Partial Differential Equations

Consider

$$u(x,t) = v(x)w(t)$$

$$\frac{\partial u}{dx} = v'(x)w(t)$$

$$\frac{\partial v}{dt} = v(x)w'(t)$$

Using separation of variables, rewrite the PDE as an ODE

$$2\frac{\delta^2 u}{\delta x^2} - 3\frac{\partial u}{dy} = 0$$

$$2\frac{\delta^2 u}{\delta x^2} = 3\frac{\partial u}{\partial y}$$

$$2[v''(x)w(y)] = 3v(x)w'(y)$$

$$2\frac{d^2v}{dx^2}w(y) = 3v(x)\frac{dw}{dy}$$

$$\frac{2}{v(x)}\frac{d^2v}{dx^2} = 3\frac{dw}{dy}w(y)^{-1} = -\lambda$$

$$\frac{2}{v(x)}\frac{d^2v}{dx^2} = -\lambda \qquad 3w(y)^{-1}\frac{dw}{dy} = -\lambda$$

$$\frac{d^2v}{dx^2} = \frac{-\lambda v(x)}{2} \qquad \frac{dw}{dy} = \frac{-\lambda}{3} w(y)$$

$$2v''(x) + \lambda v(x) = 0 \qquad 3w'(y) + \lambda w(y) = 0$$

Convert the PDE below to an ODE

$$\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial y^2} = 0$$

$$v''(x)w(y) - 9w''(y)(v(x)) = 0$$

$$v''w(y) = 9w''(y)v(x)$$

Collecting terms

$$\frac{v''(x)}{v(x)} = \frac{9w''y}{w(y)}$$

$$\frac{v''(x)}{9v(x)} = \frac{w''(y)}{w(y)} = -\lambda$$

$$v''(x) = -9\lambda v(x) \qquad w''(y) = -\lambda w(y)$$

$$v''(x) + 9\lambda v(x) \ \cancel{\&}\ w''(y) + \lambda w(y) = 0$$

$$r^2 + 9\lambda = 0 \qquad r^2 + \lambda = 0$$

$$r^2 = -9\lambda \qquad r^2 = -\lambda$$

$$r = \pm 3\sqrt{-\lambda} \qquad r = \pm\sqrt{-\lambda}$$

Recall that

$v(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ (For unequal roots)

$v(x) = c_1 e^{3\sqrt{-\lambda}x} + c_2 e^{-3\sqrt{-\lambda}x}$ $\quad w(y) = c_3 e^{\sqrt{-\lambda}y} + c_4 e^{-\sqrt{-y}}$

$u(x,y) = \left[\left(c_1 e^{3\sqrt{-\lambda}x} + c_2 e^{-3\sqrt{-\lambda}x}\right)\left(c_3 e^{\sqrt{-\lambda}y} + c_4 e^{-\sqrt{-\lambda}y}\right)\right]$

Boundary value problems

Solve the boundary value problem if $y(0) = 2$ & $y'(0) = 0$

for $y'' - 9y = 0$

$y'' - 9y = 0$

$r^2 - 9 = 0$

$r = \pm 3$

$y(x) = c_1 e^{3x} + c_2 e^{-3x}$

$y'(x) = 3c_1 e^{3x} - 3c_2 e^{-3x}$

$y(0) = 2$ $\qquad y' = 0$

$c_1 e^{3(0)} + c_2 e^{-3(0)} = 2$ $\qquad 3c_1 e^{3(0)} - 3c_2 e^{-3(0)} = 0$

$C_1 + C_2 = 2 \qquad 3C_1 - 3C_2 = 0$

$3C_1 = 3C_2$

$C_1 = C_2$

$C_1 = C_2 = 1$

$\therefore \; y = e^{3x} + e^{-3x}$

The heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{— (1)}$$

$$\frac{dw}{dt} = -k\lambda w(t) \quad \text{— (2)}$$

$$\frac{d^2 v}{dx^2} = -\lambda v(x) \quad \text{— (3)}$$

Rewriting (3)

$v'' = -\lambda v$

$v'' + \lambda v$

$r^2 + \lambda = 0$

$r^2 = -\lambda$

$r = \pm\sqrt{-\lambda}$

If $\lambda = 0$, the roots are

$v(x) = C_1 + xC_2$

$V(0) = C_1 + 0C_2$

$C_1 = 0$ & $C_2 = 0$

for $\lambda > 0$, $V(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$

$C_1 = 0$

$0 = C_2 \sin(L\sqrt{\lambda}) = 0$

$\sin L\sqrt{\lambda} = 0$ for $n\pi$ n = 1, 2, 3, 4, 5 . . . . . .

$L\sqrt{\lambda} = n\pi$

$\sqrt{\lambda} = \frac{n\pi}{L}$

$\lambda = \left(\frac{n\pi}{L}\right)^2$

$V(x) = C_2 \sin \sqrt{\left(\frac{n\pi}{L}\right)^2} x$

$V(x) = C_2 \sin\left(\frac{n\pi}{L} x\right)$

From 2

$w' + k\lambda w = 0$

$w(t) = Ce^{-k\lambda t}$

$$w(t) = Ce^{-k\left[\frac{n\pi}{L}\right]^2 t}$$

$$u(x,t) = v(x)w(t)$$

$$u(x,t) = B_n \sin\left[\frac{n\pi}{L}x\right] e^{-k\left[\frac{n\pi}{L}\right]^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left[\frac{n\pi x}{L}\right] e^{-k\left[\frac{n\pi}{L}\right]^2 t}$$