1 Characteristic classes on singular spaces – continued

After our introductory talks during the past semester, we would like to continue our seminar on Characteristic classes on singular varieties. The tentative schedule is the following.

1.1 Part 1: Characteristic classes for singular varieties

To start the semester, we want to understand MacPherson's solution of the Grothendieck-Deligne conjecture. In particular, we want to get a better feeling of the local Euler obstruction and MacPherson's graph conjecture. The talks are more or less organized as follows.

1. *Preliminaries, Nash-blowup, Mather class, local Euler obstruction*

   Explain the "Grothendieck-Deligne conjecture" and compare it to char. classes on manifolds. Introduce the Nash blowup and Nash bundle. Define the local Euler obstruction. In the best case, show a good example of a sing. variety embedded in some smooth variety. (E.g. Whitney umbrella) (Mostly 1. – 4. of [13]). Note, that there alternative, algebraic approachs to the local Euler obstruction by Gonzalez-Sprinberg, [8], which is (partly) translated to English by Jiang, [11]. See also: [12].

2. Explain the graph construction and how it helps to prove the MacPherson-Chern class theorem. (Chapters 5. – 7. of MacPherson).

To this end, we propose to start with a reading of [13] with an emphasis on constructing *explicit examples* for the theory developed.

1.2 Part 2: Riemann-Roch type theorems in singularity theory

Consider the classical Riemann-Roch theorem for a line bundle $\mathcal{L}$ on a smooth projective curve C:

$$h^0(C,\mathcal{L}) - h^0(C,\mathcal{L}^\vee \otimes \omega_C) = \deg(\mathcal{L}) + 1 - g. \tag{1}$$

Using Serre duality, one can rewrite the left hand side as

$$h^0(C,\mathcal{L}) - h^0(C,\mathcal{L}^\vee \otimes \omega_C) = h^0(C,\mathcal{L}) - h^1(C,\mathcal{L}) = \chi(\mathcal{L})$$

the *holomorphic Euler characteristic* of $\mathcal{L}$. On the right hand side, we find a purely *topological* invariant given by the degree

$$\deg(\mathcal{L}) = \int_C c_1(\mathcal{L}) = \int_C \mathrm{Eu}(\mathcal{L}),$$

which is nothing but the integral over the Euler class, and a correction term given by the *genus* g of the curve. In this spirit, whenever we speak of a "Riemann-Roch type theorem", we mean a theorem that does exactly this: Express a topological invariant as a holomorphic Euler characteristic.

Where does this occur in Singularity theory? – One example is Milnor's formula for an isolated hypersurface singularity $f\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$:

$$\mu(f) = \dim_{\mathbb{C}} \mathcal{O}_{n+1}/\mathrm{Jac}(f).$$

The left hand side is the topological obstruction to extending the 1-form $\mathrm{d}f$ from a small sphere $S^{2n+1}_\varepsilon \subset \mathbb{C}^{n+1}$ around the origin as a nowhere vanishing section to its interior. The right hand side can be interpreted as the holomorphic Euler characteristic of the *Koszul-complex* $(\Omega^*_{\mathbb{C}^{n+1},0}, \mathrm{d}f \wedge -)$ induced by the differential of f.

Why is this interesting? – Topological invariants are notoriously hard to compute. For algebraic invariants, such as holomorphic Euler characteristics, we do have a chance using computer algebra systems. Thus, whenever we can express a topological invariant in terms of the latter, we have a chance to actually get our hands on it. As a tentative goal for this seminar I propose that we reach a position in which we can effectively solve the following problem:
"Given $f \in \mathbb{Q}[x_0,\dots,x_n]$, compute a Whitney stratification for the variety $X = V(f)$ and the constructible function $\mathrm{E}u$ given by the local Euler obstruction along the strata. "

As a starting point for the reading we propose [3] to first look into the following questions:

- What is the K-group $K^0(X)$ of a topological space X?
- What is the Grothendieck group of coherent sheaves $K_0^{\mathrm{alg}}(X)$? How is it covariant? How does this relate to the derived category of X?
- What is its topological analogue $K_0^{\mathrm{top}}(X)$?
- How are the Chern characters defined in either theory? Why do the obvious diagrams commute?
- How do the classical Riemann-Roch theorems follow from this?

After this, we turn towards the local theory. Gonzalez-Sprinberg has given an algebraic formula for the local Euler obstruction via an involved blowup procedure [8] that we could start with. I propose to go through the proof and see how things can be rephrased so that one could possibly obtain another algebraic formula without the blowups – possibly through some projection formula. (Note, that there is a (partial) English translation of the Gonzales-Sprinberg paper by Jiang in [11].)

As a tentative direction, I would like to go towards algebraic residues as described in [9, Chapter 5]. To explain this, recall from [9] that the multidimensional residue of a rational $n+1$-form

$$\omega = \frac{h(z)\mathrm{d}z_0 \wedge \cdots \wedge \mathrm{d}z_n}{f_0 \cdots f_n}$$

for some regular sequence $f_0,\dots,f_n \in \mathcal{O}_{n+1}$ can be transformed into a sphere integral:

$$\frac{1}{(2\pi\sqrt{-1})^{n+1}} \int_{|f_0|=r_0} \cdots \int_{|f_n|=r_n} \omega = \int_{S^{2n+1}_\varepsilon} \eta_\omega$$

where η_ω is the *dolbeault representative* of ω. While the left hand side can be evaluated algebraically, at least for algebraic input, the right hand side has the form of a local topological obstruction: It could arise as the degree of a map of spheres. In fact, this leads to a residue formula for the Milnor number, see [6].

In the case where $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ is the affine cone of a projective variety $Y \subset \mathbb{P}^n$, we can in fact compute the local Euler obstruction of $(X, 0)$ at the origin via an algebraic residue calculus. The hope is to find an "easy" algebraic formula for the local Euler obstruction which extends this result in general.

1.3 References

- [15]: Rather broad book on characteristic classes of real and complex vector bundles.
- [10]: Touches upon obstruction theory.
- [5]: Treats, for instance, Characteristic classes with values in smooth Cech-de-Rham cohomology in an elementary way.
- [7]: The reference for Chern classes and intersection theory of algebraic schemes; mostly using Chow groups.
- [3] and [4]: Discusses Riemann-Roch-type theorems for both algebraic and topological K-theories.
- [1]: Covers the case of complex analytic vector bundles on projective manifolds using Hodge theory.
- [2] Book on intersection homology, that includes L-classes for manifolds and singular spaces and explain, why they are important wrt surgery theory. Tends to be a bit technical, but in particular the part about the motivation in Chapter 5 is good.
- [14] Another book on intersection homology, focussing on the sheaf theoretic approach. Also covers L-classes, but in a bit more streamlined (less detailed and exact way). I recommend it.

References

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