EULERIAN PATH $(G = (V, E), s, t)$

Input: A directed graph G with two vertices of s and t where $\deg^-(s) = \deg^+(s) - 1$ and $\deg^-(t) = \deg^+(t) + 1$ or $s = t$ s.t. $\deg^-(s) = \deg^+(s)$

Output: A maximal path from s to t traversing no edge twice, with all traversed edges having been removed from G.

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if $\deg^-(s) = 0$ and $\deg^+(s) = 0$, then
    return (s)

else

    Remove any edge (s, v) from G
    let P be EULERIAN PATH (G, v, t)
    if $\deg^-(s) \neq 0$ and $\deg^+(s) \neq 0$, then
        return $(s) + P$

else

return $\text{EULERIANPATH}(G, s, s) \mathbin{+\!\!+} P$

Proof:

Theorem: Let $G = (V, E)$ be a weakly connected, directed graph. If there exist:

- A vertex s s.t. $\deg^-(s) = \deg^+(s) - 1$ and a vertex t s.t. $\deg^-(t) = \deg^+(t) + 1$
- Or A vertex $s = t$ s.t $\deg^-(s) = \deg^+(s)$

... and $\deg^-(v) = \deg^+(v)$ for all other $v \in V$. then there exists a directed Eulerian Path from s to t in G.

Proof:

Let G be a directed graph that is weakly conre ted.

Basis:

Let $|E| = 0$. Then s is the only vertex in G s.t. $s = t$. Since $\deg^-(s) = \deg^+(s)$ for all vertrces, there does exist a Eulerian path from s to t in G.

Hypothesis:

Suppose $0 \le |E| \le k$, there exist a Eulerian path from vertex s to $\cancel{t}$ in G.

Induction

let $|E| = k+1$. Suppose any edge (s, v) is removed from G.

Two possibilities exist:

1. If removing e does not disconnect G. There is k edges then, which makes $\deg^+(s)$ equal $\deg^-(s)$ after removal, and $\deg^-(v) = \deg^+(v) - 1$ which is the new S. By hypothesis, there exist a Eulerian path from v to t, which means there is a eulerian path with e from s to t.

2. If removing e does disconnect G, then there is G_1 and G_2.

This means G_1 has less than k edges, and s will have $\deg^+(s) = \deg^-(s)$. By hypothesis, $(s, \dots s)$ will be a Eulerian path.

G_2 also has less than k edges, and v will have $\deg^-(v) = \deg^+(v) - 1$. This makes v the new s. By hypothesis, $(v, \dots t)$ will have a Eulerian path

Thus. s to t is a Eulerian Path in G

<u>Conclusion</u>

By PMI, there exist an Eulerian path from s to t

Cor: If $\deg^-(v) = \deg^+(v)$ for all $v \in V$, then there exist a directed <u>Eulerian cycle</u>.

Complexity Analysis

EULERIAN PATH

$$T(n) = \underbrace{1 \cdot T(E-1)}_{\text{Dominate}} + O(1)$$

$$\boxed{O(|E|)}$$