

Optimal Parameter Identification for Discrete Mechanical Systems with Application to Flexible Object Manipulation

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Abstract—We present a method for system identification of flexible objects by measuring forces and displacement during interaction with a manipulating arm. We model the object’s structure and flexibility by a chain of rigid bodies connected by torsional springs. Unlike previous work, the proposed optimal control approach using variational integrators allows identification of closed loops, which include the robot arm itself. This allows using the resulting models for planning in configuration space of the robot. In order to solve the resulting problem efficiently, we develop a novel method for fast discrete-time adjoint-based gradient calculation. The feasibility of the approach is demonstrated using full physics simulation in trep and using data recorded from a 7-DOF series elastic robot arm.

I. INTRODUCTION

The goal of this work is to use a robotic arm to identify the behavior of a flexible object through touch only. This is an important step toward manipulation of flexible objects such as rubber tubes, plants or clothes [19], [16], [3], [7]. There are many methods to model and simulate flexible objects [11], [12]. A common approach is to model the object as a lattice or collection of links of masses and springs [17], [18], [11]. This approach has been used to simulate linear object like strings, hair, and electrical cables for which the model is a series of masses linked together with springs.

Our approach is similar with the primary difference that the loop connects back onto itself. This connection restricts the loops movement and is handled using holonomic constraints. To the best of our knowledge, no one has used a robot to identify parameters of a flexible loop for manipulation.

We are using a flexible loop as running example throughout the paper. We assume the robot has already rigidly grasped the object at one end and that it is clamped at the other. The robot then bends, twists, and stretches the loop. During the manipulation, the robot measures the arm’s joint torques and joint angles. With this information, it is possible to back out mechanical properties of the loop in order to generate an accurate model for future control and manipulation. In this paper, the manner in which we model the loop is so that the underlying mechanics of the loop are the same as the robot arm, i.e., a collection of rigid bodies connected by springs, allowing us to build up on the vast

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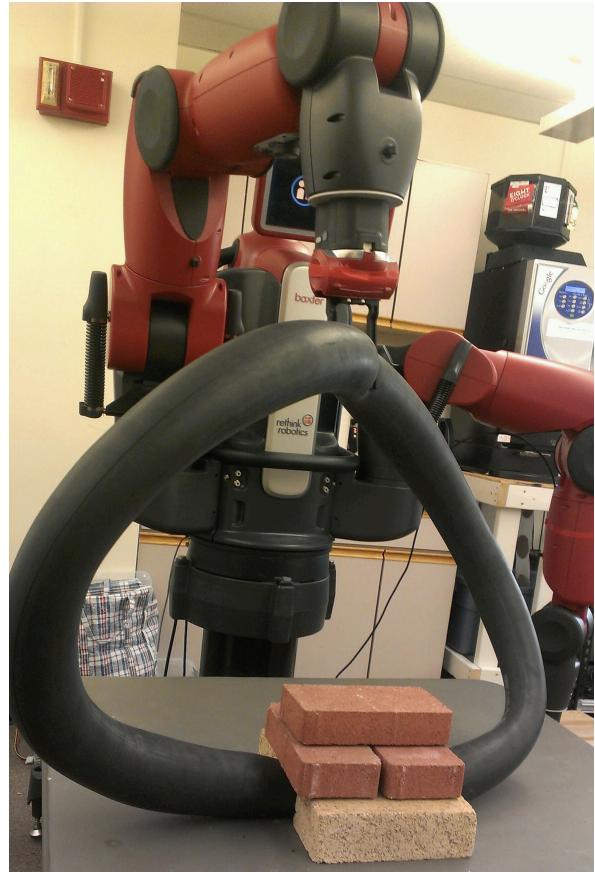


Fig. 1. Rethink Robotics’ Baxter manipulating an inflated bicycle tyre.

theory of rigid body mechanics is relevant [14]. Also, this enables planning and control to be done in the configuration space instead of the end effector or object space. We then use an optimal control approach for calculating model properties that best match the behavior of the physical loop.

We use Rethink Robotics’ Baxter [4] robot to both manipulate and measure the loop. Each of Baxter’s arms have 7 degrees of freedom. The arms are designed for compliance since each joint has series elastic actuators that allows for force sensing and control. An illustration of Baxter manipulating a 12 link approximation of a loop is in Figure 2 in Section IV.

A good representation of a flexible object is not enough to accurately model it. We also need the model simulation to be consistent with the physical behavior of the loop. We decided not to apply Euler integration or another low order integrator to the continuous dynamics, as is the case in [17], because

such integrators can introduce significant energy errors. At worst these errors will destabilize the integration and at best compromise the model's energy dissipation [8]. Instead we decided to use *variational integrators*. Variational integrators can be used to describe discrete-time equations of motion of a mechanical system. They are designed from the least action principle and have good properties that agree with known physical phenomenon like stable energy behavior [15].

Furthermore, variational integrators elegantly handle holonomic constraints. Holonomic constraints are specified as $h(q) = 0$, where q is the system's configuration. They are used to constrain positions and orientations. For the example, we use holonomic constraints to “close the loop”—i.e. to constrain the link at one end of the loop to the other end. In simulating continuous dynamics, holonomic constraints are handled with equivalent constraints on the acceleration that creep due to numerical integration error. In comparison, variational integrators apply holonomic constraints directly and do not have this issue (see [8] for a discussion).

Due to recent work by Johnson and Murphey [8], [9], it is possible to efficiently simulate mechanical systems using variational integrators in generalized coordinates. They provide a framework using a tree representation and caching that not only makes for efficient simulation—especially for articulated rigid bodies—but also efficient model-based calculations like linearizations about a trajectory. In optimal controls, linearizations are needed for gradient calculations like the gradient calculation presented in this paper.

With a model and simulation to predict the motion of a flexible object, unknown model quantities, referred to as parameters, can be identified. These model parameters may be lengths, masses, or damping coefficients. For the example, we assume the loop's stiffness is unknown. As such, the goal is to obtain the spring constants of the torsional springs at the loop model's joints. In this paper, this is done using an optimal control approach.

The parameter identification optimization problem is set up as a discrete-time Bolza problem. For the loop example, the cost functional is a summation of the error between the simulated end effector position and the experimentally-measured end effector position. The joint error directly correlates to the difference between the simulated loop displacement and the measured loop displacement, at least at the manipulator. Alternatively, the cost can be given by a maximum likelihood estimate for which the cost is lower for parameters that correspond to the simulation that is most consistent with the measurement (see [5]).

A. Contribution of this paper

The paper's contributions are twofold. First, it provides a discrete-time adjoint-based gradient calculation for optimal parameter estimation. Second, it formulates model parameter identification of flexible objects with variational integrators for greater confidence in the simulations' consistency with known physical properties.

B. Organization of this paper

This paper is organized as follows: Section II reviews continuous and discrete mechanical systems, discussing variational integrators and providing linearization calculations. Section III discusses the parameter identification optimization problem as well as provides an adjoint-based gradient calculation. Section IV provides the details of the physical experiment using Rethink Robotics' Baxter, in addition to specifics on the loop model and simulation. Moreover, it applies the parameter identification approach to estimate stiffness properties of the rubber loop model.

II. MECHANICAL SYSTEMS

In order to identify model parameters of a mechanical system, we must first be able to specify equations for simulation as well as model based calculations like linearizations. This section reviews continuous and discrete time mechanical systems, where the latter is with respect to variational integrators. A continuous mechanical system depends on its system Lagrangian which is the difference of its kinetic and potential energies. From the Lagrangian, the system's equations of motion can be derived from the Euler Lagrange or similar equations depending on whether the system's movement is constrained or is externally forced. The discrete mechanical system's equations of motion are similarly derived. For the experiment, these equations predict how the Baxter arm moves as well as how it manipulates and deforms the loop. These equations can only describe the deformation due to how we discretize the flexible behavior of the loop.

The mechanical system depends on n_ρ system parameters from the parameter space $\mathcal{P} \in \mathbb{R}^m$. The mechanical system has n_q generalized coordinates $q \in \mathbb{R}^{n_q}$. For continuous time, q varies with the continuous variable t —e.g. $q(t)$ —for t in the interval $[0, t_f]$, $t_f > 0$. Likewise, for the discrete representation, q varies with the discrete variable k —e.g. q_k —for k in the set $\mathcal{K} := \{0, 1, \dots, k_f\}$, $k_f > 0$. Each discrete time k pairs with a continuous time, labeled t_k , where $t_0 = 0$, $t_{k_f} = t_f$ and $t_k < t_{k+1}$. This section presents the continuous and discrete dynamics dependent on the parameters $\rho \in \mathcal{P}$. The continuous dynamics are provided for comparison with the discrete system, in which the paper results are given.

A. Continuous Mechanical System

We will first review continuous mechanical systems for reference with discrete mechanical systems. A mechanical system's evolution is given by the path of least action. The system's action is

$$S = \int_0^{t_f} L(q(\tau), \dot{q}(\tau), \rho) d\tau$$

where $L(q, \dot{q}, \rho) := KE(q, \dot{q}, \rho) - V(q, \rho)$, the system Lagrangian, is the difference of the system's kinetic energy, KE , with its potential energy, V . For this paper, we assume that both kinetic and potential energies depend on system

parameters $\rho \in \mathcal{P}$. Possible parameters of L include lengths, spring constants, and masses.

As is common in mechanical systems, we wish to include external forces, $F_c(q, \dot{q}, \rho, t)$. This term is the total external forcing in generalized coordinates, which we also assume depends on the parameters $\rho \in \mathcal{P}$, such as damping coefficients. By including the additional external force term F_c to the action, the Lagrange d'Alembert principle finds that the continuous dynamics of the mechanical system are given by the forced Euler-Lagrange equations [14]:

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}} L(q, \dot{q}, \rho) - \frac{\partial}{\partial q} L(q, \dot{q}, \rho) = F_c(q, \dot{q}, \rho, t).$$

Holonomic constraints can be enforced as external forces using Lagrange multipliers. The n_h holonomic constraint equations are given in the form $h(q, \rho) = [h_1, \dots, h_{n_h}]^T(q, \rho) = 0$. This is important, as it allows us to describe closed loops.

With the addition of the constraint, the forced Euler-Lagrange equations are [14]:

$$\begin{aligned} \frac{d}{dt} \frac{\partial}{\partial \dot{q}} L(q, \dot{q}, \rho) - \frac{\partial}{\partial q} L(q, \dot{q}, \rho) &= F_c(q, \dot{q}, \rho, t) + \frac{\partial}{\partial q} h^T(q, \rho) \lambda \\ \frac{\partial^2}{\partial q^2} h(q, \rho) \circ (\dot{q}, \ddot{q}) + \frac{\partial}{\partial q} h(q, \rho) \ddot{q} &= 0, \end{aligned} \quad (1)$$

where $\lambda(t) \in \mathbb{R}^{n_h}$ are Lagrange multipliers. For fixed parameters ρ , the system's evolution is integrated from Eq.(1) for q , \dot{q} and λ . It is worth noting that holonomic constraints are not enforced directly but instead an equivalent constraint is placed on the acceleration. During integration, numerical error can violate the constraint such that $h(q, \rho) \neq 0$, depending on the type of integrator.

Equation 1 can be transformed into first-order state space equations. Define the continuous state as $x = [q, \dot{q}]^T$. The state equations, dependent on the parameters $\rho \in \mathcal{P}$, are $\dot{x}(t) = f(x(t), \lambda(t), \rho, t)$ where \ddot{q} is specified by the forced Euler-Lagrange equations. In the state space representation, gradient and Hessian calculations with respect to parameters are given for parameter optimization in [13].

B. Discrete Mechanical System

The discrete mechanical system is an approximation of its continuous counterpart. For an initial configuration $q(0)$ and velocity $\dot{q}(0)$, the continuous configuration $q([0, t_f])$ is integrated from the forced Euler-Lagrange equations, Eq.(1). The discrete analog to the forced Euler-Lagrange equations instead calculates the sequence $q_k \approx q(t_k)$ using a variational integrator approach [8].

The discrete Lagrangian, labeled L_d , is an approximation of the action over a short time interval. Instead of a velocity term, the discrete Lagrangian is defined by the current and next configurations, q_k and q_{k+1} :

$$L_d(q_k, q_{k+1}, \rho) \approx \int_{t_k}^{t_{k+1}} L(q(\tau), \dot{q}(\tau), \rho) d\tau. \quad (2)$$

The integration can be approximated with a quadrature like midpoint or trapezoidal rules. Refer to [8] for details of using midpoint rule, which we use in the example.

Similarly, external forcing is included by approximating F_c with the discrete left and right forces

$F_d^-(q_k, q_{k+1}, \rho, t_k, t_{k+1})$ and $F_d^+(q_k, q_{k+1}, \rho, t_k, t_{k+1})$ using a quadrature. In addition, the n_h holonomic constraints $h(q_k, \rho)$ can be enforced with the n_h Lagrange multipliers λ_k . Given the discrete Lagrangian, discrete forces, and holonomic constraints the forced discrete Euler-Lagrange equations are [8]:¹

$$\begin{cases} D_2 L_d(q_{k-1}, q_k, \rho) + F_d^+(q_{k-1}, q_k, \rho, t_{k-1}, t_k) \\ \quad + D_1 L_d(q_k, q_{k+1}, \rho) + F_d^-(q_k, q_{k+1}, \rho, t_k, t_{k+1}) \\ \quad - D_1 h^T(q_k, \rho) \lambda_k = 0 \\ h(q_{k+1}, \rho) = 0. \end{cases} \quad (3)$$

These equations should be viewed as an implicit function on q_{k+1} . For example, given consecutive configurations q_0 and q_1 , the next configuration q_2 is found with a root solving operation on Eq.(3). Following, q_3 is obtained from q_1 and q_2 and so forth. Whereas the continuous mechanical system is solved using integration, the discrete mechanical system is solved through recursive calls to root finding Eq.(3).

As in [8], define p_k as

$$p_k := D_2 L_d(q_{k-1}, q_k, \rho) + F_d^+(q_{k-1}, q_k, \rho, t_{k-1}, t_k). \quad (4)$$

Without external forcing, p_k is the conserved momentum. The discrete state is $x_k := [q_k, p_k]^T$ which has a one-step mapping:

$$\begin{cases} x_{k+1} = f(x_k, \rho, t_k) := \\ \quad p_k + D_1 L_d(q_k, q_{k+1}, \rho) + F_d^-(q_k, q_{k+1}, \rho, t_k, t_{k+1}) \\ \quad - D_1 h^T(q_k, \rho) \lambda_k = 0 \\ h(q_{k+1}, \rho) = 0 \\ p_{k+1} = D_2 L_d(q_k, q_{k+1}, \rho) + F_d^+(q_k, q_{k+1}, \rho, t_k, t_{k+1}). \end{cases} \quad (5)$$

This equation is the state equation for discrete mechanical systems using variational integrators. The function $f(x_k, \rho, t_k)$ is implicit, but, according to the Implicit Function Theorem, it exists when

$$M_{k+1} := D_2 D_1 L_d(q_k, q_{k+1}, \rho) + D_2 F_d^-(q_k, q_{k+1}, \rho, t_k, t_{k+1})$$

is nonsingular. By assuming nonsingularity, even though f is implicit, the linearization around x_{k+1} —i.e. $\frac{\partial x_{k+1}}{\partial x_k}$ —is explicit. Letting $dx_k = [dq_k, dp_k]^T$, be the differential of x_k and da be the differential of ρ , the linearization of $f(x_k, \rho, t_k)$ is:

$$\begin{bmatrix} dq_{k+1} \\ dp_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial q_{k+1}}{\partial q_k} & \frac{\partial q_{k+1}}{\partial p_k} \\ \frac{\partial p_{k+1}}{\partial q_k} & \frac{\partial p_{k+1}}{\partial p_k} \end{bmatrix}}_{A_k} \begin{bmatrix} dq_k \\ dp_k \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial q_{k+1}}{\partial a} \\ \frac{\partial p_{k+1}}{\partial a} \end{bmatrix}}_{B_k} da \quad (6)$$

The calculations for the linearization term A_k is given in [9] and duplicated here for reference:

$$\begin{aligned} \frac{\partial q_{k+1}}{\partial q_k} &= \\ -M_{k+1}^{-1} [D_1^2 L_d(q_k, q_{k+1}, \rho) &+ D_1 F_d^-(q_k, q_{k+1}, \rho, t_k, t_{k+1}) \\ &- D_1^2 h^T(q_k, \rho) \lambda_k - D_1 h^T(q_k, \rho) \frac{\partial \lambda_k}{\partial q_k}] \end{aligned} \quad (7a)$$

¹The notation D_i slot derivatives of slot i . For example, $D_2 L_d(q_{k-1}, q_k, \rho)$ is the partial of L_d with respect to the second slot, q_k .

$$\frac{\partial q_{k+1}}{\partial p_k} = -M_{k+1}^{-1} \quad (7b)$$

$$\begin{aligned} \frac{\partial p_{k+1}}{\partial q_k} &= \\ [D_2^2 L_d(q_k, q_{k+1}, \rho) + D_2 F_d^+(q_k, q_{k+1}, \rho, t_k, t_{k+1})] \frac{\partial q_{k+1}}{\partial q_k} \\ &+ D_1 D_2 L_d(q_k, q_{k+1}, \rho) + D_1 F_d^+(q_k, q_{k+1}, \rho, t_k, t_{k+1}) \end{aligned} \quad (7c)$$

$$\begin{aligned} \frac{\partial p_{k+1}}{\partial p_k} &= \\ [D_2^2 L_d(q_k, q_{k+1}, \rho) + D_2 F_d^+(q_k, q_{k+1}, \rho, t_k, t_{k+1})] \frac{\partial q_{k+1}}{\partial p_k} \end{aligned} \quad (7d)$$

where $\frac{\partial \lambda}{\partial q_k}$ can be found in [9]. Notice the calculations for Eqs. (7c) and (7d) rely on the calculations for Eqs. (7a) and (7b) respectively. The B_k term is given by chain rule:

$$\begin{aligned} \frac{\partial q_{k+1}}{\partial a} &= \frac{\partial q_{k+1}}{\partial p_k} \frac{\partial p_k}{\partial a} + \frac{\partial q_{k+1}}{\partial q_k} \frac{\partial q_k}{\partial a} + M_{k+1}^{-1} [D_1 D_2 h^T(q_k, \rho) \lambda_k \\ &- D_3 D_1 L_d(q_k, q_{k+1}, \rho) - D_3 F_d^-(q_k, q_{k+1}, \rho, t_k, t_{k+1})] \end{aligned} \quad (8a)$$

$$\begin{aligned} \frac{\partial p_{k+1}}{\partial a} &= \\ [D_2^2 L_d(q_k, q_{k+1}, \rho) + D_2 F_d^+(q_k, q_{k+1}, \rho, t_k, t_{k+1})] \frac{\partial q_{k+1}}{\partial a} \\ &+ [D_1 D_2 L_d(q_k, q_{k+1}, \rho) + D_1 F_d^+(q_k, q_{k+1}, \rho, t_k, t_{k+1})] \frac{\partial q_k}{\partial a} \\ &+ D_3 D_2 L_d(q_k, q_{k+1}, \rho) + D_3 F_d^+(q_k, q_{k+1}, \rho, t_k, t_{k+1}) \end{aligned} \quad (8b)$$

The term B_k depends on A_k and the previous term B_{k-1} .

We use the equations of motion of discrete mechanical system, Eq.(5), as well as its linearization, Eqs.(7) and (8) for simulation and calculating the parameter identification gradient.

III. PARAMETER OPTIMIZATION

The goal of parameter optimization is to calculate the model parameters $\rho \in \mathcal{P}$ that minimize a cost functional. For the continuous problem, the cost functional is the integral of a running cost $\ell(x(t), \rho)$ plus a terminal cost $m(x(t_f), \rho)$:

Problem 1 (Continuous System Parameter Optimization): Calculate the parameters $\rho \in \mathcal{P}$ which solves:

$$\min_{\rho \in \mathcal{P}} \left[J(\rho) := \int_0^{t_f} \ell(x(t), \rho) dt + m(x(t_f), \rho) \right]$$

constrained to $\dot{x}(t) = f(x(t), \rho, t)$.

The gradient and Hessian of the continuous cost is given in [13]

For the discrete problem, it is reasonable to choose a discrete cost functional that approximates the continuous cost—i.e. $\ell_d(x_k, \rho) \approx \int_{t_{k-1}}^{t_k} \ell(x(\tau), \rho) d\tau$ and $m_d(x_{k_f}, \rho) \approx m(x(t_f), \rho)$. Alternatively, ℓ_d and m_d can be designed directly without first choosing an underlying continuous cost. The discrete parameter optimization problem is as follows:

Problem 2 (Discrete System Parameter Optimization):

Calculate the parameters $\rho \in \mathcal{P}$ which solves:

$$\min_{\rho \in \mathcal{P}} \left[J_d(\rho) := \sum_{k=1}^{k_f} \ell_d(x_k, \rho) + m_d(x_{k_f}, \rho) \right]$$

constrained to $x_{k+1} = f(x_k, \rho, t_k)$, Eq.(5).

In optimal control theory, it is common practice to solve optimization problems using iterative methods. Iterative optimization methods repeatedly reduce the cost by stepping in

a descending direction until a local optimum is found. Commonly, the step direction and step size is calculated using local derivative information [2], [10], which is practiced in this paper. In the next section, we provide an adjoint-based calculation for the gradient of the cost functional with respect to the parameters.

A. Discrete System Parameter Gradient

The gradient of the cost functional given in problem 2 is provided in the following lemma.

Lemma 1: Suppose $L_d(q_k, q_{k+1}, \rho)$, $F_d^-(q_k, q_{k+1}, \rho, t_k, t_{k+1})$, $F_d^+(q_k, q_{k+1}, \rho, t_k, t_{k+1})$, and $h(q_k, \rho)$ are C^2 with respect to q_k , q_{k+1} and ρ . Take A_k and B_k from Eq.(6) and assume M_k is always nonsingular. Then,²

$$DJ_d(\rho) = \sum_{k=1}^{k_f} \lambda_k B_{k-1} + D_2 \ell_d(x_k, \rho) + D_2 m_d(x_{k_f}, \rho) \quad (9)$$

where λ_k is the solution to the backward one-step mapping

$$\lambda_k = \lambda_{k+1} A_k + D_1 \ell_d(x_k, \rho) \quad (10)$$

starting from $\lambda_{k_f} = D_1 \ell(x_{k_f}, \rho) + D_1 m_d(x_{k_f}, \rho)$.

Proof: The derivative of the cost in the direction $\theta \in \mathbb{R}^{n_\rho}$ is

$$\begin{aligned} DJ_d(\rho) \theta &:= \sum_{k=1}^{k_f} D_1 \ell_d(x_k, \rho) \frac{\partial x_k}{\partial \theta} + D_2 \ell_d(x_k, \rho) \theta \\ &+ D_1 m_d(x_{k_f}, \rho) \frac{\partial x_{k_f}}{\partial \theta} + D_2 m_d(x_{k_f}, \rho) \theta. \end{aligned} \quad (11)$$

Label $z_k := \frac{\partial x_k}{\partial \theta}$ for convenience. Also for convenience, label

$$H := \sum_{k=1}^{k_f} D_1 \ell_d(x_k, \rho) z_k + D_1 m_d(x_{k_f}, \rho) z_{k_f} \quad (12)$$

The linearized state, z_k , is the solution to the linearized state equation, Eq.(6). In other words,

$$z_{k+1} = A_k z_k + B_k \theta$$

starting from $z_0 = 0$. The linearized state's solution depends on the discrete state transition matrix:

$$\Phi(k_2, k_1) := \prod_{j=1}^{k_2-k_1} A_{k_2-j} = A_{k_2-1} A_{k_2-2} \cdots A_{k_1}$$

for integers $k_2 > k_1$ and where $\Phi(k_1, k_1) := I$, the identity matrix. Recalling $z_0 = 0$, the linearized state's solution is

$$z_k = \Phi(k, 0) z_0 + \sum_{s=0}^{k-1} \Phi(k, s+1) B_s \theta = \sum_{s=0}^{k-1} \Phi(k, s+1) B_s \theta$$

for $k = 1, \dots, k_f$. Plugging z_k into H , Eq.(12), H becomes

$$\begin{aligned} H &= \sum_{k=1}^{k_f} D_1 \ell_d(x_k, \rho) \sum_{s=0}^{k-1} \Phi(k, s+1) B_s \theta \\ &+ D_1 m_d(x_{k_f}, \rho) \sum_{s=0}^{k_f-1} \Phi(k_f, s+1) B_s \theta \\ &= \sum_{k=1}^{k_f} \sum_{s=0}^{k-1} D_1 \ell_d(x_k, \rho) \Phi(k, s+1) B_s \theta \\ &+ \sum_{s=0}^{k_f-1} D_1 m_d(x_{k_f}, \rho) \Phi(k_f, s+1) B_s \theta \end{aligned}$$

²The notation D is a slot derivative for a function of only one slot. For example, $DJ_d(\rho)$ is the partial derivative of J_d with respect to ρ .

Switch the order of the double sum.

$$\begin{aligned} H &= \sum_{s=0}^{k_f-1} \sum_{k=s+1}^{k_f} D_1 \ell_d(x_k, \rho) \Phi(k, s+1) B_s \theta \\ &\quad + D_1 m_d(x_{k_f}, \rho) \sum_{s=0}^{k_f-1} \Phi(k_f, s+1) B_s \theta \\ &= \sum_{s=0}^{k_f-1} \left[\sum_{k=s+1}^{k_f} D_1 \ell_d(x_k, \rho) \Phi(k, s+1) \right. \\ &\quad \left. + D_1 m_d(x_{k_f}, \rho) \Phi(k_f, s+1) \right] B_s \theta \end{aligned}$$

Set λ_{s+1} as the resulting covector in the brackets so that

$$H = \sum_{s=0}^{k_f-1} \lambda_{s+1} B_s \theta = \sum_{k=1}^{k_f} \lambda_k B_{k-1} \theta.$$

The efficient calculation for λ_k is given in Eq.(10). Plugging H into Eq.(11), we find

$$DJ(\rho)\theta = \left[\sum_{k=1}^{k_f} \lambda_k B_{k-1} + D_2 \ell_d(x_k, \rho) + D_2 m_d(x_{k_f}, \rho) \right] \theta$$

and the proof is finished. ■

IV. EXPERIMENT

The goal is to identify the model parameters of a flexible loop that best match experimental measurements of a flexible loop with only touch at single point of contact. For the measuring and manipulating, we use Rethink Robotics' Baxter robot, shown in Figure 1. We simulate this interaction using a discrete model of the loop shown in Figure 2. The simulated arm joint angles and torques depend on the parameters set for a simulation. As such, we can find the parameters which correspond to simulated joint angles that best match the measured joint angles. For this experiment, the unknown parameters describe the stiffness of the loop model.

The tools for simulating Baxter manipulating a loop are given in Section II while the optimization technique for best matching the simulation to the measured data is described in Section III. The following provides the experimental setup, the model and simulation of Baxter manipulating the loop, and the results of optimally identifying loop parameters.

A. Experimental Setup

The experiment is set up as follows. Baxter's arm is positioned similarly to that in Figure 1. The top of an inflated rubber loop (bicycle tyre) is placed in Baxter's gripper and the bottom is clamped to the table. The loop is initially positioned vertically. The loop has a radius of 0.355 meters and a mass of 0.132 kilograms. The arm is run through a recorded motion that is designed to stretch, bend, and twist the tube. The arm's joint angles and torques are captured every 100Hz, which we compile into the two vectors $b_{meas}(t)$ and $T_{meas}(t)$. The measured joint angles have a position resolution of +/- 5 mm.

The measured joint angles b_{meas} correspond to a subset of the system configuration q , where the remaining configurations belong to the loop. We label this subset of q that describes the arm as b . The optimization goal is to choose model parameters so that $|b - b_{meas}|$ is small.

B. Model

We model Baxter and the loop with the same underlying rigid body mechanics. We model both as a series of rigid links connected by rotational joints. For Baxter, these joints are 7 series elastic actuators. The dimensions, inertia, and other information concerning Baxter's arm can be obtained at <https://github.com/RethinkRobotics>.

We use a discrete model of the loop. The loop is composed of 12 rigid links wherein the connection between each rigid link is a spherical joint. A single loop joint is shown in Figure 3. As seen in the figure, the orientation of the new link is given by the angles θ_i and ψ_i . These rotations are described on a frame that is rotated about the $X-axis$ by $5\pi/6$ radians so that when each $\psi_i = \theta_i = 0$, the loop approximation is a regular dodecagon. These angles, the θ_i and ψ_i , along with Baxter's joint angles, b , make up the system configuration q . Note that the choice of the discretization is arbitrary and only dependent on the required fidelity of the model as well as available computational time.

We represent Baxter grasping the loop using a tree structure of transformations. This tree structure is illustrated in Figure 2a). Starting from the world frame \mathcal{W} , the arm is specified with successive rotation and translation transformations. At Baxter's end effector, labeled H , the description branches, splitting the loop into two sides, marked with ℓ and r for left and right. The two branches meet at the base of the loop, where the loop is clamped. For arbitrary Baxter and loop joint angles, q , there is no guarantee that the two ends of the loop meet at the clamping location in space. When they do, we say that the system configuration satisfies the holonomic constraint $h(q) = 0$. There are 12 total constraints—i.e. $h(q) \in \mathbb{R}^{12}$. There are 6 constraints for each side of the loop transformation description in order to constrain the orientation and position of frames N_ℓ and N_r to the clamp location. Figure 2 shows three different configurations q that satisfy the constraints.

We model the stiffness of the loop with torsional springs on each loop configuration variable—i.e. for each θ_i and ψ_i . Due to the uniformity of the loop, we assume that all of the springs on the θ_i configuration variables have the same spring constant κ_θ . Likewise, each of the springs on ψ_i configuration variables have same spring constant κ_ψ . The objective is to identify these spring constant and as such we set $\rho = [\kappa_\theta, \kappa_\psi]^T$, where $\mathcal{P} = \mathbb{R}^+ \times \mathbb{R}^+$.

C. Simulation

In order to simulate the robot arm and loop, we turn to Section II which reviews variational integrators. The continuous state is given by q and its time derivatives—i.e. $x(t) = [q(t), \dot{q}(t)]^T$. These are the robot and loop joint angles and associated angular velocities. The system's discrete time state is $x_k = [q_k, p_k]^T$ where p_k is defined in Eq.(4). The continuous dynamics are given by the constrained, forced Euler Lagrange equations, Eq.(1), which depends on the system Lagrangian $L(q, \dot{q}, \rho)$, external forces $F_c(q, \dot{q}, \rho, t)$ and holonomic constraints $h(q, \rho)$. Techniques to derive these formulas for rigid bodies are well understood [14].

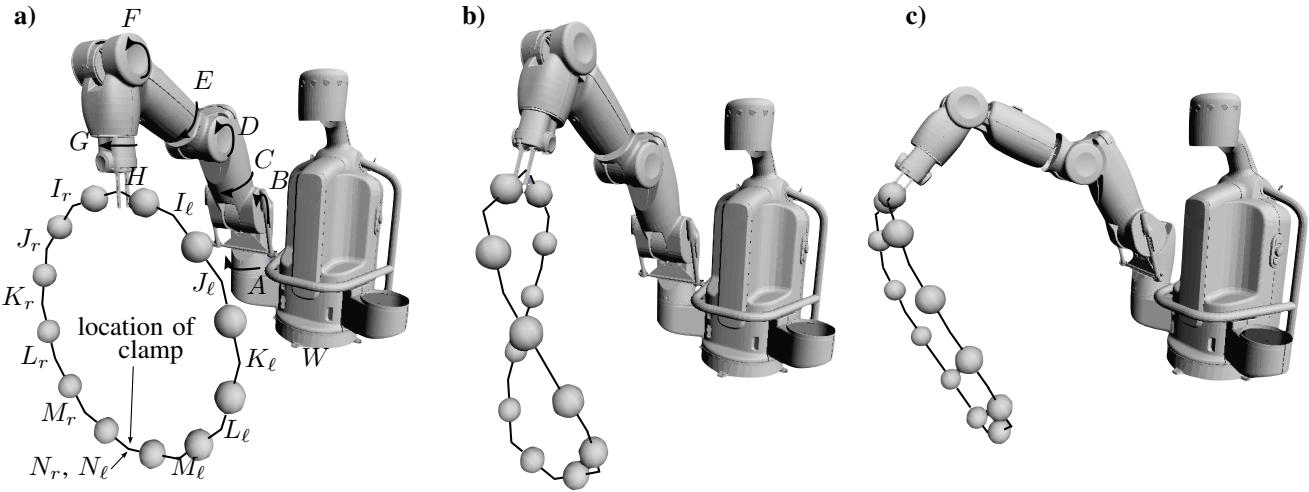


Fig. 2. Three distinct configurations. **a)** The frames for Baxter and the loop in their initial configuration. **b)** Baxter “twisting” the loop. **c)** Baxter “bending” the loop.

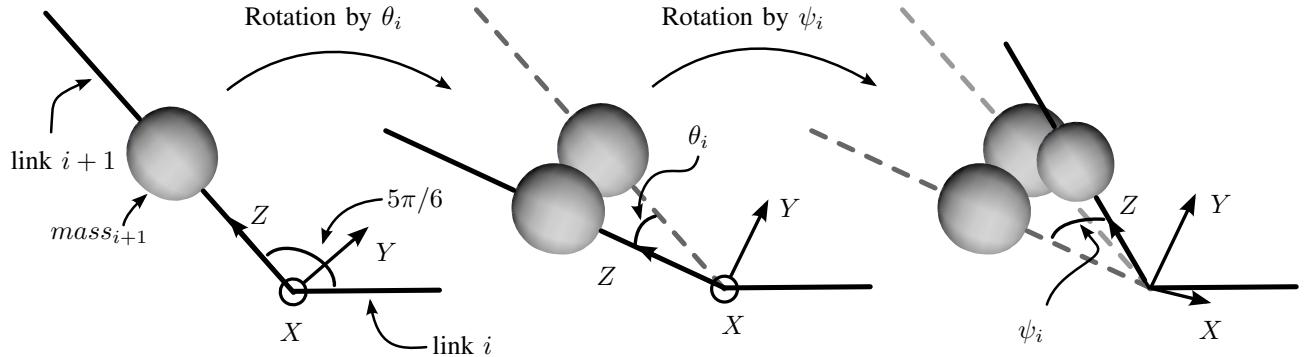


Fig. 3. Illustration of the spherical joint i connecting loop links i and $i + 1$. The joint is a rotation of θ_i radians about the X -axis followed by a rotation of ψ_i radians about the Y -axis.

From the continuous dynamics, it is straightforward to obtain the discrete dynamics, which are given in Eq.(3). The discrete dynamics depend on the discrete system Lagrangian $L_d(q_k, q_{k+1}, \rho)$, discrete external left and right forces, $F_d^-(q_k, q_{k+1}, \rho, t_k, t_{k+1})$ and $F_d^+(q_k, q_{k+1}, \rho, t_k, t_{k+1})$, and holonomic constraints $h(q_k, \rho)$.

For simulation, we chose a constant time step of $\Delta_t = 0.01$ seconds, which matches the broadcast frequency of Baxter. Further, we decided to approximate L_d from L using midpoint rule (see Eq.(2)):

$$L_d(q_k, q_{k+1}, \rho) = \Delta_t L\left(\frac{q_{k+1} + q_k}{2}, \frac{q_{k+1} - q_k}{\Delta_t}, \rho\right).$$

Similarly, using midpoint rule, we approximate F_c by F_d^- and F_d^+ where

$$\begin{cases} F_d^-(q_k, q_{k+1}, \rho, t_k, t_{k+1}) = \\ \Delta_t F_c\left(\frac{q_{k+1} + q_k}{2}, \frac{q_{k+1} - q_k}{\Delta_t}, \rho, \frac{t_{k+1} + t_k}{2}\right) \\ F_d^+(q_k, q_{k+1}, \rho, t_k, t_{k+1}) = 0. \end{cases}$$

It is a simple process to translate the continuous system to the discrete variational integrator one. Finally, in order to simulate the dynamics using the one-step mapping in Eq.(5),

we need certain partial derivatives of the Lagrangian, external forces and constraints with respect to configuration variables, which can be found in [8].

The external forces F_c are applied at the configuration variables. For Baxter manipulating the loop, the only external forces are those applied at the 7 configuration variables that define Baxter’s arm, which we have labeled as b . Due to model and sensor error, it is not reasonable to expect that the simulation will provide meaningful results, let alone be stable by directly setting $F_c = T_{meas}$, the experimentally measured joint torques. Instead, we decided to filter them using a proportional feedback loop as so:

$$F_c(t) = T_{meas}(t) - K(t)(b(t) - b_{meas}(t)),$$

where b_{meas} is Baxter’s measured joint angles as mentioned in Section IV-A and K is a feedback gain. When K is chosen correctly, the simulation is stable. We chose K from a finite horizon LQR to calculate an optimal feedback gain from the model linearized around b_{meas} and a quadratic cost functional [1].

Aside: We used the software tool trep [8] which simu-

lates articulated rigid bodies using midpoint variational integrators. It additionally provides partial derivative calculations that we need for the system linearization, Eqs.(7) and (8).

D. Linearization

The linearization of the discrete equations of motion is given by matrices A_k and B_k in Eqs.(7) and (8). We need the linearization for the gradient calculation, Lemma 1, in order to perform a gradient-based descent algorithm like steepest descent for parameter identification. Partial derivatives of L_d and F^+ with respect to q_k and p_k can be obtained from [9]. In this section, we only concern ourselves with the partials that depend on the parameters $\rho = [\kappa_\theta, \kappa_\psi]^T$ which are not included in [9].

For the loop example, we need to calculate $D_3D_1L_d(q_k, q_{k+1}, \rho)$ and $D_3D_1V_d(q_k, q_{k+1}, \rho)$ for $\rho = [\kappa_\theta, \kappa_\psi]$. Note that the potential energy of the system can be written as:

$$V(q, \rho) = V_\theta(q, \kappa_\theta) + V_\psi(q, \kappa_\psi) + V_g(q)$$

where V_θ , V_ψ and V_g are the potential energies due to the spring torques on the θ_i configuration variables, the spring torques on the ψ_i configuration variables, and gravity, respectively. Label I_θ and I_ψ as the index of the θ_i and ψ_i configuration variables in q respectively. The potential energy due to the θ_i torsional springs is $V_\theta(q, \kappa_\theta) = \sum_{i \in I_\theta} \frac{1}{2} \kappa_\theta \theta_i^2$. Approximating for the discrete time potential energy—see Eq.(2)—we find that ³

$$V_{\theta,d}(q_k, q_{k+1}, \kappa_\theta) = \sum_{i \in I_\theta} \frac{\Delta_t}{2} \kappa_\theta \left(\frac{q_{i,k+1} + q_{i,k}}{2} \right)^2.$$

Taking the needed partial derivatives to calculate $D_3D_1L_d$, the i^{th} element of $D_3D_1V_{\theta,d}$ is

$$D_3D_1V_{\theta,d}(q_k, q_{k+1}, \kappa_\theta)_i = \begin{cases} \frac{\Delta_t}{4} (q_{i,k+1} + q_{i,k}), & i \in I_\theta \\ 0 & \text{else.} \end{cases}$$

Similarly, for the needed partial derivatives of $V_{\psi,d}$,

$$D_3D_1V_{\psi,d}(q_k, q_{k+1}, \kappa_\psi)_i = \begin{cases} \frac{\Delta_t}{4} (q_{i,k+1} + q_{i,k}), & i \in I_\psi \\ 0 & \text{else.} \end{cases}$$

Since the kinetic energy does not depend on $\rho = [\kappa_\theta, \kappa_\psi]$,

$$\begin{aligned} D_3D_1L_d(q_k, q_{k+1}, \rho) = \\ -[D_3D_1V_{\theta,d}(q_k, q_{k+1}, \kappa_\theta), D_3D_1V_{\psi,d}(q_k, q_{k+1}, \kappa_\psi)]. \end{aligned}$$

Repeating the derivation for $D_3D_2L_d(q_k, q_{k+1}, \rho)$ we find that $D_3D_2L_d(q_k, q_{k+1}, \rho) = D_3D_1L_d(q_k, q_{k+1}, \rho)$.

Furthermore, we need to calculate $D_1h(q_k, \rho)$, $D_1^2h(q_k, \rho)$ and $D_1D_2h(q_k, \rho)$, the last of which is 0 since the constraints do not depend on the parameters. These partial derivatives of h are given simply by the chain rule and depend on the first and second partial derivatives of the transformations from the world frame W to frames N_r and N_ℓ (refer to Figure 2 for the frames). These partial derivatives can be found in [9].

³Here, we (poorly) index the i^{th} term of q_k as $q_{i,k}$.

E. Optimal Parameter Identification

With the linearization of the discrete state equations, Eq.(5), we can calculate the gradient of a cost from Lemma 1 in a steepest descent algorithm to identify the model parameters $\rho = [\kappa_\theta, \kappa_\psi]^T$. The experiment is set up as in Section IV-A. We program the experimental Baxter arm to stretch, bend and twist the loop over a 20 second time frame. The joint torques to displace the loop are recorded in T_{meas} while joint angles are recorded in b_{meas} . For fixed parameters ρ , the system can be simulated as discussed in Section II using T_{meas} . We wish to find the simulated Baxter arm joint angles, labelled b —recall b are the configuration variables of q that describe the Baxter arm—that best match b_{meas} .

The matching is quantified by the cost J_d . We choose J_d to be quadratic on the error from the simulated end effector position in space to the measured end effector position, which can be derived from b and b_{meas} respectively. Label $w_k(\rho)$ as this simulated position at time t_k for parameters ρ and $w_{\text{meas}}(t_k)$ as this measured position. Set $\epsilon_k := w_k(\rho) - w_{\text{meas}}(t_k)$. The cost J_d is defined by the running cost $\ell_d(q_k, \rho)$ and the terminal cost $m_d(q_{k_f}, \rho)$, set as

$$\ell_d(q_k, \rho) = \epsilon_k^T \epsilon_k \text{ and } m_d(q_{k_f}, \rho) = \epsilon_{k_f}^T \epsilon_{k_f}$$

We perform the optimization using a steepest descent algorithm with the inequality constraint that the parameters are $\kappa_\theta > 0$ and $\kappa_\psi > 0$. At each iteration of the descent, an Armijo line search updates the parameters by choosing a distance to step in the direction of the negative gradient. We used Armijo parameters $\alpha = \beta = 0.4$ [2].

We seed the steepest descent algorithm with an initial guess of $\rho = [5, 5]^T$. After 75 iterations, the algorithm terminated with gradient norm $DJ_d(\rho) = 0.001138$. The parameters were identified as $\rho^* = [4.4525252882477062, 0.96969286047029135]^T$. The convergence is shown in Fig. 4. Furthermore, a comparison of the simulated end effector’s path for each iteration of steepest descent with Baxter’s measured end effector’s path is in Fig. 5. In agreement that the cost decreased with iteration, it appears in the figure that the end effector’s path converged toward the measured end effector’s path.

For the example, optimally identifying a 12-link (31 configurations total) model from 20 seconds of manipulation activity (2000 data points recorded at 100Hz) takes 219.05 minutes on a Macbook air.

V. DISCUSSION

Through the proposed parameter identification procedure, we calculated the model parameters that best match physical phenomena within the constraint of the chosen model. This process is important since an improved model can make for better object manipulation. However, it is unclear, especially in the presented experiment, what this matching tells us about the object’s physical properties. Certainly, measuring at a single contact point provides little insight into the interior stress and strain of the rubber loop.

Furthermore, we assumed the loop has uniform stiffness. Without this assumption, multiple experiments would

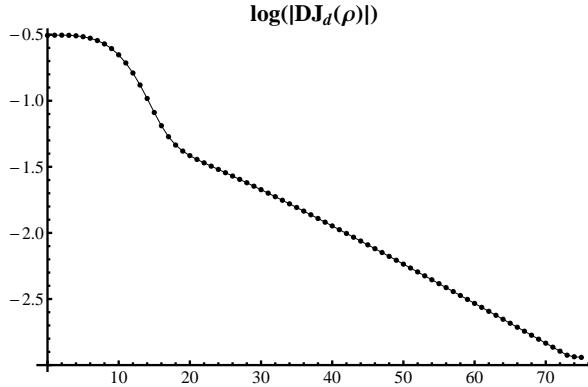


Fig. 4. Convergence of the optimization algorithm.

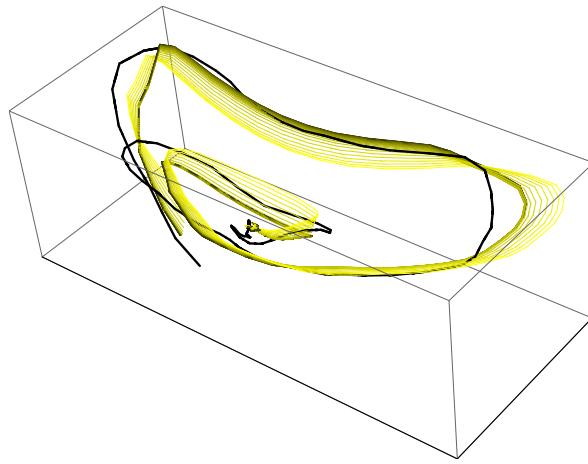


Fig. 5. The path the robot's end effector took through space. The black line is the measured end effector's path while the yellow lines are the end effector's simulated path for each iteration of the steepest descent algorithm. The iteration numbers are ordered from lighter yellow to darker yellow. It appears that the later iterations' paths are converging toward the measured path.

be needed at different contact points to identify the non-uniformity. Also, objects with more complex geometries would require additional experiments. For instance, grasping and manipulating a single leaf of a tomato plant would provide little insight for identifying the stiffness of the full plant. Multiple experiments as well as algorithms to determine appropriate actions is a topic of future work.

As the parameter identification process is informed from actual manipulation, the proposed method could also be used online, thereby improving a model estimate with the time the robot spent with an object growing. We note that identification in this paper relies exclusively on proprioceptive data that are compared with model prediction. In the future, we plan to explore the following directions: (1) using proprioception of a second arm to validate hypotheses on object movement and obtaining additional measurements and constraints, and (2) combine this data with exteroception such as cameras, 3D data, and dynamic tactile sensing [6].

Finally, we note that the fidelity of the approach heavily depends on the choice of the model. While this choice strongly depends on the desired manipulation task, we are

also interested in automatically finding appropriate model representations for given geometries and conceptual knowledge on the object, such as “plant”, “tube” or “sheet”, e.g.

VI. CONCLUSION

We provided a method for optimally identifying the model parameters of a flexible object manipulated by a robotic arm and applied it to identifying stiffness characteristics of a flexible loop. The proposed model exclusively relies on proprioception, thereby not requiring any additional sensors. We modeled the loop as a chain of rigid links connected by torsional springs and used variational integrators to simulate it. The simulation accounted for the chain of links being closed to form the loop. Additionally, since we modeled the loop with the same underlying mechanics as the robotic arm, the full system can be simulated together, allowing for planning in the configuration space. The feasibility of the approach was demonstrated using the variational integrator simulating software `treep` and with data recorded from a 7-DOF series elastic robot arm. We have identified a series of open challenges, including using multiple arms and external sensors to increase model performance, as well as identifying not only model parameters, but also the model structure itself.

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