CSC 212: Data Structures and Abstractions

04: Introduction to Analysis of Algorithms (part 2)

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Theoretical analysis

"mathematical models for analyzing time and space complexity"

Computational cost analysis

- Definition and importance
  - **computational cost**, expressed as $T(n)$, represents the resources (primarily **time**, sometimes memory) an algorithm requires to process input of a given size n
  - essential for algorithm comparison and optimization in real-world applications (without implementing/running a program)
- Mathematical framework (HW/SW independent)
  - based on counting (primitive/elementary) **operations**
    - arithmetic operations (additions, multiplications), comparisons, assignments, memory access operations, etc.
    - focuses on **asymptotic behavior**

Example

- Count the total number of operations as a function of the input size n
  - arithmetic operations, comparisons, assignments, array indexing, memory accesses, etc.

depends on specific HW

counting all operations is tricky, repetitive, and time-consuming

```
// sum of all elements in the array
int sum(int *A, int n) {
    int sum = 0;
    for (int i = 0 ; i < n ; i++) {
        sum = sum + A[i];
    }
    return sum;
}
```

Operation	Count	Time (ps)
variable declaration	2	
assignment	2 + n	
comparison (less than)	n + 1	
addition	n	
array access	n	
increment	n	

Counting operations

- Computational cost $T(n)$
  - ✓ count **elementary operations** that are **relevant** to the problem
  - ✓ express the total number of operations as a function of input size
- Examples:
  - ✓ sum of all elements in an array of length n
    - count the total number of additions => $T(n) = n$
  - ✓ finding the maximum value in an array of length n
    - count the total number of comparisons => $T(n) = n - 1$
- Formal assumptions
  - ✓ each **elementary operation** takes one time unit
  - ✓ operations execute sequentially (ignores parallelism, pipelining, and other HW optimizations)
  - ✓ infinite memory available

Practice

- Count the elementary operations (multiplications)

```
for (int i = 0 ; i < n ; i ++) {
    sum = sum * i;
}
```

Practice

- Count the elementary operations (divisions)

```
for (int i = 0 ; i < n ; i ++) {
    for (int j = 0 ; j < n ; j ++) {
        sum = sum / j;
    }
}
```

Practice

- Count the elementary operations (additions)

```
for (int i = 0 ; i < n ; i ++) {
    for (int j = 0 ; j < n ; j ++) {
        for (int k = 0 ; k < n ; k ++) {
            sum = sum + j;
        }
    }
}
```

Practice

- Count the elementary operations (multiplications)

```
for (int i = 0 ; i < n ; i ++) {
    for (int j = 0 ; j < n*n ; j ++) {
        sum = sum * j;
    }
}
```

Practice

- Count the elementary operations (multiplications)

$$\sum_{i=1}^{n-1} i = 1 + 2 + 3 + \ldots + (n-1) = \frac{n(n-1)}{2}$$

```
for (int i = 0 ; i < n ; i ++) {
    for (int j = 0 ; j < i ; j ++) {
        sum = sum * j;
    }
}
```

Some rules …

- Single loops
  - ✓ typically equals the number of iterations × the number of operations at each iteration
  - ✓ **requires careful analysis** of range and step size
- Nested loops
  - ✓ count operations from the innermost loop outward, multiplying the number of iterations at each level
  - ✓ **dependent loops** often result in operation counts that are not simply the product of the loop ranges, but rather require summation formulas to determine the exact count
- Consecutive statements
  - ✓ just add the counts

Some useful series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$$

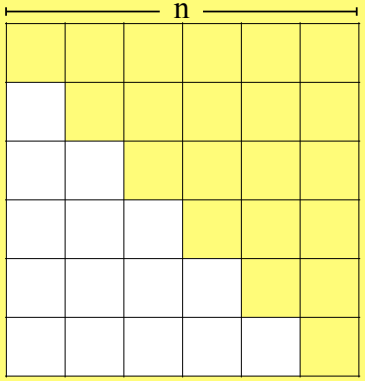
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, c \neq 1$$

https://tug.org/texshowcase/cheat.pdf

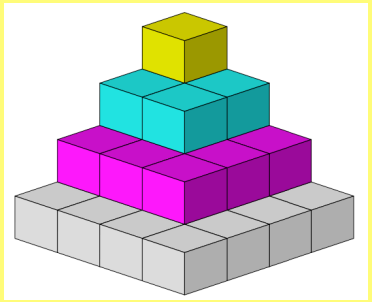
Practice



How many white squares as a function of n?

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$$
$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, c \neq 1$$

Practice



How many cubes as a function of n?

1 layer: 1
2 layers: 5
3 layers: 14
4 layers: 30
…
15 layers: ?
…
n layers: ?

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$$
$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, c \neq 1$$

Image credit: Stanford's CS 106B lectures

Case analysis

Practice

- What is T(n) for the following examples?

```
// returns the index of the last occurrence
// of the minimum value in the array
int r_argmin(int *A, int n) {
    int idx = 0;
    int current = A[idx];
    for (int i = 1 ; i < n ; i ++) {
        if (A[i] < current) {
            current = A[i];
            idx = i;
        }
    }
    return idx;
}
```

```
// returns the index of the first
// occurrence of k in the array
int l_argk(int *A, int n, int k) {
    for (int i = 0 ; i < n ; i ++) {
        if (A[i] == k) {
            return i;
        }
    }
    return -1;
}
```

Case analysis

- **best-case analysis**:
  - algorithm behavior under optimal input conditions (easiest input)
  - useful for understanding algorithm behavior, but it provides insufficient information for real-world performance
- **worst-case analysis**: (most commonly used)
  - algorithm behavior under the most unfavorable input conditions (hardest input)
  - critical for systems requiring performance guarantees
- **average-case analysis**:
  - expected algorithm behavior across all possible inputs
  - requires understanding of input probability distribution, often mathematically complex

An algorithm may run faster on some inputs than it does on others of the same size
e.g., sorting may run faster on already sorted data

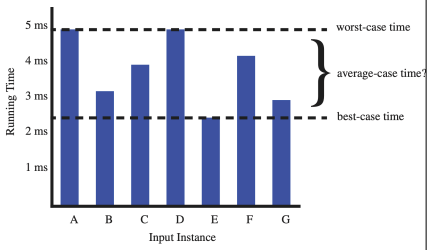


Image credit: Data Structures and Algorithms in C++, Goodrich, Tamassia, Mount

Practice

- Provide T(n) for the worst-, average-, and best-case
  - find value in an unsorted sequence (return first occurrence)
  - finding the largest element in an unsorted sequence
  - finding the largest element in a sorted sequence
  - factorial of a number — iterative algorithm