Topic 11: Electrostatics

Advanced Placement Physics C

Dr. Timothy Leung

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Olympiads School

Electrostatic Force

Review: The Charges Are

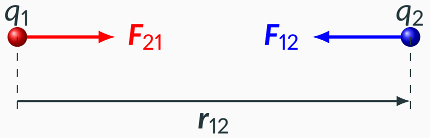
We should already know a bit about charge particles:

- A **proton** carries a **positive** charge
- An **electron** carries a **negative** charge
- A *net charge* of an object means an excess of protons or electrons
- Similar charges are repel; opposite charges attract

We start with electrostatics:

- Charges that are not moving relative to one another

Coulomb's Law for Electrostatic Force



The **electrostatic force** (or **coulomb force**) is a mutually repulsive/attractive force between all charged objects. The force that charge q_1 exerts on q_2 is given by **Coulomb's law**:

$$\boxed{\boldsymbol{F}_{12} = \frac{k q_1 q_2}{|\boldsymbol{r}_{12}|^2} \hat{\boldsymbol{r}}_{12}}$$

Coulomb's Law for Electrostatic Force

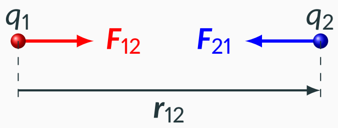
$$\boxed{\boldsymbol{F}_{12} = \frac{kq_1q_2}{|\boldsymbol{r}_{12}|^2}\hat{\boldsymbol{r}}_{12}}$$

Quantity	Symbol	SI Unit
Electrostatic force	$\boldsymbol{F}_{12}$	N
Coulomb's constant (electrostatic constant)	k	$\mathrm{N \cdot m^2 / C^2}$
Point charges 1 and 2	q_1, q_2	C
Distance between point charges	$\lvert\boldsymbol{r}_{12}\rvert$	m
Unit vector of direction between point charges	$\hat{\boldsymbol{r}}_{12}$	

Coulomb's constant $k = \dfrac{1}{4\pi\epsilon_0} = 8.99 \times 10^9\ \mathrm{N \cdot m^2 / C^2}$ where

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2 / N \cdot m^2}$ is called the "permittivity of free space"

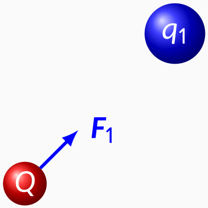
Coulomb's Law for Electrostatic Force



- If q_1 exerts an electrostatic force $\boldsymbol{F}_{12}$ on q_2, then q_2 likewise exerts a force of $\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$ on q_1. The two forces are equal in magnitude and opposite in direction (3rd law of motion).
- q_1 and q_2 are assumed to be *point charges* that do not occupy any space
- The (more familiar) scalar form is often used as well:

$$\boxed{F_q = \frac{kq_1q_2}{r^2}}$$

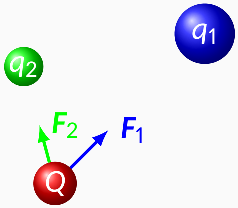
More Than One Charge



For a charge Q that is subjected to the influence of multiple discrete point charges q_i, the total electrostatic force that Q experiences is the vector sum of all the forces $\boldsymbol{F}_i$:

$$\boxed{\boldsymbol{F} = \sum_i \boldsymbol{F}_i = kQ\left(\sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{\boldsymbol{r}}_i\right)}$$

More Than One Charge



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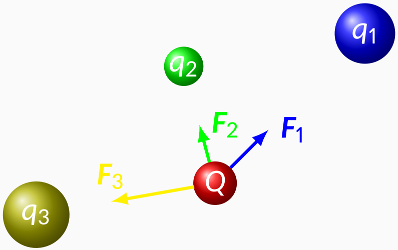

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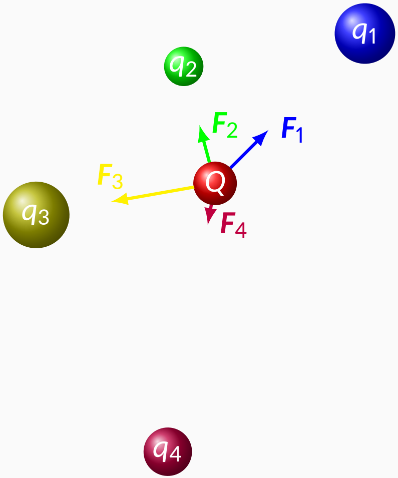

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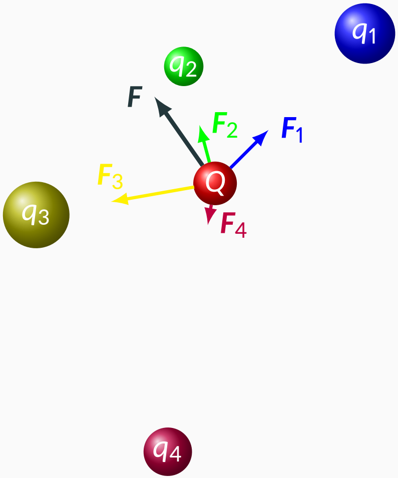

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Continuous Distribution of Charges

As $N \to \infty$, the summation becomes an integral, and can now be used to describe the force from charges with *spatial extend* i.e. charges that take up physical space (e.g. a continuous distribution of charges):

$$\boxed{\boldsymbol{F} = \int \mathrm{d}\boldsymbol{F} = kQ \int \frac{\mathrm{d}q}{r^2}\hat{\boldsymbol{r}}}$$

Electric Field

Electric Field

The expression for **electric field** is obtained by repeating the same procedure as with gravitational field, by grouping the variables in Coulomb's law:

$$F_q = \underbrace{\left[\frac{kq_1}{|\boldsymbol{r}_{12}|^2}\hat{\boldsymbol{r}}\right]}_{\boldsymbol{E}} q_2$$

The electric field $\boldsymbol{E}$ created by q_1 is a vector function (called a **vector field**) that shows how it influences other charged particles around it.

Electric Field Near a Point Charge

The electric field a distance r away from a point charge q is given by:

$$\boxed{\boldsymbol{E}(q, \boldsymbol{r}) = \frac{kq}{|\boldsymbol{r}|^2}\hat{\boldsymbol{r}}}$$

Quantity	Symbol	SI Unit
Electric field intensity	$\boldsymbol{E}$	N / C
Coulomb's constant	k	$\mathrm{N} \cdot \mathrm{m}^2 / \mathrm{C}^2$
Source charge	q	C
Distance from source charge	$\lvert\boldsymbol{r}\rvert$	m
Outward unit vector from point source	$\hat{\boldsymbol{r}}$	

The direction of $\boldsymbol{E}$ is radially outward from a positive point charge and radially inward towards a negative charge.

More Than One Charge

When multiple point charges are present, the total electric field at any position $\boldsymbol{r}$ is the vector sum of all the fields $\boldsymbol{E}_i$:

$$\boxed{\boldsymbol{E} = \sum_i \boldsymbol{E}_i = k\left(\sum_i \frac{q_i}{r_i^2}\hat{\boldsymbol{r}}_i\right)}$$

More Than One Charge

As $N \to \infty$, the summation becomes an integral, and can now be used to describe the electric field generated by charges with *spatial extend*:

$$\boxed{\boldsymbol{E} = \int \mathrm{d}\boldsymbol{E} = k \int \frac{\mathrm{d}q}{r^2}\hat{\boldsymbol{r}}}$$

This integral may be difficult to compute if the geometry of is complicated, but in general, there are usually symmetry that can be exploited.

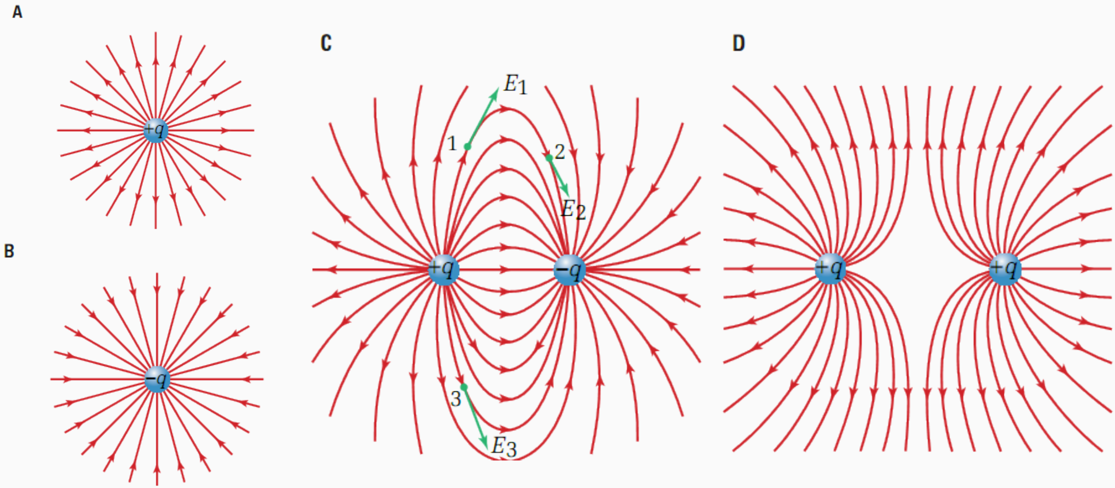
Think Electric Field

$\boldsymbol{E}$ itself *doesn't do anything* until another charge interacts with it. And when there is a charge q, the electrostatic force $\boldsymbol{F}_q$ that the charge experiences is proportional to q and $\boldsymbol{E}$, regardless of how the electric field is generated:

$$\boxed{\boldsymbol{F}_q = q\boldsymbol{E}}$$

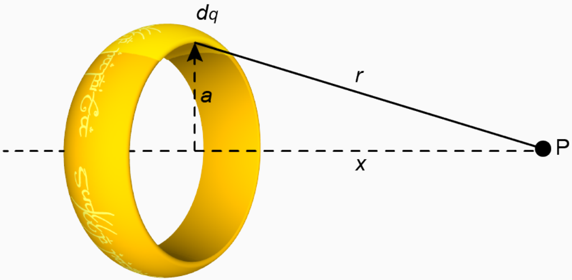
A positive charge in the electric field experiences a electrostatic force $\boldsymbol{F}$ in the same direction as $\boldsymbol{E}$.

Electric Field Lines



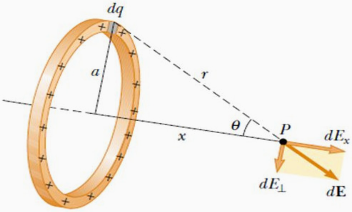
Lord of the Ring Charge

Suppose you have been given *The One Ring To Rule Them All*, and you found out that it is charged! What is its electric field at point P along its axis?



Note that calculating the electric field away from the axis is very difficult.

Electric Field Along Axis of a Ring Charge



- We can separate the electric field d$\mathbf{E}$ (generated by charge dq) into axial (dE_x) and radial (d$E_\perp$) components
- Based on symmetry, d$E_\perp$ doesn't contribute to anything; but dE_x is pretty easy to find:

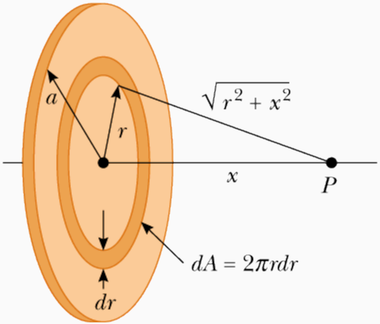
$$\mathrm{d}E_x = \frac{k\mathrm{d}q}{r^2}\cos\theta = \frac{k\mathrm{d}q}{r^2}\frac{x}{r} = \frac{kx\mathrm{d}q}{(x^2+a^2)^{3/2}}$$

Integrating this over all charges dq, we have:

$$E_x = \frac{kx}{(x^2+a^2)^{3/2}}\int \mathrm{d}q = \boxed{\frac{kQx}{(x^2+a^2)^{3/2}}}$$

Electric Field Along Axis of a Uniformly Charged Disk

Let's extend what we know to a disk of radius a and charge density σ



We start with the solution from the ring problem, and replace Q with d$q = 2\pi\sigma a$da:

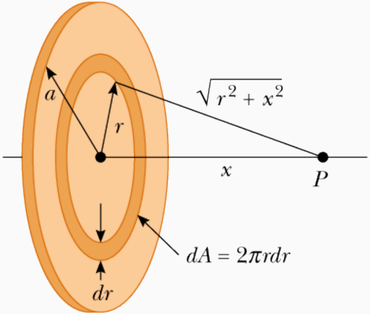
$$\mathrm{d}E_x = \frac{2\pi k x \sigma a}{(x^2 + a^2)^{3/2}}\mathrm{d}a$$

Integrating over the entire disk:

$$E_x = \pi k x \sigma \int \frac{2a}{(x^2 + a^2)^{3/2}}\mathrm{d}a$$

This is not an easy integral!

Eclectic Field Along Axis of a Uniformly Charged Disk



- Luckily for us, the integral is in the form of $\int u^n du$, with $u = x^2 + a^2$ and $n = \frac{-3}{2}$.
- You can find the integral in any math textbook:

$$\boxed{E_x = 2\pi k\sigma \left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)}$$

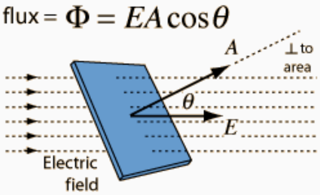
Gauss’s Law

Flux

Flux is an important concept in many disciplines in physics. The flux of a vector quantity $\boldsymbol{X}$ is the amount of that quantity flowing through a surface. In integral form:

$$\Phi = \int \boldsymbol{X} \cdot \mathrm{d}\boldsymbol{A} \quad \text{or} \quad \Phi = \int (\boldsymbol{X} \cdot \hat{\boldsymbol{n}}) \mathrm{d}A$$

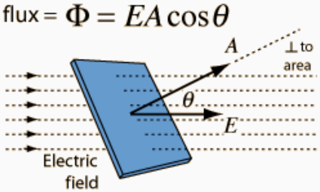
The direction of the infinitesimal area $\mathrm{d}\boldsymbol{A}$ is **outward normal** to the surface.



Flux

Φ can be something physical, like water, or bananas, or something abstract, like electric field (which is what we are interested in). We can compute a flux as long as there is a vector field i.e. $\boldsymbol{X} = \boldsymbol{X}(x, y, z)$. In the case of **electric flux**, the quantity $\boldsymbol{X}$ is just the electric field, i.e.:

$$\Phi_q = \int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A}$$



Electric Flux and Gauss's Law

Gauss's law tells us that if we have a closed surface (think of the surface of a balloon), the total electric flux is very well defined:

$$\boxed{\Phi_q = \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = \frac{Q_{\text{encl}}}{\epsilon_0}}$$

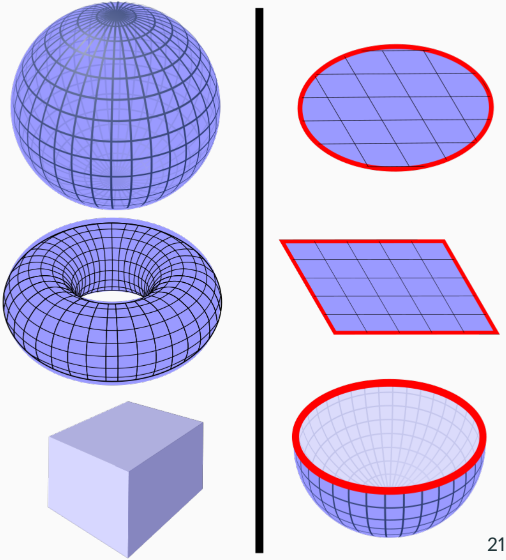
where

- Q_{encl} is the charge enclosed by the surface
- $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2/\mathrm{N} \cdot \mathrm{m}^2$ is the permittivity of free space

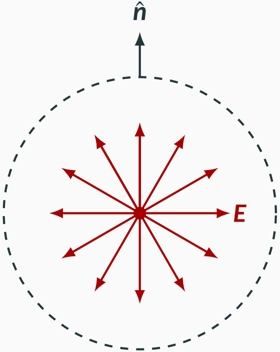
That closed surface is called a **Gaussian surface**

Closed Surfaces

A **closed surface** is one that does not have a boundary, like the sphere, toroid, and cube on the left.



Electric Field from a Positive Point Charge



By symmetry, electric field lines must be radially outward from the charge, so the integral reduces to:

$$\Phi_q = \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = EA = \frac{q}{\epsilon_0}$$

Since area of a sphere is $A = 4\pi r^2$, we recover Coulomb's law and the magnitude of the electric field from a point charge:

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} = \frac{kq}{r^2}$$

Electric Potential & Potential Energy

Electrical Potential Energy

The work done by the electrostatic force is given by:

$$W = \int \boldsymbol{F}_q \cdot \mathrm{d}\boldsymbol{r} = kq_1q_2 \int_{r_1}^{r_2} \frac{1}{r^2}\mathrm{d}r = -\frac{kq_1q_2}{r}\Bigg|_{r_1}^{r_2} = -\Delta U_q$$

where Q_q is defined as the **electric potential energy**:

$$\boxed{U_q = \frac{kq_1q_2}{r}}$$

U_q can be $(+)$ or $(-)$, because charges can be either $(+)$ or $(-)$.

How it Differs from Gravitational Potential Energy

Two positive charges:	Two negative charges:	One positive and one negative charge:
$U_q > 0$	$U_q > 0$	$U_q < 0$

- $U_q > 0$ means positive work is done to bring two charges together from $r = \infty$ to r (both charges of the same sign)
- $U_q < 0$ means negative work (the charges are opposite signs)
- For gravitational potential U_g is always < 0

Electric Potential

When I move an object of mass m against a gravitational force from one point to another, the work that I do is directly proportional to m, i.e. there is a “constant” in that scales with *any* mass, as long as they move between those same two points:

$$U_g = Km$$

In the trivial case (small changes in height, no change in g), this constant is just

$$\frac{U_g}{m} = g\Delta h$$

Electric Potential

This is also true for moving a charged particle q against an electric electric field created by q_s, and the “constant” is called the **electric potential**. For a point charge, it is defined as:

$$\boxed{V = \frac{U_q}{q} = \frac{kq_s}{r}}$$

The unit for electric potential is a *volt* which is *one joule per coulomb*:

$$1\,\mathrm{V} = 1\,\mathrm{J}/\mathrm{C}$$

We can easily the relationship between V and $\boldsymbol{E}$:

$$\boxed{\Delta V = \int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r}}$$

Potential Difference (Voltage)

The change in electric potential is called the **electric potential difference** or **voltage**:

$$\boxed{\Delta V = \frac{\Delta U_q}{q}} \quad \text{and} \quad \boxed{\mathrm{d}V = \frac{\mathrm{d}U_q}{q}}$$

Here, we can relate ΔV to an equation that we knew from Grade 11 Physics, which related to the energy dissipated in a resistor in a circuit ΔU to the voltage drop ΔV:

$$\boxed{\Delta U_q = q\Delta V}$$

Electric potential difference also has the unit *volts* (V)

Getting Those Names Right

Remember that these three scalar quantities, as opposed to electrostatic force $\boldsymbol{F_q}$ and electric field $\boldsymbol{E}$ which are vectors

- Electric potential energy:

$$U = \frac{kq_1q_2}{r}$$

- Electric potential:

$$V = \frac{kq}{r}$$

- Electric potential difference (voltage):

$$\Delta V = \frac{\Delta U_q}{q}$$

Relating U_q, F_q and E

From the fundamental theorem calculus, we can relate electrostatic force ($\boldsymbol{F}_q$) to electric potential energy (U_q) by the gradient operator, and electric field ($\boldsymbol{E}$) to the electric potential (V) the same way:

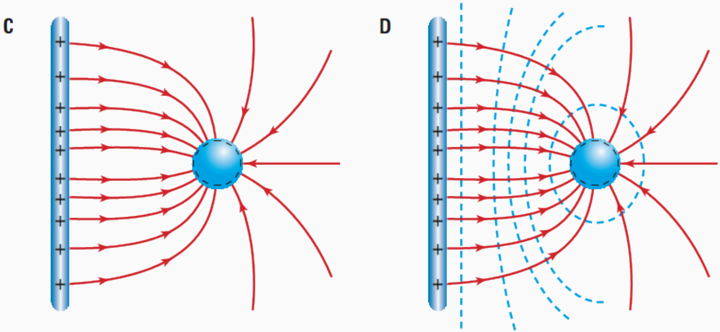
$$\boldsymbol{F}_q = -\nabla U_q = -\frac{\partial U_q}{\partial r}\hat{\boldsymbol{r}} \qquad \boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{\boldsymbol{r}}$$

- Electrostatic force $\boldsymbol{F}_q$ always points from high to low potential energy (steepest descent direction)
- Electric field can also be expressed as the change of electric potential per unit distance, which has the unit

$$1\,\mathrm{N/C} = 1\,\mathrm{V/m}$$

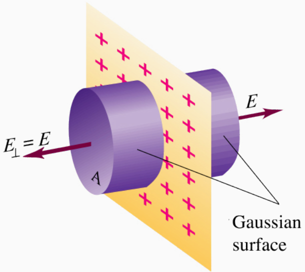
- Electric field is also called "potential gradient"

Equipotential Lines



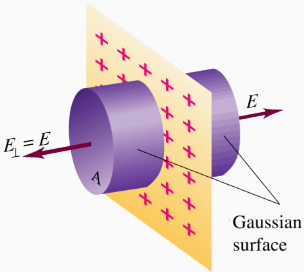
The dotted blue lines are called **equipotential lines**. They are always *perpendicular* to the electric field lines. Charges moving in the direction of the equipotential lines have constant electric potential

Electric Field Near an Infinite Plane of Charge



- Charge density (charge per unit area) σ
- By symmetry, ***E*** must be perpendicular to the plane
- Our Gaussian surface is a cylinder shown in the left with an area A; the height of the cylinder is unimportant
- Nothing "flows out" of the side of the cylinder, only at the ends
- The total flux is $\Phi_q = E(2A)$
- The enclosed charge is $Q_{\text{encl}} = \sigma A$

Electric Field Near an Infinite Plane of Charge



Gauss's law simplifies to:

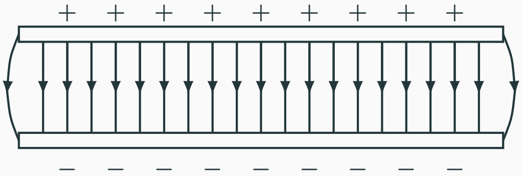
$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\mathrm{encl}}}{\epsilon_0} \quad \rightarrow \quad E(2A) = \frac{\sigma A}{\epsilon_0}$$

Solving for E, we get:

$$\boxed{E = \frac{\sigma}{2\epsilon_0}}$$

- E is a constant
- Independent of distance from the plane
- Both sides of the plane are the same

Electric Field Between Parallel Charged Plates



- Two plates, each producing an electric field pointing in the same direction
- The total electric field is twice the value of *one* infinite plane, pointing from the positively charged plate towards the negatively charged plate

$$\boxed{E = \frac{\sigma}{\epsilon_0}}$$

- ***E*** outside the plates is very low (close to zero), except for fringe effects at the edges of the plates

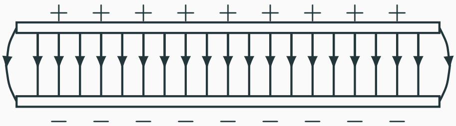
Electric Field and Electric Potential Difference

Recall the relationship between electric field ($\boldsymbol{E}$) and electric potential difference (V):

$$\boldsymbol{E} = -\frac{\partial V}{\partial r}\hat{\boldsymbol{r}}$$

This relationship holds regardless of the charge configuration.

Electric Field and Electric Potential Difference



In the case of two parallel plates, the electric field is uniform, and the relationship simplifies to:

$$\boxed{E = \frac{\Delta V}{d}}$$

Quantity	Symbol	SI Unit
Electric field intensity	E	N/C
Electric potential difference between plates	ΔV	V
Distance between plates	d	m