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Sub-regular Triangulations

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Zusammenfassung

In der vorliegenden Arbeit werden sub-reguläre Triangulierungen untersucht, eine Obermenge der regulären Triangulierungen, die man erhält, wenn man den *Down-flip Reverse Search*-Algorithmus [1] benutzt, um die Triangulierungen einer Punktmenge zu enumerieren. Es sind genau die Triangulierungen, die so enumeriert werden, wenn man die Beschränkung auf Regularität aufhebt. Die Menge der Triangulierungen einer Punktmenge bildet die Knotenmenge des sogenannten *Flipgraphen*, dessen Kanten durch lokale Modifikationen, die Flips genannt werden, gegeben sind. Eine Triangulierung heißt *regulär*, wenn sie durch eine Höhenfunktion auf den Punkten induziert wird. Es wurde gezeigt, dass der auf die regulären Triangulierungen eingeschränkte Flipgraph zusammenhängend ist, jedoch ist im Allgemeinen die Zusammenhangskomponente der regulären Triangulierungen im Flipgraph größer, das heißt es existieren nicht-reguläre Triangulierungen, die durch Flips von regulären Triangulierungen erhalten werden können. Ist eine Totalordnung auf der Menge der Triangulierungen gegeben, dann können wir die Flips charakterisieren und nennen sie *Up-Flip*, falls die erhaltene Triangulierung von höherer Ordnung und *Down-Flip*, falls sie von niederer Ordnung als die ursprüngliche Triangulierung ist.

Die Klasse der sub-regulären Triangulierungen bezieht sich wesentlich auf die Begriffe und Ideen vorgestellt in [2], hauptsächlich eine Implementation des Down-flip Reverse Search-Algorithmus namens `mptopmcom` und dessen grundlegende Konzepte. Daher ist diese Arbeit wie folgt aufgebaut: Im ersten Kapitel werden wir `mptopmcom` vorstellen und das mathematische Gerüst, auf dem `mptopmcom` aufbaut, untersuchen und anhand eines Beispiels veranschaulichen. Im zweiten Kapitel untersuchen wir das *Universelle Polytop* aus [3], das als eine "Polytopisierung" des Flipgraphen interpretiert werden kann und eine Verallgemeinerung des Sekundärpolytops ist. Im dritten Kapitel werden wir uns mit Sub-regularität auseinandersetzen und eine Beweismethode vorstellen, die zeigt, dass alle semi-regulären Triangulierungen immer auch sub-regulär sind. Anschließend werden wir uns ein Gegenbeispiel anschauen, welches zeigt, dass es nicht möglich ist, auf diesem

Wege Gleichheit zu zeigen. Danach untersuchen wir, warum dieser Ansatz fehlschlägt und wie wir die Informationen die wir auf dem Weg gesammelt haben sinnvoll nutzen können. Schlussendlich werden wir uns die Beziehung von Sub-regularität und Symmetrie anschauen.

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0 Introduction

Triangulations are an essential part of discrete geometry and moreover, they are the core tool of many algorithms and proofs. The idea is to divide a complex problem into small and simple ones, which are easy to solve. The nature of "divide and conquer" is intuitive and the applications are numerous. I recommend to have a look into the first chapter of [5] for motivating examples and introduction to the basics of this topic.

In this thesis we will examine sub-regular triangulations, a superset of the well-studied regular triangulations, that arise when using the *down-flip reverse search* algorithm to enumerate the triangulations of a point set by ordering them. More precisely, they are exactly the triangulations that are enumerated this way, when removing the restriction of regularity. The triangulations of a point set are the nodes of a graph, whose edges are local modifications, called flips. This graph is called *flip graph*. A triangulation is called *regular*, if it is induced by a height function on its vertices. It has been shown, that the flip graph restricted to the regular triangulations is connected, yet in general, the connected component of the regular triangulations in the entire flip graph is larger, i.e. there exist non-regular triangulations, which are obtainable from regular ones via flips. Given an order on the set of triangulations, we can give the flips a direction and call them *up-flip*, if the obtained triangulation is of greater order and *down-flip*, if it is of lower order than the initial. A triangulation is called *sub-regular*, if it can be obtained via down-flips from a regular one.

The class of sub-regular triangulations heavily relies on the notions that are presented in [2], mainly an implementation of the down-flip reverse search algorithm called `mptopmcom` as well as the underlying concepts that are presented in the paper. Hence this thesis is structured as follows: In chapter one we will look at `mptopmcom` and look at the underlying objects in an example. In the second chapter, we will introduce the *universal polytope* from [3], which can be interpreted as a "polytopalization" of the flip graph and an extension of the secondary polytope. In the third chapter, we will examine sub-regularity and outline a geometric road map to show that every semi-regular triangulation is also sub-regular, then we will look at a counterexample, which shows that it is not possible to show equality this way. Thereafter, we will examine why this approach fails and how we may still make use of the information gathered along the way. Finally, we will look at what happens to sub-regularity, if we take symmetry into account.

Notation and Definitions

Before we get started, let us introduce some basic notation and definitions. If the reader encounters anything that is unclear and not explained in the following, it will most likely be found in [5].

In the following, let $\mathbf{A} = (a_1, a_2, a_3, ..., a_n) \subset \mathbb{R}^d$ be a full-dimensional *point configuration*, that is $\dim(\mathit{aff}(\mathbf{A})) = d$, and by $[n] := \{1, 2, \dots, n\}$ the set of indices up to n. For the sake of simplification, we will sometimes refer to points by their indices, i.e. $i \leftrightarrow a_i$. This will be particularly useful when we look at the a_ith entry of a vector and in examples generated by `polymake` [6]. A *k-simplex* is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^d$ and a *circuit of* $\mathbf{A}$ is a minimal dependent subset $Z \subseteq \mathbf{A}$, i.e. Z is affinely dependent and $Z \setminus \{z\}$ is affinely independent for all $z \in Z$. Let us denote by $\mathbf{0}$ the zero-vector and by e_i the ith unit vector. For a matrix $M \in \mathbb{R}^{n \times k}$ define the *image of* M as $\operatorname{Im} M := \{Mx \in \mathbb{R}^n \mid x \in \mathbb{R}^k\}$ and the *kernel of* M as $\ker M := \{x \in \mathbb{R}^k \mid Mx = \mathbf{0}\}$. Points and vectors will be represented as columns.

Definition 0.1 ((Polyhedral) Subdivision)**.** A *(polyhedral) subdivision* Σ of a point configuration $\mathbf{A}$ is a set of subsets of $\mathbf{A}$, called cells, such that the following holds true:

- **CP** If $G \in \Sigma$ and F is a face of G, denoted by $F \leq G$, then $F \in \Sigma$,
- **UP** the union of the elements of Σ cover the convex hull of $\mathbf{A}$, and
- **IP** any two elements of Σ intersect in a common face (possibly the empty set as the trivial face).

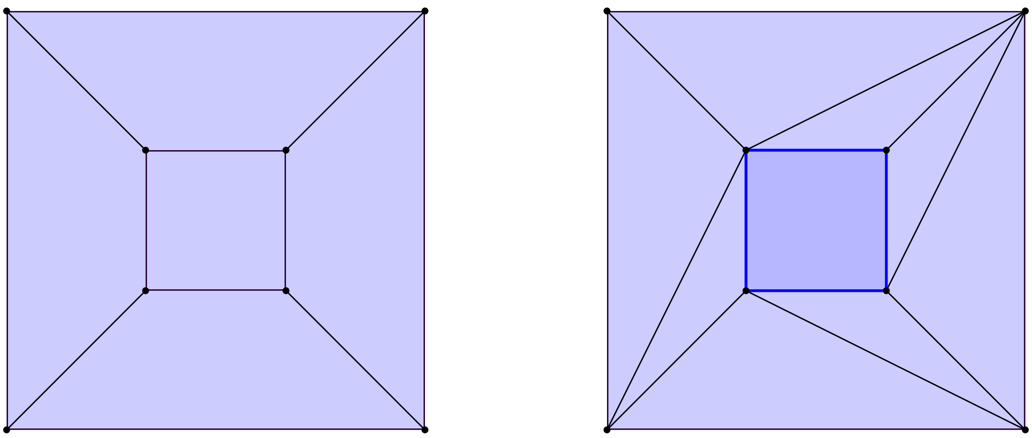
We call **CP** *closure property*, **UP** *union property* and **IP** *intersection property*. The subdivisions of a point configuration are a partially ordered set. We say $\Sigma \preceq \Sigma'$, that is Σ is a refinement of Σ', if for each $G \in \Sigma$ there is a $G' \in \Sigma'$ with $G \subseteq G'$. From this perspective, triangulations are a special case of subdivisions, in particular, they are the finest subdivisions:

Definition 0.2 (Triangulation)**.** A polyhedral subdivision of **A** consisting of simplices is called a *triangulation* of **A**.

Since simplices are simplicial, that is every proper face is a simplex, it is common to describe a triangulation as cells of cardinality $d + 1$, called maximal simplices. As mentioned before, the set of all triangulations of a point configuration is the node set of the flip graph, where two triangulations are connected by an edge, if there exist a local transformation, a flip, that transforms one into the other. For a d-dimensional point configuration consisting of n elements, define the *corank* as the number $n - d - 1$.

Definition 0.3 (Almost Triangulation)**.** A subdivision $\hat{\Delta}$ of **A** is called an *almost triangulation*, if the following holds:

1. all faces have at most corank 1 and
2. all corank-1 faces contain the same circuit.



(a) Subdivision satisfying the three conditions of Definition 0.1.

(b) Almost triangulation with designated circuit (blue) in the middle.

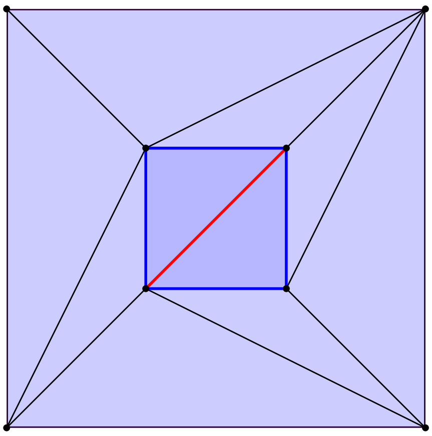
Figure 0.1: A subdivision and an almost triangulation of the nested squares.

Almost triangulations are only one step coarser than triangulations in the poset of subdivision. In other words, every refinement of an almost triangulation is a triangulation.

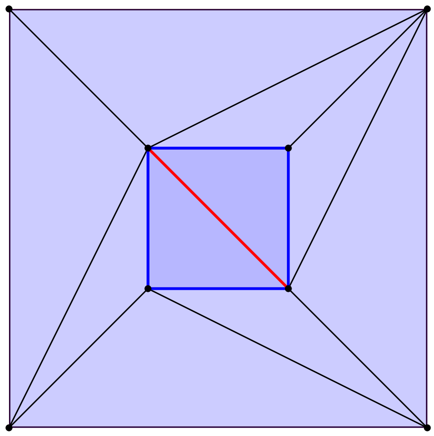
The unique circuit of Definition 0.3 is intuitively the cell that yet needs to be triangulated. We will now see, that there are exactly two ways to refine an almost triangulation.

Theorem 0.1 ([5, Lemma 2.4.2]). Let $Z = (Z^+, Z^-)$ be a circuit of $\mathbf{A}$ and $\underline{Z} = Z^+ \cup Z^-$. Then there exist exactly two triangulations of $\underline{Z}$:

$$\Delta^+ (Z) := \{\underline{Z} \setminus \{z\} \mid z \in Z^-\} \text{ and}$$
$$\Delta^- (Z) := \{\underline{Z} \setminus \{z\} \mid z \in Z^+\}.$$



(a)



(b)

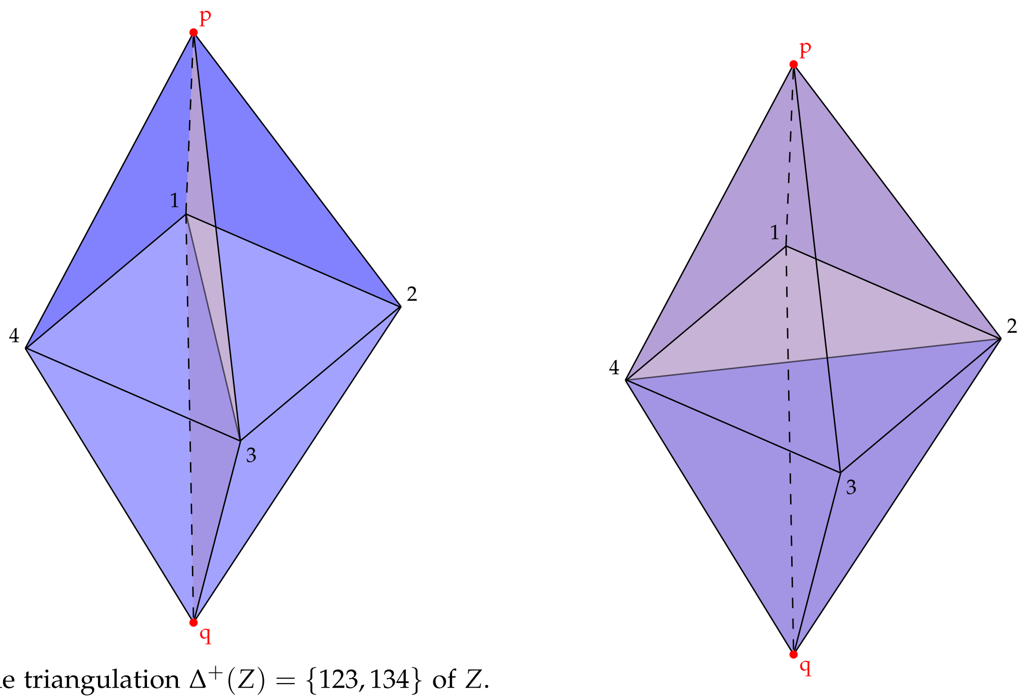
Figure 0.2: The two refinements of the almost triangulation Figure 0.1b.

The above partition $Z = (Z^+, Z^-)$ comes from Radon's theorem, where he showed that for a d-dimensional set $\underline{Z}$ containing more than $d+1$ points we can always find a partition such that $conv\,(Z^+) \cap conv\,(Z^-) \neq \emptyset$. For a circuit, this partition is unique up to orientation.

Definition 0.4 (Flip)**.** Two triangulations Δ_1 and Δ_2 are *connected by a flip*, if they are the two refinements of the same almost triangulation $\hat{\Delta}$, i.e. $\Delta_1 \preceq \hat{\Delta}$ and $\Delta_2 \preceq \hat{\Delta}$.

If Z is a full-dimensional circuit, the flip from Δ_1 to Δ_2 (and vice versa), denoted by $f = [\Delta_1 \rightsquigarrow \Delta_2]$, is exchanging $\Delta^+(Z)$ and $\Delta^-(Z)$ respectively, where Z is the unique circuit of $\hat{\Delta}$.

Remark 1. If Z is not full-dimensional we have to do some extra work: For a face $F \leq \Delta$ define the link $L_\Delta(F) := \{T \in \Delta \mid T \cap F = \emptyset, T * F \in \Delta\}$, where $\Delta * \Delta' := \{F \cup F' \mid F \in \Delta, F' \in \Delta'\}$ denotes the simplicial join. Then, if the link of all maximal simplices F of $\Delta^+(Z)$ in Δ are the same, we can exchange $\Delta^+(Z) * L_\Delta(F)$ and $\Delta^-(Z) * L_\Delta(F)$ respectively as shown in Figure 0.3. In this case we say that Δ *supports a flip at the circuit* Z. Without loss of generality we will assume, that Z is full-dimensional for the rest of this thesis, otherwise we can just replace $\Delta^+(Z)$ and $\Delta^-(Z)$ by $\Delta^+(Z) * L_\Delta(F)$ and $\Delta^-(Z) * L_\Delta(F)$ respectively.



(a) The triangulation $\Delta^+(Z) = \{123, 134\}$ of Z. Both maximal simplices 123 and 134 are linked to p and q, i.e. $L(123) = L(134) = \{p, q\}$, hence a flip is possible.

(b) The flipped triangulation $\Delta^-(Z) = \{124, 234\}$ of Z.

Figure 0.3: 2-dimensional circuit $Z = (\{13\}\{24\})$ in $\mathbb{R}^3$ linked to the two 0-simplices p and q.

Definition 0.5 (Regular Triangulation)**.** Let $\mathbf{A} = (a_1, a_2, a_3, ..., a_n) \subset \mathbb{R}^d$ be a d-dimensional point configuration and let Δ be a triangulation of $\mathbf{A}$. We say Δ is *regular*, if it is the projection of the lower envelope of a $d+1$-dimensional polytope. That is, it arises by omitting the last coordinate of all facets facing downwards. More technically, if there exists a lifting function $\omega : \mathbf{A} \mapsto \mathbb{R}$, such that for the lifted point configuration

$\mathbf{A}^\omega = ((a_1, \omega_1), (a_2, \omega_2), \ldots, (a_n, \omega_n)) \subset \mathbb{R}^{d+1}$, we have

$$\Delta = \{\pi(F) \mid F \text{ facet of } conv\left(\mathbf{A}^\omega\right), \langle \vec{n}_F, e_{d+1} \rangle \leq 0\},$$

where $\pi : \mathbb{R}^{d+1} \mapsto \mathbb{R}^d$ is a projection by ommiting the last coordinate and $\vec{n}_F$ an outer normal vector of the facet F.

Note that the above definition extends naturally to polyhedral subdivisions, but since we will be dealing with triangulations almost exclusively this definition is sufficient. It has been shown, that any two regular triangulations of a point configuration are connected by a (finite) sequence of flips [5, Theorem 5.3.7]. But it is generally also possible to obtain a non-regular triangulation from a regular one by flipping. It has been shown, that for non-regular triangulations to occur, we need at least $d + 4$ points and $d \geq 2$ [5, Theorem 5.5.1]. The smallest example of a point configuration having a non-regular triangulation is the famous *Mother of All Examples* [5, Example 2.2.5] shown in Figure 0.4.

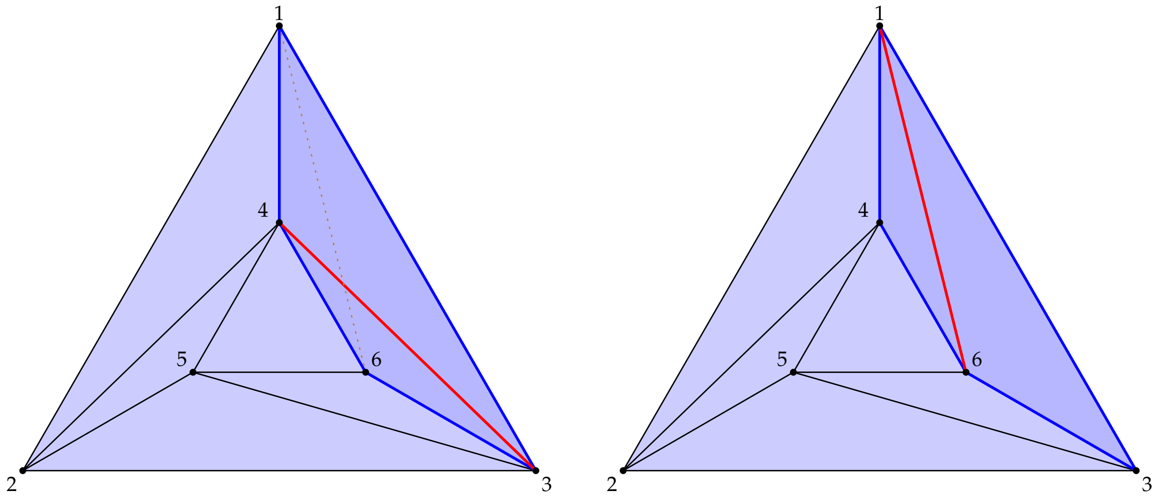


Figure 0.4: A flip between a regular (left) and a non-regular (right) triangulation over the circuit $Z = (\{1\ 6\}, \{4\ 6\})$.

Definition 0.6 (GKZ-vector, Secondary Polytope)**.** Let Δ be a triangulation of $\mathbf{A}$. The *GKZ-vector* of Δ is defined as

$$\text{gkz}_\Delta := \sum_{a \in \mathbf{A}} \text{gkz}_\Delta(a) \cdot e_a,$$

where $\mathrm{gkz}_\Delta(a) := \sum_{\substack{\sigma\in\Delta \\ a\in\sigma}} \mathrm{vol}_d(\sigma)$ and e_a the ath unit vector. In other words, the ath coordinate of gkz_Δ is the sum of the d-dimensional volumes of simplices incident to a. Then the *secondary polytope* of $\mathbf{A}$ is defined as

$$\Sigma\text{-poly}(\mathbf{A}) := conv\left(\{\,\mathrm{gkz}_\Delta \mid \Delta \text{ is a triangulation of } \mathbf{A}\}\right).$$

It has been shown, that the secondary polytope has dimension $n - d - 1$, if $\mathbf{A}$ is full-dimensional [5, Theorem 5.1.10]. The construction of the GKZ-vector and secondary polytope $\Sigma\text{-poly}(\mathbf{A})$ seems unintuitive at first, but it does have some very deep structural connections. That is, the vertices of $\Sigma\text{-poly}(\mathbf{A})$ correspond precisely to the regular triangulations of $\mathbf{A}$. But that's not all: the face lattice of $\Sigma\text{-poly}(\mathbf{A})$ corresponds to the refinement poset of subdivisions of $\mathbf{A}$, i.e. let F and G be faces of $\Sigma\text{-poly}(\mathbf{A})$ and Σ_F and Σ_G be the corresponding subdivisions of $\mathbf{A}$, then the mutual face $F \cap G$ is corresponding to $\Sigma_{F\cap G}$ with $\Sigma_{F\cap G} \preceq \Sigma_F$ and $\Sigma_{F\cap G} \preceq \Sigma_G$ and vice versa. In particular, we can interpret the edges as flips. The implication only goes one way, every edge implies a flip, the converse does not hold in the general case, since there can be flips where the corresponding edge gets "absorbed" by another face. This motivates the following definition:

Definition 0.7 (Flip Graph)**.** Let $\mathbf{A}$ be a point configuration, $\Delta_\mathbf{A}$ be the set of all triangulations of $\mathbf{A}$ and $f_{\Delta_\mathbf{A}} := \{(\Delta_1, \Delta_2) \mid \Delta_1 \text{ and } \Delta_2 \text{ are connected by a flip}\}$, the set of all flips. Then the graph $\Phi = (\Delta_\mathbf{A}, f_{\Delta_\mathbf{A}})$ is called *flip graph* of $\mathbf{A}$. We denote with Φ_{reg} the flip graph restricted to regular triangulations and with Φ^c_{reg} the connected component of the regular triangulations in Φ, that is the sub-graph consisting out of the triangulations obtainable from a regular triangulation via a sequence of flips.

Since the regular triangulations are always connected by a sequence of flips, Φ_{reg} is connected.

Lemma 0.1 ([5, Corollary 5.3.14])**.** Φ_{reg} is connected.

Proof. This follows directly from the one-to-one correspondence of regular triangulations and the vertices of the secondary polytope $\Sigma\text{-poly}(\mathbf{A})$ and the fact that every edge implies a flip, i.e. $1\text{-skeleton}(\Sigma\text{-poly}(\mathbf{A})) \subseteq \Phi_{\mathrm{reg}}$, where the $1\text{-skeleton}(\Sigma\text{-poly}(\mathbf{A}))$ is the graph consisting of the edges and vertices of $\Sigma\text{-poly}(\mathbf{A})$. □

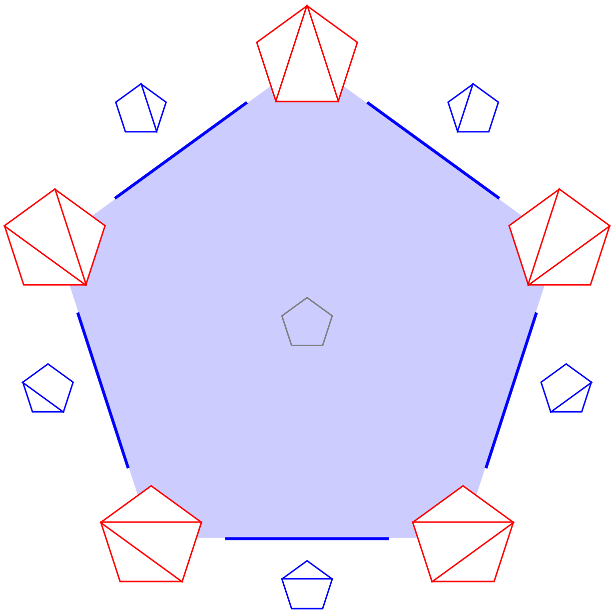


Figure 0.5: The secondary polytope of the 5-gon being a 5-gon itself. The triangulations (red) correspond to the vertices, the almost triangulations (blue) to edges and the trivial subdivision (gray) to the trivial face, i.e. the whole polytope.

By definition Φ^c_{reg} is also connected, yet Φ need not be connected in general [5, Theorem 7.4.6].

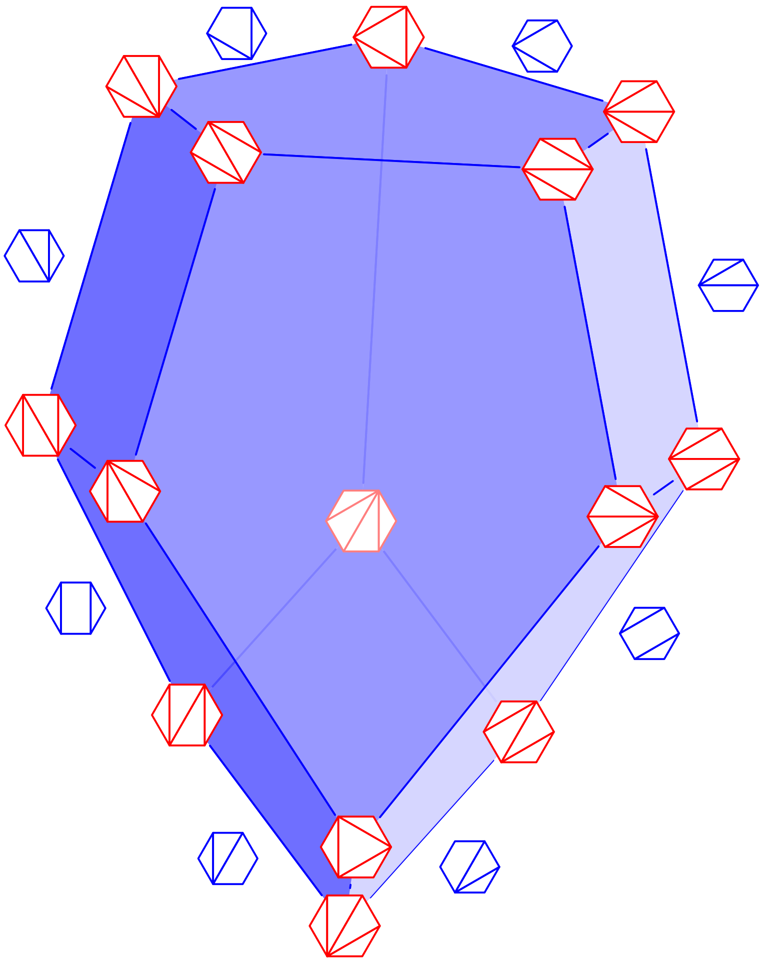


Figure 0.6: The secondary polytope of the 6-gon. All triangulations (red) and some coarser subdivisions (blue) are shown at their corresponding face.

1 The Algorithm `mptopmcom`

The way `mptopmcom` calculates the triangulations of a point configuration is creating an initial seed triangulation and then flipping through the flip graph. This notion is implemented from the algorithm `topcom` [7], which was published in 2002 by Jörg Rambau. The `mp` in `mptopmcom` originates from the `mpi` multi-thread standard, which is the interface of choice for parallelization. Since the seed is a regular triangulation, we are only visiting nodes contained in Φ^c_{reg}, i.e. we are calculating regular triangulations and triangulations that are obtainable from a regular triangulation via flips. For `mptopmcom` to enumerate in a consistent way, it uses an order on the set of triangulations:

Definition 1.1 (Total Order on $\Delta_{\mathbf{A}}$)**.** Let $\mathbf{A}$ be a point configuration and $\lambda = (M^n, M^{n-1}, \ldots, M)^\mathsf{T}$ for a sufficiently large $M \gg 0$. Then for any two triangulations $\Delta_1, \Delta_2 \in \Delta_{\mathbf{A}}$ we say Δ_2 *is larger than* Δ_1, i.e. $\Delta_2 > \Delta_1$, if

$$\langle \lambda, \text{gkz}_{\Delta_2} \rangle > \langle \lambda, \text{gkz}_{\Delta_1} \rangle$$

or in case of equality, if e_{Δ_2} is lexicographically larger than e_{Δ_1}, where e_{Δ_1} and e_{Δ_2} are the characteristic vectors of the simplices of Δ_1 and Δ_2 respectively.

It is important to note, that we can always find an $M \gg 0$ sufficiently large, such that the linear functional λ is injective on the finite set $\{\text{gkz}_\Delta \mid \Delta \in \Delta_{\mathbf{A}}\}$. Then, λ induces the lexicographic ordering on the points of Σ-poly$(\mathbf{A})$ this way. Since this correspondence is one-to-one for the regular triangulations, the case of equality can only occur, when we encounter non-regular triangulations (and it indeed does occur, see [2, Example 8]). Also note, that the e_Δ for a $\Delta \in \Delta_{\mathbf{A}}$ from the equality case does not live in the realm of Σ-poly$(\mathbf{A})$, but in a higher dimensional setting, which we will be looking at in Chapter 2. There, we will introduce the universal polytope U-poly$(\mathbf{A}) = conv\,(e_\Delta \mid \Delta \in \Delta_{\mathbf{A}})$ of $\mathbf{A}$ and see how this is an extension of Σ-poly$(\mathbf{A})$. In particular, e_{Δ_1} being lexicographically larger than e_{Δ_2}

is dependent on how we order the simplices of $\mathbf{A}$. With this order we can now introduce the terms maximal (minimal) triangulation as the triangulation of highest (lowest) order and classify flips as upward (downward), whenever the resulting triangulation is of higher (lower) order:

Definition 1.2 (Up-/Down-Flip)**.** Let $f = [\Delta_1 \rightsquigarrow \Delta_2]$ be the flip from Δ_1 to Δ_2. Then we say that f is an *up-flip* denoted as f^+, if

$$\Delta_1 < \Delta_2$$

and a *down-flip* denoted as f^- otherwise.

Let $Z = (Z^+, Z^-)$ be the corresponding circuit and without loss of generality $\Delta^-(Z) \subseteq \Delta_1$ and $\Delta^+(Z) \subseteq \Delta_2$, then f being an up-flip is equivalent to $\Delta^-(Z) < \Delta^+(Z)$. Since circuits are unique up to orientation, when we without loss of generality assume that $\Delta^+(Z) \subseteq \Delta$, then this means that the outcome would also only differ by orientation, that is a sign or change of direction of a flip.

Lemma 1.1. Let $\mathbf{A} = (a_1, a_2, a_3, ..., a_n)$ be a point configuration. There exists a unique maximal regular triangulation $\Delta^\star \in \Delta_{\mathbf{A}}$.

Proof. Since the order of a triangulation Δ is defined by evaluating its GKZ-vector gkz_Δ at the objective vector $\lambda = (M^n, M^{n-1}, \ldots, M)^\mathsf{T}$, which is lying on the moment curve for all M, λ can be considered generic and hence the optimal solution of $\max_{\Delta \in \Delta_{\mathbf{A}}} \langle \mathrm{gkz}_\Delta, \lambda \rangle$ is a vertex of Σ-poly$(\mathbf{A})$. Now the claim follows from the duality of regular triangulations and vertices of the Σ-poly$(\mathbf{A})$. □

We will see in the next section, that the existence of $\Delta^\star$ is important in order for the down-flip reverse search algorithm to work.

1.1 Down-Flip Reverse Search

The down-flip reverse search part of `mptopmcom` is based on the reverse search algorithm for enumeration in a connected graph originally introduced in [1]. The main idea is to do a top-down depth-first traversal on Φ^c_{reg}. This means finding a maximal node in the graph with respect to our evaluation λ and enumerating going down (depth-first) along the "reverse" direction of the edges, that is flipping downward. Let us dive into the details of the algorithm. The key ingredients to do a reverse search are the following:

1. A finite (node) set $\Delta_{\mathbf{A}}$ and an edge set $f^-_{\Delta_{\mathbf{A}}}$,
2. an objective function λ,
3. a predecessor function π,
4. an adjacency oracle $Adj_\Phi(\Delta, j)$.

The general environment is the directed flip graph $\Phi = (\Delta_{\mathbf{A}}, f^-_{\Delta_{\mathbf{A}}})$ of $\mathbf{A}$, where the edges are directed "reversely" downward, i.e. $(\Delta_1, \Delta_2) \in f^-_{\Delta_{\mathbf{A}}} \Leftrightarrow f^- = [\Delta_1 \rightsquigarrow \Delta_2]$. Since we always start with a regular triangulation and only move over edges, we are traversing through Φ^c_{reg}. With $\pi(\Delta)$ we will denote the neighbor of Δ with the highest value with respect to the objective function λ, that is the triangulation of highest order obtainable from Δ via an up-flip, and we will call it the *predecessor* of Δ. Let $\delta(\Delta)$ be the number of neighbors of Δ of smaller value, we call $\delta(\Delta)$ the *outdegree* of Δ. Now, given any regular triangulation, we can obtain the (unique) maximal triangulation $\Delta^\star$ by simply following the path of (regular) predecessors to "the end" (see Lemma 0.1 and Lemma 1.1), i.e. when we reach a node $\Delta^\star$ with $\pi(\Delta^\star) = null$. From there `mptopmcom` does a depth-first search through the flip graph Φ given by an adjacency oracle $Adj_\Phi(\Delta, j)$, which returns the jth neighbor (w.r.t. λ) of Δ, i.e. let $\Delta_1 < \Delta_2 < \cdots < \Delta_k$ be the neighbors of Δ obtained by a down-flip, then $Adj_\Phi(\Delta, j) = \Delta_j$, if $j \leq k$ and *null* otherwise.

The reverse search for enumeration has proven to be very well structured for parallelization [1]. The reason being, that it's easy to decompose it into independent sub-tasks. This comes at the computational costs of having to repeat some calculations like calculating the neighborhood, but these are of relatively small order and can be somewhat minimized by caching as the authors show in [2]. There are multiple ways how to split up the procedure. The naive approach is to split up the children of the root node in the search tree, which is especially good, if the height of the tree is small and tends to be bad if the tree is very unbalanced. The termination of each sub-task can easily be detected using the depth counter.

Algorithm 1: Down-Flip Reverse Search

Input: An ordered point configuration $\mathbf{A} = (a_1, a_2, a_3, ..., a_n) \subset \mathbb{R}^d$
Result: The enumerated set of (sub-)regular triangulations $\{\Delta_i \mid i \in [m], m \in \mathbb{N}\} \subseteq \Delta_{\mathbf{A}}$.

begin
  $\Delta \leftarrow$ a regular triangulation of $\mathbf{A}$ ▷ Create a seed
  while $\pi(\Delta) \neq null$ **do**
    $\Delta \leftarrow \pi(\Delta)$ ▷ Obtain maximal triangulation $\Delta^\star$
  end while
  $j \leftarrow 0$, $depth \leftarrow 0$ ▷ Start of the depth-first traverse
  repeat
    while $j < \delta(\Delta)$ **do**
      $j \leftarrow j + 1$
      if $\pi(Adj_\Phi(\Delta, j)) = \Delta$ **then**
        $\Delta \leftarrow Adj_\Phi(\Delta, j)$ ▷ Go along the jth down-flip
        $j \leftarrow 0$
        $depth \leftarrow depth + 1$
        Enumerate and output Δ
      end if
    end while
    if $depth > 0$ **then**
      $(\Delta, j) \leftarrow \pi(\Delta)$ ▷ Go one layer up
      $depth \leftarrow depth - 1$
    end if
  until $depth = 0$ *and* $j = \delta(\Delta)$
end

Example 1 (Enumerating the triangulations of the 6-gon)**.** Let us look at a run of mptopmcom with our input being the vertices

$$\mathbf{A} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

of the 6-gon. Figure 1.1 shows the flip graph Φ of $\mathbf{A}$, which is precisely the 1-skeleton of Σ-poly$(\mathbf{A})$ shown in Figure 0.6. Since $\mathbf{A}$ does allow any non-regular triangulations, every triangulation uniquely corresponds to a vertex of Σ-poly$(\mathbf{A})$.

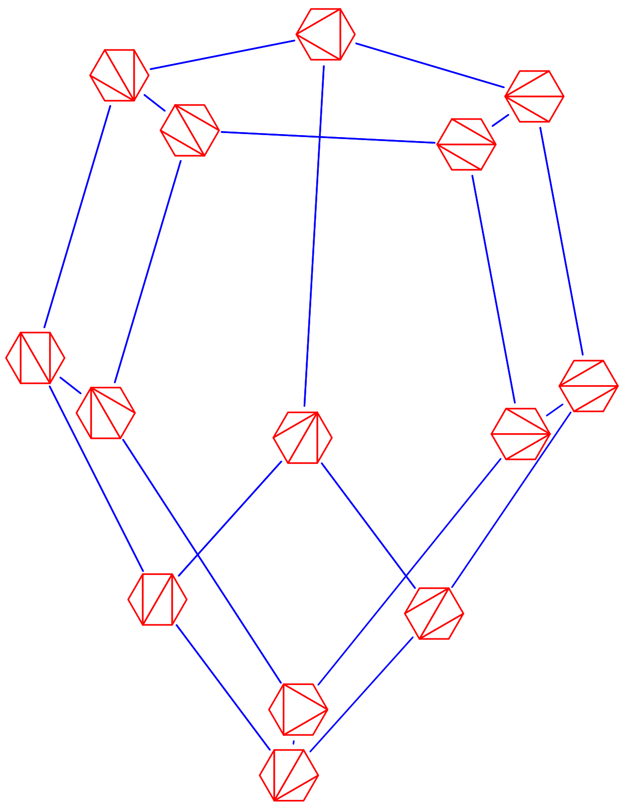


Figure 1.1: The flip graph of **A** coinciding with the graph of consisting of the vertices and edges of Σ-poly$(\mathbf{A})$.

Now we consider the total order of the triangulations of **A** and let every edge be directed upward, i.e. they point towards the node of higher order, as shown in Figure 1.2.

Given any (regular) seed triangulation we can get the maximum triangulation $\Delta^\star$ by simply following any path to the "sink"-node, i.e. until we reach the node, that does not have any outgoing edges. In this case it is the (regular) triangulation with the GKZ-vector

$$(8, 1, 4, 6, 4, 1)$$

in the top right of Figure 1.1, 1.2 and 1.3. Note that the representation of **A** in the figures is only combinatorially equivalent to **A**, since **A** is stretched more horizontally. This means that while it looks like the GKZ-vector of the top-center triangulation in Figure 1.2 should have the same entry at the coordinates 1, 3 and 5, this is only the case for the latter two. Passing **A** to `mptopmcom`, we can see below, that the seed triangulation is already $\Delta^{\star}$.

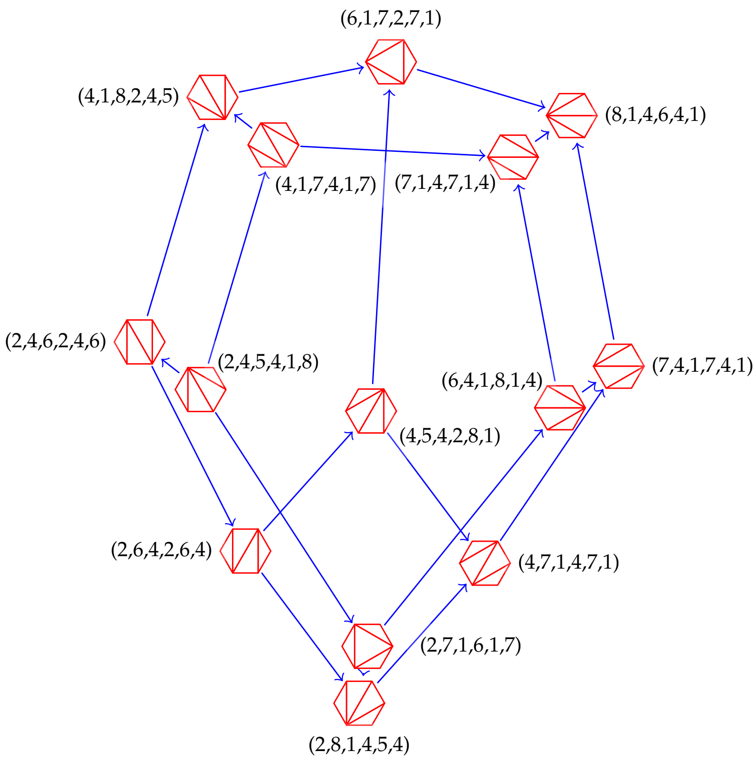


Figure 1.2: The directed flip graph of **A** with GKZ-vector at the respective node. The vertices are enumerated counter clockwise starting from the leftmost vertex.

```
mptopcom version 1.1
Evaluating Commandline Options ...
... done.
Entering no of triangs.
Initial triangulation: {0,1,2} {0,2,3} {0,3,4} {0,4,5}
Normalized volume is: 8
0: [0->6,3:{{0,1,2},{0,2,3},{0,3,4},{0,4,5}}] gkz: [8,1,4,6,4,1]
1: [0->6,3:{{0,1,2},{0,4,5},{2,3,4},{0,2,4}}] gkz: [6,1,7,2,7,1]
2: [0->6,3:{{0,1,2},{2,3,4},{2,4,5},{0,2,5}}] gkz: [4,1,8,2,4,5]
3: [0->6,3:{{2,3,4},{2,4,5},{1,2,5},{0,1,5}}] gkz: [2,4,6,2,4,6]
4: [0->6,3:{{0,4,5},{2,3,4},{1,2,4},{0,1,4}}] gkz: [4,5,4,2,8,1]
5: [0->6,3:{{2,3,4},{1,2,4},{0,1,5},{1,4,5}}] gkz: [2,6,4,2,6,4]
6: [0->6,3:{{0,1,2},{0,2,3},{3,4,5},{0,3,5}}] gkz: [7,1,4,7,1,4]
7: [0->6,3:{{0,1,2},{3,4,5},{0,2,5},{2,3,5}}] gkz: [4,1,7,4,1,7]
```

```
8: [0->6,3:{{3,4,5},{2,3,5},{1,2,5},{0,1,5}}] gkz: [2,4,5,4,1,8]
9: [0->6,3:{{0,3,4},{0,4,5},{1,2,3},{0,1,3}}] gkz: [7,4,1,7,4,1]
10: [0->6,3:{{0,4,5},{1,2,3},{0,1,4},{1,3,4}}] gkz: [4,7,1,4,7,1]
11: [0->6,3:{{1,2,3},{0,1,5},{1,4,5},{1,3,4}}] gkz: [2,8,1,4,5,4]
12: [0->6,3:{{1,2,3},{0,1,3},{3,4,5},{0,3,5}}] gkz: [6,4,1,8,1,4]
13: [0->6,3:{{1,2,3},{3,4,5},{0,1,5},{1,3,5}}] gkz: [2,7,1,6,1,7]
Hits: 103  Miss: 14
Totalcount: 14
```

The tree underlying this enumeration is visualized in Figure 1.3, where we can see its depth-first structure.

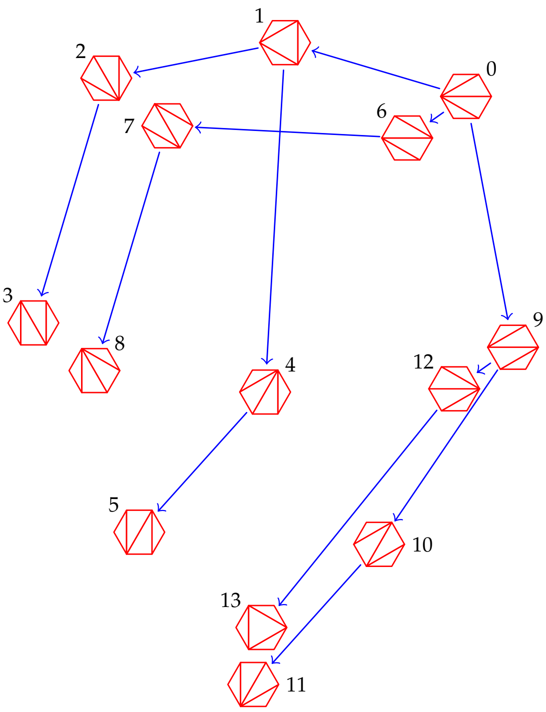


Figure 1.3: The search tree traversed by `mptopmcom`. The direction indicating how the algorithm is moving along the edges, which is the "reverse" direction of the order of Definition 1.1.

◇

1.2 Symmetry

The second part of `mptopmcom` is dealing with symmetry. This is important, since the number of triangulations grows extremely fast for increasingly complex objects, for example the cyclic polytope $C_{4n-4}(4n)$ has at least 2^n triangulations as shown in [3, Proposition 5.10]. Here, the parallelization requires some extra thought, since we can't just compare with already visited nodes. The key idea is to identify an equivalence class with a unique representative. Then, we can exchange the nodes in the flip graph with their respective equivalence class, while keeping the edges the same, and the main challenge becomes the calculation of the canonical representative of a equivalence class. Let us introduce some notions from group theory:

Let $\mathbf{A} = (a_1, a_2, a_3, ..., a_n) \subset \mathbb{R}^d$, and let

$$\mathbf{G} \leq \mathrm{SL}_d(\mathbb{R}) \rtimes \mathbb{R}^d$$

be the finite group of affine unimodular automorphisms acting on the set $\mathbf{A}$ by permutations. Then a $g \in \mathbf{G}$ acts on a subset of $\mathbf{A}$ by $g \cdot \{a_{i_1}, \ldots, a_{i_m}\} = \{g(a_{i_1}), \ldots, g(a_{i_m})\}$. This induces how $\mathbf{G}$ acts on a triangulation Δ of $\mathbf{A}$: $g \cdot \Delta = \{g \cdot \sigma \mid \sigma \in \Delta\}$.

Lemma 1.2 ([2, Lemma 7]). Let $g \in \mathbf{G}$ and $\Delta \in \Delta_{\mathbf{A}}$, then

$$\mathrm{gkz}_{g \cdot \Delta}(g(a)) = \mathrm{gkz}_{\Delta}(a)$$

for all points $a \in \mathbf{A}$.

In particular we get

Corollary 1.1. The group $\mathbf{G}$ leaves the regular triangulations invariant, in other words $g \cdot \Delta$ is regular (for all $g \in \mathbf{G}$) if and only if Δ is regular.

Now the symmetry is induced by $\mathbf{G}$ with the following definition:

Definition 1.3 ($\mathbf{G}$-orbit). Let $\mathbf{G}$ be a group acting on $\Delta_{\mathbf{A}}$ and $\Delta \in \Delta_{\mathbf{A}}$. Then the $\mathbf{G}$-orbit of Δ is defined as

$$\mathbf{G} \cdot \Delta := \{g \cdot \Delta \in \Delta_{\mathbf{A}} \mid g \in \mathbf{G}\}$$

The **G**-orbits are the equivalence classes of the triangulations. Two triangulations are equivalent, if and only if they belong to the same **G**-orbit. Now let us define the representative of a **G**-orbit:

Definition 1.4 (Canonical Representative of an **G**-orbit)**.** Let **G** be a group acting on $\Delta_{\mathbf{A}}$ and $\Delta \in \Delta_{\mathbf{A}}$ a triangulation. Then the *canonical representative* of the **G**-orbit $\mathbf{G} \cdot \Delta$ is defined as

$$\rho(\Delta) := \max_{\Delta^\star \in \mathbf{G} \cdot \Delta} \Delta^\star$$

where the maximum is taken with respect to the order as in Definition 1.1, i.e. it has the lexicographically maximal GKZ-vector and if this is not unique, it also has the lexicographically maximal characteristic vector.

The canonical representative is unique, since the order from Definition 1.1 is strict. It follows from the definition, that $\rho(\Delta) = \rho(\Delta')$ if and only if $\mathbf{G} \cdot \Delta = \mathbf{G} \cdot \Delta'$ and $\rho(\rho(\Delta)) = \rho(\Delta)$. This means that for any given triangulation, calculating the representative of the respective **G**-orbit is the same as membership testing. The calculation of the representative is done by a variation of the Schreier-Sims algorithm, which we will not look into in detail, as it would go beyond the scope of this thesis.

Example 2 (Enumerating the triangulations of the 6-gon modulo symmetry)**.** Let us continue with the 6-gon $\mathbf{A} = \{(0,0)^\top, (1,-1)^\top, (2,-1)^\top, (3,0)^\top, (2,1)^\top, (1,1)^\top\}$. The group **G** has order 12, meaning it contains 12 elements, and is given in permutation cycle notation by

$$\begin{aligned}\mathbf{G} &= \langle (123456), (16)(25)(34) \rangle \\ &= \{(1)(2)(3)(4)(5)(6), \\ &\quad (1)(4)(26)(35), \\ &\quad (2)(5)(13)(46), \\ &\quad (3)(6)(15)(24), \\ &\quad (12)(36)(45), \\ &\quad (14)(23)(56), \\ &\quad (16)(25)(34), \\ &\quad (135)(246), \\ &\quad (153)(264), \\ &\quad (123456), \\ &\quad (165432)\}.\end{aligned}$$

Figure 1.4 illustrates the actions of **G** on **A**. The lines going through represent mirroring the vertices along that line. The inner triangles shown in green illustrate rotating twice.

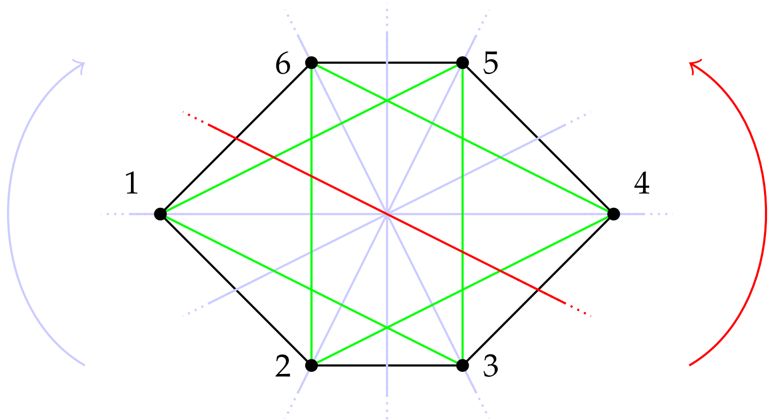


Figure 1.4: The 6-gon and some actions of **G**. The generators given above are indicated in red.

Note that we are stretching the idea of symmetry a little bit in this example, since some of these actions do not preserve volume of the cells of the triangulations contrary to the requirement of **G** (see also entries of the GKZ-vectors of the triangulations and their rotations in Figure 1.2). This is due to the implementation of `mptopmcom` only allowing integral points and for the sake of simplicity of the coordinates. We can restore this by modifying **A** so that all outer edges are the same length.

Passing **A** and **G** to `mptopmcom` outputs the following:

```
mptopcom version 1.1
Evaluating Commandline Options ...
... done.
Entering no of triangs.
Initial triangulation: {0,1,2} {0,2,3} {0,3,4} {0,4,5}
Normalized volume is: 8
0: [0->6,3:{{0,1,2},{0,2,3},{0,3,4},{0,4,5}}] gkz: [8,1,4,6,4,1]
1: [0->6,3:{{0,1,2},{0,4,5},{2,3,4},{0,2,4}}] gkz: [6,1,7,2,7,1]
2: [0->6,3:{{0,3,4},{0,4,5},{0,1,3},{1,2,3}}] gkz: [7,4,1,7,4,1]
Hits: 13  Miss: 4
Totalcount: 3

```

The nodes of the flip graph now consist of **G**-orbits of triangulations, i.e. the node $\rho(\Delta)$ represents all triangulations $g \cdot \Delta$ for $g \in \mathbf{G}$.

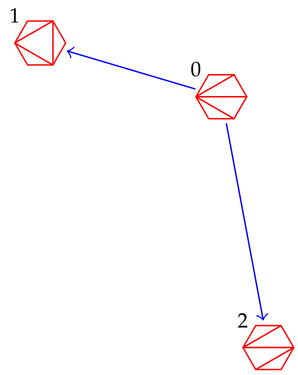


Figure 1.5: The flip graph of the 6-gon modulo symmetry consists of only three nodes, the canonical representatives of the corresponding **G**-orbits.

◇

2 The Universal Polytope

It has been shown, that GKZ-vectors cannot distinguish between non-regular triangulations in general as shown in [2, Example 8]. To have the order in Definition 1.1 be strict is achieved by introducing an order on a structure, which extends the duality of regular triangulations and the vertices of the secondary polytope to all triangulations:

Definition 2.1 (Universal Polytope [3])**.** Let $\mathbf{A} = (a_1, a_2, a_3, ..., a_n) \subset \mathbb{R}^d$ be a point configuration and let $S_{\mathbf{A}}$ denote the set of all d-simplices of $\mathbf{A}$. Then the *universal polytope* of $\mathbf{A}$ is defined as

$$\text{U-poly}(\mathbf{A}) := conv\left(e_\Delta \in \{0,1\}^{S_{\mathbf{A}}} \mid \Delta \in \Delta_{\mathbf{A}}\right) \subset \mathbb{R}^{S_{\mathbf{A}}},$$

where $e_\sigma \in \{0,1\}^{S_{\mathbf{A}}}$ is the σth unit vector, i.e. $e_\sigma = (0, \dots, 0, \overbrace{1}^{\sigma\text{th position}}, 0, \dots, 0)$ for $\sigma \in S_{\mathbf{A}}$ and for a subset $\Sigma \subseteq S_{\mathbf{A}}$ set $e_\Sigma := \sum_{\sigma \in \Sigma} e_\sigma$. In other words, it is the convex hull of the incidence vectors of the triangulations of $\mathbf{A}$ in $\mathbb{R}^{S_{\mathbf{A}}}$.

Remark 2. The name "universal polytope" was originally used in [8] for a polytope $\mathscr{U}_{\mathbf{A}}$ living in the exterior algebra $\Lambda_\star \mathbb{R}^n$. Since $\text{U-poly}(\mathbf{A})$ is isomorphic to $\mathscr{U}_{\mathbf{A}}$ ([3]), we will be using the name interchangeably.

From this perspective one can also introduce the secondary polytope of $\mathbf{A}$ as a projection from $\text{U-poly}(\mathbf{A})$ via $\pi : \mathbb{R}^{S_{\mathbf{A}}} \to \mathbb{R}^{\mathbf{A}}, \pi(e_\sigma) = \text{vol}_d(\sigma) \sum_{a \in \sigma} e_a$. This generalization comes with an increased complexity: The dimension of the secondary polytope is $\dim(\Sigma\text{-poly}(\mathbf{A})) = n - d - 1$ [5, Theorem 5.1.10] and if $\mathbf{A}$ is in general position, the dimension of the universal polytope is $\dim(\text{U-poly}(\mathbf{A})) = \binom{n-1}{d+1}$ [8]. This is important since running linear programs on $\text{U-poly}(\mathbf{A})$ is one of its main applications. These can solve the enumeration of all triangulation and finding the maximal/minimal triangulation with respect to the number of simplices.

2.1 Outer Description

Since running a linear problem on a lattice polytope is $\mathcal{NP}$-hard, that is "very very"-hard, the following relaxation arises naturally.

Definition 2.2 ([3]). Let U-poly$(\mathbf{A})$ be the universal polytope of $\mathbf{A}$. Define

$$\text{Q-poly}(\mathbf{A}) := \mathit{aff}(\text{U-poly}(\mathbf{A})) \cap \mathbb{R}_+^{S_\mathbf{A}},$$

the *linear programming relaxation of* U-poly$(\mathbf{A})$.

In order to find equations describing Q-poly$(\mathbf{A})$, the connection between the properties of triangulations and the matroid of $\mathbf{A}$ is used as introduced in [7]. There, cocircuits are used in order to check, if a set of simplices violates the intersection property.

Definition 2.3 (Cocircuit Form). Let τ be a $(d-1)$-simplex of $\mathbf{A}$ and H_τ be the hyperplane spanned by τ. Consider the partition $(\mathbf{A} \cap H_\tau^+, \mathbf{A} \cap H_\tau, \mathbf{A} \cap H_\tau^-)$ of $\mathbf{A}$. Then we call

$$Co_\tau := \sum_{\sigma = \tau \cup \{a\}, a \in \mathbf{A} \cap H_\tau^+} x_\sigma - \sum_{\sigma = \tau \cup \{a\}, a \in \mathbf{A} \cap H_\tau^-} x_\sigma \tag{2.1}$$

the *cocircuit form* associated with τ.

The following theorem gives us the outer description of Q-poly$(\mathbf{A})$, that is it is given as inequalities.

Theorem 2.1 ([3]). Let $\mathbf{A}$ be a point configuration.

(i) The affine span $\mathit{aff}(\text{U-poly}(\mathbf{A})) \subset \mathbb{R}^{S_\mathbf{A}}$ is defined by the linear equations $Co_\tau = 0$ for every $(d-1)$-simplex τ, together with one non-homogeneous linear equation valid on U-poly$(\mathbf{A})$ (see Equations 2.2, 2.3 and 2.4).

(ii) U-poly$(\mathbf{A})$ coincides with the integral hull of Q-poly$(\mathbf{A})$, i.e. the lattice points in Q-poly$(\mathbf{A})$ are precisely the incidence vectors of triangulations of $\mathbf{A}$.

(iii) Two triangulations Δ_1 and Δ_2 of $\mathbf{A}$ are neighbors in the edge graph of U-poly$(\mathbf{A})$ if and only if they are neighbors in the edge graph of Q-poly$(\mathbf{A})$.

(iv) For the case of the n-gon and configurations with at most $d+3$ points, we have Q-poly$(\mathbf{A}) =$ U-poly$(\mathbf{A})$. This is not true in general for $n \geq d+4 \geq 6$.

Note, that the case of (iv). does not allow any non-regular triangulation, whereas for the general case, $n \geq d+4 \geq 6$ are the minimal requirements for n and d in order for a non-regular triangulation to occur. The following are the non-homogeneous equations defining *aff* (U-poly$(\mathbf{A})$) from (i). For this, let τ be a non-interior $(d-1)$-simplex and *conv* (τ) a facet of *conv* $(\mathbf{A})$, then Co_τ has constant value ± 1 on U-poly$(\mathbf{A})$, since τ can be contained in exactly one cell of a triangulation. This yields a set of valid equations for *aff* (U-poly$(\mathbf{A})$), the *boundary cocircuit equations*:

$$\sum_{\sigma=\tau\cup\{a\},\, a\in\mathbf{A}\setminus\tau} x_\sigma = 1. \tag{2.2}$$

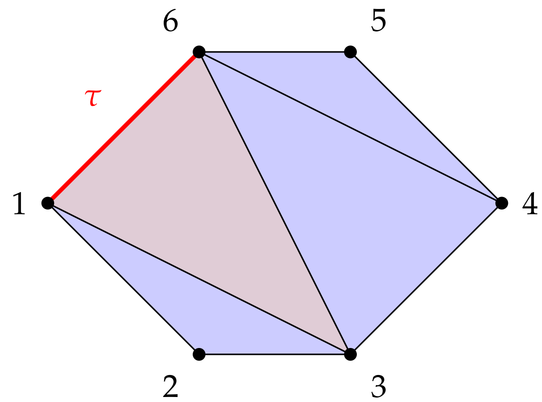


Figure 2.1: A non-interior 1-simplex τ and its unique cell in the triangulation.

We get one equation for each $(d-1)$-simplex lying inside the boundary facets of *conv* $(\mathbf{A})$. It is easy to see, that a triangulation satisfies all of these equations, since it has the union and intersection property (see Definition 0.1), meaning that a τ like above is the facet of exactly one cell: the union property tells us, that there has to be at least one d-simplex containing τ. The intersection property makes sure, that it is exactly one, since d-simplices can only intersect in common faces and τ is contained in a boundary facet of *conv* $(\mathbf{A})$.
Let $p \in$ *conv* $(\mathbf{A})$ be a point not lying in the convex hull of any $(d-1)$-simplex, then every

triangulation of **A** satisfy the *chamber equations*:

$$\sum_{\sigma \in S_{\mathbf{A}},\ p \in conv(\sigma)} x_\sigma = 1. \tag{2.3}$$

This gives us one equation for each chamber.

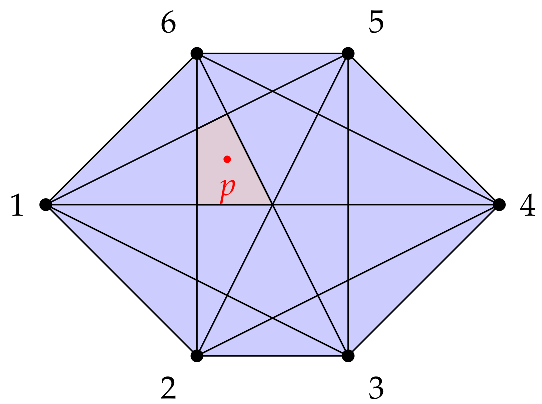


Figure 2.2: The chambers of the 6-gon and a point p lying in the interior of a chamber, i.e it is not contained in the convex hull of any 1-simplex, which are line segments between vertices.

For any triangulation of **A**, we have the *volume equation*:

$$\sum_{\sigma \in S_{\mathbf{A}}} \mathrm{vol}_d(conv\,(\sigma))x_\sigma = \mathrm{vol}_d(conv\,(\mathbf{A})). \tag{2.4}$$

This description enables us to run linear programs more efficiently, though the authors also mention, that the enumeration of all triangulations via enumerating the vertices of U-poly$(\mathbf{A})$ has two major drawbacks: The outer description of U-poly$(\mathbf{A})$ is hard to get and might be very complex and while this is easier for Q-poly$(\mathbf{A})$, as seen above, using Q-poly$(\mathbf{A})$ instead, yields the enumeration of many fractional vertices, which do not correspond to triangulations of **A**. In the proof of Theorem 2.1 (i), the authors surprisingly show that the affine span *aff*(U-poly$(\mathbf{A})$) is spanned by the regular triangulations only [3, Corollary 2.3]. This once more justifies U-poly$(\mathbf{A})$ as an extension of the secondary polytope Σ-poly$(\mathbf{A})$.

Example 3 (Universal Polytope of the 6-gon)**.** Let us look at the universal polytope of $\mathbf{A} = \{(0,0)^\mathsf{T}, (1,-1)^\mathsf{T}, (2,-1)^\mathsf{T}, (3,0)^\mathsf{T}, (2,1)^\mathsf{T}, (1,1)^\mathsf{T}\} \subset \mathbb{R}^2$. We have U-poly$(\mathbf{A}) \subset \mathbb{R}^{\binom{6}{3}} = \mathbb{R}^{20}$ and since the points of $\mathbf{A}$ are in general position, we have $\dim(\text{U-poly}(\mathbf{A})) = \binom{6-1}{2+1} = 10$. Using `polymake` the order of the simplices (and hence the coordinates of U-poly$(\mathbf{A})$) can be printed with `MAX_INTERIOR_SIMPLICES`.

```
  polytope > print $A->MAX_INTERIOR_SIMPLICES;
  {0 1 2}
  {0 1 3}
  {0 1 4}
  {0 1 5}
  {0 2 3}
  {0 2 4}
  {0 2 5}
  {0 3 4}
  {0 3 5}
  {0 4 5}
  {1 2 3}
  {1 2 4}
  {1 2 5}
  {1 3 4}
  {1 3 5}
  {1 4 5}
  {2 3 4}
  {2 3 5}
  {2 4 5}
  {3 4 5}

```

This means that a lattice point $p \in$ U-poly$(\mathbf{A})$ with $p_1 = 1$ corresponds to a triangulation containing the cell `012`. Every cocircuit equation is given by `polymake` as follows:

```
polytope > print universal_polytope($A)->EQUATIONS;
-8 1 3 3 2 3 4 3 3 3 1 1 2 2 3 4 2 2 3 2 1
(21) (1 -1) (5 1) (6 1) (7 1)
(21) (2 -1) (5 -1) (8 1) (9 1)
(21) (3 -1) (6 -1) (8 -1) (10 1)
(21) (2 1) (11 -1) (14 1) (15 1)
(21) (3 1) (12 -1) (14 -1) (16 1)
(21) (4 -1) (13 1) (15 1) (16 1)
(21) (6 -1) (12 -1) (17 1) (19 -1)
(21) (7 -1) (13 -1) (18 1) (19 1)
```

```
(21) (9 -1) (15 -1) (18 -1) (20 1)

```

Each row of tuples $(i \quad \mu_i)$, where $i \in [20]$ and $\mu_i \in \{-1, 1\}$, corresponds to the cocircuit equation $\sum \mu_i x_i = 0$. The leading (21) indicates the total number of parameters of the equation consisting of $x_1, \ldots, x_{20}$, the coefficients corresponding to the 20 simplices, and one constant. Using the result of Theorem 2.1 (i) `polymake` uses the volume equation (Equation (2.4)) and a non-negativity equation to describe Q-poly$(\mathbf{A})$. Theorem 2.1 (iv) tells us, that in our case Q-poly$(\mathbf{A})$ = U-poly$(\mathbf{A})$. Since there are 14 triangulations of $\mathbf{A}$ and they are all regular, U-poly$(\mathbf{A})$ is spanned by 14 vertices.

```
polytope > print universal_polytope($A)->VERTICES;
1 0 1 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 1
1 0 1 0 0 0 0 0 1 0 1 1 0 0 0 0 0 0 0 0 0
1 0 0 1 0 0 0 0 0 0 1 0 1 0 0 0 0 1 0 0 0
1 0 0 1 0 0 0 0 0 0 1 1 0 0 1 0 0 0 0 0 0
1 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 1
1 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 1 0 1 0
1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 1 0 0 0
1 0 0 0 1 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 1
1 0 0 0 1 0 0 0 0 0 0 1 0 0 1 0 1 0 0 0 0
1 1 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1
1 1 0 0 0 1 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0
1 1 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 1 0 0 0
1 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 0 1
1 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 1 0

```

◇

3 Sub-Regularity

This section is dedicated to examine the property of sub-regularity. Recall, that a triangulation is called *sub-regular*, if it is obtainable from a regular triangulation via down-flips. As mentioned before, this definition depends on the ordering of the points. Running `mptopmcom` on $\mathbf{A} = (a_1, a_2, a_3, ..., a_n)$ without the restriction of regularity, yields precisely the sub-regular triangulations with respect to the lexicographic ordering of the GKZ-vectors, which is equivalent to the order given by evaluating with $\lambda = (M^n, M^{n-1}, \ldots, M)^\intercal$ for a sufficiently large $M \gg 0$. In case they have the same GKZ-vector, we compare them as points of U-poly$(\mathbf{A})$ from the previous section (see Definition 1.1). The lemma below follows from definition:

Lemma 3.1. It holds that

$$\Phi_{\text{reg}} \subseteq \Phi_{\text{sub-reg}} \overset{\text{(a)}}{\subseteq} \Phi^c_{\text{reg}} \overset{\text{(b)}}{\subseteq} \Phi. \tag{3.1}$$

It has long been unclear, if the flip graph is always connected or in other words if (b) is an equality [3]. In general, this is not the case as shown in [5, Theorem 7.3.3], where they prove the existence of a 5-dimensional point configuration whose flip graph consists of at least 13 connected components. This motivates the following section, where we will look at the relation (a) closer.

Remark 3. In the following, we will be using the objective function $\lambda = (M^n, M^{n-1}, \ldots, M)^\intercal$ to identify the induced lexicographic order on $\mathbb{R}^{\mathbf{A}}$, where $\mathbf{A} = (a_1, a_2, a_3, ..., a_n)$ is our point configuration, to employ its geometric properties. This will be useful since $\mathbb{R}^{\mathbf{A}}$ is the environment, where the secondary polytope Σ-poly$(\mathbf{A})$ lives. Let $\mathbf{S}_n$ be the symmetric group of degree n and $\sigma \in \mathbf{S}_n$ a permutation. Then the objective function inducing the lexicographic order on $\mathbb{R}^{\sigma(\mathbf{A})}$, where $\sigma(\mathbf{A}) = (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, \ldots, a_{\sigma(n)})$, is

$$\lambda^\sigma := \sigma^{-1}(\lambda) = \left(M^{n+1-\sigma^{-1}(1)}, M^{n+1-\sigma^{-1}(2)}, M^{n+1-\sigma^{-1}(3)}, \dots, M^{n+1-\sigma^{-1}(n)}\right)^\mathsf{T}.$$

This is easy to see since for any $x \in \mathbb{R}^n$

$$\begin{aligned}\langle \sigma(x), \lambda \rangle &= \sum_{i=1}^{n} x_{\sigma(i)} \cdot M^{n+1-i} \\ &= \sum_{i=1}^{n} x_{\sigma(i)} \cdot M^{n+1-\sigma(\sigma^{-1}(i))} \\ &= \sum_{i=1}^{n} x_i \cdot M^{n+1-\sigma^{-1}(i)} \\ &= \langle x, \lambda^\sigma \rangle.\end{aligned}$$

Note that λ^σ is λ permuted by the inverse of σ. With this, we can examine what happens when we change the order of the points of $\mathbf{A}$ without having to go from $\mathbb{R}^{\mathbf{A}} \to \mathbb{R}^{\sigma(\mathbf{A})}$ and $\Sigma\text{-poly}(\mathbf{A}) \to \Sigma\text{-poly}(\sigma(\mathbf{A}))$. Instead, we stay in $\mathbb{R}^{\mathbf{A}}$, use $\Sigma\text{-poly}(\mathbf{A})$ and go from $\lambda \to \lambda^\sigma$. For the rest of this thesis, we will use σ and λ^σ to refer to the lexicographic order on $\mathbb{R}^{\sigma(\mathbf{A})}$.

3.1 Sub-Regular and Semi-Regular

A triangulation is called *semi-regular* if it is contained in Φ^c_{reg}, i.e. it is obtainable via a series of flips from a regular triangulation. Since the definition of sub-regularity is dependent on the order of the vertices, two questions arise:

Q1 Does there always exist a permutation $\sigma \in \mathbf{S}_n$ such that any semi-regular triangulation is sub-regular with respect to σ?

Q2 Do we have equality in Equation (3.1) (a), i.e. is every sub-regular triangulation semi-regular independent of how we order the vertices or does there exist a semi-regular triangulation, which is non-sub-regular for some $\sigma \in \mathbf{S}_n$?

Clearly, if we can show that the equality case of **Q2** is true, **Q1** is also answered. This motivates the following sections, where we are trying to find an answer to **Q2**.

3.2 A Geometric Roadmap to Equality (or not)

So far all semi-regular triangulations turned out to be sub-regular (with respect to the "canonical" ordering of the vertices). In this section we will look at a roadmap of how to show the equality of both sets. Then Example 4 is a counterexample and shows us, that it is not possible to show equality this way in general. After that, we will examine the underlying structure further to find weakened conditions from which the equality still follows.

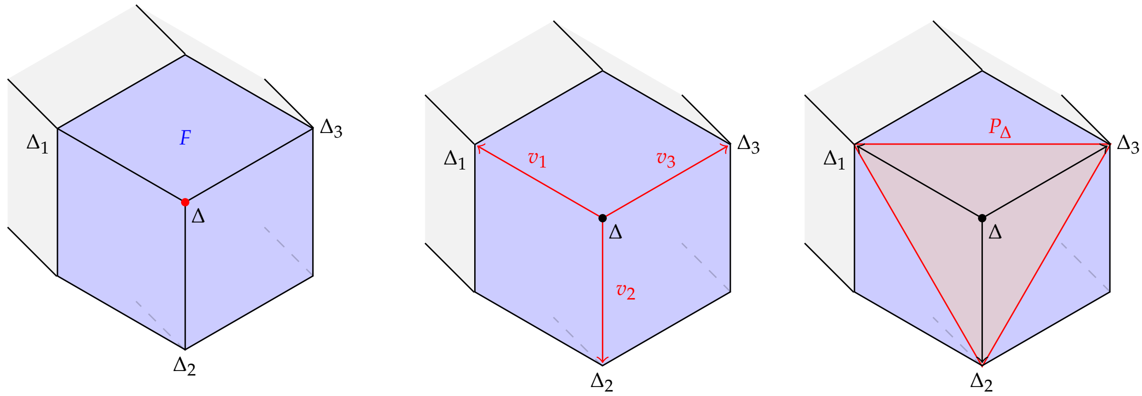
Remark 4. All regular triangulations are both sub- and semi-regular. This means in particular, that when we talk about finding a semi-regular triangulation, which is non-sub-regular, it is enough to look at non-regular triangulations, contained in the connected component Φ^c_{reg}. We will call these *strictly semi-regular* triangulations. Analogously we will call sub-regular triangulations, which are non-regular, *strictly sub-regular* triangulations.

3.2.1 The Neighborhood Cone

Let us try to formalize some conditions, under which the sets of sub- and semi-regular triangulations are equal independent from the order $\sigma \in \mathbf{S}_n$. Let Δ be a strictly semi-regular triangulation of $\mathbf{A} = (a_1, a_2, a_3, ..., a_n)$. We want to show that Δ is sub-regular for all $\sigma \in \mathbf{S}_n$. Recall, that the lexicographic ordering of GKZ-vectors corresponds to the evaluation with the objective function $\lambda = (M^n, M^{n-1}, \dots, M)^\mathsf{T}$ for a sufficient large $M \gg 0$. Since the secondary polytope does not contain the information about all the flips, we are geometrically embedding Φ^c_{reg} in $\Sigma\text{-poly}(\mathbf{A}) \subset \mathbb{R}^\mathbf{A}$ by identifying nodes and vertices and equipping the secondary polytope with the edge set of Φ^c_{reg}.
Let

$$F := \bigcap_{\substack{G \leq \Sigma\text{-poly}(\mathbf{A}) \\ \text{gkz}_\Delta \in G}} G$$

be the smallest face containing gkz_Δ and $\Delta_1, \Delta_2, \dots, \Delta_k$ be the neighbors of Δ in Φ^c_{reg} lying in F. This means, they are precisely the triangulations, such that $(\Delta, \Delta_i) \in \Phi^c_{\text{reg}}$ and $\text{gkz}_{\Delta_i} \in F$ for $i = 1, \dots, k$. Since Δ is non-regular, we have $\text{gkz}_\Delta \in \text{relint}(F)$.



(a) Face F containing gkz_Δ and its neighbors.

(b) The construction of the v_i.

(c) The polytope P_Δ.

Figure 3.1: The construction of P_Δ. Note, that the v_i and P_Δ are translated by gkz_Δ in the above figure.

For each neighbor Δ_i set

$$v_i := \mathrm{gkz}_{\Delta_i} - \mathrm{gkz}_\Delta \tag{3.2}$$

and define

$$P_\Delta := conv\,(v_1, \ldots, v_k)\,. \tag{3.3}$$

The v_i are the geometric realization of the corresponding edge (Δ, Δ_i) translated to the origin and P_Δ their convex hull. Further, let $Z_i = \left(Z_i^+, Z_i^-\right)$ be the circuit inducing the flip to the neighbor Δ_i, that is to say either $\Delta^+(Z)$ or $\Delta^-(Z)$ is contained in Δ.

Definition 3.1. Let $Z = (Z^+, Z^-)$ be a circuit of $\mathbf{A}$, then for a triangulation $\Delta \in \Delta_\mathbf{A}$ supporting a flip at Z and $\Delta^+(Z) \subseteq \Delta$, define

$$\mathrm{gkz}_{Z,\Delta} := \mathrm{gkz}_{\Delta^-(Z)} - \mathrm{gkz}_{\Delta^+(Z)}\,,$$

the *GKZ-vector of Z with respect to Δ.*

Note that the above definition is independent of the triangulation Δ, if and only if Z is full-dimensional and otherwise dependent on the link of Z in Δ (see Remark 1). We will be examining the GKZ-vector of a circuit more in Section 3.4, where we will see this in the prove of Lemma 3.9. Then since

$$\begin{aligned}\mathrm{gkz}_{\Delta_i} =&\mathrm{gkz}_{\Delta} + \mathrm{gkz}_{\Delta^-(Z_i)} - \mathrm{gkz}_{\Delta^+(Z_i)}\\ =&\mathrm{gkz}_{\Delta} + \mathrm{gkz}_{Z_i,\Delta}\end{aligned}$$

we can represent our v_i via the GKZ-vectors of the circuits Z_i with respect to Δ

$$v_i = \mathrm{gkz}_{\Delta_i} - \mathrm{gkz}_{\Delta} = \mathrm{gkz}_{Z_i,\Delta}. \tag{3.4}$$

Set

$$V := \begin{bmatrix} v_1 & v_2 & \dots & v_k \end{bmatrix} \in \mathbb{R}^{n\times k}$$

and define the *neighborhood cone of* Δ as the convex polyhedral cone

$$\mathbf{C}^{\mathrm{nh}}(\Delta) := \{x \in \mathbb{R}^n \mid V^{\mathsf{T}} x \leq \mathbf{0}\} \tag{3.5}$$

as the intersection of closed half spaces induced by the v_i. The elements of $\mathbf{C}^{\mathrm{nh}}(\Delta)$ are exactly the objective functions, which will maximize Δ compared to its neighbors. Further let

$$L := \ker V^{\mathsf{T}} = \{x \in \mathbb{R}^n \mid V^{\mathsf{T}} x = \mathbf{0}\}$$

be the *lineality space* of $\mathbf{C}^{\mathrm{nh}}(\Delta)$. Since the L is a closed subspace of $\mathbb{R}^n$, we can decompose $\mathbb{R}^n = L^{\perp} \oplus L$, where $L^{\perp} = \{x \in \mathbb{R}^n \mid \langle x, y\rangle = 0\, \forall y \in L\}$ denotes the orthogonal complement of L. We have $L^{\perp} = \operatorname{Im} V$, since for $x \in \ker V^{\mathsf{T}}$ and $Vy \in \operatorname{Im} V$

$$\langle x, Vy\rangle = \langle V^{\mathsf{T}} x, y\rangle = \langle \mathbf{0}, y\rangle = 0.$$

Setting $K := \mathbf{C}^{\mathrm{nh}}(\Delta)|_{L^{\perp}}$ yields

$$\mathbf{C}^{\mathrm{nh}}(\Delta) = K \oplus L. \tag{3.6}$$

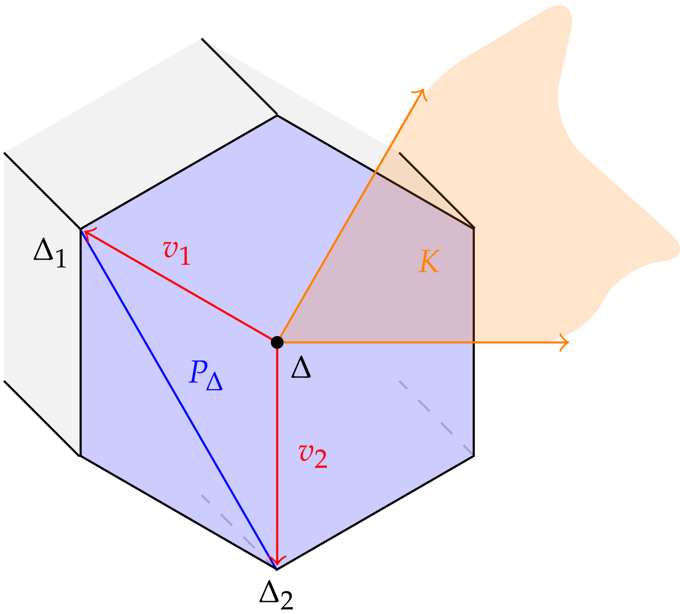


Figure 3.2: The cone K living on $\operatorname{Im} V$, the hyperplane spanned by the v_i.

Remark 5. The neighborhood cone $\mathbf{C}^{\mathrm{nh}}(\Delta)$ is precisely the polar cone of P_Δ. Since for a regular triangulation Δ^{reg} the smallest face containing it is $F = \{\mathrm{gkz}_{\Delta^{\mathrm{reg}}}\}$, we have $P_{\Delta^{\mathrm{reg}}} = \emptyset$ and thus $\mathbf{C}^{\mathrm{nh}}(\Delta^{\mathrm{reg}}) = \mathbb{R}^n$. There are similarities to the notion of the secondary cone for regular triangulations, which is defined as $C(\Delta) := \{\omega \in \mathbb{R}^n \mid \Delta \preceq \Sigma_\omega\}$ ([5, Definition 5.2.1]), where Σ_ω is the regular polyhedral subdivision induced by ω (see Definition 0.5). Although the definition of the secondary cone does not make it apparent that it even is a cone, it indeed is [5, Corollary 5.2.8]. Not only that, it is even one way to define the secondary polytope, since the set $\mathcal{C}(\mathbf{A})$ of secondary cones of all subdivisions of $\mathbf{A}$ is a complete fan in $\mathbb{R}^n$, $\Sigma\text{-poly}(\mathbf{A})$ is defined to be the unique polyhedron (up to scaling) having $\mathcal{C}(\mathbf{A})$ as its outer normal fan. This implies, that for a regular triangulation, the secondary cone is the outer normal cone of its GKZ-vector in $\Sigma\text{-poly}(\mathbf{A})$, in particular $\lambda^\sigma \in C(\Delta^\star)$. Similarly for a strictly semi-regular triangulation, the neighborhood cone is the outer normal cone of its GKZ-vector in the subpolytope $\mathit{conv}\left(\mathrm{gkz}_\Delta, \mathrm{gkz}_{\Delta_1}, \ldots, \mathrm{gkz}_{\Delta_k}\right) = \mathrm{gkz}_\Delta + \mathit{conv}\,(\mathbf{0}, P_\Delta)$.

With these notions, we can formalize a connection between sub-regularity of a triangulation and λ^σ:

Lemma 3.2. Let $\Delta \in \Phi^c_{\mathrm{reg}}$ be non-regular. If Δ is sub-regular w.r.t. $\sigma \in \mathbf{S}_n$, then

$\lambda^\sigma \notin \mathbf{C}^{\mathrm{nh}}(\Delta)$ for a sufficient large $M \gg 0$.

Proof. We are going to show the inverse of this. Let $\sigma \in \mathbf{S}_n$ such that $\lambda^\sigma \in \mathbf{C}^{\mathrm{nh}}(\Delta)$ and $\Delta_1, \dots, \Delta_k$ be the neighbors of Δ in Φ^c_{reg}. Since λ lies on the moment curve for all $M \in \mathbb{N}$, we can assume that $\lambda^\sigma \in \mathrm{int}(\mathbf{C}^{\mathrm{nh}}(\Delta))$. This implies $\langle v_i, \lambda^\sigma \rangle < 0$ and thus

$$\begin{aligned}\langle \mathrm{gkz}_{\Delta_i}, \lambda^\sigma \rangle &= \langle \mathrm{gkz}_{\Delta}, \lambda^\sigma \rangle + \langle \mathrm{gkz}_{Z_i,\Delta}, \lambda^\sigma \rangle \\ &= \langle \mathrm{gkz}_{\Delta}, \lambda^\sigma \rangle + \langle v_i, \lambda^\sigma \rangle \\ &< \langle \mathrm{gkz}_{\Delta}, \lambda^\sigma \rangle,\end{aligned}$$

i.e. $\Delta > \Delta_i$ for all $i \in [k]$. Since all neighbors of Δ are of lower order, flipping towards Δ is always an up-flip, and hence, it cannot be sub-regular with respect to $\sigma \in \mathbf{S}_n$. □

The main idea is somewhat of a "globalized" version of the converse of the above lemma:

Lemma 3.3. Let $\mathbf{A} = (a_1, a_2, a_3, ..., a_n)$ be a point configuration. Assume, that $\lambda^\sigma \notin \mathbf{C}^{\mathrm{nh}}(\Delta)$ for all non-regular triangulations $\Delta \in \Phi^c_{\mathrm{reg}}$ and all $\sigma \in \mathbf{S}_n$, then they are all sub-regular, i.e. $\Phi_{\text{sub-reg}} = \Phi^c_{\mathrm{reg}}$, for all $\sigma \in \mathbf{S}_n$.

Proof. Let $\Delta \in \Phi^c_{\mathrm{reg}}$ be a non-regular triangulation. Since $\lambda^\sigma \notin \mathbf{C}^{\mathrm{nh}}(\Delta)$ for all $\sigma \in \mathbf{S}_n$ and M sufficient large. Hence analogously to above, for all $\sigma \in \mathbf{S}_n$ there exists a v_i such that $\langle v_i, \lambda^\sigma \rangle > 0$. Then for the corresponding neighbor Δ_i we have

$$\begin{aligned}\langle \mathrm{gkz}_{\Delta_i}, \lambda^\sigma \rangle &= \langle \mathrm{gkz}_{\Delta}, \lambda^\sigma \rangle + \langle \mathrm{gkz}_{Z_i,\Delta}, \lambda^\sigma \rangle \\ &= \langle \mathrm{gkz}_{\Delta}, \lambda^\sigma \rangle + \langle v_i, \lambda^\sigma \rangle \\ &> \langle \mathrm{gkz}_{\Delta}, \lambda^\sigma \rangle.\end{aligned}$$

Now the claim follows since the maximal triangulation is always regular (Lemma 1.1). □

So the main idea is trying to show that $\mathbf{C}^{\mathrm{nh}}(\Delta)$ cannot contain λ^σ. We can see in Figure 3.1c, that this can be achieved by showing the following:

Lemma 3.4. Let $K = \{\mathbf{0}\}$. Then $\lambda^\sigma \notin \mathbf{C}^{\mathrm{nh}}(\Delta)$ for all $\sigma \in \mathbf{S}_n$ and a sufficient large $M \gg 0$.

Proof. Let $\sigma \in \mathbf{S}_n$ be an arbitrary permutation. Since λ^σ is a point on the (permuted) moment curve, for a fixed $n-1$-hyperplane H we can assume that $\lambda^\sigma \notin H$ for a sufficient large $M \gg 0$ (otherwise pick a larger M). Since $\Delta \in \Phi^c_{\text{reg}}$, we know that there exists a neighbor Δ_1, i.e. $k \geq 1$. This means that $V \in \mathbb{R}^{n \times k}$ consists of at least one row $v_1 = \text{gkz}_{Z_1,\Delta} \neq \mathbf{0}$, where the last inequality follows from the definition of GKZ-vectors for circuits. The rank-nullity theorem gives us

$$\dim(L) = \dim(\ker V^\mathsf{T}) = n - \dim(\operatorname{Im} V^\mathsf{T}) = n - \underbrace{\dim(\operatorname{Im} V)}_{\geq 1} < n.$$

Now $K = \{\mathbf{0}\}$ implies $\mathbf{C}^{\text{nh}}(\Delta) = L$ and with the above $\lambda^\sigma \notin \mathbf{C}^{\text{nh}}(\Delta)$. □

In order to compute this more easily with `polymake` and essentially letting us skip the computation of $\mathbf{C}^{\text{nh}}(\Delta)$ (at least for the purpose of proving Lemma 3.3), the following equivalence proves to be useful:

Lemma 3.5. The following holds

$$\mathbf{0} \in \operatorname{relint}(P_\Delta) \Leftrightarrow K = \{\mathbf{0}\}.$$

Proof. By definition $\mathbf{0} \in K$. If $x \in K$, $x \neq \mathbf{0}$, then for all $\mu > 0$

$$V\mu x = \mu \underbrace{Vx}_{<\mathbf{0}} < \mathbf{0}$$

and hence $\mu x \in K$. Note, that the relations in the above equation are strict since for $x \neq \mathbf{0}$ such that $Vx = \mathbf{0}$ follows, that $x \in L$. So either K is unbounded or $K = \{\mathbf{0}\}$. Let us denote by $*|_{L^\perp}$ the orthogonal projection onto $L^\perp$.
"$\Rightarrow$" Let $\mathbf{0} \in \operatorname{relint}(P_\Delta)$ and $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{R}^n$, then the polar polyhedron $P^\circ_\Delta|_{L^\perp} = \{x \in \mathbb{R}^n \mid Vx \leq \mathbf{1}\}|_{L^\perp}$ is bounded in $L^\perp$, since P_Δ is full-dimensional in $L^\perp$, and we have

$$\begin{aligned} K =& \{x \in \mathbb{R}^n \mid Vx \leq \mathbf{0}\}|_{L^\perp} \\ \subseteq& \{x \in \mathbb{R}^n \mid Vx \leq \mathbf{1}\}|_{L^\perp} \\ =& P^\circ_\Delta|_{L^\perp}. \end{aligned}$$

"$\Leftarrow$" Conversely, let $K = \{\mathbf{0}\}$ and assume $P_\Delta^\circ|_{L^\perp}$ is unbounded, i.e. we can find a $x \in P_\Delta^\circ|_{L^\perp}$, $x \neq \mathbf{0}$ such that $\mu x \in P_\Delta^\circ|_{L^\perp}$ for all $\mu > 0$ and similarly to the above we get

$$V\mu x = \mu V x < \mathbf{1}$$

which implies $Vx < \mathbf{0}$, where the relations are again strict, because we are in the subspace $L^\perp$. ↯ □

Trivially the following holds

$$\mathbf{0} \in \operatorname{relint}(P_\Delta) \Rightarrow \mathit{conv}\,(P_\Delta \setminus \{\mathbf{0}\}, x_1, \ldots, x_l) = \mathit{conv}\,(P_\Delta, \mathbf{0}, x_1, \ldots, x_l) \tag{3.7}$$

for all $x_1, \ldots, x_l \in \mathbb{R}^n$. In particular it follows, that $\mathbf{0}$ is not a vertex of P_Δ. The above equation seems unnecessary at first, but gets useful quickly as the complexity of the secondary polytope Σ-poly$(\mathbf{A})$ increases rapidly with the number of points and its computation becomes very challenging. Instead, we will be looking at local neighborhoods of a given triangulation Δ in Σ-poly$(\mathbf{A})$, which comes to the price that we can't always know the smallest face F containing Δ (this happens, when there are flips from Δ going through the interior of Σ-poly$(\mathbf{A})$). In this case, Equation (3.7) can be used as a necessary criteria.

3.2.2 A Counterexample

It turns out, that there exists a triangulation, which violates the right hand side of Equation (3.7), which implies, that it is not possible to show this property generally.

Example 4 (Counterexample)**.** This example shows, that it is not possible to generally show $\mathbf{0} \in \operatorname{relint}(P_\Delta)$ for all non-regular $\Delta \in \Phi^c_{\text{reg}}$. Let $I^4 = [0,1]^4$ be the 4-dimensional 0/1-cube with canonical ordering of the vertices. Let us identify the 16 vertices with the hexadecimal digits `0` through `F`, as done in [2, Example 8]. Look at the following strictly sub-regular triangulation Δ_1^{ex} of I^4

```
0457C  0467E  0267E  089CE  089AE  09AEF  09CEF  59CDF  0135F
0357F  019CF  015CF  057CF  159CF  19ABF  019AF  13ABF  013AF
047CE  07CEF  023AE  03AEF  037EF  0237E
```

with GKZ-vector

$$(20, 8, 3, 8, 3, 7, 2, 9, 2, 9, 8, 2, 10, 1, 12, 16).$$

The neighborhood of Δ_1^{ex} in Σ-poly(I^4) consists of 14 neighbors, 3 regular and 11 non-regular ones:

Regular neighbors		
gkz_Δ	v_i	$Z_i = (Z_i^+, Z_i^-)$
(20 9 3 7 3 6 2 10 2 9 8 2 10 1 12 16)	(0 1 0 -1 0 -1 0 1 0 0 0 0 0 0 0 0)	- ({1 7} {3 5})
(20 8 3 8 5 7 2 7 2 9 8 2 8 1 12 18)	(0 0 0 0 2 0 0 -2 0 0 0 0 -2 0 0 2)	- ({4 15} {7 12})
(20 8 3 8 3 9 2 7 2 9 8 2 8 1 14 16)	(0 0 0 0 0 2 0 -2 0 0 0 0 -2 0 2 0)	- ({5 14} {7 12})

Non-regular neighbors		
gkz_Δ	v_i	$Z_i = (Z_i^+, Z_i^-)$
(22 6 3 8 3 7 2 9 2 9 6 4 10 1 12 16)	(2 -2 0 0 0 0 0 0 0 0 -2 2 0 0 0 0)	- ({0 11} {1 10})
(21 6 3 8 3 8 2 9 2 10 8 2 9 1 12 16)	(1 -2 0 0 0 1 0 0 0 1 0 0 -1 0 0 0)	- ({0 5 9} {1 12})
(20 9 3 8 3 6 2 9 2 8 8 2 10 2 12 16)	(0 1 0 0 0 -1 0 0 0 -1 0 0 0 1 0 0)	- ({1 13} {5 9})
(20 8 5 6 3 7 2 9 2 9 8 2 10 1 10 18)	(0 0 2 -2 0 0 0 0 0 0 0 0 0 0 -2 2)	- ({2 15} {3 14})
(20 8 3 8 3 7 2 9 4 7 8 2 10 1 10 18)	(0 0 0 0 0 0 0 0 2 -2 0 0 0 0 -2 2)	- ({8 15} {9 14})
(20 8 3 8 3 7 2 9 1 9 9 2 11 1 11 16)	(0 0 0 0 0 0 0 0 -1 0 1 0 1 0 -1 0)	+ ({8 14} {10 12})
(20 8 3 8 2 7 3 9 2 9 8 2 11 1 11 16)	(0 0 0 0 -1 0 1 0 0 0 0 0 1 0 -1 0)	+ ({4 14} {6 12})
(20 8 3 6 3 7 2 11 2 9 10 2 10 1 10 16)	(0 0 0 -2 0 0 0 2 0 0 2 0 0 0 -2 0)	+ ({3 14} {7 10})
(20 8 2 9 3 7 3 8 2 9 8 2 10 1 12 16)	(0 0 -1 1 0 0 1 -1 0 0 0 0 0 0 0 0)	+ ({2 7} {3 6})
(20 7 3 9 3 7 2 9 2 10 8 1 10 1 12 16)	(0 -1 0 1 0 0 0 0 0 1 0 -1 0 0 0 0)	+ ({1 11} {3 9})
(19 8 4 8 4 7 1 9 2 9 8 2 10 1 12 16)	(-1 0 1 0 1 0 -1 0 0 0 0 0 0 0 0 0)	+ ({0 6} {2 4})

Table 3.1: GKZ-vectors of the neighboring triangulations of Δ_1^{ex}, the difference vectors $v_i = \mathrm{gkz}_{Z_i, \Delta_1^{ex}}$ and the circuits Z_i. The sign in front of the circuit indicates whether $\Delta^+(Z)$ or $\Delta^-(Z)$ is contained in Δ_1^{ex}.

In `polymake`, let us define the polytope `$P_Tri` as the convex hull of the v_i:

```
polytope > print $P_Tri->VERTICES;
1 0 1 0 -1 0 -1 0 1 0 0 0 0 0 0 0 0
1 0 0 0 0 2 0 0 -2 0 0 0 0 -2 0 0 2
1 0 0 0 0 0 2 0 -2 0 0 0 0 -2 0 2 0
1 2 -2 0 0 0 0 0 0 0 0 -2 2 0 0 0 0
```

```
1 1 -2 0 0 0 1 0 0 0 1 0 0 -1 0 0 0
1 0 1 0 0 0 -1 0 0 0 -1 0 0 0 1 0 0
1 0 0 2 -2 0 0 0 0 0 0 0 0 0 0 -2 2
1 0 0 0 0 0 0 0 0 0 2 -2 0 0 0 0 -2 2
1 0 0 0 0 0 0 0 0 0 -1 0 1 0 1 0 -1 0
1 0 0 0 0 -1 0 1 0 0 0 0 0 1 0 -1 0
1 0 0 0 -2 0 0 0 2 0 0 2 0 0 0 -2 0
1 0 0 -1 1 0 0 1 -1 0 0 0 0 0 0 0 0
1 0 -1 0 1 0 0 0 0 0 0 1 0 -1 0 0 0 0
1 -1 0 1 0 1 0 -1 0 0 0 0 0 0 0 0 0

polytope > print $P_Tri->DIM;
11
```

As mentioned above, this is a case, where we don't know the F, since `$P_Tri` is 11-dimensional, i.e. there are flips from Δ_1^{ex} going through the interior of Σ-poly$(\mathbf{A})$. It is possible to guess F by looking at how the neighbors are lying in the facets of `$P_Tri` (remember that $\mathrm{gkz}_{\Delta_1^{\mathrm{ex}}} \in \mathrm{relint}(F)$). Fortunately, this is not necessary, since with Equation (3.7) we can easily compute that $\mathbf{0} \notin \mathrm{relint}(P_{\Delta_1^{\mathrm{ex}}})$. $\diamond$

We can see from Table 3.1, that it is exactly the flips induced by circuits Z with $\Delta^-(Z)$ being contained in Δ_1^{ex}, that are up-flips. This comes to no surprise, as `polymake` is giving us the circuits ordered lexicographically, i.e. $\Delta^+(Z) > \Delta^-(Z)$.

3.3 What it takes to contain a λ^σ

Having seen Example 4, we must examine further, if it is possible to weaken our conditions and still get a general result or if Δ_1^{ex} is a semi-regular triangulation, which is non-sub-regular for some λ^σ. For this, we are first going to continue examining the objects from Example 4. Then we will formulate some conditions on $\mathbf{C}^{\mathrm{nh}}(\Delta)$, that have to be fulfilled in order for it to contain a λ^σ.

Remark 6. For clarity's sake, we will indicate the $M \in \mathbb{N}$ from the definition of λ^σ by changing the notation to $\lambda_M^\sigma = (M^n, M^{n-1}, \dots, M)$ if needed. This will be useful when testing with `polymake`. In cases where it is not needed, it will be omitted like before.

3.3.1 Some Ideas following the Counterexample

It turns out, that even though $\mathbf{0} \notin \mathrm{relint}(P_{\Delta_1^{\mathrm{ex}}})$, we have that Δ_1^{ex} is sub-regular for all $\sigma \in \mathbf{S}_n$. To see this, let us add $\mathbf{0}$ as a point to `$P_Tri`:

```
polytope > print $P_Tri->VERTICES;
1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 0 1 0 -1 0 -1 0 1 0 0 0 0 0 0 0 0
1 0 0 0 0 2 0 0 -2 0 0 0 0 -2 0 0 2
1 0 0 0 0 0 2 0 -2 0 0 0 0 -2 0 2 0
1 2 -2 0 0 0 0 0 0 0 0 -2 2 0 0 0 0
1 1 -2 0 0 0 1 0 0 0 1 0 0 -1 0 0 0
1 0 1 0 0 0 -1 0 0 0 -1 0 0 0 1 0 0
1 0 0 2 -2 0 0 0 0 0 0 0 0 0 0 -2 2
1 0 0 0 0 0 0 0 0 2 -2 0 0 0 0 -2 2
1 0 0 0 0 0 0 0 0 -1 0 1 0 1 0 -1 0
1 0 0 0 0 -1 0 1 0 0 0 0 0 1 0 -1 0
1 0 0 0 -2 0 0 0 2 0 0 2 0 0 0 -2 0
1 0 0 -1 1 0 0 1 -1 0 0 0 0 0 0 0 0
1 0 -1 0 1 0 0 0 0 0 1 0 -1 0 0 0 0
1 -1 0 1 0 1 0 -1 0 0 0 0 0 0 0 0 0
```

As we can see, $\mathbf{0}$ is a vertex of `$P_Tri`, which violates the right hand side of Equation (3.7). The outer normal cone of $\mathbf{0}$ in `$P_Tri` is exactly $\mathbf{C}^{\mathrm{nh}}(\Delta_1^{\mathrm{ex}})$. Since `polymake` is giving us the inner normal cone, we have to mirror the cone at the origin by multiplying the generating rays by -1 (see line 2). In line 3 we are requesting the outer description via `$C_Tri->FACETS`, which is by definition a subset of the vertices of `$P_Tri`.

```
1  polytope > $C_Tri = normal_cone($P_Tri, 0);
2  polytope > $C_Tri = new Cone<Rational>(INPUT_RAYS=>-$C_Tri->RAYS,
     INPUT_LINEALTY=>-$C_Tri->LINEALITY_SPACE);
3  polytope > print $C_Tri->FACETS;
4  0 -1 0 1 0 1 0 -1 0 0 0 0 0 0 0 0
5  0 0 0 0 0 -1 0 1 0 0 0 0 1 0 -1 0
6  1 0 -1 0 -1 0 1 0 0 0 0 0 0 0 0 0
7  0 0 1 -1 0 0 -1 1 0 0 0 0 0 0 0 0
8  0 0 0 0 0 0 0 0 1 0 -1 0 -1 0 1 0
9  0 1 0 -1 0 0 0 0 0 -1 0 1 0 0 0 0
10 -1 1 0 0 0 0 0 0 0 0 1 -1 0 0 0 0
11 0 0 0 0 1 0 -1 0 0 0 0 0 -1 0 1 0
12 0 -1 0 0 0 1 0 0 0 1 0 0 0 -1 0 0
13 0 0 0 1 0 0 0 -1 0 0 -1 0 0 0 1 0
```

```
0 0 0 0 0 0 0 0 -1 1 0 0 0 0 1 -1

polytope > print $C_Tri->LINEALITY_SPACE;
-1 -1 -1 -1 -1 -1 -1 -1 0 0 0 0 0 0 0 0
-1 -1 -1 -1 1 1 1 1 -2 -2 -2 -2 0 0 0 0
-1 -1 2 2 -2 -2 1 1 -2 -2 1 1 -3 -3 0 0
-1 3 -2 2 -2 2 -3 1 -2 2 -3 1 -3 1 -4 0
1 1/3 1/3 -1/3 1/3 -1/3 -1/3 -1 1/3 -1/3 -1/3 -1 -1/3 -1 -1 -5/3

polytope > print $C_Tri->RAYS;
-1 0 -1 0 -1 0 -1 0 0 0 0 0 0 0 0 0
-1 -1/2 -1 -1/2 -1/2 0 -1/2 0 -1/2 -1/2 -1/2 0 0 0 0 0
-1 -2/3 -1 -1/3 -2/3 -1/3 -2/3 0 -1/3 -1/3 -1/3 0 0 0 0 0
1 1 1/2 1/2 1/2 1/2 0 0 1/2 1/2 0 0 1/2 0 0 0
0 0 -1 0 -1 0 -1 0 0 0 0 0 0 0 0 0
-1 -2/3 -2/3 -1/3 -2/3 -1/3 -1/3 0 -1/3 -1/3 -1/3 0 -1/3 0 0 0
-1 -2/3 -2/3 -1/3 -2/3 -1/3 -1/3 0 -2/3 -1/3 -1/3 0 -1/3 0 0 0
-1 -3/4 -3/4 -1/4 -3/4 -1/4 -1/2 0 -1/2 -1/2 -1/4 0 -1/4 0 0 0
0 0 -1 0 0 0 -1 0 0 0 0 0 0 0 0 0
0 0 0 0 -1 0 -1 0 0 0 0 0 0 0 0 0
1 1 0 0 1 1 0 0 1 1 0 0 1 0 0 0
```

Note, that we didn't need to invert the lineality space, but did it for the sake of orientation. Let us look at the separating hyperplane of **0** in `$P_Tri`:

```
polytope > print $P_Tri->VERTEX_NORMALS->[0];
-1 21/2 6 23/2 7/2 9 3/2 9 0 4 7/2 4 0 1 0 0 0
```

Since `polymake` is giving us the inner normal vector, analogously to above we have to mirror it at the origin and get the outer normal vector

$$\vec{n}_0 = -(21, 12, 23, 7, 18, 3, 18, 0, 8, 7, 8, 0, 2, 0, 0, 0).$$

Then $\vec{n}_0$ is lying in `$C_Tri` and induces the lexicographical ordering permuted via

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 8 & 12 & 14 & 15 & 16 & 13 & 6 & 4 & 10 & 9 & 11 & 2 & 5 & 7 & 1 & 3 \end{pmatrix}.$$

i.e. we are ordering the points as follows $(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(16)}) = (a_8, a_{12}, \dots, a_3)$. Note, that there exist more permutations, that are induced by $\vec{n}_0$, since the lineality space of

$\mathbf{C}^{\mathrm{nh}}(\Delta_1^{\mathrm{ex}})$ has dimension > 0. Then the inverse is given by

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 15 & 12 & 16 & 8 & 13 & 7 & 14 & 1 & 10 & 9 & 11 & 2 & 6 & 3 & 4 & 5 \end{pmatrix} \in \mathbf{S}_{16},$$

which gives us the λ-vector

$$\lambda_M^{\sigma} = (M^2, M^5, M^1, M^9, M^4, M^{10}, M^3, M^{16}, M^7, M^8, M^6, M^{15}, M^{11}, M^{14}, M^{13}, M^{12}).$$

With `polymake` we can test, if λ_{10}^{σ} is contained in `$C_Tri`:

```
polytope > print $lambda_perm;
100 100000 10 1000000000 10000 10000000000 1000 10000000000000000 10000000
    100000000 1000000 1000000000000000 100000000000 100000000000000
    10000000000000 1000000000000

polytope > print $C_Tri->contains($lambda_perm);
false
```

Testing further, we get that $\lambda_M^{\sigma} \notin \mathbf{C}^{\mathrm{nh}}(\Delta_1^{\mathrm{ex}})$ for all $M > 1$. This means, that while in this case $\mathbf{0} \notin P_{\Delta_1^{\mathrm{ex}}}$, the induced ordering, which maximizes Δ_1^{ex} with respect to its neighbors, is still not making it non-sub-regular. Taking a step back, this is the "geometricalization" of just looking at the entries of $\mathrm{gkz}_{\Delta_1^{\mathrm{ex}}}$ and comparing them to the respective entry of its neighbors. There are two main points making sense of this result:

1. The way that λ^{σ} changes, when we change the order of the elements of $\mathbf{A}$ is by being mirrored at particular hyperplanes. Let $\sigma \in \mathbf{S}_n$ be a permutation represented as a product of transpositions $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_m = \sigma$. Each π_i corresponds geometrically to mirroring λ at the hyperplane $H_i = \{x \mid x_k = x_l\}$, where $k \neq l \in [n]$ are precisely the indices being transposed by π_i.

2. A cone which contains λ^{σ} for all $M \gg 0$ sufficiently large needs to fulfill some requirements. This is discussed later in Section 3.3.3.

3.3.2 More Counterexamples

Is there a property that we can use to construct a counterexample to equality in Equation (3.1) (a)? It turns out that there are 865 strictly sub-regular triangulations of I^4 (modulo symmetry), which violate the right hand side of Equation (3.7).

	Neighbors	regular Neighbors	non-regular Neighbors
Average	11.88	2.17	9.71
Maximum	14	5	13
Minimum	11	1	6

Table 3.2: Neighborhood of the 865 strictly sub-regular triangulations of I^4 violating the right hand side of Equation (3.7).

	Neighbors	regular Neighbors	non-regular Neighbors
Average	12.80	3.13	9.47
Maximum	20	8	16
Minimum	8	0	2

Table 3.3: Neighborhood of the 12174 strictly sub-regular triangulations of I^4.

Notably, since it is possible for strictly sub-regular triangulations to have no regular neighbors, all of the counterexamples have a regular neighbor. This might only be a correlation though, since the number of counterexamples (865) and strictly sub-regular triangulations adjacent to no regular triangulation (607) is relatively small. When looking at the left side of Table A.4, we can see, that this subset of non-regular triangulations is almost normally distributed. From Example 4 we can conclude, that showing $K = \{\mathbf{0}\}$ is not possible in general. We weren't able to construct a counterexample to equality in Equation (3.1) (a), since $\mathbf{C}^{\mathrm{nh}}(\Delta_1^{\mathrm{ex}})$ did not contain λ_M^σ for $M > 1$ and all $\sigma \in \mathbf{S}_n$. It turns out, that for all these triangulations we have that $\lambda_M^\sigma \notin \mathbf{C}^{\mathrm{nh}}(\Delta)$ for $M > 1$ and all permutation $\sigma \in \mathbf{S}_n$. It is therefore not possible to create a counterexample on I^4 this way. We will look at what is necessary for $\mathbf{C}^{\mathrm{nh}}(\Delta)$ to contain a λ_M^σ for all $M \gg 0$ sufficiently large and a $\sigma \in \mathbf{S}_n$ in Section 3.3.3.

Looking for counterexamples somewhere else

A quick test on the 4×4-lattice $L(4,4) = 4 \cdot I^2 \cap \mathbb{Z}^2$ showed, that there does not exist a single triangulation violating the right hand side of Equation (3.7). Presumably this is due to it being highly symmetrical. Though this was not further investigated, it was mentioned fo the sake of being thorough. The triangulations are also added in the repository (see Appendix A.2).

3.3.3 A λ-containing Cone

Let us now try to construct some constraints for a cone to contain λ^σ. For this, we are going to look at the canonical order, i.e. $\lambda^{\mathrm{Id}} = \lambda$ and use the ideas from Remark 3 to generalize. First, recall that

$$\mathbf{C}^{\mathrm{nh}}(\Delta) = \{x \in \mathbb{R}^n \mid V^\top x \leq \mathbf{0}\} = \bigcap_{i=1}^{k} \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \leq 0\},$$

where $v_i = \mathrm{gkz}_{Z_i,\Delta}$ and Z_i are the circuits of $\mathbf{A} = (a_1, a_2, a_3, ..., a_n) \subset \mathbb{R}^d$, which induce a flip in Δ. What does it require for a cone to contain λ? Let us state a necessary condition:

Lemma 3.6. Let C be a convex polyhedral cone. If $\lambda \in C$ for all $M \gg 0$, then $e_1 \in C$.

Proof. Assume $e_1 \notin C$. Let

$$C = \{x \in \mathbb{R}^n \mid Ax \leq \mathbf{0}\}$$

be an outer description of C. Then there exists an $i \in [n]$ such that

$$\langle \mathrm{row}_i(A), e_1 \rangle = a_{i,1} > 0,$$

where $\mathrm{row}_i(A)$ denotes the ith row of A. But then for all $M \gg 0$

$$\begin{aligned}\langle \mathrm{row}_i(A), \lambda\rangle &= a_{i,1}M^n + a_{i,2}M^{n-1} + \cdots + a_{i,n}M \leq 0\\ &\Leftrightarrow \underbrace{a_{i,1} + a_{i,2}\frac{1}{M} + \cdots + a_{i,n}\frac{1}{M^{n-1}}}_{\circledast} \leq 0.\end{aligned}$$

Now $\circledast \to a_{i,1} > 0$ as $M \to \infty$, i.e. $\lambda \notin C$ for all $M \gg 0$. $\square$

Taking a step back once more, the result of Lemma 3.6 is, that there has to exist a coordinate $j \in [n]$ such that $(v_i)_j \leq 0$ for all $i \in [k]$. This is trivially necessary, when thinking in terms of lexicographic order. Looking at Table 3.1, we can verify, that this is not the case for Δ_1^{ex}.

Lemma 3.7. Let $e_j \in \mathrm{int}(\mathbf{C}^{\mathrm{nh}}(\Delta))$ for some $j \in [n]$. Then Δ is non-sub-regular for all $\sigma \in \mathbf{S}_n$, such that $\sigma(1) = j$.

Proof. Since e_j is in the interior of $\mathbf{C}^{\mathrm{nh}}(\Delta)$, by definition, the jth coordinate of gkz_Δ is strictly larger than the jth coordinate of any of its neighbors. $\square$

Note that in that case, there might be more orders for which Δ is non-sub-regular: If $e_m \in \partial\mathbf{C}^{\mathrm{nh}}(\Delta)$, then any $\sigma' \in \mathbf{S}_n$, such that $\sigma'(1) = m$ and $\sigma'(2) = j$ also makes Δ non-sub-regular, where $\partial\mathbf{C}^{\mathrm{nh}}(\Delta) = \mathbf{C}^{\mathrm{nh}}(\Delta) \setminus \mathrm{int}(\mathbf{C}^{\mathrm{nh}}(\Delta))$ is the boundary of $\mathbf{C}^{\mathrm{nh}}(\Delta)$. When we look at the decomposition $\mathbf{C}^{\mathrm{nh}}(\Delta) = K \oplus L$ from Equation (3.6) again, then by definition these e_m are lying in the lineality space L and since we are mostly interested what happens with K this might suffice. Now let us look at a cone containing λ for all M.

Lemma 3.8. Let

$$C = \mathrm{pos}\{\underbrace{(1,0,\ldots,0)}_{=:c_1}, \underbrace{(1,1,0,\ldots,0)}_{=:c_2}, \ldots, \underbrace{(1,\ldots,1)}_{=:c_n}\} \subset \mathbb{R}^n.$$

Then $\lambda \in C$ for all $M \in \mathbb{N}$.

Proof. Let $M \in \mathbb{N}$, we have

$$\lambda = M \cdot c_n + \sum_{i=1}^{n-1} (M^{i+1} - M^i) \cdot c_{n-i}.$$

$\square$

3.4 Circuit GKZ-Vectors

Let us take a closer look at the circuits of a point configuration $\mathbf{A}$, since it might give us more insight on how we can traverse the flip graph Φ^c_{reg}. Let $\mathcal{Z}_{\mathbf{A}}$ be the set of all circuits of $\mathbf{A}$. Generally, the GKZ-vector of a circuit $Z \in \mathcal{Z}_{\mathbf{A}}$ from Definition 3.1 varies depending on the triangulation $\Delta \in \Delta_{\mathbf{A}}$. This is the case, if Z is not full-dimensional and dependent on the link of $\Delta^+(Z)$ in Δ.

Lemma 3.9. Let $Z = (Z^+, Z^-) \in \mathcal{Z}_{\mathbf{A}}$ be a circuit and $\Delta \in \Delta_{\mathbf{A}}$ a triangulation supporting a flip over Z. Set $v := \text{gkz}_{Z,\Delta}$ with $v = (v_1, \ldots, v_n)$. Then following holds

(i) $v_i \neq 0$ for $i \in \underline{Z}$ and $v_i = 0$ for $i \notin \underline{Z}$,

(ii) $\#\{i \mid v_i \neq 0\} \geq 3$,

(iii) $\sum_{i \in \underline{Z}} v_i = 0$,

(iv) $\text{sign}(v_i) = -\text{sign}(v_j)$ for $i \in Z^+, j \in Z^-$.

(v) If $\Delta^+(Z) \subseteq \Delta$, we have $v_i < 0$ for $i \in Z^+$ and $v_i > 0$ for $i \in Z^-$.

Proof. (i),(iv) and (v): Let Δ' be the triangulation obtained from Δ by flipping via Z. We have $\text{gkz}_{Z,\Delta} = \text{gkz}_{\Delta'} - \text{gkz}_{\Delta}$. Since Δ and Δ' are connected by a flip, there exists a almost triangulation $\hat{\Delta}$ whose two refinements are precisely Δ and Δ' and whose unique circuit is

Z. Without loss of generality let $\Delta^+(Z) \subseteq \Delta$. We have

$$\begin{aligned}
\mathrm{gkz}_{Z,\Delta} &= \mathrm{gkz}_{\Delta'} - \mathrm{gkz}_{\Delta} \\
&= \sum_{a\in\mathbf{A}} \left(\mathrm{gkz}_{\Delta'}(a) - \mathrm{gkz}_{\Delta}(a)\right)\cdot e_a \\
&= \sum_{a\in\mathbf{A}} \left(\sum_{\substack{\sigma\in\Delta' \\ a\in\sigma}} \mathrm{vol}_d(\sigma) - \sum_{\substack{\sigma\in\Delta \\ a\in\sigma}} \mathrm{vol}_d(\sigma) \right)\cdot e_a \\
&\overset{\circledast}{=} \sum_{a\in\mathbf{A}} \left(\sum_{\substack{\sigma\in\hat\Delta \\ a\in\sigma}} \mathrm{vol}_d(\sigma) - \sum_{\substack{\sigma\in\hat\Delta \\ a\in\sigma}} \mathrm{vol}_d(\sigma) + \sum_{\substack{\sigma\in\Delta'\setminus\hat\Delta \\ a\in\sigma}} \mathrm{vol}_d(\sigma) - \sum_{\substack{\sigma\in\Delta\setminus\hat\Delta \\ a\in\sigma}} \mathrm{vol}_d(\sigma) \right)\cdot e_a \\
&\overset{\odot}{=} \sum_{a\in\mathbf{A}} \left(\sum_{\substack{\sigma\in\Delta'\cap\underline{Z} \\ a\in\sigma}} \mathrm{vol}_d(\sigma) - \sum_{\substack{\sigma\in\Delta\cap\underline{Z} \\ a\in\sigma}} \mathrm{vol}_d(\sigma) \right)\cdot e_a \\
&= \sum_{a\in\underline{Z}} \left(\sum_{\substack{\sigma\in\Delta^-(\mathrm{Z}) \\ a\in\sigma}} \mathrm{vol}_d(\sigma) - \sum_{\substack{\sigma\in\Delta^+(\mathrm{Z}) \\ a\in\sigma}} \mathrm{vol}_d(\sigma) \right)\cdot e_a,
\end{aligned}$$

where we used at $\circledast$ that $\Delta, \Delta' \preceq \hat\Delta$ and at $\odot$ that Z is the unique circuit of $\hat\Delta$. Note that we also could have started with $\mathrm{gkz}_{Z,\Delta} = \mathrm{gkz}_{\Delta^-(Z)} - \mathrm{gkz}_{\Delta^+(Z)}$ as a shortcut, but this way we can build up more intuition. For $a \notin \underline{Z}$ the above equation shows us, that $v_a = 0$. Without loss of generality let $a \in Z^-$ then the definition of $\Delta^+(Z)$ and $\Delta^-(Z)$ yields

$$\begin{aligned}
v_a &= \sum_{\substack{\sigma\in\Delta^-(\mathrm{Z}) \\ a\in\sigma}} \mathrm{vol}_d(\sigma) - \sum_{\substack{\sigma\in\Delta^+(\mathrm{Z}) \\ a\in\sigma}} \mathrm{vol}_d(\sigma) \\
&= \mathrm{vol}_d(\mathit{conv}\,(\underline{Z})) - \sum_{\substack{\sigma\in\Delta^+(\mathrm{Z}) \\ a\in\sigma}} \mathrm{vol}_d(\sigma) \\
&= \mathrm{vol}_d(\mathit{conv}\,(\underline{Z})) - [\mathrm{vol}_d(\mathit{conv}\,(\underline{Z})) - \mathrm{vol}_d(\mathit{conv}\,(\underline{Z}\setminus\{a\}))] \\
&= \mathrm{vol}_d(\mathit{conv}\,(\underline{Z}\setminus\{a\})) > 0.
\end{aligned}$$

Note, that we assumed Z to be full-dimensional, since otherwise we use the link L_Δ as mentioned in Remark 1. This means we replace $\mathrm{vol}_d(conv\,(\underline{Z} \setminus \{a\}))$ with

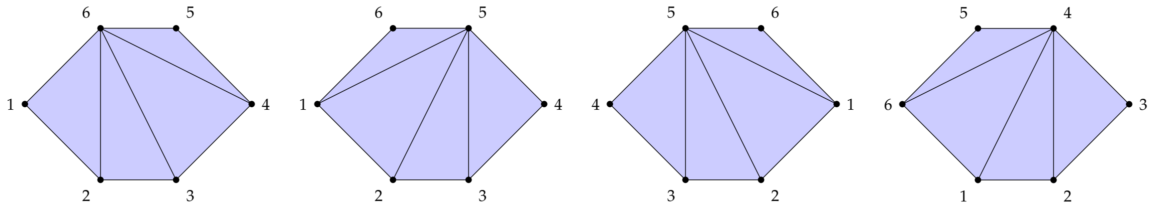
$$\mathrm{vol}_d(conv\,((\underline{Z} \setminus \{a\}) * L_\Delta)) = \sum_{F \in L_\Delta} \mathrm{vol}_d(conv\,((\underline{Z} \cup F) \setminus \{a\}))$$

and thus v_a might change depending on what the link of Z is in the respective triangulation (as mentioned after Definition 3.1). In particular, this does not change the sign. With this (ii) follows from the fact that the number of elements of any circuit is ≥ 3 and (iii) follows from $\sum_{i=1}^{n}(\mathrm{gkz}_{\Delta_1})_i = \sum_{i=1}^{n}(\mathrm{gkz}_{\Delta_2})_i = (d+1) \cdot \mathrm{vol}_d(conv\,(\mathbf{A}))$. $\square$

Further it might be interesting to examine what it means for a cone C to be generated by these, since the constraints of the neighborhood cone are given by GKZ-vectors of circuits.

3.5 Symmetry and Sub-Regularity

Let us take symmetry into account. We already know from Corollary 1.1, that $g \cdot \Delta$ is regular for all $g \in \mathbf{G}$ if and only if $\rho(\Delta)$ is regular, which implies that the same is true for the non-regular case. Figure 3.3 gives some intuition for how $\mathbf{G}$ is acting and what happens, when we interpret the elements of $\mathbf{G}$ as an order of $\mathbf{A}$. In this section, we will mostly refer to the points of $\mathbf{A}$ by their indices, i.e. $a_i \leftrightarrow i$ for $i = 1, \ldots, n$.



(a) Δ on $\mathbf{A}$. **(b)** $g \cdot \Delta$ on $\mathbf{A}$. **(c)** $g \cdot \Delta$ on $g \cdot \mathbf{A}$. **(d)** $g^2 \cdot \Delta$ on $g \cdot \mathbf{A}$.

Figure 3.3: The element $g = (14)(23)(56)$ of $\mathbf{G}$ applied to Δ as a group action (b) is mirroring horizontally. If we interpret g as an order of vertices the result is shown in (c). On the right (d) illustrates applying g as both, a group action and an order of $\mathbf{A}$. One can easily confirm, that applying $g' = (165432)$ results in the same picture (b), which is Δ rotated clockwise. Then the corresponding figure (c) would be precisely (b) rotated counterclockwise.

Lemma 3.10. Let Δ be sub-regular for some order of $\mathbf{A}$ and let Δ be the lexicographically largest among all triangulations with the same GKZ-vector. Then, there exists an order $\sigma \in \mathbf{S}_n$, such that $\rho(\Delta) = \Delta$ with respect to σ.

Proof. Let $g \in \mathbf{G}$, such that $\rho(\Delta) = g \cdot \Delta$ and set $\sigma = g^{-1}$ and since $\mathbf{G} \subseteq \mathbf{S}_n$ the claim follows. □

Since $\mathbf{G}$ is defined to be structure preserving, it comes to no surprise that $g \cdot Z$ is a circuit of $\mathbf{A}$ for all $g \in \mathbf{G}$ and circuits Z of $\mathbf{A}$. For a flip f this implies that $g \cdot f$ is also a flip.

Definition 3.2. Let $g \in \mathbf{G}$, we will say that g *changes the direction of a flip* f, if $g \cdot f$ directed reversely to f, in other words if f is an up-flip, then $g \cdot f$ is a down-flip and vice versa.

The above definition will be useful, when looking at series of flips from a regular triangulation, in order to decide if g changes the monotonicity of the series, i.e. if for a series of down-flips $(f_1, \dots, f_m)$ their image $(g \cdot f_1, \dots, g \cdot f_m)$ is also a series of down-flips or if there exists an $i \in [m]$, such that $g \cdot f_i$ is an up-flip.

Lemma 3.11. Let $f = [\Delta_1 \rightsquigarrow \Delta_2]$ be a flip via the circuit Z, set $z^* := \min_{z \in \underline{Z}} z$ and without loss of generality assume $\Delta^+(Z) \subseteq \Delta_1$. Then f is an up-flip if and only if $z^* \in Z^-$.

Proof. It holds that

$$
\begin{aligned}
&\langle \lambda, \mathrm{gkz}_{\Delta_2} - \mathrm{gkz}_{\Delta_1} \rangle \\
&= \langle \lambda, \mathrm{gkz}_{Z,\Delta_1} \rangle \\
&\overset{3.9}{=} \sum_{z \in Z^-} M^{n+1-z} \cdot (\mathrm{gkz}_{Z,\Delta_1})_z + \sum_{z \in Z^+} M^{n+1-z} \cdot (\mathrm{gkz}_{Z,\Delta_1})_z \\
&= M^{n+1-z^*} \cdot \left[\sum_{z \in Z^-} M^{z^*-z} \cdot (\mathrm{gkz}_{Z,\Delta_1})_z + \sum_{z \in Z^+} M^{z^*-z} \cdot (\mathrm{gkz}_{Z,\Delta_1})_z \right] \\
&= M^{n+1-z^*} \cdot \left[(\mathrm{gkz}_{Z,\Delta_1})_{z^*} + \underbrace{\sum_{z \in Z^- \setminus \{z^*\}} M^{z^*-z} \cdot (\mathrm{gkz}_{Z,\Delta_1})_z + \sum_{z \in Z^+ \setminus \{z^*\}} M^{z^*-z} \cdot (\mathrm{gkz}_{Z,\Delta_1})_z}_{\odot} \right].
\end{aligned}
$$

We have $z^* - z < 0 \quad \forall z \in Z \setminus \{z^*\}$ and therefore $\odot \to 0$ for $M \to \infty$. This means for an arbitrarily large $M \gg 0$ the whole expression gets arbitrarily close to $M^{n+1-z^*} \cdot (\mathrm{gkz}_{Z,\Delta_1})_{z^*}$.
"$\Rightarrow$" Let f be an up-flip. Then we have that $0 < \langle \lambda, \mathrm{gkz}_{\Delta_2} - \mathrm{gkz}_{\Delta_1} \rangle$. Now, towards a contradiction assume $z^* \in Z^+$. Using Lemma 3.9 we get that $(\mathrm{gkz}_{Z,\Delta_1})_{z^*} < 0$ and thus $M^{n+1-z^*} \cdot (\mathrm{gkz}_{Z,\Delta_1})_{z^*} < 0$ for an sufficiently large $M \gg 0$ $\lightning$.

"$\Leftarrow$" Suppose $z^* \in Z^-$. We need to show that $f = [\Delta_1 \rightsquigarrow \Delta_2]$ is an up-flip, but analogously to above we already know that for $M \gg 0$ sufficiently large, we have

$$\langle \lambda, \mathrm{gkz}_{\Delta_2} - \mathrm{gkz}_{\Delta_1} \rangle = M^{n+1-z^*} \cdot \underbrace{(\mathrm{gkz}_{Z,\Delta_1})_{z^*}}_{>0} > 0$$

and hence the claim follows. $\square$

Corollary 3.1. Let $f = [\Delta_1 \rightsquigarrow \Delta_2]$ be an up-flip via the circuit Z and without loss of generality assume $\Delta^+(Z) \subseteq \Delta_1$. For any $g \in \mathbf{G}$ set $z^* := \operatorname{argmin}_{z \in \underline{Z}} g(z)$. Then g changes the flip direction of f, if $g(z^*) \in g \cdot Z^+$ or equivalently $z^* \in Z^+$.

Confirming that the monotonicity of a path is being lost under g, does not guarantee for $g \cdot \Delta$ to not be sub-regular, since there might exist another series of down-flips from a regular triangulation resulting in $g \cdot \Delta$. Nonetheless, this allows us to only examine the extended neighborhood of Δ to decide, whether there is going to be a triangulation not being sub-regular in its orbit. Let us elaborate what this extended neighborhood is: Let $\mathcal{P}_\Delta$ be the set of all minimal paths p from a regular triangulation to Δ, where minimal means, that along these paths only the initial triangulation is regular. More technically, for a series of flips $(f_1, f_2, \ldots, f_m)_\Delta$ towards Δ let $\Delta_1 \overset{f_1}{\rightsquigarrow} \Delta_2 \overset{f_2}{\rightsquigarrow} \ldots \overset{f_{m-1}}{\rightsquigarrow} \Delta_m \overset{f_m}{\rightsquigarrow} \Delta$ be the path in Φ, then define the *minimal regular paths to* Δ as

$$\mathcal{P}_\Delta := \left\{ p = (f_1, f_2, \ldots, f_m)_\Delta \;\middle|\; \Delta_1 \text{ is the only regular triangulation of the path.} \right\}.$$

With this, we can only examine $\mathcal{P}_\Delta$ in order to decide whether there exists a triangulation not being sub-regular in $\mathbf{G} \cdot \Delta$.

Lemma 3.12. There holds

$$\mathcal{P}_{g \cdot \Delta} = g \cdot \mathcal{P}_\Delta.$$

Proof. For $g \in \mathbf{G}$ and a series of flips $(f_1, f_2, \dots, f_m)_\Delta$, we have that

$$g \cdot (f_1, f_2, \dots, f_m)_\Delta = (g \cdot f_1, g \cdot f_2, \dots, g \cdot f_m)_{g \cdot \Delta}$$

is a series of flips to $g \cdot \Delta$. This is because $g \cdot Z$ is a circuit of $\mathbf{A}$ for all $g \in \mathbf{G}$ and circuits Z of $\mathbf{A}$. The corresponding path is

$$g \cdot \Delta_1 \overset{g \cdot f_1}{\rightsquigarrow} g \cdot \Delta_2 \overset{g \cdot f_2}{\rightsquigarrow} \dots \overset{g \cdot f_{m-1}}{\rightsquigarrow} g \cdot \Delta_m \overset{g \cdot f_m}{\rightsquigarrow} g \cdot \Delta.$$

Then

$$\begin{aligned} \mathcal{P}_{g \cdot \Delta} &= \left\{ (f_1, f_2, \dots, f_m)_{g \cdot \Delta} \mid \Delta_1 \text{ is the only regular triangulation of the path.} \right\} \\ &= \left\{ g \cdot (g^{-1} \cdot f_1, g^{-1} \cdot f_2, \dots, g^{-1} \cdot f_m)_\Delta \mid g^{-1} \cdot \Delta_1 \ \begin{smallmatrix} \text{is the only regular} \\ \text{triangulation of the path.} \end{smallmatrix} \right\} \\ &= g \cdot \left\{ (\hat{f}_1, \hat{f}_2, \dots, \hat{f}_m)_\Delta \mid \hat{\Delta}_1 \text{ is the only regular triangulation of the path.} \right\} \\ &= g \cdot \mathcal{P}_\Delta \end{aligned}$$

where we used that $\mathbf{G}$ is a group, in particular, that $g^{-1} \in \mathbf{G}$. □

Let us go back to the ideas of Section 3.3.3 and incorporate the notion of the neighborhood cone once more.

Lemma 3.13. Let $g \in \mathbf{G}$, there holds

$$e_j \in \mathbf{C}^{\mathrm{nh}}(\Delta) \iff e_{g(j)} \in \mathbf{C}^{\mathrm{nh}}(g \cdot \Delta), \quad j \in [n].$$

Proof. Let $F \leq \Sigma\text{-poly}(\mathbf{A})$ be the smallest face containing Δ and $\Delta_1, \dots, \Delta_k$ be the neighbors of Δ contained in F. Let $g \in \mathbf{G}$, there holds

$$\begin{aligned}
e_j \in \mathbf{C}^{\mathrm{nh}}(\Delta) &\iff \mathrm{gkz}_\Delta(j) \geq \mathrm{gkz}_{\Delta_i}(j) \\
&\iff \sum_{\substack{\sigma\in\Delta \\ j\in\sigma}} \mathrm{vol}_d(\sigma) \geq \sum_{\substack{\sigma\in\Delta_i \\ j\in\sigma}} \mathrm{vol}_d(\sigma) \\
&\overset{\circledast}{\iff} \sum_{\substack{\sigma\in\Delta \\ g(j)\in g\cdot\sigma}} \mathrm{vol}_d(\sigma) \geq \sum_{\substack{\sigma\in\Delta_i \\ g(j)\in g\cdot\sigma}} \mathrm{vol}_d(\sigma) \\
&\iff \sum_{\substack{\sigma\in g\cdot\Delta \\ g(j)\in\sigma}} \mathrm{vol}_d(\sigma) \geq \sum_{\substack{\sigma\in g\cdot\Delta_i \\ g(j)\in\sigma}} \mathrm{vol}_d(\sigma) \\
&\iff \mathrm{gkz}_{g\cdot\Delta}(g(j)) \geq \mathrm{gkz}_{g\cdot\Delta_i}(g(j)) \\
&\iff e_{g(j)} \in \mathbf{C}^{\mathrm{nh}}(g\cdot\Delta),
\end{aligned}$$

for all $j \in [k]$, where at $\circledast$ we used that g is an affine unimodular automorphism of $\mathbf{A}$. □

The following is another corollary from Lemma 1.2.

Corollary 3.2. Δ is sub-regular with respect to $\sigma \in \mathbf{S}_n$ if and only if $g \cdot \Delta$ is sub-regular with respect to $g \circ \sigma \in \mathbf{S}_n$ for all $g \in \mathbf{G}$.

Proof. For any triangulation Δ we have

$$\begin{aligned}
\langle \mathrm{gkz}_\Delta, \lambda^\sigma \rangle &= \sum_{j\in[n]} \mathrm{gkz}_\Delta(j) \cdot M^{n+1-\sigma^{-1}(j)} \\
&= \sum_{j\in[n]} \mathrm{gkz}_{g\cdot\Delta}(g(j)) \cdot M^{n+1-\sigma^{-1}(j)} \\
&= \sum_{j\in[n]} \mathrm{gkz}_{g\cdot\Delta}(j) \cdot M^{n+1-\sigma^{-1}(g^{-1}(j))} \\
&= \sum_{j\in[n]} \mathrm{gkz}_{g\cdot\Delta}(j) \cdot M^{n+1-(g\circ\sigma)^{-1}(j)} \\
&= \langle \mathrm{gkz}_{g\cdot\Delta}, \lambda^{g\circ\sigma} \rangle.
\end{aligned}$$

□

A Appendix

A.1 Triangulations of I^4

Let us take a closer look at the triangulations of the 4-dimensional cube $I^4 := [0,1]^4$. All triangulations of I^4 are sub-regular [2, Section 7.1]. We have **247451** triangulations modulo symmetry, **12174** of which are strictly sub-regular. Even though we started on a subset of triangulations, that is the canonical representatives of the **G**-orbits w.r.t. the canonical order of vertices, the number of neighbors is not modulo symmetry. This is in order to get a general intuition about the connectedness of the triangulations in Φ^c_{reg}.

Triangulations	#	Average # Nb.	Maximum # Nb.	Minimum # Nb.
Regular	235277	12.81	24	11
Non-Regular	12174	12.60	20	8
All	247451	12.80	24	8

Table A.1: Number of neighbors of the triangulations of I^4 in the flip graph.

Triangulations	Average # Regular Nb.	Maximum # Regular Nb.	Minimum # Regular Nb.
Regular	12.65	24	11
Non-Regular	3.13	8	0
All	12.18	24	0

Table A.2: Number of regular neighbors of the triangulations of I^4 in the flip graph.

Triangulations	Average # Non-regular Nb.	Maximum # Non-regular Nb.	Minimum # Non-regular Nb.
Regular	0.16	6	0
Non-Regular	9.47	16	2
All	0.62	16	0

Table A.3: Number of non-regular neighbors of the triangulations of I^4 in the flip graph.

# Regular Triangulations adjacent to		x	# Non-regular Triangulations adjacent to	
x Regular Tri.	x Non-regular Tri.		x Regular Tri.	x Non-regular Tri.
0	205159	0	607	0
0	24080	1	1205	0
0	5037	2	2615	4
0	822	3	2564	16
0	117	4	2908	27
0	40	5	1382	159
0	22	6	822	457
0	0	7	32	1129
0	0	8	39	1869
0	0	9	0	2292
0	0	10	0	2659
57630	0	11	0	1874
58300	0	12	0	1133
63917	0	13	0	443
31225	0	14	0	111
16522	0	15	0	0
5307	0	16	0	1
1835	0	17	0	0
361	0	18	0	0
140	0	19	0	0
30	0	20	0	0
7	0	21	0	0
0	0	22	0	0
2	0	23	0	0
1	0	24	0	0
235277		Total	12174	

Table A.4: Distribution of the triangulations of I^4, when looking at the number of regular/non-regular neighbors.

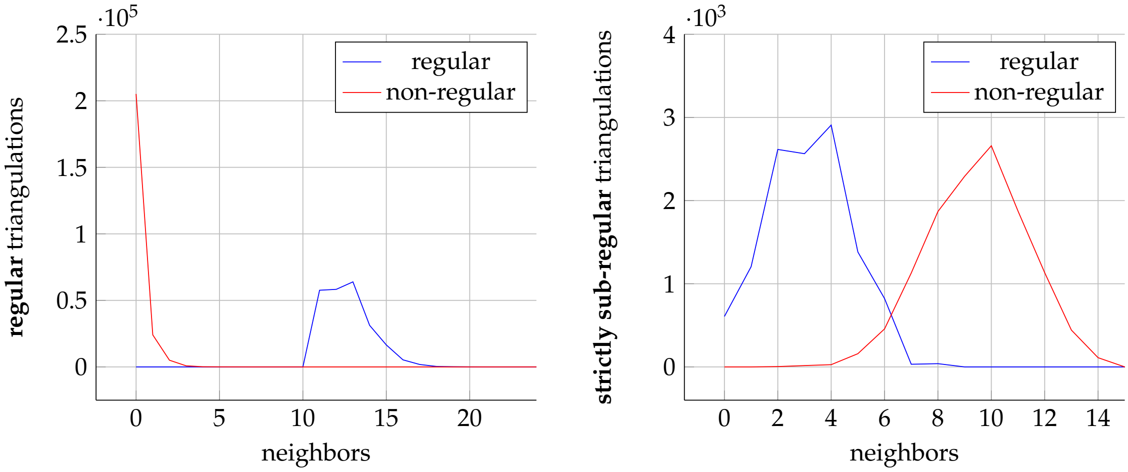


Figure A.1: Visualizing Table A.4.

A.2 `polymake`: script and data

All of the files created for and used in this thesis, including the thesis itself, can be found here:

`https://github.com/timo-y/MA-Sub-regular_Triangulations`

The repository contains the following files:

- `Abschlussarbeit_340985.pdf`
  This thesis.
- `Timo_MA_polymake.script`
  The documented script containing all functions that I used for this thesis.
- `triangulations/`
  The triangulations calculated with `mptopmcom`.
  - ↳ `i4_triangulations.txt`
    The triangulations of I^4.
  - ↳ `l44_triangulations.txt`
    The triangulations of the 4×4-lattice.
- `neighborhood/`
  These files contain the data of the neighborhood of the triangulations of I^4.

↳ `i4-all_neighborhoods.csv`
Table containing the information of the neighborhood of all triangulations of I^4 shown in Appendix A.1.

↳ `i4-counterexamles_neighborhoods.csv`
Table containing the information of the neighborhood of those triangulations of I^4 being the counterexamples mentioned in Section 3.3.2.

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Statement of Authentication

I hereby declare that I have written the present thesis independently, without assistance from external parties and without use of other resources than those indicated. The ideas taken directly or indirectly from external sources (including electronic sources) are duly acknowledged in the text. The material, either in full or in part, has not been previously submitted for grading at this or any other academic institution.

Place, Date

Signature