CSC 212: Data Structures and Abstractions

09: Priority Queues

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THINK BIG WE DO℠

Practice

- A server runs tasks in order. Each day, the server can run for at most T minutes. Each task has a duration. Given n tasks, write a program that outputs how many tasks can the server finish before exceeding T?
- Input:
  - ✓ first line: $n\ T$, $\quad n \leq 50$, $T \leq 500$
  - ✓ second line: n task times
- Output:
  - ✓ number of tasks that can be completed

```
sample input:
6 180
45 30 55 20 90 20

sample output:
4
```


Practice

- A server runs tasks in a specific order. Each day, the server can run for at most T minutes. Each task has a duration. The server runs tasks by alternating between the first and last remaining tasks. Given n tasks, write a program that outputs how many tasks can the server finish before exceeding T?
- Input:
  - ✓ first line: $n\ T$, $\quad n \leq 50$, $T \leq 500$
  - ✓ second line: n task times
- Output:
  - ✓ number of tasks that can be completed

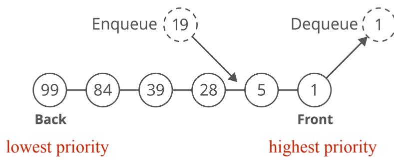
```
sample input:
6 180
45 30 55 20 90 20

sample output:
3
```


Priority queues

Priority queues

- Definition
  - a **priority queue** is a data structure similar to a queue, but where each element has an associated priority
  - elements with **higher priority** are removed before elements with lower priority
- Main Operations
  - **enqueue**: add element with an associated priority
  - **dequeue**: remove element with highest priority
- Applications
  - algorithms for graphs
  - event-driven simulation
  - search methods in artificial intelligence
  - job scheduling in operating systems, etc.



Implementation

- Representation
  - elements in a priority queue can be implemented as a collection of **<key,value>** pairs
    - **key**: determines the priority, used for comparison
    - **value**: actual data/payload associated with that priority

Operations (min-pq)	Return value
enqueue(5, A)	
enqueue(10, D)	
enqueue(3, B)	
dequeue()	(3, B)
enqueue(7, C)	
dequeue()	(5, A)
dequeue()	(7, C)
size()	1
isEmpty()	FALSE

Practice

- What is the output of the following code?

```
#include <iostream>
#include <queue>

int main() {
    // by default, std::priority_queue is a max-priority queue
    std::priority_queue<std::pair<int, std::string>> pq;

    pq.push({3, "Low priority"});
    pq.push({9, "High priority"});
    pq.push({5, "Medium priority"});

    while (!pq.empty()) {
        std::cout << pq.top().first << ": " << pq.top().second << "\n";
        pq.pop();
    }

    return 0;
}
```


Practice

- What is the output of this code?

```
#include <iostream>
#include <queue>
#include <utility> // for std::pair

int main() {
    // default priority_queue – max-priority queue behavior
    std::priority_queue<std::pair<int, std::string>> pq;

    pq.push(std::make_pair(3, "Job 1"));
    pq.push(std::make_pair(1, "Job 2"));
    pq.push(std::make_pair(5, "Job 3"));
    pq.pop();
    pq.push(std::make_pair(2, "Job 4"));
    pq.pop();
    pq.push(std::make_pair(7, "Job 5"));
    pq.pop();
    pq.pop();
    pq.push(std::make_pair(7, "Job 6"));
    pq.push(std::make_pair(7, "Job 7"));

    while (! pq.empty()) {
        std::pair<int, std::string> top = pq.top();
        std::cout << top.second << std::endl;
        pq.pop();
    }

    return 0;
}
```

Implementation

- Using arrays
  - ensure `enqueue` and `dequeue` work efficiently
  - array can be fixed-length or a dynamic array
- Considerations
  - highest priority can be defined in different ways
    - in a **max-priority queue**, highest priority refers to largest key value
    - in a **min-priority queue**, highest priority refers to smallest key value
  - for equal priorities, the order is determined by the underlying implementation
    - in some implementations, equal priority elements are served in FIFO order
    - in others, the order of elements with the same priority is undefined
  - underflow: throw an error when calling dequeue on an empty pq
  - overflow: throw an error when calling enqueue on a full pq

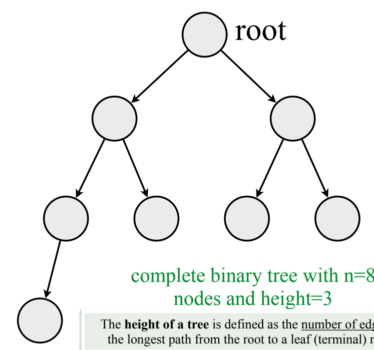
Implementation

- Array-based (unsorted array)
  - enqueue at the end — $\Theta(1)$ cost (amortized cost if using a dynamic array)
  - dequeue (extract max/min) — $\Theta(n)$ cost
    - requires searching the entire array
- Array-based (sorted array)
  - enqueue at position — $\Theta(n)$ cost
    - requires finding position for insertion and shifting elements
  - dequeue (extract max/min) — $\Theta(1)$ cost
- Binary heap (array-based)
  - most common and efficient
  - enqueue — $\Theta(\log n)$ cost
  - dequeue (extract max/min) — $\Theta(\log n)$ cost
  - can also build a binary heap from an unsorted array in $\Theta(n)$ cost (heapify)

Binary heaps

Complete binary tree

- Binary tree
  - tree data structure in which each **node** has at most two children, referred to as the left child and the right child
- Complete binary tree
  - binary tree in which every level, except possibly the last, is completely filled
    - all nodes in the last level are as far left as possible



complete binary tree with n=8 nodes and height=3

The **height of a tree** is defined as the number of edges on the longest path from the root to a leaf (terminal) node.

The height of a complete binary tree with n nodes is $\lfloor \log_2 n \rfloor$

Practice

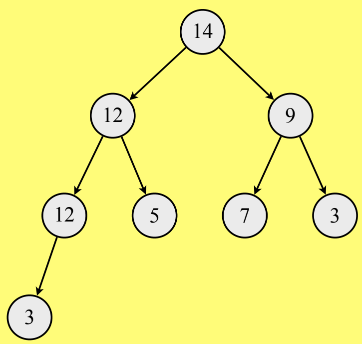
- Consider a complete binary tree of height h
  - what is n_{max}, the max number of nodes in the tree as a function of h?
    - hint: use a summation formula
  - what is n_{min}, the min number of nodes in the tree as a function of h?
- For a complete binary tree the following inequality holds: $n_{min} \leq n < n_{max} + 1$
  - take the logarithm (base 2) of this inequality and express h in terms of n

Binary heap

- Definition
  - **structure property**: a binary heap is a **complete binary tree**
  - **heap property**: a binary heap can be:
    - **max-heap**: each node's value is greater than or equal to its children's values
    - **min-heap**: each node's value is smaller than or equal to its children's values
- Considerations
  - the height of a binary heap is $\lfloor \log_2 n \rfloor$
  - the number of nodes at each level h is at most 2^h
  - the number of nodes in a heap is at most: $\sum_{i=0}^{h} 2^i = 2^{h+1} - 1$

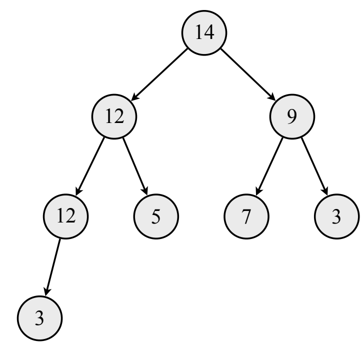
Practice

- Check:
  - structure property
  - heap-order property (max-heap)
- Add 3 elements
  - without violating properties
- Change 2 values
  - that violate the heap property



Array representation

- A binary heap can be represented as an array
  - **root** is at index 0
  - **last element** is at index $n - 1$
- For any node at index i:
  - **left child** is at index $2i + 1$
  - **right child** is at index $2i + 2$
  - **parent** is at index $(i - 1)$ // 2



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
14	12	9	12	5	7	3	3								

`n=8, capacity=16`

```cpp
template <typename T>
class PriorityQueue {
    private:
        T *arr;
        size_t capacity;
        size_t size;

        size_t parent(size_t i) { return (i-1) / 2; }
        size_t left(size_t i) { return 2*i + 1; }
        size_t right(size_t i) { return 2*i + 2; }

        void upHeap(size_t i);
        void downHeap(size_t i);

    public:
        PriorityQueue(size_t cap);
        ~PriorityQueue();

        void enqueue(const T& val);
        void dequeue();

        T& front();
        size_t get_size() { return size; }
        size_t get_capacity() { return capacity; }
        bool empty() { return size == 0; }
};
```

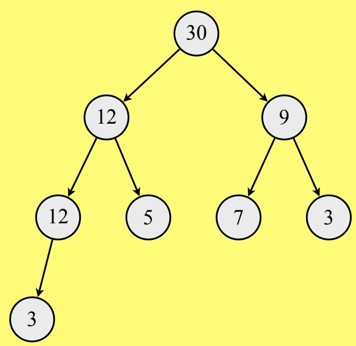

Enqueue (max-heap)

- Algorithm (min-heap is analogous)
  1. append element to the end of the array
  2. "bubble up" (a.k.a. `upHeap`)
     - **set** `idx` to the newly added element's position
     - **while** `idx` is not the root **and** element at `idx` is greater than its parent
       - swap them and update `idx` to the parent's position
- Time complexity
  - ✓ how many swaps are necessary in the worst case? $\Theta(\log n)$

https://visualgo.net/en/heap

Practice

- Enqueue 20
  - ✓ show resulting array
- Enqueue 1
  - ✓ show resulting array
- Enqueue 50
  - ✓ show resulting array



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
30	12	9	12	5	7	3	3								

size

Enqueue

```cpp
template <typename T>
void PriorityQueue<T>::enqueue(const T& val) {
    if (size == capacity) {
        throw std::out_of_range("PriorityQueue is full");
    }
    arr[size] = val;
    size ++;
    upHeap(size-1);
}


template <typename T>
void PriorityQueue<T>::upHeap(size_t idx) {
    while (idx > 0) {
        size_t p = parent(idx);
        if (arr[idx] > arr[p]) {
            std::swap(arr[idx], arr[p]);
            idx = p;
        } else {
            break;
        }
    }
}
```

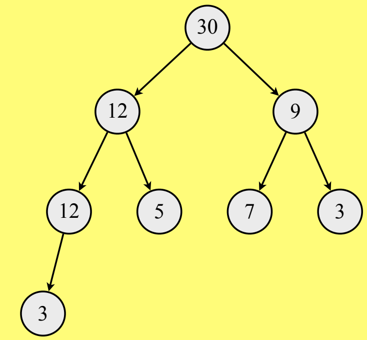
Dequeue (max-heap)

- Algorithm (min-heap is analogous)
  1. swap the root with the last element in the array
  2. remove the last element from the array (value to be returned)
  3. "bubble down" (a.k.a. `downHeap`)
     - **set** `idx` to the root element's position
     - **while** `idx` is not a leaf **and** element at `idx` is smaller than its largest child
       - swap them and update `idx` to that child's position
- Time complexity
  - ✓ how many swaps are necessary in the worst case? $\Theta(\log n)$

https://visualgo.net/en/heap

Practice

- Dequeue
  - ✓ show resulting array
- Dequeue
  - ✓ show resulting array
- Dequeue
  - ✓ show resulting array



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
30	12	9	12	5	7	3	3								

`size`

Dequeue

```
template <typename T>
void PriorityQueue<T>::dequeue() {
    if (size == 0) {
        throw std::out_of_range("PriorityQueue is empty");
    }
    arr[0] = arr[size-1];
    size --;
    downHeap(0);
}

template <typename T>
void PriorityQueue<T>::downHeap(size_t i) {
    while (true) {
        size_t largest = i;
        size_t l = left(i);
        size_t r = right(i);
        if (l < size && arr[l] > arr[largest]) {
            largest = l;
        }
        if (r < size && arr[r] > arr[largest]) {
            largest = r;
        }
        if (largest != i) {
            std::swap(arr[i], arr[largest]);
            i = largest; // move down to largest
        } else {
            break;
        }
    }
}
```


Performance (priority queues)

Method	Unsorted Array	Sorted Array	Binary Heap
Enqueue	O(1)	O(n)	O(log n)
Dequeue	O(n)	O(1)	O(log n)
Max/Min	O(n)	O(1)	O(1)
Size	O(1)	O(1)	O(1)
IsEmpty	O(1)	O(1)	O(1)

Practice

- Given a stream of integers, design an algorithm to efficiently report the **k-th largest element** seen so far at any time.
  - ✓ as new numbers arrive, your algorithm must update its data structure and return the current k-th largest number
- Sample input:
  - ✓ k = 3
  - ✓ values = {4, 5, 8, 2, 3, 5, 10, 9, 4}
- Output:
  - ✓ show k-th largest element after every insertion

Practice

- Given an array of integers and a window size k, design an algorithm that returns a new array where each element represents the **maximum value** within a sliding window of size k, as it moves from left to right across the input array
- Sample input:
  - ✓ k = 4
  - ✓ values = {1, 3, -1, -3, 5, 3, 6, 7}
- Output:
  - ✓ {3, 5, 5, 6, 7}

This sliding-window maximum is directly analogous to **max pooling** in neural networks