Topic 14: Circuit Analysis

AP Physics C

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Olympiads School

Electric Current

Current

The **electric current** is defined as the rate at which **charge carriers** pass through a point in a circuit:

$$\boxed{I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t}}$$

Expanding the expression:

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{Q}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = (ne)(Av_d)$$

- Q/V is the amount of charge carriers *per volume*, which is just the **charge carrier density** (number of charge carriers per volume) n times the **elementary charge** e
- $\mathrm{d}V/\mathrm{d}t$ is the rate the volume of charges moves through the conductor, give by the wire's cross-section area A times the **drift velocity** v_d of the charge carrier

Current Through the Conductor

Combining the terms:

$$\boxed{I = \frac{\mathrm{d}Q}{\mathrm{d}t} = neAv_d}$$

Quantity	Symbol	SI Unit
Current	I	A
Charge carrier density	n	$1/\mathrm{m}^3$
Elementary charge	e	C
Cross-section area of the conductor	A	m^2
Drift velocity of the charge carriers	v_d	m/s

The calculation for the charge carrier density n requires some additional thoughts.

Charge Carrier Density

Calculating the charge carrier density in a *metal* conductor involves some physical information about the metal:

1. Divide the metal's density ρ by its molar mass M to find the *number of moles of atoms per unit volume*
2. Multiply by Avogadro's number $N_A = 6.0221 \times 10^{23}\ /\mathrm{mol}$ to find *number of atoms per unit volume*
3. Multiply by the number of free electrons per atom k for that particular metal

Charge Carrier Density

Collecting all the terms from the last slide, we have:

$$\boxed{n = \frac{\rho k N_A}{M}}$$

Quantity	Symbol	SI Unit
Charge carrier density	n	$1/\mathrm{m}^3$
Density of material	ρ	kg/m^3
Free electrons per atom	k	
Avogadro's number	N_A	$1/\mathrm{mol}$
Molar mass	M	kg/mol

For copper, $M = 63.54 \times 10^{-3}\,\mathrm{kg}/\mathrm{mol}$, $\rho = 8.96 \times 10^{3}\,\mathrm{kg}/\mathrm{m}^3$, $k = 1$ and therefore $n = 8.5 \times 10^{28}\,/\mathrm{m}^3$. The drift velocity is in the order of $\approx 1\,\mathrm{mm}/\mathrm{s}$.

Current

Another alternate description of the electric current is to express it in terms of the **current density** J, with a unit of of *ampère per meters squared* ($\mathrm{A/m^2}$).

$$\boxed{I(t) = J(t)A}$$

It is obvious from the previous expression that the current density is the product of the charge carrier density, elementary charge, and the drift velocity:

$$\boxed{J = nev_d}$$

Current Through the Conductor

Combining the terms:

$$\boxed{I = \frac{\mathrm{d}Q}{\mathrm{d}t} = neAv_d}$$

Quantity	**Symbol**	**SI Unit**
Current	I	A
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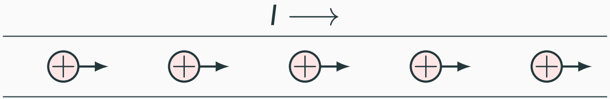
Cross-section area of the conductor	A	m^2
Drift velocity of the charge carriers	v_d	m/s

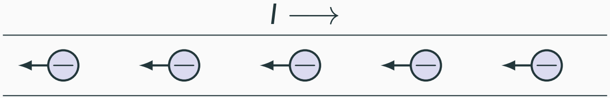
The calculation for the charge carrier density n requires some additional thoughts.

Electric Current: Conventional vs. Electron Flow

The flow of electric current assumes the flow of *positive* charges. We call this the **conventional current**:



In a conducting wire, however, negatively charged electrons flow in the opposite direction. We call this the **electron current**:



Resistors

Resistivity and Electric Field

The resistivity of a material is proportional to the electric field and current density:

$$\boxed{\boldsymbol{E} = \rho \boldsymbol{J}} \quad \text{or} \quad \boxed{\rho = \left|\frac{E}{J}\right|}$$

Quantity	Symbol	SI Unit
Electric field	$\boldsymbol{E}$	N/C
Current density	$\boldsymbol{J}$	$\mathrm{A/m^2}$
Resistivity	ρ	$\Omega \cdot \mathrm{m}$

- In a conductor, the electrons are free to move, and the electric field tend to be weak, and the resistivity is low.
- In an insulator, electrons cannot move easily, therefore the electric field are generally strong, and the resistivity is high.

Resistance of a Conductor

The resistance of a conductor is proportional to the resistivity ρ and its length L, and inversely proportional to the cross-sectional area A:

$$\boxed{R = \rho \frac{L}{A}}$$

Quantity	Symbol	SI Unit
Resistance	R	Ω
Resistivity	ρ	$\Omega \cdot \mathrm{m}$
Length of conductor	L	m
Cross-sectional area	A	m^2

Resistance of a Conductor

$$\boxed{R = \rho\frac{L}{A}}$$

Gauge	Diameter (mm)	R/L ($10^{-3}\,\Omega/\mathrm{m}$)
0	9.35	0.31
10	2.59	2.20
14	1.63	8.54
18	1.02	21.90
22	0.64	51.70

Material	Resistivity ρ ($\Omega \cdot \mathrm{m}$)
silver	1.6×10^{-8}
copper	1.7×10^{-8}
aluminum	2.7×10^{-8}
tungsten	5.6×10^{-8}
Nichrome	100×10^{-8}
carbon	3500×10^{-8}
germanium	0.46
glass	10^{10} to 10^{14}

Ohm’s Law

Ohm's Law

The electric potential difference V across a "load" (resistor) equals the product of the current I through the load and the resistance R of the load.

$$\boxed{V = IR}$$

Quantity	Symbol	SI Unit
Potential difference	V	V
Current	I	A
Resistance	R	Ω

A resistor is considered "ohmic" if it obeys Ohm's law

Power Dissipated by a Resistor

Power is the rate at which work W is done, and from electrostatics, the change in electric potential energy ΔE_q is proportional to the amount of charge q and the voltage V. This gives a very simple expression for power through a resistor:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}E_q}{\mathrm{d}t} = \frac{\mathrm{d}(qV)}{\mathrm{d}t} = \left(\frac{\mathrm{d}q}{\mathrm{d}t}\right) V \rightarrow \boxed{P = IV}$$

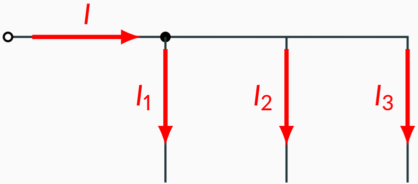
Combining Ohm's law with the above equation gives two additional expressions for power through a resistor:

$$\boxed{P = \frac{V^2}{R}} \quad \boxed{P = I^2 R}$$

Kirchhoff’s Laws

Kirchhoff’s Current Law

The electric current that flows into any junction in an electric circuit must be equal to the current which flows out.



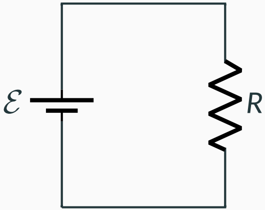
In the example on the left, with I going into the junction, and I_1, I_2 and I_4 coming out, the current law says that

$$I = I_1 + I_2 + I_3$$

Basically, it means that there cannot be any accumulation of charges anywhere in the circuit. The law is a consequence of conservation of energy.

Kirchhoff's Voltage Law

The voltage changes around any closed loop in the circuit must sum to zero, no matter what path you take through an electric circuit.

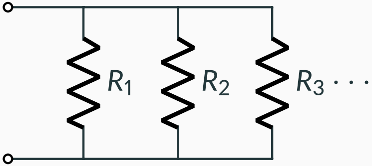


Assume that the current flows clockwise and we draw a clockwise loop, we get

$$\mathcal{E} - V_R = 0 \quad \rightarrow \quad \mathcal{E} - IR = 0$$

Resistors in Circuits

Resistors in Parallel



The total current is the current through all the resistors, which can be rewritten in terms of voltage and resistance using Ohm's law:

$$I = I_1 + I_2 + I_3 \cdots = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} \cdots$$

Since $V_1 = V_2 = V_3 = \cdots = V$ from the voltage law, we can re-write as

$$I = \frac{V}{R_p} = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \cdots\right)$$

Equivalent Resistance of Resistors in Parallel

The reciprocal of the equivalent resistance for resistors connected in parallel is the sum of the inverses of the individual resistances.

$$\boxed{\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}}$$

Quantity	**Symbol**	**SI Unit**
Equivalent resistance in parallel	R_p	Ω
Resistance of individual loads	$R_{1,2,3,\cdots,N}$	Ω

Resistors in Series

The analysis for resistors in series is similar (but easier). From the current law, the current through each resistor is the same:

$$I_1 = I_2 = I_3 = \cdots = I$$

And the total voltage drop across all resistor is therefore:

$$V = V_1 + V_2 + V_3 + \cdots = I\left(R_1 + R_2 + R_3 + \cdots\right)$$

Equivalent Resistance: Resistors in Series

The equivalent resistance of loads is the sum of the resistances of the individual loads.

$$\boxed{R_s = \sum_{i=1}^{N} R_i}$$

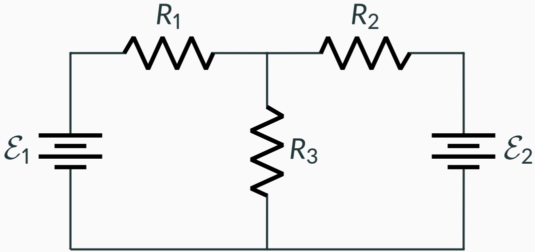
Quantity	Symbol	SI Unit
Equivalent resistance in series	R_s	Ω
Resistance of individual loads	$R_{1,2,3,\cdots,N}$	Ω

Tips for Solving "Simple" Circuit Problems

1. Identify groups of resistors that are in parallel or in series, and find their equivalent resistance.
2. Gradually reduce the entire circuit to one voltage source and one resistor.
3. Using Ohm's law, find the current out of the battery.
4. Using Kirchhoff's laws, find the current through each of the resistors.

Circuits Aren't Always Simple

Some of these problems require you to solve a system of linear equations. The following is a simple example with two voltage sources:



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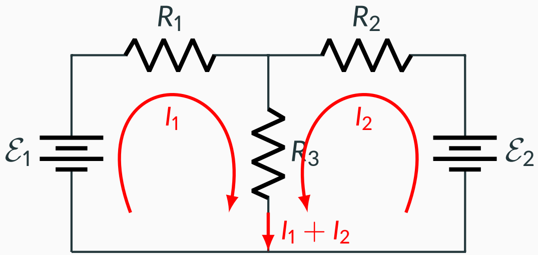
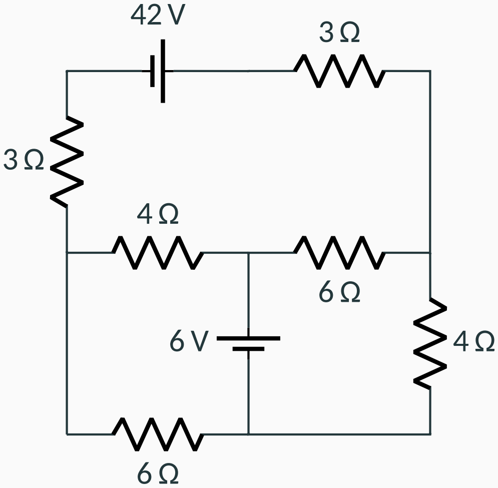


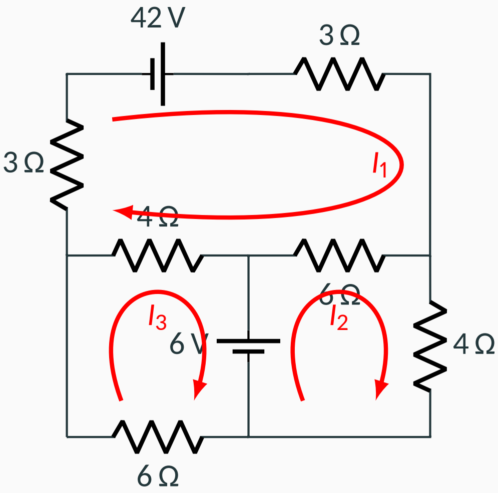
In this case, we have to draw two loops of current.

As Difficult As It Gets



- To solve this problem, we define a few “loops” around the circuit: one on top, one on bottom left, and one on bottom right.

As Difficult As It Gets



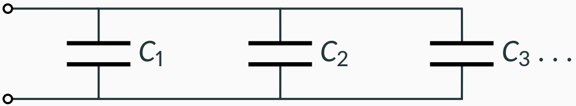
- To solve this problem, we define a few "loops" around the circuit: one on top, one on bottom left, and one on bottom right.
- Apply the voltage law in the loops. For example, in the lower left:

$$4(I_1 - I_3) - 6 - 6I_3 = 0$$

- Solve the linear system to find the current. If the current that you worked out is negative, it means that you have the direction wrong.

Capacitors in Circuit

Capacitors in Parallel



From the voltage law, we know that the voltage across all the capacitors are the same, i.e. $V_1 = V_2 = V_3 - \cdots = V$. We can express the total charge Q_{tot} stored across all the capacitors in terms of capacitance and this common voltage V:

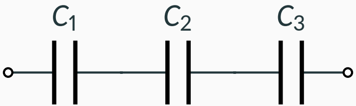
$$Q_{\text{tot}} = Q_1 + Q_2 + Q_3 + \cdots = C_1 V + C_2 V + C_3 V + \cdots$$

Factoring out V from each term gives us the equivalent capacitance:

$$\boxed{C_p = \sum_i C_i}$$

Capacitors in Series

Likewise, we can do a similar analysis to capacitors connected in series.



The total voltage across these capacitors are the sum of the voltages across each of them, i.e. $V_{\text{tot}} = V_1 + V_2 + V_3 + \cdots$

The charge stored on all the capacitors must be the same! The total voltage in terms of capacitance and charge is:

$$V_{\text{tot}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \cdots$$

Equivalent Capacitance in Series

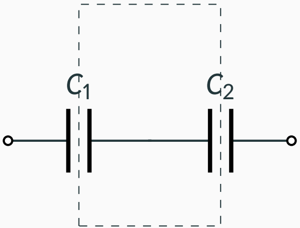
The inverse of the equivalent capacitance for N capacitors connected in series is the sum of the inverses of the individual capacitance.

$$\boxed{\frac{1}{C_s} = \sum_i \frac{1}{C_i}}$$

Make sure we don't confuse ourselves with resistors.

How Do We Know That Charges Are The Same?

It's simple to show that the charges across all the capacitors are the same

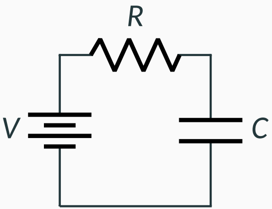


The capacitor plates and the wire connecting them are really one piece of conductor. There is nowhere for the charges to leave the conductor, therefore when charges are accumulating on C_1, C_2 must also have the same charge because of conservation of charges.

R-C Circuits

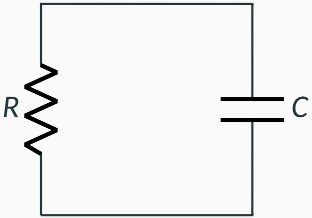
Circuits with Resistors and Capacitors

An **RC circuit** is one that has both resistors and capacitors. The simplest form is a resistor and capacitor connected in series, and then connect to a voltage source.



Because of the nature of capacitors, the current through the circuit will not be steady as were the case with only resistors.

Discharging a Capacitor



The analysis starts with something simpler. There is no voltage source, and the capacitor is already charged to $V_c = Q_{\text{tot}}/C$. What happens when the current begin to flow?

As current starts to flow, the charge on the capacitor decreases. Over time the current decreases, until the capacitor is fully discharged, and current stops flowing.

Now we apply the voltage law for the circuit. In this case, as the current flow in the circuit *decreases* the total charge in the capacitor, $I = -\mathrm{d}Q/\mathrm{d}t$, while the voltage across a capacitor is $V_c = Q/C$:

$$V_c - IR = 0 \quad \rightarrow \quad \frac{Q}{C} + R\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

Discharging a Capacitor

Separating the variable gives the first-order linear differential equation:

$$\frac{\mathrm{d}Q}{Q} = \frac{-\mathrm{d}t}{RC}$$

which we can now integrate and “exponentiate”:

$$\int \frac{\mathrm{d}Q}{Q} = \int \frac{-\mathrm{d}t}{RC} \;\rightarrow\; \ln Q = \frac{-t}{RC} + K \;\rightarrow\; Q = e^K e^{-t/RC}$$

The constant of integration K is the initial charge on the capacitor Q_{tot}:

$$e^K = Q_{\text{tot}}$$

Discharging a Capacitor

The expression of charge across the capacitor is time-dependent:

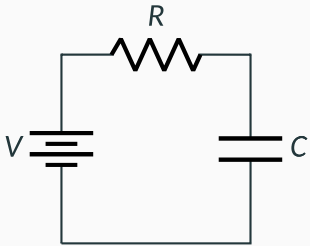
$$\boxed{Q(t) = Q_0 e^{-t/\tau}}$$

where $Q_0 = Q_{\text{tot}}$ is the initial charge on the capacitor, and $\tau = RC$ is called the **time constant**. Taking the time derivative of $Q(t)$ gives us the current through the circuit:

$$\boxed{I(t) = \frac{dQ}{dt} = I_0 e^{-t/\tau}}$$

where the initially current at $t = 0$ is given by $I_0 = Q_{\text{tot}}/\tau$.

Charging a Capacitor



In charging up the capacitor, we go back to our original circuit, and apply the voltage law, then substitute the expression for current and voltage across the capacitor:

$$V - R\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{Q}{C} = 0$$

Again, separating variables, and integrating, we get:

$$\int \frac{\mathrm{d}Q}{CV - Q} = \int \frac{\mathrm{d}t}{RC} \quad \rightarrow \quad -\ln(CV - Q) = \frac{t}{RC} + K$$

Charging a Capacitor

"Exponentiating" both sides, we have

$$CV - Q = e^K e^{-t/RC}$$

To find the constant of integration K, we note that at $t = 0$, the charge across the capacitor is 0, and we get

$$e^K = CV = Q_{\text{tot}}$$

which is the charge stored in the capacitor at the end. Substitute this back into the equation:

$$\boxed{Q(t) = Q_{\text{tot}}(1 - e^{-t/RC})}$$

Capacitors

$$\boxed{Q(t) = Q_{\text{tot}}\left(1 - e^{-t/\tau}\right)}$$

Charging a capacitor has the same time constant $\tau = RC$ as during discharge. We can also differentiate to find the current through the circuit; it is identical to the equation for discharge:

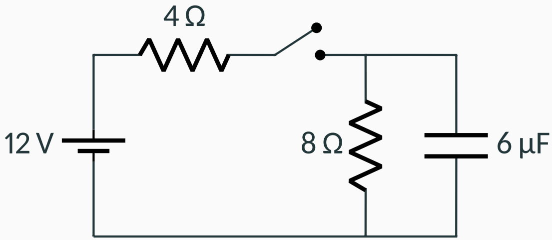
$$\boxed{I(t) = \frac{dQ}{dt} = I_0 e^{-t/\tau}}$$

where the initial current is given by $I_o = Q_{\text{tot}}/\tau = V/R$. This makes sense because $V_C(t = 0) = 0$, and all of the energy must be dissipated through the resistor. Similarly at $I(t = \infty) = 0$.

Two Small Notes

1. When a capacitor is uncharged, there is no voltage across the plate, it acts like a short circuit.
2. When a capacitor is charged, there is a voltage across it, but no current flows *through* it. Effectively it acts like an open circuit.

A Slightly More Difficult Problem



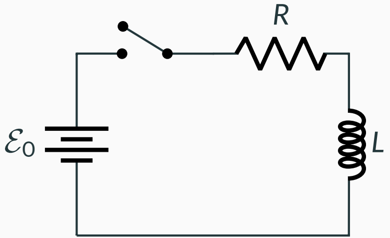
Example: The capacitor in the circuit is initially uncharged. Find the current through the battery

1. Immediately after the switch is closed
2. A long time after the switch is closed

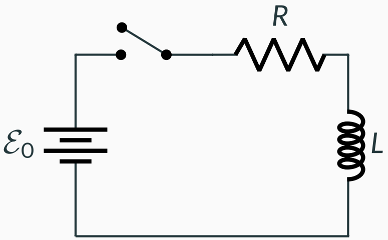
LR Circuits

Circuits with Inductors

- Coils and solenoids in circuits are known as “inductors” and have large self inductance L
- Self inductance prevents currents rising and falling instantaneously
- A basic circuit containing a resistor and an inductor is called an ***LR* circuit**:



Analyzing LR Circuits



Applying Kirchhoff's voltage law:

$$\mathcal{E}_0 - IR - L\frac{\mathrm{d}I}{\mathrm{d}t} = 0$$

Following the same procedure as charging a capacitor, the time-dependent current is found to be:

$$I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-t/\tau}\right)$$

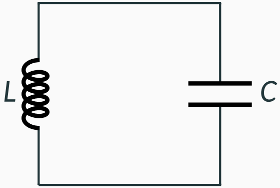
Where the time constant τ is defined as:

$$\tau = \frac{L}{R}$$

LC Circuit

LC Circuit

The final type of circuit in AP Physics is the LC circuit. In its simplest form, the circuit has an inductor and capacitor connected in series:



We apply the Kirchhoff's voltage law:

$$-V_L - V_C = 0 \quad \rightarrow \quad L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{Q}{C} = 0$$

LC Circuits

Since both terms are continuously differentiable, we can differentiate both sides of the equation, which gives:

$$L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{1}{C}\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

In fact, the above equation a second-order ordinary differential equation with constant coefficients.

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{1}{LC} I = 0$$

The solution to such an equation is the simple harmonic motion.

$$I(t) = I_0 \sin(\omega t + \varphi) \quad \text{where} \quad \omega = \frac{1}{\sqrt{LC}}$$