# Maths @ 27/04/2023

## First order equations

When solving for y, we can apply the formula

$$\frac{dy}{dx} + \underbrace{8x^3}_{p(x)} y = \underbrace{12x^3}_{Q(x)}$$

$$I(x) = \int p(x)\,dx$$

$$I(x) = e^{\frac{8x^4}{4}} = e^{2x^4}$$

Multiply the integrating factor $I(x)$

$$e^{2x^4}\left[\frac{dy}{dx} + 8x^3 y\right] = e^{2x^4}(12x^3)$$

$$e^{2x^4}\frac{dy}{dx} + e^{2x^4} 8x^3 y = e^{2x^4} 12x^3$$

$$\frac{d}{dx}\left[I(x) \times y\right] = 12x^3 e 2x^4$$

Integrate both sides

$$\int \frac{d}{dx} e^{2x^4} y = \int 12x^3 e^{2x^4} dx$$

let $u = 2x^4$ $\quad \frac{du}{dx} = 8x^3$

$dx = \frac{du}{8x^3}$ $\qquad dx\, x^3 = \frac{du}{8}$

$$e^{2x^4} y = \cancel{12}^{\cancel{6}\,3} \int \frac{du}{\cancel{8}\,\cancel{4}\,2} e^u$$

$$e^{2x^4} y = \frac{3}{2}\int e^u \, du$$

$$e^{2x^4} y = \frac{3}{2} e^u + c$$

$$e^{2x^4} y = \frac{3}{2} e^{2x^4} + c$$

$$y = \frac{3}{2} + ce^{-2x^4}$$

Q2

Solving LDE

$$xy' + x^2 - x^2 y = 0$$

$$xy' - x^2 y = x^2$$

$$y' - xy = x$$

$P(x) = x$
$Q(x) = x$

The integrating factor $I(x)$
$= e^{\int P(x)}$

$I(x) = e^{\int x} = e^{-x^2/2}$

$\left[e^{-x^2/2}\right] y = \left[e^{-x^2/2}\right] x$

$\times \frac{d}{dx}$

$\frac{d}{dx}[y]\left[e^{-x^2/2}\right] = xe^{-x^2/2}$

Integrate both sides

$$\int \frac{d}{dx} e^{-x^2/2} = \int x e^{-x^2/2}$$

$$u = -x^2/2 \qquad \frac{du}{dx} = -x$$

$$du = -x\,dx$$

$$e^{x^2} \qquad y = -\frac{1}{2}\int e^{u}\,du$$

$$e^{-x^2/2} y = e^{u} + C$$

$$e^{-x^2/2} y = e^{-x^2/2} + C$$

$$Y = e^{-x^2/2} + C$$

$$\overline{e^{-x^2/2}}$$

$$y = 1 + ce^{x^2/2}$$

<u>separable diff eqn</u>

Generally in the form

$$y' = F(ax+by)$$

Consider

$$y' = x + y$$

$$u = x + y$$

$$\therefore u' = 1 + y'$$

$$y' = u' - 1$$

Rewriting the eqn gives,

$$(u' - 1) = u$$

$$\frac{du}{dx} - 1 = u$$

$$\frac{du}{dx} = u + 1$$

$$du = (u+1)\,dx$$

$$\left[\frac{1}{u+1}\right] du = dx$$

$$\int \frac{1}{u+1}\, du = \int dx$$

$$\ln|u+1| = x + c$$

$$e^{\ln|u+1|} = e^{x+c}$$

$$u+1 = e^{x} e^{c}$$

$$u+1 = e^{x} c$$

$$u+1 = ce^{x}$$

$$u = ce^{x} - 1$$

$$x+y = ce^{x} - 1$$

$$\underline{y = ce^{x} - x - 1}$$

This method is known as substitution

Consider

$$y' = 3x + y$$

$$u = 3x + y$$

$$u' = 3 + y'$$

$$\therefore \quad y' = u' - 3$$

$$u' - 3 = u$$

$$\frac{du}{dx} - 3 = u$$

$$\frac{du}{dx} = u + 3$$

$$du = (u + 3)\,dx$$

$$\frac{du}{u+3} = dx$$

$$\int \frac{1}{u+3}\, du = \int dx$$

$$\ln|u+3| = x + c$$

$$e^{\ln(u+3)} = e^{x+c}$$

$$u+3 = e^{x} \times e^{c}$$

$$u+3 = ce^{x}$$

$$u = ce^{x} - 3$$

$$3x + y = ce^{x} - 3$$

$$y = ce^{x} - 3x - 3$$

## Bernoulli equations

They are usually given as

$$y' + p(x) = q(x)y^n$$

If $n = 0$

$$y' + p(x) = q(x)y^0$$

$$y' + p(x) = q(x)$$ — Linear eq$^n$

## Questions

Find the solution to the linear equation

$$y' - y\sin x = 0$$

Let $p(x) = \sin x$

The integrating factor

$$I = e^{\int p(x)}$$

$$\therefore I(x) = e^{\int \sin x}$$

$$= e^{\cos x}$$

$$e^{\cos x} y = 0$$

Reversing the product rule and integrating both sides

$$\frac{d}{dx}\left(e^{\cos x} y\right) = 0$$

$$y \int e^{\cos x} = \int 0$$

$$ye^{\cos x} = c$$

$$y = ce^{-\cos x}$$

② Solve the differential equation

$$y' \cos x + y \sin x = 1$$

$y' \cos x + y \sin x = 1$ $\neq$ standard form of linear eqn

$$\frac{y' \cancel{\cos x}}{\cancel{\cos x}} + y \frac{\sin x}{\cos x} = \frac{1}{\cos x}$$

$$y' + y\tan x = \sec x$$
$$p(x) = \tan x$$
The integrating factor $I(x) = e^{\int p(x)}$

$$I(x) = e^{\int \tan x}$$
$$I(x) = e^{\ln|\sec x|}$$
$$I(x) = \sec x$$
$$(\sec x \times y) = \sec x \times \sec x$$
$$y \sec x = \sec^2 x$$

$$y \int \sec x \frac{d}{dx} = \int \sec^2 x$$

$$y \sec x = \tan x + c$$
$$y = \frac{\tan x + c}{\sec x}$$
$$y = \frac{\left(\frac{\sin x}{\cos x}\right)}{\frac{1}{\cos x}} + \frac{c}{\frac{1}{\cos x}}$$

$$y = \sin x \times \cancel{\cos x} + c(\cos x)$$

$1 \quad \cancel{\cos x} \quad 1$

$y = \sin x + c\cos x$

(3) Find a solution to the Bernoulli eqn

$(y^4 + x^4 y)\,dx - 3x^5\,dy = 0$

$y^4 + x^4 y\,dx = 3x^5\,dy$

$y^4 + x^4 y = 3x^5 \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{y^4 + x^4 y}{3x^5}$

$\frac{dy}{dx} = \frac{y^4}{3x^5} + \frac{y}{3x}$

$y' - \frac{y}{3x} = \left(\frac{1}{3x^5}\right) y^4$

Divide both sides by $y^4$

$y^{-4}y' - y^{-4}\left(\frac{y}{3x}\right) = \frac{1}{3x^5}$

$y^{-4}y' - \frac{y^{-3}}{\ } = \frac{1}{\ }$

$1 \qquad 1 \qquad 3x \qquad 3x^5$

Let $u = y^{-3}$

$$u' = -3(y)^{-4}y'$$

$$u' = -3y^{-4}y'$$

$$y^{-4}y' = -\frac{u'}{3} \qquad\qquad y^{-4}y' = -\frac{u'}{3}$$

$$-\frac{u'}{3} - \frac{1}{3x}[u] = \frac{1}{3x^5}$$

$$-\cancel{\frac{1}{3}}\left[u' + \frac{1}{x}u\right] = \cancel{\frac{1}{3}} \times \frac{1}{x^5}$$

$$u' + \frac{1}{x}u = -\frac{1}{x^5}$$

Let $1/x$ be $p(x)$

$$I(x) = e^{\int p(x)} = e^{\int 1/x}$$

$$= e^{\ln x} = x$$

Multiply through by $I(x)$

$$xv' + \frac{1}{x} xv = -\frac{1}{x^5} x$$

$$xv' + v = -\frac{1}{x^4}$$

Reversing the derivatives for the LHS

$$\frac{d}{dx}(xv) = -\frac{1}{x^4}$$

$$\int \frac{d}{dx}(xv) = \int -\frac{1}{x^4}\, dx$$

$$xv = \frac{1}{3x^3} + C$$

$$v = \frac{\frac{1}{3x^3} + C}{x}$$

$$v = \frac{1}{3x^4} + \frac{C}{x}$$

As $v = y^{-3}$

$$y^{-3} = \frac{1}{3x^4} + \frac{c}{x}$$

$$\frac{1}{y^3} = \frac{1+3x^3c}{3x^4}$$

$$y^3 = \frac{3x^4}{1+3x^3c}$$

$$y = \sqrt{\frac{3x^4}{1+3x^3c}}$$

4. Use the substitution method to solve the eqn

$$y' = \sin(x+y)$$

let $u = x+y$

$u' = 1 + y'$

$y' = u' - 1$

$$u' - 1 = \sin u$$

$$\frac{du}{dx} = \sin u + 1$$

$$du = (\sin u + 1)\,dx$$

$$\left(\frac{1}{\sin u + 1}\right) du = dx$$

$$\int \frac{1}{\sin u + 1}\, du = \int dx$$

$$\frac{2}{\cot \frac{u}{2} + 1} = x + c$$

$$\frac{2}{x+c} = \cot u/2 + 1$$

$$\frac{2}{x+c} - 1 = \cot \frac{u}{2}$$

$$\frac{2}{x+c} - 1 = \cot\left(\frac{x+y}{2}\right)$$

5. Find the solution to the exact equation

$$(6y + xe^{-y})\,dy - e^{-y}\,dx = 0$$

$$\delta m = \delta N$$

$$\delta_{6y+xe^{-y}} = \delta -e^{-y}$$

$$\frac{\delta y}{dx} = e^{-y} \qquad \frac{\partial}{\partial y} = e^{-y}$$

$$\psi = \int m(x,y)\,dx + h(y)$$

$$\psi = \int -e^{-y}\,dx + h(y)$$

$$\psi = -xe^{-y} + h(y)$$

$\psi_y = xe^{-y} + h'(y)$ (After differentiating)

$\psi_y = N(x,y)$ as its an exact eqn

$$\cancel{xe^{-y}} + h'(y) = 6y + \cancel{xe^{-y}}$$

$$h'(y) = 6y$$

$$\int h'(y) = \int 6y$$

$$h(y) + k_1 = 3y^2 + k_2$$

$$h(y) = 3y^2 + k_2 - k_1$$

$h(y) = 3y^2 + k$

Recall that

$$\psi = -xe^{-y} + h(y)$$

$$\psi = -xe^{-y} + 3y^2 + k$$

$$c_1 - k = -xe^{-y} + 3y^2$$

$$C = -xe^{-y} + 3y^2$$