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PROJECT NAME ME 564

PROJECT NUMBER HW 1

BY Tim Yoder

SHEET NUMBER 1 OF 6

1-1

a) $\frac{d}{dx}\left(f(x) = \cos(x^3)\right)$

$\boxed{f'(x) = -3x^2\sin(x^3)}$

b) $\frac{d}{dx}\left(2^x\right)$

$f' = \boxed{\log(2)\,2^x}$

c) $\frac{d}{dx}\left(e^{x^2}\sin x\right)$

$f' = 2xe^{x^2}\sin x + e^{x^2}\cos x$

$\boxed{f' = e^{x^2}(2x\sin x + \cos x)}$

d) $\frac{d}{dx}\left(\tan x^2\right)$

$\boxed{f' = 2x\sec^2(x^2)}$

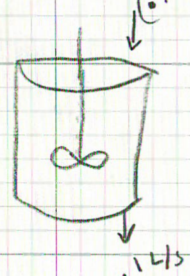
e) $f(x,y) = \cos(x^2)\sin(y^2)$

$\frac{d}{dt}\left(f(x,y) = -\sin(x^2)\,2x\frac{dx}{dt}\sin(y^2) + \cos(y^2)\,2y\frac{dy}{dt}\cos(x^2)\right)$

$\boxed{\frac{df}{dt} = 2y\frac{dy}{dt}\cos(x^2)\cos(y^2) - 2x\frac{dx}{dt}\sin(x^2)\sin(y^2)}$

1-2

(.1 L/s) @ (0.1 kg/L)



.1 L/s

t_0; $\frac{5\text{ kg salt}}{100\text{ L wa}\text{tr}} = \frac{1}{20}\frac{\text{kg}}{\text{L}} = 0.05\ \text{kg/L}$

$x(t)$: Mass of salt

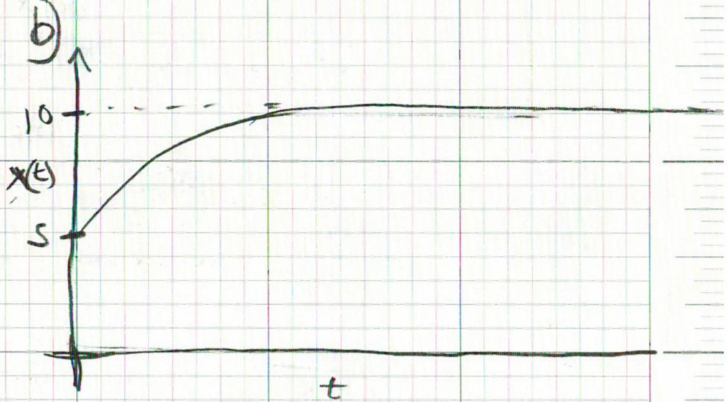
$x(0) = 5$ kg

$\frac{dx}{dt}$ = rate in − rate out

$\frac{dx}{dt} = (.1)\frac{L}{s}(.1)\frac{kg}{L} - (.1)\frac{L}{s}\frac{x}{100}\frac{kg}{L}$

a) $\boxed{\frac{dx}{dt} = .01 - .001x}$

b)



$\int \frac{dx}{(.01-.001x)} = \int dt$

$u = 0.01 - 0.001x$

$du = -0.001x$

$-1000\int \frac{1}{u}du = t + c$

$-1000\ln(u) + c = t + c$

$-1000\ln(.01 - .001x) = t + c$

$-1000\ln\left(\frac{10-x}{1000}\right) = t + c$

undefined $-1000\ln\left(\frac{-1}{1000}\right) - 1000\ln(x-10) = t + c$

$-1000\ln(x-10) = t + c$

$e^{[\ln(x-10)]} = e^{\left[\frac{-t}{1000} + \frac{c}{1000}\right]}$

$x(0) = 5$

$x - 10 = e^{\left[\frac{-t}{1000} + \frac{c}{1000}\right]}$

$x = 10 + e^{\left[-\frac{t}{1000} + \frac{c}{1000}\right]}$

$5 = 10 + e^{(c/1000)}$

$-5 = e^{c/1000}$

$\ln(-5) = c/1000$

$c = 1000\ln(-5)$

$x = 10 + e^{\left[t/1000 + \frac{1000}{1000}\ln(-5)\right]}$

$x = 10 + e^{-t/1000}e^{\ln(-5)} \rightarrow \boxed{x(t) = 10 - 5e^{\left[\frac{-t}{1000}\right]}}$

See next Page

1-2

c) $7 = 10 - 5e^{(-t/1000)}$

$-3 = -5e^{(-t/1000)}$

$\frac{3}{5} = e^{-t/1000}$

$\ln\left(\frac{3}{5}\right) = \frac{-t}{1000}$

$t = -1000\ln\left(\frac{3}{5}\right)$

$\boxed{t \approx 510.8 \text{ seconds}}$

d) $x = 10 - 5e^{\left(-\frac{30}{1000}\right)}$

$= 10 - 4.852$

$\boxed{x \approx 5.148 \text{ kg}}$

e) $\boxed{X(\infty) = 10 \text{ kg}}$

1-3 Taylor Series around $a=0$

a) $f(x) = x\cos(x)$

$f(x) = \cos(x) \approx \cos(0) - \cancel{\sin(0)}\,x - \cos(0)\frac{x^2}{2} + \cancel{\sin(0)}\frac{x^3}{3!} + \frac{\cos(0)}{4!}x^4 - \frac{\cancel{\sin(0)}}{\cancel{5!}}x^5 - \frac{\cos(0)x^6}{6!} + \dots$

$f(x) = x\cos(x) = x\left[\underset{n=1}{1} - \underset{n=2}{\frac{x^2}{2!}} + \underset{n=3}{\frac{x^4}{4!}} - \frac{x^6}{6!} + \dots \frac{x^{2(n-1)}}{2(n-1)!} + \dots\right]$

$$\boxed{f(x) \approx x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \dots \frac{x^{2(n-1)+1}}{[2(n-1)]!}(-1)^{n+1}}$$

see Attached for Plot.

b) $f(x) = \frac{\cos(x)}{x^2}$ @ $x=0$

$\cos(x) \approx \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right]$

$f(x) = \frac{1}{x^2}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right]$

$$\boxed{f(x) = \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + (-1)^{n+1}\frac{x^{2n-4}}{[2(n-1)]!} + \dots \text{ for } x \neq 0}$$

See Attached for Plot

c) $f(x) = e^{x^2}$ | $f(0) = 1$

$\frac{df}{dx} = 2xe^{x^2}$ | $\frac{df}{dx}(0) = 0$

$\frac{d^2f}{dx^2} = 2e^{x^2} + 4x^2e^{x^2}$ | $\frac{d^2f}{dx^2}(0) = 2$

$\frac{d^3f}{dx^3} = 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2}$ | $\frac{d^3f}{dx^3}(0) = 0$

$\frac{d^4f}{dx^4} = 4e^{x^2} + 8x^2e^{x^2} + 8e^{x^2} + 16x^2e^{x^2} + 24x^2e^{x^2} + 16x^4e^{x^2}$ | $\frac{d^4f}{dx^4}(0) = 12$

$f(x) = 1 + \frac{2x^2}{2!} + \frac{12x^4}{4!} + O(x^6)$

$$\boxed{f(x) = 1 + x^2 + \frac{x^4}{2} + O(x^6)}$$

See Attached for plot

$\boxed{1-3}$

@ $a=0$

(d) $f(x) = \frac{1}{1-x}$

$$f(x) = \frac{1}{1-x} + \frac{1}{(1-x)^2}x + \frac{2}{(1-x)^3}\frac{x^2}{2} + \frac{6}{(1-x)^4}\frac{x^3}{3!} + \cdots$$

$$\boxed{f(x) = 1 + x + x^2 + x^3 + \cdots x^n + \cdots}$$

See Attached for Plot

The series converges when $|x|<1$.

(e) @ $a=1$

$f(x) = \sin\left(\frac{1}{x}\right)$

$$\frac{df(x)}{dx} = \frac{-1}{x^2}\cos\left(\frac{1}{x}\right) \quad \Big| \quad \frac{df(1)}{dx} = -\cos(1)$$

$$\frac{d^2f(x)}{dx} = \frac{2}{x^3}\cos\left(\frac{1}{x}\right) - \frac{1}{x^4}\sin\left(\frac{1}{x}\right) \quad \Big| \quad \frac{d^2f(1)}{dx} = 2\cos(1) - \sin(1)$$

$$\frac{d^3f}{dx^3} = -\frac{6}{x^4}\cos\left(\frac{1}{x}\right) + \frac{2}{x^5}\sin\left(\frac{1}{x}\right) + \frac{4}{x^5}\sin\left(\frac{1}{x}\right) + \frac{1}{x^6}\cos\left(\frac{1}{x}\right) \quad \Big| \quad \frac{d^3f(1)}{dx^3} = -5\cos(1) + 6\sin(1)$$

$$\boxed{f(x) = \sin(1) - \cos(1)(x-1) + \frac{(x-1)^2}{2}(2\cos(1) - \sin(1)) + \frac{(x-1)^2}{3!}[6\sin(1) - 5\cos(1)] + \cdots}$$

See Attached for plot, plotted first 4 terms

Series is Valid $x \neq 0$. I used $a=1$ so $x \leq 0$ is not to be trusted either.

$\boxed{1-4}$

$\ddot{x} - \lambda x = 0$

$\ddot{x} = \lambda x$

$x(t) = e^{\lambda t}$

$\ddot{x}(t) = \lambda^2 e^{\lambda t}$

$\lambda^2 \cancel{e^{\lambda t}} = \lambda \cancel{e^{\lambda t}}$

$\lambda^2 = \lambda$

$\lambda = \pm\sqrt{\lambda}$

$x(t) = c_1 e^{\sqrt{\lambda}\, t} + c_2 e^{-\sqrt{\lambda}\, t}$

$x(0) = c_1 + c_2$ $\qquad$ $\dot{x}(0) = \sqrt{\lambda}\, c_1 - \sqrt{\lambda}\, c_2 = 0$

$\sqrt{\lambda}\, c_1 = \cancel{\sqrt{\lambda}}\, c_2$

$c_1 = c_2$

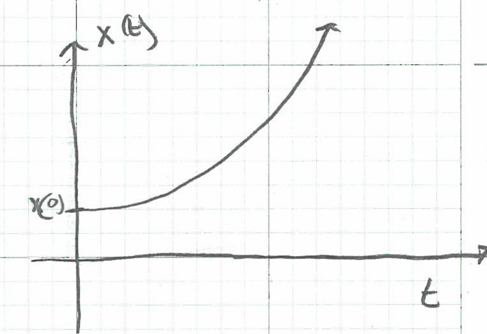
$c_1 = c_2 = \frac{x(0)}{2}$

$$\boxed{x(t) = \frac{x(0)}{2}\left[e^{\sqrt{\lambda}\, t} + e^{-\sqrt{\lambda}\, t}\right]}$$

if $\lambda > 0$: $x(t) = \frac{x(0)}{2}\left[e^{\sqrt{\lambda}\, t} + e^{-\sqrt{\lambda}\, t}\right]$

$\to 0$ as $t \to -\infty$
$\to \infty$ as $t \to \infty$

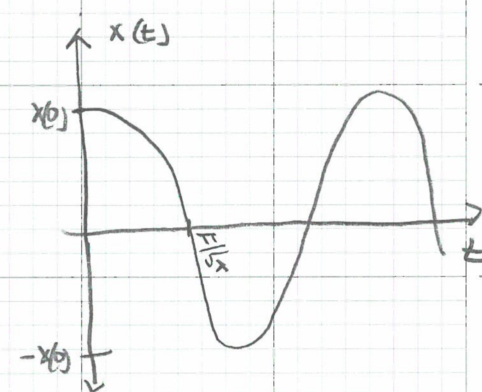
$\to \infty$ as $t \to -\infty$
$\to 0$ as $t \to \infty$



$\lambda < 0$: $x(t) = \frac{x(0)}{2}\left[e^{i\sqrt{\lambda}\, t} + e^{-i\sqrt{\lambda}\, t}\right]$

$= \frac{x(0)}{2}\left[\cos\sqrt{\lambda}\, t + \cancel{i\sin\sqrt{\lambda}\, t} + \cos\sqrt{\lambda}\, t - \cancel{i\sin\sqrt{\lambda}\, t}\right]$

$x(t) = x(0)\cos\sqrt{\lambda}\, t$



1-5

$\ddot{x} + 4\dot{x} + 3x = 0 \qquad x(0) = 0 \quad \dot{x}(0) = 4$

a)

$x(t) = e^{\lambda t}$

$\dot{x}(t) = \lambda e^{\lambda t} \qquad [\lambda^2 + 4\lambda + 3]e^{\lambda t} = 0$

$\ddot{x}(t) = \lambda^2 e^{\lambda t}$

$\lambda^2 + 4\lambda + 3 = 0$

$(\lambda + 3)(\lambda + 1) = 0$

$\lambda = -3, -1$

$x(t) = C_1 e^{-3t} + C_2 e^{-t}$

$x(0) = 0 \Rightarrow C_1 + C_2 = 0$

$C_1 = -C_2$

$\dot{x}(0) = 4$

$\dot{x}(t) = -3C_1 e^{-3t} - C_2 e^{-t}$

$-3C_1 - C_2 = 4$

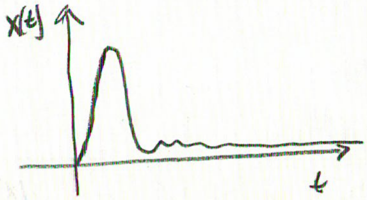
$3C_2 - C_2 = 4$

$2C_2 = 4$

$C_2 = 2 \Rightarrow C_1 = -2$

$$\boxed{x(t) = 2e^{-t} - 2e^{-3t}}$$

Damping > Spring



long term behavior will trend toward stationary equilibrium after initial positive pulse (negative eigenvalues ⇒ stable)

b) $\ddot{x} - 4x = 0 \quad x(0) = 4 \quad \dot{x}(0) = -4$

$x(t) = e^{\lambda t}$

$\ddot{x} = \lambda^2 e^{\lambda t} \qquad [\lambda^2 - 4]e^{\lambda t} = 0$

$\lambda^2 - 4 = 0$

$\lambda = \pm 2$

$x(t) = C_1 e^{2t} + C_2 e^{-2t}$

$x(0) = 4 \Rightarrow C_1 + C_2 = 4 \quad C_1 = 4 - C_2$

$\dot{x} = 2C_1 e^{2t} - 2C_2 e^{-2t}$

$\dot{x}(0) = -4 \Rightarrow 2C_1 - 2C_2 = -4$

$2(4 - C_2) - 2C_2 = -4$

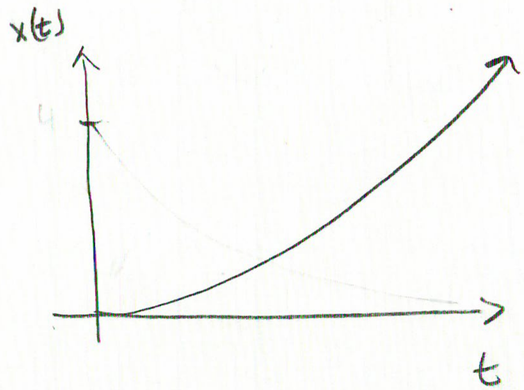
$8 - 2C_2 - 2C_2 = -4$

$-4C_2 = -12$

$C_2 = 3$

$C_1 = 1$

$$\boxed{x(t) = e^{2t} + 3e^{-2t}}$$



long term behavior will trend towards unstable solution. There is no "damping" term, only "spring".

Positive eigenvalue ⇒ unstable solution

1-6 $\ddot{x} + k_1\dot{x} + k_2 x = 0$ so $\lambda_1 = C$, $\lambda_2 = D$ $x(t) = Ae^{ct} + Be^{Dt}$ IC?

Characteristic eq:

$$(\lambda - C)(\lambda - D) = 0$$

$$(\lambda^2 - C\lambda - D\lambda + CD)e^{\lambda t} = 0$$

$$[\lambda^2 - (C+D)\lambda + CD]e^{\lambda t} = 0$$

$$\lambda^2 e^{\lambda t} \underbrace{-(C+D)}_{K_1}\lambda e^{\lambda t} + \underbrace{CD}_{K_2}e^{\lambda t} = 0$$

$$\boxed{\ddot{X} - (C+D)\dot{X} + CD(X) = 0}$$

$$\boxed{\begin{aligned} X(0) &\Rightarrow A + B \\ \dot{X}(0) &\Rightarrow AC + BD \end{aligned}}$$

1-7

a) $\boxed{f(t) = e^{-t}}$ → already a real function $Im\, f(t) = 0$ $\boxed{Re\, f(t) = e^{-t}}$

See Attached for Plot

b) $\boxed{f(t) = e^{t}}$ → already a real function $Im\, f(t) = 0$ $\boxed{Re\, f(t) = e^{t}}$

See Attached for Plot

c) $f(t) = e^{2\pi i t} = \boxed{\cos(2\pi t) + i\sin(2\pi t)}$ See attached for Plot

d) $f(t) = e^{i(2\pi t + \pi/2)} = e^{\frac{\pi}{2}i} e^{i2\pi t} = e^{\pi/2}\cos(2\pi t) + ie^{\pi/2}\sin(2\pi t)$ $e^{\pi/2} = i$

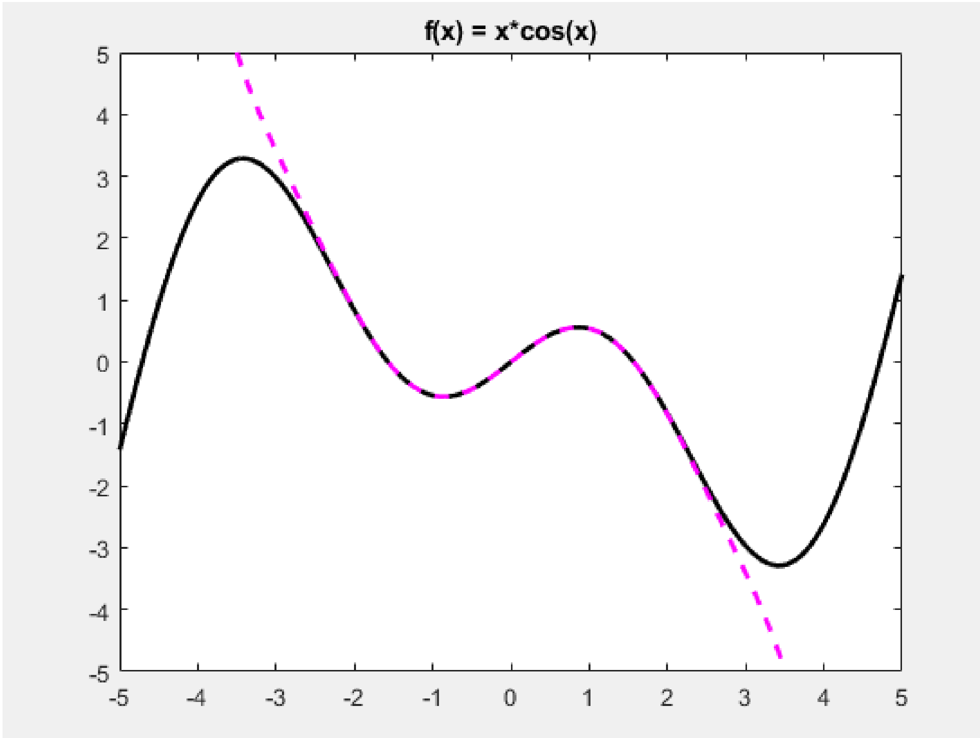
$\boxed{f(t) = -\sin(2\pi t) + i\cos(2\pi t)}$ see Attached for Plot

e) $e^{t(-1+i)} = e^{-t}e^{it} = \boxed{e^{-t}\cos(t) + ie^{-t}\sin(t)}$

f) $e^{t(1+i)} = e^{t}e^{it} = \boxed{e^{t}\cos(t) + ie^{t}\sin(t)}$

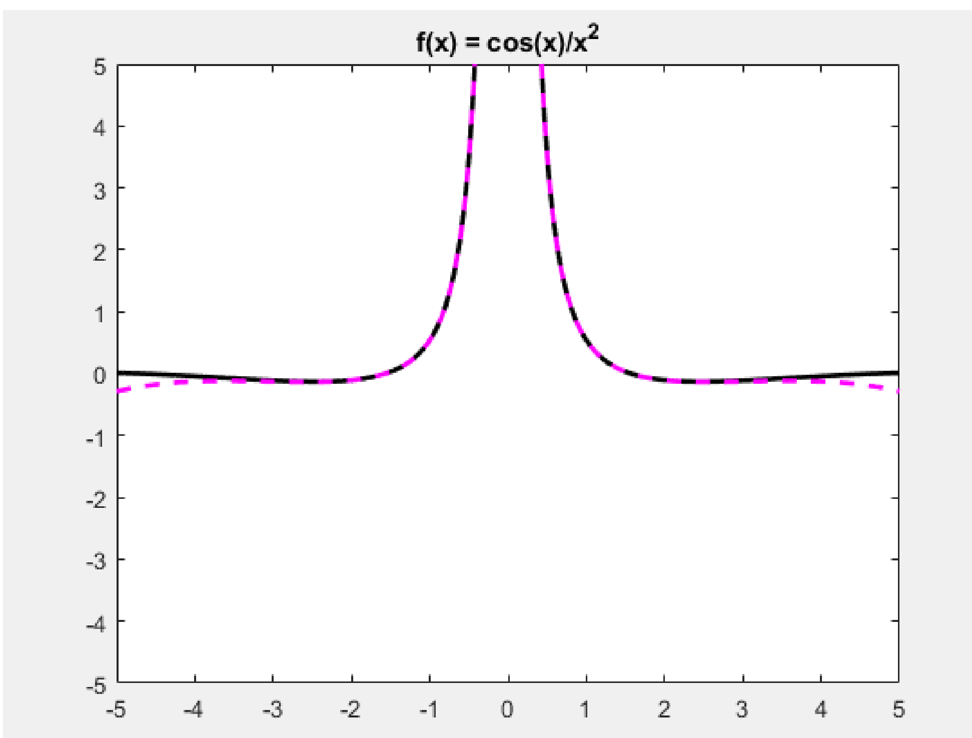
See Attached for Plot

1-3 (a)



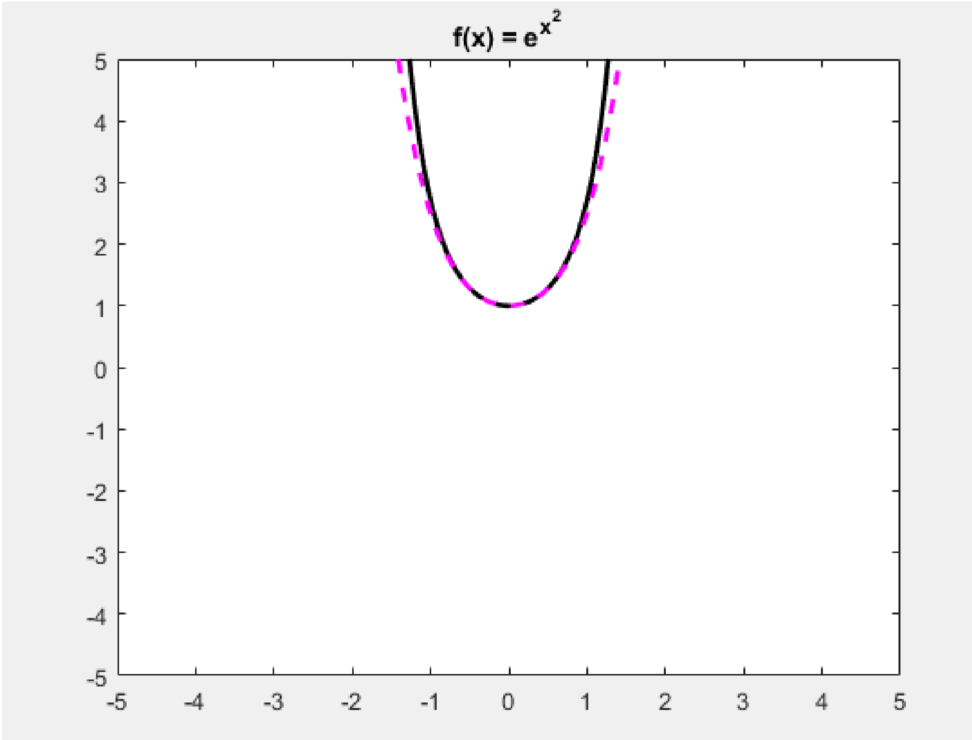
f(x) = x*cos(x)
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

1-3 (b)



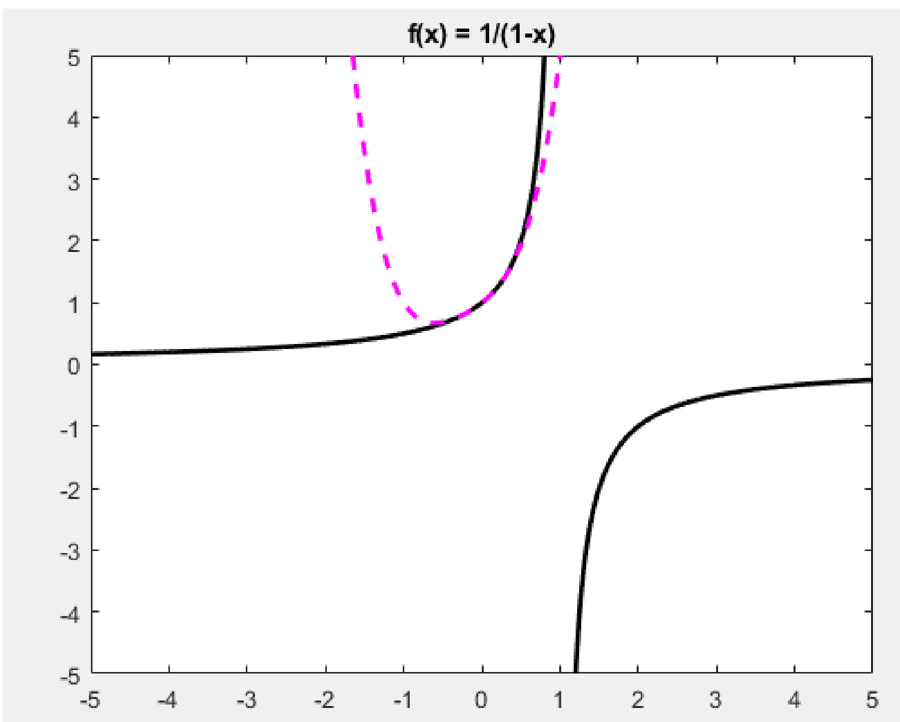
f(x) = cos(x)/x^2
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

1-3 (c)



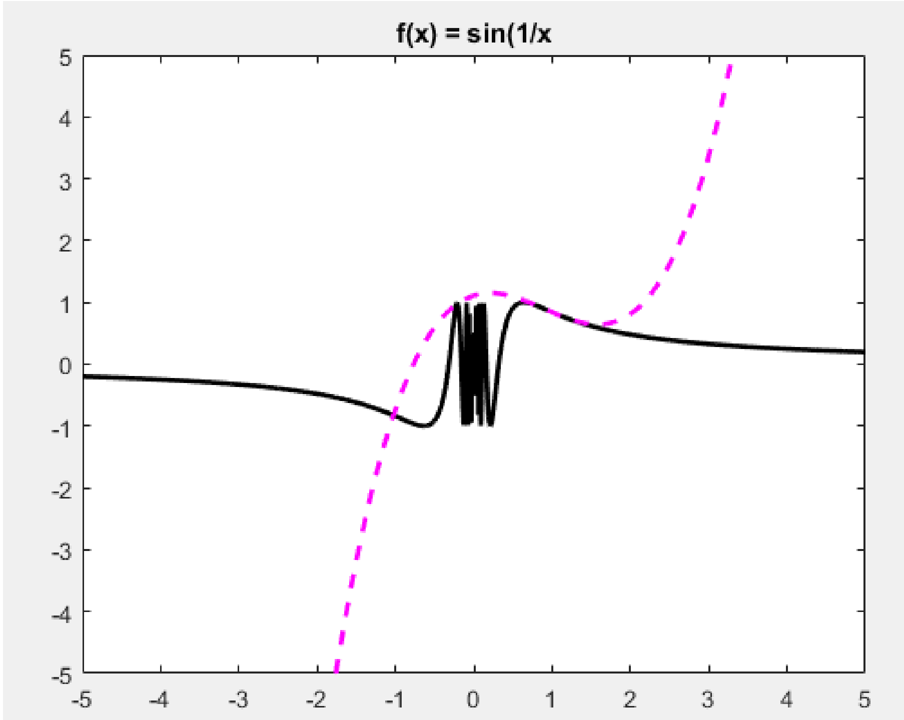
f(x) = e^{x^2}
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

1-3 (d)



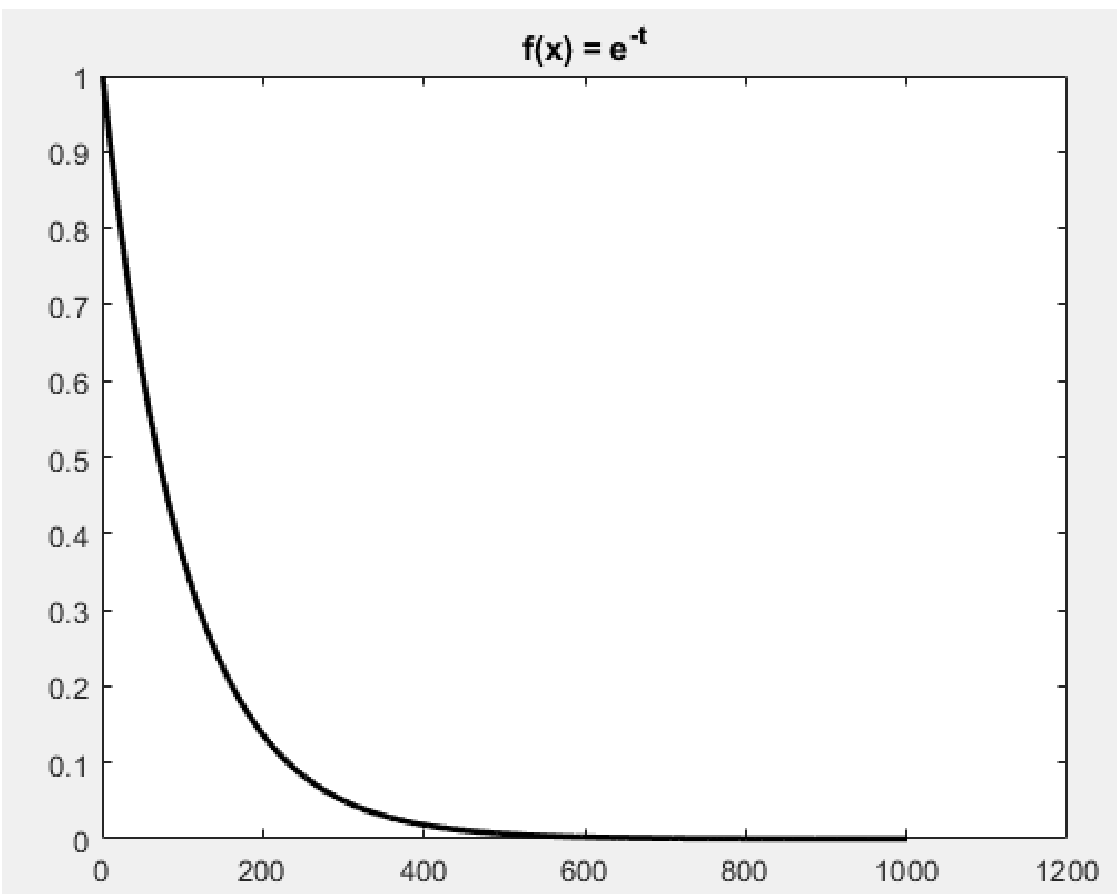
f(x) = 1/(1-x)
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

1-3 (e)

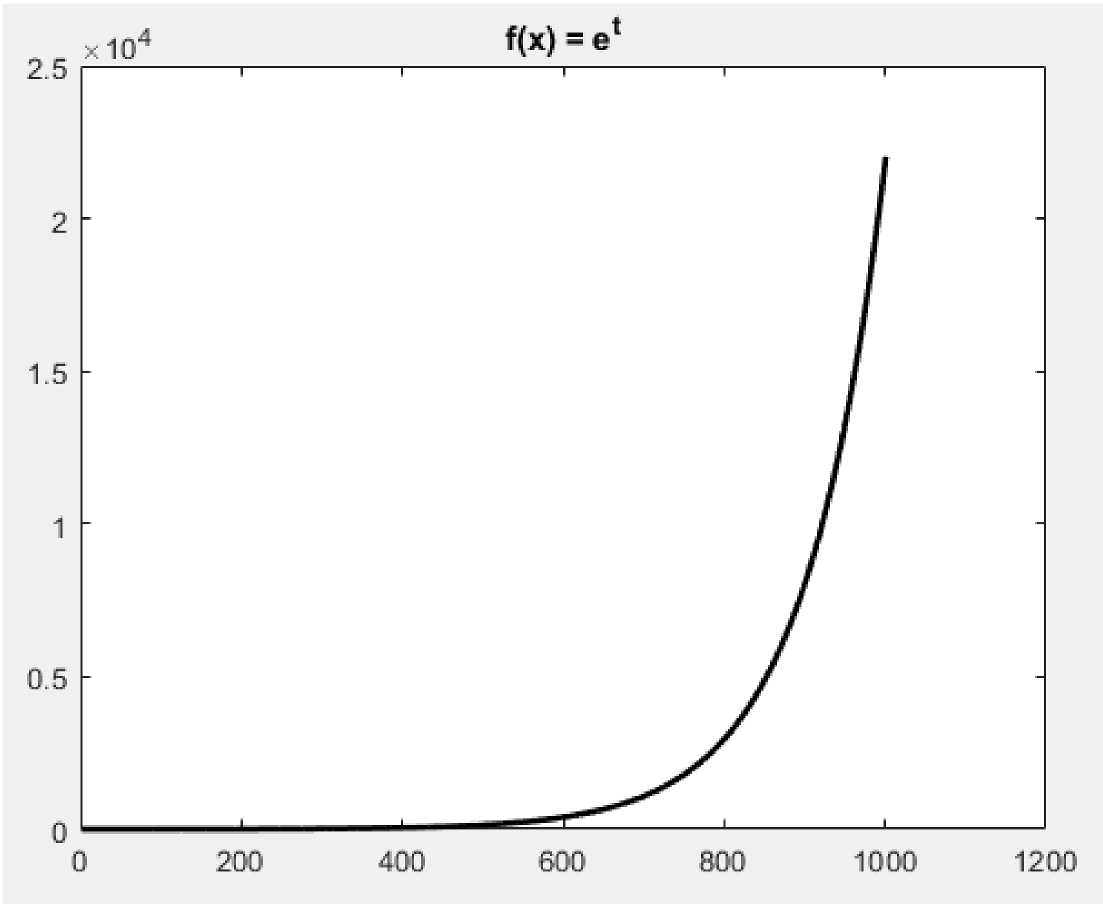


First four non-zero terms were plotted.

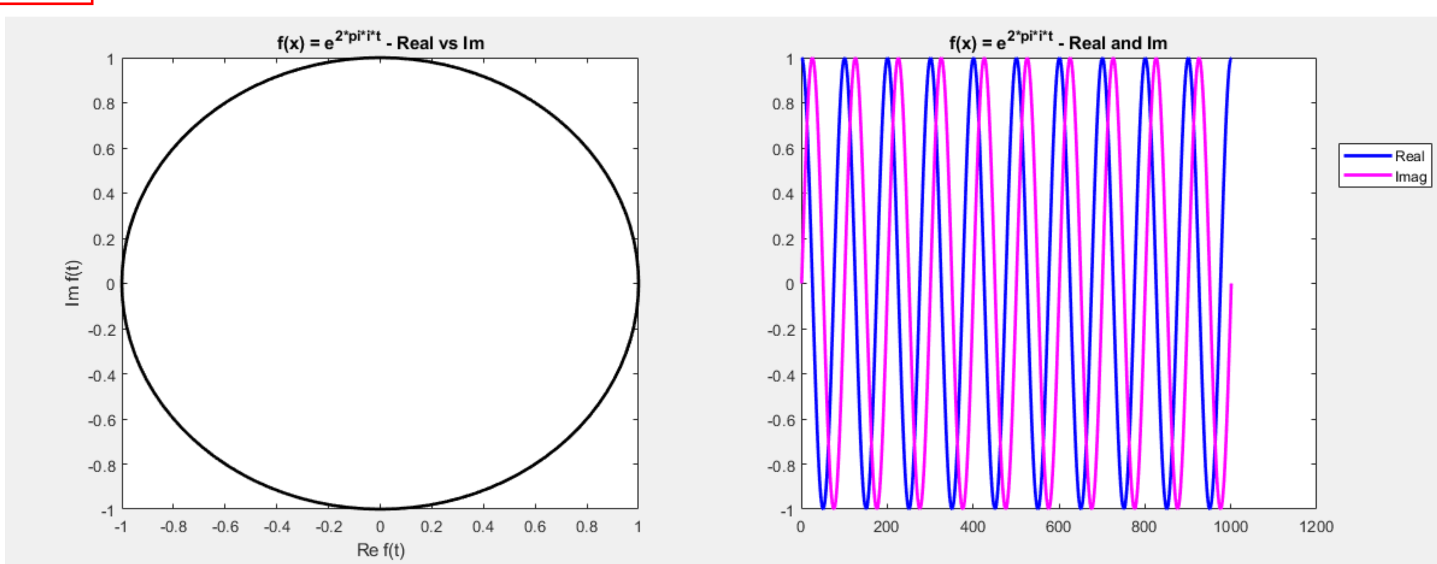
1-7 (a)



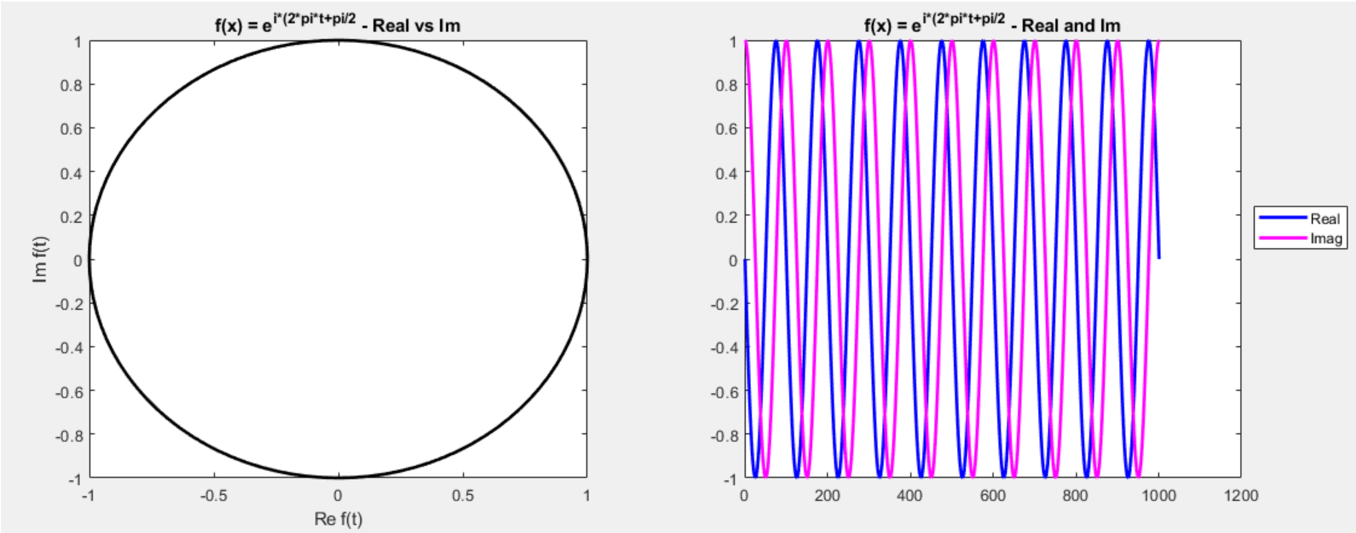
1-7 (b)


f(x) = e^t
×10^4
2.5
2
1.5
1
0.5
0
0
200
400
600
800
1000
1200

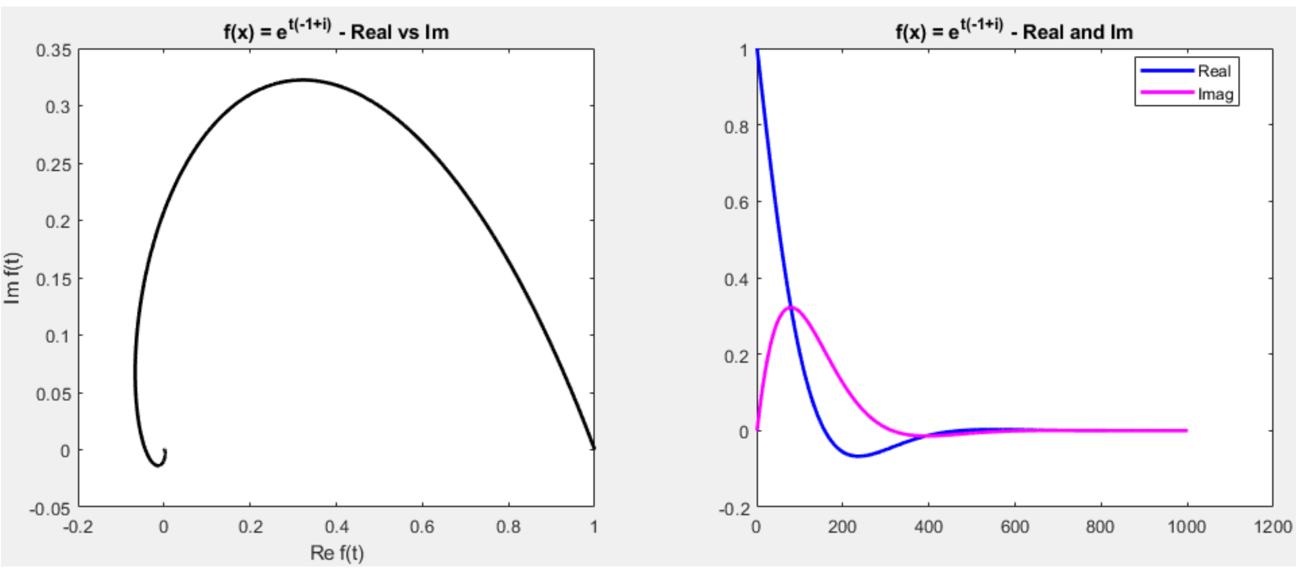
1-7 (c)


f(x) = e^2*pi*i*t - Real vs Im
Im f(t)
Re f(t)
f(x) = e^2*pi*i*t - Real and Im
Real
Imag

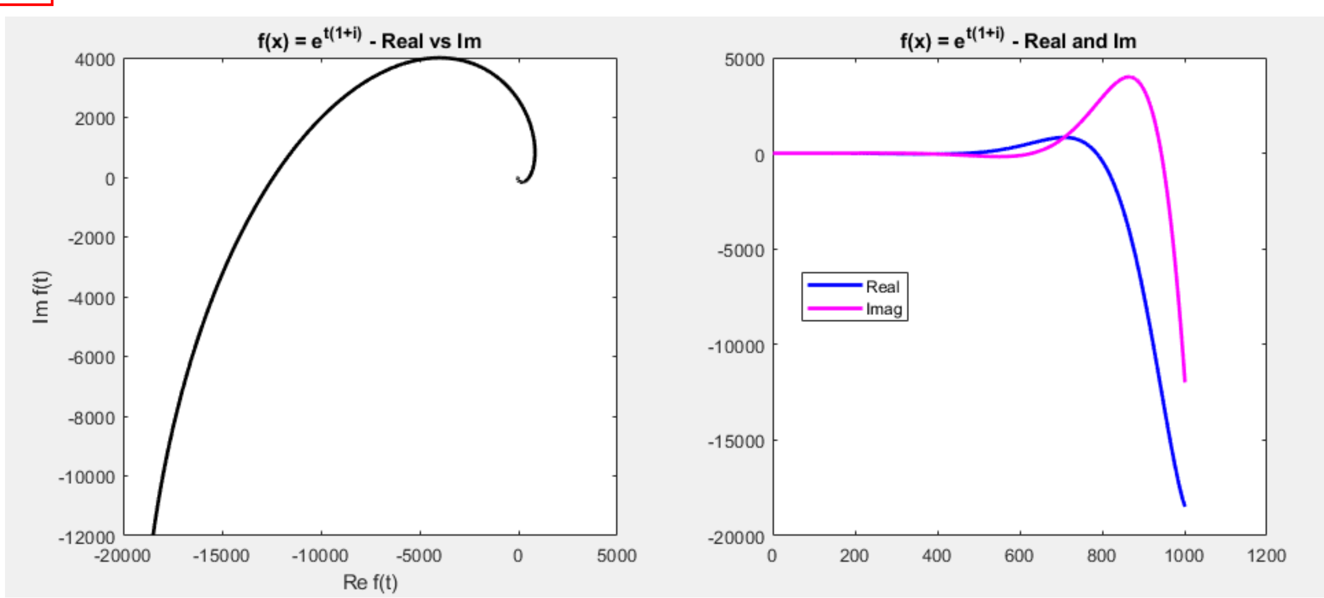
1-7 (d)


f(x) = e^i*(2*pi*t+pi/2 - Real vs Im
Im f(t)
Re f(t)
f(x) = e^i*(2*pi*t+pi/2 - Real and Im
Real
Imag

1-7 (e)


f(x) = e^t(-1+i) - Real vs Im
Im f(t)
Re f(t)
f(x) = e^t(-1+i) - Real and Im
Real
Imag

1-7 (f)


f(x) = e^t(1+i) - Real vs Im
Im f(t)
Re f(t)
f(x) = e^t(1+i) - Real and Im
Real
Imag