Second Order Linear Equations

Roots

Distinct real roots $\quad y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Equal real roots $\quad y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

Complex conjugate roots $\quad y(x) = e^{ax}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

where $r = \alpha + \beta i$

Q1 Find the general solution to 2nd OE

$$y'' + 5y + 4 = 0$$

$$r^2 + 5r + 4$$

$$r^2 + r + 4r + 4 = 0$$

$$r(r+1) + 4(r+1) = 0$$

$$(r+1)(r+4) = 0$$

$$r = -1, r = -4$$

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$y(x) = c_1 e^{(-1)x} + c_2 e^{(-4)x}$$

$$y(x) = c_1 e^{-x} + c_2 e^{-4x}$$

Reduction of Orders

$y'' - 2y' + y = 0$ if $y_1 = e^x$

Assuming that $y_2 = vy_1$

$y_2 = ve^x$

$y_2' = v'e^x + ve^x$

$y_2'' = v''e^x + v'e^x + v'e^x + ve^x$

$y_2'' = v''e^x + 2v'e^x + ve^x$

Plugging this into the equation

$v''e^x + 2v'e^x + ve^x - 2[v'e^x + ve^x] + ve^x = 0$

$v''e^x + \cancel{2v'e^x} + \cancel{ve^x} - \cancel{2v'e^x} + \cancel{2ve^x} + \cancel{ve^x} = 0$

$v''e^x = 0$

Substituting $w = v'$, $\therefore$ $w' = v''$

$w'e^x = 0$

$\frac{dw}{dx}[e^x] = 0$

$\frac{dw}{dx} = 0$

$w = C$

If $w = c$, recall that $w = v'$

$\therefore$ Integrating w should give v

$\int w\,dx = \int c\,dx = cx + k$

Assuming constants (w) $c = 1$ & $k = 0$

$(1)x + 0 = x$

$v = x$

$\therefore y_2 = vy_1$

$y_2 = xy_1 \quad y_2 = xe^x$

$\therefore y_2 = c_1e^x + c_2xe^x$

Undetermined coefficients for homogenous equation

Find the general solution to the equation

$y'' + 3y' = 2x - 6e^{-3x}$

Firstly solving the LHS

$r^2 + 3r = 0$

$r(r+3) = 0$

$\boxed{r = 0,\ r = -3}$

As the values are unequal

$$Y_c = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$Y_c = C_1 e^{0x} + C_2 e^{-3x}$$

$$Y_c = C_1(1) + C_2 e^{-3x}$$

$$Y_c = C_1 + C_2 e^{-3x}$$

Matching the RHS to deduce a particular guess from $2x - 6e^{-3x}$ becomes

$$Y_p = Ax + B + Ce^{-3x}$$

$$Y_p = (Ax+B)x + Ce^{-3x}(x)$$

$$Y_p = Ax^2 + Bx + Cxe^{-3x}$$

$$Y_p' = 2Ax + B + Ce^{-3x} - 3Cxe^{-3x}$$

$$Y_p'' = 2A + 0 - (0) - (+3Ce^{-3x}) - [(-3)[-3Cxe^{-3x}] + [-3Ce^{-3x}]]$$

$$Y_p'' = 2A - 6Ce^{-3x} + 9Cxe^{-3x}$$

$$2A - 6Ce^{-3x} + 9Cxe^{-3x} + 3[2Ax + B + Ce^{-3x} - 3Cxe^{-3x}]$$

$$= 2A - 6Ce^{-3x} + \cancel{9Cxe^{-3x}} + 6Ax + 3B + 3Ce^{-3x} - \cancel{9Cxe^{-3x}}$$

$$= 2A + 3B + 6Ax - 3Ce^{-3x}$$

$$2A + 3B + 6Ax - 3Ce^{-3x} = 2x - 6e^{-3x}$$

$$2A + 3B = 0 \quad \text{—— } (1)$$

$$6A = 2 \quad \text{—— } (2)$$

$$-3C = -6 \quad \text{—— } (3)$$

From (2)

$$A = 1/3 \quad \therefore \quad 2\left[\frac{1}{3}\right] + 3B = 0$$

$$\frac{2}{3} + 3B = 0$$

$$3B = -2/3$$

$$B = -\frac{2}{9}$$

$$-3C = -6$$

$$C = +2$$

$$Y_p = Ax^2 + Bx + Cxe^{-3x}$$

$$Y_p = \frac{1}{3}x^2 - \frac{2}{9}x + 2xe^{-3x}$$

$$Y = Y_p + Y_c$$

$$Y = \frac{1}{3}x^2 - \frac{2}{9}x + 2xe^{-3x} + C_1 + C_2e^{-3x}$$

Variation of parameters for non homogenous eqn

Find the solution of the differential eqn

$$y'' - 2y' + y = \frac{3e^x}{2x}$$

Rewrite the LHS

$r^2 - 2r + 1$

$r^2 - r - r + 1$

$r(r-1) - 1(r-1)$

$(r-1)(r-1) \quad r = 1 \text{ (twice)}$

Since the roots are equal

$$y_c(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} x$$

$$y_c(x) = c_1 e^x + c_2 x e^x$$

Recall that

$$u_1' y_1 + u_2' y_2 = 0 \ \& \ u_1' y_1' + u_2' y_2' = g(x)$$

From $y_c(x)$

$[y_1, y_2] = [e^x, xe^x]$

$[y_1', y_2'] = [e^x, e^x + xe^x]$

$\therefore \ u_1' e^x + u_2' xe^x = 0 \quad — (1)$

$u_1' e^x + u_2' (e^x + xe^x) = \frac{3e^x}{2x} \quad — (2)$

Taking 2 from 1

$$u_2' e^x + u_2' \cancel{x e^x} - \cancel{u_2'} x e^x = \frac{3e^x}{2x}$$

$$u_2' \cancel{e^x} = \frac{3\cancel{e^x}}{2x}$$

$$u_2' = \frac{3}{2x}$$

$$u_1' e^x + \frac{3}{2x}(xe^x) = 0$$

$$u_1' \cancel{e^x} = \frac{-3}{2\cancel{x}} \cancel{x} \cancel{e^x}$$

$$u_1' = \frac{-3}{2}$$

$$u_2 = \int u_2' \ \$ \ u_1 = \int u_1'$$

$$u_2 = \int \frac{3}{2x} \quad \$ \ u_1 = \int -3/2$$

$$u_2 = \frac{3}{2} \ln|x| \quad u_1 = -3/2 \, x$$

$$y_p(x) = u_1 y_1 + u_2 y_2$$

$$y_p(x) = \frac{-3}{2} x (e^x) + \frac{3}{2} \ln|x| (xe^x)$$

$$y_p(x) = -\frac{3}{2} x e^x + \frac{3}{2} \ln|x| x e^x$$

$y = y_c(x) + y_p(x)$

$$\boxed{y = c_1 e^{x} + c_2 x e^{x} - \frac{3}{2} x e^{x} + \frac{3}{2} x e^{x} \ln|x|}$$