THE UNIVERSITY OF HONGKONG
DEPARTMENT OF PHYSICS

Assignment 2

Course: *Computational Physics (PHYS4150/8150)* – Professor: *Dr. Ziyang Meng*
Tutor: *Mr. Menghan Song, Mr. Ting-Tung Wang*
Due date: *Oct. 17th, 2022*

This assignment is also a project, you will need to form group of 4 to 5 people and present your work on **question 1**, during the class on **Oct. 17th (Mon)**. The presentation should contain your answer to the questions, as well as how you solve them.

1. Poisson Equation
   In the lectures, we have learned to use three different relaxation methods to solve Poisson Equation $\Delta\phi(\vec{x}) = -\frac{1}{\epsilon_0}\rho(\vec{x})$ with Dirichlet boundary conditions.
   In this question, the error bound is set to be 10^{-4}, h and ϵ_0 are set to be 1.

   (a) Please use the Jacobi Relaxation method to solve the Poisson Equation with Neumann boundary condition. Unlike Dirichlet boundary condition, which specifies the exact function value on the boundary, Neumann boundary condition specifies the derivative of the function on the boundary along the normal direction.
   Other than updating every interior point using the function value of all neighbours and the on-site charge density, we then need to update the boundary value (including corners) using the given derivative. For example, for $\vec{x}$ on the right vertical edge, we have $\phi(\vec{x}) = \phi(\vec{x} - h\hat{i}) + h\frac{\partial\phi}{\partial x}$, and that on the left vertical edge is calculated as $\phi(\vec{x}) = \phi(\vec{x} + h\hat{i}) - h\frac{\partial\phi}{\partial x}$. For the corner points, we take average on the effect of the derivatives on the two directions. For the top left corner we have $\phi(\vec{x}) = \frac{1}{2}[\phi(\vec{x} + h\hat{i}) - h\frac{\partial\phi}{\partial x} + \phi(\vec{x} - h\hat{j}) + h\frac{\partial\phi}{\partial y}]$
   One may readily see that the solution is unique up to a constant, i.e. if $\phi_0(\vec{x})$ is a solution, so will $\phi_0(\vec{x}) + C$, where C is an arbitrary constant.
   Here we consider a two-dimensional 21×21 square lattice. Charge density is uniform in a triangular region as shown in the Figure 1. We consider an **isosceles right** triangle, whose hypotenuse lies in the middle of the 14th row and has length of 16. The points inside or on the edge of the triangle have charge 1, while those outside have no charge. And the boundary condition is $\nabla\phi \cdot \hat{n} = -\frac{81}{76}$, where $\hat{n}$ is the direction normal to the boundary pointing out, i.e. on the right vertical edges, we have $\frac{\partial\phi}{\partial x} = -\frac{81}{76}$ while that on the left vertical edges is $\frac{\partial\phi}{\partial x} = \frac{81}{76}$. This number is chosen to fulfil the *compatibility condition of Neumann problem*.

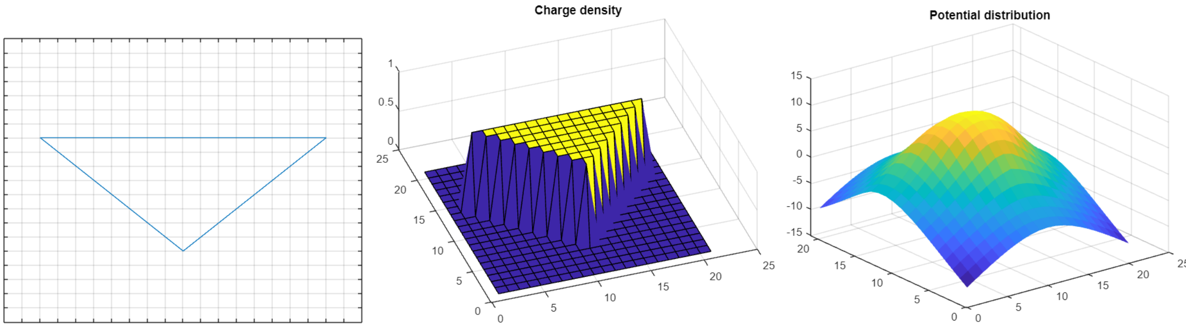

Figure 1: 1(a)

(b) Then we consider the Dirichlet boundary condition, $\phi(\vec{x})|_{boundary} = 0$. On the 2d plane, we consider a **regular** hexagon, with one of the vertexes lies in the middle of the second row, and length of the vertical diagonal equals to N (crosses **N+1** points), as shown in the Figure 2. The points inside or on the edge of the hexagon have charge 1, while those outside have no charge. The plane is a square lattice with size (N+3)×(N+3). Please consider N = 10, 20, 40 and use Jacobi Relaxation, Gauss-Seidel Relaxation and Successive Overrelaxation scheme. Compare the numerical result on the numbers of iterations for each method to the theoretical prediction on computational complexity introduced in the lecture notes (Chap1_3). Comment on the leading order, as well as its coefficient.

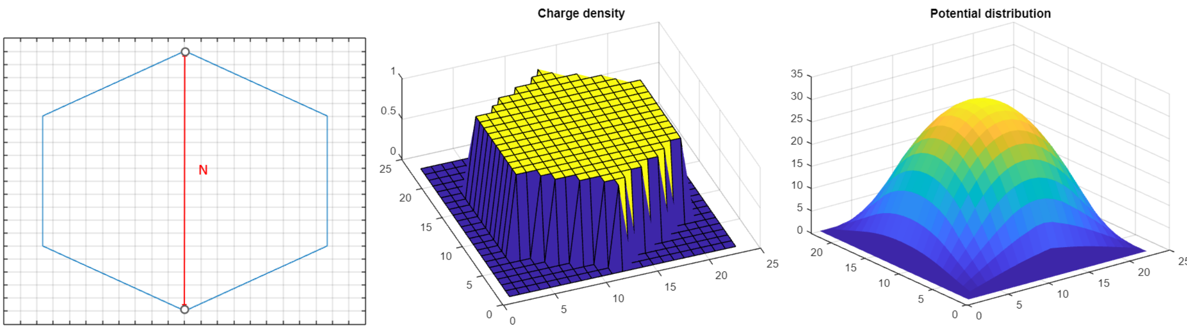

Figure 2: 1(b)

Complete code and necessary figures with analysis are required.

2. Perform the von Neumann stability analysis on FTCS (forward time derivative&centerd space derivative), FTFS, CTCS (which is the Leap-Frog method), and Lax-Wendroff schemes for advection equation $\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$ respectively. You need to show the complete analysis process.