Class 11

~~Topic 9~~: Universal Gravitation

Advanced Placement Physics C

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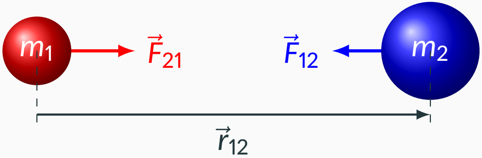
Fall 2021

Olympiads School

Gravitational Force

Law of Universal Gravitation



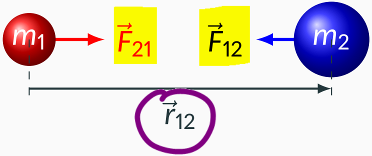
In classical mechanics, **gravity** is a mutually attractive force between all massive objects, given by the law of universal gravitation:

$$\boxed{\vec{F}_{12} = -G\frac{m_1 m_2}{|\vec{r}_{12}|^2}\hat{r}_{12}}$$

$$F_g = \frac{G m_1 m_2}{r^2}$$

where $G = 6.674 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2}$ is the **universal gravitational constant**, $r = |\vec{r}_{12}|$ is the distance between the centers of the masses, and $\hat{r}_{12} = \vec{r}_{12}/|\vec{r}_{12}|$ is the unit vector pointing in the direction from m_1 to m_2.

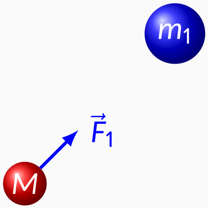
Law of Universal Gravitation



- If m_1 exerts a gravitational force $\vec{F}_{12}$ on m_2, then m_2 likewise also exerts a force of $\vec{F}_{21} = -\vec{F}_{12}$ on m_1. The two forces are equal in magnitude and opposite in direction (third law of motion).
- m_1 and m_2 are *point masses* that do not occupy any space
- The (more familiar) scalar form is often used as well:

$$\boxed{F_g = G\frac{m_1 m_2}{r^2}}$$

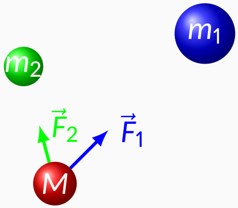
More Than One Mass



For a mass that is subjected to the influence of multiple discrete point masses m_i, the total gravitational force that M experiences is the vector sum of all the forces $\vec{F}_i$:

$$\boxed{\vec{F} = \sum_i \vec{F}_i = GM\left(\sum_{i=1}^{N} \frac{m_i}{r_i^2}\hat{r}_i\right)}$$

More Than One Mass



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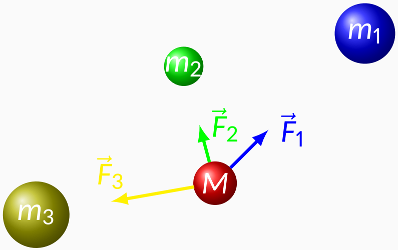

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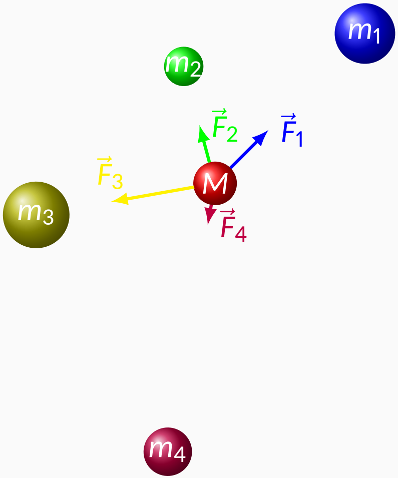

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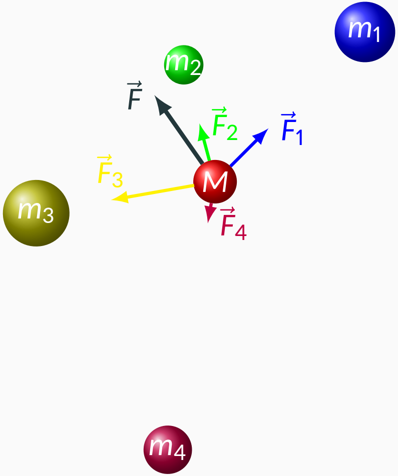

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Continuous Distribution of Mass

At the limit $N \to \infty$, the summation becomes an integral, and can now be used to describe the gravitational force from objects with *spatial extend* i.e. masses that take up space (e.g. a continuous distribution of mass):

$$\boxed{\vec{F} = \int d\vec{F} = \underbrace{GM \int \frac{dm}{r^2}\hat{r}}}$$

3D: $\rho \, dV$

$\rho \, dx \, dy \, dz$

Objects that are symmetrically spherical (e.g. planets are stars in our solar system) can be treated as point masses, and integration can be avoided. However, this is not necessarily the case for some celestial objects.

Gravitational Field

Gravitational Field

We generally describe the gravitational force (weight) as:

$$\boxed{\vec{F}_g = m\vec{g}}$$

To find $\vec{g}$, we group the variables in the law of universal gravitation:

$$\vec{F}_g = \underbrace{\left[-\frac{Gm_1}{|\vec{r}|^2}\hat{r}\right]}_{=\vec{g}} m_2 = m_2\vec{g}$$

The vector field function $\vec{g}$ is known as the **acceleration due to gravity** in kinematics, and **gravitational field** in field theory.

Gravitational Field

On/near the surface of Earth, we can use

$$m_1 = m_E = 5.972 \times 10^{24}\,\mathrm{kg}$$
$$r = r_E = 6.371 \times 10^{6}\,\mathrm{m}$$

to compute the commonly known value of

$$g \approx 9.81\,\mathrm{m/s^2}$$
$$g \approx 9.81\,\mathrm{N/kg}$$

both units are equivalent

Gravitational Field

The **gravitational field** $\vec{g}$ generated by point mass m shows how it influences the gravitational forces on other masses:

$$\boxed{g(m, \vec{r}) = -\frac{Gm}{|\vec{r}|^2}\hat{r}}$$

Quantity	Symbol	SI Unit
Gravitational field	$\vec{g}$	N / kg
Universal gravitational constant	G	$\mathrm{N \cdot m^2 / kg^2}$
Source mass	m	kg
Distance from source mass	$\lvert\vec{r}\rvert$	m
Outward radial unit vector from source	$\hat{r}$	N/A

The negative sign indicates that *direction* of the gravitational field is toward m.

More Than One Mass

When there are multiple point masses present, the total gravitational field at any position $\vec{r}$ is the vector sum of all the ~~forces $\vec{F}_i$:~~ fields $\vec{g}_i$:

$$\boxed{\vec{g} = \sum_i \vec{g}_i = G\left(\sum_{i=1}^{N} \frac{m_i}{r_i^2}\hat{r}_i\right)}$$

At the limit $N \to \infty$, the summation becomes an integral, and can now be used to describe the gravitational field generated by objects with *spatial extend*:

$$\boxed{\vec{g} = \int d\vec{g} = G\int \frac{dm}{r^2}\hat{r}}$$

This integral may be difficult to compute, if the geometry is complicated.

Relating Gravitational Field & Gravitational Force

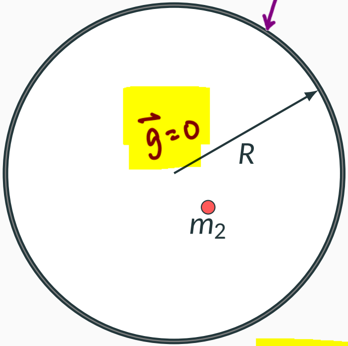
$\vec{g}$ itself doesn't *do* anything unless/until another mass m enters the field. Then, m experiences a gravitational force $\vec{F}_g$ proportional to m and $\vec{g}$, regardless of how the field is created:

$$\boxed{\vec{F}_g = m\vec{g}}$$

Quantity	Symbol	SI Unit
Gravitational force on a mass	$\vec{F}_g$	N
Mass inside the gravitational field	m	kg
Gravitational field	$\vec{g}$	N/kg

Note: A point mass is not affected by the gravitational field that itself generates.

What If You Are Inside Another Mass?



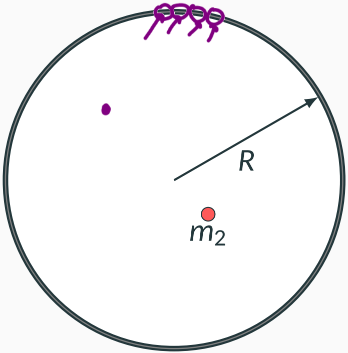
Newton used the **shell theorem** to show that if a mass m_2 is *inside* a spherical shell of mass m_1, the gravitational force that it experiences is *zero*.

$$\boxed{\vec{F}_g = \begin{cases} \vec{0} & \text{if } r < R \\ -Gm_1m_2/r^2\hat{r} & \text{otherwise} \end{cases}}$$

It also means that gravitational field is also *zero*

$\vec{g} = \frac{Gm_1}{r^2}$

What If You Are Inside Another Mass?



That $\vec{g}_{\text{inside}} = \vec{0}$ can be calculated by:

- Integrating the fields created by infinitesimal mass elements *dm* at any point inside the shell, or
- Using **Gauss's law** for gravity, similar to finding the electric field inside a charged conducting sphere:

$$\oint \vec{g} \cdot d\vec{A} = -4\pi G \underbrace{M_{\text{encl}}}_{=0}$$

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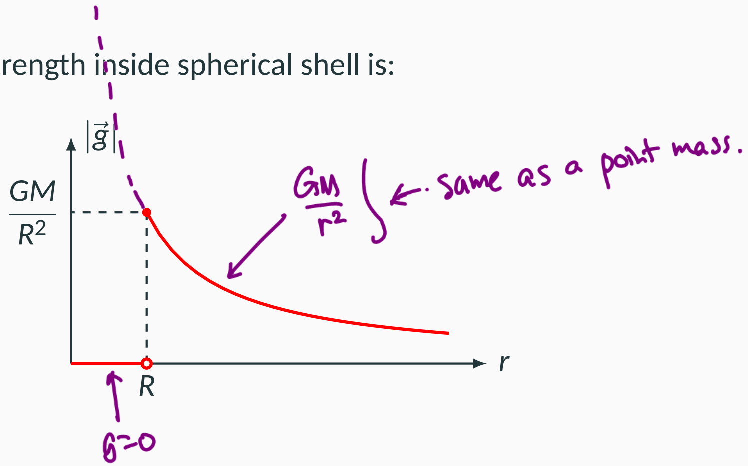
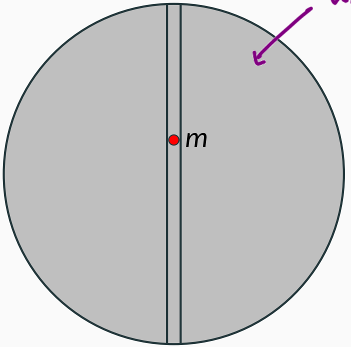
Gravitational Field Inside a Spherical Shell

The gravitational field strength inside spherical shell is:



What If You Are Inside Another Mass?

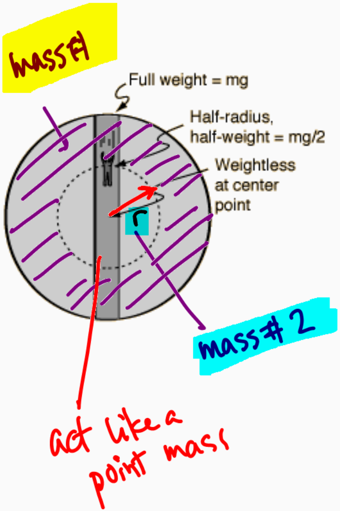
assume constant density.



Suppose you could drill a hole through the Earth and then jump into it. How long would it take you to emerge on the other side of the Earth?

To calculate this, we need to know how the gravitational force changes as you fall through Earth.

Falling Toward the Center of Earth



As you fall through Earth, we can separate the part of Earth that is "above" you, and the part that is "below" you

- The part that is "above" you is like the spherical shell, and does not contribute to the gravitational field, and therefore does not exert any force $\vec{g}=0$
- The part that is "below" you gets smaller as you fall toward the center

Falling Toward the Center of Earth

Assuming that Earth's density is uniform, and neglecting air resistance and other factors, the value of g as the person falls through Earth ($r < R$) is given by finding how much mass is still "below" the person, $M(r)$:

$$g(r) = \frac{GM(r)}{r^2} \qquad M(r) = \frac{4}{3}\rho\pi r^3 \qquad \rho = \frac{3M_E}{4\pi R^3}$$

$\rho = \frac{M_E}{V_E}$ $\qquad \frac{3M_E}{4\pi R^3}$

where M_E is the mass of Earth, R is the radius of Earth, ρ is the (constant) density, and r is the distance from Earth's center. Then $M(r)$ is the amount of mass "below" the person as he/she falls toward the center.

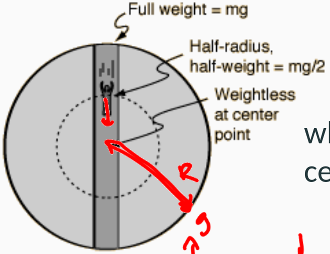
$$g(r) = \frac{GM(r)}{r^2} = \frac{G}{r^2}\left(\frac{4}{3}\rho\pi r^3\right) = \frac{4}{3}\pi G\rho r \quad \frac{4}{3}\pi G r\left(\frac{3M_E}{4\pi R^3}\right) = \left[\frac{GM_E}{R^3}\right] r$$

Falling Toward the Center of Earth

The gravitational field strength inside this hypothetical Earth is a linear function of distance r from the center:

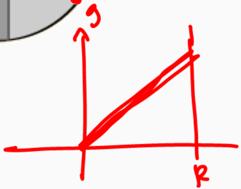
$$g(r) = \frac{GM_E r}{R_E^3} = \left(\frac{r}{R_E}\right) g_0$$

towards center.



where $g_0 = 9.81\,\mathrm{N/kg}$ is the field strength at the surface. At the center ($r = 0$), $g = 0$. The gravitational force is:

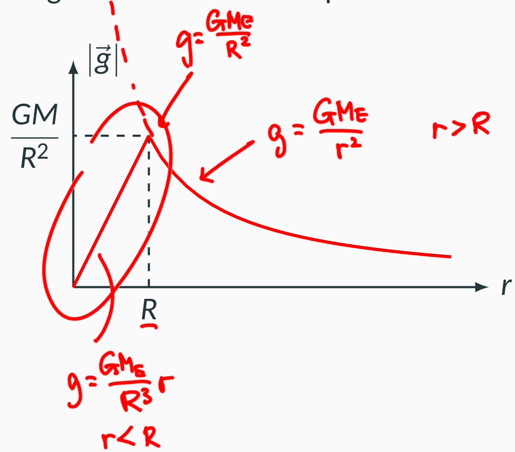
$$F_g(r) = -\underbrace{\left[\frac{mg_0}{R_E}\right]}_{k} r$$



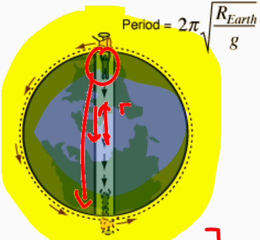
$F_g = mg$

Gravitational Field Strength Inside a Uniform Sphere

The gravitational field strength inside a uniform sphere is:



Falling Toward the Center of Earth



A satellite at the Earth's radius would have the same period as one falling through the Earth.

The gravitational force has the same form as Hooke's law: it is proportional to displacement from the center, but in the opposite direction:

$$F_g(r) = -kr$$

$\frac{mg_0}{R}$

The motion is a simple harmonic motion. The traveler will oscillate through Earth with a period of:

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{R}{g_0}}$$

6.4×10^6 m

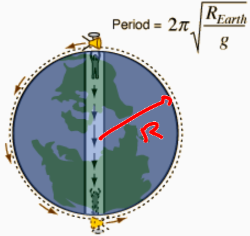
9.81

For Earth, $T = 5068$ s. The traveler would pop up on the opposite side every 42 min.

Falling Toward the Center of Earth

orb speed $V = \sqrt{\frac{GM}{R}} = \frac{2\pi R}{T} \rightarrow \frac{GM}{R} = \frac{4\pi^2 R^2}{T^2}$

$T^2 = \frac{4\pi^2}{GM} R^3$



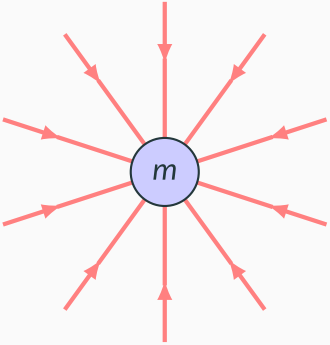
A satellite at the Earth's radius would have the same period as one falling through the Earth.

Since simple harmonic motion is a projection of a uniform circular motion, if a satellite is in a circular orbit just above the surface, and passes overhead just above the traveler as he/she popped up out of the hole. The period of such an orbit would be the same as oscillating traveler.

$T^2 = \frac{4\pi^2}{GM} R^3 = 4\pi^2 \underbrace{\frac{R^2}{GM}}_{\frac{1}{g_0}} R = 4\pi^2 \frac{R}{g_0}$

$\boxed{T = 2\pi\sqrt{\frac{R}{g_0}}}$

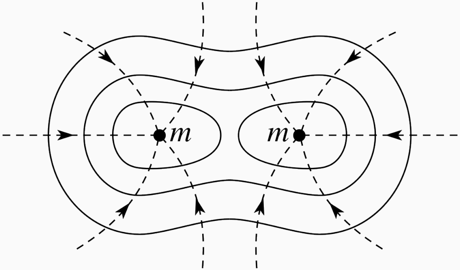
Gravitational Field Lines



- The direction of $\vec{g}$ is toward the center of the object that created it
- Field lines do not tell the intensity (i.e. magnitude) of $\vec{g}$, only the direction

Gravitational Field Lines

When there are multiple masses, the total gravitational field (dotted line) is the vector sum of all the individual fields.



The solid lines are called **equipotential lines**, where the potential energy is constant. Equipotential lines are perpendicular to gravitational field lines.

Gravitational Potential Energy

Gravitational Potential Energy

Gravitational potential energy is found by integrating the work equation and using the law of universal gravitation:

$$W = \int \vec{F}_g \cdot d\vec{r} = -\int_{r_1}^{r_2} \frac{Gm_1m_2}{r^2}\hat{r} \cdot d\vec{r}$$

$$= -\int_{r_1}^{r_2} \frac{Gm_1m_2}{r^2} dr = \frac{Gm_1m_2}{r}\bigg|_{r_1}^{r_2} = -\Delta U_g$$

where

$$\boxed{U_g = -G\frac{m_1m_2}{r}}$$

- U_g is the work required to move two objects from r to ∞
- $U_g = 0$ at $r = \infty$ and *decrease* as r decreases

Relating Gravitational Potential Energy to Force

The fundamental theorem of calculus shows that gravitational force ($\vec{F}_g$) is the negative gradient of the gravitational potential energy (U_g):

$$\vec{F}_g = -\nabla U_g = -\frac{\partial U_g}{\partial r}\hat{r}$$

$$F_c = -\frac{dU}{dx}\hat{i}$$

$$F_c = -\frac{dU}{dx}\hat{i} - \frac{dU}{dy}\hat{j} \cdots$$

$$= -\nabla U$$

The direction of $\vec{F}_g$ always points from high to low potential energy

- A free-falling object is always decreasing in U_g
- "Steepest descent": the direction of $\vec{F}_g$ is the shortest path to decrease U_g
- Objects traveling perpendicular to $\vec{F}_g$ has constant U_g

Relating U_g, $\vec{F}_g$ and $\vec{g}$

Knowing that $\vec{F}_g$ and $\vec{g}$ only differ by a constant (mass m), we can also relate gravitational field to potential energy by the gradient operator:

$$\vec{g} = -\nabla V_g = -\frac{\partial V_g}{\partial r}\hat{r} \quad \text{where} \quad V_g = \frac{U_g}{m}$$

We already know that the direction of $\vec{g}$ is the same as $\vec{F}_g$, i.e.

- The direction of $\vec{g}$ is the shortest path to decrease U_g
- Objects traveling perpendicular to $\vec{g}$ has constant U_g
- V_g is called the **gravitational potential** but it is rarely used