Supplemental Material for Stochastic Structural Analysis for Context-Aware Design and Fabrication

1 Derivative of Reduced Basis Vectors

Here we show how we take the derivative of our reduced space basis vectors $\bar{F}$. These basis vectors are the eigenvectors of the covariance matrix of our samples, stored in JF^{rigid}. Thus we wish to compute $\frac{\partial \bar{F}}{\partial \boldsymbol{w}}$ where $J\Sigma J^T = \bar{F}\lambda\bar{F}^T$, Σ is the covariance matrix of F and λ are its eigenvalues. In order to compute $\frac{\partial \bar{F}}{\partial \boldsymbol{w}}$ we follow the procedure of Xu et. al. [2015]. As they do, we begin by differentiating the eigenvalue decomposition to arrive at

$$\frac{\partial J\Sigma J^T}{\partial w^e} = \frac{\partial \bar{F}}{\partial w^e}\lambda\bar{F}^T + \bar{F}\frac{\partial \lambda}{\partial w^e}\bar{F}^T + \bar{F}\lambda\frac{\partial \bar{F}^T}{\partial w^e}. \tag{1}$$

Again following Xu et al. [2015], we rearrange this equation to the form

$$B^e = \omega^e\lambda + \frac{\partial \lambda}{\partial w^e} + \lambda\omega^{eT}, \tag{2}$$

where $B^e = \bar{F}^T\frac{\partial J^T\Sigma J}{\partial w^e}\bar{F}$ and $\omega^e = \bar{F}^T\frac{\partial \bar{F}}{\partial w^e}$ which is skew-symmetric. Because of this, $\frac{\partial \lambda}{\partial w^e} = diag\left(B^e\right)$ and so we arrive at

$$B^e - diag\left(B^e\right) = \omega^e\lambda + \lambda\omega^{eT}. \tag{3}$$

Here we diverge from Xu et al. [2015] somewhat. By exploiting the skew-symmetry of ω^e and diagonal structure of λ we can derive a simple update equation given as

$$\left(\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j\right)\omega^e_{ij} = B^*_{ij}, \tag{4}$$

where $\boldsymbol{\lambda}$ is the vector of eigenvalues and $B^* = B^e - diag\left(B^e\right)$. This equation can be solved in parallel for each ω_{ij}. Finally we can compute the approximate gradient $\frac{\partial \bar{F}}{\partial w^e} = \bar{F}\omega^e$. We note that J has a simple closed form derivative and the derivative of Σ can be computed using finite differences of the rigid-body force samples and so $\frac{\partial J\Sigma J^T}{\partial w^e}$ is straightforward to evaluate. Due to our assumption of a fixed contact surface, our simulator is entirely parameterized by the mass properties of our object: it's center of mass and its rigid body inertia tensor which is a total of 9 parameters. Again, because rigid body sampling is relatively quick, we can compute this gradient using finite differencing. Thus we are left with solving Equation 4 for every cell in our finite element system which can be done in an embarrassingly parallel fashion. One note of robustness, the eigenvector derivative can be singular if two or more eigenvalues are the same. In these cases we add a small random perturbation to the matrix to alleviate this problem [Xu et al. 2015].

2 Constraint Derivative Evaluation

In detail, we will have N_s stress samples ${}^i\boldsymbol{\sigma}$, each with a set of ${}^i\boldsymbol{\alpha}$ values. Using the method layed out in § 4.3, 4.4, and 6.2 of the main text , we can approximate this distribution and its derivative as

$$\Psi(\tilde{\sigma})N^{-1}\mathbf{b} \qquad \Psi(\tilde{\sigma})N^{-1}\frac{d\mathbf{b}}{d\,{}^i\boldsymbol{\sigma}}$$

We need to take a derivative w.r.t our design variables, i.e. ω_e. We can write the constraint as

$$\begin{aligned} J &= \boldsymbol{P}\left(S<1\right) \\ &= \int_0^1 \Psi(y)N^{-1}\mathbf{b}\,dy \\ &= \int_0^1 \Psi(y)\,dy\,N^{-1}\mathbf{b} \end{aligned}$$

Then stacking all our stress samples into a vector $\mathbf{s} = \begin{bmatrix}{}^1\sigma & {}^2\sigma & \ldots\end{bmatrix}^T$, we have

$$\begin{aligned} \frac{dJ}{d\boldsymbol{\omega}} &= \int_0^1 \Psi(y)N^{-1}\frac{\partial \mathbf{b}(\mathbf{s})}{\partial \mathbf{s}}\frac{\partial \mathbf{s}}{\partial \boldsymbol{\omega}}\,dy \\ &= \left[\int_0^1 \Psi(y)\,dy\right]N^{-1}\frac{\partial \mathbf{b}(\mathbf{s})}{\partial \boldsymbol{\omega}} \\ &= \mathbf{N}^*\frac{\partial \mathbf{b}(\mathbf{s})}{\partial \boldsymbol{\omega}} \\ &= \sum_{l=1}^{k} N^*_l(s)\frac{\partial b_l(\mathbf{s})}{\partial \boldsymbol{\omega}} \end{aligned}$$

, i.e., it boils down to taking derivatives of our samples w.r.t the density. This can be done with $3*(m+1)$ FEM solves as follows.

We have

$$b_l(s) = \frac{1}{N_s}\sum_{i=1}^{N_s}\psi_l(s_i) \qquad s_i = S = {}^i\tilde{\sigma} = \left[\sum_{e=1}^{N_{el}} S^{ep}\right]^{1/p}$$

So

$$\frac{\partial b_l(\mathbf{s})}{\partial \boldsymbol{\omega}} = \frac{1}{N_s}\sum_{i=1}^{N_s}\frac{\partial \psi_l(s_i)}{\partial s_i}\frac{\partial s_i}{\partial \boldsymbol{\omega}}$$

Now we have

$$\begin{aligned} \frac{\partial S}{\partial \boldsymbol{\omega}} &= \frac{1}{p}\left[\sum_{e=1}^{N_{el}} S^{ep}\right]^{\frac{1}{p}-1}\Bigg[p\sum_{e=1}^{N_{el}} S^{e(p-1)} \\ &\qquad \left(\frac{\partial S^e}{\partial \boldsymbol{\omega}}\left({}^i\boldsymbol{\sigma}^e\right) + \frac{\partial S^e}{\partial \tilde{\sigma}^e}\frac{\partial \tilde{\sigma}^e}{\partial\,{}^i\boldsymbol{\sigma}}\frac{\partial\,{}^i\boldsymbol{\sigma}}{\partial \boldsymbol{\omega}}\right)\Bigg] \\ &= \left[\sum_{e=1}^{N_{el}} S^{ep}\right]^{\frac{1}{p}-1}\left[\sum_{j=1}^{N_{el}} S^{e(p-1)}\frac{\partial S^e}{\partial \boldsymbol{\omega}}\right] \\ &\quad + \left[\sum_{e=1}^{N_{el}} S^{ep}\right]^{\frac{1}{p}-1}\left[\sum_{e=1}^{N_{el}} S^{e(p-1)}\frac{\partial S^e}{\partial \tilde{\sigma}^e}\frac{\partial \tilde{\sigma}^e}{\partial\,{}^i\boldsymbol{\sigma}}\right]\frac{\partial\,{}^i\boldsymbol{\sigma}}{\partial \boldsymbol{\omega}} \\ &= \left[\sum_{e=1}^{N_{el}} S^{ep}\right]^{\frac{1}{p}-1}\left[\sum_{e=1}^{N_{el}} S^{e(p-1)}\frac{\partial S}{\partial \boldsymbol{\omega}}\right] + \mathbf{c}_i^T\frac{\partial\,{}^i\boldsymbol{\sigma}}{\partial \boldsymbol{\omega}} \\ &= \mathbf{a}_i^T + \mathbf{c}_i^T\frac{\partial\,{}^i\boldsymbol{\sigma}}{\partial \boldsymbol{\omega}} \end{aligned}$$

where $\mathbf{c}_i$ is a vector. Then, rewriting Equation 21 from the main text in indicial notation (using b to index $\boldsymbol{\omega}$) we have

$$\begin{aligned}
\frac{\partial\, {}^i\boldsymbol{\sigma}}{\partial \boldsymbol{\omega}} = {}^i\sigma_{j,b} &= C_{jk}B_{k\ell}K^{-1}_{\ell m,b}\, {}^i f_m + C_{jk}B_{k\ell}K^{-1}_{\ell m}\, {}^i f_{m,b} \\
&= -C_{jk}B_{k\ell}K^{-1}_{\ell m}K_{mn,b}K^{-1}_{no}\, {}^i f_o + C_{jk}B_{k\ell}K^{-1}_{\ell m}\, {}^i f_{m,b} \\
&= C_{jk}B_{k\ell}K^{-1}_{\ell m}\left[-K_{mn,b}\, {}^i u_n + {}^i f_{m,b}\right] \\
&= C_{jk}B_{k\ell}K^{-1}_{\ell m}\left[-K_{mn,b}\left(\sum_{o=0}^{N_b}\bar{u}_{no}\, {}^i\alpha_o\right)\right. \\
&\quad \left. + \left(\sum_{o=0}^{N_b}\bar{f}_{mo,b}\, {}^i\alpha_o + \bar{f}_{mo}\, {}^i\alpha_{o,b}\right)\right] \\
&= -C_{jk}B_{k\ell}K^{-1}_{\ell m}K_{mn,b}\sum_{o=0}^{N_b}\bar{u}_{no}\, {}^i\alpha_o \\
&\quad + C_{jk}B_{k\ell}K^{-1}_{\ell m}\sum_{o=0}^{N_b}\bar{f}_{mo,b}\, {}^i\alpha_o \\
&\quad + C_{jk}B_{k\ell}K^{-1}_{\ell m}\sum_{o=1}^{N_b}\bar{f}_{mo}\, {}^i\alpha_{o,b} \\
&= -\underbrace{C_{jk}B_{k\ell}K^{-1}_{\ell m}K_{mn,b}\bar{u}_{no}\, {}^i\alpha_o}_{\mathrm{I}} \\
&\quad + \underbrace{C_{jk}B_{k\ell}K^{-1}_{\ell m}\bar{f}_{mo,b}\, {}^i\alpha_o}_{\mathrm{II}} \\
&\quad + \underbrace{C_{jk}B_{k\ell}\bar{u}_{\ell o}\, {}^i\alpha_{o,b}}_{\mathrm{III}}
\end{aligned}$$

Note the slight abuse of notation, in term II $o = 0$ corresponds to the mean vector. In this case $\alpha_0 = 1$. So now we have

$$\frac{\partial b_l(\mathbf{s})}{\partial \boldsymbol{\omega}} = \frac{1}{N_s}\sum_{i=1}^{N_s}\frac{\partial \psi_l(s_i)}{\partial s_i}\left[\mathbf{a}_i^T + \mathbf{c}_i^T\left(-\mathrm{I}^i + \mathrm{II}^i + \mathrm{III}^i\right)\right]$$

$$\begin{aligned}
\frac{\partial J}{\partial \boldsymbol{\omega}} &= \sum_{l=1}^{N_{pdf}} N_l^*(\mathbf{s})\frac{1}{N_s}\sum_{i=1}^{N_s}\frac{\partial n_l(s_i)}{\partial s_i} \\
&\quad \left[\mathbf{a}_i^T + \mathbf{c}_i^T\left(-\mathrm{I}^i + \mathrm{II}^i + \mathrm{III}^i +\right)\right] \\
&= \frac{1}{N_s}\sum_{l=1}^{N_{pdf}} N_l^*(\mathbf{s})\sum_{i=1}^{N_s}\left(-y^l_{i\ell}K^{-1}_{\ell m}K_{mn,b}\bar{u}_{no}\, {}^i\alpha_o\right. \\
&\quad \left. + y^l_{i\ell}K^{-1}_{\ell m}\bar{f}_{mo,b}\, {}^i\alpha_o + y^l_{i\ell}\bar{u}_{\ell o}\, {}^i\alpha_{o,b}\right) \\
&= \frac{1}{N_s}\left(\sum_{l=1}^{N_{pdf}} N_l^*(\mathbf{s})\sum_{i=1}^{N_s}\frac{\partial n_l(s_i)}{\partial s_i}a_{ij}\right. \\
&\quad - \sum_{l=1}^{N_{pdf}} N_l^*(\mathbf{s})\sum_{i=1}^{N_s} y^l_{i\ell}K^{-1}_{\ell m}K_{mn,b}\bar{u}_{no}\, {}^i\alpha_o \\
&\quad + \sum_{l=1}^{N_{pdf}} N_l^*(\mathbf{s})\sum_{i=1}^{N_s} y^l_{i\ell}K^{-1}_{\ell m}\bar{f}_{mo,b}\, {}^i\alpha_o \\
&\quad \left. + \sum_{l=1}^{N_{pdf}} N_l^*(\mathbf{s})\sum_{i=1}^{N_s} y^l_{i\ell}\bar{u}_{\ell o}\, {}^i\alpha_{o,b}\right)
\end{aligned}$$

where $y^l_{i\ell} = \frac{\partial n_l(s_i)}{\partial s_i}c_{ij}C_{jk}B_{k\ell}$. III is fast to compute. I and II need more attention. For I, we can rearrange to

$$\left(\underbrace{\bar{u}_{no}\sum_{l=1}^{N_{pdf}} N_l^*(\mathbf{s})\sum_{i=1}^{N_{samp}} y^l_{i\ell}\, {}^i\alpha_o\, K^{-1}_{\ell m}}_{Y_{n\ell}}\right) : K_{mn,b}$$

Now we can just compute

$$\left(K^{-1}_{m\ell}Y_{\ell n}\right) : K_{mn,b} \qquad Y_{\ell n} = \sum_{o=1}^{N_b} y_o\bar{u}_o^T$$

which requires N_b solves.

For II, we can rearrange to

$$\left(\underbrace{\sum_{l=1}^{N_{pdf}} N_l^*(\mathbf{s})\sum_{i=1}^{N_{samp}} y^l_{i\ell}\, {}^i\alpha_o\, K^{-1}_{\ell m}}_{X_{o\ell}}\right) : \bar{f}_{mo,b}$$

and compute

$$\left(K^{-1}_{m\ell}X_{\ell o}\right) : \bar{f}_{mo,b}$$

which again requires N_b solves.

The calculation of $\bar{f}_{m,b}$ can be done as described in § 1. The last piece is $\frac{\partial\, {}^i\alpha}{\partial \boldsymbol{\omega}}$. This is again a distribution, but composed of force samples projected onto this basis vector instead of stress samples. However, we cannot differentiate the individual samples, so we'll take a different approach.

Note that a sample of ${}^i\alpha$ can be thought of as being generated from a uniform random variable u as $\alpha = CDF^{-1}(u)$. Since $CDF(CDF^{-1}(u)) = u$, we have

$$\frac{\partial CDF(CDF^{-1}(u))}{\partial \boldsymbol{\omega}} = \frac{\partial u}{\partial \boldsymbol{\omega}} = 0$$

So

$$\frac{\partial CDF(CDF^{-1}(u))}{\partial \boldsymbol{\omega}} = \left.\frac{\partial CDF}{\partial \alpha}\right|_{\alpha}\left.\frac{\partial CDF^{-1}}{\partial \boldsymbol{\omega}}\right|_{u} + \left.\frac{\partial CDF}{\partial \boldsymbol{\omega}}\right|_{\alpha} = 0$$

$$\begin{aligned}
\left.\frac{\partial CDF^{-1}}{\partial \boldsymbol{\omega}}\right|_{u} &= -\left(\left.\frac{\partial CDF}{\partial \alpha}\right|_{\alpha}\right)^{-1}\left.\frac{\partial CDF}{\partial \boldsymbol{\omega}}\right|_{\alpha} \\
&= \frac{-1}{PDF(\alpha)}\left.\frac{\partial CDF}{\partial \boldsymbol{\omega}}\right|_{\alpha}
\end{aligned}$$

To calculate the derivative of the CDF of α w.r.t the densities, we can use finite differences of the rigid body simulation.

$$\frac{\partial CDF}{\partial \boldsymbol{\omega}} = \frac{\partial CDF}{\partial M}\frac{\partial M}{\partial \boldsymbol{\omega}}$$

where again M is the object parameters. $\frac{\partial CDF}{\partial M}$ will require 9 extra rounds of rigid body sampling (because there are 6 independent entries of the mass matrix, and 3 for the center of mass) or 18 extra rounds if centered differences are used. $\frac{\partial M}{\partial \boldsymbol{\omega}}$ can be calculated analytically.

3 Additional Results

Here we include additional probability computations as well as drop test results.

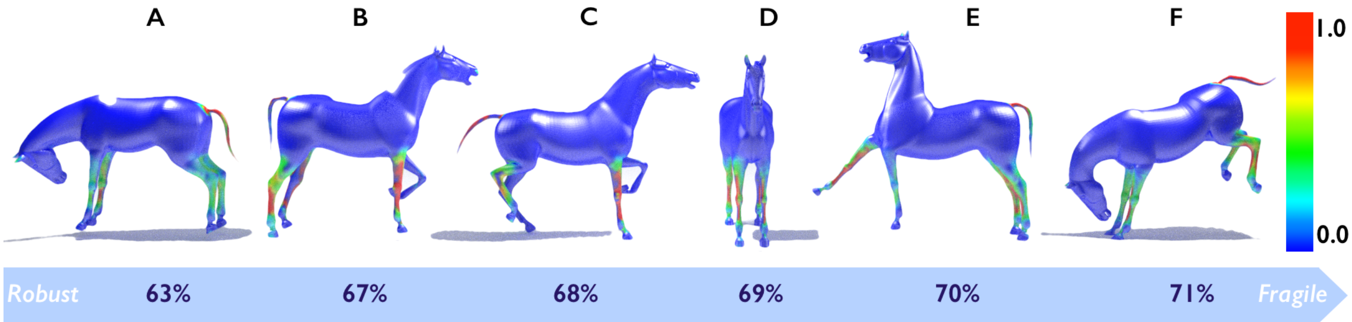

Figure 1: *Computed failure probabilities for a series of horse characters. We order the characters from most robust (left) to most fragile (right) which allows a toy designer to balance structural soundness with intended expressivity.*

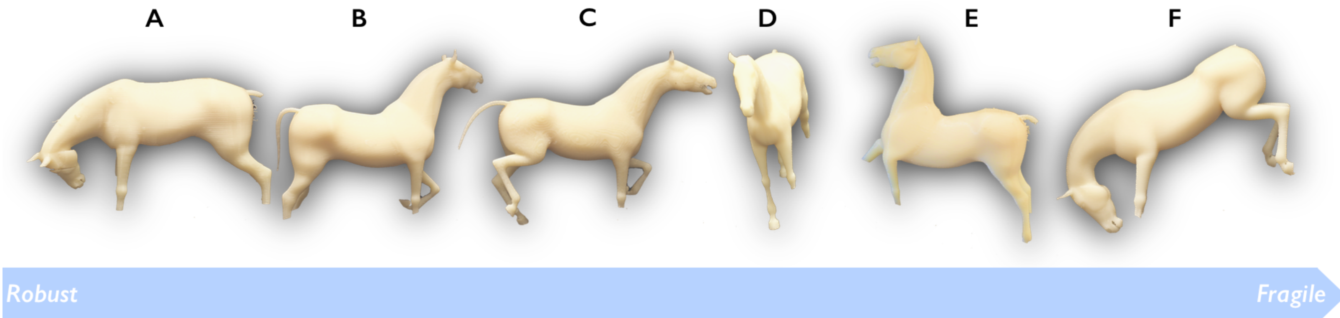

Figure 2: *Representative drop test results for the horse character. Notice that objects fracture where predicted by our algorithm.*

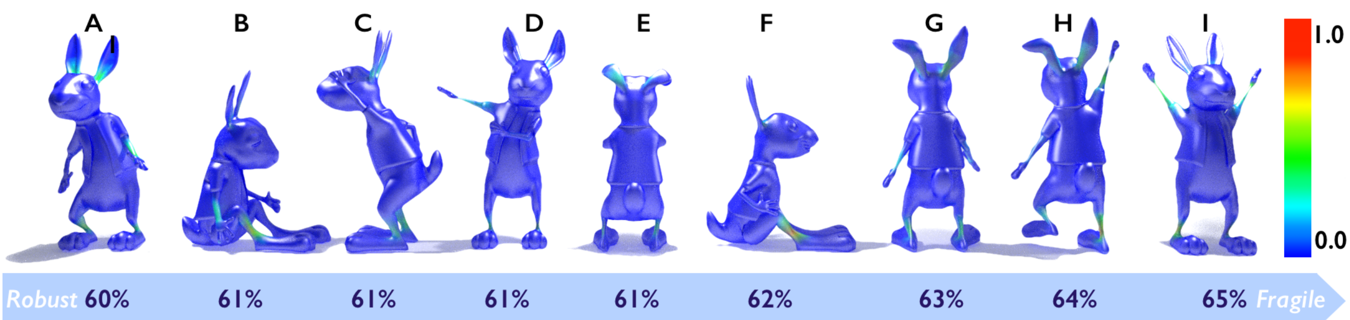

Figure 3: *Computed failure probabilities for a series of bunny characters using our Drop Test scenario. We order the characters from most robust (left) to most fragile (right) which allows a toy designer to balance structural soundness with intended expressivity.*

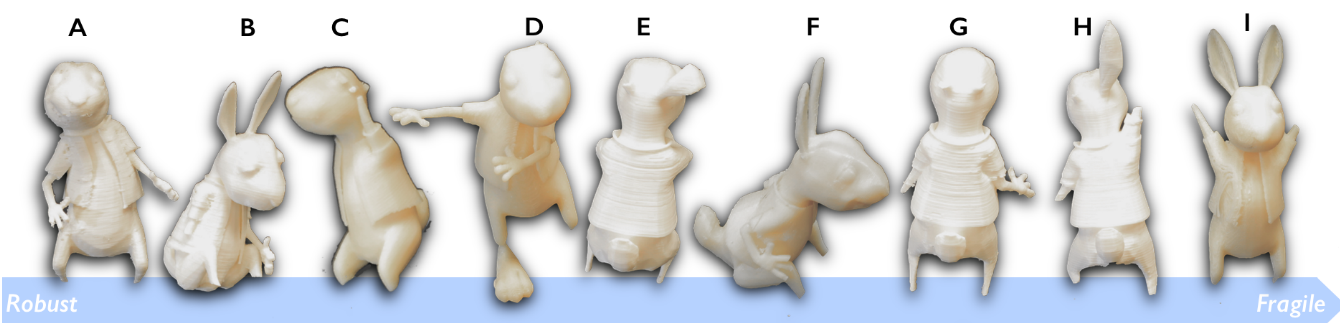

Figure 4: *Representative drop test results for the bunny character. Notice that objects fracture where predicted by our algorithm. As important, in no tests do we observe fractures in areas which are predicted to be perfectly robust (0% failure). The green box show's the outiline of the exterior container.*

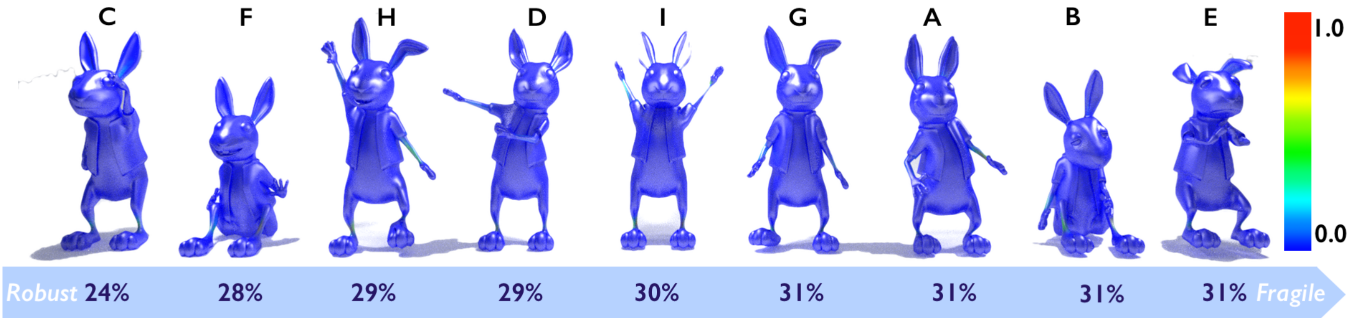

Figure 5: *Computed failure probabilities for a series of bunny characters using our Stair Test scenario. We order the characters from most robust (left) to most fragile (right) which allows a toy designer to balance structural soundness with intended expressivity.*

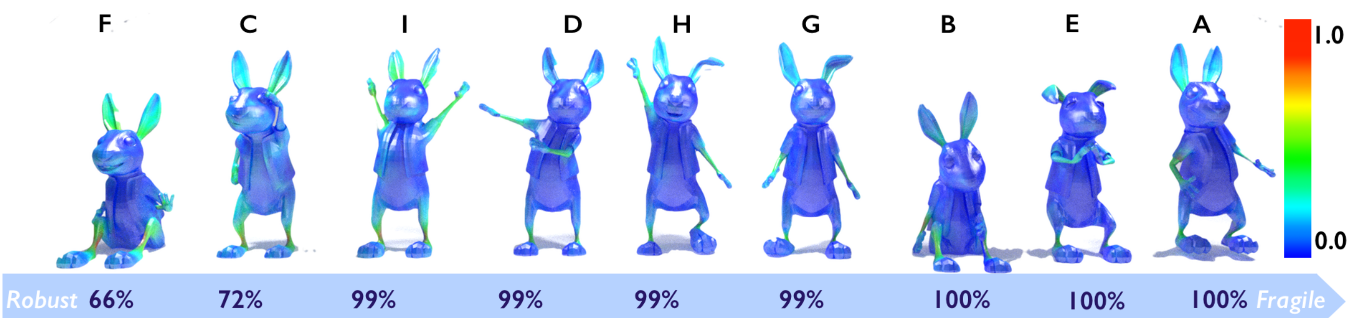

Figure 6: *Computed failure probabilities for a series of bunny characters using our Two Object Smash scenario. We order the characters from most robust (left) to most fragile (right) which allows a toy designer to balance structural soundness with intended expressivity.*

References

XU, H., SIN, F., ZHU, Y., AND BARBIČ, J. 2015. Nonlinear material design using principal stretches. *ACM Trans. Graph. 34*, 4 (July), 75:1–75:11.