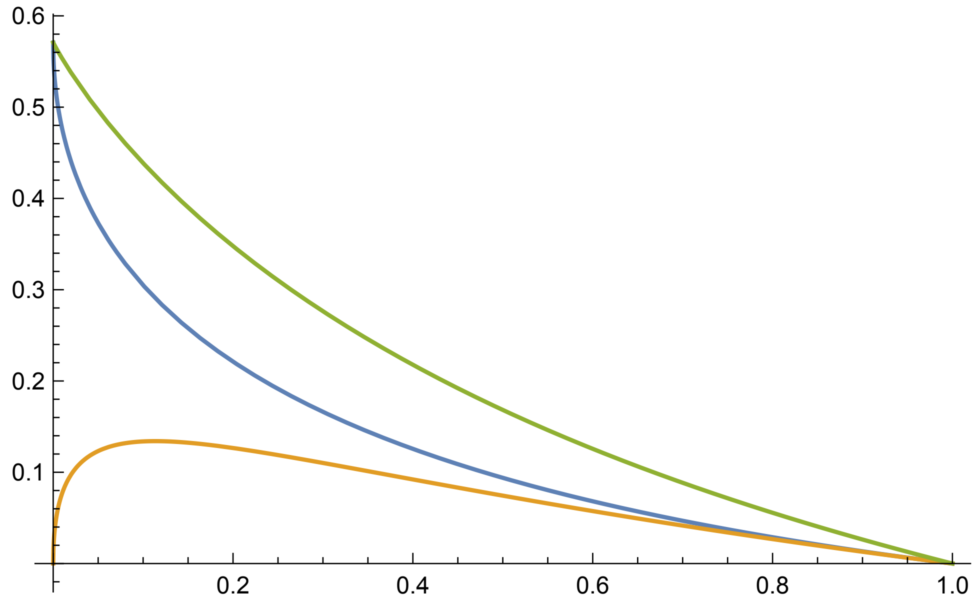


Figure 9.1: The lowest plot is $x_{\text{dec}}(m_f)$, the middle plot is $x_{\text{acc}}(m_f)$, and the top plot is the sum of the two.

(a) Show from Equation (9.19) that the maximum attainable speed is $\frac{1}{2}m_0$.

(b) From the expansion of (9.21) show that $x_{\text{acc}} \to \frac{1}{8}m_0$.

(c) From (9.22) conclude that $x_{\text{dec}} \to \frac{1}{8}m_0$.

Solution:

(a) With $m_f = 1 - m_0$, Equation (9.19) becomes

$$\beta_0 = \frac{m_0}{2 - m_0} \approx \tfrac{1}{2}m_0 \text{ for } \ m_0 \to 0.$$

(b) Since the denominator of (9.21) is m_0, we expand the numerator up to m_0^2. First look at each term of the numerator separately:

$$\sqrt{m_f} = (1 - m_0)^{1/2} \approx 1 - \tfrac{1}{2}m_0 + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2}m_0^2 = 1 - \tfrac{1}{2}m_0 - \tfrac{1}{8}m_0^2.$$

From

$$\tan^{-1}(1 - x) \approx \frac{\pi}{4} - \frac{x}{2} - \frac{x^2}{4}$$

and the result above, we get

$$\begin{aligned}\tan^{-1}(\sqrt{m_f}) = \tan^{-1}\left(1 - \tfrac{1}{2}m_0 - \tfrac{1}{8}m_0^2\right) &= \frac{\pi}{4} - \frac{1}{2}(\tfrac{1}{2}m_0 + \tfrac{1}{8}m_0^2) - \frac{1}{4}(\tfrac{1}{2}m_0 + \tfrac{1}{8}m_0^2)^2 \\ &= \frac{\pi}{4} - \frac{m_0}{4} - \frac{m_0^2}{16} - \frac{m_0^2}{16} = \frac{\pi}{4} - \frac{m_0}{4} - \frac{m_0^2}{8},\end{aligned}$$

where I kept terms up to m_0^2 in the final expression. Now substitute these in (9.21) and obtain

$$x_{\mathrm{acc}} = \frac{1 \overbrace{-\tfrac{1}{2}m_0 - \tfrac{1}{8}m_0^2}^{\sqrt{m_f}} \overbrace{-\frac{\pi}{2} + \frac{m_0}{2} + \frac{m_0^2}{4}}^{-2\tan^{-1}(\sqrt{m_f})} + \pi/2 - 1}{m_0} = \frac{m_0}{8}.$$

(c) I leave this part for you.

■

9.17. Problem 9.16 is based on the assumption of uniform depletion of the fuel. The result, however, turns out to be general, at least for the case of a laser-propelled rocket as discussed on page 206.

(a) Suppose you use half the fuel (saving the other half for landing), i.e., the lasing photons, on a massless rocket ($M_f = 0$) to accelerate the rocket from rest to a speed β_f. Using the first equation in (9.7), show that β_f is given by

$$\frac{1+0.5\kappa}{1+\kappa} = \frac{1}{\gamma_f(\beta_f+1)}, \quad \kappa << 1.$$

(b) Solve this equation for β_f in terms of κ to show that $\beta_f \approx 0.5\kappa$.

Solution:

(a) The fuel mass at time t is $\kappa M_{\mathrm{las}}(t)$, where $M_{\mathrm{las}}(t)$ is the mass of the lasing material at t. Define $r(t) \equiv [M_{\mathrm{las}}(t)/M_{\mathrm{las}}]$ as the ratio of the fuel mass at t to the initial fuel mass. Then the fuel mass at time t is $\kappa r(t) M_{\mathrm{las}}$, and with $M_f = 0$, the ratio of the mass at t to the initial mass is

$$m(t) \equiv \frac{M(t)}{M_0} = \frac{M_{\mathrm{las}} + \kappa r(t) M_{\mathrm{las}}}{M_{\mathrm{las}} + \kappa M_{\mathrm{las}}} = \frac{1+\kappa r(t)}{1+\kappa}.$$

For $r(t) = 0.5$, the first equation in (9.7) gives

$$\frac{1+0.5\kappa}{1+\kappa} = \frac{1}{\gamma_f(\beta_f+1)}, \quad \kappa << 1.$$

(b) Approximate the left-hand side to first order in κ:

$$\frac{1+0.5\kappa}{1+\kappa} = (1+0.5\kappa)(1+\kappa)^{-1} \approx (1+0.5\kappa)(1-\kappa) \approx 1 - 0.5\kappa;$$

rewrite the right-hand side as $\sqrt{(1-\beta_f)/(1+\beta_f)}$ and square both sides:

$$(1-0.5\kappa)^2 = \frac{1-\beta_f}{1+\beta_f} \iff 1-\kappa = \frac{1-\beta_f}{1+\beta_f} \iff \beta_f = \frac{\kappa}{2-\kappa} \approx \frac{\kappa}{2}.$$

■

9.18. Lorentz transform both the energy E of a photon and the sum $E_{\text{sail}} + P_{\text{sail}}$ of the energy and momentum of the light sail to another RF and show that you get the same equality (9.25) in the new RF.

Solution: Write Equation (9.24) as

$$\begin{aligned} E - E_{\text{ref}} &= E'_{\text{sail}} - E_{\text{sail}} = \delta E_{\text{sail}} \\ p + p_{\text{ref}} &= P'_{\text{sail}} - P_{\text{sail}} = \delta P_{\text{sail}}, \end{aligned}$$

which is more general than (9.24) because I am not assuming that M is constant. Add these two equations to obtain

$$2E = \delta(E_{\text{sail}} + P_{\text{sail}}).$$

Let O be the old RF and assume it moves in the positive x direction of the new frame O'. Then the left-hand side of the equation above transforms to

$$2E' = 2\gamma(E + \beta p) = 2\gamma(E + \beta E) = \gamma(1+\beta)(2E).$$

The right-hand side transforms to

$$\begin{aligned} \delta(E'_{\text{sail}} + P'_{\text{sail}}) &= \delta[\gamma(E_{\text{sail}} + \beta P_{\text{sail}}) + \gamma(P_{\text{sail}} + \beta(E_{\text{sail}})] \\ &= \delta[\gamma(1+\beta)(E_{\text{sail}} + P_{\text{sail}})] = \gamma(1+\beta)\delta(E_{\text{sail}} + P_{\text{sail}}). \end{aligned}$$

Thus, $2E' = \delta(E'_{\text{sail}} + P'_{\text{sail}})$. ■

9.19. In this problem you are going to verify Equation (9.27).

(a) Lorentz transform the event of the emission of the ith photon to the instantaneous reference frame of the spacecraft and show that

$$(t_i^{\text{nano}}, x_i^{\text{nano}}) = \gamma t_i(1, -\beta).$$

(b) How long does it take for this photon to reach the craft? Adding t_i^{nano} to this time, find $t_i^{\text{nano, ref}}$.

(c) From (a) and (b) and the fact that reflections occur at the spacecraft, conclude that the events of the reflection of the ith and $i+1$th photons in the spacecraft's RF are

$$\big(\gamma(1+\beta)t_i, 0\big), \quad \text{and} \quad \big(\gamma(1+\beta)t_{i+1}, 0\big).$$

(d) Now Lorentz transform this back to the laser RF and derive (9.27).

Solution:

(a) The spacecraft is moving in the positive direction of laser's frame. So, the laser is moving in the negative direction of the spacecraft frame. Assuming that the laser is at the origin, and the origin of time is the same for both RFs, the coordinates of the ith photon in the laser frame is $(t_i, 0)$. Lorentz transforming to the nanocraft's RF, we get

$$x_i^{\text{nano}} = \gamma(0 - \beta t_i) = -\gamma\beta t_i, \quad t_i^{\text{nano}} = \gamma(-0 + t_i) = \gamma t_i.$$

(b) The distance from which the photon is emitted is x_i^{nano} (note that it doesn't matter that the laser doesn't remain at that position). Therefore, the time at which the photon reaches the craft is

$$t_i^{\text{nano, ref}} = t_i^{\text{nano}} + |x_i|^{\text{nano}} = \gamma(1+\beta)t_i.$$

(c) Since the craft is assumed to be at the origin of its frame, $x_i^{\text{nano, ref}} = 0$.

(d) Lorentz transforming the event of the reflection of the ith photon to the laser frame yields

$$\begin{aligned} t_i^{\text{ref}} &= \gamma\left(t_i^{\text{nano, ref}} + 0\right) = \gamma^2(1+\beta)t_i = \frac{t_i}{1-\beta} \\ x_i^{\text{ref}} &= \gamma\left(0 + \beta t_i^{\text{nano, ref}}\right) = \gamma^2\beta(1+\beta)t_i = \frac{\beta t_i}{1-\beta}, \end{aligned}$$

with similar equations for $i+1$th photon. Equation (9.27) now follows. Note that for (9.27), we don't need x_i^{ref}.

■

9.20. Why reflection coefficient R makes no sense in relativity! Write Equation (8.18) as

$$\begin{aligned} E - E_{\text{ref}} &= E'_{\text{sail}} - E_{\text{sail}} = \delta E_{\text{sail}} \\ p + p_{\text{ref}} &= P'_{\text{sail}} - P_{\text{sail}} = \delta P_{\text{sail}}, \end{aligned} \tag{9.1}$$

where no approximation is made regarding the mass of the spacecraft.

(a) Using R, show that these equations yield

$$\frac{1-R}{1+R} = \frac{\delta E_{\text{sail}}}{\delta P_{\text{sail}}}.$$

(b) Use $E_{\text{sail}} = \sqrt{P_{\text{sail}}^2 + M^2}$ to show that

$$\frac{1-R}{1+R} = \beta,$$

where β is the instantaneous speed of the craft.

Solution:

(a) With $E_{\text{ref}} = p_{\text{ref}} = Rp = RE$ and taking the ratio of the first to the second equation above, we get

$$\frac{1-R}{1+R} = \frac{\delta E_{\text{sail}}}{\delta P_{\text{sail}}}.$$

(b) From $E_{\text{sail}} = \sqrt{P_{\text{sail}}^2 + M^2}$, we obtain

$$\delta E_{\text{sail}} = \delta\sqrt{P_{\text{sail}}^2 + M^2} = \frac{P_{\text{sail}}\delta P_{\text{sail}}}{\sqrt{P_{\text{sail}}^2 + M^2}} = \frac{P_{\text{sail}}}{E_{\text{sail}}}\delta P_{\text{sail}} = \beta\delta P_{\text{sail}}.$$

This gives the result we are after.

■

9.21. Recall that the solid angle subtended at point P_0 by an infinitesimal area da located at point P is given by

$$d\Omega = \frac{|\hat{\mathbf{e}}_n \cdot (\vec{r} - \vec{r}_0)|}{|\vec{r} - \vec{r}_0|^3}\, da,$$

where $\hat{\mathbf{e}}_n$ is the unit vector normal to da and $\vec{r}$ and $\vec{r}_0$ are the position vectors of P and P_0, respectively. Now center a square of side L in the xy-plane and let P_0 have coordinates $(0, 0, z_0)$.

(a) Show that the total solid angle Ω subtended at P_0 by the square is

$$\Omega = 4\tan^{-1}\left(\frac{L^2}{2z_0\sqrt{2L^2 + 4z_0^2}}\right).$$

(b) Show that if $z_0 >> L$, then

$$\Omega \approx \frac{L^2}{z_0^2}.$$

Solution: In this problem, $\hat{\mathbf{e}}_n = \pm\hat{\mathbf{e}}_z$. The ambiguity is due to the fact that a surface has two sides! Let's take $\hat{\mathbf{e}}_n = -\hat{\mathbf{e}}_z$ (away from P_0). We also have

$$\vec{r} = \langle x, y, 0\rangle, \quad \vec{r}_0 = \langle 0, 0, z_0\rangle \quad \vec{r} - \vec{r}_0 = \langle x, y, -z_0\rangle.$$

Thus,

$$d\Omega = \frac{|-\hat{\mathbf{e}}_z \cdot (\vec{r} - \vec{r}_0)|}{|\vec{r} - \vec{r}_0|^3}\, dxdy = \frac{z_0}{(x^2 + y^2 + z_0^2)^{3/2}}\, dxdy,$$

and

$$\Omega = z_0 \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \frac{dxdy}{(x^2 + y^2 + z_0^2)^{3/2}} = 4\tan^{-1}\left(\frac{L^2}{2z_0\sqrt{2L^2 + 4z_0^2}}\right).$$

(b) If $z_0 >> L$, then

$$\frac{L^2}{2z_0\sqrt{2L^2 + 4z_0^2}} \approx \frac{L^2}{2z_0(2|z_0|)} = \frac{L^2}{4z_0^2}, \quad z_0 > 0.$$

Now we know that $\tan^{-1}(x) \approx x$ to first order in x. Thus,

$$\Omega \approx 4\left(\frac{L^2}{4z_0^2}\right) = \frac{L^2}{z_0^2}.$$

■

9.22. This problem treats the first stage of the journey of a spacecraft nonrelativistically. Assume perfect reflection of a beam of photons from the light sail.

(a) What is the momentum transfer to the light sail by a beam of photons carrying energy ΔE? Show that the force exerted on it is $2P/c$, where P is the fraction of laser power hitting the light sail. What is the acceleration if the light sail has a mass M?

(b) For the first leg of the trip, show that the acceleration is uniform. Find the value of this acceleration for a nanocraft with a mass of 10 grams driven by a 100 GW ground laser.

Solution:

(a) The momentum carried by the beam of photons is $\Delta E/c$. The momentum transfer is $\Delta p = 2\Delta E/c$, assuming perfect reflection. The force is

$$F = \frac{\Delta p}{\Delta t} = \frac{2\Delta E}{c\Delta t} = \frac{2P}{c} \iff a = \frac{F}{M} = \frac{2P}{Mc}.$$

(b) For the first stage, $P = P_{\text{laser}}$. Therefore the acceleration is constant. For a nanocraft with a mass of 10 grams driven by a 100 GW ground laser, the acceleration is

$$a = \frac{2P_{\text{laser}}}{Mc} = \frac{2 \times 10^{11}}{0.01 \times 3 \times 10^{8}} = 6.7 \times 10^{4}\ \text{m/s}^2.$$

■

9.23. This problem treats the second stage of the journey of a spacecraft nonrelativistically. Assume perfect reflection of a beam of photons from the light sail.

(a) For the second leg of the trip, show that the acceleration is

$$a = \frac{2P_{\text{laser}}}{Mc}\frac{x_0^2}{x^2}.$$

(b) Write dv/dt as vdv/dx and integrate the equation in (a) to get

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \frac{2P_{\text{laser}}}{Mc}x_0\left(1 - \frac{x_0}{x}\right)$$

assuming that the speed is v_0 at x_0.

(c) Show that the result in (b) can also be written as

$$v = v_0\sqrt{2 - \frac{x_0}{x}},$$

giving $v_\infty = \sqrt{2}\,v_0$. Hint: Find x_0 in terms of v_0 from Problem 9.22.

(d) Integrate (c) to find t as a function of x with the constant of integration determined by the condition that $x = x_0$ at $t = 0$.

Solution: The difference between this and the previous problem is the fraction of laser power delivered. In this problem, $P = P_{\text{laser}}x_0^2/x^2$.

(a) Therefore, the acceleration is

$$a = \frac{2P_{\text{laser}}}{Mc}\frac{x_0^2}{x^2}.$$

(b) Write the equation in (a) as

$$\frac{dv}{dt} = v\frac{dv}{dx} = \frac{2P_{\text{laser}}}{Mc}\frac{x_0^2}{x^2},$$

or

$$vdv = \frac{2x_0^2P_{\text{laser}}}{Mc}\frac{dx}{x^2}.$$

Integrate this to get

$$\int_{v_0}^{v} u du = \frac{2x_0^2 P_{\text{laser}}}{Mc} \int_{x_0}^{x} \frac{du}{u^2},$$

or

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \frac{2x_0^2 P_{\text{laser}}}{Mc}\left(\frac{1}{x_0} - \frac{1}{x}\right) = \frac{2x_0 P_{\text{laser}}}{Mc}\left(1 - \frac{x_0}{x}\right).$$

(c) The speed v_0 is also the speed at the end of the first stage. Thus, we can write

$$v_0^2 = 2ax_0 \iff \tfrac{1}{2}v_0^2 = ax_0 = \frac{2P_{\text{laser}}}{Mc}x_0.$$

Therefore, the last equation in (b) becomes

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \tfrac{1}{2}v_0^2\left(1 - \frac{x_0}{x}\right)$$

or

$$v^2 = v_0^2\left(2 - \frac{x_0}{x}\right) \iff v = v_0\sqrt{2 - \frac{x_0}{x}}.$$

(d) We have

$$\frac{dx}{dt} = v_0\sqrt{2 - \frac{x_0}{x}} \iff \frac{dx}{\sqrt{2 - \frac{x_0}{x}}} = v_0 dt$$

or

$$\sqrt{\frac{x}{2x - x_0}}\, dx = v_0 dt.$$

Integrating this, we obtain

$$\int_{x_0}^{x} \sqrt{\frac{u}{2u - x_0}}\, du = v_0 \int_0^t du = v_0 t.$$

The integration on the left gives the following complicated and uninteresting final result:

$$\frac{\sqrt{x(2x - x_0)} - x_0}{2} + \frac{\sqrt{2}\,x_0}{4} \ln\left|\frac{2\sqrt{x} + \sqrt{4x - 2x_0}}{(2 + \sqrt{2})\sqrt{x_0}}\right| = v_0 t.$$

Note that if $x = x_0$, the left-hand side gives zero, as it should.

■

CHAPTER 10

A Painless Introduction to Tensors

Problems With Solutions

10.1. Using indices, show that the divergence of the curl of any 3-vector is zero. Similarly, show that the curl of the gradient of any function is zero.

Solution:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = \partial_i(\boldsymbol{\nabla}\times\mathbf{A})_i = \partial_i\epsilon_{ijk}\partial_j A_k = \epsilon_{ijk}\partial_i\partial_j A_k = 0,$$

by Note 10.1.6 and the fact that $\partial_i\partial_j = \partial_j\partial_i$.

$$(\boldsymbol{\nabla}\times\boldsymbol{\nabla}f)_i = \epsilon_{ijk}\partial_j(\partial_k f) = \epsilon_{ijk}\partial_j\partial_k f = 0,$$

again by Note 10.1.6 and the fact that $\partial_j\partial_k = \partial_k\partial_j$. ■

10.2. Using the elementary determinant way of calculating cross products, show that

$$(\mathbf{A}\times\mathbf{C})\cdot(\mathbf{B}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}).$$

Now use this vector identity and Note 10.1.10 to prove Equation (10.8).

Solution:

$$\mathbf{A}\times\mathbf{C} = \det\begin{pmatrix}\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z\\ A_x & A_y & A_z\\ C_x & C_y & C_z\end{pmatrix} = (A_yC_z - A_zC_y)\hat{\mathbf{e}}_x - (A_xC_z - A_zC_x)\hat{\mathbf{e}}_y + (A_xC_y - A_yC_x)\hat{\mathbf{e}}_z.$$

Similarly,

$$\mathbf{B}\times\mathbf{D} = (B_yD_z - B_zD_y)\hat{\mathbf{e}}_x - (B_xD_z - B_zD_x)\hat{\mathbf{e}}_y + (B_xD_y - B_yD_x)\hat{\mathbf{e}}_z.$$

Therefore,

$$\begin{aligned}(\mathbf{A}\times\mathbf{C})\cdot(\mathbf{B}\times\mathbf{D}) &= (A_yC_z - A_zC_y)(B_yD_z - B_zD_y) + (A_xC_z - A_zC_x)(B_xD_z - B_zD_x)\\ &\quad + (A_xC_y - A_yC_x)(B_xD_y - B_yD_x)\\ &= A_yB_y(C_zD_z + C_xD_x) + A_zB_z(C_yD_y + C_xD_x) + A_xB_x(C_yD_y + C_zD_z)\\ &\quad - A_yD_y(B_zC_z + B_xC_x) - A_zD_z(B_yC_y + B_xC_x) - A_xD_x(B_zC_z + B_yC_y).\end{aligned}$$

Note that I can write the first three terms as

$$A_y B_y \mathbf{C}\cdot\mathbf{D} - A_y B_y C_y D_y + A_z B_z \mathbf{C}\cdot\mathbf{D} - A_z B_z C_z D_z + A_x B_x \mathbf{C}\cdot\mathbf{D} - A_x B_x C_x D_x,$$

or as

$$(\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\cdot\mathbf{D}) - A_y B_y C_y D_y - A_z B_z C_z D_z - A_x B_x C_x D_x, \tag{10.1}$$

and the last three terms as

$$-A_y D_y \mathbf{B}\cdot\mathbf{C} + A_y D_y B_y C_y - A_z D_z \mathbf{B}\cdot\mathbf{C} + A_z D_z B_z C_z - A_x D_x \mathbf{B}\cdot\mathbf{C} + A_x D_x B_x C_x$$

or as

$$-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) + A_y D_y B_y C_y + A_z D_z B_z C_z + A_x D_x B_x C_x.$$

Combining this with (10.1) above gives the final result.

To prove Equation (10.8), multiply it on both sides by $A_j C_k B_m D_n$. On the left, you get

$$\epsilon_{ijk}\epsilon_{imn} A_j C_k B_m D_n = (\epsilon_{ijk} A_j C_k)(\epsilon_{imn} B_m D_n) = (\mathbf{A}\times\mathbf{C})_i (\mathbf{B}\times\mathbf{D})_i = (\mathbf{A}\times\mathbf{C})\cdot(\mathbf{B}\times\mathbf{D})$$

and on the right you get

$$\begin{aligned}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) A_j C_k B_m D_n &= \delta_{jm} A_j B_m \delta_{kn} C_k D_n - \delta_{jn} A_j D_n \delta_{km} C_k B_m \\ &= A_j B_j C_k D_k - A_j D_j C_k B_k = (\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}).\end{aligned}$$

Since Equation (10.8) leads to a true vector identity, it holds. ■

10.3. Express $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ in index form. Then using it show the cyclic property of this triple product:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}).$$

Solution: Just use the properties of the Levi-Civita symbol, for example, $\epsilon_{ijk} = \epsilon_{kij}$.

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= A_i(\mathbf{B}\times\mathbf{C})_i = A_i \epsilon_{ijk} B_j C_k = \epsilon_{ijk} A_i B_j C_k \\ &= \epsilon_{kij} C_k A_i B_j = \epsilon_{lmn} C_l A_m B_n = C_l (\mathbf{A}\times\mathbf{B})_l = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}).\end{aligned}$$

In the last line I changed the dummy indices from ijk to lmn. I leave the remaining cyclic property for you. ■

10.4. Using indices, prove the following vector identities:

$$\begin{aligned}\boldsymbol{\nabla}\cdot(f\mathbf{A}) &= (\boldsymbol{\nabla} f)\cdot\mathbf{A} + f\boldsymbol{\nabla}\cdot\mathbf{A} \\ \boldsymbol{\nabla}\times(f\mathbf{A}) &= (\boldsymbol{\nabla} f)\times\mathbf{A} + f\boldsymbol{\nabla}\times\mathbf{A} \\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}\end{aligned}$$

Solution:

$$\boldsymbol{\nabla}\cdot(f\mathbf{A}) = \partial_i(f\mathbf{A})_i = \partial_i(fA_i) = (\partial_i f)A_i + f(\partial_i A_i) = (\boldsymbol{\nabla} f)\cdot\mathbf{A} + f\boldsymbol{\nabla}\cdot\mathbf{A}$$

$$\begin{aligned}[\boldsymbol{\nabla}\times(f\mathbf{A})]_i &= \epsilon_{ijk}\partial_j(fA_k) = \epsilon_{ijk}[(\partial_j f)A_k + f\partial_j A_k] \\ &= \epsilon_{ijk}(\partial_j f)A_k + f\epsilon_{ijk}\partial_j A_k = [(\boldsymbol{\nabla} f)\times\mathbf{A}]_i + f(\boldsymbol{\nabla}\times\mathbf{A})_i\end{aligned}$$

$$\begin{aligned}[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A})]_i &= \epsilon_{ijk}\partial_j(\boldsymbol{\nabla} \times \mathbf{A})_k = \epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_l A_m) = \epsilon_{kij}\epsilon_{klm}\partial_j\partial_l A_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l A_m = \delta_{il}\delta_{jm}\partial_j\partial_l A_m - \delta_{im}\delta_{jl}\partial_j\partial_l A_m \\ &= \partial_m\partial_i A_m - \partial_j\partial_j A_i = \partial_i(\partial_m A_m) - (\partial_j\partial_j)A_i \\ &= \partial_i(\boldsymbol{\nabla} \cdot \mathbf{A}) - (\nabla^2)A_i = [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}]_i - (\nabla^2\mathbf{A})_i.\end{aligned}$$

■

10.5. Show that the determinant of a 3×3 matrix A with elements a_{ij} is given by

$$\det \mathsf{A} = \epsilon_{ijk}a_{1i}a_{2j}a_{3k}.$$

Solution: Let

$$\mathsf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Remember the summation convention:

$$\epsilon_{ijk}a_{1i}a_{2j}a_{3k} = \epsilon_{1jk}a_{11}a_{2j}a_{3k} + \epsilon_{2jk}a_{12}a_{2j}a_{3k} + \epsilon_{3jk}a_{13}a_{2j}a_{3k}.$$

Now I'll calculate each term separately by summing over the remaining indices, using the properties of the Levi-Civita symbol. The first term is

$$\begin{aligned}\epsilon_{1jk}a_{11}a_{2j}a_{3k} &= \epsilon_{12k}a_{11}a_{22}a_{3k} + \epsilon_{13k}a_{11}a_{23}a_{3k} \\ &= \epsilon_{123}a_{11}a_{22}a_{33} + \epsilon_{132}a_{11}a_{23}a_{32} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32});\end{aligned}$$

the second term is

$$\begin{aligned}\epsilon_{2jk}a_{12}a_{2j}a_{3k} &= \epsilon_{21k}a_{12}a_{21}a_{3k} + \epsilon_{23k}a_{12}a_{23}a_{3k} \\ &= \epsilon_{213}a_{12}a_{21}a_{33} + \epsilon_{231}a_{12}a_{23}a_{31} \\ &= -a_{12}(a_{21}a_{33} - a_{23}a_{31});\end{aligned}$$

and the last term is

$$\begin{aligned}\epsilon_{3jk}a_{13}a_{2j}a_{3k} &= \epsilon_{31k}a_{13}a_{21}a_{3k} + \epsilon_{32k}a_{13}a_{22}a_{3k} \\ &= \epsilon_{312}a_{13}a_{21}a_{32} + \epsilon_{321}a_{13}a_{22}a_{31} \\ &= a_{13}(a_{21}a_{32} - a_{22}a_{31}).\end{aligned}$$

When you add all the terms, you get the familiar expansion of the determinant according to its first row. ■

10.6. Let $\hat{e}_i$, $i = 1, 2, 3$ be the components of a unit vector. Let

$$a_{ijk} = \epsilon_{ijm}\hat{e}_k\hat{e}_m + \epsilon_{imk}\hat{e}_j\hat{e}_m + \epsilon_{mjk}\hat{e}_i\hat{e}_m.$$

Show that $a_{123} = 1$, $a_{jik} = -a_{ijk}$, and $a_{ikj} = -a_{ijk}$. Therefore, $a_{ijk} = \epsilon_{ijk}$.

Solution: Without loss of generality, choose your axes so that the unit vector lies along the x-axis. Then $\hat{e}_i = \delta_{i1}$ and

$$\begin{aligned} a_{ijk} &= \epsilon_{ijm}\delta_{k1}\delta_{m1} + \epsilon_{imk}\delta_{j1}\delta_{m1} + \epsilon_{mjk}\delta_{i1}\delta_{m1} \\ &= \epsilon_{ij1}\delta_{k1} + \epsilon_{i1k}\delta_{j1} + \epsilon_{1jk}\delta_{i1}. \end{aligned}$$

This implies that

$$a_{123} = \epsilon_{121}\delta_{k1} + \epsilon_{11k}\delta_{21} + \epsilon_{123}\delta_{11} = \epsilon_{123}$$

and

$$\begin{aligned} a_{jik} &= \epsilon_{ji1}\delta_{k1} + \epsilon_{j1k}\delta_{i1} + \epsilon_{1ik}\delta_{j1} \\ &= -\epsilon_{ij1}\delta_{k1} - \epsilon_{1jk}\delta_{i1} - \epsilon_{i1k}\delta_{j1} = -a_{ijk}, \end{aligned}$$

and

$$\begin{aligned} a_{ikj} &= \epsilon_{ik1}\delta_{j1} + \epsilon_{i1j}\delta_{k1} + \epsilon_{1kj}\delta_{i1} \\ &= -\epsilon_{i1k}\delta_{j1} - \epsilon_{ij1}\delta_{k1} - \epsilon_{1jk}\delta_{i1} = -a_{ijk}. \end{aligned}$$

■

10.7. By substituting the components of $d\vec{r}_1$, $d\vec{r}_2$, and $d\vec{r}_3$ parallel and perpendicular to $\vec{\beta}$ in $dV = d\vec{r}_1 \cdot (d\vec{r}_2 \times d\vec{r}_3)$,

(a) show that

$$\begin{aligned} dV = &|(d\vec{r}_1)_{||}||(d\vec{r}_2)_\perp \times (d\vec{r}_3)_\perp| + |(d\vec{r}_2)_{||}||(d\vec{r}_3)_\perp \times (d\vec{r}_1)_\perp| \\ &+ |(d\vec{r}_3)_{||}||(d\vec{r}_1)_\perp \times (d\vec{r}_2)_\perp|. \end{aligned} \tag{10.2}$$

This shows that the volume element is the product of an area perpendicular to the direction of motions [terms like $|(d\vec{r}_2)_\perp \times (d\vec{r}_3)_\perp|$] times a height in the direction of motion [terms like $|(d\vec{r}_1)_{||}|$].

(b) From (6.17) and $dt' = 0$, show that $|(d\vec{r}\,'_1)_{||}| = |(d\vec{r}_1)_{||}|/\gamma$, with similar relations for $|(d\vec{r}\,'_2)_{||}|$ and $|(d\vec{r}\,'_3)_{||}|$.

(c) Insert these results in the expression for dV' as given by (10.2) and derive the formula $dV' = dV/\gamma$.

Solution:

(a) First calculate the cross product

$$\begin{aligned} d\vec{r}_2 \times d\vec{r}_3 &= [(d\vec{r}_2)_{||} + (d\vec{r}_2)_\perp] \times [(d\vec{r}_3)_{||} + (d\vec{r}_3)_\perp] \\ &= (d\vec{r}_2)_{||} \times (d\vec{r}_3)_\perp + (d\vec{r}_2)_\perp \times (d\vec{r}_3)_{||} + (d\vec{r}_2)_\perp \times (d\vec{r}_3)_\perp. \end{aligned}$$

As you dot $d\vec{r}_1 = (d\vec{r}_1)_{||} + (d\vec{r}_1)_\perp$ with the expression above, note that the dot product of $(d\vec{r}_1)_{||}$ with any cross product containing parallel components gives zero. Also note that $(d\vec{r}_2)_\perp \times (d\vec{r}_3)_\perp$ is parallel to the direction of motion, so it gives zero when dotted with $(d\vec{r}_1)_\perp$. With these remarks in mind, we get

$$\begin{aligned} dV &= (d\vec{r}_1)_{||} \cdot [(d\vec{r}_2)_\perp \times (d\vec{r}_3)_\perp] + (d\vec{r}_1)_\perp \cdot [(d\vec{r}_2)_{||} \times (d\vec{r}_3)_\perp + (d\vec{r}_2)_\perp \times (d\vec{r}_3)_{||}] \\ &= (d\vec{r}_1)_{||} \cdot [(d\vec{r}_2)_\perp \times (d\vec{r}_3)_\perp] + (d\vec{r}_2)_{||} \cdot [(d\vec{r}_3)_\perp \times (d\vec{r}_1)_\perp] \\ &\quad + (d\vec{r}_3)_{||} \cdot [(d\vec{r}_1)_\perp \times (d\vec{r}_2)_\perp], \end{aligned}$$

where in the last step, I used the cyclic property of the mixed dot and cross product. Finally note that the participants in the dot product are in the same direction, so you can replace the dot product with the product of absolute values.

(b) With $dt' = 0$, the first equation of (6.17) gives $dt = -\left|\vec{\beta}\right| \left|\vec{r}_{||}\right|$, and the second equation yields

$$d\vec{r}_{||}' = \gamma \left[d\vec{r}_{||} - \vec{\beta}\left|\vec{\beta}\right| \left|d\vec{r}_{||}\right| \right] = \gamma \left(d\vec{r}_{||} - \left|\vec{\beta}\right|^2 d\vec{r}_{||} \right) = \frac{d\vec{r}_{||}}{\gamma}.$$

(c) When you prime (10.2), the cross products don't change [see the third equation of (6.17)], and by (b), the parallel components get divided by γ. Therefore, the entire expression is divided by γ.

■

CHAPTER 11

Relativistic Electrodynamics

Problems With Solutions

11.1. From $B_i = \epsilon_{ijk}\partial_j A_k$ show that $\partial_j A_k - \partial_k A_j = \epsilon_{jkm} B_m$.

Solution: Multiply both sides of $B_i = \epsilon_{ijk}\partial_j A_k$ by ϵ_{lmi} to obtain

$$\begin{aligned}\epsilon_{lmi}B_i &= \epsilon_{lmi}\epsilon_{ijk}\partial_j A_k = \epsilon_{ilm}\epsilon_{ijk}\partial_j A_k = (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj})\partial_j A_k \\ &= \delta_{lj}\delta_{mk}\partial_j A_k - \delta_{lk}\delta_{mj}\partial_j A_k = \partial_l A_m - \partial_m A_l.\end{aligned}$$

■

11.2. Go through the details of the manipulation of the last line of (11.10) and derive (11.11).

Solution: Multiply the two square brackets and note that the product of the two β terms will contain symmetric indices which give zero when multiplied by the Levi-Civita symbol.

$$\begin{aligned}\mathcal{W} &= \tfrac{1}{2}\epsilon_{ijk}\epsilon_{lmn}[\delta_{km} + (\gamma-1)\hat{\beta}_k\hat{\beta}_m][\delta_{jl} + (\gamma-1)\hat{\beta}_j\hat{\beta}_l]B_n \\ &= \tfrac{1}{2}\epsilon_{ijk}\epsilon_{lmn}\delta_{km}[\delta_{jl} + (\gamma-1)\hat{\beta}_j\hat{\beta}_l]B_n + \tfrac{1}{2}(\gamma-1)\epsilon_{ijk}\epsilon_{lmn}\delta_{jl}\hat{\beta}_k\hat{\beta}_m B_n \\ &= \tfrac{1}{2}\epsilon_{ijk}\epsilon_{lkn}[\delta_{jl} + (\gamma-1)\hat{\beta}_j\hat{\beta}_l]B_n + \tfrac{1}{2}(\gamma-1)\epsilon_{ijk}\epsilon_{jmn}\hat{\beta}_k\hat{\beta}_m B_n.\end{aligned}$$

Now use Note 10.1.7 to rewrite the products of the Levi-Civita symbols. You have to rearrange indices to bring them to the same order as they appear in the Note. Once you do that, you'll end up with the following expression:

$$\begin{aligned}\mathcal{W} &= -\tfrac{1}{2}(\delta_{il}\delta_{jn} - \delta_{in}\delta_{jl})\delta_{jl}B_n - \tfrac{1}{2}(\gamma-1)(\delta_{il}\delta_{jn} - \delta_{in}\delta_{jl})\hat{\beta}_j\hat{\beta}_l B_n \\ &\quad - \tfrac{1}{2}(\gamma-1)(\delta_{im}\delta_{kn} - \delta_{in}\delta_{km})\hat{\beta}_k\hat{\beta}_m B_n \\ &= -\tfrac{1}{2}(\delta_{ij}\delta_{jn} - \delta_{in}\delta_{jj})B_n - \tfrac{1}{2}(\gamma-1)(\hat{\beta}_j\hat{\beta}_i B_j - \hat{\beta}_j\hat{\beta}_j B_i) \\ &\quad - \tfrac{1}{2}(\gamma-1)(\hat{\beta}_k\hat{\beta}_i B_k - \hat{\beta}_k\hat{\beta}_k B_i).\end{aligned}$$

Now note that $\delta_{jj} = 3$ and $\hat{\beta}_k\hat{\beta}_k = \hat{\beta}\cdot\hat{\beta} = 1$. Then

$$\mathcal{W} = B_i - (\gamma-1)(\hat{\beta}_i\hat{\beta}\cdot\vec{B} - B_i) = \gamma B_i - (\gamma-1)\hat{\beta}_i(\hat{\beta}\cdot\vec{B}).$$

■

11.3. Show that the sum of the first two terms of (11.12) gives $\gamma\epsilon_{ijk}\beta_j E_k$.

Solution: This involves simply renaming the dummy indices. I'll work on the first term:

$$\gamma\epsilon_{imk}\beta_k\partial_m A_0 = \gamma\epsilon_{i\clubsuit\spadesuit}\beta_\spadesuit\partial_\clubsuit A_0 = -\gamma\epsilon_{i\spadesuit\clubsuit}\beta_\spadesuit\partial_\clubsuit A_0 = -\gamma\epsilon_{ijk}\beta_j\partial_k A_0.$$

Now I add it to the second term to get

$$\gamma\epsilon_{ijk}\beta_j(-\partial_k A_0 + \partial_0 A_k) = \gamma\epsilon_{ijk}\beta_j E_k.$$

■

11.4. Derive Equation (11.14) from (11.8) and (11.13).

Solution: Since (11.8) and (11.13) are identical (except for the change in sign in the first term), I'll do (11.8). Inserting the parallel and perpendicular components on both sides of the equation, I get

$$\vec{E}_{||}' + \vec{E}_\perp' = \gamma[\vec{E}_{||} + \vec{E}_\perp - \vec{\beta}\times(\vec{B}_{||}+\vec{B}_\perp)] - (\gamma-1)\hat{\beta}[\hat{\beta}\cdot(\vec{E}_{||}+\vec{E}_\perp).$$

Now note that

$$\vec{\beta}\times\vec{B}_{||} = 0 = \hat{\beta}\cdot\vec{E}_\perp, \quad \text{and} \quad \hat{\beta}\hat{\beta}\cdot\vec{E}_{||} = \vec{E}_{||}.$$

Hence,

$$\begin{aligned}\vec{E}_{||}' + \vec{E}_\perp' &= \gamma\vec{E}_{||} + \gamma\vec{E}_\perp - \gamma\vec{\beta}\times\vec{B}_\perp - (\gamma-1)\vec{E}_{||}\\ &= \vec{E}_{||} + \gamma(\vec{E}_\perp - \vec{\beta}\times\vec{B}_\perp).\end{aligned}$$

The first term on the right-hand side is parallel to the direction of motion, so it must be equal to $\vec{E}_{||}'$ and the second term is perpendicular to the direction of motion, so it must be equal to $\vec{E}_\perp'$. ■

11.5. Derive Equation (11.18).

Solution: Dotting both sides of (11.17) by $\hat{\beta}$ and rewriting the resulting equation slightly differently yields

$$\begin{aligned}\hat{\beta}\cdot\vec{r} &= \hat{\beta}\cdot\vec{r}' - \gamma\beta\left(t' - \frac{\gamma\beta}{\gamma+1}\hat{\beta}\cdot\vec{r}'\right)\\ &= -\gamma\beta t' + \left(1 + \frac{\gamma^2\beta^2}{\gamma+1}\right)\hat{\beta}\cdot\vec{r} = -\gamma\beta t' + \gamma\hat{\beta}\cdot\vec{r},\end{aligned}$$

because $\gamma^2\beta^2 = \gamma^2 - 1 = (\gamma-1)(\gamma+1)$. ■

11.6. Derive Equation (11.19).

Solution: The equation before (11.19) can be expressed as

$$\vec{S} = \gamma\vec{r}' - \gamma\vec{\beta}t' + \gamma\underbrace{\left(\frac{\gamma^2\beta^2}{\gamma+1} - \gamma + 1\right)}_{=0}\hat{\beta}\hat{\beta}\cdot\vec{r}'.$$

The fact that the expression in parentheses is zero can be readily shown. ■

11.7. Derive Equation (11.21). Now take the dot product of the equation with itself to derive (11.22).

Solution: Rewrite (11.17) as

$$\vec{r} = \vec{r}\,' - \gamma\vec{\beta}t' + \frac{\gamma^2\beta^2}{\gamma+1}\hat{\beta}\hat{\beta}\cdot\vec{r}\,' = \vec{r}\,' - \gamma\vec{\beta}t' + (\gamma-1)\hat{\beta}\hat{\beta}\cdot\vec{r}\,'.$$

Now use (11.19) and substitute for $\vec{r}\,'$:

$$\begin{aligned}
\vec{r} &= \vec{r}\,'_{\text{inst}} + \vec{\beta}t' - \gamma\vec{\beta}t' + (\gamma-1)\hat{\beta}\hat{\beta}\cdot(\vec{r}\,'_{\text{inst}} + \vec{\beta}t') \\
&= \vec{r}\,'_{\text{inst}} - (\gamma-1)\vec{\beta}t' + (\gamma-1)\hat{\beta}\hat{\beta}\cdot\vec{r}\,'_{\text{inst}} + (\gamma-1)\underbrace{\hat{\beta}\hat{\beta}\cdot\vec{\beta}}_{=\vec{\beta}}t' \\
&= \vec{r}\,'_{\text{inst}} + (\gamma-1)\hat{\beta}\hat{\beta}\cdot\vec{r}\,'_{\text{inst}}.
\end{aligned}$$

Now square (dot with itself) both sides

$$\begin{aligned}
r^2 &= r'^2_{\text{inst}} + (\gamma-1)^2(\hat{\beta}\cdot\vec{r}\,'_{\text{inst}})^2 + 2(\gamma-1)(\hat{\beta}\cdot\vec{r}\,'_{\text{inst}})^2 \\
&= r'^2_{\text{inst}} + \underbrace{(\gamma-1)(\gamma+1)}_{=\gamma^2-1=\gamma^2\beta^2}(\hat{\beta}\cdot\vec{r}\,'_{\text{inst}})^2 \\
&= r'^2_{\text{inst}} + \gamma^2(\beta\hat{\beta}\cdot\vec{r}\,'_{\text{inst}})^2 = r'^2_{\text{inst}} + \gamma^2(\vec{\beta}\cdot\vec{r}\,'_{\text{inst}})^2.
\end{aligned}$$

■

11.8. Derive Equation (11.25).

Solution: From the equation before (11.25), plus $\vec{\beta}\times\vec{r} = \vec{\beta}\times\vec{r}\,'_{\text{inst}}$ and (11.22), we get

$$\vec{B}' = \frac{k_e q\gamma}{\left[r'^2_{\text{inst}} + \gamma^2(\vec{\beta}\cdot\vec{r}\,'_{\text{inst}})^2\right]^{3/2}}\vec{\beta}\times\vec{r}\,'_{\text{inst}},$$

with

$$\begin{aligned}
r'^2_{\text{inst}} + \gamma^2(\vec{\beta}\cdot\vec{r}\,'_{\text{inst}})^2 &= r'^2_{\text{inst}} + \gamma^2\beta^2 r'^2_{\text{inst}}\cos^2\theta_{\text{inst}} = r'^2_{\text{inst}} + (\gamma^2-1)r'^2_{\text{inst}}\cos^2\theta_{\text{inst}} \\
&= r'^2_{\text{inst}}\sin^2\theta_{\text{inst}} + \gamma^2 r'^2_{\text{inst}}\cos^2\theta_{\text{inst}} = r'^2_{\text{inst}}(\sin^2\theta_{\text{inst}} + \gamma^2\cos^2\theta_{\text{inst}}).
\end{aligned}$$

■

11.9. Consider an infinite plate charged uniformly with surface density σ. The plate is moving uniformly with speed β perpendicular to its surface. There are two ways to calculate the electric and magnetic fields of this charge distribution.

(a) Calculate the fields $\vec{E}$ and $\vec{B}$ in the rest frame of the plate using the elementary method of Gauss's law. Now transform those fields to the frame in which the plate is moving.

(b) Use (11.24) and (11.25) to write the contribution from each element of charge on the surface and integrate over the plate to get the fields.

Solution: I'll assume that the plate is infinitely thin. This precludes a conductor because a conductor is really *two* infinite sheets with the same surface charge density (that's how the field inside becomes zero). Let O be the rest frame of the plate and place it in the yz-plane of O. Consider the side facing the positive x direction.

(a) From Gauss's law $\vec{E} = 2\pi k_e \sigma \hat{\beta}$ and $\vec{B} = 0$ in O. Then (11.14) gives $\vec{E}' = 2\pi k_e \sigma \hat{\beta}$ and $\vec{B}' = 0$ in O' as well.

(b) I'll calculate the field at the moment that the two origins coincide, so that the plate is in the $y'z'$-plane of O'. Let $P_0 = (x_0', y_0', z_0')$ be the field point, and $P_{\text{inst}} = (0, y', z')$ the instantaneous location of the charge element. Then

$$\vec{r}'_{\text{inst}} = \langle x_0', y_0', z_0' \rangle - \langle 0, y', z' \rangle = \langle x_0', y_0' - y', z_0' - z' \rangle$$

and (11.22) becomes

$$\begin{aligned} r^2 &= x_0'^2 + (y_0' - y')^2 + (z_0' - z')^2 + \gamma^2 \beta^2 x_0'^2 \\ &= \gamma^2 x_0'^2 + (y_0' - y')^2 + (z_0' - z')^2 \end{aligned}$$

using the indispensable identity $\gamma^2 \beta^2 = \gamma^2 - 1$. Plugging this in (11.23) and integrating, we get the electric field at P_0:

$$\vec{E}' = k_e \sigma \gamma \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\langle x_0', y_0' - y', z_0' - z' \rangle}{[\gamma^2 x_0'^2 + (y_0' - y')^2 + (z_0' - z')^2]^{3/2}} dy' dz'.$$

Integration over y' gives $E_y' = 0$ and integration over z' gives $E_z' = 0$. So, the only surviving component is E_x', and that is given by

$$E_x' = k_e \sigma \gamma x_0' \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dy' dz'}{[\gamma^2 x_0'^2 + (y_0' - y')^2 + (z_0' - z')^2]^{3/2}}}_{=2\pi/(\gamma x_0')} = 2\pi k_e \sigma$$

or $\vec{E}' = 2\pi k_e \sigma \hat{\beta}$. Since $\vec{B}' = \vec{\beta} \times \vec{E}'$, we get $\vec{B}' = 0$, in agreement with (a).

■

11.10. Verify that the elements of the matrix (11.27) are indeed the electric and magnetic fields as given there.

Solution: Compare (11.5) with $F_{0i} \equiv \partial_0 A_i - \partial_i A_0$ and conclude that $F_{0i} = -E_i$. This verifies the first row and the first column. Furthermore, from (11.2) we get

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = B_3, \quad F_{31} = \partial_3 A_1 - \partial_1 A_3 = B_2, \quad F_{23} = \partial_2 A_3 - \partial_3 A_2 = B_1.$$

These verify the rest of the matrix elements. Note how the three indices in $F_{ij} = B_k$ are cyclically permuted. ■

11.11. Let F^{sup} and F_{sub} denote the matrices of $F^{\mu\nu}$ and $F_{\mu\nu}$, respectively. Show that the matrix form of Equation (11.28) is $\mathsf{F}^{\text{sup}} = \eta \mathsf{F}_{\text{sub}} \eta$. Now carry out the multiplication of the three matrices to come up with Equation (11.29).

Solution: Look at a general element of F^{sup}, and since the matrix is antisymmetric, assume that $\mu < \nu$:

$$
\begin{aligned}
F^{\mu\nu} &= \eta^{\mu\alpha}F_{\alpha\beta}\eta^{\beta\nu} = \eta^{\mu 0}F_{0\beta}\eta^{\beta\nu} + \eta^{\mu i}F_{i\beta}\eta^{\beta\nu} \\
&= \eta^{\mu 0}F_{0i}\eta^{i\nu} + \eta^{\mu i}F_{i\beta}\eta^{\beta\nu} = \eta^{\mu 0}F_{0i}\eta^{i\nu} + \eta^{\mu i}F_{ij}\eta^{j\nu}, \qquad i < j.
\end{aligned}
$$

This shows that $F^{0i} = -F_{0i}$ because $\eta^{00} = 1 = -\eta^{ii}$ (no sum over i) and that $F^{ij} = F_{ij}$ because $\eta^{ii}\eta^{jj} = 1$ (no sum over i or j). ■

11.12. Verify Equation (11.30) by going through each step of its derivation.

Solution: There's really nothing to verify because all the steps are in the equation. The only thing you need to know is that $\widetilde{\mathsf{F}} = -\mathsf{F}$. ■

11.13. Noting that F' is given by the same matrix as (11.29) except that its elements carry a prime, use Equation (11.31) to derive the transformation rules of (11.8) and (11.13) for electric and magnetic fields.

Solution: It is convenient to write the matrices in block form. $\mathsf{\Lambda}$ is given by (6.14). With obvious notation, we can also define the block form of F:

$$
\mathsf{\Lambda} = \begin{pmatrix} \gamma & \gamma\widetilde{\vec{\beta}} \\ \gamma\vec{\beta} & \overleftrightarrow{\Lambda} \end{pmatrix}, \quad \mathsf{F} = \begin{pmatrix} 0 & \widetilde{\vec{E}} \\ -\vec{E} & \mathsf{B} \end{pmatrix}.
$$

Now calculate the product of the three matrices in (11.31). The first two give

$$
\mathsf{\Lambda F} = \begin{pmatrix} \gamma & \gamma\widetilde{\vec{\beta}} \\ \gamma\vec{\beta} & \overleftrightarrow{\Lambda} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{\vec{E}} \\ -\vec{E} & \mathsf{B} \end{pmatrix} = \begin{pmatrix} -\gamma\vec{\beta}\cdot\vec{E} & \gamma(\widetilde{\vec{E}} + \widetilde{\vec{\beta}}\mathsf{B}) \\ -\overleftrightarrow{\Lambda}\vec{E} & \gamma\vec{\beta}\widetilde{\vec{E}} + \overleftrightarrow{\Lambda}\mathsf{B} \end{pmatrix}
$$

and multiplying this by $\mathsf{\Lambda}$ on the right gives

$$
\mathsf{F}' = \begin{pmatrix} -\gamma\vec{\beta}\cdot\vec{E} & \gamma(\widetilde{\vec{E}} + \widetilde{\vec{\beta}}\mathsf{B}) \\ -\overleftrightarrow{\Lambda}\vec{E} & \gamma\vec{\beta}\widetilde{\vec{E}} + \overleftrightarrow{\Lambda}\mathsf{B} \end{pmatrix} \begin{pmatrix} \gamma & \gamma\widetilde{\vec{\beta}} \\ \gamma\vec{\beta} & \overleftrightarrow{\Lambda} \end{pmatrix}.
$$

Rather than multiplying out matrices, I'll look at the elements $\mathsf{F}'_{(i,j)}$ (with $i, j = 1, 2$) of the block matrix that I need. Although I don't need $\mathsf{F}'_{(1,1)}$, I'll calculate it to make sure I get zero.

$$
\mathsf{F}'_{(1,1)} = -\gamma^2\vec{\beta}\cdot\vec{E} + \gamma^2(\widetilde{\vec{E}} + \widetilde{\vec{\beta}}\mathsf{B})\vec{\beta} = \gamma^2\widetilde{\vec{\beta}}\,\mathsf{B}\,\vec{\beta}.
$$

But, since the matrix B is antisymmetric,

$$
\widetilde{\vec{\beta}}\,\mathsf{B}\,\vec{\beta} = \beta_i(\mathsf{B}\,\vec{\beta})_i = \beta_i\mathsf{B}_{ij}\beta_j = -\beta_i\mathsf{B}_{ji}\beta_j = -\beta_\clubsuit\mathsf{B}_{\spadesuit\clubsuit}\beta_\spadesuit = -\beta_\spadesuit\mathsf{B}_{\spadesuit\clubsuit}\beta_\clubsuit = -\widetilde{\vec{\beta}}\,\mathsf{B}\,\vec{\beta},
$$

and $\mathsf{F}'_{(1,1)} = 0$ as expected. Next, I'll calculate $\mathsf{F}'_{(2,1)}$:

$$
\mathsf{F}'_{(2,1)} = -\gamma\overleftrightarrow{\Lambda}\vec{E} + \left(\gamma\vec{\beta}\widetilde{\vec{E}} + \overleftrightarrow{\Lambda}\mathsf{B}\right)\gamma\vec{\beta}.
$$

This is a column vector; so we calculate its ith component:

$$\left(\mathsf{F}'_{(2,1)}\right)_i = -\gamma\Lambda_{ij}E_j + \gamma^2\beta_i\vec{E}\cdot\vec{\beta} + \gamma\Lambda_{ij}\mathsf{B}_{jk}\beta_k.$$

I'll evaluate each term separately [see (B.9) for Λ_{ij}]:

$$\begin{aligned}
\text{term } 1 &= -\gamma\Lambda_{ij}E_j = -\gamma[\delta_{ij} + (\gamma-1)\hat{\beta}_i\hat{\beta}_j]E_j = -\gamma E_i - \gamma(\gamma-1)\hat{\beta}_i\hat{\beta}\cdot\vec{E} \\
\text{term } 2 &= \gamma^2\beta_i\vec{E}\cdot\vec{\beta} = \gamma^2\beta^2\hat{\beta}_i\vec{E}\cdot\hat{\beta} = (\gamma^2-1)\hat{\beta}_i\vec{E}\cdot\hat{\beta} \\
\text{term } 3 &= \gamma\Lambda_{ij}\mathsf{B}_{jk}\beta_k = \gamma[\delta_{ij} + (\gamma-1)\hat{\beta}_i\underbrace{\hat{\beta}_j]\mathsf{B}_{jk}\beta_k}_{=0} \\
&= \gamma\mathsf{B}_{ik}\beta_k = \gamma\epsilon_{ikm}B_m\beta_k = \gamma(\vec{\beta}\times\vec{B})_i.
\end{aligned}$$

Adding the three terms, we obtain

$$\left(\mathsf{F}'_{(2,1)}\right)_i = -\gamma E_i - \gamma(\gamma-1)\hat{\beta}_i\hat{\beta}\cdot\vec{E} + (\gamma^2-1)\hat{\beta}_i\vec{E}\cdot\hat{\beta} + \gamma(\vec{\beta}\times\vec{B})_i$$

or

$$-E'_i = -\gamma(E - \vec{\beta}\times\vec{B})_i + (\gamma-1)\hat{\beta}_i\hat{\beta}\cdot\vec{E},$$

which agrees with (11.8).

To obtain the transformation of the magnetic field, we'll have to look at $\mathsf{F}'_{(2,2)}$.

$$\mathsf{F}'_{(2,2)} = -\gamma\overleftrightarrow{\Lambda}\vec{E}\widetilde{\vec{\beta}} + \left(\gamma\vec{\beta}\widetilde{\vec{E}} + \overleftrightarrow{\Lambda}\mathsf{B}\right)\overleftrightarrow{\Lambda}.$$

This is a 3×3 matrix; so let's find it's ijth element:

$$\left(\mathsf{F}'_{(2,2)}\right)_{ij} = -\gamma\Lambda_{ik}E_k\beta_j + \gamma\beta_iE_k\Lambda_{kj} + \Lambda_{ik}\mathsf{B}_{km}\Lambda_{mj}.$$

Again I'll evaluate each term separately:

$$\begin{aligned}
\text{term } 1 &= -\gamma\Lambda_{ik}E_k\beta_j = -\gamma[\delta_{ik} + (\gamma-1)\hat{\beta}_i\hat{\beta}_k]E_k\beta_j = -\gamma E_i\beta_j - \gamma(\gamma-1)\hat{\beta}\cdot\vec{E}\hat{\beta}_i\beta_j \\
\text{term } 2 &= \gamma\beta_iE_k\Lambda_{kj} = \gamma\beta_iE_k[\delta_{kj} + (\gamma-1)\hat{\beta}_k\hat{\beta}_j] = \gamma\beta_iE_j + \gamma(\gamma-1)\hat{\beta}\cdot\vec{E}\beta_i\hat{\beta}_j \\
\text{term } 3 &= \Lambda_{ik}\mathsf{B}_{km}\Lambda_{mj} = [\delta_{ik} + (\gamma-1)\hat{\beta}_i\hat{\beta}_k]\mathsf{B}_{km}[\delta_{mj} + (\gamma-1)\hat{\beta}_m\hat{\beta}_j] \\
&= [\delta_{ik} + (\gamma-1)\hat{\beta}_i\hat{\beta}_k][\mathsf{B}_{kj} + (\gamma-1)\hat{\beta}_m\hat{\beta}_j\mathsf{B}_{km}] \\
&= \mathsf{B}_{ij} + (\gamma-1)\hat{\beta}_m\hat{\beta}_j\mathsf{B}_{im} + (\gamma-1)\hat{\beta}_i\hat{\beta}_k\mathsf{B}_{kj}.
\end{aligned}$$

In the third term, the product of the two $\gamma - 1$ terms gives zero because $\hat{\beta}_k\hat{\beta}_m\mathsf{B}_{km} = 0$. Adding the three terms and noting that $\mathsf{F}'_{(2,2)} = \mathsf{B}'$, we get

$$\mathsf{B}'_{ij} = \gamma\beta_iE_j - \gamma E_i\beta_j + \mathsf{B}_{ij} + (\gamma-1)\hat{\beta}_m\hat{\beta}_j\mathsf{B}_{im} + (\gamma-1)\hat{\beta}_i\hat{\beta}_k\mathsf{B}_{kj}.$$

The relation between the matrix B and the magnetic field $\vec{B}$ is $B_n = \frac{1}{2}\epsilon_{nij}\mathsf{B}_{ij}$ and $\mathsf{B}_{ij} = \epsilon_{ijn}B_n$. So, multiply both sides of the last equation by $\frac{1}{2}\epsilon_{nij}$ to obtain

$$\begin{aligned}
B'_n &= \gamma\tfrac{1}{2}\epsilon_{nij}(\beta_iE_j - E_i\beta_j) + B_n + (\gamma-1)\tfrac{1}{2}\epsilon_{nij}(\hat{\beta}_m\hat{\beta}_j\epsilon_{imk}B_k + \hat{\beta}_i\hat{\beta}_k\epsilon_{kjl}B_l) \\
&= \gamma(\vec{\beta}\times\vec{E})_n + B_n + \tfrac{1}{2}(\gamma-1)(\underbrace{\epsilon_{nij}\epsilon_{imk}}_{=\delta_{nk}\delta_{jm}-\delta_{nm}\delta_{jk}}\hat{\beta}_m\hat{\beta}_jB_k + \underbrace{\epsilon_{nij}\epsilon_{kjl}}_{=\delta_{nl}\delta_{ik}-\delta_{nk}\delta_{il}}\hat{\beta}_i\hat{\beta}_kB_l) \\
&= \gamma(\vec{\beta}\times\vec{E})_n + B_n + \tfrac{1}{2}(\gamma-1)(B_n - \hat{\beta}_n\hat{\beta}\cdot\vec{B} + B_n - \hat{\beta}_n\hat{\beta}\cdot\vec{B})
\end{aligned}$$

or

$$B'_n = \gamma(\vec{B} + \vec{\beta} \times \vec{E})_n - (\gamma - 1)\hat{\beta}_n(\hat{\beta} \cdot \vec{B}).$$

This agrees with Equation (11.13). ■

11.14. Show that $F^{\mu\nu}F_{\mu\nu} = 2(|\vec{B}|^2 - |\vec{E}|^2)$. Hint: Prove that $F^{\mu\nu}F_{\mu\nu}$ is the negative of the trace (the sum of the diagonal elements) of the product of the two matrices in (11.29) and (11.27). Then find that trace.

Solution: I let you calculate $F^{\mu\nu}F_{\mu\nu}$ via trace. I'll calculate it by index manipulation as more exercise in "index gymnastics."

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= F^{0\nu}F_{0\nu} + F^{i\nu}F_{i\nu} = F^{0i}F_{0i} + F^{i0}F_{i0} + F^{ij}F_{ij} = -F^{0i}F^{0i} - F^{i0}F^{i0} + F^{ij}F^{ij} \\ &= -E_iE_i - (-E_i)(-E_i) + (\epsilon_{ijk}B_k)(\epsilon_{ijm}B_m) = -2|\vec{E}|^2 + (\delta_{jj}\delta_{jm} - \delta_{kj}\delta_{km})B_kB_m \\ &= -2|\vec{E}|^2 + (3\delta_{jm} - \delta_{jm})B_kB_m = -2|\vec{E}|^2 + 2B_kB_k = -2|\vec{E}|^2 + 2|\vec{B}|^2. \end{aligned}$$

■

11.15. Show that $\det \mathsf{F} = (\vec{E} \cdot \vec{B})^2$, where F is as given by (11.29).

Solution: You can use the evaluation of the determinant of the matrix according to one of its rows or columns, or use indices. I'll use the latter.

$$\begin{aligned} \det \mathsf{F} &= \epsilon_{\mu\nu\alpha\beta}F^{0\mu}F^{1\nu}F^{2\alpha}F^{3\beta} = \epsilon_{i\nu\alpha\beta}F^{0i}F^{1\nu}F^{2\alpha}F^{3\beta} \\ &= \epsilon_{i0\alpha\beta}F^{0i}F^{10}F^{2\alpha}F^{3\beta} + \epsilon_{ij\alpha\beta}F^{0i}F^{1j}F^{2\alpha}F^{3\beta} \\ &= \epsilon_{i0j\beta}F^{0i}F^{10}F^{2j}F^{3\beta} + \epsilon_{ij0\beta}F^{0i}F^{1j}F^{20}F^{3\beta} + \epsilon_{ijk\beta}F^{0i}F^{1j}F^{2k}F^{3\beta} \\ &= \epsilon_{i0jk}F^{0i}F^{10}F^{2j}F^{3k} + \epsilon_{ij0k}F^{0i}F^{1j}F^{20}F^{3k} + \epsilon_{ijk0}F^{0i}F^{1j}F^{2k}F^{30}. \end{aligned}$$

Here, I used the properties of the 4-dimensional Levi-Civita symbol, which a trivial generalization of the properties of the 3-dimensional Levi-Civita symbol. Now note that $\epsilon_{0ijk} = \epsilon_{ijk}$ and $F^{0i} = E_i$ to get

$$\begin{aligned} \det \mathsf{F} &= E_i\epsilon_{ijk}(-F^{10}F^{2j}F^{3k} + F^{1j}F^{20}F^{3k} - F^{1j}F^{2k}F^{30}) \\ &= E_i\epsilon_{ijk}(E_1F^{2j}F^{3k} - E_2F^{1j}F^{3k} + E_3F^{1j}F^{2k}). \end{aligned}$$

I'll evaluate the first term in detail, leaving the other two terms—which are obtained in exactly the same way—for you.

$$\begin{aligned} \epsilon_{ijk}F^{2j}F^{3k} &= \epsilon_{ijk}\epsilon_{2jm}B_m\epsilon_{3kn}B_n = \epsilon_{jik}\epsilon_{j2m}\epsilon_{3kn}B_mB_n \\ &= (\delta_{i2}\delta_{km} - \delta_{im}\delta_{k2})\epsilon_{3kn}B_mB_n = (\delta_{i2}\epsilon_{3mn} - \delta_{im}\epsilon_{32n})B_mB_n \\ &= \delta_{i2}\underbrace{\epsilon_{3mn}B_mB_n}_{=0} - \epsilon_{321}B_1B_i = B_1B_i. \end{aligned}$$

Similarly, you should show that

$$\epsilon_{ijk}F^{1j}F^{2k} = B_3B_i, \qquad \epsilon_{ijk}F^{1j}F^{3k} = -B_2B_i.$$

Plugging these results in the expression for the determinant, we get

$$\det \mathsf{F} = E_i(E_1B_1B_i + E_2B_2B_i + E_3B_3B_i) = (\vec{E} \cdot \vec{B})^2.$$

■

11.16. The second cyclotron that Lawrence's group built could accelerate a proton to a kinetic energy of 1 MeV in a magnetic field of 1.26 Tesla.

(a) What is γ_{α_0} and v_0 for such a proton?

(b) What was the diameter of the cyclotron?

Solution: The energy of the proton is

$$E = KE + m_p = 1 + 938.272 = 939.272 \text{ MeV}.$$

(a) $\gamma_{\alpha_0} = E/m_p = 1.001$, and

$$\gamma_{\alpha_0}\alpha_0 = \sqrt{\gamma_{\alpha_0}^2 - 1} = 0.0462 \iff \alpha_0 = \frac{0.0462}{1.001} = 0.0461,$$

and $v_0 = c\alpha_0 = 1.38 \times 10^7$ m/s.

(b) From $R = p_0/qB$, and $p_0 = \sqrt{E^2 - m_p^2} = 43.33 \text{ MeV}/c = 6.9 \times 10^{-12}$ J/c, we get

$$R = \frac{6.93 \times 10^{-12}}{cqB} = \frac{6.93 \times 10^{-12}}{(3 \times 10^8)(1.6 \times 10^{-19})1.26} = 11.5 \text{ cm},$$

or a little over four inches.

■

11.17. A K^0 meson at rest decays into two charged pions π^+ and π^- in a bubble chamber in which a magnetic field $B = 1.25$ T is present. The pions have equal mass $m_\pi = 139.57$ MeV. If the radius of curvature of the pions is 55 cm, determine the momenta and speeds of the pions and the mass of the K^0.

Solution: First calculate the momentum:

$$p_0 c = qBRc = (1.6 \times 10^{-19})(1.25)(0.55)(3 \times 10^8) = 3.3 \times 10^{-11} \text{ J} = 206 \text{ MeV}.$$

Then the energy: $E = \sqrt{206^2 + 139.57^2} = 249$ MeV. The mass of the K^0 meson is twice this or 498 MeV, and the speed of each pion is $\alpha_0 = p_0/E = 0.827$. ■

11.18. A Λ particle at rest decays into a proton and a π^- in a bubble chamber in which a magnetic field $B = 0.5$ T is present. The mass of the pion is $m_\pi = 139.57$ MeV and that of the proton is $m_p = 938.27$ MeV. If the radius of curvature of the pion is 67 cm, determine its speed, the radius of curvature of the proton, the speed of the proton, and the mass of Λ.

Solution: The momentum of the pion is

$$p_0 c = qBRc = (1.6 \times 10^{-19})(0.5)(0.67)(3 \times 10^8) = 1.61 \times 10^{-11} \text{ J} = 100.5 \text{ MeV}.$$

Since the Λ particle is at rest, this is also the momentum of the proton. Therefore, the radius of curvature of the proton is also 67 cm. The energies of the pion and proton are

$$E_\pi = \sqrt{100.5^2 + 139.57^2} = 172 \text{ MeV}, \quad E_p = \sqrt{100.5^2 + 938.27^2} = 943.6 \text{ MeV},$$

and the mass of the Λ particle is the sum of these two energies: $m_\Lambda = 943.6 + 172 = 1115.6$ MeV. The speed of the pion is $\alpha_\pi = 100.5/172 = 0.54$, and that of the proton is $\alpha_p = 100.5/943.6 = 0.107$. ■

11.19. Verify Equation (11.40) for $i = 2$ and $i = 3$.

Solution: I'll verify it for general i as more practice in index manipulation.

$$\partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_j F^{ji} = \partial_0 E_i + \partial_j(\epsilon_{jik} B_k) = \partial_0 E_i - \epsilon_{ijk}\partial_j B_k = \frac{\partial E_i}{\partial t} - (\vec{\nabla} \times \vec{B})_i.$$

■

11.20. Show that $\epsilon^{\sigma\xi\mu i}\partial_\sigma F_{\xi\mu} = 2[\partial_0 B_i + (\vec{\nabla} \times \vec{E})_i]$.

Solution: Use $F_{0k} = -E_k$, $F_{jk} = \epsilon_{jkm} B_m$, $\delta_{kk} = 3$, and $\epsilon^{0ijk} = \epsilon_{ijk}$:

$$\begin{aligned}
\epsilon^{\sigma\xi\mu i}\partial_\sigma F_{\xi\mu} &= \epsilon^{0\xi\mu i}\partial_0 F_{\xi\mu} + \epsilon^{j\xi\mu i}\partial_j F_{\xi\mu} = \epsilon^{0j\mu i}\partial_0 F_{j\mu} + \epsilon^{j0\mu i}\partial_j F_{0\mu} + \epsilon^{jk\mu i}\partial_j F_{k\mu} \\
&= \epsilon^{0jki}\partial_0 F_{jk} + \epsilon^{j0ki}\partial_j F_{0k} + \epsilon_{jk0i}\partial_j F_{k0} = \epsilon^{0ijk}(\partial_0 F_{jk} - \partial_j F_{0k} + \partial_j F_{k0}) \\
&= \epsilon_{ijk}(\partial_0 F_{jk} - 2\partial_j F_{0k}) = \epsilon_{ijk}[\partial_0(\epsilon_{jkm} B_m) + 2\partial_j E_k] \\
&= \partial_0(\epsilon_{jki}\epsilon_{jkm} B_m) + 2\epsilon_{ijk}\partial_j E_k = \partial_0[\underbrace{(\delta_{kk}\delta_{im} - \delta_{km}\delta_{ik})}_{=2\delta_{im}} B_m] + 2(\vec{\nabla} \times \vec{E})_i
\end{aligned}$$

■

11.21. In this problem, you'll show that Maxwell's equations imply the conservation of the electric charge.

(a) Write the continuity equation (see Note A.1.2) in terms of indices in four-dimensional spacetime.

(b) Suppose that $S^{\mu\nu}$ and $A^{\mu\nu}$ are, respectively symmetric and antisymmetric under the interchange of their indices. Show that $S^{\mu\nu}A_{\mu\nu} = 0$.

(c) Differentiate the first equation in Note 11.3.2 and use (b) to prove that Maxwell's equations imply charge conservation.

Solution:

(a) $\partial_\mu J^\mu = 0$.

(b) Let SA stand for $S^{\mu\nu}A_{\mu\nu} = 0$. Then

$$\mathsf{SA} = S^{\mu\nu}A_{\mu\nu} = -S^{\mu\nu}A_{\nu\mu} = -S^{\clubsuit\spadesuit}A_{\spadesuit\clubsuit} = -S^{\spadesuit\clubsuit}A_{\spadesuit\clubsuit} = -\mathsf{SA}.$$

So, $\mathsf{SA} = 0$.

(c) Differentiate the first equation in Note 11.3.2 with respect to x^ν to get

$$\partial_\nu\partial_\mu F^{\mu\nu} = \mu_0 \partial_\nu J^\nu.$$

Now note that $\partial_\nu\partial_\mu$ is symmetric and $F^{\mu\nu}$ antisymmetric. So, by (b) the left-hand side is zero.

■

CHAPTER 12

Early Universe

Problems With Solutions

12.1. Derive (12.6) from (12.4) and (12.9) from (12.7).

Solution: Substitute $\omega = 2\pi c/\lambda$ and $d\omega = -2\pi c d\lambda/\lambda^2$ in (12.4) to obtain

$$\begin{aligned} u_\gamma &= \frac{\hbar}{\pi^2 c^3} \int_\infty^0 \frac{(2\pi c/\lambda)^3}{\exp(2\pi\hbar c/\lambda k_B T) - 1} (-2\pi c d\lambda/\lambda^2) \\ &= \int_0^\infty \frac{8\pi h c}{\lambda^5} \frac{1}{\exp(hc/\lambda k_B T) - 1} d\lambda. \end{aligned}$$

Equation (12.9) followd from (12.7) in a similar way. ■

12.2. Consider the reaction $\gamma + \gamma \longrightarrow p + \bar{p}$, which can occur if the initial photons are extremely energetic. Here p is a generic particle and $\bar{p}$ its antiparticle.

(a) Use Equation (8.24) to show that

$$E_1 E_2 = \frac{m_p^2}{\sin^2(\theta/2)}$$

where m_p is the mass of p (or $\bar{p}$), E_1 and E_2 are the energies of the photons, and θ is the angle between the direction of motion of the initial photons.

(b) What value of θ minimizes $E_1 E_2$? If $E_1 > E_2$, what is the total momentum of the pair on the right-hand side (in terms of E_1 and m_p) for the θ that minimizes $E_1 E_2$?

(c) Using the θ in (b), show that the value of E_1 that minimizes the total energy $E_1 + E_2$ is $E_1 = m_p$. What is the total momentum of the pair on the right-hand side now?

Solution: With $m_1 = m_2 = 0$ and $m_3 = m_4 = m_p$, Equation (8.24) yields

$$\mathbf{p}_1 \bullet \mathbf{p}_2 = 2m_p^2, \iff E_1 E_2 - |\vec{p}_1||\vec{p}_2| \cos\theta = 2m_p^2.$$

(a) Since for photons $E_1 = |\vec{p}_1|$ and $E_2 = |\vec{p}_2|$ (speed of light is 1), we get

$$E_1E_2(1-\cos\theta) = 2m_p^2 \iff E_1E_2\sin^2(\theta/2) = m_p^2.$$

(b) E_1E_2 is minimum when $\sin^2(\theta/2) = 1$ or $\theta = \pi/2$.

(c) $E_1 + E_2 = E_1 + m_p^2/E_1$. Set the derivative of this expression equal to zero and solve for E_1:

$$1 - \frac{m_p^2}{E_1^2} = 0 \iff E_1 = m_p.$$

Part (a) now gives $m_pE_2 = m_p^2$, or $E_2 = m_p$. So both particles are produced at rest. Hence, the total momentum is zero.

■

12.3. In Example 12.1.3, I calculated the surface temperature of the Sun assuming that it was (approximately) a black body radiator. Using the result of that example and Equation (12.17),

(a) find the brightness of the Sun.

(b) How many 100-Watt light bulbs do you have to put on a square meter to give this brightness?

(c) The radius of the Sun is 700,000 km. What is the power output of the Sun?

(d) If the mass of the Sun is the source of this energy, how many kilograms of its mass does the Sun lose every second?

(e) The mass of the Sun is 2×10^{30} kg. Assuming that its mass depletion rate is proportional to its mass, find the present proportionality constant.

(f) Assuming that the constant in (e) does not change with time, find $m(t)$, the Sun's mass as a function of time.

(g) How many years do you have to wait before the Sun loses half of its mass (and is no longer a shining star)? This is a huge overestimate! The Sun dies far sooner than this due to nuclear processes that give off colossal explosive (and implosive) energies.

Solution: The temperature was found to be 5800 K.

(a) The brightness is given by (12.17), which, in conjunction with (12.18), gives

$$I_\gamma = \sigma_B T^4 = 5.67 \times 10^{-8}(5800)^4 = 6.4 \times 10^7 \text{ W/m}^2.$$

(b) $N = 6.4 \times 10^7/100 = 640,000$. The Sun is really bright!

(c) Brightness is power per square meter. So, the total power output of the Sun is

$$P_{tot} = 4\pi R^2 I_\gamma = 4\pi(7 \times 10^8)^2(6.4 \times 10^7) = 3.94 \times 10^{26} \text{ W}.$$

(d) Divide (c) by c^2:

$$\frac{dm}{dt} = \frac{3.94 \times 10^{26}}{9 \times 10^{16}} = 4.4 \times 10^9 \text{ kg/s}.$$

(e)

$$k = \frac{dm/dt}{M_\odot} = \frac{4.4 \times 10^9}{2 \times 10^{30}} = 2.2 \times 10^{-21}.$$

(f)

$$\frac{dm}{dt} = -km \iff m(t) = m_0 e^{-kt} = M_\odot e^{-kt}.$$

(g)

$$\frac{M_\odot}{2} = M_\odot e^{-kt_{1/2}} \iff \ln 2 = kt_{1/2}$$

or

$$t_{1/2} = \frac{\ln 2}{k} = \frac{0.693}{2.2 \times 10^{-21}} = 3.15 \times 10^{20} \text{ s},$$

or about 10^{13} years.

■

12.4. Using the present temperature of the universe and Equations (12.27) and (12.36), calculate Ω_γ and compare it with the value given in (12.29).

Solution: From (12.36), we get

$$\rho_\gamma = 8.38 \times 10^{-33} T^4 \text{ kg/m}^3 = (8.38 \times 10^{-33}) 2.725^4 \text{ kg/m}^3 = 4.62 \times 10^{-33} \text{ kg/m}^3,$$

and from (12.27), we obtain

$$\Omega_\gamma = \frac{4.62 \times 10^{-33} \text{ kg/m}^3}{8.5 \times 10^{-27} \text{ kg/m}^3} = 5.4 \times 10^{-5},$$

which compares very well with (12.29). ■

12.5. In this problem you are asked to compare the present number of photons and baryons in the universe.

(a) Using the present temperature of the universe, calculate n_γ, the present photon number density.

(b) From (12.27) and the value of Ω_b in (12.29), calculate the present ρ_b.

(c) Ignore the negligible mass difference between a proton and a neutron and assume that only protons contribute to ρ_b. Moreover, since protons are not moving fast, you can assume that their energy is just $m_p c^2$. With $m_p = 1.67 \times 10^{-27}$ kg, find n_b, the present baryon number density.

(d) How many photons are there in the universe for each baryon?

Solution:

(a) Equation (12.11) yields

$$n_\gamma = 2 \times 10^7 T^3 \text{ photons/m}^3 = (2 \times 10^7) 2.725^3 \text{ photons/m}^3 = 4 \times 10^8 \text{ photons/m}^3$$

for the present number density of photons.

(b)

$$\rho_b = \Omega_b \rho_{\text{crit}} = 0.0484(8.5 \times 10^{-27}\ \text{kg/m}^3) = 4.1 \times 10^{-28}\ \text{kg/m}^3.$$

(c)

$$n_b = \frac{\rho_b}{m_p} = \frac{4.1 \times 10^{-28}\ \text{kg/m}^3}{1.67 \times 10^{-27}\ \text{kg}} = 0.25 \quad \text{baryons/m}^3.$$

(d) Take the ratio of n_γ over n_b:

$$\frac{n_\gamma}{n_b} = \frac{4 \times 10^8\ \text{photons/m}^3}{0.25 \quad \text{baryons/m}^3} = 1.6 \times 10^9 \quad \text{photons/baryons.}$$

■

12.6. Decide which particles of Table 12.1 were present at the beginning of the second epoch.

(a) Calculate α assuming that whatever particle you have is relativistic.

(b) How old was the universe at the beginning of the second epoch?

Solution: All the particles were present at the beginning of the second epoch, because all the particles in Table 12.1 have threshold temperatures below 10^{15} K.

(a) So, α is the sum of all the α's in Table 12.1: $\alpha = 32.375$.

(b) Now that we have α, we can calculate the age of the universe using (12.49):

$$t_\alpha = \frac{2.32 \times 10^{20}}{\sqrt{32.375}\, 10^{30}}\ \text{s} = 4 \times 10^{-11}\ \text{s} = 40\ \text{ps}.$$

■

12.7. In this problem, you'll calculate the density of the universe at the beginning of the second epoch.

(a) Find α and from that, the density of the universe.

(b) The Earth has a mass of 5.97×10^{24} kg. What would the radius of the Earth be if it were composed of such a dense material?

(c) At the confinement temperature, the density of the universe dropped to $6.5 \times 10^{17}\ \text{kg/m}^3$. What would the radius of the Earth be if it were composed of this material?

Solution: All the particles were present at the beginning of the second epoch, because all the particles in Table 12.1 have threshold temperatures below 10^{15} K.

(a) So, α is the sum of all the α's in Table 12.1: $\alpha = 32.375$. The density is

$$\rho_\alpha = \alpha \rho_\gamma = 32.375(8.38 \times 10^{-33} T^4) = 2.7 \times 10^{29}\ \text{kg/m}^3.$$

(b)

$$\rho_\alpha = \frac{M_\oplus}{\frac{4}{3}\pi R_\oplus^3} \iff R_\oplus = \left(\frac{3M_\oplus}{4\pi\rho_\alpha}\right)^{1/3} = 5.3 \times 10^{-6}\ \text{m} = 5.3\ \mu\text{m}.$$

(c)

$$R_\oplus = \left(\frac{3M_\oplus}{4\pi\rho_{\rm conf}}\right)^{1/3} = \left(\frac{3(5.97\times10^{24})}{4\pi(6.5\times10^{17})}\right)^{1/3} = 0.04 \text{ m} = 4 \text{ cm}.$$

■

12.8. In this problem you'll calculate the age of the universe when quarks confined at a temperature of 2×10^{12} K.

(a) Look at Table 12.1 and decide which particles were present just before that temperature was reached. Calculate α.

(b) Now use (12.49) to calculate the age of the universe.

Solution:

(a) All particles with a threshold temperature less that 2×10^{12} K were present in the universe. This gives $\alpha = 20.275$.

(b) Now from (12.49), we get

$$t_\alpha = \frac{2.32\times10^{20}}{\sqrt{20.275}\,(2\times10^{12})^2} \text{ s} = 1.3\times10^{-5} \text{ s} = 13\ \mu\text{s}.$$

■

12.9. Assume that at a temperature of 2×10^{12} K, quarks and gluons are bagged into hadrons, the lightest of which are pions with spin zero (therefore $\alpha = 0.5$, just like the Higgs boson). There are three kinds of pion: neutral pions π^0 (with no antiparticle) with a mass of 135 MeV and two charged pions (one being the antiparticle of the other) $\pi^\pm$ with a mass of 139.57 MeV. The next lightest hadron has a mass of approximately 500 MeV.

(a) What is the threshold temperature for π^0? For $\pi^\pm$? For the next lightest hadron?

(b) Determine the particle content of the universe at the beginning of the third epoch, and from that, α.

(c) Now use (12.49) to calculate the age of the universe at the beginning of the third epoch.

(d) The mean life of π^0 is 8.5×10^{-17} s, and that of $\pi^\pm$ is 2.6×10^{-8} s. What fraction of charged pions decay between their production at confinement time and the beginning of the third epoch? What fraction of neutral pions?

(e) In light of (d), do you have to reconsider (b) and (c)? If so, recalculate them and find the new value for the age of the universe at the beginning of the third epoch.

Solution:

(a) Using $T_{\rm th} = 4.3\times10^9\, mc^2$ MeV^{-1} K, we have

$$\begin{aligned} T_{\pi^0} &= 4.3\times10^9(135) = 5.8\times10^{11} \text{ K},\\ T_{\pi^\pm} &= 4.3\times10^9(139.57) = 6\times10^{11} \text{ K},\\ T_3 &= 4.3\times10^9(500) = 2.15\times10^{12} \text{ K}. \end{aligned}$$

(b) The third epoch begins with a temperature below quark confinement. So, no free quarks or gluons are there, and (a) shows that out of all hadrons, only pions will be present. From Table 12.1, only photons, neutrinos, electrons and muons have temperatures below 10^{12} K. The α for each pion is 0.5. Thus the three of them contribute 1.5 to the total α; and the contribution of Table 12.1 is 7.125. Hence, $\alpha = 8.625$.

(c) From (12.49), we get

$$t_\alpha = \frac{2.32 \times 10^{20}}{\sqrt{8.625}\,(10^{12})^2} \text{ s} = 7.9 \times 10^{-5} \text{ s} = 79 \ \mu\text{s}.$$

(d) From Problem 12.8, quarks confine 13 μs after the big bang. From (c), the third epoch occurs 79 μs after the big bang. So, pions have 66 μs to decay. The fraction r of any decaying particle remaining at time t is given by $r = e^{-t/\tau}$, where τ is the mean life of the particle. Thus, the fraction of pions remaining is

$$r_{\pi^\pm} = \exp\left(-\frac{66 \times 10^{-6}}{2.6 \times 10^{-8}}\right) = e^{-2538.5} = 2.75 \times 10^{-1102}$$
$$r_{\pi^0} = \exp\left(-\frac{66 \times 10^{-6}}{8.5 \times 10^{-17}}\right) = 5.75 \times 10^{-337216891830}.$$

In other words, zero pion will see the light of the third epoch!

(e) We have to subtract the contribution of pions to α. So, $\alpha = 7.125$ and

$$t_\alpha = \frac{2.32 \times 10^{20}}{\sqrt{7.125}\,(10^{12})^2} \text{ s} = 8.69 \times 10^{-5} \text{ s} = 86.9 \ \mu\text{s}.$$

■

12.10. Right after μ^+-μ^- annihilation, the temperature is 10^{11} K.

(a) Which particles are present then? Ignore the baryon "contamination." What is the value of α?

(b) How old is the universe?

(c) What is the density of the universe assuming that it is all ρ_{rel}? Do you have to worry about the contribution from matter density? Use (12.47) to find out!

(d) Assume that a grain of sand is a cube of side one millimeter. What would the mass of this grain be if it had a density you found in (c)?

Solution:

(a) The particles present are photons, neutrinos, and electrons. Therefore, From Table 12.1, $\alpha = 5.375$.

(b) From (12.49),

$$t_\alpha = \frac{2.32 \times 10^{20}}{\sqrt{5.375}\,(10^{11})^2} \text{ s} = 0.01 \text{ s}.$$

(c)

$$\rho_{\text{rel}} = \alpha\rho_\gamma = 5.375(8.38 \times 10^{-33}T^4)\ \text{kg/m}^3$$
$$= 5.375(8.38 \times 10^{-33}(10^{11})^4)\ \text{kg/m}^3 = 4.5 \times 10^{12}\ \text{kg/m}^3.$$

Using (12.47), the contribution from matter density is

$$\rho_m(T) = 1.32 \times 10^{-28}\,T^3\ \text{kg/m}^3 = 1.32 \times 10^{-28}\,(10^{11})^3\ \text{kg/m}^3 = 13200\ \text{kg/m}^3,$$

much smaller than ρ_{rel}.

■

12.11. Derive Equation (12.51) from the equations preceding it.

Solution: On the right-hand side of the equation for neutron production we have

$$m_n^2 = (m_p + \Delta m)^2 \approx m_p^2 + 2m_p\Delta m.$$

Substituting this and letting $m_p = m$, we get

$$m\mathcal{E}_1 = m\Delta m + \mathcal{E}_3(\mathcal{E}_1 + m) - \mathcal{E}_1\mathcal{E}_3\cos\theta,$$

or

$$\mathcal{E}_1(m - \mathcal{E}_3 + \mathcal{E}_3\cos\theta) = m\Delta m + m\mathcal{E}_3,$$

which is the first equation in (12.51). The only difference between the two equations is that for $\mathcal{E}_1^p$, the roles of m_n and m_p are switched, introducing a minus sign for Δm. ■

12.12. Things happen very quickly when the universe is very young. To see how quickly, suppose two points of the universe are one millimeter apart 10^{-30} second after the big bang.

(a) Which particles of Table 12.1 are present at this time?[1] Use (12.48) to find $a_\alpha(t)$.

(b) One second after the big bang, the fourth epoch starts. What is α now? What is $a_\alpha(t)$?

(c) How far apart are the two points now? Recall that the ratio of distances is the same as the ratio of the scale factors.

(d) Compare the distance in (c) with the Earth-Sun distance. From 1 mm to this distance in just one second! Now you can appreciate why we call it a "bang," a "BIG" bang!

Solution:

(a) All particles in Table 12.1 are present. Thus, $\alpha = 32.375$, and (12.48) gives

$$a_\alpha(t) = \sqrt[4]{4\alpha\Omega_\gamma}\left(\frac{t}{t_H}\right)^{1/2} = \sqrt[4]{4(32.375)(0.0000538)}\left(\frac{10^{-30}}{4.59 \times 10^{17}}\right)^{1/2}$$

or $a_\alpha(t) = 4.26 \times 10^{-26}$.

[1]Hint: How old is the universe at the beginning of the second epoch?

(b) The universe consists of electrons, positrons, neutrinos, and photons. So, $\alpha = 5.375$, and

$$a_\alpha(t) = \sqrt[4]{4(5.375)(0.0000538)}\left(\frac{1}{4.59 \times 10^{17}}\right)^{1/2} = 2.7 \times 10^{-10}.$$

(c)

$$\frac{R}{0.001} = \frac{2.7 \times 10^{-10}}{4.26 \times 10^{-26}} \iff R = 6.4 \times 10^{12} \text{ m}.$$

(d) The Earth-Sun distance is 1.5×10^{11} m, so this distance is more than 42 times the Earth-Sun distance!

■

12.13. In this problem, you are asked to estimate the helium-proton abundance if $e^- + D \to 2n + \nu$ were responsible for deuteron dissociation. I have already calculated the minimum energy for the electron. It is 3.517 MeV.

(a) What is the wavelength associated with this energy?

(b) Assuming that electrons (and positrons) outnumber protons and neutrons 1.6 billion to 1, use Example 12.1.2 to show that the threshold temperature for deuteron formation is 1.5×10^9 K. However, there is a problem here. This temperature is less than e^+-e^- annihilation temperature. At 1.5×10^9 K, the electrons do not outnumber baryons 1.6 billion to 1. This means that 3.517 MeV is not sufficient for deuteron production. Let's assume that the minimum energy corresponds to the e^+-e^- annihilation temperature of 2.2×10^9 K.

(c) If the content of the universe is electron, positron, photons, and neutrinos, what is α? Use (12.49) to estimate the age of the universe when deuterons start to form at 2.2×10^9 K.

(d) Show that from the end of the third epoch, about 2.2% of the neutrons decay into protons. Even if the other contributions to neutron depletion remain at 28% (the actual number is smaller than this because 28% corresponds to the entire fifth epoch), the total depletion rate is 30.2%. Now show that the helium-proton abundance is 34.9% and 65.1%, respectively.

Solution:

(a) The energy is $3.517(1.6 \times 10^{-13}) = 5.63 \times 10^{-13}$ J.

$$\lambda = \frac{hc}{E} = \frac{(6.626 \times 10^{-34})(3 \times 10^8)}{5.63 \times 10^{-13}} = 3.53 \times 10^{-13} \text{ m}.$$

(b) Equation (12.16) gives the relation between wavelength and temperature:

$$\lambda_2 T = 5.33 \times 10^{-4} \text{ m K} \iff T = \frac{5.33 \times 10^{-4}}{3.53 \times 10^{-13}} = 1.51 \times 10^9 \text{ K}.$$

(c) Table 12.1 gives $\alpha = 5.375$, and (12.49) yields

$$t_\alpha = \frac{2.32 \times 10^{20}}{\sqrt{5.375}\,(2.2 \times 10^9)^2} = 20.68 \text{ s}.$$

(d) From the end of the third epoch until t_α, the neutrons have 19.68 seconds to decay. Therefore the fraction remaining is

$$\frac{N(t_\alpha)}{N_0} = e^{-t_\alpha/\tau} = e^{-19.68/880.3} = 0.978.$$

Therefore, about 2.2% of the neutrons decay into protons. Add this to the other decay contribution of 28% to get 30.2%. So, the fraction of neutrons reduces from 25% to $25 \times 0.698 = 17.45\%$. These neutrons combine with an equal number of protons to form helium. So, 34.9% of the baryons are in the form of helium nuclei and 65.1% in the form of protons.

■

12.14. Calculate the photon and neutrino mass densities at the beginning of the fifth epoch and compare them with matter density. What was the ratio ρ_γ/ρ_m then? Do the same for the beginning of the last epoch.

Solution: Table 12.1 gives $\alpha_\nu = 0.681$ after electron-positron annihilation. Therefore,

$$\rho_\gamma = 8.38 \times 10^{-33}(10^9)^4 = 8380 \text{ kg/m}^3$$

and

$$\rho_\nu = 0.681(8.38 \times 10^{-33})(10^9)^4 = 5707 \text{ kg/m}^3.$$

For ρ_m we use (12.47):

$$\rho_m(T) = 1.32 \times 10^{-28}\,(10^9)^3 = 0.132 \text{ kg/m}^3.$$

This gives a ratio of

$$\frac{\rho_\gamma}{\rho_m} = \frac{8380}{0.132} = 63485.$$

In general,

$$\frac{\rho_\gamma(T)}{\rho_m(T)} = \frac{8.38 \times 10^{-33}\,T^4}{1.32 \times 10^{-28}\,T^3} = 6.35 \times 10^{-5}\,T.$$

For the last epoch, $T = 10^8$ K; thus, the ratio is 6350, which is a tenth of the ratio at the beginning of the fifth epoch. ■

APPENDIX A

Maxwell's Equations

Problems With Solutions

A.1. Derive the differential form of the fourth equation in (A.1).

Solution: Use Stokes' Theorem on the left and the definition of current in terms of current density on the right:

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{a}.$$

This must be true for all S. Therefore, the integrands must equal: $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$. ■

A.2. Starting with Maxwell's equations, show that the magnetic field satisfies the same wave equation as the electric field. In particular, that it, too, propagates with the same speed.

Solution: Take the curl of the 4th equation in (A.12) and use the last equation in (A.9). You'll get

$$\text{LHS of (4)} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \boldsymbol{\nabla} \underbrace{(\boldsymbol{\nabla} \cdot \mathbf{B})}_{=0 \text{ by } (2)} - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B},$$

and

$$\text{RHS} = \mu_0 \epsilon_0 \boldsymbol{\nabla} \times \left(\frac{\partial \mathbf{E}}{\partial t} \right) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\boldsymbol{\nabla} \times \mathbf{E}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\frac{\partial \mathbf{B}}{\partial t} \right).$$

Now set the two sides equal and obtain

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

■

A.3. Consider $\mathbf{E} = \mathbf{E}_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}$ and $\mathbf{B} = \mathbf{B}_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}$, where $i = \sqrt{-1}$, $\mathbf{E}_0$, $\mathbf{B}_0$, k, and ω are constants. The $\mathbf{E}$ and the $\mathbf{B}$ so defined represent *plane waves* moving in the direction of the vector $\mathbf{k}$.

(a) Show that they satisfy Maxwell's equations in free space if:

$$(1)\ \mathbf{k}\cdot\mathbf{E}_0 = 0; \qquad (2)\ \mathbf{k}\cdot\mathbf{B}_0 = 0;$$
$$(3)\ \mathbf{k}\times\mathbf{E}_0 = \omega\mathbf{B}_0; \quad (4)\ \mathbf{k}\times\mathbf{B}_0 = -\frac{\omega}{c^2}\mathbf{E}_0.$$

(b) In particular, show that $\mathbf{k}$, the propagation direction, and $\mathbf{E}$ and $\mathbf{B}$ form a mutually perpendicular set of vectors.

(c) By taking the cross product of $\mathbf{k}$ with an appropriate equation, show that $|\mathbf{k}| = \omega/c$.

Solution: The problem becomes a lot easier if we first derive two more general identities. If $\mathbf{C}$ is a constant vector and f is a function, then

$$\boldsymbol{\nabla}\cdot(\mathbf{C}f) = \mathbf{C}\cdot(\boldsymbol{\nabla}f) \quad \text{and} \quad \boldsymbol{\nabla}\times(\mathbf{C}f) = (\boldsymbol{\nabla}f)\times\mathbf{C}.$$

The first identity is derived as follows:

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\mathbf{C}f) &= \frac{\partial}{\partial x}(C_x f) + \frac{\partial}{\partial y}(C_y f) + \frac{\partial}{\partial z}(C_z f)\\ &= C_x\frac{\partial f}{\partial x} + C_y\frac{\partial f}{\partial y} + C_z\frac{\partial f}{\partial z} = \mathbf{C}\cdot(\boldsymbol{\nabla}f).\end{aligned}$$

For the second identity use (A.8) and note that the derivatives operate only on f:

$$\boldsymbol{\nabla}\times(\mathbf{C}f) = \det\begin{pmatrix}\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ C_x f & C_y f & C_z f\end{pmatrix} = \det\begin{pmatrix}\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z\\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z}\\ C_x & C_y & C_z\end{pmatrix} = (\boldsymbol{\nabla}f)\times\mathbf{C}.$$

(a) Let's apply the general results obtained above to the specific function at hand. From $\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z$, we get

$$\frac{\partial}{\partial x}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} = -ik_x e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})},$$

with similar results for y and z. Therefore,

$$\boldsymbol{\nabla}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} = -i\mathbf{k}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}.$$

Now let's apply what we have obtained so far to Maxwell's equation in free space. The first equation yields

$$\begin{aligned}0 = \boldsymbol{\nabla}\cdot\mathbf{E} &= \boldsymbol{\nabla}\cdot[\mathbf{E}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}] = \mathbf{E}_0\cdot[\boldsymbol{\nabla}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}]\\ &= \mathbf{E}_0\cdot[-i\mathbf{k}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}] = -i\mathbf{E}_0\cdot\mathbf{k}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} \iff \mathbf{k}\cdot\mathbf{E}_0 = 0,\end{aligned}$$

with a similar result for the magnetic field. It is obvious that time differentiation introduces a multiplicative $i\omega$. So, the third Maxwell equation becomes

$$\boldsymbol{\nabla}\times[\mathbf{E}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}] = [\boldsymbol{\nabla}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}]\times\mathbf{E}_0 = [-i\mathbf{k}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}]\times\mathbf{E}_0 = -i\omega\mathbf{B}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})},$$

or $\mathbf{k}\times\mathbf{E}_0 = \omega\mathbf{B}_0$. The fourth equation follows in the same way.

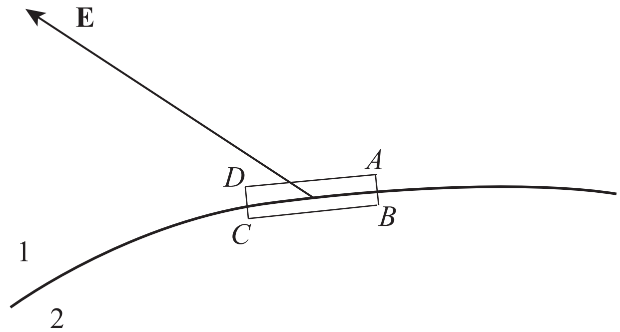


Figure A.1: The long sides of the rectangle $ABCD$ are infinitesimals. The short sides are infinitesimals compared to the long sides.

(b) (3) shows that $\mathbf{B}_0$ is perpendicular to both $\mathbf{E}_0$ and $\mathbf{k}$. (4) shows that $\mathbf{E}_0$ is perpendicular to both $\mathbf{B}_0$ and $\mathbf{k}$. So, all three vector are mutually perpendicular to each other.

(c) Take the cross product of (3) with $\mathbf{k}$. Then, using the "bac cab rule," on the left you get

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0) = \mathbf{k}\underbrace{(\mathbf{k} \cdot \mathbf{E}_0)}_{=0} - (\mathbf{k} \cdot \mathbf{k})\mathbf{E}_0 = -|\mathbf{k}|^2\mathbf{E}_0,$$

and on the right, using (4), you get

$$\omega \mathbf{k} \times \mathbf{B}_0 = -\frac{\omega^2}{c^2}\mathbf{E}_0.$$

The last two equations yield $|\mathbf{k}| = \omega/c$.

■

A.4. Take a rectangular loop with two long sides on either side of a boundary surface. All sides are infinitesimal, but the short side is infinitesimal even compared to the long side. Apply the third Maxwell equation in integral form (A.1) to this loop. Since everything is infinitesimal, you can forget about the integrals. The magnetic flux is zero because the area is triply infinitesimal. The contribution from the two short sides to the left-hand side is zero. Now complete the proof of the equality of the tangential components of the electric field on the two sides of the boundary.

Solution: Figure A.1 of the manual shows a rectangle with the long sides in two different media. Applying the third Maxwell equation in (A.1) to this loop, for the right-hand side we get zero because the area is infinitesimal to the third power, and therefore, the flux, which is the product of $\mathbf{B}$ and this area, can be neglected. The left-hand side can be evaluated as follows:

$$\oint_C \mathbf{E} \cdot d\mathbf{r} \approx \mathbf{E}_2 \cdot \hat{\mathbf{e}}_t \overline{BC} + \mathbf{E}_1 \cdot (-\hat{\mathbf{e}}_t)\overline{DA} = (\mathbf{E}_2 \cdot \hat{\mathbf{e}}_t - \mathbf{E}_1 \cdot \hat{\mathbf{e}}_t)\overline{BC},$$

where $\hat{\mathbf{e}}_t$ is a unit vector tangent to the boundary, which I have taken to be pointing from right to left in the figure. I have neglected the contributions from the smaller sides. Since $\overline{BC} \neq 0$, we have to assume that $\mathbf{E}_2 \cdot \hat{\mathbf{e}}_t = \mathbf{E}_1 \cdot \hat{\mathbf{e}}_t$, i.e., that the tangential components of the electric field are equal on both sides. ■

APPENDIX B

Derivation of 4D Lorentz transformation

Problems With Solutions

B.1. Take the transpose of $\mathsf{A}^{-1}\mathsf{A} = \mathsf{A}\mathsf{A}^{-1} = 1$ to show that the inverse of the transpose is the transpose of the inverse. You need both matrix products because an inverse must be a left inverse as well as a right inverse.

Solution: Taking the transpose of $\mathsf{A}^{-1}\mathsf{A} = \mathsf{A}\mathsf{A}^{-1} = 1$, we get

$$\widetilde{\mathsf{A}^{-1}\mathsf{A}} = \widetilde{\mathsf{A}\mathsf{A}^{-1}} = \tilde{1} \iff \widetilde{\mathsf{A}}\widetilde{\mathsf{A}^{-1}} = \widetilde{\mathsf{A}^{-1}}\widetilde{\mathsf{A}} = 1.$$

Therefore, $\widetilde{\mathsf{A}^{-1}}$ is the inverse of $\widetilde{\mathsf{A}}$. ∎

B.2. Show that $\widetilde{\mathsf{A}}\mathsf{A} = \mathsf{A}\widetilde{\mathsf{A}} = 1$ for a 3×3 matrix implies that each row (or column) of A has a unit length and the dot product of any two different rows (or different columns) vanishes.

Solution: Consider the rows and columns of A as vectors. Let $\vec{a}_i$ denote the ith row and $\vec{b}_i$ the ith columns of A. The ijth element of $\widetilde{\mathsf{A}}\mathsf{A}$ is the product of the ith row of $\widetilde{\mathsf{A}}$ and the jth column of A. But the ith row of $\widetilde{\mathsf{A}}$ is the ith column of A. Thus the ijth element of $\widetilde{\mathsf{A}}\mathsf{A}$ is $\vec{b}_i \cdot \vec{b}_j$. This is equal to the ijth element of the unit matrix. Therefore, if $i = j$, we get $\vec{b}_i \cdot \vec{b}_i = 1$ and if $i \neq j$, we get $\vec{b}_i \cdot \vec{b}_j = 0$. Evaluating the ijth element of $\mathsf{A}\widetilde{\mathsf{A}}$ implies that if $i = j$, we get $\vec{a}_i \cdot \vec{a}_i = 1$ and if $i \neq j$, we get $\vec{a}_i \cdot \vec{a}_j = 0$. Note that the assumption that A is 3×3 never entered the derivation. So, the conclusion holds for any $n \times n$ matrix satisfying $\widetilde{\mathsf{A}}\mathsf{A} = \mathsf{A}\widetilde{\mathsf{A}} = 1$. ∎

B.3. By writing out the 4×4 matrices, verify that $\Lambda_{\mathrm{rot}}^{-1}$ as given by Equation (B.4) is the inverse of Λ_{rot} as given by Equation (B.3).

Solution: Given that $\widetilde{\mathsf{A}}\mathsf{A} = \mathsf{A}\widetilde{\mathsf{A}} = 1$, this is a simple exercise in matrix multiplication. ∎

B.4. Using relations such as $\hat{\beta}_x\hat{\beta}_y + a_{12}a_{22} + a_{13}a_{23} = 0$ and $a_{12}^2 + a_{13}^2 = 1 - (\hat{\beta}_x)^2$, obtained from Problem B.2, derive Equation (B.5).

Solution: I'll calculate the first two rows of Equation (B.5) from the equation preceding it. You provide the calculation for the third and fourth row. From the equation preceding (B.5), for the first row, we have

$$\Lambda_{11} = \begin{pmatrix}\gamma & \gamma\beta & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \gamma$$

$$\Lambda_{12} = \begin{pmatrix}\gamma & \gamma\beta & 0 & 0\end{pmatrix}\begin{pmatrix}0\\ \hat{\beta}_x\\ a_{12}\\ a_{13}\end{pmatrix} = \gamma\beta\hat{\beta}_x = \gamma\beta_x$$

$$\Lambda_{13} = \begin{pmatrix}\gamma & \gamma\beta & 0 & 0\end{pmatrix}\begin{pmatrix}0\\ \hat{\beta}_y\\ a_{22}\\ a_{23}\end{pmatrix} = \gamma\beta\hat{\beta}_y = \gamma\beta_y$$

$$\Lambda_{14} = \begin{pmatrix}\gamma & \gamma\beta & 0 & 0\end{pmatrix}\begin{pmatrix}0\\ \hat{\beta}_z\\ a_{32}\\ a_{33}\end{pmatrix} = \gamma\beta\hat{\beta}_z = \gamma\beta_z.$$

The second row is only slightly more complicated:

$$\Lambda_{21} = \begin{pmatrix}\gamma\beta_x & \gamma\hat{\beta}_x & a_{12} & a_{13}\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \gamma\beta_x$$

$$\Lambda_{22} = \begin{pmatrix}\gamma\beta_x & \gamma\hat{\beta}_x & a_{12} & a_{13}\end{pmatrix}\begin{pmatrix}0\\ \hat{\beta}_x\\ a_{12}\\ a_{13}\end{pmatrix} = \gamma\hat{\beta}_x^{\,2} + \underbrace{a_{12}^2 + a_{13}^2}_{=1-\hat{\beta}_x^2} = 1 + \hat{\beta}_x^{\,2}(\gamma - 1),$$

because the first row of the submatrix A in (B.3) has unit length.

$$\Lambda_{23} = \begin{pmatrix}\gamma\beta_x & \gamma\hat{\beta}_x & a_{12} & a_{13}\end{pmatrix}\begin{pmatrix}0\\ \hat{\beta}_y\\ a_{22}\\ a_{23}\end{pmatrix} = \gamma\hat{\beta}_x\hat{\beta}_y + \underbrace{a_{12}a_{22} + a_{13}a_{23}}_{=-\hat{\beta}_x\hat{\beta}_y} = \hat{\beta}_x\hat{\beta}_y(\gamma - 1),$$

because the first row of the submatrix A in (B.3) is orthogonal to its second row.

$$\Lambda_{24} = \begin{pmatrix}\gamma\beta_x & \gamma\hat{\beta}_x & a_{12} & a_{13}\end{pmatrix}\begin{pmatrix}0\\ \hat{\beta}_z\\ a_{32}\\ a_{33}\end{pmatrix} = \gamma\hat{\beta}_x\hat{\beta}_z + \underbrace{a_{12}a_{32} + a_{13}a_{33}}_{=-\hat{\beta}_x\hat{\beta}_z} = \hat{\beta}_x\hat{\beta}_z(\gamma - 1),$$

because the first row of the submatrix A in (B.3) is orthogonal to its third row. ■

B.5. Multiply the matrix of Equation (B.5) by the column 4-vector of $\mathbf{r}$ to obtain Equation (B.6).

Solution: This is a trivial exercise in matrix multiplication. ■

B.6. Show directly that the matrices of (B.5) and (B.11) satisfy $\Lambda^{-1}\Lambda = \Lambda\Lambda^{-1} = 1$.

Solution: First write Λ^{-1} in block form:

$$\Lambda^{-1} = \begin{pmatrix} \gamma & -\gamma\tilde{\vec{\beta}} \\ & \\ -\gamma\vec{\beta} & \overleftrightarrow{\Lambda} \end{pmatrix}.$$

Now multiply it on the right by the block form of Λ:

$$\Lambda^{-1}\Lambda = \begin{pmatrix} \gamma & -\gamma\tilde{\vec{\beta}} \\ & \\ -\gamma\vec{\beta} & \overleftrightarrow{\Lambda} \end{pmatrix} \begin{pmatrix} \gamma & \gamma\tilde{\vec{\beta}} \\ & \\ \gamma\vec{\beta} & \overleftrightarrow{\Lambda} \end{pmatrix} = \begin{pmatrix} \gamma^2 - \gamma^2\beta^2 & \gamma^2\tilde{\vec{\beta}} - \gamma\tilde{\vec{\beta}}\,\overleftrightarrow{\Lambda} \\ & \\ -\gamma^2\vec{\beta} + \gamma\overleftrightarrow{\Lambda}\,\vec{\beta} & -\gamma^2\vec{\beta}\tilde{\vec{\beta}} \; + \overleftrightarrow{\Lambda}\overleftrightarrow{\Lambda} \end{pmatrix}.$$

Let's look at each block element of $\Lambda^{-1}\Lambda$:

$$\left(\Lambda^{-1}\Lambda\right)_{11} = \gamma^2(1-\beta^2) = 1.$$

For $\overleftrightarrow{\Lambda}$, I'll use (B.8). Then

$$\begin{aligned}\left(\Lambda^{-1}\Lambda\right)_{12} = \gamma^2\tilde{\vec{\beta}} - \gamma\tilde{\vec{\beta}}\,\overleftrightarrow{\Lambda} &= \gamma^2\begin{pmatrix}\beta_x & \beta_y & \beta_z\end{pmatrix} - \gamma\begin{pmatrix}\beta_x & \beta_y & \beta_z\end{pmatrix}\begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix} \\ &\quad - \gamma(\gamma-1)\begin{pmatrix}\beta_x & \beta_y & \beta_z\end{pmatrix}\begin{pmatrix}\hat\beta_x^2 & \hat\beta_x\hat\beta_y & \hat\beta_x\hat\beta_z \\ \hat\beta_x\hat\beta_y & \hat\beta_y^2 & \hat\beta_y\hat\beta_z \\ \hat\beta_x\hat\beta_z & \hat\beta_y\hat\beta_z & \hat\beta_z^2\end{pmatrix} \\ &= \gamma^2\begin{pmatrix}\beta_x & \beta_y & \beta_z\end{pmatrix} - \gamma\begin{pmatrix}\beta_x & \beta_y & \beta_z\end{pmatrix} - \gamma(\gamma-1)\begin{pmatrix}\beta_x & \beta_y & \beta_z\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\end{pmatrix}.\end{aligned}$$

Similarly,

$$\left(\Lambda^{-1}\Lambda\right)_{21} = \begin{pmatrix}0 \\ 0 \\ 0\end{pmatrix}.$$

For the last element, first write (B.8) as $\Lambda = 1 + (\gamma-1)\mathsf{B}$, with obvious definition for B. Now, let's evaluate $\overleftrightarrow{\Lambda}\overleftrightarrow{\Lambda}$ first:

$$\overleftrightarrow{\Lambda}\overleftrightarrow{\Lambda} = [1 + (\gamma-1)\mathsf{B}]\,[1 + (\gamma-1)\mathsf{B}] = 1 + 2(\gamma-1)\mathsf{B} + (\gamma-1)^2\mathsf{B}^2,$$

with

$$\mathsf{B}^2 = \begin{pmatrix}\hat\beta_x^2 & \hat\beta_x\hat\beta_y & \hat\beta_x\hat\beta_z \\ \hat\beta_x\hat\beta_y & \hat\beta_y^2 & \hat\beta_y\hat\beta_z \\ \hat\beta_x\hat\beta_z & \hat\beta_y\hat\beta_z & \hat\beta_z^2\end{pmatrix}\begin{pmatrix}\hat\beta_x^2 & \hat\beta_x\hat\beta_y & \hat\beta_x\hat\beta_z \\ \hat\beta_x\hat\beta_y & \hat\beta_y^2 & \hat\beta_y\hat\beta_z \\ \hat\beta_x\hat\beta_z & \hat\beta_y\hat\beta_z & \hat\beta_z^2\end{pmatrix} = \begin{pmatrix}\hat\beta_x^2 & \hat\beta_x\hat\beta_y & \hat\beta_x\hat\beta_z \\ \hat\beta_x\hat\beta_y & \hat\beta_y^2 & \hat\beta_y\hat\beta_z \\ \hat\beta_x\hat\beta_z & \hat\beta_y\hat\beta_z & \hat\beta_z^2\end{pmatrix} = \mathsf{B}.$$

You should go through the matrix multiplication and verify this equation. Now we have

$$\overleftrightarrow{\Lambda}\overleftrightarrow{\Lambda} = 1 + 2(\gamma-1)\mathsf{B} + (\gamma-1)^2\mathsf{B} = 1 + (\gamma^2-1)\mathsf{B} = 1 + \gamma^2\beta^2\mathsf{B}.$$

This is one of the terms in $\left(\mathsf{\Lambda}^{-1}\mathsf{\Lambda}\right)_{22}$. The other term is

$$-\gamma^2\vec{\beta}\widetilde{\vec{\beta}} = -\gamma^2\beta^2\begin{pmatrix}\hat{\beta}_x\\ \hat{\beta}_y\\ \hat{\beta}_z\end{pmatrix}\begin{pmatrix}\hat{\beta}_x & \hat{\beta}_y & \hat{\beta}_z\end{pmatrix} = -\gamma^2\beta^2\mathsf{B}.$$

Adding this to the previous equation, we get

$$\left(\mathsf{\Lambda}^{-1}\mathsf{\Lambda}\right)_{22} = \mathsf{1} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

This completes the proof that $\mathsf{\Lambda}^{-1}\mathsf{\Lambda} = \mathsf{1}$. You should now show that $\mathsf{\Lambda}\mathsf{\Lambda}^{-1} = \mathsf{1}$ as well. ∎

APPENDIX C

Relativistic Photography Formulas

Problems With Solutions

C.1. Verify that Equation (C.1) is a line passing through Q and the pinhole of the camera.

Solution: The equation gives $\mathbf{r}_1(0) = \mathbf{r}_q$, the location of Q and $\mathbf{r}_1(1) = b\hat{\mathbf{e}}_z$, the location of the pinhole. ■

C.2. Derive Equation (C.2).

Solution: From (C.1), we get

$$\mathbf{r}_{q'} \equiv \mathbf{r}_1(t_{q'}) = (1 - t_{q'})\mathbf{r}_q + t_{q'}b\hat{\mathbf{e}}_z \iff \mathbf{r}_{q'} - b\hat{\mathbf{e}}_z = \mathbf{r}_q - b\hat{\mathbf{e}}_z - t_{q'}\mathbf{r}_q + t_{q'}b\hat{\mathbf{e}}_z$$

or

$$\mathbf{r}_{q'} - b\hat{\mathbf{e}}_z = (1 - t_{q'})(\mathbf{r}_q - b\hat{\mathbf{e}}_z).$$

I can now take the absolute value of both sides and note that $|1 - t_{q'}| = 1 - t_{q'}$ because $1 - t_{q'} > 0$. This leads to Equation (C.2). ■

C.3. Substitute (C.1), (C.2), and (C.4) in (C.5) to show that

$$t_{p'} = 1 - \frac{d|\mathbf{r}_q - b\hat{\mathbf{e}}_z|}{(\mathbf{r}_p - b\hat{\mathbf{e}}_z) \cdot (\mathbf{r}_q - b\hat{\mathbf{e}}_z)}.$$

Solution: Rewrite Equation (C.5) as

$$[(1 - t_{p'})\mathbf{r}_p + t_{p'}b\hat{\mathbf{e}}_z - (1 - t_{q'})\mathbf{r}_q - t_{q'}b\hat{\mathbf{e}}_z] \cdot (\mathbf{r}_q - b\hat{\mathbf{e}}_z) = 0.$$

Adding and subtracting $b\hat{\mathbf{e}}_z$, we can re-express this equation as

$$[(\mathbf{r}_p - b\hat{\mathbf{e}}_z)(1 - t_{p'}) - (\mathbf{r}_q - b\hat{\mathbf{e}}_z)(1 - t_{q'})] \cdot (\mathbf{r}_q - b\hat{\mathbf{e}}_z) = 0,$$

or

$$(1 - t_{p'})(\mathbf{r}_p - b\hat{\mathbf{e}}_z) \cdot (\mathbf{r}_q - b\hat{\mathbf{e}}_z) - \underbrace{(1 - t_{q'})}_{=d/|\mathbf{r}_q - b\hat{\mathbf{e}}_z|} |\mathbf{r}_q - b\hat{\mathbf{e}}_z|^2 = 0,$$

or

$$(1 - t_{p'})(\mathbf{r}_p - b\hat{\mathbf{e}}_z) \cdot (\mathbf{r}_q - b\hat{\mathbf{e}}_z) = d|\mathbf{r}_q - b\hat{\mathbf{e}}_z|,$$

which yields

$$1 - t_{p'} = \frac{d|\mathbf{r}_q - b\hat{\mathbf{e}}_z|}{(\mathbf{r}_p - b\hat{\mathbf{e}}_z) \cdot (\mathbf{r}_q - b\hat{\mathbf{e}}_z)}.$$

■

C.4. Derive Equation (C.7).

Solution: Add and subtract $b\hat{\mathbf{e}}_z$:

$$\begin{aligned}
\mathbf{s} = \mathbf{r}_2(t_{p'}) - \mathbf{r}_1(t_{q'}) &= (1 - t_{p'})\mathbf{r}_p + t_{p'}b\hat{\mathbf{e}}_z - (1 - t_{q'})\mathbf{r}_q - t_{q'}b\hat{\mathbf{e}}_z \\
&= (1 - t_{p'})\mathbf{r}_p + t_{p'}b\hat{\mathbf{e}}_z \underbrace{-b\hat{\mathbf{e}}_z + b\hat{\mathbf{e}}_z}_{=0} -(1 - t_{q'})\mathbf{r}_q - t_{q'}b\hat{\mathbf{e}}_z \\
&= (1 - t_{p'})(\mathbf{r}_p - b\hat{\mathbf{e}}_z) - (1 - t_{q'})(\mathbf{r}_q - b\hat{\mathbf{e}}_z).
\end{aligned}$$

Now use (C.2) and (C.6) to get the final answer. ■

C.5. Derive Equations (C.8) and (C.9).

Solution: When Q is on the x-axis, then $\mathbf{r}_q - b\hat{\mathbf{e}}_z = \langle x_q, 0, -b\rangle$, and with $\mathbf{r}_p = \langle x, y, z\rangle$, we get

$$|\mathbf{r}_q - b\hat{\mathbf{e}}_z| = \sqrt{x_q^2 + b^2}, \quad \text{and} \quad (\mathbf{r}_p - b\hat{\mathbf{e}}_z) \cdot (\mathbf{r}_q - b\hat{\mathbf{e}}_z) = xx_q - b(z - b).$$

Plugging these in (C.2) and (C.6) yields (C.8). Plugging them in (C.7) gives

$$\mathbf{s} = \frac{d\sqrt{x_q^2 + b^2}}{xx_q - b(z - b)}\langle x, y, z - b\rangle - \frac{d}{\sqrt{x_q^2 + b^2}}\langle x_q, 0, -b\rangle,$$

whose components are expressed in (C.9). ■

C.6. Derive Equation (C.12) from Equation (C.11).

Solution: The only thing you need to remember is that x' is the Lorentz transform of the x coordinate of the event whose time t is the negative of the distance between the location of the event and the pinhole. Thus,

$$t = -|\mathbf{r}_p - b\hat{\mathbf{e}}_z| = -\sqrt{x^2 + y^2 + (z - b)^2}.$$

■

C.7. In this problem you'll find x'_q in a different way.

(a) If the event of the explosion is at $(x_q, 0)$ in O, what is it in O'?

(b) What distance does Q travel in O' before C' takes the picture of the object?

(c) From (a) and (b) determine where Q is according to O' when C' takes a picture and show that $x'_q = x_q/\gamma$.

Solution: (a) Assume that the object is approaching the origin from negative values of x. Thus, $x_q < 0$, and the initial location of the object when the explosion occurs and its time of occurrence are

$$x'_{qi} = \gamma(x_q - 0) = \gamma x_q, \quad t'_q = \gamma(0 - \beta x_q) = -\gamma\beta x_q.$$

(b) C' takes the picture of the object at $t' = 0$. Therefore, the object travels for t'_q before the picture is taken. The distance is $\beta|t'_q| = \gamma\beta^2|x_q|$.

(c) So, the final location, i.e., the location at which the picture is taken is

$$x'_{qf} \equiv x'_q = x'_{qi} + \beta|t'_q| = \gamma x_q + \gamma\beta^2|x_q| = \gamma x_q - \gamma\beta^2 x_q = \frac{x_q}{\gamma}.$$

■