# Topic 9: Universal Gravitation

Advanced Placement Physics C

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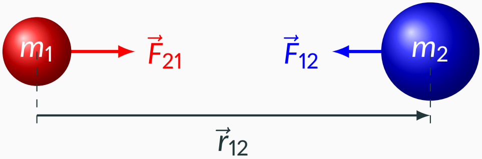
Fall 2021

Olympiads School

# Gravitational Force

# Law of Universal Gravitation

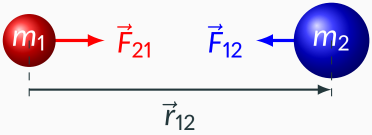



In classical mechanics, **gravity** is a mutually attractive force between all massive objects, given by the law of universal gravitation:

$$\boxed{\vec{F}_{12} = -G\frac{m_1 m_2}{\left|\vec{r}_{12}\right|^2}\hat{r}_{12}}$$

where $G = 6.674 \times 10^{-11}\,\mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2$ is the **universal gravitational constant**, $r = \left|\vec{r}_{12}\right|$ is the distance between the centers of the masses, and $\hat{r}_{12} = \vec{r}_{12}/\left|\vec{r}_{12}\right|$ is the unit vector pointing in the direction from $m_1$ to $m_2$.

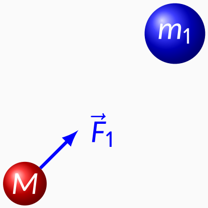
# Law of Universal Gravitation




- If $m_1$ exerts a gravitational force $\vec{F}_{12}$ on $m_2$, then $m_2$ likewise also exerts a force of $\vec{F}_{21} = -\vec{F}_{12}$ on $m_1$. The two forces are equal in magnitude and opposite in direction (third law of motion).
- $m_1$ and $m_2$ are *point masses* that do not occupy any space
- The (more familiar) scalar form is often used as well:

$$\boxed{F_g = G\frac{m_1 m_2}{r^2}}$$

# More Than One Mass




For a mass that is subjected to the influence of multiple discrete point masses $m_i$, the total gravitational force that $M$ experiences is the vector sum of all the forces $\vec{F}_i$:

$$\boxed{\vec{F} = \sum_i \vec{F}_i = GM\left(\sum_{i=1}^{N} \frac{m_i}{r_i^2}\hat{r}_i\right)}$$

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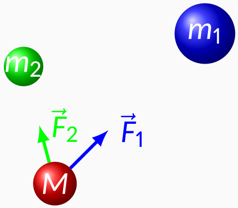


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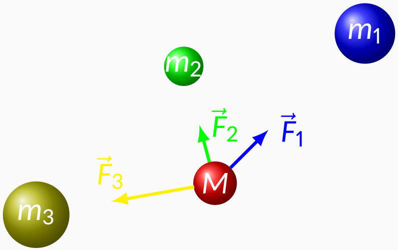


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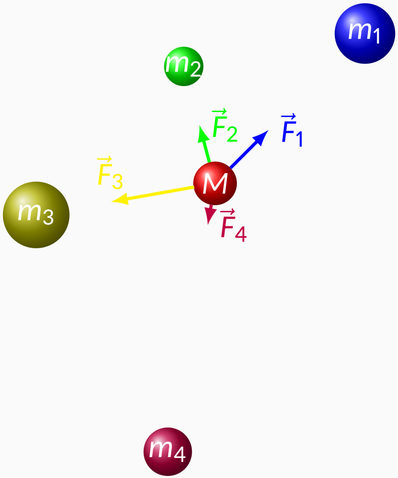


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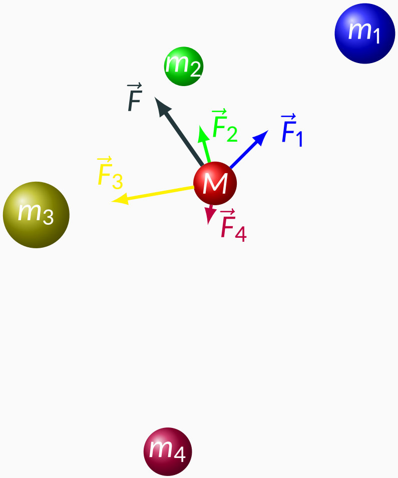


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# Continuous Distribution of Mass

At the limit $N \to \infty$, the summation becomes an integral, and can now be used to describe the gravitational force from objects with *spatial extend* i.e. masses that take up space (e.g. a continuous distribution of mass):

$$\boxed{\vec{F} = \int \mathrm{d}\vec{F} = GM \int \frac{\mathrm{d}m}{r^2}\hat{r}}$$

Objects that are symmetrically spherical (e.g. planets are stars in our solar system) can be treated as point masses, and integration can be avoided. However, this is not necessarily the case for some celestial objects.

# Gravitational Field

# Gravitational Field

We generally describe the gravitational force (weight) as:

$$\boxed{\vec{F}_g = m\vec{g}}$$

To find $\vec{g}$, we group the variables in the law of universal gravitation:

$$\vec{F}_g = \underbrace{\left[-\frac{Gm_1}{|\vec{r}|^2}\hat{r}\right]}_{=\vec{g}} m_2 = m_2\vec{g}$$

The vector field function $\vec{g}$ is known as the **acceleration due to gravity** in kinematics, and **gravitational field** in field theory.

# Gravitational Field

On/near the surface of Earth, we can use

$$m_1 = m_E = 5.972 \times 10^{24}\,\mathrm{kg}$$
$$r = r_E = 6.371 \times 10^{6}\,\mathrm{m}$$

to compute the commonly known value of

$$g \approx 9.81\,\mathrm{m/s^2}$$
$$g \approx 9.81\,\mathrm{N/kg}$$

both units are equivalent

# Gravitational Field

The **gravitational field** $\vec{g}$ generated by point mass $m$ shows how it influences the gravitational forces on other masses:

$$\boxed{g(m, \vec{r}) = -\frac{Gm}{|\vec{r}|^2}\hat{r}}$$

| Quantity | Symbol | SI Unit |
|---|---|---|
| Gravitational field | $\vec{g}$ | N / kg |
| Universal gravitational constant | $G$ | $\mathrm{N} \cdot \mathrm{m}^2 / \mathrm{kg}^2$ |
| Source mass | $m$ | kg |
| Distance from source mass | $\lvert\vec{r}\rvert$ | m |
| Outward radial unit vector from source | $\hat{r}$ | N/A |

The negative sign indicates that *direction* of the gravitational field is toward $m$.

# More Than One Mass

When there are multiple point masses present, the total gravitational field at any position $\vec{r}$ is the vector sum of all the forces $\vec{F}_i$:

$$\boxed{\vec{g} = \sum_i \vec{g}_i = G\left(\sum_{i=1}^{N} \frac{m_i}{r_i^2}\hat{r}_i\right)}$$

At the limit $N \to \infty$, the summation becomes an integral, and can now be used to describe the gravitational field generated by objects with *spatial extend*:

$$\boxed{\vec{g} = \int \mathrm{d}\vec{g} = G \int \frac{\mathrm{d}m}{r^2}\hat{r}}$$

This integral may be difficult to compute, if the geometry is complicated.

# Relating Gravitational Field & Gravitational Force

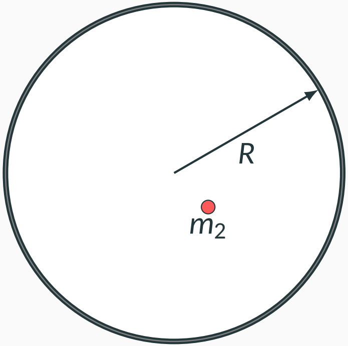
$\vec{g}$ itself doesn't *do* anything unless/until another mass $m$ enters the field. Then, $m$ experiences a gravitational force $\vec{F}_g$ proportional to $m$ and $\vec{g}$, regardless of how the field is created:

$$\boxed{\vec{F}_g = m\vec{g}}$$

| **Quantity** | **Symbol** | **SI Unit** |
|---|---|---|
| Gravitational force on a mass | $\vec{F}_g$ | N |
| Mass inside the gravitational field | $m$ | kg |
| Gravitational field | $\vec{g}$ | N/kg |

Note: A point mass is not affected by the gravitational field that itself generates.

# What If You Are Inside Another Mass?

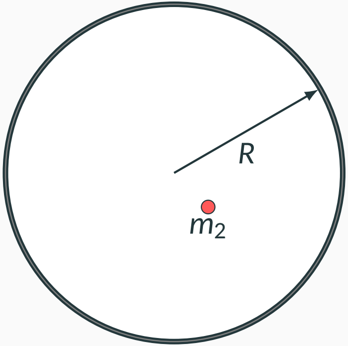



Newton used the **shell theorem** to show that if a mass $m_2$ is *inside* a spherical shell of mass $m_1$, the gravitational force that it experiences is *zero*.

$$\boxed{\vec{F}_g = \begin{cases} \vec{0} & \text{if } r < R \\ -Gm_1m_2/r^2\hat{r} & \text{otherwise} \end{cases}}$$

It also means that gravitational field is also *zero*

# What If You Are Inside Another Mass?

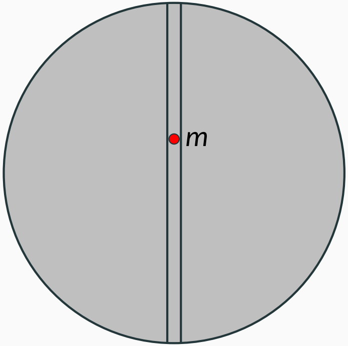



That $\vec{g}_{\text{inside}} = \vec{0}$ can be calculated by:

- Integrating the fields created by infinitesimal mass elements *dm* at any point inside the shell, or
- Using **Gauss's law** for gravity, similar to finding the electric field inside a charged conducting sphere:

$$\oint \vec{g} \cdot \mathrm{d}\vec{A} = -4\pi G M_{\text{encl}}$$

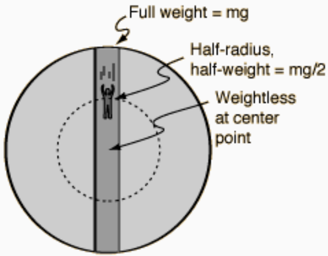
# What If You Are Inside Another Mass?




Suppose you could drill a hole through the Earth and then jump into it. How long would it take you to emerge on the other side of the Earth?

To calculate this, we need to know how the gravitational force changes as you fall through Earth.

# Falling Toward the Center of Earth




As you fall through Earth, we can separate the part of Earth that is “above” you, and the part that is “below” you

- The part that is “above” you is like the spherical shell, and does not contribute to the gravitational field, and therefore does not exert any force
- The part that is “below” you gets smaller as you fall toward the center

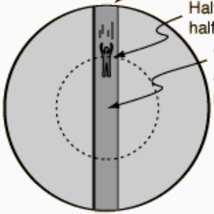
# Falling Toward the Center of Earth

Assuming that Earth's density is uniform, and neglecting air resistance and other factors, the value of $g$ as the person falls through Earth ($r < R$) is given by finding how much mass is still "below" the person, $M(r)$:

$$g(r) = \frac{GM(r)}{r^2} \quad M(r) = \frac{4}{3}\rho\pi r^3 \quad \rho = \frac{3M_E}{4\pi r_E^3}$$

where $M_E$ is the mass of Earth, $r_E$ is the radius of Earth, $\rho$ is the (constant) density, and $r$ is the distance from Earth's center. Then $M(r)$ is the amount of mass "below" the person as he/she falls toward the center.

# Falling Toward the Center of Earth



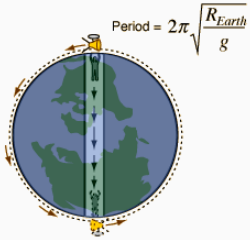

The gravitational field strength inside this hypothetical Earth is a linear function of distance $r$ from the center:

$$g(r) = \frac{GM_E r}{r_E^3} = \left(\frac{r}{r_E}\right) g_0$$

where $g_0 = 9.81\,\mathrm{N/kg}$ is the field strength at the surface. At the center ($r = 0$), $g = 0$. The gravitational force is:

$$F_g(r) = -\underbrace{\left[\frac{mg_0}{r_E}\right]}_{k} r$$

# Falling Toward the Center of Earth




A satellite at the Earth's radius would have the same period as one falling through the Earth.

The gravitational force has the same form as Hooke's law: it is proportional to displacement from the center, but in the opposite direction:

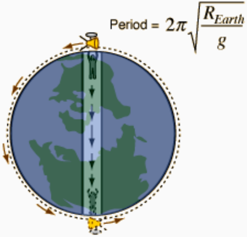
$$F_g(r) = -kr$$

The motion is a simple harmonic motion. The traveler will oscillate through Earth with a period of:

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{r_E}{g_0}}$$

For Earth, $T = 5068\,\text{s}$. The traveler would pop up on the opposite side every 42 min.

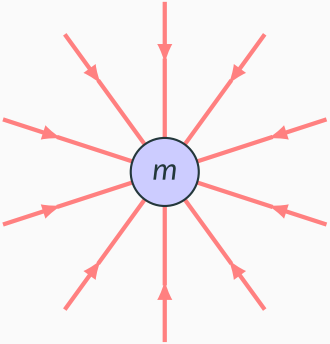
# Falling Toward the Center of Earth




A satellite at the Earth's radius would have the same period as one falling through the Earth.

Since simple harmonic motion is a projection of a uniform circular motion, if a satellite is in a circular orbit just above the surface, and passes overhead just above the traveler as he/she popped up out of the hole. The period of such an orbit would be the same as oscillating traveler.

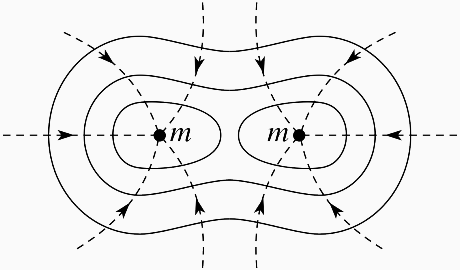
# Gravitational Field Lines




- The direction of $\vec{g}$ is toward the center of the object that created it
- Field lines do not tell the intensity (i.e. magnitude) of $\vec{g}$, only the direction

# Gravitational Field Lines

When there are multiple masses, the total gravitational field (dotted line) is the vector sum of all the individual fields.




The solid lines are called **equipotential lines**, where the potential energy is constant. Equipotential lines are perpendicular to gravitational field lines.

# Gravitational Potential Energy

# Gravitational Potential Energy

**Gravitational potential energy** is found by integrating the work equation and using the law of universal gravitation:

$$W = \int \vec{F}_g \cdot \mathrm{d}\vec{r} = -\int_{r_1}^{r_2} \frac{Gm_1 m_2}{r^2}\hat{r} \cdot \mathrm{d}\vec{r}$$

$$= -\int_{r_1}^{r_2} \frac{Gm_1 m_2}{r^2} dr = \left.\frac{Gm_1 m_2}{r}\right|_{r_1}^{r_2} = -\Delta U_g$$

where

$$\boxed{U_g = -G\frac{m_1 m_2}{r}}$$

- $U_g$ is the work required to move two objects from $r$ to $\infty$
- $U_g = 0$ at $r = \infty$ and *decrease* as $r$ decreases

# Relating Gravitational Potential Energy to Force

The fundamental theorem of calculus shows that gravitational force ($\vec{F}_g$) is the negative gradient of the gravitational potential energy ($U_g$):

$$\vec{F}_g = -\nabla U_g = -\frac{\partial U_g}{\partial r}\hat{r}$$

The direction of $\vec{F}_g$ always points from high to low potential energy

- A free-falling object is always decreasing in $U_g$
- "Steepest descent": the direction of $\vec{F}_g$ is the shortest path to decrease $U_g$
- Objects traveling perpendicular to $\vec{F}_g$ has constant $U_g$

# Relating $U_g$, $\vec{F}_g$ and $\vec{g}$

Knowing that $\vec{F}_g$ and $\vec{g}$ only differ by a constant (mass $m$), we can also relate gravitational field to potential energy by the gradient operator:

$$\vec{g} = -\nabla V_g = -\frac{\partial V_g}{\partial r}\hat{r} \quad \text{where} \quad V_g = \frac{U_g}{m}$$

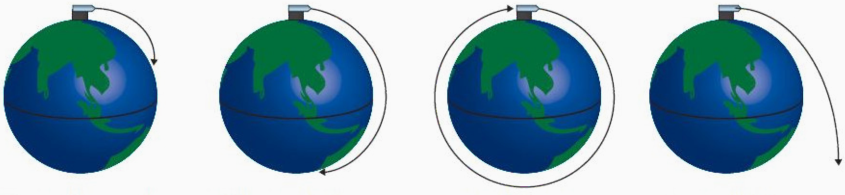
We already know that the direction of $\vec{g}$ is the same as $\vec{F}_g$, i.e.

- The direction of $\vec{g}$ is the shortest path to decrease $U_g$
- Objects traveling perpendicular to $\vec{g}$ has constant $U_g$
- $V_g$ is called the **gravitational potential** but it is rarely used

# Orbits

# Newton's Thought Experiment

In *Treatise of the System of the World*, the third book in *Principia*, Newton presented this thought experiment:



- How fast does the cannonball have to travel before it goes around Earth without falling? (i.e. goes into orbit)
- How fast does the cannonball have to travel before it never comes back?

# Relating Gravitational and Centripetal Force

Assuming a small mass $m$ in circular orbit around a much larger mass $M$. The required centripetal force is supplied by the gravitational force:

$$F_g = F_c \quad \longrightarrow \quad \frac{GMm}{r^2} = \frac{mv^2}{r}$$

Solving for $v$, we obtain the **orbital velocity** $v_{\text{orbit}}$, which does not depend on the mass of the small object in orbit:

$$\boxed{v_{\text{orbit}} = \sqrt{\frac{GM}{r}}}$$

This equation is only applicable for perfectly circular orbits.

# Escape Speed

An object can leave the surface of Earth at any speed. But when all the kinetic energy of that object is converted to gravitational potential energy, it will return back to the surface of the earth. There is, however, a *minimum* velocity at which the object *would not* fall back to Earth.

# Escape Speed

The calculation for escape is a simple exercise in conservation of energy, since gravity is a conservative force, i.e.:

$$K + U_g = K' + U_g'$$

- Initial gravitational potential energy at the surface is:

$$U_g = -\frac{GMm}{r_i}$$

- The final gravitational potential energy is at the other side of the universe ($r_f = \infty$), where $U_g' = 0$. At this point, the object has *escaped* the gravitational pull of the planet/star
- The minimum kinetic energy at $r = \infty$ is $K' = 0$

# Escape Speed from Circular Orbits

Set $K$ to equal to $-U_g$:

$$\frac{1}{2}mv_i^2 = \frac{GMm}{r_i}$$

We can then solve for the initial speed $v_i = v_{\text{esc}}$ (**escape speed** or **escape velocity**):

$$\boxed{v_{\text{esc}} = \sqrt{\frac{2GM}{r_i}}}$$

where $r_i$ is the initial distance from the center of the planet/star. There is a simple relationship between orbital speed and escape speed:

$$v_{\text{esc}} = \sqrt{2}v_{\text{orbit}}$$

# Example Problem

**Example:** Determine the escape velocity and energy for a $1.60 \times 10^4$ kg rocket leaving the surface of Earth.

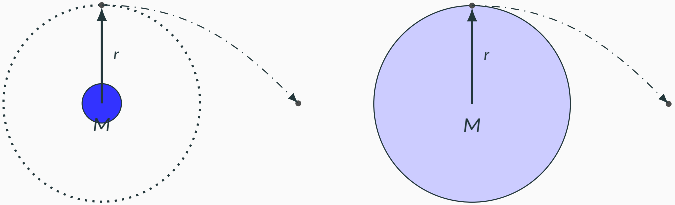
# Example Problem

**Example:** Determine the escape velocity and energy for a $1.60 \times 10^4$ kg rocket leaving the surface of Earth.

Note: The equation for the escape speed is based on the object have a *constant* mass, which is *not* the case for a rocket going into space.

# What if I'm not escaping from the surface?

Both objects have the same escape velocity:

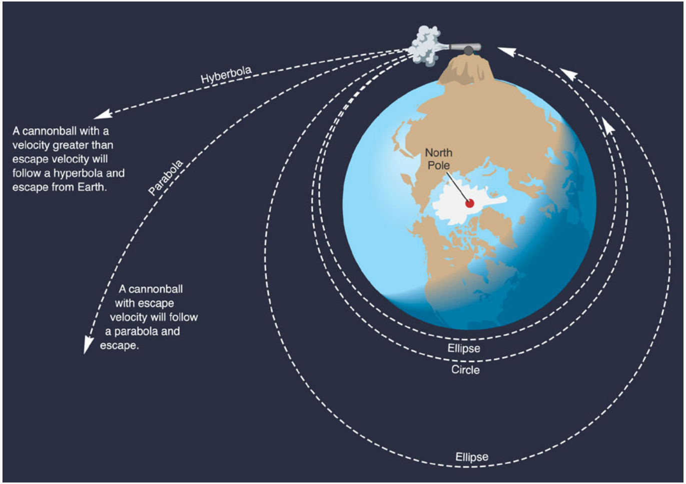



The difference is that the object in orbit (left) already has orbital speed $v_{\mathrm{orbit}}$, so escaping from that orbit requires only an additional speed of

$$\Delta v = v_{\mathrm{esc}} - v_{\mathrm{orbit}} = (\sqrt{2} - 1)v_{\mathrm{orbit}}$$

- What if $v_{\mathrm{orbit}} < v < v_{\mathrm{esc}}$?
- What if $v < v_{\mathrm{orbit}}$?

# Non-Circular Orbits



# Orbital Energies

We can obtain the **orbital kinetic energy** in a perfectly circular orbit by using the orbital speed in our expression of kinetic energy:

$$K_{\text{orbit}} = \frac{1}{2}mv_{\text{orbit}}^2 = \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2 = \boxed{\frac{GMm}{2r}}$$

We already have an expression for **gravitational potential energy**:

$$U_g = -\frac{GMm}{r} = -2K_{\text{orbit}}$$

The **total orbital energy** is the sum of $K$ and $U_g$:

$$E_T = K_{\text{orbit}} + U_g = -\frac{GMm}{2r} = -K_{\text{orbit}}$$

# Orbital Mechanics

# Orbital Mechanics

We turn our attention to applying the law of universal gravitation to the orbital motion of planets and stars in our solar system.

# Properties of Gravitational Force

Two properties of gravity are crucial to understanding of orbital mechanics:

1. Gravity is a *conservative force*, in that
   - The total mechanical energy of objects under gravity is constant
   - Work done by gravity converts gravitational potential energy $U_g$ into kinetic energy $K$; work against gravity converts $K$ into $U_g$
2. Gravity is a *central force*, in that
   - Gravitational force $\vec{F}_g$ is always in the $-\hat{r}$ direction, i.e. $\vec{F} \times \vec{r} = \vec{0}$
   - Therefore gravity doesn't generate any torque
   - And therefore angular momentum $\vec{L}$ is constant

These two properties are true regardless of the shape of the orbit, and even for objects that are not in orbit at all!

# Kepler's Laws of Planetary Motion

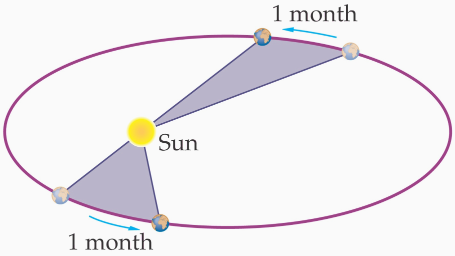
Johannes Kepler (1571–1630) formulated the **laws of planetary motion** between 1609 to 1619, by interpreting planetary motion data from his teacher, Tycho Brahe. It is an improvement over the heliocentric theory of Nicolaus Copernicus. Expressed in modern language:

1. **Law of ellipses:** The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. **Law of equal areas:** A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time
3. **Law of periods:** The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

(For anyone who is interested, there is a handout with the proofs of Kepler's laws using Newton's laws of motion.)

# Kepler’s Second Law: Law of Equal Areas

**Law of Equal Areas: A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time**




The second law of planetary motion is the easiest to proof, by applying the conservation of angular momentum $\vec{L} = m(\vec{r} \times \vec{v})$ (gravity is a central force).

# Kepler's Second Law: Law of Equal Areas

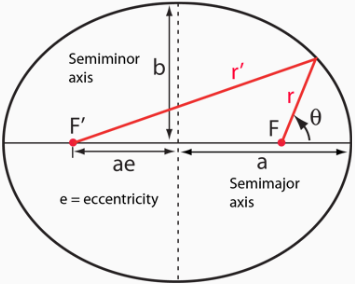
The rate of change of the area ($dA/dt$) swept out by a planet (called the **areal velocity**) is given by:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{L}{2m} = \text{constant}$$

The rate a planet sweeps out the area in orbit is its angular momentum around the sun divided by twice its mass.

# Kepler's First Law: Law of Ellipses

Proofing Kepler's first law requires some understanding the ellipse. If the law is true, then orbital motion must agree with the equations of an ellipse.



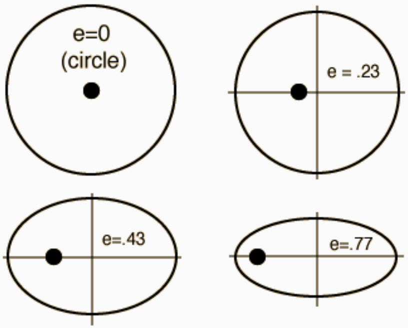

- $r' + r = 2a$
- The area of the ellipse is $A = \pi ab$
- The relationship between $r$ and $\theta$ given by:

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \quad \text{where} \quad 0 \leq e < 1$$

- when $e = 0$ it's a circle: $a = b = r$
- When $e = 1$ it's no longer an ellipse

# Kepler’s First Law: Law of Ellipses

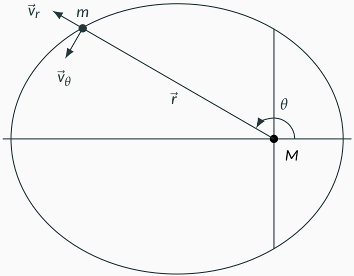
Most planets in the solar system have very small eccentricity, so their orbits are fairly close to being circular, but comets are much more eccentric




| **Object** | *e* |
|---|---|
| Mercury | 0.206 |
| Venus | 0.0068 |
| Earth | 0.0167 |
| Mars | 0.0934 |
| Jupiter | 0.0485 |
| Saturn | 0.0556 |
| Uranus | 0.0472 |
| Neptune | 0.0086 |
| Pluto | 0.25 |
| Halley’s Comet | 0.9671 |
| Comet Hale-Bopp | 0.9951 |
| Comet Ikeya-Seki | 0.9999 |

# Kepler's First Law: Law of Ellipses

As $m$ orbits around $M$, there are two velocity components: **radial velocity** $\vec{v}_r$ and **angular velocity** $\vec{v}_\theta$.




- $\vec{v}_\theta$ means a centripetal acceleration toward $M$
- Changes in $\vec{v}_r$ (i.e. accceleration in the radial direction) also means a force along $\hat{r}$
- Both components of acceleration are due entirely to gravitational force toward $M$
- Applying second law of motion gives a complicated (at least for students new to the concept) ordinary differential equation.

A full description for solving the differential equation is presented in the accompanied handout for anyone interested.

# Kepler's First Law: Law of Ellipses

The solution to the ODE is the expression for $r(\theta)$, with eccentricity $e$ determined by a constant $B$ based on initial condition (how the planet is formed):

$$r = \left[\frac{L^2}{GMm^2}\right]\frac{1}{1+e\cos\theta} \quad \text{where} \quad e = \frac{BL^2}{GMm^2}$$

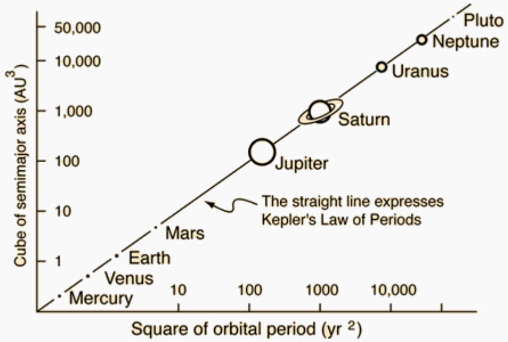
The semi-major axis is the average value between the minimum and maximum values of $r$:

$$a = \frac{1}{2}(r_{\min} + r_{\max}) = \left[\frac{L^2}{GMm^2}\right]\frac{1}{1-e^2}$$

We can rearrange the terms to see that this is the equation for an ellipse.

# Kepler’s Third Law: The Law of Periods

**Law of Periods: The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.**



# Kepler's Third Law: The Law of Periods

The area swept by the planet through one orbital period is the areal velocity (constant!) integrated by time, from $t = 0$ to $t = T$:

$$A = \int \mathrm{d}A = \int_0^T \frac{\mathrm{d}A}{\mathrm{d}t}\mathrm{d}t = \frac{L}{2m}\int_0^T \mathrm{d}t = \frac{L}{2m}T$$

But this area is an ellipse, given by the equation based on $a$ (semi-major axis), $b = a\sqrt{1-e^2}$ (semi-minor axis):

$$A = \pi ab = \pi a^2\sqrt{1-e^2}$$

Equating two equations above and squaring both sides give this expression:

$$T^2 = \frac{m^2}{L^2}4\pi^2 a^4(1-e^2)$$

# Kepler's Third Law: The Law of Periods

But we also (from proving the first law) have:

$$a(1-e^2) = \frac{L^2}{GMm^2}$$

Substituting this expression into the equation for the period, and after some simple algebra, we end up with this expression:

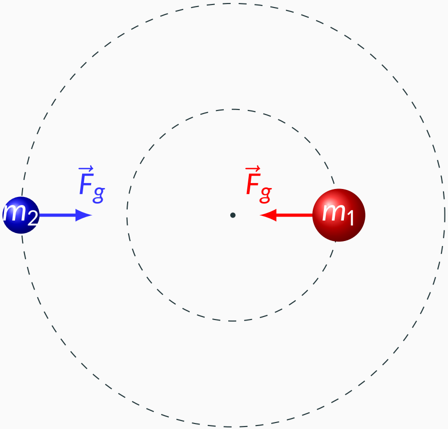
$$\boxed{T^2 = \left[\frac{4\pi^2}{GM}\right] a^3}$$

# Reality of Orbital Motion

As always, nothing is as simple as is first seems

- Most AP Problems will be circular instead of elliptical, but you must know the nature of gravitational force (conservative, central)
- The analysis on the slides shown assumes a small mass $m$ orbiting around a large mass $M$. In reality:
  - Just as planets experience a gravitational force by the Sun, the Sun experiences a gravitational force from the planets
  - The smaller mass $m$ does not actually orbit about the center of $M$, but rather, the *center of mass between M and m*
  - Especially important when the two objects orbiting each other has similar masses (e.g. a binary star system)

# Binary System




In a binary star system, two stars orbit around their center of mass. Both have the same period, and the gravitational force provides the centripetal force, but this time, the distance to the center of motion is empty space.