Class 5: Center of Mass

Advanced Placement Physics C

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Olympiads School

Center of Mass

Finding an object's center of mass is important, because

- The laws of motion are formulated by treating an objects as point masses (for real-life objects, we let the forces apply to the center of mass)
- Objects can have *rotational* motion in addition to *translational* motion as well (we will examine that a bit more in a very-important topic later)

Start with a Definition

The **center of mass** (“CM”) is the *weighted average of the masses in a system.* The “system” may be:

- A collection of individual particles
- A continuous distribution of mass with constant density. In this case, CM is also the geometric center (**centroid**) of the object
- A continuous distribution of mass with varying density
- If the masses are inside of a gravitational field, then the CM is also its **center of gravity** (“CG”)

Simple Example

Start with a very simple example: two equal masses m along the x-axis, located at x_1 and x_1. What is the center of mass of the system?

The answer is simple: the half way point between them:

$$x_{\text{cm}} = \frac{x_1 + x_2}{2}$$

Multiply both numerator and denominator by mass m (for generalization later), the equation becomes:

$$x_{\text{cm}} = \frac{mx_1 + mx_2}{2m}$$

Slightly More Challenging

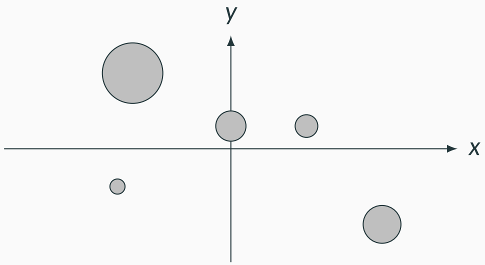
What if one of the masses are increased to $2m$? This is still not a difficult problem; you can still *guess* the right answer without knowing the equation for center of mass.

The answer is still simple. The CM is no longer half way between the two masses, but now $\frac{1}{3}$ the total distance from the larger masses. We can show using a weighted average:

$$x_{\mathrm{cm}} = \frac{m x_1 + (2m) x_2}{m + 2m}$$

Complicating Things Further

The weighted average concept can now be applied to cases when there are masses in 2D or 3D:



An Equation Helps

The center of mass is defined for discrete number of masses with the weighted average:

$$\boxed{\vec{x}_{\text{cm}} = \frac{\sum \vec{x}_i m_i}{\sum m_i}}$$

Quantity	Symbol	SI Unit
Position of center of mass (vector)	$\vec{x}_{\text{cm}}$	m
Position of point mass i (vector)	$\vec{x}_i$	m
Point mass i	m_i	kg

In components:

$$x_{\text{cm}} = \frac{\sum x_i m_i}{\sum m_i} \qquad y_{\text{cm}} = \frac{\sum y_i m_i}{\sum m_i} \qquad z_{\text{cm}} = \frac{\sum z_i m_i}{\sum m_i}$$

An Example

Example: Consider the following masses and their coordinates which make up a "discrete mass" rigid body"

$$
\begin{aligned}
m_1 &= 5.0\,\mathrm{kg} & \vec{x}_1 &= 3\hat{\imath} - 2\hat{k} \\
m_2 &= 10.0\,\mathrm{kg} & \vec{x}_2 &= -4\hat{\imath} + 2\hat{\jmath} + 7\hat{k} \\
m_3 &= 1.0\,\mathrm{kg} & \vec{x}_3 &= 10\hat{\imath} - 17\hat{\jmath} + 10\hat{k}
\end{aligned}
$$

What are the coordinates for the center of mass of this system?

Continuous Mass Distribution

When the number of masses approaches infinity, this becomes a continuous distribution of mass. Taking the limit of masses $N \to \infty$ gives the integral form of our equation:

$$\boxed{\vec{x}_{\mathrm{cm}} = \frac{\int \vec{x}\mathrm{d}m}{\int \mathrm{d}m}}$$

What is the infinitesimal mass dm then?

Densities

Linear density (for 1D problems)

$$\gamma = \frac{\mathrm{d}m}{\mathrm{d}L} \quad \rightarrow \quad \mathrm{d}m = \gamma \mathrm{d}L$$

Surface area density (for 2D problems)

$$\sigma = \frac{\mathrm{d}m}{\mathrm{d}A} \quad \rightarrow \quad \mathrm{d}m = \sigma \mathrm{d}A$$

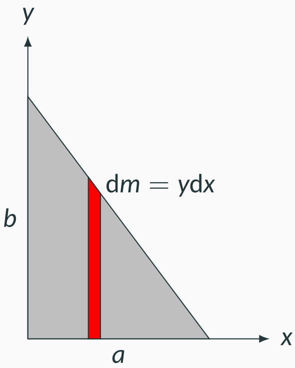
Volume density (for 3D problems)

$$\rho = \frac{\mathrm{d}m}{\mathrm{d}V} \quad \rightarrow \quad \mathrm{d}m = \rho \mathrm{d}V$$

The densities do not have to be constant

An Example with Integrals

Example 2: A triangular plate is placed in a Cartesian coordinate system with two of its edges along the x and y-axis. The length of the edges along the axes are a and b respectively. Assuming that the surface area density σ is uniform, determine the coordinate of its center of mass.

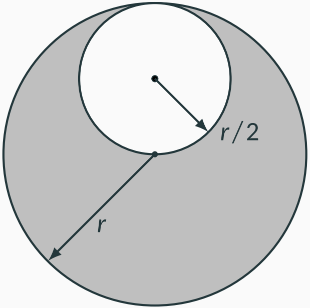


Symmetry

- Any plane of symmetry, mirror line, axis of rotation, point of inversion *must* contain the center of mass.
- Caveat: only works if the density distribution is also symmetric
- Again: if density is uniform, CM is also geometric center (centroid)

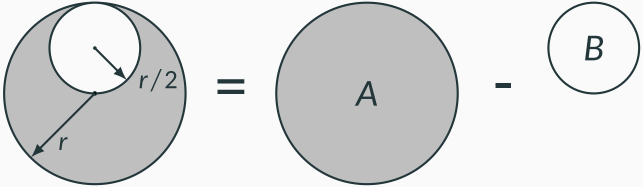
“Negative Mass”

- Where there is a “hole” in the geometry, treat it as having negative mass density $-\sigma$ in that region.
- Negative masses don’t exist, so this is really just a trick.
- **Example:** What is the center of mass of this shape?



Negative Mass Example

- This is how we would think of it:



- Let the origin of the coordinate system to located at the center of A
- Based on symmetry: $x_{\text{cm}} = 0$; only have to find y-coordinate.

$$y_{\text{cm}} = \frac{\sum y_i m_i}{\sum m_i} = \frac{m_A(0) + m_B(r/2)}{m_A + m_B} = \frac{-\sigma\pi\,(r/2)^2\,(r/2)}{\sigma\pi r^2 - \sigma\pi\,(r/2)^2} = \frac{-r}{6}$$

Velocity of the Center of Mass

Take time derivative of the equation for $\vec{x}_{\mathrm{cm}}$ to get the velocity at the center of mass:

$$\vec{v}_{\mathrm{cm}} = \frac{\mathrm{d}\vec{x}_{\mathrm{cm}}}{\mathrm{d}t} = \frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \vec{x}\mathrm{d}m\right) = \frac{1}{m}\int \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\mathrm{d}m = \frac{\int \vec{v}\mathrm{d}m}{m}$$

Or, in the form that is familiar, the velocity at the center of mass is the weighted sum of the velocities of the distribution of mass:

$$\vec{v}_{\mathrm{cm}} = \frac{\int \vec{v}\mathrm{d}m}{m}$$

Velocity and Momentum

We can also rearrange the equation for the velocity of the center of mass to relate it to momentum, because the term $\int \vec{v}\mathrm{d}m$ is the net momentum of the mass distribution p_{net}:

$$\vec{v}_{\text{cm}} = \frac{\int \vec{v}\mathrm{d}m}{m} \quad \longrightarrow \quad \vec{p}_{\text{net}} = m\vec{v}_{\text{cm}}$$

During a collision, there is no change in the net momentum[1], the center of mass will continue to move at the same velocity before/after the collision, as if the collision never occurred.

[1]Because there are are no external forces

Acceleration of the Center of Mass

Finding the rate of change of the net momentum (i.e applying the 2nd law of motion to this distribution of masses):

$$\frac{\mathrm{d}\vec{p}_{\text{net}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\vec{v}_{\text{cm}})$$

If the system mass is constant, then this equation reduces to:

$$\frac{\mathrm{d}\vec{p}_{\text{net}}}{\mathrm{d}t} = m\frac{\mathrm{d}\vec{v}_{\text{cm}}}{\mathrm{d}t} \quad \longrightarrow \quad \boxed{\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}}}$$

We can see that when a net force is applied to an object, the object's acceleration is evaluated at the center of mass.