Università degli Studi di Padova
Dipartimento di Ing. Civile, Edile e Ambientale

Corso di Laurea in Mathematical Engineering

Project in Dynamical Systems

Study of the paper on Poverty trap and global indeterminacy in a growth model with open-access natural resources

Gennaio 2023

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Anno Accademico 2023-24

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1 Introduction

This report is based mainly on a couple of papers, which are correlated: the first one is [5], in which I have chosen a continuous time model and in particular the model briefly described in the subsection 4.1 called *"The Antoci-Russu-Galeotti Model"*, and the second one is [1], the one containing the full description. Nonetheless, other third part papers could be involved and cited.

As Antoci [1] pointed out in the paper, equilibrium selection in dynamic optimization models with externalities may depend on expectations; that is, given the initial values of the state variables (history), the path followed by the economy may be determined by the choice of the initial values of the jumping variables. This implies that expectations play a key role in equilibrium selection and in fact global indeterminacy may occur: that is, starting from the same initial values of the state variables, different equilibrium paths can approach different ω-limit sets (for example, different attractive stationary states). In this context, local stability analysis may be misleading, in that it refers to a neighborhood of a stationary state, whereas the initial values of jumping variables do not have to belong to such a neighborhood. In words of Matsuyama [11] p. 619: *"Knowing the local dynamics is not enough, because, for example, demonstrating the uniqueness of the perfect foresight path in a neighborhood of a stationary state does not necessarily rule out the existence of other perfect foresight paths in the large."*

Although some works on indeterminacy focus on global dynamics and stress the relevance of global analysis, the literature on indeterminacy is almost exclusively based on local analysis, due to the fact that dynamic models exhibiting indeterminacy are often highly nonlinear and difficult to be analyzed globally. Few papers study global indeterminacy in environmental dynamics, see references in [1].

In the present work the authors were not dealing with another important problem in economic dynamics, namely the existence of indifference points in an optimal control problem. Starting from these points, several optimal solutions exist, giving rise to the same value of the objective function. Vice versa, in authors' context, as it happens to be the case in the literature on indeterminacy, the trajectories followed by the economy are Nash equilibria but do not represent optimal solutions, being the dynamics conditioned by externalities. Therefore, when multiple Nash equilibrium trajectories exist, starting from the same initial values of the state variables, economic agents may select one which is Pareto-dominated by others, due to coordination problems.

The objective of the paper [1] is to highlight the relevance of global indeterminacy in a context in which economic activity depends on the exploitation of a free-access natural resource. They analyzed a growth model with environmental externalities, giving rise to a three-dimensional nonlinear dynamic system (the framework is that introduced by Wirl in [14]). In particular they studied the equilibrium growth dynamics of an economy constituted by a continuum of identical agents. At each instant of time t, the representative agent produces the output $Y(t)$ by labor $L(t)$, by the accumulated physical capital $K(t)$ and by the stock $E(t)$ of an open-access renewable natural resource. The economy-wide aggregate production $Y(t)$ negatively affects the stock of the natural resource; however, the value of $Y(t)$ is considered as exogenously determined by the representative agent, so that economic dynamics is affected by negative environmental externalities.[1]

They assumed that the representative agent's instantaneous utility, depending on leisure $1-L(t)$ and consumption $C(t)$ of the output $Y(t)$, is represented by the additively non-separable function $\frac{[C(1-L)^{\epsilon}]^{1-\eta}-1}{1-\eta}$. Moreover, they assumed that the production technology is represented by the Cobb–Douglas function $[K(t)]^{\alpha}[L(t)]^{\beta}[E(t)]^{\gamma}$, with $\alpha+\beta<1$ and α, β, $\gamma>0$.

In this context, I am going to show that, if $\alpha+\gamma<1$, the dynamics can admit a locally attracting stationary state (also equilibria or fixed point) $P_1^*=(K_1^*,E_1^*,L^*)$, in fact a *poverty trap*, coexisting with another stationary state $P_2^*=(K_2^*,E_2^*,L^*)$, where $K_1^*<K_2^*$ and $E_1^*<E_2^*$, possessing saddle-point stability.
Global analysis shows that, under some conditions on the parameters, if the economy starts from initial values K_0 and E_0 sufficiently close to K_1^* and E_1^*, then there exists a continuum of initial values L_0^1 such that the trajectory from (K_0,E_0,L_0^1) approaches P_1^* and a locally unique initial value L_0^2 such that the trajectory from (K_0,E_0,L_0^2) approaches P_2^* (see Figure 6). Therefore, their model exhibits not only local indeterminacy (i.e. there exists a continuum of trajectories leading to P_1^*), but also global indeterminacy, since either P_1^* or P_2^* may be selected according to agents' expectations. Along the trajectories belonging to the basin of attraction of P_1^*, over-exploitation of the natural resource drives the economy towards a *tragedy of commons* scenario (later explained).

These results are obtained through a partial description of the shape of the saddle-point two dimensional stable manifold. A full description of such a manifold is usually very difficult or impossible. However, in a three-dimensional system, two-dimensional stable (or unstable) manifolds are separatrices between different regimes of the trajectories. Therefore, if one is able to detect, in a significant region of the phase space, a separatrix between two sets of points whose trajectories show different behavior, that may lead to relevant information on the manifold of interest. Also it must be stressed that gaining information on separatrices is paramount to any global analysis: in particular it can lead to information on size and/or shape of attractive basins (which several authors in different works cited in [1] consider the main goal of global indeterminacy analysis).
For example Wirl [15] analyzes a separatrix problem in an optimal growth model where, differently from this case, the only production factor is physical capital K and a renewable environmental resource R enters only into the utility function. The analytical context is also quite different from the underlying work. In fact Wirl considers optimal solutions for a four dimensional system exhibiting two saddle-point stable stationary states. It turns out that in the two-dimensional state space (K,R) the basins of attraction of the two stable stationary states are separated by a curve, whose corresponding optimal trajectories lie on the one-dimensional stable manifold of a third conditionally stable stationary point. Moreover, Wirl shows the possible existence of limit cycles around the stationary states, arisen through Hopf bifurcations. Vice versa, in this work the authors analyzed a three-dimensional system whose trajectories are sub-optimal Nash solutions of a dynamical control problem. In such a con-

[1] Environmental externalities can affect economic activities especially in developing countries, where property rights tend to be ill-defined and ill-protected, environmental institutions and regulations are weak and natural resources are more fragile than in developed countries, which are located in temperate areas instead than in tropical and sub-tropical regions.

text the two-dimensional stable manifold of a saddle-point stable stationary state may separate, in the three-dimensional phase space, the basin of attraction of another Pareto-dominated stationary state (a poverty trap) from a region whose trajectories tend to a boundary point where the economy collapses (i.e. physical capital and labor tend to zero, while the environmental resource tends to its carrying capacity). The possible existence of limit cycles, generated by Hopf bifurcations, is shown also in this work.

The paper [1] also contains some results from numerical simulations, which provide further insights about the dynamics of the model (which I will newly perform):

- Global indeterminacy may occur also in the context which P_1^* has a stable manifold of dimension one and P_2^* is either saddle-point stable or repelling (three positive real part eigenvalues), and thus no locally indeterminate stationary state exists. In such a context (with Hopf bifurcation already occurred), numerical simulations show that, in case P_2^* is a saddle-point, then, starting from the same initial values of K and E, the economy may approach either the determinate stationary state P_2^* or an attracting limit cycle surrounding P_1^* (see Figure 9). Vice versa, when P_2^* is a source (i.e. repelling or repulsive), simulations indicate the possible existence of two limit cycles (see Figure 10), so that, starting from the same initial values of K and E, the economy may approach either an attracting limit cycle surrounding P_1^* or a limit cycle endowed with a two-dimensional stable manifold surrounding P_2^*.

- Numerical simulations suggest that the stable manifold of the locally determinate point P_2^* bounds the basin of attraction of the locally indeterminate point P_1^* or of the attracting cycle around P_1^* (see Figures 6, 9). This implies that even if the economy starts very close to the locally determinate point P_2^*, it can move quite far away from it, in particular toward a poverty trap.

This analysis focuses on global indeterminacy of dynamics, but it also gives sufficient conditions for local indeterminacy. There exists an enormous literature on local indeterminacy in economic growth models. Although in the article [1] they did not have room for a review, they pointed out the place that their results occupy in the current research. In fact, even if the main body of the literature on local indeterminacy concerns economies with increasing social returns, a growing proportion of articles deals with models where indeterminacy is obtained under the assumption of social constant return technologies. In this paper [1], local indeterminacy can occur with social constant or decreasing returns and is generated by negative environmental externalities of production activity affecting the natural resource. Other works focus on the role played by negative externalities in producing local indeterminacy. For examples see the paper [1] p. 573.

The present work has the following structure. Section 2 defines the set-up of the model and the associated dynamic system. Section 3 deals with the existence and local stability of stationary states and with Hopf bifurcations arising from stability changes. Section 4 is devoted to the global analysis of dynamics and provides the main results of the paper.

2 The model and dynamics

The model. The economy analyzed is constituted by a continuum of identical economic agents; the size of the population of agents is normalized to unity. At each instant of time $t \in [0, +\infty)$, the representative agent produces an output $Y(t)$ by the following Cobb–Douglas technology

$$Y(t) = [K(t)]^{\alpha}[L(t)]^{\beta}[E(t)]^{\gamma}, \quad \text{with } \alpha + \beta < 1 \text{ and } \alpha, \beta, \gamma > 0,$$

where $K(t)$ is the stock of physical capital or monetary value (necessary for buildings, machinery, and equipment) accumulated by the representative agent, $L(t)$ is the agent's labor input that represents the productive effort of the workforce, measured in person-hours, and $E(t)$ is the stock of an open-access renewable natural resource (typically $L, E \in (0, 1)$).[2] The exponent α measures the responsiveness or elasticity of output with respect to capital.[3] The other exponent, β, is the elasticity of output with respect to labor. And the same for the last exponent γ. The authors have assumed that the representative agent's instantaneous utility function depends on leisure $1 - L(t)$ and consumption $C(t)$ of the output $Y(t)$; precisely, they have considered the following additively non-separable function (type of function also used, among the others, by Bennett [3] and Itaya [10])

$$U\Big(C(t), L(t)\Big) = \frac{[C(t)(1 - L(t))^{\epsilon}]^{1-\eta} - 1}{1 - \eta}$$

where $\epsilon, \eta > 0$ and $\eta \neq 1$. Moreover, we assume that the utility function is concave in C and in $1 - L$, i.e. $\eta > \frac{\epsilon}{1+\epsilon}$. The parameter ϵ denotes the weight on utility toward leisure and η the inverse of the intertemporal elasticity of substitution in consumption.[4] This utility function displays a constant intertemporal elasticity of substitution and possesses the property that income and substitution effects exactly balance each other in the labor supply equation.

The evolution of $K(t)$ (assuming, for simplicity, the depreciation of K to be zero) is represented by the differential equation

$$\dot{K} = K^{\alpha} L^{\beta} E^{\gamma} - C. \tag{1}$$

In order to model the dynamics of E they started from the well-known logistic equation[5] and augment it by considering the negative impact due to the production process

$$\dot{E} = E(\bar{E} - E) - \delta \bar{Y} \tag{2}$$

[2]In modeling production activity based on open-access natural resources (for example, fishery, forestry and tourism), the stock $E(t)$ of the environmental resource very often enters as an input in the production function: see, for example, Berck and Perloff [4], Ayong Le Kama [2]. Some authors use the Cobb–Douglas production function introduced by Gordon [8] and Schaefer [12], with all the exponents equal to one. In the considered paper for this work the authors have chosen, instead, to work with a more general Cobb–Douglas function, allowing to analyze the case with constant social returns to scale.

[3]For example, an elasticity of $\alpha = 0.5$, or 50%, means that every dollar of capital investment translates to an increase in production valued at \$0.50.

[4](From Wikipedia) In economics, elasticity of intertemporal substitution (or intertemporal elasticity of substitution, EIS, IES) is a measure of responsiveness of the growth rate of consumption to the real interest rate.

[5]The logistic function has been extensively used as a growth function of renewable resources; see, for example, Elìasson and Turnovsky [7].

where the parameter $\bar{E}$ represents the carrying capacity of the natural resource, $\bar{Y}$ is the economy-wide average output and the parameter $\delta > 0$ measures the negative impact of $\bar{Y}$ on E.[6] Under the specification (2) of the environmental dynamics, the production process in our economy can be interpreted as an extractive activity. Its impact on the natural resource is given by the rate of harvest which is proportional to $\bar{Y}$. This assumption is usual in models of economic dynamics depending on open-access resources (see, for example, Berck and Perloff [4], Wirl [13], D'Alessandro [6]) and has been also introduced in economic growth models where a natural resource-intensive sector is considered (see, for example, Ayong Le Kama [2]).

We assume that the representative agent chooses the functions $C(t)$ and $L(t)$ (control variables) in order to solve the following problem

$$\max_{C,L} \int_0^{+\infty} \frac{[C(1-L)^\epsilon]^{1-\eta} - 1}{1-\eta} e^{-\theta t} \, dt \tag{3}$$

subject to the two dynamic equations (1) and (2), that are

$$\dot{K} = K^\alpha L^\beta E^\gamma - C,$$

$$\dot{E} = E(\bar{E} - E) - \delta \bar{Y}$$

with $K(0)$ and $E(0)$ given, $K(t)$, $E(t)$, $C(t) \geq 0$ and $0 \leq L(t) \leq 1$ for every $t \in [0, +\infty)$; $\theta > 0$ is the discount rate. $K(t)$ is also called state variable.

The authors have assumed in the paper under study that capital K is reversible, i.e., they allowed for disinvestment ($\dot{K} < 0$) at some instants of time.[7] Furthermore they have assumed that, in solving problem (3), the representative agent considers $\bar{Y}$ as exogenously determined since, being economic agents a continuum, the impact on $\bar{Y}$ of each one is null. However, since agents are identical, ex post $\bar{Y} = Y$ holds. This implies that the trajectories resulting from their model are not socially optimal but Nash equilibria, because no agent has an incentive to modify his choices if the others do not modify theirs.

Dynamics. The current value Hamiltonian function associated to problem (3) is (see Wirl [14])

$$H = \frac{[C(1-L)^\epsilon]^{1-\eta} - 1}{1-\eta} + \Omega \cdot (K^\alpha L^\beta E^\gamma - C)$$

where Ω is the co-state variable associated to K. By applying the Maximum Principle, the dynamics of the economy is described by the system

$$\begin{aligned} \dot{K} &= \frac{\partial H}{\partial \Omega} = K^\alpha L^\beta E^\gamma - C \\ \dot{\Omega} &= \theta \Omega - \frac{\partial H}{\partial K} = \Omega \cdot (\theta - \alpha K^{\alpha-1} L^\beta E^\gamma) \end{aligned} \tag{4}$$

[6] Notice that $\bar{E}$ is the value that E would reach, as $t \to +\infty$, in absence of the negative impact due to economic activity.

[7] This amounts to assume that the economy they have been analyzing is a small open economy that can sell or buy capital goods abroad at a fixed price.

with the constraint (equation (2)) $\dot{E} = E(\bar{E} - E) - \delta\bar{Y}$, where C and L satisfy the following conditions[8]

$$\begin{aligned} \frac{\partial H}{\partial C} &= C^{-\eta}(1-L)^{\epsilon(1-\eta)} - \Omega = 0 \\ \frac{\partial H}{\partial L} &= 0, \quad \text{i.e. } \beta(1-L)\Omega K^{\alpha} L^{\beta-1} E^{\gamma} - \epsilon C^{1-\eta}(1-L)^{\epsilon(1-\eta)} = 0 \end{aligned} \tag{5}$$

Since their system meets the Mangasarian hypotheses[9], the above conditions plus the limit transversality condition $\lim_{t\to+\infty} \Omega(t)K(t)e^{-\theta t} = 0$ are sufficient for solving problem (3). This is the case also if $\alpha + \beta + \gamma > 1$ (remember I assumed $\alpha + \beta < 1$), because the stock E is considered as a positive externality in the decision problem of the representative agent.[10]

By replacing $\bar{Y}$ with $K^{\alpha}L^{\beta}E^{\gamma}$, the Maximum Principle conditions yield a dynamic system with two state variables, K and E, and one jumping variable, Ω. Notice that, from

$$\epsilon C \frac{\partial H}{\partial C} + \frac{\partial H}{\partial L} = 0$$

one obtains

$$\begin{aligned} C &= \frac{\beta}{\epsilon}(1-L)L^{\beta-1}K^{\alpha}E^{\gamma} \\ f(L) := \frac{\epsilon}{\beta}(1-L)^{\frac{\epsilon-\eta(1+\epsilon)}{\eta}} L^{1-\beta} &= K^{\alpha}E^{\gamma}\Omega^{\frac{1}{\eta}} \end{aligned} \tag{6}$$

Hence one can write the following system, equivalent to (4)

$$\begin{aligned} \dot{K} &= \frac{1}{\epsilon}K^{\alpha}E^{\gamma}L^{\beta-1}[(\beta+\epsilon)L - \beta] \\ \dot{E} &= E(\bar{E} - E) - \delta K^{\alpha}L^{\beta}E^{\gamma} \\ \dot{L} &= \frac{f(L)}{f'(L)}\Big[\frac{\alpha}{\epsilon}K^{\alpha-1}E^{\gamma}L^{\beta-1}[(\beta+\epsilon)L - \beta] + \gamma(\bar{E} - E - \delta K^{\alpha}L^{\beta}E^{\gamma-1}) \\ &\quad + \frac{1}{\eta}(\theta - \alpha K^{\alpha-1}L^{\beta}E^{\gamma})\Big] \end{aligned} \tag{7}$$

In such a context, the jumping variable is L, instead of Ω (L and Ω are related by (6)). As a consequence, given the initial values of the state variables, K_0 and E_0, the representative agent has to choose the initial value L_0 of L. Instead, the values of the parameters in the model (7) have been estimated but the authors of the paper did not specify how they did it, even though I am going to use them, in particular in the numerical simulations. They are $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.58$, $\delta = 0.05$(object of study later in this work with $\bar{E}$), $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$.

[8]Notice that the adopted utility function implies $C > 0$ and $0 < L < 1$.

[9]Are some particular condition (also known as Mangasarian-Fromowitz constraint qualification MFCQ) of the general KKT ones. For details search the KKT conditions into the website `https://www.wikiwand.com`.

[10]The procedure applied till here is the common one for deterministic optimization problems: Hamiltonian setting and maximum principle. Given an optimization problem subject to some conditions (here are the dynamics of K and E), this technical procedure is based on the application of the Lagrangian function that is taking into account some multipliers (also called co-state variables) then one can rewrite down the problem in terms of the Hamiltonian function. Furthermore, one has to derive the first order conditions, and using the dynamic constraints, simplify those first order conditions. This gives a system of differential equations.

3 Fixed points, stability and Hopf bifurcations

Recall the conditions on the parameters: they are all positive, with $\alpha+\beta < 1$ and $1 \neq \eta > \frac{\epsilon}{1+\epsilon}$. The following theorem that the authors postulated in the main paper [1] deals with the problem of the existence and numerosity of stationary states (also known as fixed points) of the dynamic system (7). I mention also the proof because it includes some formulas used in a second moment.

Theorem 1. *System* (7) *has one stationary state if* $\alpha + \gamma > 1$*; one or zero stationary states if* $\alpha + \gamma = 1$*; one or two stationary states if* $\alpha + \gamma < 1$.

Proof. I shall consider the system of equations in (7), and put them equal to zero, i.e. $\dot{K} = \dot{E} = \dot{L} = 0$. Then, the authors of the paper suggest to assign $L = L^* = \frac{\beta}{\beta+\epsilon}$ (in Section 4 one can understand why). Consequently, the first equation becomes zero and one eventually obtains
$L^* = \frac{\beta}{\beta+\epsilon}$
$0 = E(\bar{E} - E) - \delta K^\alpha (L^*)^\beta E^\gamma$
$0 = \frac{f(L^*)}{f'(L^*)} \left[\frac{1}{\eta} (\theta - \alpha K^{\alpha-1} (L^*)^\beta E^\gamma) \right]$.
And observing that $\frac{f(L^*)}{f'(L^*)} = \frac{\eta L^*(1-L^*)}{\eta - \epsilon L^*} \neq 0$ always since $\eta > 0$ and $L^* \in (0,1)$, one can rewrite the relations above as
$L^* = \frac{\beta}{\beta+\epsilon}$
$0 = E(\bar{E} - E) - \delta K^\alpha (L^*)^\beta E^\gamma$
$0 = \theta - \frac{\alpha}{\delta} K^{-1} E(\bar{E} - E)$.
From the third equation one can obtain an explicit form of $K = K^*$, and substituting in the second equation one gets this last one in the form
$0 = E(\bar{E} - E) - \delta (\frac{\alpha}{\delta\theta})^\alpha E^\alpha (\bar{E} - E)^\alpha (L^*)^\beta E^\gamma$,
and then at last one can find $E = E^*$.

Therefore, a stationary state $P^* = (K^*, E^*, L^*)$ of (7) have to satisfy the following relations

$$\begin{aligned} L^* &= \frac{\beta}{\beta+\epsilon} \\ K^* &= \frac{\alpha}{\delta\theta} E^* (\bar{E} - E^*) \\ g(E^*) &:= E^* + \delta \Big(\frac{\beta}{\beta+\epsilon} \Big)^{\frac{\beta}{1-\alpha}} \Big(\frac{\alpha}{\theta} \Big)^{\frac{\alpha}{1-\alpha}} (E^*)^{\frac{\alpha+\gamma-1}{1-\alpha}} = \bar{E} \end{aligned} \tag{8}$$

or equivalently,

$$\begin{aligned} L^* &= \frac{\beta}{\beta+\epsilon} \\ K^* &= \Big(\frac{\beta}{\beta+\epsilon} \Big)^{\frac{\beta}{1-\alpha}} \Big(\frac{\alpha}{\theta} \Big)^{\frac{1}{1-\alpha}} (E^*)^{\frac{\gamma}{1-\alpha}} \\ g(E^*) &:= E^* + \delta \Big(\frac{\beta}{\beta+\epsilon} \Big)^{\frac{\beta}{1-\alpha}} \Big(\frac{\alpha}{\theta} \Big)^{\frac{\alpha}{1-\alpha}} (E^*)^{\frac{\alpha+\gamma-1}{1-\alpha}} = \bar{E} \end{aligned} \tag{9}$$

Hence the graph of $g(E)$ intersects the line $E = \bar{E}$ exactly at one point if $\alpha + \gamma > 1$, at most at one point if $\alpha + \gamma = 1$, at zero, one or two points if $\alpha + \gamma < 1$, while K^* is an increasing function of E^* (see Figure 1). $\square$

Observe that, if $\alpha + \gamma < 1$, then there exists one stationary state only if the minimum of the function $g(E^*)$ coincides with the value $\bar{E}$ (assumed to be the carrying capacity of the natural resource); so, generically, the stationary states are zero or two.

By (9), when two stationary states exist, $P_1^* = (K_1^*, E_1^*, L^*)$ and $P_2^* = (K_2^*, E_2^*, L^*)$, then $K_1^* < K_2^*$ and $E_1^* < E_2^*$; so P_2^* Pareto-dominates[11] P_1^*. If the economy approaches the latter, then a *tragedy of commons*[12] scenario emerges, characterized by over-exploitation of the natural resource and by low physical capital accumulation (labor input is equal to $L^* = \frac{\beta}{\beta+\epsilon}$ at both stationary states).[13]

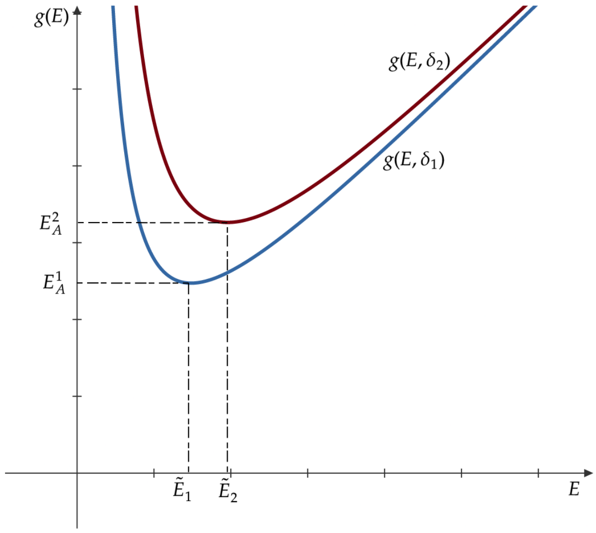

Figure 1: The graph of $g(E)$, $E > 0$, is drawn for values $\delta_1 < \delta_2$ of δ (clearly in the case of $\alpha + \gamma < 1$), when δ increases then both the coordinates of the minimum point, i.e. $(\tilde{E}, E_A)$ with $E_A = g(\tilde{E}) = (\frac{2-2\alpha-\gamma}{1-\alpha-\gamma})\tilde{E}$, increase.

Now, let $P^* = (K^*, E^*, L^*)$ be a stationary state of the system (7) and consider the Jacobian matrix of the same system evaluated at P^*

$$Jac(P^*) = J^* = \begin{bmatrix} 0 & 0 & \frac{\partial \dot{K}}{\partial L} \\ \frac{\partial \dot{E}}{\partial K} & \frac{\partial \dot{E}}{\partial E} & \frac{\partial \dot{E}}{\partial L} \\ \frac{\partial \dot{L}}{\partial K} & \frac{\partial \dot{L}}{\partial E} & \frac{\partial \dot{L}}{\partial L} \end{bmatrix}$$

[11] The meaning of the term "Pareto-domination" is described on web at `https://en.wikipedia.org/wiki/Pareto_efficiency`.

[12] Term that refers to a situation in which individuals with access to a public resource (also called a *common*) act in their own interest and, in doing so, ultimately deplete the resource. Examples are coffee consumption, over-fishing, fast fashion, groundwater use, etc. For more details take a look at `https://en.wikipedia.org/wiki/Tragedy_of_the_commons`.

[13] It is worth to stress that, even if a trajectory approaches P_2^*, it does not represent an optimal growth path, since environmental externalities are not internalized by economic agents.

where, by straightforward computations

$$
\begin{aligned}
\frac{\partial \dot{K}}{\partial L} &= \frac{\beta+\epsilon}{\delta\epsilon} E^*(\bar{E}-E^*) \\
\frac{\partial \dot{E}}{\partial K} &= -\delta\theta \\
\frac{\partial \dot{E}}{\partial E} &= \bar{E}(1-\gamma) - E^*(2-\gamma) \\
\frac{\partial \dot{E}}{\partial L} &= -(\beta+\epsilon)E^*(\bar{E}-E^*) \\
\frac{\partial \dot{L}}{\partial K} &= \frac{f(L^*)}{f'(L^*)}\frac{\delta\theta}{E^*}\left[\frac{\theta(1-\alpha)}{\alpha\eta(\bar{E}-E^*)} - \gamma\right] \\
\frac{\partial \dot{L}}{\partial E} &= \frac{f(L^*)}{f'(L^*)}\frac{\gamma}{E^*}\left[(1-\gamma)(\bar{E}-E^*) - E^* - \frac{\theta}{\eta}\right] \\
\frac{\partial \dot{L}}{\partial L} &= \frac{f(L^*)}{f'(L^*)}(\beta+\epsilon)\left[\frac{\theta(\beta+\epsilon)}{\epsilon} - \frac{\theta}{\eta} - \gamma(\bar{E}-E^*)\right].
\end{aligned}
\tag{10}
$$

Another important theorem from the paper holds:

Theorem 2. *If the stationary state is unique with $\alpha+\gamma \geq 1$, or, in case of two stationary states, is the one with the larger E^*, then J^* has an odd number of positive eigenvalues; instead, if, in case of two stationary states, P^* corresponds to the one with the smaller E^*, then J^* has an odd number of negative eigenvalues.*

Proof. By computing $det(J^*)$, one can check that

$$
sign[det(J^*)] = sign[(2-2\alpha-\gamma)E^* - (1-\alpha-\gamma)\bar{E}] \tag{11}
$$

It follows that, when the stationary state is unique and $\alpha+\gamma \geq 1$, then $det(J^*) > 0$. Vice versa, when two stationary states exist, implying $\alpha+\gamma < 1$, then it follows from (11), by observing Figure 1, that $det(J^*)$ has the same sign of $g'(E^*)$, which proves the theorem. □

Considering a non-generic case when a unique stationary state exists under the condition $\alpha+\gamma < 1$, then $det(J^*) = 0$ holds and the stationary state is not hyperbolic (in fact, a saddle-node bifurcation occurs). Consequently, if one looks for an attracting stationary state, they have to restrict their analysis to the case when, under the assumption $\alpha+\gamma < 1$, two stationary states exist, P_1^* and P_2^*, with $E_1^* < E_2^*$ and $K_1^* < K_2^*$. Here I aim to show that, in such a context, P_1^* can be attracting for suitable values of the parameters (see Figures 2 and 3). Along the trajectories belonging to the basin of attraction of P_1^* the over-exploitation of the natural resource drives the economy towards a *tragedy of commons* scenario.

First of all, again if $\alpha+\gamma < 1$, a necessary and sufficient condition for the existence of two stationary states is

$$
\bar{E} > E_A := g(\tilde{E}) = \left(\frac{2-2\alpha-\gamma}{1-\alpha-\gamma}\right)\tilde{E} \tag{12}
$$

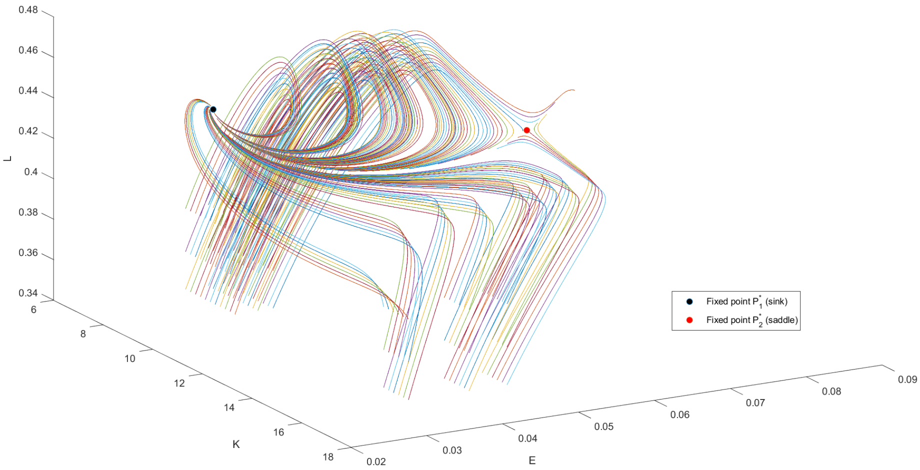

Figure 2: Phase portrait showing global indeterminacy in the space (K, E, L); parameter values: $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.58$, $\delta = 0.05$, $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$, $\bar{E} = 0.17$. Considering in particular two trajectories, one approaching the sink $P_1^* = (8.802396759, 0.031860545540, 0.4444)$ staring from the point $(K_1^*, E_1^*, L_0^1 = 0.374591)$, while the other approaching the saddle $P_2^* = (13.11009169, 0.0591165274446, 0.4444)$ starting from the point $(K_1^*, E_1^*, L_0^2 = 0.34591)$

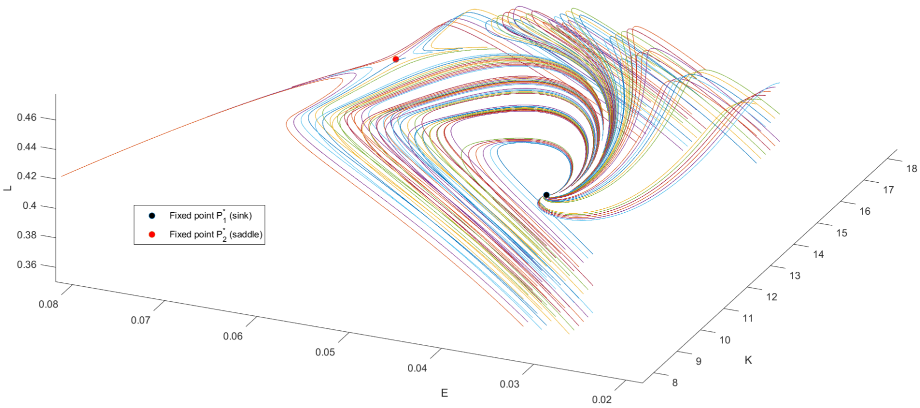

Figure 3: Different view of the phase portrait.

where $\tilde{E}$ is the only positive value satisfying $g'(\tilde{E}) = 0$.
Straightforward computations yield

$$E_A = (2 - 2\alpha - \gamma)\left[\frac{\delta^{1-\alpha}}{(1-\alpha)^{1-\alpha}(1-\alpha-\gamma)^{1-\alpha-\gamma}}\left(\frac{\beta}{\beta+\epsilon}\right)^{\beta}\left(\frac{\alpha}{\theta}\right)^{\alpha}\right]^{\frac{1}{2-2\alpha-\gamma}} \tag{13}$$

Thus, $E_1^* < \tilde{E} < (\frac{1-\alpha-\gamma}{2-2\alpha-\gamma})\bar{E}$. From now on the index 1 will be omitted.

The well-known Routh–Hurwitz Criterion[14] (see Hurwitz [9]) yields that J^*,

[14] Technique used a lot in control systems which include essentially the calculation of $det(J^* - \lambda I_3) = 0$ and derivation of the characteristic equation (in the form of a polynomial). Without calculating directly the eigenvalues of J^*, that may be quite complicate in general, one can study only the coefficients of the characteristic equation. Indeed, create the so called Routh table containing the coefficients and a combination of them. Finally, consider some conditions on those terms in the table according to a rule.

the Jacobian matrix at P^*, has three eigenvalues with negative real part if and only if

$$det(J^*) < 0 \tag{14}$$

$$\begin{aligned} \sigma(J^*) &= \frac{\partial \dot{E}}{\partial E}\frac{\partial \dot{L}}{\partial L} - \frac{\partial \dot{E}}{\partial L}\frac{\partial \dot{L}}{\partial E} - \frac{\partial \dot{K}}{\partial L}\frac{\partial \dot{L}}{\partial K} > 0 \\ \rho(J^*) &= -\sigma(J^*) \cdot trace(J^*) + det(J^*) > 0 \end{aligned} \tag{15}$$

The last inequality, in particular, guarantees the non-existence of complex eigenvalues with non-negative real part. In fact, when $\rho(J^*)$ crosses the origin value 0, the real part of two complex conjugate eigenvalues changes sign, causing, generically, a Hopf bifurcation.Remember that the condition (14) is always verified at P^* (see (11)).

In fact, as an example, I performed a couple of numerical simulations (the code is available in the appendix A) using the values of parameters as defined in Figures 2 and 5, i.e. $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.58$, $\delta = 0.05$, $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$, and with different values of $\bar{E}$.

In the first case of the attracting limit cycle with $\bar{E} = 0.21$ (hence Hopf bifurcation has already occurred), $\bar{E} > 0.167159 = E_A$, where E_A comes from (13), so one observes two fixed points $P_1^* = (4.5625, 0.0115, 0.4444)$ and $P_2^* = (21.2463, 0.12505, 0.4444)$. Considering P_1^* (that is a saddle node with one dimensional stable manifold) and $J^* = Jac(P_1^*)$, I am going to show the conditions coming from the simulation, that are

$det(J^*) = -2.2387e-07$, $\quad trace(J^*) = -1.2789e-04$,

$\sigma(J^*) = 7.1892e-05$, $\quad \rho(J^*) = -2.1468e-07$.

The simulation shows also the three eigenvalues, that are one real negative and the other two complex conjugate with positive real part, and this result is correct since $\rho(J^*) < 0$.

In the second case (shown in the Figure 2) with the same parameter values but with $\bar{E} = 0.17$ one can see again two different fixed points, one a sink and one a saddle. For the sink P_1^* the results are

$det(J^*) = -6.5390e-08$, $\quad trace(J^*) = -0.0237$,

$\sigma(J^*) = 3.1802e-05$, $\quad \rho(J^*) = 6.8923e-07$,

with 3 eigenvalues, one real negative and the other two complex conjugate with negative real part. Finally, for the saddle P_2^* the results are

$det(J^*) = 7.4620e-08$, $\quad trace(J^*) = -0.0524$,

$\sigma(J^*) = 7.7007e-07$, $\quad \rho(J^*) = 1.1501e-07$,

with 3 eigenvalues, two real negative and the third real positive.

As for the condition (15), the authors of the paper stated the following lemma.

Lemma 3. *If*

$$\eta \geq \frac{\epsilon}{\epsilon + \alpha\beta} \quad and \quad \bar{E} > E_B = \frac{\theta(\beta+\epsilon)(2-2\alpha-\gamma)}{\alpha\beta\gamma\eta} \tag{16}$$

then the condition $\sigma(J^*) > 0$ is verified.

Proof. By recalling (10), straightforward computations lead to

$$sign[\sigma(J^*)] = sign\left[\left(\frac{\beta+\epsilon}{\epsilon} - \frac{1}{\eta}\right)(\bar{E} - 2E^*) - \frac{\theta(1-\alpha)(\beta+\epsilon)}{\alpha\epsilon\eta}\right]$$

So, since $E^* < (\frac{1-\alpha-\gamma}{2-2\alpha-\gamma})\bar{E}$, the assumptions of the lemma imply $\sigma(J^*) > 0$. □

I shall now compute $trace(J^*)$; observing that $\frac{f(L^*)}{f'(L^*)} = \frac{\beta\epsilon\eta}{(\beta+\epsilon)[\eta(\beta+\epsilon)-\beta\epsilon]}$, one obtains

$$trace(J^*) = a(\bar{E} - E^*) - E^* + b$$

where

$$a := \frac{\eta[(1-\gamma)(\beta+\epsilon) - \beta\gamma\epsilon] - \beta\epsilon(1-\gamma)}{\eta(\beta+\epsilon) - \beta\epsilon}, \qquad b := \frac{\beta\theta[\eta(\beta+\epsilon) - \epsilon]}{\eta(\beta+\epsilon) - \beta\epsilon}. \tag{17}$$

Then the results from the above analysis, which aim at detecting an attracting stationary state, are summarized by the following theorem.

Theorem 4. *Let $\alpha+\gamma < 1$ and $\bar{E} > E_A$, so that system (7) has two stationary states, P_1^* and P_2^*, with $E_1^* < E_2^*$ and $K_1^* < K_2^*$. Then there exist values of the parameters for which P_1^* is a sink, while P_2^* is a saddle with a two-dimensional stable manifold. Moreover, in such a case, take $\bar{E}$ as a bifurcation parameter. As $\bar{E}$ is increased, P_2^* does not change its nature (i.e. it remains a saddle with a two-dimensional stable manifold), whereas P_1^* can undergo one, two or no Hopf bifurcations.*

Proof. Omitted. See Appendix A of the usual paper [1]. □

Notice that the sufficient conditions for local indeterminacy given above depend on the intertemporal elasticity of substitution[15] and can be satisfied both in case $\eta < 1$ (i.e. elasticity of substitution greater than 1) and in case $\eta > 1$ (i.e. elasticity of substitution lower than 1): in fact, previously in the lemma they have assumed $\eta \geq \frac{\epsilon}{\epsilon+\alpha\beta}$. Furthermore, those conditions require that the impact of the production process of output (measured by δ) is high enough and/or the subjective discount rate θ is low enough. Finally, the elasticity γ of the production function with respect to the natural capital E must be not too high, that is $\alpha+\gamma < 1$, while social returns to scale can be constant or decreasing, that is $\alpha+\beta+\gamma \leq 1$.

At this point, the authors have made an analysis about the relation of the coordinates of the fixed points with respect to the parameter $\bar{E}$ (the carrying capacity of the environmental resource). I have performed the numerical simulation fixing the parameters to estimated values, and making sure to have $\alpha+\gamma < 1$ (see appendix A). Figure 4 shows how the stationary state values of K and E change by varying the value of $\bar{E}$. The coordinates of P_1^* are indicated by a bold line if P_1^* is a sink and by a dash–dot line if it is a saddle with a one-dimensional stable manifold; the coordinates of P_2^* (which, in the numerical example, is a saddle with a two dimensional stable manifold) are indicated by a dotted line. Notice that a Hopf bifurcation occurs when the parameter $\bar{E}$ crosses the value 0.2 (the bifurcation point is indicated by H or the marker $'*'$ in Figure 4).

Remark. The authors in the paper have also shown the relationship among K, E and the parameter δ (fixing $\bar{E}$), which measure the environmental impact

[15] In economics, intertemporal elasticity of substitution, IES, is a measure of responsiveness of the growth rate of consumption to the real interest rate. For more details see `https://en.wikipedia.org/wiki/Elasticity_of_intertemporal_substitution`.

of the production process.[16] Notice that a Hopf bifurcation occurs also in this example and that indeterminacy is observed when δ is high enough.

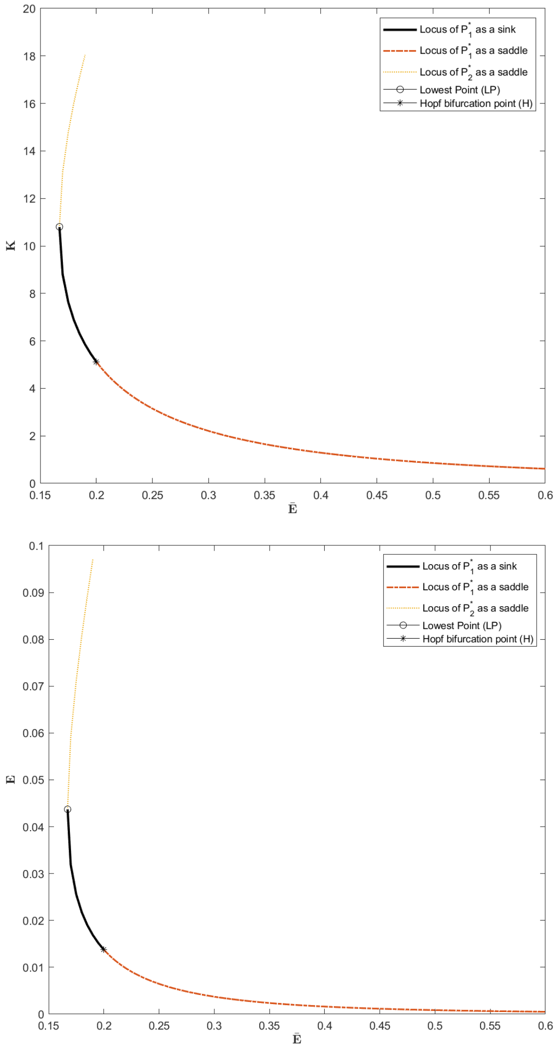

Figure 4: The fixed point values of K and E, varying $\bar{E}$; parameter values: $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.58$, $\delta = 0.05$, $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$.

After performed the numerical simulations of the model using four-stage Runge-Kutta method (see appendix A), in Figure 5 is shown a locally attracting limit cycle around P_1^* (which has a one-dimensional stable manifold) arisen via the Hopf bifurcation of Figure 4. In such a case (with $\alpha + \gamma < 1$), local indeterminacy occurs, since for every initial point (K_0, E_0) close to the projection of the cycle in the plane (K, E), there exists a continuum of initial values L_0 of L such that the trajectory starting from (K_0, E_0, L_0) approaches the cycle. In particular, even starting from the fixed point P_1^* one can observe the attraction

[16] By observing Figure 1, it follows that the effect of reducing δ, when $\bar{E}$ is fixed, is qualitatively analogous to that of increasing $\bar{E}$, when δ is fixed. In fact, as δ increases, both $\tilde{E}$ and $E_A = g(\tilde{E})$ increase.

to the limit cycle (trajectory in red color in Figure 5).

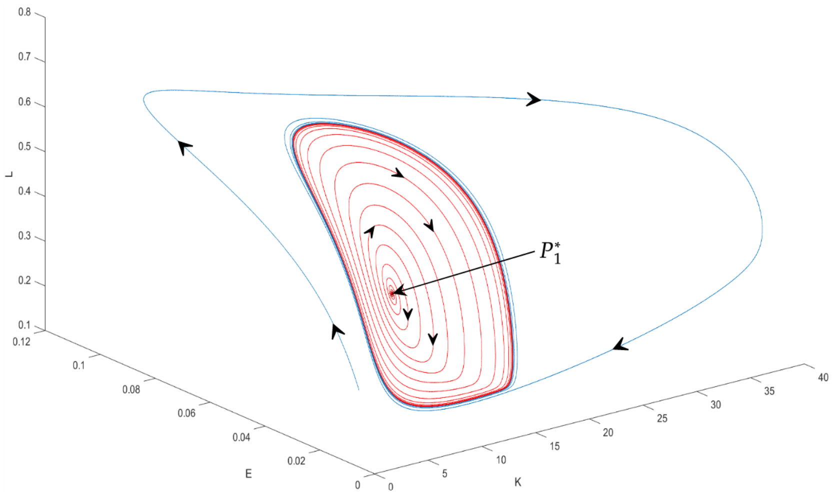

Figure 5: Locally attracting limit cycle "around" P_1^*; parameter values: $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.58$, $\delta = 0.05$, $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$.

Now to complete the local analysis I shall discuss the case $\alpha + \gamma > 1$, when the stationary state P^* is unique. Since $det(J^*) > 0$ (see Theorem 2), P^* is not attracting. If, for some value of $\bar{E}$, $trace(J^*) = a(\bar{E} - E^*) - E^* + b < 0$, with a and b defined in (17), then P^* is a saddle with a two-dimensional stable manifold. By the equilibrium condition $g(E^*) = \bar{E}$, one can write $trace(J^*)$ as

$$trace(J^*) = r(E^*)^{\frac{\alpha+\gamma-1}{1-\alpha}} - E^* + b$$

with $b > 0$, $\alpha + \gamma - 1 > 0$ and $sign(r) = sign(a)$. By the same arguments developed in Theorem 4, the following one is easily proved.

Theorem 5. *Let $\alpha + \gamma > 1$ and P^* denote the only stationary state of system* (7). *Write $trace(J^*) = a(\bar{E} - E^*) - E^* + b$, with a and b defined in* (17). *Assume $trace(J^*) < 0$ for some value of $\bar{E}$ and let $\bar{E}$ increase. Then: if $a \leq 0$, no bifurcation occurs and P^* remains a saddle with a two-dimensional stable manifold; if $a > 0$ and $2\alpha + \gamma > 2$, eventually P^* becomes a source and one Hopf bifurcation generically takes place; if $a > 0$ and $2\alpha + \gamma < 2$, P^* is a saddle with a two-dimensional stable manifold for sufficiently large values of $\bar{E}$ and Hopf bifurcations are, generically, zero or two.*

According to the above theorem, indeterminacy cannot be observed in the context of a unique stationary state: i.e., the stationary state can possess saddle-type stability (two eigenvalues with negative real part) but cannot be a sink.

4 Global analysis and Conclusion

In this section, as the authors did for the sake of convenience in representing the figures, I will change the order of the variables from (K, E, L) into (E, K, L).

In the following I take into consideration the case where, for $\alpha + \gamma < 1$, two stationary states exist, $P_1^* = (E_1^*, K_1^*, L^*)$ and $P_2^* = (E_2^*, K_2^*, L^*)$, with

$E_1^* < E_2^*$, $K_1^* < K_2^*$, $L^* = \frac{\beta}{\beta+\epsilon}$, and P_1^* is a sink (when $\bar{E}$ is chosen properly in order to avoid Hopf bifurcation).

Hence the basin of attraction of P_1^* can be considered a *poverty trap* and I wonder if, given a point $P_0 = (E_0, K_0, L_0)$ belonging to such a basin, it is possible to modify the initial choice of labor L_0 in such a way that the positive semi-trajectory starting from the *new* point $\widetilde{P}_0 = (E_0, K_0, \widetilde{L}_0)$ can tend to the saddle P_2^* (having a two-dimensional stable manifold).

In fact, I will give a (partially) affirmative answer to the above question (Theorem 10) and, moreover, I will conduct some numerical simulations that are aiming at detecting trajectories leading to the desirable equilibrium, i.e. to the saddle P_2^* (formalization in Lemma 7).

These results will be obtained through a partial description of the shape of the saddle-point two-dimensional stable manifold. A full description of such a manifold is usually very difficult or impossible. However, in a three-dimensional system, two-dimensional stable (or unstable) manifolds are separatrices between different regimes of the trajectories. Therefore, if one is able to detect, in a significant region of the phase space (in our case a sub-region of the plane $L = L^*$ where $\dot{L} > 0$), a separatrix between two sets of points whose trajectories show different behavior, that may lead to relevant information on the manifold of interest. As I will show, this is, in fact, our case.

Consider the following theorem.

Theorem 6. *Consider a point $P_0 = (E_0, K_0, L_0)$, $0 < E_0 < \bar{E}$, $0 < K_0$, $0 < L_0 < 1$. Then, if L_0 is small enough, the positive semi-trajectory from P_0 tends to $(\bar{E}, 0, 0)$.*

Proof. See in the paper [1], pg. 582. □

As I said (and authors of the paper said), we want to see if, given a point $P_0 = (E_0, K_0, L_0)$ belonging to the basin of attraction of P_1^* and sufficiently close to P_1^*, it is possible to modify the initial choice of labor in such a way that the positive semi-trajectory starting from the *new* point $\widetilde{P}_0 = (E_0, K_0, \widetilde{L}_0)$ can tend to the saddle P_2^*. To this aim, as suggested by the above theorem I will consider values $\widetilde{L} < L_0$. For instance, let us start, in order to fix ideas, from P_1^* itself. Moving downward along the half-line $E = E_1^*$, $K = K_1^*$, $L < L^*$, we cross, as shown in the above theorem, the basin of attraction of P_1^* at a certain point, say $\widetilde{P} = (E_1^*, K_1^*, \widetilde{L})$, $\widetilde{L} < L^*$. I wonder if the positive semi-trajectory starting from $\widetilde{P}$ tends to the saddle P_2^*. As $K_1^* < K_2^*$ and $K(t)$ decreases when $L < L^*$, the trajectory from $\widetilde{P}$, in order to tend to P_2^*, must cross, first, the plane $L = L^*$. In such a case, that will take place at a point where $K < K_1^*$ and $\dot{L} > 0$.

Furthermore, observe that, being $\widetilde{L} < L^*$, $\dot{E} > 0$ at $\widetilde{P}$. Should the trajectory *go back*, before crossing $L = L^*$, to a point where $E = E_1^*$, at such a point it would be again $\dot{E} > 0$, since $K < K_1^*$ and $L < L^*$. So our hypothetical trajectory must cross $L = L^*$ at a point where $E > E_1^*$ and $K < K_1^*$.

The following lemma allows, precisely, to detect the points with the above described features, i.e. such that $E > E_1^*$, $K < K_1^*$, $L = L^*$, $\dot{L} > 0$, belonging to the stable manifold of P_2^*.

Lemma 7. *Let $\alpha+\gamma < 1$ and assume that two equilibria exist, $P_1^* = (E_1^*, K_1^*, L^*)$ and $P_2^* = (E_2^*, K_2^*, L^*)$, $E_1^* < E_2^*$, $K_1^* < K_2^*$, $L^* = \frac{\beta}{\beta+\epsilon}$. Moreover, assume that* *the conditions of Lemma 3 of the previous section, i.e.*

$$\eta \geq \frac{\epsilon}{\epsilon + \alpha\beta}, \quad \bar{E} > E_B = \frac{\theta(\beta+\epsilon)(2-2\alpha-\gamma)}{\alpha\beta\gamma\eta}$$

are satisfied and that P_1^* is a sink. Consider, in the plane $L = L^*$, the open set

$$A = \{P = (E, K, L^*) : E > E_1^*, K < K_1^*, \dot{L}(P) > 0\}.$$

Then A can be partitioned into three non-empty pair-wise disjoint subsets, $A = A_1 \cup A_2 \cup A_3$, where A_1 and A_2 are open, while A_3 is a unidimensional set belonging to the stable manifold of P_2^*. More precisely:

A_1 is the subset of points $P \in A$ such that the positive semi-trajectory starting from P crosses $\dot{E} = 0$ before crossing again $L = L^*$.

A_2 is the subset of points $Q \in A$ such that the positive semi-trajectory starting from Q crosses again $L = L^*$ before crossing $\dot{E} = 0$.

Proof. See in the paper [1], Appendix B. □

An immediate consequence of the previous lemma is the following:

Corollary 8. *Let $\alpha + \gamma < 1$ and two equilibria, P_1^* and P_2^*, exist, P_1^* being a sink. Moreover, assume $\eta \geq \frac{\epsilon}{\epsilon+\alpha\beta}$ and $\bar{E} > E_B = \frac{\theta(\beta+\epsilon)(2-2\alpha-\gamma)}{\alpha\beta\gamma\eta}$. Then P_2^* is a saddle with a two-dimensional stable manifold and the eigenvalues of its Jacobian matrix are all real (one positive and two negative).*

Remark 9. The above corollary suggests that, in the case of two fixed points, whenever P_1^* is a sink, the eigenvalues at P_2^* are all real: one positive and two negative. However, it happens that, when P_1^* is not attracting, P_2^* can be either a saddle-point with two negative real part complex eigenvalues or a source (all eigenvalues have positive real part).

For example, I shall adopt the above notations and I recall from the proof of the Theorem 4 that $E_C := \frac{b(2-2\alpha-\gamma)}{1-\alpha-\gamma}$ where b is defined in (17), and E_A from (13). I assume, firstly, $E_A = E_C > \widehat{E} := \frac{\theta(1-\alpha)(\beta+\epsilon)(2-2\alpha-\gamma)}{\alpha\gamma[\eta(\beta+\epsilon)-\epsilon]}$, $a = 0$.
Then it is easily checked that, if $0 < \bar{E} - E_A \ll 1$,
$trace(J(P_2^*)) < 0 < trace(J(P_1^*))$, $det(J(P_1^*)) < 0 < det(J(P_2^*))$ and
$|trace(J(P_{1,2}^*))| \ll \sigma(J(P_{1,2}^*))$ hold.
This implies that P_1^* has one negative and two positive real part complex eigenvalues, while P_2^* has one positive and two negative real part complex eigenvalues.

Vice versa, let, with the same notations, $E_C > E_A > \widehat{E}$, $a = 0$. Then, if $0 < \bar{E} - E_A \ll 1$, both $trace(J(P_1^*))$ and $trace(J(P_2^*))$ are positive,
$det(J(P_1^*)) < 0 < det(J(P_2^*))$ and $\rho(J(P_2^*)) < 0$.
Hence in this case P_1^* has again a one-dimensional stable manifold, while P_2^* is a source (three positive real part eigenvalues). Moreover, when, posed $E_A = \lambda\widehat{E} + (1-\lambda)E_C$, $0 < \lambda < 1$, and λ sufficiently small, then both P_1^* and P_2^* have two complex conjugate eigenvalues.

At this point, one is able to prove the main theorem:

Theorem 10. *Given the assumptions of Lemma 7, there exists a neighborhood N of the sink P_1^*, such that, for any $(E_0, K_0, L_0) \in N$, the half-line $\{E = E_0, K = K_0, L < L_0\}$ intersects the stable manifold of P_2^*.*

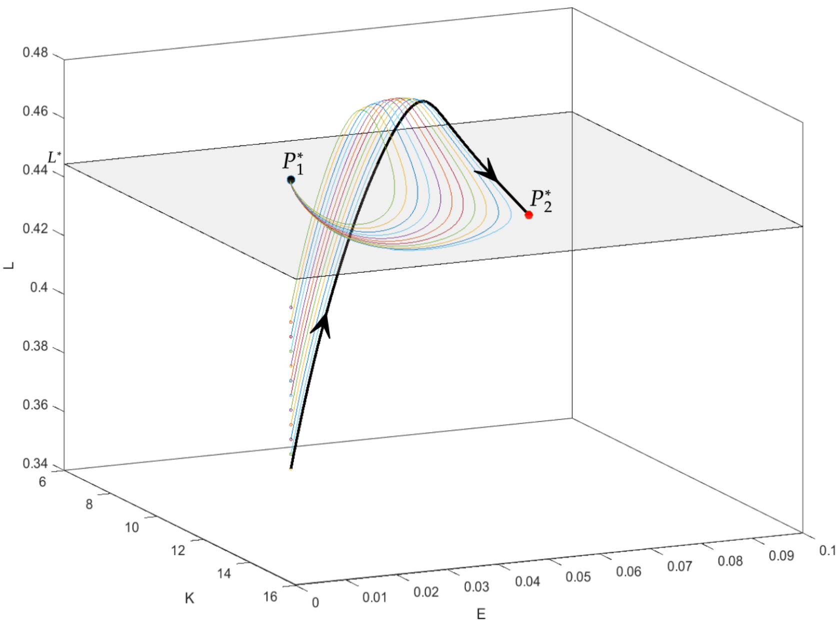

Figure 6: Global indeterminacy in the space (E, K, L); the black trajectory (in bold) approaching $P_2^* = (E_2^*, K_2^*, L^* = 0.444444)$ starts from the point $(E_1^*, K_1^*, L_0 \simeq 0.345910588598)$; parameter values: $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.58$, $\delta = 0.05$, $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$, $\bar{E} = 0.17$.

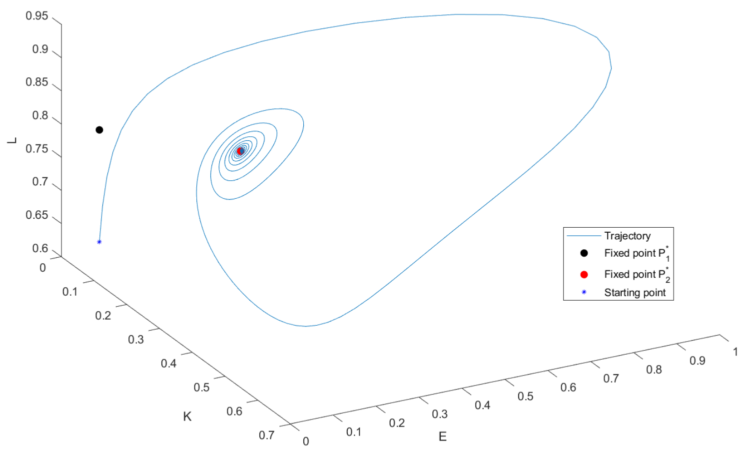

Figure 7: A trajectory spiraling toward P_2^* starting from the point $(E_1^*, K_1^*, L_0 \simeq 0.6316546)$ plotted with a low end time $(T = 100)$; parameter values: $\alpha = \theta = 0.19$, $\beta = \gamma = 0.8$, $\epsilon = 0.2$, $\eta = \frac{4}{9}$, $\delta = 12.8620934285269$, $\overline{E} = 10.75$.

Proof. See in the paper [1], pg. 584. □

Therefore, under the assumptions of the theorem, for any initial point (E_0, K_0) sufficiently close to (E_1^*, K_1^*), there exists a continuum of initial values L_0^1 such that the trajectory starting from (E_0, K_0, L_0^1) approaches P_1^*, and a locally unique value L_0^2 such that the trajectory starting from (E_0, K_0, L_0^2) converges to P_2^*. So global indeterminacy occurs, since, from the initial position (E_0, K_0),

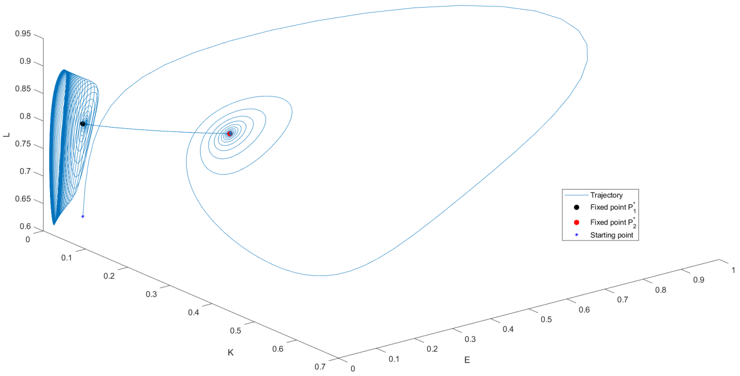

Figure 8: The same trajectory spiraling toward P_2^* starting from the point $(E_1^*, K_1^*, L_0 \simeq 0.6316546)$ as in Figure 7 but plotted with a high end time ($T = 1000$); the same parameter values.

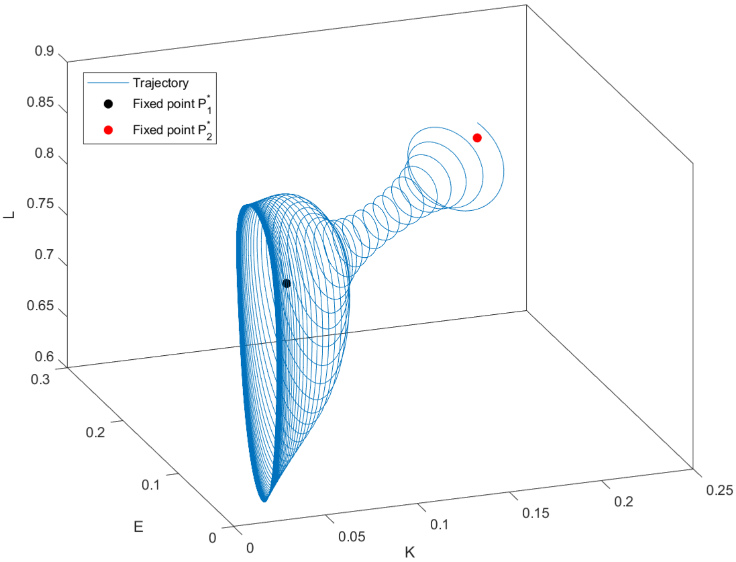

Figure 9: Starting from the point $(E_2^*, K_2^*, L_0 \simeq 0.815)$, "very close" to $P_2^* = (E_2^*, K_2^*, L^* = 0.8)$, the trajectory approaches the attracting limit cycle surrounding P_1^*; the parameter values are those used in Figure 7.

the economy may follow one of the trajectories belonging to the basin of attraction of the 'poverty trap' P_1^* but it may also follow a trajectory lying on the stable manifold of the stationary state P_2^*.

In Figure 6 a numerical simulation is shown. The starting points of the trajectories are chosen along the half-line $\{E = E_1^*, K = K_1^*, L < L^*\}$. The trajectory in bold lies on the stable manifold of P_2^* and consequently converges to P_2^*, while all the others approach P_1^*. Notice that some trajectories approaching P_1^* are characterized by an initial phase where the values of K and L are higher than along the trajectory converging to P_2^*; however, the higher values of K and L give rise to over-exploitation of the natural resource and the consequent reduction of the stock E drives the economy towards the undesirable equilibrium P_1^*, where the values of K and E are lower than in P_2^*.

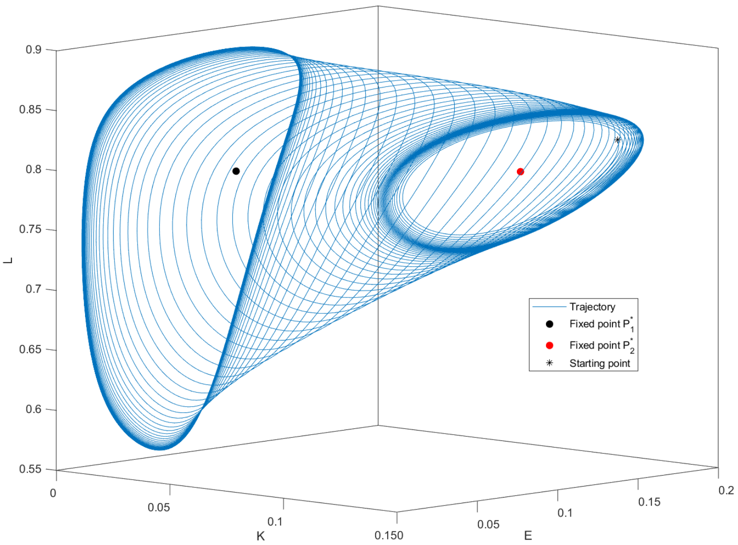

Figure 10: Two limit cycles: an attractive one "around" P_1^* and a repulsive one "around" P_2^*; parameter values: $\alpha = \theta = 0.19$, $\beta = \gamma = 0.8$, $\epsilon = 0.2$, $\eta = \frac{4}{9}$, $\delta = 8.42228$, $\overline{E} = 7.06$.

In the main model studied in this work, global indeterminacy can occur even if the assumptions of Lemma 7 are not satisfied, as the trajectories drawn in Figures 7-9 suggest.

In those figures, $P_1^* = (E_1^*, K_1^*, L^*)$ has one negative and two positive real part complex eigenvalues, while P_2^* has one positive and two negative real part complex eigenvalues (computed with the already implemented code setting properly the parameters, see appendix A). Numerical simulations suggest the existence of an attracting limit cycle surrounding P_1^*.
In Figure 7 the point (E_1^*, K_1^*, L_0), lying on the half-line $\{E = E_1^*,\ K = K_1^*,\ L < L^*\}$, appears to belong to the stable manifold of P_2^* (i.e. the trajectory starting from it approaches P_2^*). In case one performs the simulation for particularly long time (e.g. $T = 1000$), it can be observed that the spiraling around P_2^* goes to a limit cycle around P_1^* as shown in the Figure 8.
Using the same parameter values of Figure 7, the simulation in Figure 9 shows a trajectory starting from a point on the half-line $\{E = E_2^*, K = K_2^*, L > L^*\}$, very close to the saddle-point P_2^*, which approaches the attracting limit cycle surrounding P_1^*.

Figure 10 shows the occurrence of two limit cycles. In this case P_1^* has one negative and two positive real part complex eigenvalues, while P_2^* is a source (three positive real part eigenvalues). The simulation shows an attracting limit cycle surrounding P_1^* and a not-attracting one arisen around P_2^* through a Hopf bifurcation and endowed with a two-dimensional stable manifold, which bounds, as a separatrix, the basin of the attracting limit cycle.

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A Implementations in Matlab

In this section I would like to show the implemented codes in Matlab of the developed theory above and some additional plots of dynamics. The line below I arranged in thin paragraphs where each of them shows the name of the Matlab file and some comments about it. These files are found in the folder of this work. I shall start with the publication of a code related to Section 3 which finds the fixed points of the main model (7) performing a solver based on the equations found in the proof of the Theorem 1, i.e. the (9). This code is defined in the file `'section3_find_fixed_points_dyn_model.m'`.

The code which generates the Figures 2 and 3 is defined in the file `'section3_phase_portrait_two_equilibria_sink_saddle.m'`. This code yields the simulations of the phase portrait, i.e. a set of trajectories starting from different regions of the space in order to observe the behavior.

The code which computes the Jacobian matrix of a specific fixed point, the determinant, the σ and ρ functions coming from the Routh-Hurwitz Criterion (as seen in section 3 at (15)) for stability is defined in `'section3_eigenvalues_of_Jacobian_and_stability.m'`. This code was used to generate the results discussed in the paragraph just above the Lemma 3.

The next code shown in the file `'section3_K_and_E_vs_Ebar.m'` is referred to the plot of a curve, which is divided in three parts, that indicates how the fixed point values of K and E change by varying the value of $\bar{E}$. The results of that are shown in the Figure 4.

The code yielding the locally attracting limit cycle in Figure 5 is defined in the file `'section3_loc_attract_limit_cycle.m'`. The file contains the dynamic equations of the main model (7) and the trajectories worked out using the 4th Runge-Kutta method.

TO DO: To be implemented figures and results for section 4

Now the code developed for the Section 4 is indicated in the following lines. For the generation of Figure 6, i.e. the representation of the global indeterminacy, the code can be found in the file `'section4_global_indeterminacy.m'`. As in others codes, I have used 4th Runge-Kutta method to plot the trajectories of the main dynamical model.

The Figures 7 and 8 were generated by the implementation defined in `'section4_spiraling_on_P2_and_limit_cycle.m'` using different values of $T = 100, 1000$, i.e. the time horizon.

Then the Figure 9 was created by the code `'section4_limit_cycle_P1_from_close_P2.m'`. This view is similar to that of the 8 when the spiraling around P_2^* tend to a limit cycle.

Finally, the code `'section4_two_limit_cycles_P1_and_P2.m'` yields the Figure 10 of the two limit cycles, one attractive and the other one repulsive.