BALLINGER

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PROJECT NAME: ME 564

PROJECT NUMBER: HW #3

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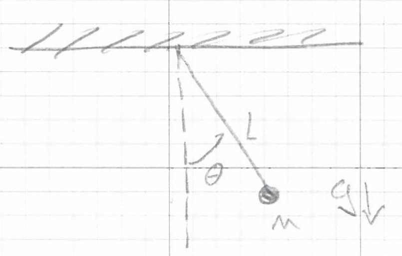
SHEET NUMBER 1 OF 4

a) FP
b) linearized eq
linear stab.
c) eigen vector/values

(3-1)



$T = \frac{1}{2} m L^2 \dot{\theta}^2$

$V = -gmL\cos(\theta)$

$\mathcal{L} = T - V$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$

$\mathcal{L} = \frac{1}{2} m L^2 \dot{\theta}^2 + gmL\cos(\theta)$

$\frac{\partial L}{\partial \dot{\theta}} = mL^2\dot{\theta}$

$\frac{\partial L}{\partial \theta} = -gmL\sin\theta$

$\frac{d}{dt} mL^2\dot{\theta} + gmL\sin\theta = 0$

$mL^2\ddot{\theta} + gmL\sin\theta = 0$

$L = m = g = 1$

$\boxed{\ddot{\theta} = -\frac{g}{L}\sin\theta}$

$\dot{\theta} = \omega$

$\dot{\omega} = -\cancel{\frac{g}{L}}\sin\theta$

$\boxed{\text{F.P.s} \; \cdots \begin{bmatrix}-\pi\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pi\\0\end{bmatrix}, \begin{bmatrix}2\pi\\0\end{bmatrix}, \cdots \begin{bmatrix}n\pi\\0\end{bmatrix}}$ $n = \ldots -2, -1, 0, 1, 2 \ldots$

pendulum up and pendulum down

$\dot{\theta} = \omega = 0$

$\dot{\omega} = -\sin\theta = 0$

$\theta = \pi n, \; n = \ldots -1, 0, 1, 2 \ldots$

Taylor Series $\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$

small $\theta \Rightarrow \sin\theta = \theta$
approximation

b) small θ $\quad \theta = 0, 2\pi n \quad n = 0, 1, 2 \ldots$

$\boxed{\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= -\theta\end{aligned}}$

$\frac{d}{dt}\begin{bmatrix}\theta\\\omega\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}\theta\\\omega\end{bmatrix}$

$\dot{x} = A \; x$

$\begin{bmatrix}\theta\\\omega\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \quad \frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2\\-\sin x_1\end{bmatrix}$

for $\theta = 0, 2\pi n \ldots$ using Jacobian $\rightarrow$ $\frac{Df}{Dx} = \begin{bmatrix}0 & 1\\-\cos x_1 & 0\end{bmatrix}$

$\boxed{\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2\\-x_1\end{bmatrix} \Rightarrow \frac{d}{dt}\begin{bmatrix}\theta\\\omega\end{bmatrix} = \begin{bmatrix}\omega\\-\theta\end{bmatrix}}$ ✓ matches small angle!

$\left.\frac{Df}{Dx}\right|_{\bar{x}=\begin{bmatrix}0\\0\end{bmatrix}}$ $\quad A = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$

$\left.\frac{Df}{Dx}\right|_{\bar{x}=\begin{bmatrix}\pi\\0\end{bmatrix}} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = A \quad \frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2\\x_1\end{bmatrix} \Rightarrow \boxed{\frac{d}{dt}\begin{bmatrix}\theta\\\omega\end{bmatrix} = \begin{bmatrix}\omega\\\theta\end{bmatrix}}$

$\theta = \pi, \; \pi(2n+1) \quad n = 0, 1, 2 \ldots$

(c) "Down Position" $\theta = 0, 2\pi n$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$$

$\det(A - \lambda I) = 0$ $\qquad \lambda^2 + 1 = 0$

$\lambda = \pm i$

$\mathrm{Re}\,\lambda = 0$

$\mathrm{Im}\,\lambda = \pm i \Rightarrow$ $\boxed{\text{neutrally stable center}}$

$\lambda = -i$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \boxed{\lambda = -i \quad \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}}$$

$i V_1 + V_2 = 0 \Rightarrow V_1 = i$

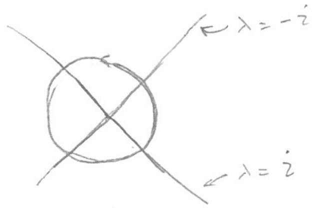
$V_2 = 1$

$\lambda = i$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad -i V_1 + V_2 = 0$$

$$\boxed{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \lambda = i}$$

d)



"up" $\theta = \pi, \pi(2n+1)$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (A - \lambda I) = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$\det(A - \lambda I) = 0$ $\qquad \lambda^2 - 1 = 0$

$\lambda = \pm 1$

$\mathrm{Im}\,\lambda = 0$

$\mathrm{Re}\,\lambda > 0 \Rightarrow$ $\boxed{\text{Saddle point}}$

$\lambda = -1$

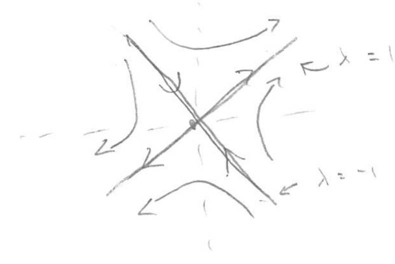
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \lambda = -1}$$

$V_1 + V_2 = 0$

$\lambda = 1$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda = 1}$$

d)



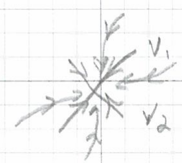
e) The results match what one would expect around the points (pendulum tends toward down) if there is no friction for the down position. For up position, It matches local behavior but the physical system will not be able to go to infinity so long term prediction is poor.

3-2 $\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -0.01 & 0 \\ 1 & -0.011 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$

a) $\lambda_1 = -0.01$ $\quad \mathrm{Re}\,\lambda < 0$ $\Rightarrow$ Stable source

$\lambda_2 = -0.011$ $\quad \mathrm{Im}\,\lambda = 0$

LOWER TRIANGULAR



b) see Matlab Plots Attached.
Yes, the solution matches answer in Part A. Both x, v decay to zero after initial bump of velocity

c). See Matlab Plot Attached

3-3

$\ddot{x} + 6\dot{x} + 8x = f(t)$ $\qquad$ IC: $x(0) = 2$, $\dot{x}(0) = -6$

a) Assume
$x(t) = e^{\lambda t}$
$x'(t) = \lambda e^{\lambda t}$
$x''(t) = \lambda^2 e^{\lambda t}$
$f(t) = 0$

$(\lambda^2 + 6\lambda + 8)e^{\lambda t} = 0$

$(\lambda+4)(\lambda+2) = 0$

$\lambda = -4, -2$

$x(t) = c_1 e^{-2t} + c_2 e^{-4t}$

$x(0) = [c_1 + c_2 = 2]\,2$

$\dot{x}(0) = -2c_1 - 4c_2 = -6$

$0 + -2c_2 = -2$

$c_2 = 1 \Rightarrow c_1 = 1$

$\boxed{x(t) = e^{-2t} + e^{-4t}}$

b) $f(t) = 6e^{-t}$

$\ddot{x} + 6\dot{x} + 8x = 6e^{-t}$

complimentary solution $x_c(t) = \longrightarrow$

$x_p(t) = a_1 e^{-t}$

$\dot{x}_p(t) = -a_1 e^{-t}$

$\ddot{x}_p(t) = a_1 e^{-t}$

$\ddot{x}_p + 6\dot{x}_p + 8x_p = 6e^{-t}$

$a_1 e^{-t} - 6a_1 e^{-t} + 8a_1 e^{-t} = 6e^{-t}$

$3a_1 e^{-t} = 6e^{-t}$

$3a_1 = 6$

$a_1 = 2 \Rightarrow x_p(t) = 2e^{-t}$

$x(t) = x_c + x_p = c_1 e^{-2t} + c_2 e^{-4t} + 2e^{-t}$

$x(0) = c_1 + c_2 + 2 = 2 \Rightarrow c_1 = -c_2$

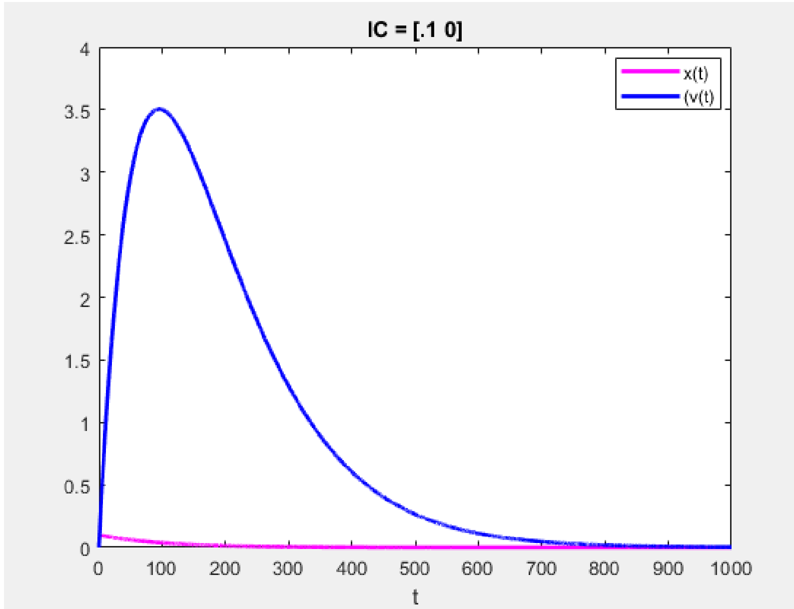
$\dot{x}(0) = -2c_1 - 4c_2 - 2 = -6$

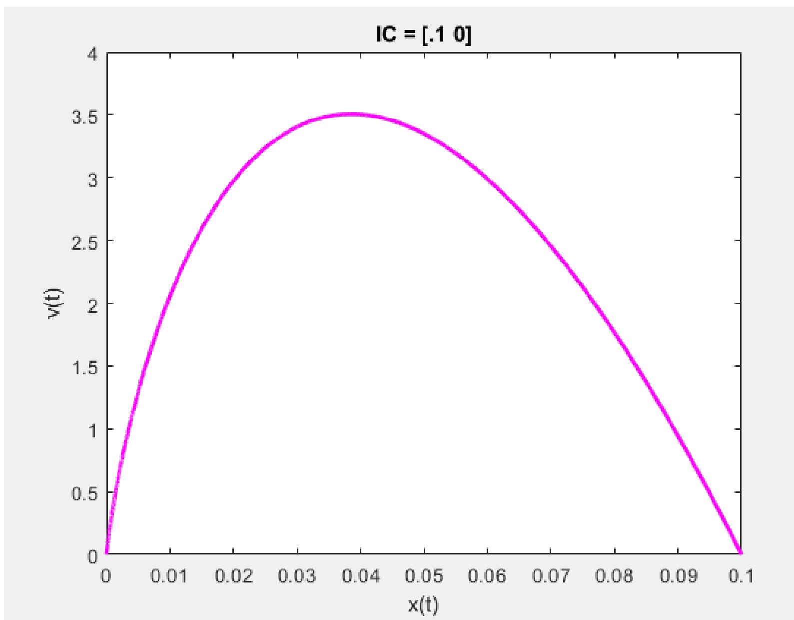
$2c_2 - 4c_2 = -4$

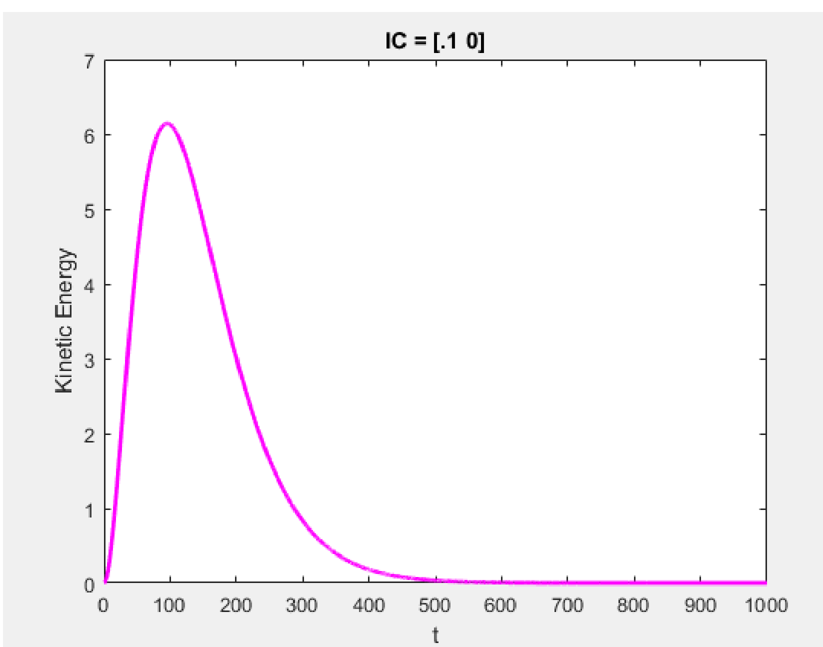
$-2c_2 = -4$

$c_2 = 2 \Rightarrow c_1 = -2$

$\boxed{x(t) = -2e^{-2t} + 2e^{-4t} + 2e^{-t}}$


IC = [.1 0]
x(t)
(v(t)
t
0
0.5
1
1.5
2
2.5
3
3.5
4
100
200
300
400
500
600
700
800
900
1000

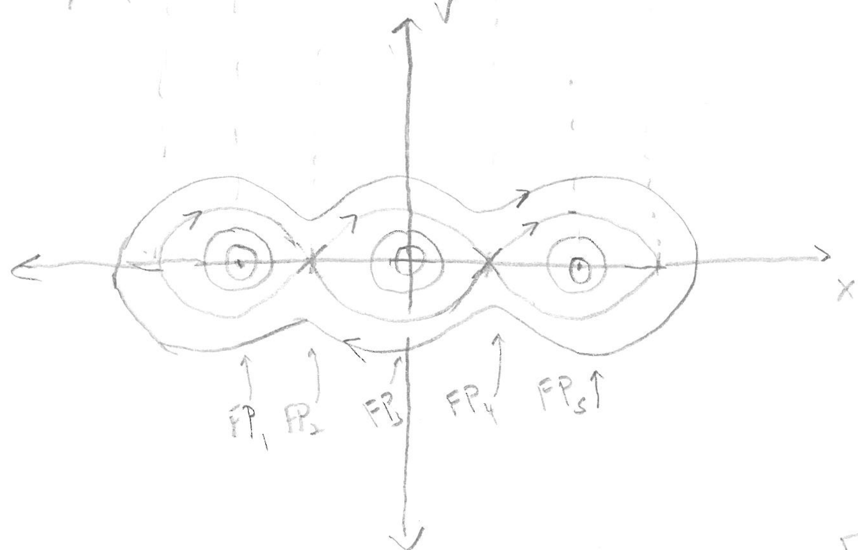

IC = [.1 0]
v(t)
x(t)
0
0.5
1
1.5
2
2.5
3
3.5
4
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1


IC = [.1 0]
Kinetic Energy
t
0
1
2
3
4
5
6
7
100
200
300
400
500
600
700
800
900
1000

$V(x)$

$v = \dot{x}$

w/o Damping



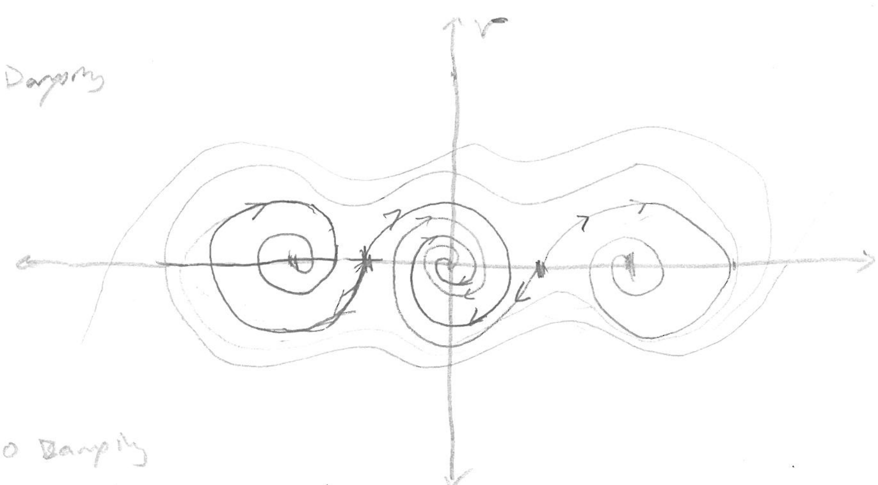
W/ Damping There a 5 fixed points

FP_1, FP_3, FP_5 Neutrally stable centers.

FP_2, FP_4 are unstable saddle points

If friction is turned on, The neutrally stable centers (FP_1, FP_3, FP_5) will becom stable spirals. The saddle points (FP_2, FP_4) will Remain saddle points

W/ Damping



w/o Damping

eigenvalues around the centers (FP_1, FP_3, FP_5) have the form $\pm bi$ ($\text{Re}\,\lambda = 0$)

eigenvalues around the saddle points (FP_2, FP_4) have the form $\pm a$ ($\text{Im}\,\lambda = 0$)

W/ Damping

$FP_1, FP_3, FP_5 \quad \lambda = a \pm bi \qquad \text{Im}\,\lambda \neq 0 \quad \text{Re}\,\lambda < 0$

$FP_2, FP_4 \quad \lambda = \pm a \quad (\text{Im}\,\lambda = 0)$