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Economic growth model with open-access natural resources

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Contents

1 Introduction

This report is based mainly on a couple of papers, which are correlated: the first one is [5], in which I have chosen the a continuous time model and in particular the model briefly described in the subsection 4.1 called *"The Antoci-Russu-Galeotti Model"*, and the second one is [1], the one containing the full description. Nonetheless other third part papers could be involved and cited.

As Antoci [1] pointed out in the paper, equilibrium selection in dynamic optimization models with externalities may depend on expectations; that is, given the initial values of the state variables (history), the path followed by the economy maybe determined by the choice of the initial values of the jumping variables. This implies that expectations play a key role in equilibrium selection and in fact global indeterminacy may occur: that is, starting from the same initial values of the state variables, different equilibrium paths can approach different ω-limit sets (for example, different attractive stationary states). In this context, local stability analysis may be misleading, in that it refers to a neighborhood of a stationary state, whereas the initial values of jumping variables do not have to belong to such a neighborhood. In words of Matsuyama [11] p. 619: *"Knowing the local dynamics is not enough, because, for example, demonstrating the uniqueness of the perfect foresight path in a neighborhood of a stationary state does not necessarily rule out the existence of other perfect foresight paths in the large."*

Although some works on indeterminacy focus on global dynamics and stress the relevance of global analysis, the literature on indeterminacy is almost exclusively based on local analysis, due to the fact that dynamic models exhibiting indeterminacy are often highly nonlinear and difficult to be analyzed globally. Few papers study global indeterminacy in environmental dynamics, see references in [1].

In the present work the authors were not dealing with another important problem in economic dynamics, namely the existence of indifference points in an optimal control problem. Starting from these points, several optimal solutions exist, giving rise to the same value of the objective function. Vice versa, in our context, as it happens to be the case in the literature on indeterminacy, the trajectories followed by the economy are Nash equilibria but do not represent optimal solutions, being the dynamics conditioned by externalities. Therefore, when multiple Nash equilibrium trajectories exist, starting from the same initial values of the state variables, economic agents may select one which is Pareto-dominated by others, due to coordination problems.

The objective of the paper [1] is to highlight the relevance of global indeterminacy in a context in which economic activity depends on the exploitation of a free-access natural resource. They analyzed a growth model with environmental externalities, giving rise to a three-dimensional nonlinear dynamic system (the framework is that introduced by Wirl in [14]). In particular they studied the equilibrium growth dynamics of an economy constituted by a continuum of identical agents. At each instant of time t, the representative agent produces the output $Y(t)$ by labor $L(t)$, by the accumulated physical capital $K(t)$ and by the stock $E(t)$ of an open-access renewable natural resource. The economy-wide aggregate production $Y(t)$ negatively affects the stock of the natural resource; however, the value of $Y(t)$ is considered as exogenously determined by the representative agent, so that economic dynamics is affected by negative environmental externalities.[1]

They assumed that the representative agent's instantaneous utility, depending on leisure $1 - L(t)$ and consumption $C(t)$ of the output $Y(t)$, is represented by the additively non-separable function $\frac{[C(1-L)^{\epsilon}]^{1-\eta}-1}{1-\eta}$. Moreover, we assume that the production technology is represented by the Cobb–Douglas function $[K(t)]^{\alpha}[L(t)]^{\beta}[E(t)]^{\gamma}$, with $\alpha + \beta < 1$ and α, β, $\gamma > 0$.

In this context, we show that, if $\alpha + \beta < 1$, the dynamics can admit a locally attracting stationary state (also equilibria or fixed point) $P_1^* = (K_1^*, E_1^*, L^*)$, in fact a *poverty trap*, coexisting with another stationary state $P_2^* = (K_2^*, E_2^*, L^*)$, where $K_1^* < K_2^*$ and $E_1^* < E_2^*$, possessing saddle-point stability.
Global analysis shows that, under some conditions on the parameters, if the economy starts from initial values K_0 and E_0 sufficiently close to K_1^* and E_1^*, then there exists a continuum of initial values L_0^1 such that the trajectory from (K_0, E_0, L_0^1) approaches P_1^* and a locally unique initial value L_0^2 such that the trajectory from (K_0, E_0, L_0^2) approaches P_2^* (see Fig. **??**). Therefore, their model exhibits local indeterminacy (i.e. there exists a continuum of trajectories leading to P_1^*), but also global indeterminacy, since either P_1^* or P_2^* may be selected according to agents' expectations. Along the trajectories belonging to the basin of attraction of P_1^*, over-exploitation of the natural resource drives the economy towards a *tragedy of commons* scenario.

These results are obtained through a partial description of the shape of the saddle-point two dimensional stable manifold. A full description of such a manifold is usually very difficult or impossible. However, in a three-dimensional system, two-dimensional stable (or unstable) manifolds are separatrices between different regimes of the trajectories. Therefore, if one is able to detect, in a significant region of the phase space, a separatrix between two sets of points whose trajectories show different behavior, that may lead to relevant information on the manifold of interest. Also it must be stressed that gaining information on separatrices is paramount to any global analysis: in particular it can lead to information on size and/or shape of attractive basins (which several authors in different works cited in [1] consider the main goal of global indeterminacy analysis).
For example Wirl [15] analyzes a separatrix problem in an optimal growth model where, differently from this case, the only production factor is physical capital K and a renewable environmental resource R enters only into the utility function. The analytical context is also quite different from the underlying work. In fact Wirl considers optimal solutions for a four dimensional system exhibiting two saddle-point stable stationary states. It turns out that in the two-dimensional state space (K, R) the basins of attraction of the two stable stationary states are separated by a curve, whose corresponding optimal trajectories lie on the one-dimensional stable manifold of a third conditionally stable stationary point. Moreover, Wirl shows the possible existence of limit cycles around the stationary states, arisen through Hopf bifurcations. Vice versa, in this work they analyzed a three-dimensional system whose trajectories are sub-optimal Nash solutions of a dynamical control problem. In such a context the two-dimensional stable manifold of a saddle-point stable stationary state may separate, in the three-

[1] Environmental externalities can affect economic activities especially in developing countries, where property rights tend to be ill-defined and ill-protected, environmental institutions and regulations are weak and natural resources are more fragile than in developed countries, which are located in temperate areas instead than in tropical and sub-tropical regions.

dimensional phase space, the basin of attraction of another Pareto-dominated stationary state (a poverty trap) from a region whose trajectories tend to a boundary point where the economy collapses (i.e. physical capital and labor tend to zero, while the environmental resource tends to its carrying capacity). The possible existence of limit cycles, generated by Hopf bifurcations, is shown also in this work, through numerical simulations.

The paper [1] also contains some numerical simulations, which provide further insights about the dynamics of the model (which I will newly perform):

- Global indeterminacy may occur also in the context which P_1^* has a stable manifold of dimension one and P_2^* is either saddle-point stable or repelling (three positive real part eigenvalues), and thus no locally indeterminate stationary state exists. In such a context, numerical simulations show that, in case P_2^* is a saddle-point, then, starting from the same initial values of K and E, the economy may approach either the determinate stationary state P_2^* or an attracting limit cycle surrounding P_1^* (see Fig. **??**). Vice versa, when P_2^* is a source (i.e. repelling), simulations indicate the possible existence of two limit cycles (see Fig. **??**), so that, starting from the same initial values of K and E, the economy may approach either an attracting limit cycle surrounding P_1^* or a limit cycle endowed with a two-dimensional stable manifold surrounding P_2^*.

- Numerical simulations suggest that the stable manifold of the locally determinate point P_2^* bounds the basin of attraction of the locally indeterminate point P_1^* or of the attracting cycle around P_1^* (see Figs. **??**, **??**). This implies that even if the economy starts very close to the locally determinate point P_2^*, it can move quite far away from it, in particular toward a poverty trap.

This analysis focuses on global indeterminacy of dynamics, but it also gives sufficient conditions for local indeterminacy. There exists an enormous literature on local indeterminacy in economic growth models. Although in the article [1] they did not have room for a review, they pointed out the place that their results occupy in the current research. In fact, even if the main body of the literature on local indeterminacy concerns economies with increasing social returns, a growing proportion of articles deals with models where indeterminacy is obtained under the assumption of social constant return technologies. In this paper [1], local indeterminacy can occur with social constant or decreasing returns and is generated by negative environmental externalities of production activity affecting the natural resource. Other works focus on the role played by negative externalities in producing local indeterminacy. For examples see the paper [1] p. 573.

The present work has the following structure. Section 2 defines the set-up of the model and the associated dynamic system. Section 3 deals with the existence and local stability of stationary states and with Hopf bifurcations arising from stability changes. Section 4 is devoted to the global analysis of dynamics and provides the main results of the paper.

2 The model and dynamics

The model. The economy analyzed is constituted by a continuum of identical economic agents; the size of the population of agents is normalized to unity. At each instant of time $t \in [0, +\infty)$, the representative agent produces an output $Y(t)$ by the following Cobb–Douglas technology

$$Y(t) = [K(t)]^{\alpha}[L(t)]^{\beta}[E(t)]^{\gamma}, \quad \text{with } \alpha + \beta < 1 \text{ and } \alpha, \beta, \gamma > 0,$$

where $K(t)$ is the stock of physical capital accumulated by the representative agent, $L(t)$ is the agent's labor input and $E(t)$ is the stock of an open-access renewable natural resource.[2] The authors have assumed that the representative agent's instantaneous utility function depends on leisure $1 - L(t)$ and consumption $C(t)$ of the output $Y(t)$; precisely, they have considered the following additively non-separable function (type of function also used, among the others, by Bennett [3] and Itaya [10])

$$U\Big(C(t), L(t)\Big) = \frac{[C(t)(1 - L(t))^{\epsilon}]^{1-\eta} - 1}{1 - \eta}$$

where $\epsilon, \eta > 0$ and $\eta \neq 1$. Moreover, we assume that the utility function is concave in C and in $1 - L$, i.e. $\eta > \frac{\epsilon}{1+\epsilon}$. The parameter ϵ denotes the weight on utility toward leisure and η the inverse of the intertemporal elasticity of substitution in consumption.[3] This utility function displays a constant intertemporal elasticity of substitution and possesses the property that income and substitution effects exactly balance each other in the labor supply equation.

The evolution of $K(t)$ (assuming, for simplicity, the depreciation of K to be zero) is represented by the differential equation

$$\dot{K} = K^{\alpha} L^{\beta} E^{\gamma} - C. \tag{1}$$

In order to model the dynamics of E they started from the well-known logistic equation[4] and augment it by considering the negative impact due to the production process

$$\dot{E} = E(\bar{E} - E) - \delta\bar{Y} \tag{2}$$

where the parameter $\bar{E}$ represents the carrying capacity of the natural resource, $\bar{Y}$ is the economy-wide average output and the parameter $\delta > 0$ measures the negative impact of $\bar{Y}$ on E.[5] Under the specification (2) of the environmental dynamics, the production process in our economy can be interpreted as an

[2] In modeling production activity based on open-access natural resources (for example, fishery, forestry and tourism), the stock $E(t)$ of the environmental resource very often enters as an input in the production function: see, for example, Berck and Perloff [4], Ayong Le Kama [2]. Some authors use the Cobb–Douglas production function introduced by Gordon [8] and Schaefer [12], with all the exponents equal to one. In the considered paper for this work the authors have chosen, instead, to work with a more general Cobb–Douglas function, allowing to analyze the case with constant social returns to scale.

[3] (From Wikipedia) In economics, elasticity of intertemporal substitution (or intertemporal elasticity of substitution, EIS, IES) is a measure of responsiveness of the growth rate of consumption to the real interest rate.

[4] The logistic function has been extensively used as a growth function of renewable resources; see, for example, Elìasson and Turnovsky [7].

[5] Notice that $\bar{E}$ is the value that E would reach, as $t \to +\infty$, in absence of the negative impact due to economic activity.

extractive activity. Its impact on the natural resource is given by the rate of harvest which is proportional to $\bar{Y}$. This assumption is usual in models of economic dynamics depending on open-access resources (see, for example, Berck and Perloff [4], Wirl [13], D'Alessandro [6]) and has been also introduced in economic growth models where a natural resource-intensive sector is considered (see, for example, Ayong Le Kama [2]).

We assume that the representative agent chooses the functions $C(t)$ and $L(t)$ (control variables) in order to solve the following problem

$$\max_{C,L} \int_0^{+\infty} \frac{[C(1-L)^\epsilon]^{1-\eta} - 1}{1-\eta} e^{-\theta t}\, dt \tag{3}$$

subject to the two dynamic equations (1) and (2), that are

$$\dot{K} = K^\alpha L^\beta E^\gamma - C,$$

$$\dot{E} = E(\bar{E} - E) - \delta \bar{Y}$$

with $K(0)$ and $E(0)$ given, $K(t)$, $E(t)$, $C(t) \geq 0$ and $0 \leq L(t) \leq 1$ for every $t \in [0, +\infty)$; $\theta > 0$ is the discount rate. $K(t)$ is also called state variable.

The authors have assumed in the paper under study that capital K is reversible, i.e., they allowed for disinvestment ($\dot{K} < 0$) at some instants of time.[6] Furthermore they have assumed that, in solving problem (3), the representative agent considers $\bar{Y}$ as exogenously determined since, being economic agents a continuum, the impact on $\bar{Y}$ of each one is null. However, since agents are identical, ex post $\bar{Y} = Y$ holds. This implies that the trajectories resulting from their model are not socially optimal but Nash equilibria, because no agent has an incentive to modify his choices if the others do not modify theirs.

Dynamics. The current value Hamiltonian function associated to problem (3) is (see Wirl [14])

$$H = \frac{[C(1-L)^\epsilon]^{1-\eta} - 1}{1-\eta} + \Omega \cdot (K^\alpha L^\beta E^\gamma - C)$$

where Ω is the co-state variable associated to K. By applying the Maximum Principle, the dynamics of the economy is described by the system

$$\begin{aligned} \dot{K} &= \frac{\partial H}{\partial \Omega} = K^\alpha L^\beta E^\gamma - C \\ \dot{\Omega} &= \theta\Omega - \frac{\partial H}{\partial K} = \Omega \cdot (\theta - \alpha K^{\alpha-1} L^\beta E^\gamma) \end{aligned} \tag{4}$$

with the constraint (equation (2)) $\dot{E} = E(\bar{E} - E) - \delta\bar{Y}$, where C and L satisfy the following conditions[7]

$$\begin{aligned} &\frac{\partial H}{\partial C} = C^{-\eta}(1-L)^{\epsilon(1-\eta)} - \Omega = 0 \\ &\frac{\partial H}{\partial L} = 0, \quad \text{i.e. } \beta(1-L)\Omega K^\alpha L^{\beta-1} E^\gamma - \epsilon C^{1-\eta}(1-L)^{\epsilon(1-\eta)} = 0 \end{aligned} \tag{5}$$

[6]This amounts to assume that the economy they have been analyzing is a small open economy that can sell or buy capital goods abroad at a fixed price.

[7]Notice that the adopted utility function implies $C > 0$ and $0 < L < 1$.

Since their system meets the Mangasarian hypotheses[8], the above conditions plus the limit transversality condition $\lim_{t\to+\infty}\Omega(t)K(t)e^{-\theta t}=0$ are sufficient for solving problem (3). This is the case also if $\alpha+\beta+\gamma>1$ (remember I assumed $\alpha+\beta<1$), because the stock E is considered as a positive externality in the decision problem of the representative agent.[9]

By replacing $\bar{Y}$ with $K^{\alpha}L^{\beta}E^{\gamma}$, the Maximum Principle conditions yield a dynamic system with two state variables, K and E, and one jumping variable, Ω. Notice that, from

$$\epsilon C\frac{\partial H}{\partial C}+\frac{\partial H}{\partial L}=0$$

one obtains

$$\begin{aligned} C &= \frac{\beta}{\epsilon}(1-L)L^{\beta-1}K^{\alpha}E^{\gamma} \\ f(L) &:= \frac{\epsilon}{\beta}(1-L)^{\frac{\epsilon-\eta(1+\epsilon)}{\eta}}L^{1-\beta}=K^{\alpha}E^{\gamma}\Omega^{\frac{1}{\eta}} \end{aligned} \tag{6}$$

Hence one can write the following system, equivalent to (4)

$$\begin{aligned} \dot{K} &= \frac{1}{\epsilon}K^{\alpha}E^{\gamma}L^{\beta-1}[(\beta+\epsilon)L-\beta] \\ \dot{E} &= E(\bar{E}-E)-\delta K^{\alpha}L^{\beta}E^{\gamma} \\ \dot{L} &= \frac{f(L)}{f'(L)}\Big[\frac{\alpha}{\epsilon}K^{\alpha-1}E^{\gamma}L^{\beta-1}[(\beta+\epsilon)L-\beta]+\gamma(\bar{E}-E-\delta K^{\alpha}L^{\beta}E^{\gamma-1}) \\ &\quad +\frac{1}{\eta}(\theta-\alpha K^{\alpha-1}L^{\beta}E^{\gamma})\Big] \end{aligned} \tag{7}$$

In such a context, the jumping variable is L, instead of Ω (L and Ω are related by (6)). As a consequence, given the initial values of the state variables, K_0 and E_0, the representative agent has to choose the initial value L_0 of L.

3 Fixed points, stability and Hopf bifurcations

Recall the conditions on the parameters: they are all positive, with $\alpha+\beta<1$ and $1\neq\eta>\frac{\epsilon}{1+\epsilon}$. The following theorem that the authors postulated in the main paper [1] deals with the problem of the existence and numerosity of stationary states (also known as fixed points) of the dynamic system (7). I mention also the proof because it includes some formulas used in a second moment.

Theorem 1. *System* (7) *has one stationary state if* $\alpha+\gamma>1$*; one or zero stationary states if* $\alpha+\gamma=1$*; one or two stationary states if* $\alpha+\gamma<1$.

[8]Are some particular condition (also known as Mangasarian-Fromowitz constraint qualification MFCQ) of the general KKT ones. For details search the KKT conditions into the website `https://www.wikiwand.com`.

[9]The procedure applied till here is the common one for deterministic optimization problems: Hamiltonian setting and maximum principle. Given an optimization problem subject to some conditions (here are the dynamics of K and E), this technical procedure is based on the application of the Lagrangian function that is taking into account some multipliers (also called co-state variables) then one can rewrite down the problem in terms of the Hamiltonian function. Furthermore, one has to derive the first order conditions, and using the dynamic constraints, simplify those first order conditions. This gives a system of differential equations.

Proof. A stationary state $P^* = (K^*, E^*, L^*)$ of (7) have to satisfy the following relations

$$\begin{aligned} L^* &= \frac{\beta}{\beta+\epsilon} \\ K^* &= \frac{\alpha}{\delta\theta} E^*(\bar{E} - E^*) \\ g(E^*) &:= E^* + \delta\Big(\frac{\beta}{\beta+\epsilon}\Big)^{\frac{\beta}{1-\alpha}} \Big(\frac{\alpha}{\theta}\Big)^{\frac{\alpha}{1-\alpha}} (E^*)^{\frac{\alpha+\gamma-1}{1-\alpha}} = \bar{E} \end{aligned} \tag{8}$$

or equivalently,

$$\begin{aligned} L^* &= \frac{\beta}{\beta+\epsilon} \\ K^* &= \Big(\frac{\beta}{\beta+\epsilon}\Big)^{\frac{\beta}{1-\alpha}} \Big(\frac{\alpha}{\theta}\Big)^{\frac{1}{1-\alpha}} (E^*)^{\frac{\gamma}{1-\alpha}} \\ g(E^*) &:= E^* + \delta\Big(\frac{\beta}{\beta+\epsilon}\Big)^{\frac{\beta}{1-\alpha}} \Big(\frac{\alpha}{\theta}\Big)^{\frac{\alpha}{1-\alpha}} (E^*)^{\frac{\alpha+\gamma-1}{1-\alpha}} = \bar{E} \end{aligned} \tag{9}$$

Hence the graph of $g(E)$ intersects the line $E = \bar{E}$ exactly at one point if $\alpha + \gamma > 1$, at most at one point if $\alpha + \gamma = 1$, at zero, one or two points if $\alpha + \gamma < 1$, while K^* is an increasing function of E^* (see Fig. 1). □

Observe that, if $\alpha + \gamma < 1$, then there exists one stationary state only if the minimum of the function $g(E^*)$ coincides with the value $\bar{E}$ (assumed to be the carrying capacity of the natural resource); so, generically, the stationary states are zero or two.
By (9), when two stationary states exist, $P_1^* = (K_1^*, E_1^*, L^*)$ and $P_2^* = (K_2^*, E_2^*, L^*)$, then $K_1^* < K_2^*$ and $E_1^* < E_2^*$; so P_2^* Pareto-dominates[10] P_1^*. If the economy approaches the latter, then a *tragedy of commons*[11] scenario emerges, characterized by over-exploitation of the natural resource and by low physical capital accumulation (labor input is equal to $L^* = \frac{\beta}{\beta+\epsilon}$ at both stationary states).[12]

Now, let $P^* = (K^*, E^*, L^*)$ be a stationary state of (7) and consider the Jacobian matrix of the same system (7) evaluated at P^*

$$Jac(P^*) = J^* = \begin{bmatrix} 0 & 0 & \frac{\partial \dot{K}}{\partial L} \\ \frac{\partial \dot{E}}{\partial K} & \frac{\partial \dot{E}}{\partial E} & \frac{\partial \dot{E}}{\partial L} \\ \frac{\partial \dot{L}}{\partial K} & \frac{\partial \dot{L}}{\partial E} & \frac{\partial \dot{L}}{\partial L} \end{bmatrix}$$

[10]The meaning of the term "Pareto-domination" is described on web at `https://en.wikipedia.org/wiki/Pareto_efficiency`.

[11]Term that refers to a situation in which individuals with access to a public resource (also called a *common*) act in their own interest and, in doing so, ultimately deplete the resource. Examples are coffee consumption, over-fishing, fast fashion, groundwater use, etc. For more details take a look at `https://en.wikipedia.org/wiki/Tragedy_of_the_commons`.

[12]It is worth to stress that, even if a trajectory approaches P_2^*, it does not represent an optimal growth path, since environmental externalities are not internalized by economic agents.

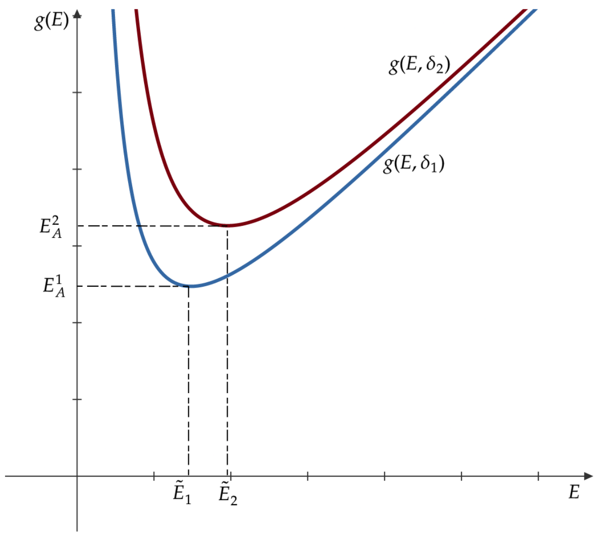

Figure 1: The graph of $g(E)$, $E > 0$, is drawn for values $\delta_1 < \delta_2$ of δ (clearly in the case of $\alpha + \gamma < 1$), when δ increases then both the coordinates of the minimum point, i.e. $(\tilde{E}, E_A)$ with $E_A = g(\tilde{E}) = (\frac{2-2\alpha-\gamma}{1-\alpha-\gamma})\tilde{E}$, increase.

where, by straightforward computations

$$\begin{aligned}
\frac{\partial \dot{K}}{\partial L} &= \frac{\beta+\epsilon}{\delta\beta} E^*(\bar{E} - E^*) \\
\frac{\partial \dot{E}}{\partial K} &= -\delta\theta \\
\frac{\partial \dot{E}}{\partial E} &= \bar{E}(1-\gamma) - E^*(2-\gamma) \\
\frac{\partial \dot{E}}{\partial L} &= -(\beta+\epsilon)E^*(\bar{E} - E^*) \\
\frac{\partial \dot{L}}{\partial K} &= \frac{f(L^*)}{f'(L^*)} \frac{\delta\theta}{E^*} \left[\frac{\theta(1-\alpha)}{\alpha\eta(\bar{E} - E^*)} - \gamma \right] \\
\frac{\partial \dot{L}}{\partial E} &= \frac{f(L^*)}{f'(L^*)} \frac{\gamma}{E^*} \left[(1-\gamma)(\bar{E} - E^*) - E^* - \frac{\theta}{\eta} \right] \\
\frac{\partial \dot{L}}{\partial L} &= \frac{f(L^*)}{f'(L^*)} (\beta+\epsilon) \left[\frac{\theta(\beta+\epsilon)}{\epsilon} - \frac{\theta}{\eta} - \gamma(\bar{E} - E^*) \right].
\end{aligned} \tag{10}$$

Another important theorem from the paper holds:

Theorem 2. *If the stationary state is unique with $\alpha + \gamma \geq 1$, or, in case of two stationary states, is the one with the larger E^*, then J^* has an odd number of positive eigenvalues; instead, if, in case of two stationary states, P^* corresponds to the one with the smaller E^*, then J^* has an odd number of negative eigenvalues.*

Proof. By computing $det(J^*)$, one can check that

$$sign[det(J^*)] = sign[(2 - 2\alpha - \gamma)E^* - (1 - \alpha - \gamma)\bar{E}] \tag{11}$$

It follows that, when the stationary state is unique and $\alpha+\gamma \geq 1$, then $det(J^*) > 0$. Vice versa, when two stationary states exist, implying $\alpha + \gamma < 1$, then it follows from (11), by observing Fig. 1, that $det(J^*)$ has the same sign of $g'(E^*)$, which proves the theorem. □

Considering a non-generic case when a unique stationary state exists under the condition $\alpha + \gamma < 1$, then $det(J^*) = 0$ holds and the stationary state is not hyperbolic (in fact, a saddle-node bifurcation occurs). Consequently, if one looks for an attracting stationary state, that one have to restrict his/her analysis to the case when, under the assumption $\alpha + \gamma < 1$, two stationary state exist, P_1^* and P_2^*, with $E_1^* < E_2^*$ and $K_1^* < K_2^*$. Here the scientist aim to show that, in such a context, P_1^* can be attracting for suitable values of the parameters. Along the trajectories belonging to the basin of attraction of P_1^* the over-exploitation of the natural resource drives the economy towards a *tragedy of commons* scenario.

First of all, if $\alpha+\gamma < 1$, a necessary and sufficient condition for the existence of two stationary states is

$$\bar{E} > E_A := g(\tilde{E}) = \left(\frac{2 - 2\alpha - \gamma}{1 - \alpha - \gamma}\right)\tilde{E} \tag{12}$$

where $\tilde{E}$ is the only positive value satisfying $g'(\tilde{E}) = 0$.
Straightforward computations yield

$$E_A = (2 - 2\alpha - \gamma)\left[\frac{\delta^{1-\alpha}}{(1-\alpha)^{1-\alpha}(1-\alpha-\gamma)^{1-\alpha-\gamma}}\left(\frac{\beta}{\beta+\epsilon}\right)^{\beta}\left(\frac{\alpha}{\theta}\right)^{\alpha}\right]^{\frac{1}{2-2\alpha-\gamma}} \tag{13}$$

Thus, $E_1^* < \tilde{E} < (\frac{1-\alpha-\gamma}{2-2\alpha-\gamma})\bar{E}$. <u>From now the index 1 will be omitted</u>.

The well-known Routh–Hurwitz Criterion (see Hurwitz [9]) yields that J^*, the Jacobian matrix at P^*, has three eigenvalues with negative real part if and only if

$$det(J^*) < 0, \tag{14}$$

$$\begin{aligned} \sigma(J^*) &= \frac{\partial \dot{E}}{\partial E}\frac{\partial \dot{L}}{\partial L} - \frac{\partial \dot{E}}{\partial L}\frac{\partial \dot{L}}{\partial E} - \frac{\partial \dot{K}}{\partial L}\frac{\partial \dot{L}}{\partial K} > 0 \\ \rho(J^*) &= -\sigma(J^*) \cdot trace(J^*) + det(J^*) > 0 \end{aligned} \tag{15}$$

The last inequality, in particular, guarantees the non-existence of complex eigenvalues with non-negative real part. In fact, when $\rho(J^*)$ crosses the origin value 0, the real part of two complex conjugate eigenvalues changes sign, causing, generically, a Hopf bifurcation. Remember that the condition (14) is always verified at P^* (see (11)). As for the condition (15), the authors of the paper stated the following lemma.

Lemma 3. *If*

$$\eta \geq \frac{\epsilon}{\epsilon + \alpha\beta} \quad \textit{and} \quad \bar{E} > E_B = \frac{\theta(\beta+\epsilon)(2-2\alpha-\gamma)}{\alpha\beta\gamma\eta} \tag{16}$$

then the condition $\sigma(J^*) > 0$ is verified.

Proof. By recalling (10), straightforward computations lead to

$$sign[\sigma(J^*)] = sign\left[\left(\frac{\beta+\epsilon}{\epsilon} - \frac{1}{\eta}\right)(\bar{E} - 2E^*) - \frac{\theta(1-\alpha)(\beta+\epsilon)}{\alpha\epsilon\eta}\right]$$

So, since $E^* < (\frac{1-\alpha-\gamma}{2-2\alpha-\gamma})\bar{E}$, the assumptions of the lemma imply $\sigma(J^*) > 0$. □

One shall now compute $trace(J^*)$; observing that $\frac{f(L^*)}{f'(L^*)} = \frac{\beta\epsilon\eta}{(\beta+\epsilon)[\eta(\beta+\epsilon)-\beta\epsilon]}$, one obtain

$$trace(J^*) = a(\bar{E} - E^*) - E^* + b$$

where

$$a := \frac{\eta[(1-\gamma)(\beta+\epsilon) - \beta\gamma\epsilon] - \beta\epsilon(1-\gamma)}{\eta(\beta+\epsilon) - \beta\epsilon}, \qquad b := \frac{\beta\theta[\eta(\beta+\epsilon) - \epsilon]}{\eta(\beta+\epsilon) - \beta\epsilon}. \tag{17}$$

Then the results of the so far done analysis, aimed at detecting an attracting stationary state(or fixed point), are summarized by the following theorem.

Theorem 4. *Let $\alpha + \gamma < 1$ and $\bar{E} > E_A$, so that system (7) has two stationary states, P_1^* and P_2^*, with $E_1^* < E_2^*$ and $K_1^* < K_2^*$. Then there exist values of the parameters for which P_1^* is a sink, while P_2^* is a saddle with a two-dimensional stable manifold. Moreover, in such a case, take $\bar{E}$ as a bifurcation parameter. As $\bar{E}$ is increased, P_2^* does not change its nature (i.e. it remains a saddle with a two-dimensional stable manifold), whereas P_1^* can undergo one, two or no Hopf bifurcations.*

Proof. Omitted. See Appendix A of the usual paper [1]. □

4 Global analysis and Conclusion

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