Computer exercise for WI4201
Spectral Considerations on Two-grid methods for the discrete Poisson equation.

Students: Group:

Supervisor: Kees Vuik

Guidelines

- The deadline to hand in the report on this assignment is **January 12th, 2024**.
- Make sure to schedule an intermediate video-call appointment with your supervisor Calin Georgescu (email: C.A.Georgescu@tudelft.nl) in the **first week of December (4th of December till 8th of December)** where you can discuss your progress with this take-home exam. The minimal implementation results you have to show are: the convergence to the exact solution and the implementation of the first solver required in your take home exam. It is allowed that only one of you comes to the appointment if the other is not available at the same time.
- Put your group number, names and student numbers on the front page of your report.
- Please upload your report digitally **via brightspace**.
- For the computer implementation, any language can be used (Matlab, Python or other).
- The implementation itself will **NOT** be graded. All intelligence and wisdom in realizing the implementation should therefore be reflected in the report.
- Write a concise and neatly written report in which you answer the questions one by one. Both the structure of the report and the correctness of the answers will be taken into account in the grade. Do include the source code of your implementation in this report.

Let the unit interval be denoted by $\Omega = (0, 1)$. Let $x \in \Omega$ denote the dependent variable. Let us assume that the numbers c, α and β are given numbers. Our goal then is to numerically approximate the function $u(x)$ that is the solution to the following boundary value problem for the diffusion equation:

$$-\frac{d^2u}{dx^2} + c\,u = f(x) \text{ for } x \in \Omega \tag{1}$$

supplied with the following non-homogeneous Dirichlet boundary conditions:

$$u(x = 0) = \alpha \text{ and } u(x = 1) = \beta\,. \tag{2}$$

For testing purposes, choose c, α, β and the function $f(x)$ such that a given function $u_{ex}(x)$ (e.g. $u_{ex}(x, y) = e^x\,(1 - x)$) is the exact solution to the problem (1)-(2).

The objective of this assignment is to compare the computational efficiency of the coarse grid correction procedure in two settings. The first is the use of coarse grid correction as a component of a two-grid method. The second is its use as an accelerator for the conjugate gradient method.

Throughout this assignment different solution methods are used. First, the Gauss-Seidel Method as a Solver and Symmetric Gauss-Seidel Method as a Preconditioner are considered. By *Symmetric* Gauss-Seidel method we mean the Symmetric Successive Over Relaxation Method SSOR(ω) with relaxation parameter $\omega = 1$. Verify that the resulting preconditioner can be written as $M_{SGS} = (D - E)D^{-1}(D - F)$.

Second, the Coarse Grid Correction as a Solver and Coarse Grid Correction as a Projection are considered. For the Coarse Grid Correction we introduce an auxiliary coarser grid problem by discretizing (1)-(2) using the meshwidth $H = 2\,h$. Let I_H^h denote the linear interpolation mapping from coarse to fine grid. Let I_h^H denote the restriction by half weighting mapping from fine to coarse grid. Let A^H denote the coarse grid matrix obtained by Galerkin coarsening, i.e., set

$$A^H = I_h^H A^h I_H^h$$

Let the inverse of the coarse grid correction operator be defined as

$$M_{CGC}^{-1} = I_H^h (A^H)^{-1} I_h^H$$

Let the coarse grid correction error propagation matrix be defined as

$$B_{CGC} = I - M_{CGC}^{-1} A = I - I_H^h (A^H)^{-1} I_h^H A$$

B_{CGC} is formed explicitly by restricting each column of A, applying to coarse grid solve to each of the resulting vectors, and bringing the results back to the fine grid.

For the Coarse Grid Correction as a Projection let P be the matrix defined by $P = I - AM_{CGC}^{-1}$. Then P defines a projection operator, that is, $P\,P = P^2 = P$. The original linear system $A\mathbf{u} = \mathbf{f}$ can then be transformed into the projected linear system $PA\mathbf{u} = P\,\mathbf{f}$. The coefficient matrix $\widehat{A} = P\,A$ of this projected linear system is singular. One indeed has that $\widehat{A}\,I_H^h = P\,A\,I_H^h = O$, where O denotes the zero matrix of appropriate size. Stated differently, all the columns of I_H^h are eigenvectors of $\widehat{A}$ corresponding to the zero eigenvalue. Although $\widehat{A}$ is singular, the projected linear system can be solved because the projected right-hand vector $P\,\mathbf{f}$ belongs to the column space of the projected coefficient matrix $\widehat{A}$. The projected linear system has infinitely many solutions. A solution found by application of a numerical procedure is thus determined up to an additive component. This means that if $\hat{\mathbf{u}}$ is a solution found, then $\hat{\mathbf{u}}+\mathbf{u}^\dagger$ is a solution as well as long as $\widehat{A}\mathbf{u}^\dagger = \mathbf{0}$.

Lastly, Multigrid as a Solver and Multigrid as a Preconditioner are considered. For Multigrid as a Solver one forward GS sweep, the coarse grid operator, and one backward GS sweep are combined in a multiplicative fashion to obtain the two-grid grid operator B_{TGM} and the corresponding splitting M_{TGM}. B_{TGM} can be formed explicitly by a triple matrix product corresponding to post-smoother, coarse grid correction and pre-smoother.

Turning the Multigrid Solver into a preconditioner can be done in a similar way as the symmetric Gauss-Seidel method was turned from a solver into a preconditioner.

Pen and Paper Assignments (48 points)

1. *(10 pt)* Write a (preconditioned) CG solution algorithm that enables
   - plotting the Ritz values of each iteration;
   - plotting the convergence at the end of the solution process.

   This implementation will be employed in the remainder of this assignment.
2. *(8 pt)* Perform a finite difference discretization of (1) - (2).
3. *(8 pt)* Give the stencil representation of I_h^H and I_H^h and verify that some constant c exists such that $I_h^H = c\,\left(I_H^h\right)^T$;
4. *(4 pt)* Verify under which conditions the Galerkin approach and rediscretization approach to construct A^H give the same result.
5. *(6 pt)* Verify that P indeed defines a projection.
6. *(6 pt)* Verify that $\widehat{A}\,I_H^h = O$ and therefore that $\widehat{A}$ is singular. Give the size of O.
7. *(6 pt)* Verify that $P\,\mathbf{f}$ is an element of the column space of $\widehat{A}$.

Implementation Assignments (points 52)

1. *(4 pt)* You have performed a finite difference discretization of (1) - (2). Solve for the discrete unknowns using a direct solution method to verify that the discretization error scales with the mesh width as theoretically predicted.

   Answer the following questions for various values of the meshwidth h and the parameter c.

2. *(3 pt)* Compute and plot the eigenvalues of the Gauss-Seidel error propagation matrix $B_{GS} = I - M_{GS}^{-1}A$. Do not be concerned about the computation time here.
3. *(3 pt)* Compute and tabulate the spectral radius $\rho(B_{GS})$.
4. *(3 pt)* Run Gauss-Seidel as a solver. Determine the asymptotic rate of convergence. Compare with the spectral radius.
5. *(3 pt)* Compute and tabulate the condition number of the Symmetric Gauss-Seidel preconditioned operator $M_{SGS}^{-1}A$.
6. *(3 pt)* Run the Symmetric Gauss-Seidel Method as a preconditioner for the CG method. You can do so by proceeding along one of the two possible paths. The first is to explicitly form the preconditioned system. The second is to use a function that given the residual vector $\mathbf{r}^k$ at the k-th iteration, forms the new vector $\mathbf{z}^k$ as $\mathbf{z}^k = M_{SGS}^{-1}\,\mathbf{r}^k$. Plot the convergence history for various values of the meshwidth h and the parameter c. Discuss the relationship with the condition number.
7. *(3 pt)* Choose a value of the parameter c and the meshwidth for which the superlinear convergence of CG is (clearly) visible. Plot the eigenvalues of the preconditioner operator for this problem. Also, plot the converge of the Ritz values towards the eigenvalues.
8. *(3 pt)* Compute and plot the eigenvalues of the coarse grid correction error propagation matrix B_{CGC}.
9. *(3 pt)* Compute and tabulate the spectral radius $\rho(B_{CGC})$.
10. *(3 pt)* Run the coarse grid correction as a solver. Determine the asymptotic rate of convergence. Compare with the spectral radius.
11. *(3 pt)* Solve the projected system using CG. You can so do by explicitly forming the deflated system and applying CG to this newly formed system. Plot convergence history for various values of the meshwidth h and the parameter c. Discuss the relationship with the effective condition number that excludes the zero eigenvalue.
12. *(9 pt)* Repeat the questions 2, 3, and 4 using Multigrid as a solver with M_{SGS} replaced by M_{TGM} and B_{SGS} by B_{TGM}.
13. *(9 pt)* Repeat the questions 5, 6, and 7 using Multigrid as a preconditioner with M_{SGS} now replaced by M_{TGM}.