MacPherson-Chern-Classes

21.11.22

Overview:
1. What is it?
2. Why do we use this language?
3. How do we construct it?

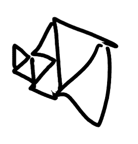
1. Setting

X, Y, V_i, V = (possibly singular) complex alg. varieties

$f: X \to Y$ is a proper morphism of alg varieties

H_*^{BM} = Borel-Moore-Homology = Homology of the chain complex of (poss. infinite) sing. chains $\sum_\sigma c_\sigma \sigma$ (σ sing. simplex) s.t. for any $K \subseteq X$ compact: $c_\sigma \neq 0$ only for a finite # of the σ's with $\sigma^{-1}(K) \neq \emptyset$

Some properties:
1. Covariant wrt. proper maps $f: X \to Y$: $f_*: H_*^{BM}(X) \to H_*^{BM}(Y)$
2. Contravariant wrt. open inclusions:
   $U \hookrightarrow X \rightsquigarrow H_*^{BM}(X) \to H_*^{BM}(U)$
   This gives a localization sequence:
   For X loc compact, $F \subseteq X$ closed, $U := X \setminus F$ open
   $$\cdots \to H_i^{BM}(F) \xrightarrow{1} H_i^{BM}(X) \xrightarrow{2} H_i^{BM}(U) \to H_{i-1}^{BM}(F) \to \cdots$$
   Application: If X sing variety $\Rightarrow$ $X_{reg} \subseteq X$ smooth, X_{sing} has codim ≥ 2
   $$\Rightarrow \quad \to 0 = H_n^{BM}(X_{sing}) \to H_n^{BM}(X) \xrightarrow{\cong} H_n^{BM}(X_{reg}) \to 0 \to$$
   $$[X] \mapsto [X_{reg}]$$



   Q: For $X = \mathbb{R}^2$, $X_{reg} = \mathbb{R}^2 \setminus \{0\}$ how is $[X]$ represented if one uses an infinite oriented Δ-ion of $\mathbb{R}^2 - \{0\}$ to define $[X_{reg}]$?
3. For any smooth/PL-manifold M^n, one has a fundamental class $[M] \in H_n^{BM}(M)$ and Poincaré-duality
   $$H^i(M) \xrightarrow{\cong} H_{n-i}^{BM}(M)$$
   reg cohom
   $$\varphi \mapsto \varphi \cap [M]$$

MacPherson Chern-class: $c_*: CF(X) \to H_*^{BM}(X)$, $\alpha \mapsto c_*(\alpha)$ s.t.

(1) $f_* c_*(\alpha) = c_* f_*(\alpha)$ NATURALITY

(2) $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$ ADDITIVITY

(3) X = smooth $\Rightarrow$ $c_*(\mathbb{1}_X) = c^*(TX) \cap [X]$ NORMALIZATION

2. Constructible Functions

- $\alpha: X \to \mathbb{C}$ is called constructible, if there is a decomposition of X into locally closed subvarities $\{V_i\}_{1 \leq i \leq k}$: $\alpha = \sum_{i=1}^k n_i \mathbb{1}_{V_i}$, $n_i \in \mathbb{C}$, $\mathbb{1}_{V_i}(x) = \begin{cases} 0 & x \in V_i^c \\ 1 & x \in V_i \end{cases}$

Example $V = \{x^3 - y^2 = 0\} \subseteq \mathbb{C}^2 = X$

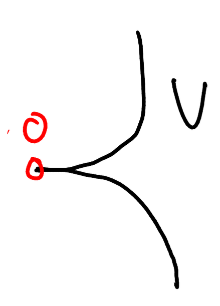
$$\alpha = 3 \cdot \mathbb{1}_{V \setminus \{0\}} + 2 \cdot \mathbb{1}_{\{0\}}$$
$$3 \cdot \mathbb{1}_V - \mathbb{1}_{\{0\}} \in CF(X)$$

- $CF(X)$ is a ring with pointwise $+$ & $\cdot$
- For $f: X \to Y$ proper, $W \subseteq X$ locally closed subvar. we define
  $$f_*(\mathbb{1}_W)(p) := \chi(f^{-1}(p) \cap W), \quad p \in Y$$

  Claim: This is a constructible function.

THM: $\exists!\ c_*$

Part 1: Uniqueness of c_*



Example $\alpha = 3\, \mathbb{1}_V - \mathbb{1}_0$

$V \neq$ smooth

take blowup/resolution $p: \tilde{V} \to V$ proper. $p_* \mathbb{1}_{\tilde{V}} = \mathbb{1}_V$

$\Rightarrow c_* \mathbb{1}_{\tilde{V}} = c^*(\tilde{V}) \cap [\tilde{V}]$ ($\tilde{V}$ smooth) $\Rightarrow c_* \mathbb{1}_V = p_* c_* \mathbb{1}_{\tilde{V}}$

$\Rightarrow c_* \alpha = 3\, p_* c_* \mathbb{1}_{\tilde{V}} - c_* \mathbb{1}_0$

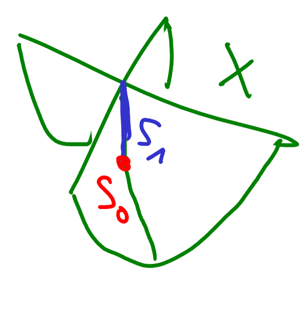
General: By induction $X = \bigcup_{i=1}^N V_i$, $\alpha = \sum n_i \mathbb{1}_{V_i}$

$k-1 \to k$ Ind: Let $W_k \subseteq \mathrm{supp}\left(\alpha - \sum_{i=1}^{k-1} n_i g_{i*} \mathbb{1}_{\tilde{V}_i}\right)$ an irr. subvariety of max dimension

$g_i: \tilde{V}_i \to V_i$ is a res. of sing.

Then take resolution $g_k: \tilde{V}_k \to W_k$ and replace $\alpha - \sum_{i=1}^{k-1} n_i g_{i*} \mathbb{1}_{\tilde{V}_i}$ by $\alpha - \sum_{i=1}^{k} n_i g_{i*} \mathbb{1}_{\tilde{V}_i}$ $\Rightarrow$ $\alpha = \sum_{i=1}^{n} n_i g_{i*} \mathbb{1}_{\tilde{V}_i}$

Ex Whitney-umbrella



$$\alpha = \mathbb{1}_X + 2\, \mathbb{1}_{\bar{S}_1} + 3 \cdot \mathbb{1}_{S_0}$$

$V_1 = X$ $\quad \alpha = p_{1*} \mathbb{1}_{\tilde{X}} + \ldots$

$\Rightarrow \mathrm{supp}(\alpha - p_{1*} \mathbb{1}_{\tilde{X}}) = \bar{S}_1$

PART 2 : Existence of c_*

Construct: $c_* \alpha = c_M(\overbrace{T^{-1}\alpha}^{\text{alg. cycle ass. with } \alpha})$

$\uparrow$ Chern-Mather-class of alg. cycle

For the proof of naturality, it suffices to prove naturality for

$f : X \xrightarrow{\text{proper}} Y$ with X smooth and $\alpha = 1\!\!1_X$

Argument: Let now $f: W \to Y$ proper, $\alpha \in CF(W)$

$$f_* c_*(\alpha) = \sum n_i f_* c_* g_{i*}(1\!\!1_{\tilde{V}_i})$$

$\uparrow$

$\sum n_i g_{i*}(1\!\!1_{\tilde{V}_i})$

resolution maps from before; smooth

$g_i : \tilde{V}_i \to W_i$

$$= \sum n_i \underbrace{f_* g_{i*}}_{\underbrace{(f\circ g_i)_*}_{\tilde{V}_i \to Y}} c_*(1\!\!1_{\tilde{V}_i}) = \sum n_i c_* (f\circ g_i)_* (1\!\!1_{\tilde{V}_i})$$

$$= c_* f_*(\alpha)$$

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3. Definition of C_M

An alg. cycle on X is a formal sum $\sum n_i V_i$ with $V_i \subseteq X$ irr. subvariety

(1) Define $c_M(V_i)$

(2) $c_M(\sum n_i V_i) := \sum n_i c_M(V_i)$ Linear extension

Def of c_M

Let $\underset{\text{smooth}}{V_{reg}} \subseteq V^{v} \subseteq N \leftarrow$ non-sing (Assumption)

$G_v(TN) = \{ (p, W_p) : W_p \leq T_pN, \dim W = v \}$ Grassmannian bundle over V

$s: V_{reg} \xrightarrow{\text{section}} G_v(TN)$

$p \mapsto (p, T_pV_{reg})$

$$\begin{tikzcd}
\hat{V} = \overline{\mathrm{im}(s)} \arrow[drr, "\nu"'] & \subseteq & G_v(TN|_V) \arrow[r] \arrow[d] & G_v(TN) \arrow[d] \\
 & & V \arrow[r] & N
\end{tikzcd}$$

Nash-blowup

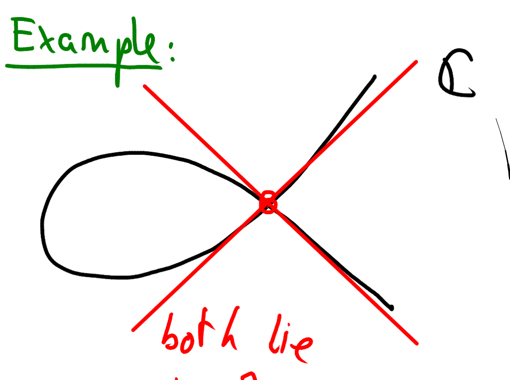
Tautological bundle: $\{ (U_p, x_p) \in G_v(TN) \times TN : x_p \in U_p \} \subseteq G_v(TN) \times TN$

$\downarrow$ $G_v(TN)$; $\mapsto U_p$

$TV := \{ (U_p, x_p) \in \hat{V} \times TN : x_p \in U_p \} \to \hat{V}$

$(U_p, x_p) \mapsto U_p$

Example:



<u>Definition:</u> $c_M(V) := \nu_*(\underbrace{c^*(TV)}_{\in H^*(\hat{V})} \cap [\hat{V}])$