# Topic 4: Momentum, Impulse and Collisions

Advanced Placement Physics

Dr. Timothy Leung

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Olympiads School

# Momentum

# Linear Momentum

**Linear momentum** (or **translational momentum**, or just **momentum**) is a quantity of motion defined as:

$$\boxed{\boldsymbol{p} = m\boldsymbol{v}}$$

| Quantity | Symbol | SI Unit |
|---|---|---|
| Momentum | $\boldsymbol{p}$ | $\mathrm{kg \cdot m/s}$ |
| Mass | $m$ | $\mathrm{kg}$ |
| Velocity | $\boldsymbol{v}$ | $\mathrm{m/s}$ |

For rotational motion of a rigid body, there is also **angular momentum** which will be studied in a later topic.

# General Form of Second Law of Motion

Taking the time derivative of the momentum vector (from an inertial frame of reference) using the chain rule:

$$\frac{d\boldsymbol{p}}{dt} = \frac{d(m\boldsymbol{v})}{dt} = m\frac{d\boldsymbol{v}}{dt} + \frac{dm}{dt}\boldsymbol{v} = m\boldsymbol{a} + \dot{m}\boldsymbol{v}$$

For constant mass $m$ (i.e. $\dot{m} = 0$), this right-hand-side reduces to the familiar form of the second law of motion, $m\boldsymbol{a}$. In fact, the *general* form of the second law of motion is that **the net external force on an object is the time rate of change of its momentum**, i.e.:

$$\boxed{\boldsymbol{F}_{net}(t) = \frac{d\boldsymbol{p}(t)}{dt}}$$

# Laws of Motion & Conservation of Momentum

$$\boxed{\boldsymbol{F}_{net} = \frac{d\boldsymbol{p}}{dt}}$$

**First law of motion:** The momentum state of an object is conserved unless a net unbalanced external force acts on it

- Momentum is conserved (i.e. $\sum \boldsymbol{p}$ constant) when the net *external* force on an object or a system of objects is zero
- Internal forces do not contribute to net force
- In the absence of external forces, we have **conservation of momentum** for a collection of objects:

$$\sum \boldsymbol{p}_i = \sum \boldsymbol{p}_i'$$

# Impulse

Rearranging the variables in the general form of the second law of motion:

$$\boldsymbol{F}_{net} = \frac{d\boldsymbol{p}}{dt} \;\rightarrow\; \boldsymbol{F}_{net}dt = d\boldsymbol{p}$$

Integrating both sides, we get the **impulse-momentum theorem**:

$$\boxed{\boldsymbol{J}_{net} = \int_{t_1}^{t_2} \boldsymbol{F}_{net}dt = \int_{p_1}^{p_2} d\boldsymbol{p} = \Delta\boldsymbol{p}}$$

The quantity $\boldsymbol{J}_{net}$ is called the **net impulse**.

# Impulse

***F***, ***p*** and ***J*** are all vectors, so the integral can be evaluated in each of the *x*, *y* and *z* axis, i.e., for the *x* direction:

$$J_x = \int_{t_1}^{t_2} F_x dt = \int dp_x = \Delta p_x$$

Note that impulse from each individual force does not depend on whether the object moves. The change in momentum only depends on *net* impulse.

# Average Force

**Average force** $\bar{\boldsymbol{F}}$ is the time-averaged force vector that gets the same impulse. It is used extensively in introductory physics courses to avoid integration:

$$\boldsymbol{J} = \int_{t_1}^{t_2} \boldsymbol{F} dt = \bar{\boldsymbol{F}} \Delta t$$

# Impulse: An Example

**Example 1:** Jim pushes a box with mass 1.0 kg with a 5.0 N force for 10 s while the box stays on the same place. Find the impulse of the pushing force, friction force, the gravitational force, and the net force.

# Rocket Propulsion Problem

**Example 2:** A rocket generates a thrust force by ejecting hot gases from an engine. If it takes 1 ms to combust 1.0 kg of fuel, ejecting it at a speed of 1000 m/s, what thrust is generated?

A. 1000 N
B. 10 000 N
C. 100 000 N
D. 1 000 000 N

# Another Space Example

**Example 3:** A rocket for mining the asteroid belt is designed like a large scoop. It is approaching asteroids at a velocity of $10^4$ m/s. The asteroids are much smaller than the rocket. If the rocket scoops asteroids at a rate of 100 kg/s, what thrust (force) must the rocket's engine provide in order for the rocket to maintain constant velocity? Ignore any variation in the rocket's mass due to the burning fuel.

A. $10^3$ N
B. $10^6$ N
C. $10^9$ N
D. $10^{12}$ N

# Collisions

# Conservation of Momentum

- From the third law of motion, we know that the action and reaction forces are always equal in magnitude and in opposite direction. Thus, their total impulse would be zero.
- When there is no external force, the momentum of the total system will always be constant. We saw that a few slides ago:

$$\boxed{\sum \boldsymbol{p}_i = \sum \boldsymbol{p}_i'}$$

# Classifications of Collisions

- Elastic Collision:
  - Total kinetic energy is conserved
  - Momentum is conserved
- Inelastic collision:
  - Kinetic energy is **not** conserved
  - Momentum is conserved
- Completely inelastic collision:
  - “Perfectly inelastic collision”
  - A special case of inelastic collision
  - The objects move together after the collision
  - Kinetic energy is **not** conserved
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# How to Solve Conservation of Momentum Problem

1. Check whether the condition for the conservation of momentum is satisfied (i.e. are there any external forces?)
2. If so, write out expressions for initial momentum and final momentum, and equate the two. You will get 1 to 3 equations (one for each direction).
3. Solve these equations, find the quantity you need to find.

Remember that momentum is a vector. If there is no external force component in some direction, then the momentum component in this direction is still conserved.

# Before We Dive Into Some Exercises

The most typical applications of momentum conservation are collision and explosions

- **Collision: A hits B**
  - Regardless of whether they move together or not afterwards, momentum is conserved
  - Head-on collisions are usually 1D
  - Glancing collisions are usually 2D or 3D
- **Explosion: A explodes and becomes B and C (and D and E...)**
  - A perfectly inelastic collision in reverse
  - Total momentum of B and C (and D and E...) is the same as A in the beginning
  - Usually a 2D or 3D problem

# Collision Problem

**Example 5:** Two objects with equal mass are heading toward each other with equal speeds, undergo a head-on collision. Which one of the following statement is correct?

A. Their final velocities are zero
B. Their final velocities may be zero
C. Each must have a final velocity equal to the other's initial velocity
D. Their velocities must be reduced in magnitude

# Conservation of Momentum Example

**Example 6:** Two astronauts, each of mass 75 kg, are floating next to each other in space, outside the space shuttle. One of them pushes the other through a distance of 1.0 m (about an arm's length) with a force of 300 N. What is the final relative velocity of the two?

A. 2.0 m/s
B. 2.83 m/s
C. 4.0 m/s
D. 16.0 m/s

# Glancing Collision

**Example 7:** A billiard ball of mass 0.155 kg ("cue ball") moves with a velocity of 1.25 m/s towards a stationary billiard ball ("eight ball") of identical mass and strikes it with a glancing blow. The cue ball moves off at an angle of 29.7° clockwise from its original direction, with a speed of 0.956 m/s. What is the final velocity of the eight ball?

# Elastic Collisions

# Elastic Collision Problems

In elastic collisions, *both* momentum and kinetic energy is conserved. In a 1D collision, both equations below have to be satisfied:

$$\sum m_i v_i = \sum m_i v_i'$$
$$\sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i v_i'^2$$

**How kinetic energy is conserved:** In an elastic collision, energy is first converted into a potential energy (e.g. elastic potential energy in a spring), and then all the energy is released back as kinetic energy.

# Conservation of Momentum & Energy in Elastic Collisions

For collision of two objects, the conservation of momentum equation can be expressed as:

$$\boxed{m_1(v_1 - v_1') = m_2(v_2' - v_2)} \tag{1}$$

By moving $m_1$ terms to the left, and $m_2$ terms to the right. Likewise, the conservation of energy can also be arranged as:

$$\boxed{m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2)} \tag{2}$$

By multiplying every term by 2, and again, moving $m_1$ terms to the left, and $v_2$ terms to the right.

# Conservation of Momentum & Energy in Elastic Collisions

Dividing the equations (2) by (1) from the last slide, we get:

$$\frac{(2)}{(1)} \quad \to \quad \frac{m_1(v_1^2 - v_1'^2)}{m_1(v_1 - v_1')} = \frac{m_2(v_2'^2 - v_2^2)}{m_2(v_2' - v_2)}$$

$m_1$ and $m_2$ terms cancel out, while the terms in the numerator can be expanded as the difference of two squares which is then simplified:

$$\frac{(v_1 + v_1')(v_1 - v_1')}{(v_1 - v_1')} = \frac{(v_2' + v_2)(v_2' - v_2)}{(v_2' - v_2)}$$

Leading to the final expression, which is substituted back into (1)

$$v_1 + v_1' = v_2 + v_2'$$

# Final Velocities in an Elastic Collision

When two objects 1 and 2 of mass $m_1$ and $m_2$ and collide elastically, their final velocities will be determined by the initial velocities $v_1$ and $v_2$:

$$v_1' = \frac{m_2 - m_1}{m_2 + m_1} v_1 + \frac{2m_2}{m_2 + m_1} v_2$$
$$v_2' = \frac{2m_2}{m_2 + m_1} v_1 + \frac{m_2 - m_1}{m_2 + m_1} v_2$$

These equations are *not* provided in the AP exam equation sheet, which means that we are more interested in the behavior qualitatively rather than quantitatively.

# Special Cases

If both objects have equal mass ($m_1 = m_2 = m$) and the second object is initially at rest ($v_2 = 0$), then the equations simplifies to

$$v_1' = \frac{v_1(m - m) + 2mv_2}{m + m} = 0$$
$$v_2' = \frac{v_2(m - m) + 2mv_1}{m + m} = v_1$$

All the momentum and energy from $m_1$ is transferred to $m_2$. Object 1 stops all together, while object 2 continues with the initial momentum and velocity of Object 1.

# Special Cases

Another special case is when $m_1 \gg m_2$ and $v_2 = 0$ (i.e. a large object colliding with a small stationary object) then we can effectively "ignore" $m_2$:

$$v_1' = \frac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} \approx \frac{m_1v_1}{m_1} = v_1$$

$$v_2' = \frac{v_2(m_2 - m_1) + 2m_1v_1}{m_1 + m_2} \approx \frac{2m_1v_1}{m_1} = 2v_1$$

Object 1 continues to move like nothing happened, but object 2 is pushed to move at *twice* the initial speed of object 1.

# Special Cases

In the reverse case, if $m_1 \ll m_2$, and $v_2 = 0$ (a small object colliding with a large stationary object), then we can “ignore” the $m_1$ term:

$$v_1' = \frac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} \approx \frac{-m_2v_1}{m_2} = -v_1$$

$$v_2' = \frac{v_2(m_2 - m_1) + 2m_1v_1}{m_1 + m_2} \approx 0$$

Object 1 bounces off object 2, and travels in the opposite direction with the same velocity magnitude, while object 2 does not move.

# Elastic Collision Example

**Example 8:** Blocks A and B have the same mass; A hits B with a speed of $5.0\,\mathrm{m/s}$ while B is initially at rest. If the collision is elastic, what would be the final speed of these two objects?

# Elastic Collision Example

**Example 9:** Blocks A and B with the same mass; A has a velocity 3.0 m / s to the east while B has 2 m / s to the west. If the collision is elastic, after the collision, what would the velocity of the two blocks be?

# Elastic Collision Example

**Example 10:** Throw a ball to a really big wall, when the ball reaches the wall, it has a velocity $10\,\mathrm{m/s}$ toward the wall. If the collision is elastic, what would the final velocity of the ball be?

# Elastic Collision Example

**Example 11:** Throw a ball with a velocity 4.0 m/s toward a train with a velocity 40 m/s toward the ball. If the collision is elastic, what would the final velocity of the ball be?

## Inelastic Collision: Calculating Energy Loss

**Example 12:** Two blocks A and B with mass 2.0 kg, block A hits B with velocity 4.0 m / s while B is at rest.

(a) Suppose the collision is completely inelastic, what would the final velocity of A and B be?

(b) What is the loss of energy?