Topic 5: Circular Motion

Advanced Placement Physics 1

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Olympiads School

Files to Download

Please download the following files from the school website if you have not already done so:

- `PhysAP1-05-circMotion-print.pdf`—The "print version" of the class slides for this topic.
- `PhysAP1-05-Homework.pdf`—Homework problems for this topic.

Please download/print the PDF file for the class slides before each class. There is no point copying notes that are already on the slides. Instead, focus on things that aren't necessarily on the slides. If you wish to print the slides, we recommend printing 4 slides per page.

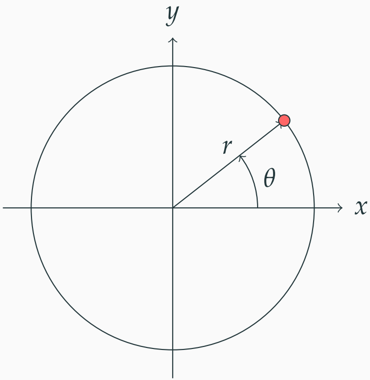
Review of Circular Motion

In a **circular motion**, an object of mass m moves in a circular path about a fixed center. In Grade 12 Physics, you should have studied *uniform* circular motion, where:

- the object's speed (magnitude of velocity) is constant
- the object's **centripetal acceleration** is toward the center
- the object's acceleration is caused by a **centripetal force**

Polar Coordinates

Polar Coordinate System in 2D



In the Cartesian coordinate system for rectilinear motion, an object's position is described by its x and y coordinates:

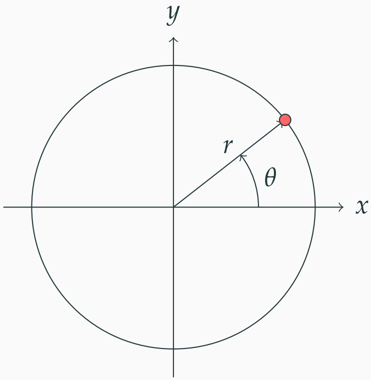
$$\mathbf{x}(x, y)$$

For circular motion, the **polar coordinate system** is preferred. The position of an object can also be described by:

$$\mathbf{r}(r, \theta)$$

Where r is distance from the origin, and θ is the standard angle, measured counterclockwise from the x axis in *radians*

Polar Coordinate System in 2D



- This is consistent with how position vectors are expressed in Grade 11/12 Physics: magnitude (r) and direction (θ)
- Cartesian coordinates can be obtained from the polar coordinates:

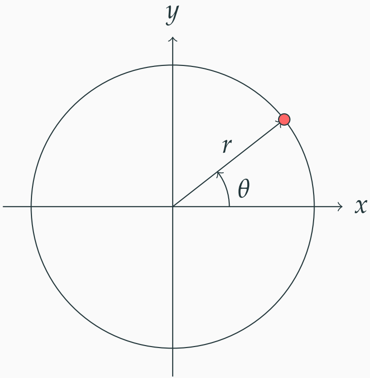
$$x = r\cos\theta \quad y = r\sin\theta$$

- While polar coordinates can be obtained from Cartesian coordinates:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

Rigid-Body Circular Motion

Angular Position and Angular Velocity



For constant radius r, the **angular position** θ fully describes an object's position:

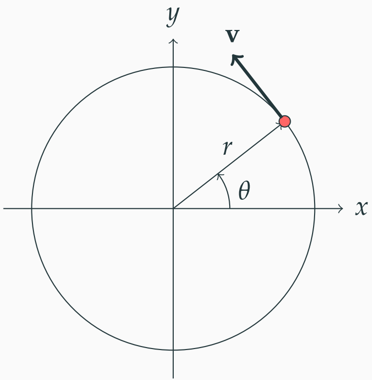
$$\boxed{\theta = \theta(t)}$$

Average angular velocity ω (or **angular frequency**) is the rate of change in angular position over a finite time interval:

$$\boxed{\omega(t) = \frac{\Delta\theta}{\Delta t}}$$

ω is measured in rad/s

Velocity and Angular Velocity

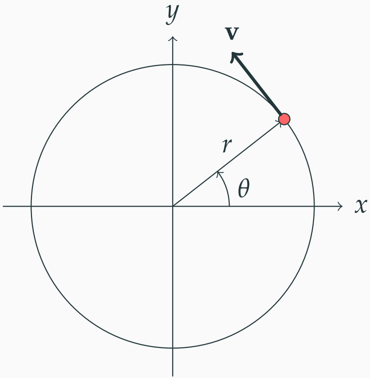


The actual velocity of an object in circular motion is related to the angular velocity by:

$$v = r\omega$$

- The direction of $\mathbf{v}$ is tangent to circle
- If $\omega > 0$, the motion is counter-clockwise
- If $\omega < 0$, the motion is clockwise

Period & Frequency



For constant ω (uniform circular motion), the motion is periodic, with its **frequency** f and **period** T given by:

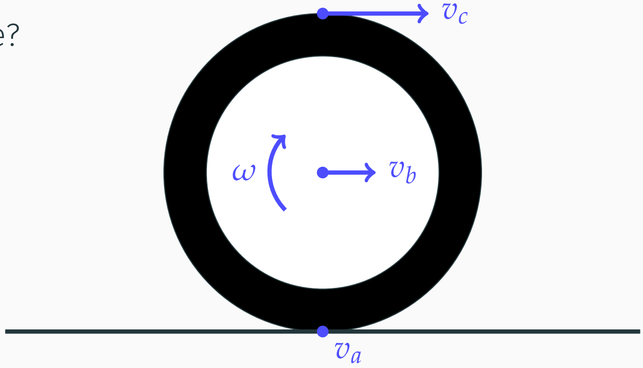
$$f = \frac{\omega}{2\pi} \quad T = \frac{2\pi}{\omega} \quad f = \frac{1}{T}$$

T is measured in **seconds** (s) and f in **hertz** (Hz)

Rotating Object Without Slipping

A tire with radius r rolls along the road with an angular velocity ω *without slipping*. (This is a very common case for analysis.) What is its velocity v

a. at the contact between the ground and the tire?
b. at the center?
c. at the top of the tire?



Angular Acceleration

The change in ω over a finite time Δt is **average angular acceleration**, which has a unit of rad/s^2:

$$\boxed{\bar{\alpha} = \frac{\Delta \omega}{\Delta t}}$$

Similar to the relationship between velocity and angular velocity, **tangential acceleration** a_θ is related to angular acceleration α by the radius r:

$$\bar{a}_\theta = \frac{\Delta v}{\Delta t} = \frac{r \Delta \omega}{\Delta t} = r\bar{\alpha}$$

For *uniform* circular motion, ω is constant, therefore $\alpha_\theta = 0$

Kinematics in the Angular Direction

For constant angular acceleration α, the kinematic equations are just like in rectilinear motion, but with the θ replaces x, ω replaces v, and α replaces a:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \theta_0 + \frac{\omega_0 + \omega}{2} t$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

If α is *not* constant, calculus will be required.

A Simple Example

Example 1: An object moves in a circle with angular acceleration 3.0 rad/s^2. The radius is 2.0 m and it starts from rest. How long does it take for this object to finish a circle?

Centripetal Acceleration & Centripetal Force

From Grade 12 Physics, you should know that there is also a component of acceleration toward the center of the rotation, called the **centripetal acceleration** a_r:

$$\boxed{a_r = -\frac{v^2}{r} = \omega^2 r}$$

The force that causes the centripetal acceleration is called the **centripetal force**, also toward the center of rotation:

$$\boxed{\mathbf{F}_r = ma_r = -\frac{mv^2}{r}}$$

Centripetal Acceleration for Uniform Circular Motion

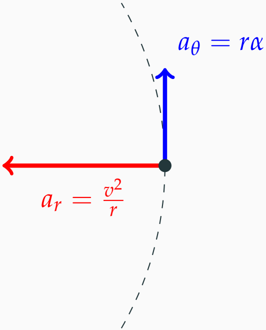
In uniform circular motion ($\alpha = 0$) problems where the period or frequency are known, the speed of the object is:

$$v = \frac{2\pi r}{T} = 2\pi r f$$

Centripetal acceleration can therefore be expressed based on T or f:

$$a_r = -\frac{v^2}{r^2} \quad \rightarrow \quad \boxed{a_r = \frac{4\pi^2 r}{T^2} = 4\pi^2 r f^2}$$

Acceleration: The General Case



- In general circular motion, there are two components of acceleration:
  - Centripetal acceleration a_r depends on radius of curvature r and instantaneous speed v. The direction of the acceleration is toward the center of the circle.
  - Tangential acceleration a_θ depends on radius r and angular acceleration α. The direction of the acceleration is tangent to the circle

How to Solve Circular Motion Problems

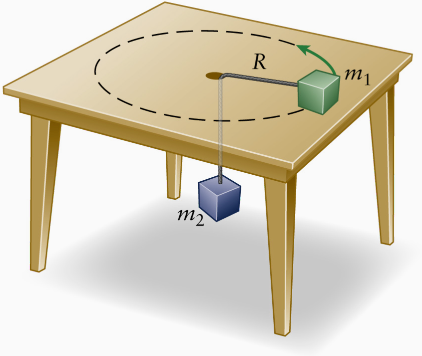
1. Is there any circular motion?
2. If so, the condition for circular motion is:

$$\mathbf{F}_{\text{provided}} = \mathbf{F}_{\text{required}}$$

   - The *provided* force comes from FBD
   - The *required* force comes from the centripetal force equation

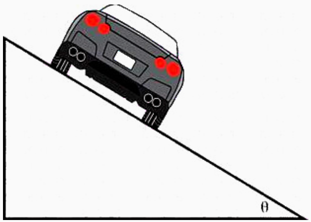
3. If the net force also has a tangential component, then there is also a change in angular velocity

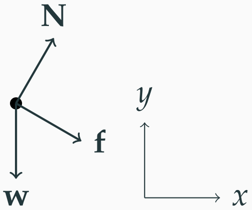
Example: Horizontal Motion



Example 2: In the figure on the left, a mass $m_1 = 3.0\,\text{kg}$ is rolling around a frictionless table with radius $R = 1.0\,\text{m}$. with a speed of $2.0\,\text{m/s}$. What is the mass of the weight m_2?

Banked Curves on Highways and Racetracks





No motion in the y direction, i.e. no net force:

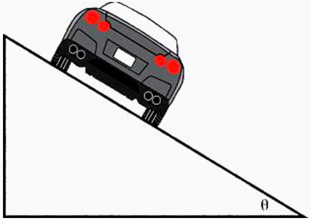
$$\sum F_y = N\cos\theta - f\sin\theta - w = 0$$

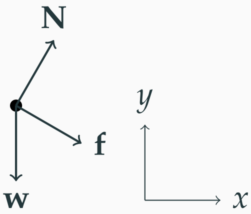
Net force in the x direction is the centripetal force:

$$\sum F_x = N\sin\theta + f\cos\theta = \frac{mv^2}{r}$$

Friction force $\mathbf{f}$ may be static or kinetic.

Banked Curves on Highways and Racetracks



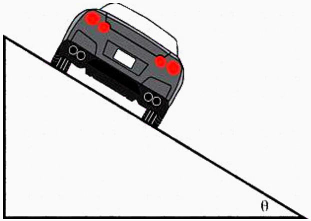


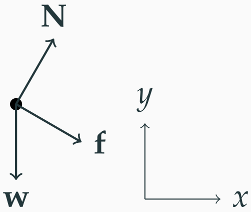
For analysis, use the simplified equation for friction $f = \mu N$ (i.e. assume either kinetic friction or maximum static friction), and weight $w = mg$, the equations on the previous slides can be arranged as:

$$N\left(\cos\theta - \mu\sin\theta\right) = mg$$

$$N\left(\sin\theta + \mu\cos\theta\right) = \frac{mv^2}{r}$$

Banked Curves on Highways and Racetracks





Dividing the two equations removes both the normal force and mass terms:

$$\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg}$$

The *maximum* velocity v_{max} can be expressed as:

$$\boxed{v_{\text{max}} = \sqrt{rg\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}}}$$

Note that v_{max} does not depend on mass.

Banked Curves on Highways and Racetracks

In the limit of $\mu = 0$ (frictionless case), the equation reduces to:

$$\boxed{v_{\text{max}} = \sqrt{rg\tan\theta}}$$

And in the limit of a flat roadway with no banking ($\theta = 0$, $\sin\theta = 0$ and $\cos\theta = 1$), the equation reduces to:

$$\boxed{v_{\text{max}} = \sqrt{\mu rg}}$$

Vertical Circles

Vertical Circles

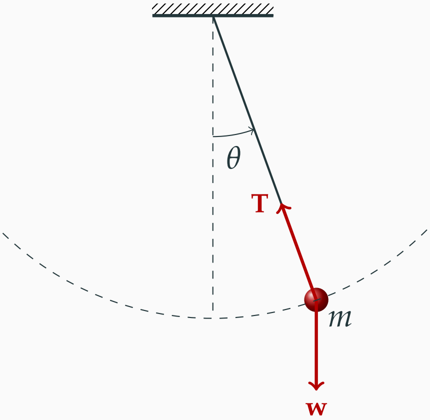
Circular motion with a horizontal path is straightforward. However, for vertical motion:

- Generally difficult to solve by dynamics and kinematics
- Instead, use conservation of energy to solve for speed v
- Then use the equation for centripetal force to find other forces

If it is impossible to get the required centripetal force, then it could not continue the circular motion

What About a Pendulum?

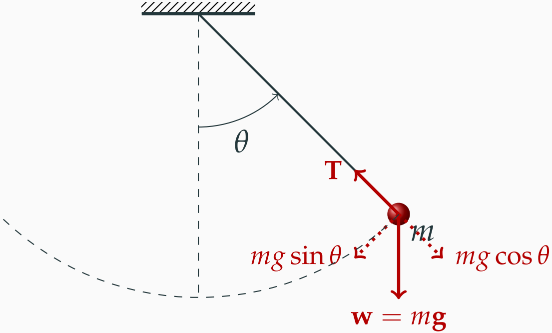
A simple pendulum is also like a vertical circular motion problem.



- There are two forces act on the pendulum: weight $\mathbf{w} = m\mathbf{g}$, and tension $\mathbf{T}$
- Speed of the pendulum at any height is found using conservation of energy
    - Tension $\mathbf{T}$ is always $\perp$ to motion, therefore it does not do any work
    - Work is done by gravity (a conservative force) alone
- Tangential and centripetal accelerations are based on the net force along the angular and radial directions

Simple Pendulum

At the top of the swing, velocity v is zero, therefore:



Centripetal acceleration is also zero:

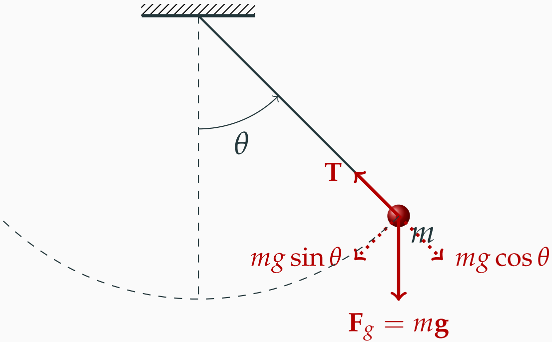
$$a_r = \frac{v^2}{r} = 0$$

and therefore the net force along the radial direction is zero. The tension force T can be calculated:

$$T = mg\cos\theta$$

At the highest point when θ is largest, tension is the lowest.

Simple Pendulum



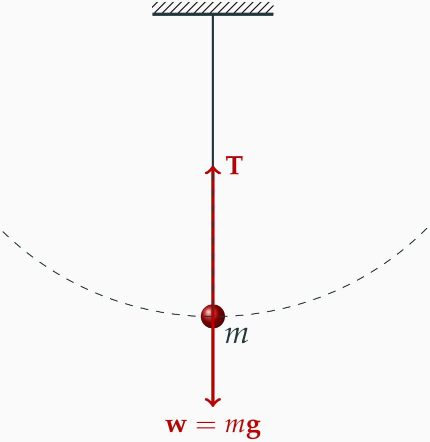
In the radial direction , there is still a net force of $mg\sin\theta$, therefore, there is a tengential acceleration along that direction, with a magnitude of:

$$a_\theta = g\sin\theta$$

This is the same acceleration as an object sliding down a frictionless ramp at an angle of θ.

Simple Pendulum

At the bottom of the swing, the velocity is at its maximum value,



- Maximum centripetal acceleration:

$$a_r = \frac{v^2}{r}$$

- No tangential acceleration:

$$a_\theta = 0$$

- At the lowest point, tension is the highest:

$$T = w + F_r = m\left(g + \frac{v^2}{r}\right)$$

Example Problem

Example 4: You are playing with a yo-yo with a mass of 225 g. The full length of the string is 1.2 m. You decide to see how slowly you can swing it in a vertical circle while keeping the string fully extended, even when the yo-yo is at the top of its swing.

A. Calculate the minimum speed at which you can swing the yo-yo while keeping it on a circular path.
B. Find the tension in the string when the yo-yo is at the side and at the bottom of its swing.

Example: Roller Coaster

Example 5: A roller coaster car is on a track that forms a circular loop, of radius R, in the vertical plane. If the car is to maintain contact with the track at the top of the loop (generally considered to be a good thing), what is the minimum speed that the car must have at the bottom of the loop. Ignore air resistance and rolling friction.

(a) $\sqrt{2gR}$

(b) $\sqrt{3gR}$

(c) $\sqrt{4gR}$

(d) $\sqrt{5gR}$

Example 6: A stone of mass m is attached to a light strong string and whirled in a *vertical* circle of radius r. At the exact bottom of the path, the tension of the string is three times the weight of the stone. The stone's speed at that point is given by:

(a) $2\sqrt{gR}$
(b) $\sqrt{2gR}$
(c) $\sqrt{3gR}$
(d) $4gR$