Fibered Canonical Calculus

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Abstract

We introduce a deduction system which given a source $F(X)$ consisting of a functor applied to an object and a target Y produces instructions how to construct a *canonical map* $F(X) \to Y$, if it exists. We have the following goals

UNIQUENESS There should be at most one canonical map $F(X) \to Y$.

VERIFIABILITY To check whether there is a canonical map $F(X) \to Y$ should be a finite process, which hopefully could be done by a computer in the future.

COMPOSITION If there are canonical maps $1X \to Y$ and $1Y \to Z$ and $1X \to Z$ then they should make the respective triangle commute.

NATURALITY A natural transformation is canonical iff all the components are canonical.

for COMPOSITION it looks like the functor F has to be the identity.

Let $\mathbb{E} \to \mathbb{B}$ be a cocartesian fibration equipped with a cocleavage together with a partial cleavage (i.e. a choice of cartesian lifts, whenever they exist).

Example. *We have the universal cocartesian fibration of small categories, where $\mathbb{B}$ is* Cat *and the fiber over $C \in \mathbb{B}$ is C itself. The Cocleavage is given by functor application. A cartesian lift of $Y \in D$ over $F : C \to D$ is just a universal morphism from Y to F, i.e. a representing object of* $\mathrm{Hom}_D(F(\_), Y)$.

Notation. $\mathbb{B}$ looks like $C, D \in \mathbb{B}$ or $c \xrightarrow{f} d$. Objects of $\mathbb{E}$ are x, y, z, X, Y, Z. We write cartesian morphisms as $x \xrightarrow{\square} y$. cocartesian lifts of x along f are sometimes written $x \to fx \equiv f_! x$. For a morphism f in $\mathbb{B}$, **we write $x \underset{f}{\to} y$ for a morphism $x \to y$ in $\mathbb{E}$ over f.** The reason is, that almost every morphism in $\mathbb{E}$ we write down is canonical, hence unspecified and does not deserve a name on its own.

Example (in Cat)**.** *Let C : Cat consider $\sqcup : C \times C \to C$. Then for any $X : C$ we have that the codiagonal is a cartesian map $(X, X) \underset{\sqcup}{\xrightarrow{\square}} X$.*

We think of this calculus as : There is a canonical $x \to y$ over $f : c \to d$ if there is a canonical $fx \to y$ in d. The punchline of this framework is, that we want to talk about the outer most functor of the domain, hence we may put this as datum in the base category of the cocartesian fibration.

Definition 0.1 (Canonical morphisms in cocartesian fibrations)**.** We introduce the following list of deduction rules of how to cook up canonical morphisms from other canonical morphisms.

Name	ID	CLVG	FCT	CCT	CRT
want to show	$x \xrightarrow[1]{} x$	$f^*x \xrightarrow[f]{\square} x$	$x \xrightarrow[fg]{} f_!y$	$x \xrightarrow[gf]{} y$	$x \xrightarrow[f]{} y; y \xrightarrow[g]{\square} z$
suffices to show			$x \xrightarrow[g]{} y$	$f_!x \xrightarrow[g]{} y$	iff $x \xrightarrow[gf]{} z; y \xrightarrow[g]{\square} z$

If $\mathbb{B}$ is cartesian closed and the cocleavage splits (i.e. a split choice of cocartesian lifts) the following makes sense to add :

Name	EXT	EXT2	NAT
want to show	$x \xrightarrow[gh]{} y$	$x \xrightarrow[gh]{} y$	$f \xrightarrow[1]{} g$
suffices to show	$h \xrightarrow[g*]{} f; x \xrightarrow[f]{} y$	$g \xrightarrow[h*]{} f; x \xrightarrow[f]{} y$	iff $\forall x, x \xrightarrow[f]{} g_!x$ & Naturality

The extension rules allow you to get rid of free variables and just produce natural transformations. The NAT rule allows you to construct a canonical natural transformation from its canonical levels, provided you can show naturality.

0.1 Results

Lemma 0.2. *We have a canonical map $x \xrightarrow[f]{} f_!x$, namely the preferred cocartesian lift.*

Proof. There are two different proofs. One could either argue by FCT , ID, or by CCT , ID □

Notation. For $I, C : \mathsf{Cat}$ we write $\Delta_I : C \to C^I$.

Example. *For $X : C^I$, the cartesian lift $\Delta^* X \xrightarrow[\Delta]{} X$ is the limitcone of X.*

Because when constructing a canonical map one only knows source and target (over which morphism in $\mathbb{B}$ it lies), we proceed backwards.
We use the first column to denote free variables.
If CRT is applied to a standart cartesian lift we write CRT(& CLVG).

Canonical Morphism 1.

$X : \mathcal{C}$	$X \to \Delta^*\Delta X$	*CRT (& CLVG)*
	$X \xrightarrow[\Delta]{} \Delta X$	*0.2*

Example. $X \to X \times X$.

Definition 0.3. There is a canonical isomorphism $X \to Y$ iff there is a canonical morphism $X \to Y$ that is an isomorphism.
We write $X = Y$ if there is a canonical isomorphism in (at least) one of the directions.
A triangle of canonical isomorphisms commutes if starting in any edge the loop composite is the identity. There is a bicanonical morphism $X \leftrightarrow Y$ if there is a canonical morphism in any direction.

Remark 1. If COMPOSITION holds, then a bicanonical morphism is an isomorphism

Canonical Morphism 2. *There is a bicanonical isomorphism*

$$f^*g^*X = (g \circ f)^*X.$$

$f^*g^*X \underset{1}{\to} (g \circ f)^*X$	*CRT (& CLVG)*	$(g \circ f)^*X \underset{1}{\to} f^*g^*X$	*CRT (& CLVG)*
$f^*g^*X \underset{g \circ f}{\to} X$	CRT^{-1}	$(g \circ f)^*X \underset{f}{\to} g^*X$	*CRT (& CLVG)*
$f^*g^*X \underset{f}{\to} g^*X \underset{g}{\to} X$	*CLVG ; CLVG*	$(g \circ f)^*X \underset{gf}{\to} X$	*CLVG*

Notation. If there are many goals at the same time we use $';'$ to seperate the solutions.

Example. For C a category with pullback, the above canonical isomorphism in the cocartesian fibration $C/\bullet \to C$ is pullback pasting.

Definition 0.4. A terminal object $\top_{\mathbb{B}}$ of $\mathbb{B}$ is strictly terminal, if the functor $(a_c)_!$ is constant.

Lemma 0.5. *If $\mathbb{B}$ has a strictly terminal object, then an object $X : C$ is terminal in C if the canonical map $X \underset{a_C}{\to} (a_c)_! X$ is cartesian.*

Canonical Morphism 3. *Assume $\mathbb{B}$ has a strictly terminal object $\top_{\mathbb{B}}$. If $C : \mathbb{B}$ has a terminal object $\top_C$, then*

$X : C$	$X \underset{1}{\to} \top_C$	
	$X \underset{1}{\to} \top_C \overset{\square}{\underset{a_C}{\to}} (a_C)_! \top_C = (a_C)_! X$	*CRT ; FCT*
	$X \underset{a_C}{\to} (a_C)_! X$	*0.2*

Remark 2. Why is there no composition rule like $\dfrac{x \underset{gf}{\to} z}{x \underset{f}{\to} y \underset{g}{\to} z}$?

Answer: If C is a pointed category, then

$x, z : C$	$x \underset{1}{\to} z$	COMPOSITION
	$x \underset{1}{\to} 0 \underset{1}{\to} z$	3

Canonical Morphism 4. *This canonical morphism compares F applied to a limit and the limit applied to the postcomposition with F.*

$X : C^I$	$\Delta^* X \underset{F}{\to} \Delta^* F_* X$	*CRT (& CLVG)*
	$\Delta^* X \underset{\Delta \circ F = F_* \circ \Delta}{\longrightarrow} F_* X$	*FCT*
	$\Delta^* X \underset{\Delta}{\to} X$	*CLVG*

Canonical Morphism 5. *In a symmetric monoidal category C viewed as a cocartesian fibration, the flip $X \otimes Y \cong Y \otimes X$ is canonical*

$X, Y : C$	$(X, Y) \underset{\mu = \mu \circ flip}{\to} Y \otimes X = \mu(Y, X)$	*FCT*
	$(X, Y) \underset{flip}{\to} (Y, X)$	*0.2*

where $flip : 2 \to 2$ swaps the two elements and $\mu : 2 \to 1$ sends everything not to the basepoint.

0.2 Cartesian fibrations

For any cocartesian fibration $E \to B$ we have an associated opposite cocartesian fibration $E^{\odot} \to B$. Its automatically equipped with a cocleavage. However, the partial cleavage we have to choose.

Notation. Given a cocartesian fibration $E \to B$ and $x : E_c, y : E_d$, $f : c \to d$, we write $y \underset{f}{\to} x := x^{op} \underset{f}{\to} y^{op}$, where $y^{op} \in E^{\odot}_d, x^{op} \in E^{\odot}_c$. It gets interpreted as $y \to fx$.

One can also mirror all the axioms for cartesian fibrations $p : \mathbb{E} \to \mathbb{B}$.

Definition 0.6 (Opposite structure)**.**

Name	OP	OP2 (TODO)
want to show	$y \underset{f}{\to} x$	$F \underset{G_*}{\to} H$
suffices to show	$y^{op} \underset{f}{\to} x^{op}$ in $E^{\odot}$	$F^{op} \underset{G_*^{op}}{\to} H^{op}$

Canonical Morphism 6. *if $\perp : I$ is an initial object, then*

$X : \mathcal{C}^I$	$\perp \underset{X}{\to} \Delta^* X$	*CRT (& CLVG)*
	$\perp \underset{\Delta \circ X = X_* \circ \Delta}{\longrightarrow} X = X_*(\mathrm{id})$	*FCT*
	$\perp \underset{\Delta}{\to} \mathrm{id}_I$	*CCT*
	$\perp = \Delta \perp \underset{1}{\to} \mathrm{id}_I$	*OP*
	$\mathrm{id}_{I^{op}} \underset{1}{\to} \top$	*3*

Canonical Morphism 7. *Dually, we have* $\operatorname{colim}_I X \underset{X}{\to} \top$.

1 Morphisms between cocartesian fibrations

In (closed) (symmetric) monoidal categories, things like the associators have to be assumed to be canonical. We switch now to ∞-categories Let $\mathcal{U}' \to \mathcal{U}$ denote the universal cocartesian fibration, i.e. for any small category B the map given by pullback

$$\mathsf{Map}(B,\mathcal{U}) \to \mathsf{Cocart}(B)$$

into small cocartesian fibrations over B is an equivalence.
Let B be a category (e.g. FinSet_*). Consider a natural transformation ϕ between two (pseudo)-functors $\chi_1, \chi_2 : B \to \mathcal{U}$. This datum is classified by some $f : c_1 \to c_2$ in $\mathcal{U}$, let $\tilde{\chi}_i : C_i \to \mathcal{U}'$ be the morphisms making

$$\begin{tikzcd} C_i \arrow[r, "\bar{\chi}_i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U}' \arrow[d] \\ B \arrow[r, "\chi_i"] & \mathcal{U} \end{tikzcd}$$

into pullbacks.

Example. Given a symmetric monoidal ∞-category $C^{\otimes} \to \mathsf{Finset}_*$, we may choose χ_E such that $\chi_E([n]) = C^n$.

This gives us $C_i \to B$ cocartesian fibrations such that $f_!$ is a cocartesian functor B.

Let $\mu : m_1 \to m_2$ in B be any morphism. Let $x_i : (C_i)_{m_i}$.
Then

Definition 1.1. We write $x_1 \underset{f,\mu}{\to} x_2$ for the proposition, that we have a canonical

$$x_1 \underset{f}{\to} x_2 \textit{ in } \mathcal{U}',$$

such that the induced map $f_! x_1 \to x_2$ *in* C_2 lies over μ.

Theorem 1.2. *The canonical calculus goes through for this notion, where one has to compose the different μ's whenever necessary to make sense.*

Reduction 1. We introduce the change of base and the internalization axiom:

	CHNGBS	INT
want to show	$x_1 \underset{1,\mu}{\to} x_2$ in $\mathcal{U}'$	$\bar{\chi}_1(x_1) \underset{\phi(\mu)}{\to} \bar{\chi}_2(x_2)$
suffices to show	iff $x_1 \underset{\mu}{\to} x_2$ in $C_1 = C_2$	iff $x_1 \underset{f,\mu}{\to} x_2$

where

$$\phi(\mu) : \chi_1(m_1) \to \chi_2(m_2)$$

denotes the up right composite in the naturality square

$$\begin{tikzcd}
\chi_1(m_1) \arrow[r, "\phi_{m_1}"] \arrow[d, "\chi_1(\mu)"'] & \chi_2(m_1) \arrow[d, "\chi_2(\mu)"] \\
\chi_1(m_2) \arrow[r, "\phi_{m_2}"'] & \chi_2(m_2)
\end{tikzcd}$$

Compare the previous two axioms to the following one, which is some interplay between rewriting along some canonical natural isomorphisms $\phi(\mu) \cong \tilde{\chi}_2(\mu) \circ \phi\mu$ and decomposing the latter into its two parts

	NEW
want to show	$\bar{\chi}_1(x_1) \underset{\phi(\mu)}{\to} \bar{\chi}_2(x_2)$
suffices to show	$\bar{\chi}_1 y \underset{\phi(m_2)}{\to} \bar{\chi}_2(x_2)$, $x_1 \underset{\mu}{\to} y$ in C_2

The NEW axiom works as follows

$$\phi(\mu)_!(\tilde{\chi}_1)(x_1) \simeq (\phi_{m_2})_! \circ (\chi_1(\mu))_!(\tilde{\chi}_1(x_1)) \overset{\star}{=} (\phi_{m_2})_!(\tilde{\chi}_1\mu_!x_1) \to (\phi_{m_2})_! \circ (\tilde{\chi}_1(y))$$

where we used

$$\chi(\mu)_!(\tilde{\chi}x) = \tilde{\chi}(\mu_!x) \qquad (\star)$$

which follows from the construction of the funcor $\chi(\mu)$.
Observe that NEW implies CHNGBS by putting $f = \mathrm{id}, y = x_2$.

Notation.

$$C^n \overset{\sim}{\leftarrow} C_n$$
$$(X_1, \dots, X_n) \mapsfrom \langle X_1, \dots, X_n \rangle$$

Canonical Morphism 8. *Let C be a symmetric monoidal cat. Let $g \circ f = h : i \to j$ be a commuting triangle of morphisms in FinSet_* (witnessing associativity or symmetry). Those induce functors $h_\# := \chi(h)_! : C^i \to C^j$.*

	$h_\# \xrightarrow[1]{} g_\# \circ f_\#$	*NAT*
$x : C^i$	$x \underset{h_\#}{\to} g_\#(f_\# x)$	*INT*
$x : C_i$	$x \underset{1,h}{\to} g_!(f_! x)$	*CHNGBS*
	$x \underset{h=g\circ f}{\to} g_!(f_!(x))$	*FCT , FCT , ID*

With NEW axiom

	$h_\# \xrightarrow[1]{} g_\# \circ f_\#$	NAT
$x : C^i$	$x \underset{h_\#}{\to} g_\#(f_\# x)$	NEW
$x : C^i$	$\tilde{\chi} g_!(f_! x) \to g_\# f_\# x$; $x \underset{h}{\to} g_! f_! x$	ID ; FCT , FCT , ID

Example. If $X, Y : C$ are free, we have a canonical $(X, Y) \underset{\otimes}{\to} Y \otimes X$.

Remark 3 (WARNING)**.** If X and Y are not free, it does not work. To see this, Plug in $X = Y$ to contradict UNIQUENESS. The flip is a natural transformation $\_ \otimes + \to + \otimes \_$, but the identity not.

Remark 4. In a symmetric monoidal cat, for all X, do we have a canonical $(X, X) \underset{\mu}{\to} X \otimes X$? We need a way to exclude the flip. One way out would be to say, that the left hand side depends on the context in a way, that is not expressed by the language of the cocartesian fibration $C^{\otimes} \to \mathsf{FinSet}_*$: The diagonal $C_1 \to C_2$ is not induced by something in the base.

Canonical Morphism 9. *Let $F^{\otimes} : C^{\otimes} \to D^{\otimes}$ be a strong symmetric monoidal functor between symmetric monoidal cats C, D.*

$X, Y : C$	$(X, Y) \underset{\otimes\circ F^2}{\to} F(X \otimes Y)$	*INT*
	$\langle X, Y\rangle \underset{F^{\otimes},\mu}{\to} F^{\otimes}(X \otimes Y)$	*FCT*
	$\langle X, Y\rangle \underset{1,\mu}{\to} X \otimes Y$	*CHNGBS*
	$\langle X, Y\rangle \underset{\mu}{\to} X \otimes Y$ *in* $C^{\otimes}$	*CCT*

The same with the NEW axiom

$X, Y : C$	$(X, Y) \underset{\otimes\circ F^2}{\to} F(X \otimes Y)$	*NEW*
	$X \otimes Y \underset{F}{\to} F(X \otimes Y)$; $\langle X, Y\rangle \underset{\mu}{\to} X \otimes Y$	*FCT,ID;CCT*

2 Extra structure

Furthermore the tensor-hom adjunction has to be equipped with canonical (co-) units. Then one has things like

Reduction 2. Let $C_\bullet$ be a symmetric monoidal category. If $\langle U, \_, W\rangle$ denote the composite

$$C \overset{(U,\_,W)}{\to} C^3 \overset{\sim}{\leftarrow} C_3$$

Notation: $\nu_1 = \mu \circ (\mu \times C) : C_3 \to C, \nu_2 = \mu \circ (C \times \mu)$. then we have

$U' \underset{U\otimes(\_\otimes W)=\langle U,\_,W\rangle\circ\nu_2}{\to} V$	EXT2
$\nu_2 \underset{\langle U,\_,W\rangle^*}{\to} \nu_1 \circ \langle U, \_, W\rangle$; $U' \underset{(U\otimes\_)\otimes W}{\to} V$	FCT , 8 ;
$U' \underset{(U\otimes\_)\otimes W}{\to} V$	

Canonical Morphism 10.

$[U, V] \underset{\_\otimes W}{\to} [U, V \otimes W]$	*CRT (& CLVG)*
$[U, V] \underset{U\otimes(\_\otimes W)}{\to} V \otimes W$	*2*
$[U, V] \underset{(U\otimes\_)\otimes W}{\to} V \otimes W$	*FCT*
$[U, V] \underset{U\otimes\_}{\to} V$	*CRT (& CLVG)*

where in the last line I assumed, that $[U, V]$ is the preferred cartesian lift from the cleavage.

Remark 5. One could think of formulating that inside the cocartesian fibration over FinSet_* associated to the symmetric monoidal category. The problem is that $[U, V]$ does not have an approiate universal property, i.e. there is no cartesian lift like $U, [U, V] \to U, V$, because U has to be used twice in contrast to linear type theory.

3 Adjunctions

Definition 3.1. An adjunction is a bifibration over $[1] = \{0 \xrightarrow{\mu} 1\}$, i.e. a cocartesian fibration with a choice of a (full) cleavage.

Lemma 3.2. *Let $\mathcal{A} = (F : C \rightleftarrows D : G)$ be an adjunction.*

Reduction 3. $\dfrac{X : C, Y : D}{\ } \left| \dfrac{X \underset{F}{\to} Y}{\text{iff } X \underset{1}{\to} F^*Y = G_!Y} \right|$

Proof. □

4 Fully faithfullness in B

Definition 4.1. $f : C \to D$ is fully faithfull iff any cocartesian lift of f is cartesian.

Example (For Cat)**.** *The direction $\Rightarrow$ is true. Conversely if any $x \in C$ has a cocartesian lift $x \to y$ such that y has a cartesian lift and f is fully faithfull then any cocartesian lift is cartesian*

Proof. We can prove both directions by three out of four:

$$\begin{array}{ccc} \mathrm{Hom}(t, f^*y) & = & \mathrm{Hom}(t, x) \\ \| & & \| \\ \mathrm{Hom}(f_!t, y) & = & \mathrm{Hom}(f_!t, f_!x) \end{array}$$

□

Theorem 4.2. *To satisfy uniqueness, the following condition is necessary: Choose the standart cartesian lift to be standart cocartesian, if possible. ($\star$)*

Proof. Assume there exists a standart cocartesian lift of some z which is cartesian. So Let $x : c$, such that $x \underset{f}{\to} f_!x = z$ is cartesian. Let $\alpha : x \cong y$ be a nontrivial automorphism (i.e. $x = y$). I claim, one can choose the partial cleavage, such that α is canonical but the condition from the theorem is not satisfied. Indeed if we choose $y \cong x \to f_!x$ to be the standart cartesian lift, then by CRT α is canonical. But this can not be standart cocartesian: If it would then we find two different maps $y \to x$ making the following square commute

$$\begin{tikzcd} y \arrow[r] \arrow[d, shift right, "\alpha^{-1}"'] \arrow[d, shift left, "\mathrm{id}"] & f_! y \arrow[d, equal] \\ x \arrow[r, "\square"'] & f_! x \end{tikzcd}$$

which contradicts the assumption that $x \to f_! x$ is cartesian. □

This is important whenever $f : c \to d$ is fully faithfull, because then the condition is satisfied for all $x : c$: Every standart cocartesian morphism is cartesian.

Example (USING ($\star$))**.** The limitcone of the constant diagram at X has to be judgementally equal to $\mathrm{id} : \Delta X \to \Delta X$.

Lemma 4.3 (using ($\star$))**.** *If the cocleavage splits, , to satisfy uniqueness, the standart cartesian lift along the identity is the identity.*

Proof. The identity will be a standart cocartesian lift. □

Example. In the cocartesian fibration $\mathcal{C}_{/\bullet}$, If the cocleavage splits, to satisfy uniqueness, for any A/X the canonical isomorphism $A \cong A \times_X X$ is the identity.

5 Negative results

Remark 6 (Warning)**.** Why do we even need the fibered setup? Let $f : x \to y$ in C. Then this gives a functor $F : [1] = \{0 < 1\} \to C$ in $\mathbb{B}$ and a canonical $0 \underset{F}{\to} F1 = y$. But this does not mean that there is a canonical map $x = F0 \to F1 = y$, because the axiom CCT only holds in one direction. Morally, every pathology we use to get a canonical map over F is stored in F.

Remark 7 (Warning)**.** Not every cartesian lift is canonical! any two cartesian lifts differ by an automorphism of the domain. Down to earth example: Consider Cat. For $X : C$ let $i : \mathrm{End}\, X \hookrightarrow C$. Then every automorphism of X induces a cartesian map $* \underset{i}{\to} X$ but in general it is not canonical. The only canonical cartesian map $* \underset{i}{\to} X$ is the one we get from the cleavage.

Remark 8 (WARNING)**.** We shouldn't expect a canonical morphism $ff^*x \underset{1}{\to} x$. Because f could be i as above, end then for any $y \in [x] \in \pi_0(C)$ we have a canonical isomorphism $x = i(i^*y) \to y$.

Remark 9 (WARNING)**.** We shouldn't expect canonical morphisms $i \underset{X}{\to} \operatorname{colim} X_\bullet$. Assume otherwise, that the inclusions would be canonical . Let $X \in C$, let $\mathrm{id} \neq \alpha \in \mathrm{Aut}(X)$. Consider the diagram $X_0 \xrightarrow{\alpha} X_1$ with $X_i := X$. Lets say the cleavage tells us $\operatorname{colim}_i X_i = X_1$ Then to conclude a canonical map

$0 \underset{X}{\to} \operatorname*{colim}_{i} X_i = X_1$, one could use CCT and then ID, which does not give the canonical inclusion α.
How does this observation play along with NAT? Fortunately nothing breaks: Lets talk about limits instead of colimits. Why cant we produce a canonical map $\Delta^* X \to Xi, \forall i$? By NAT, this is equivalent to a map $\Delta_! \Delta^* X \to X$. By the observation 8, we dont want such a canonical map. We only have the canonical *weakening* $\Delta^* X \underset{\Delta}{\to} X$.

Remark 10. Cisinski pointed out, that two objects, that happens to have a universal property, are canonically isomorphic. But I say they are only canonically isomorphic in a category whose objects memorize making a certain diagram of cartesian lifts commute.
Lets say we have a functor $F : \mathcal{C}^{op} \to \mathsf{Set}$. Then two representing objects $(X, x), (Y, y)$ are bicanonically isomorphic in $\int F$, because they are terminal objects 3. But its not reasonable to say there is a canonical isomorphism $X \to Y$. Any such isomorphism together with a representation of F by X induces a representation of F by Y.

Remark 11. We are not assuming that any cocartesian lift is canonical! Otherwise any isomorphism $FX \to Y$ would be canonical.

Remark 12. We cant hope for *[STRONG NATURALITY]: There is a canonical map $F(X) \to G(X)$ for any $X : C$ iff there is a canonical map $F \to G$.* Namely there are evil functors: If $\mathcal{C}$ is a pointed category, then there is the functor

$$\begin{aligned} Z : \mathcal{C} &\to \mathcal{C} \\ X &\mapsto X \\ f &\mapsto 0 \end{aligned}$$

Altough we have a canonical map $\operatorname{id} X \to ZX$ for all X, there is no (canonical) map $\operatorname{id} \to Z$.

Remark 13. Recall *COMPOSITION*: If there are canonical maps $X \to Y$ and $Y \to Z$ and $X \to Z$ then they should make the respective triangle commute. In any symmetric monoidal category $C^{\otimes} \to \mathsf{FinSet}_*$ we have the following non commutative triangle of canonical morphisms

$$\begin{tikzcd} (X, X) \arrow[rr, "\mu"'] \arrow[dr, "flip"'] & & X \otimes X \\ & (X, X) \arrow[ur, "\mu"] & \end{tikzcd}$$

Actually they are standart cocartesian lifts. What happened is, that for some stupid reason, $g \circ f = h$ and $g_!(f_! x) = h_! x$, but $g_! \circ f_! \neq h_!$, so one can accidently ask that the diagram commutes.

So if we want to have *COMPOSITION* we also the context is natural enough, the object (X, X) cannot be obtained as a morphisms only depending on a free variable $X : \mathcal{C}_1$.

Question 1. Is it okay to require *COMPOSITION*1? If there are canonical maps $X \underset{1}{\to} Y$ and $Y \underset{1}{\to} Z$ and $X \underset{1}{\to} Z$ then they should make the respective triangle commute.