Examples of Rigid-Body Rotational Motion

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The case of a rolling sphere is a standard example of combining the dynamics of translational and rotational motions of a rigid body. In this handout, two typical examples are be presented for a non-slip (i.e. pure rolling) case, while one example is presented for a case with slippage.

1 Pure Rolling of Rigid Body on Flat Surface

In the first and simplest case, a smooth solid sphere of constant density rolls along a smooth surface without slipping (called **pure rolling**). We assume that the sphere and the surface are both perfectly rigid, in that they do not deform. We also assume that the sphere and the surface are both perfectly smooth without defects even at the microscopic level. The free-body diagram is shown in Figure 1. Notice that *there is no friction between the sphere and the surface*. Since there is neither a net force nor a net torque acting on the sphere, the translational and rotational states are constant in time. In theory (if our assumptions are correct), the sphere with translational velocity $\vec{v}$ and angular velocity $\vec{\omega}$ (where $\vec{v} = \vec{\omega} \times R$) will roll along forever.

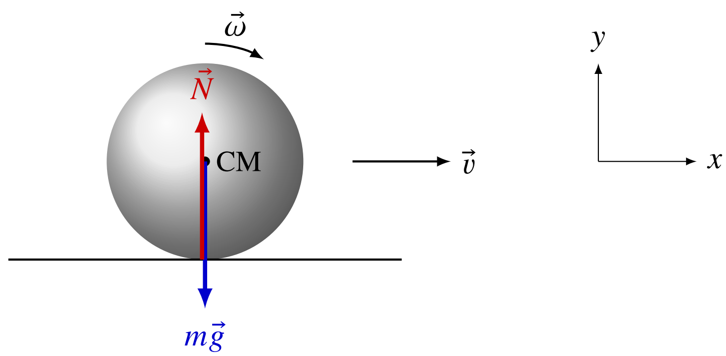

Figure 1: Free-body diagram on a uniform density solid sphere rolling on a smooth flat surface without slipping.

But of course, we are very observant. Even a casual observer will notice that in reality, a ball slows down and eventually come to a stop. A steel ball bearing on a track can roll over a much longer distance and much longer time than a soccer ball on a grassy field, but neither will roll forever. So what causes this? Specifically, what is missing in the free-body diagram in Figure 1?

Our assumptions aren't quite correct. There are two major oversights in our initial assumptions.

1. **Nothing is perfectly smooth.** Firstly, our original assumptions mean that the contact area is infinitesimal small, and the normal force, by basic geometry, points straight toward the CM. However, we should recognize that neither the ball bearing nor the rail are perfectly smooth. When a non-smooth ball rolls over a non-smooth surface, their surface roughness means that the contact point is finite in size, and that the normal force does not necessarily point toward the CM of the ball. This means that unlike in Figure 1, there is a net force and net torque that will slow down the motion of the ball.

2. **Nothing is perfectly rigid.** Secondly, we must recognize that there is no such thing as a perfectly rigid body[1]. Both the ball and the surface deform as they make contact. A perfect illustration is how a tire flattens when it makes contact with the ground, shown in Figure 2. The normal force is large in magnitude on the front side is large in magnitude than on the other, and therefore exerts both a resistive force to slow down the wheel, as well as negative torque to slow down the tire. Also, the normal force does not point toward the CM. This is called **rolling resistance**.

Now that we have understood the basic problem, we can tackle the next problem that involves friction.

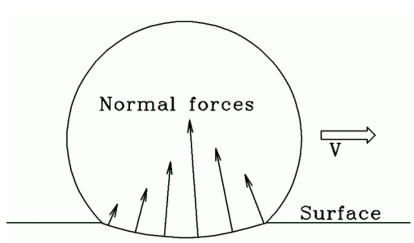

Figure 2: Deformation of a tire under load as it rolls over a surface without slipping.

2 Pure Rolling on an Inclined Surface

But what if the sphere rolls without slippage down a ramp of angle θ instead? The free-body diagram for that sphere shown in Figure 3. The radius of the sphere is R. This time, there is also a static friction $\vec{f_s}$ acting up the ramp at at the point of contact between the sphere and the surface.

Be careful what forces are acting on it. The weight of the sphere acts at the center of gravity, while the normal force acts at the point of contact. Neither forces generate any torque about the CM, therefore,

[1]This should be obvious, but in the pursuit of learning physics, this is a detail that may be lost. When two objects collide in any collision, it takes a finite amount of time for either objects to accelerate to the new velocities. If both objects are perfectly rigid, then the collision will occur over an infinitely small time interval, with infinitely large forces acting on them.

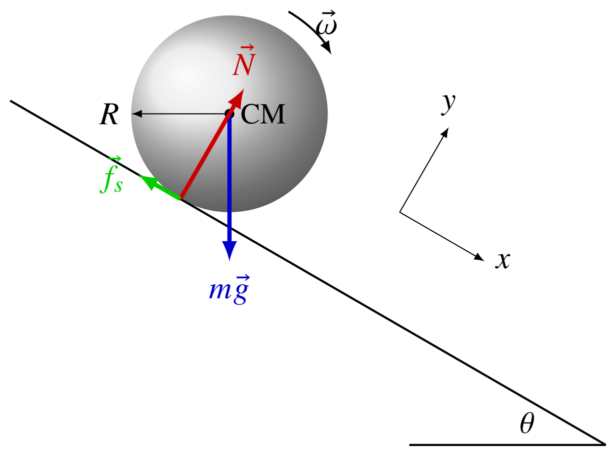

Figure 3: Free-body diagram on a smooth solid sphere of radius R rolling down a smooth ramp without slipping. The ball travels distance d to the bottom of the ramp.

without friction, the sphere will just *slide* down the ramp without rotation. To solve this problem, we have three dynamic equations along the three axes[2]:

$$\sum F_x = mg\sin\theta - f_s = ma \tag{1}$$

$$\sum F_y = N - mg\cos\theta = 0 \tag{2}$$

$$\sum \tau_z = r f_s = I_z\alpha \tag{3}$$

Don't be so sure about what μ_s tells us. At this stage, the actual static friction force is not known and is a quantity that needs to be solved. Knowledge of the coefficient of static friction μ_s may not be useful, because it only tells you the *maximum* static friction, not the *actual* friction that exists. However, we can use it to double check to see if the answer makes sense.

Relating rotational and translational motions. Inserting the expression for the moment of inertia of the solid sphere $I_z = \frac{2}{5}mR^2$ and recognizing that for pure rolling, $\alpha = \frac{a}{R}$, we can use Eq. 3 to express static friction in terms of linear acceleration a:

$$f_s = \frac{I_z\alpha}{R} = \left(\frac{2}{5}mR^2\right)\left(\frac{a}{R}\right)\left(\frac{1}{R}\right) = \frac{2}{5}ma \tag{4}$$

Substituting the expression in Eq. 4 into Eq. 2, the force equation in the x-direction becomes:

$$mg\sin\theta - \frac{2}{5}ma = ma \tag{5}$$

Cancelling the mass terms and solving for acceleration, we find a constant acceleration of :

$$a = \frac{5}{7}g\sin\theta \tag{6}$$

[2]The $\hat{k}$-axis points *out* of the page. Counter-clockwise rotational motion is positive, while clockwise rotational motion is negative

Compare the results in Eq. 6 to that of an object *sliding* without friction down the same ramp, the acceleration for the sliding block is $a = g \sin\theta$ which is higher than the pure rolling case.

Since acceleration is constant, kinematic equation can be used to compute the speed of the sphere when it reaches the bottom of the ramp, a distance d away. If the sphere starts from rest:

$$v = \sqrt{2ad} = \sqrt{\frac{10}{7} g d \sin\theta} \tag{7}$$

There is, of course, one "sanity check" that must be done, that is to make sure that the friction calculated in Eq. 4 has not exceeded the maximum static friction, given by

$$f_s \leq \mu_s N \tag{8}$$

Combining the expression for f_s in Eq. 4, the acceleration in Eq. 6, and the normal force on an incline, Eq. 8 becomes:

$$f_s \leq \mu_s N \tag{9}$$

$$\frac{2}{7} mg \sin\theta \leq \mu_s mg \cos\theta \tag{10}$$

$$\frac{2}{7} \tan\theta \leq \mu_s \tag{11}$$

If the ramp angle is too steep, then the friction will transition from static to kinetic, which is a much more difficult problem.

Energy is conserved, of course. There is a much simpler way to find v, that is, by using the conservation of energy. In this case, kinetic energy is split between translational kinetic energy K_t and rotational kinetic energy K_r:

$$\begin{aligned}
\Delta U_g &= K_t + K_r \\
mg\Delta h &= \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 \\
mgd \sin\theta &= \frac{1}{2} m v^2 + \frac{1}{2}\left(\frac{2}{5} m R^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{2} m v^2 + \frac{1}{5} m v^2 = \frac{7}{10} m v^2
\end{aligned}$$

Cancelling mass terms on both sides, and solving for v, we arrive at the same expression as using dynamics and kinematics equations:

$$v = \sqrt{\frac{10}{7} g d \sin\theta} \tag{12}$$

But why is energy conserved? That the total system energy is conserved even when there is friction should be a significant insight for the novice physics student. Clearly, static friction is *non-conservative*; surely it would have done some non-conservative external work. However, a careful look at the work done by the static friction reveals how energy is transformed:

- Static friction does *positive rotational work*. It is the only force that generates a torque, therefore the positive work done by $\vec{f_s}$ increases the rotational kinetic energy, i.e. $W = \Delta K_r > 0$ by the work-energy theorem. This is supported by the fact that as the ball rolls, the angular velocity increases.

- At the same time, static friction also does *negative translational work*, as the direction of the frictional force is in the opposite direction to the translational motion of the center of gravity. Again, by the work-energy theorem, i.e. $W = \Delta K_t < 0$. This is supported by the fact that the (translational) velocity for the rolling case is *lower* than the sliding case with no friction.

The work done by static friction essentially converts some translational kinetic energy into rotational kinetic energy[3]. In this case, the system remains isolated from the surroundings, and therefore, although static friction is non-conservative, the work done is not external to the system, and therefore the total energy of the system is conserved.

3 Rolling on Flat Surface with Slippage

We return to the flat-surface problem, but this time, we allow slippage at the point of contact between the sphere and the surface. In this case, because there is relative motion between them, there is *kinetic* friction $f_k = \mu_k N$ at the point of contact, as shown in Figure 4. At that point, the sphere slides to the left relative to the surface, and therefore the force of friction is toward the right.

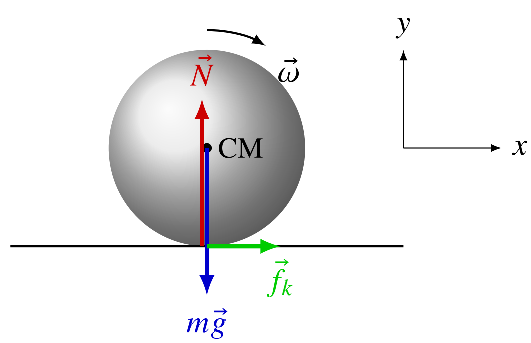

Figure 4: Force diagram on a smooth solid sphere rolling on a flat surface with slippage.

There is now a net force in the $+\hat{\imath}$ direction, and a positive net torque in the $+\hat{k}$ direction (i.e. net torque is counter-clockwise). The consequences are that:

1. The net force caused by kinetic friction f_k causes the sphere to accelerate toward the $+\hat{\imath}$ direction. Since f_k is constant, the acceleration is also constant as well (for as long as the sphere slips). At first

[3]This should be obvious, but it may not be: only conservative forces convert kinetic energy into the related potential energy, so static friction cannot directly convert from gravitational potential energy to rotational kinetic energy.

glance, this may seem counter intuitive, but, we know that a car with its tires spinning on ice will still have a small acceleration.

2. The net torque in the $+\hat{k}$ (counter clockwise) direction causes the angular velocity $\vec{\omega}$ to decrease over time.

It is important to note that, unlike the previous no-slip cases where we can relate angular acceleration α with linear acceleration a by the radius of the sphere, for the slippage case, there is *no relationship between α and a*. The velocity of the sphere $\vec{v}$ toward the right can be expressed with a simple kinetic equation:

$$v_x = v_0 + at \tag{13}$$

while the angular velocity of the sphere is given by:

$$\omega = \omega_0 + \alpha t \tag{14}$$

Note that ω_0 is negative, since the rotation is clockwise. At some time t there will be a point in time where $v = \omega r$. When this happens, the sphere stops slipping, and the problem returns to the no-slip case that was discussed in Section 1.

4 How To Solve Rotational Problems

When solving for rotational problems like the ones described in the previous sections, it is imperative to carefully draw a free-body diagram to account for all the forces and torques acting on an object, as we have done in the previous examples. A few things to keep in mind:

- The direction of friction force is not always obvious.
- The magnitude of any static friction force cannot be assumed to be at maximum.
- If the object is to change its rotational state, there must be a net torque causing it.

Once the free-body diagram is complete, we can breaks down the *forces* into $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ components. We have now a set of three equations from the second law of motion:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z$$

It is likely that only *one* direction will have acceleration.[4] In the problems that are presented in this handout, there are no forces in the $\hat{k}$ direction. We have only needed to use the $\hat{\jmath}$ direction in the third (with slippage) problem to calculate the normal force, so that the kinetic friction f_s can be calculated.

Because the motion is rotational in nature, we will also have to sum the net torque along those same three axes as well:

$$\sum \tau_x = I_x\alpha_x \qquad \sum \tau_y = I_y\alpha_y \qquad \sum \tau_z = I_z\alpha_z$$

[4] In fact, whenever possible, it is a good practice to orient the Cartesian coordinate system such that acceleration only occurs in one direction.

In simpler cases like the ones presented here, net torque will only be along the $\hat{k}$ direction, and there were no torque by any of the forces along the $\hat{\imath}$ and $\hat{\jmath}$ directions (although for more complicated problems, there can be net torque in all three directions). Note that the moments of inertia are not equal ($I_x \neq I_y \neq I_z$) if the rolling object is not a sphere.

Depending on whether an object rolls with or without slipping, there may be no relationship between angular velocity ω and translational velocity $\vec{v}$, or between angular acceleration α and translational acceleration $\vec{a}$. But in any case, you will be left with a system of equations with equal number of unknowns to be solved. Use whatever method that you are comfortable with to solve for the answers.