THE CHERN-MATHER UNNATURAL TRANSFORMATION

$$Z_* \overset{c_M}{\twoheadrightarrow} H_*$$

(***)

$$\begin{tikzcd}
Z_*X \arrow[r, "c_{M,X}"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\cancel{\circlearrowleft}"] & H_*X \arrow[d, "f_*"] \\
Z_*Y \arrow[r, "c_{M,Y}"'] & H_*Y
\end{tikzcd}$$

(***) $\quad Z_*X \xrightarrow{c_{M,X}} H_*X$

$$V \longmapsto incl_*\,\underline{\nu_*\left(c(\tilde{T}_V)\cap[\tilde{V}]\right)}$$

$= c(V)\cap[V]$ if V smooth

$$\tilde{V} \subseteq G(M)$$

image of obvious section on regular part of V

Grassmanian dim V subspaces of tangent of ambient space of V

$$\begin{tikzcd}
\tilde{V} \arrow[r, hook] \arrow[d, "\nu"'] & G(M) \arrow[d] \\
V \arrow[r, hook] & M
\end{tikzcd}$$

$\tilde{T}_V$ = restriction of tautological bundle of $G(M)$

= Nash bundle of $\tilde{V}$

Violation of naturality:

$$f : \mathbb{P}^1 \longrightarrow C \subseteq \mathbb{P}^2, \quad (t,s) \longmapsto (t^2 s, t^3, s^3)$$

$$\begin{tikzcd}
Z_*\mathbb{P}^1 \arrow[r, "c_M"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\cancel{\circlearrowleft}"] & H_*X \arrow[d, "f_*"] \\
Z_*C \arrow[r, "c_M"'] & H_*Y
\end{tikzcd}$$

$$c_M(f_*\mathbb{P}^1) = c_M(C) = 3 + [C] \in H_*(C)$$

$$c_M(\mathbb{P}^1) = \mathrm{Dual}(\underbrace{c(\mathbb{P}^1)}) = \mathrm{Dual}((1+a)^2) = \mathrm{Dual}(1+2a)$$

$$\in H^*(\mathbb{P}^1) = \langle a \rangle / a^2$$

$$\in H^2(\mathbb{P}^1), \quad a = c_1(\mathcal{O}(1))$$

$$= (1+2a) \cap [\mathbb{P}^1] = [\mathbb{P}^1] + 2\underbrace{(a \cap [\mathbb{P}^1])}_{=1}$$

$$= [\mathbb{P}^1] \pm 2$$

$$f_*([\mathbb{P}^1] + 2) = [C] \pm 2 \neq [C] + 3 = c_M(C)$$

$$c_M(\mathbb{P}^n) = ((1+a)^{n+1}) \cap [\mathbb{P}^n]$$

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$$= (1+(n+1)a+\ldots) \cap [\mathbb{P}^n]$$

use $\quad a^k \cap [\mathbb{P}^n] = [\mathbb{P}^{n-k}]$

$$\mathbb{P}^{n-k} \hookrightarrow \mathbb{P}^n$$

THE CHERN-MACPHERSON NATURAL TRANSFORMATION

$$F \overset{c_*}{\Longrightarrow} H_*$$

$$\begin{tikzcd}
F(X) \arrow[r, "c_*^X"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_*(X) \arrow[d, "f_*"] \\
F(Y) \arrow[r, "c_*^Y"'] & H_*(Y)
\end{tikzcd}$$

Properties:

(1) $c_*\left(\sum n_i \mathbb{1}_{V_i}\right) = \sum n_i \, c_*(\mathbb{1}_{V_i})$

(2) $f_*(c_*\alpha) = c_*(f_*\alpha)$

(3) X smooth $\Rightarrow$ $c_*(\mathbb{1}_X) = c^*(X) \cap [X]$

Construction:

$$c_* = c_M \circ T^{-1}$$

Mather class
UNNATURAL

local Euler obstruction
UNNATURAL
LEO

THE LOCAL EULER OBSTRUCTION

UNNATURAL TRANSFORMATION $Z_* \overset{T}{\Rightarrow} F$

$$\begin{tikzcd}
Z_*X \arrow[r, "T_X\ (***)"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\cancel{\circlearrowleft}"] & FX \arrow[d, "f_*"] \\
Z_*Y \arrow[r, "T_Y"'] & FY
\end{tikzcd}$$

(***) $T_X : Z_*X \longrightarrow FX$

$V \longmapsto T_V := (p \mapsto Eu_p(V))$

$V \longmapsto 1_X V \cdot (p \mapsto \mathrm{Eu}_p(V))$

MAC PHERSON:

Reduction. It is enough to show for a map $f: X \to Y$, where X is smooth, that there exists an algebraic cycle $\sum n_i V_i$ on Y such that

(1) $$f_* \,\mathrm{Dual}\; c(X) = \sum n_i \,\mathrm{incl}_{i*} c_M(V_i)$$

and

(2) $$\chi f^{-1}(p) = \sum n_i \,\mathrm{Eu}_p(V_i) \qquad \text{for all } p.$$

While the spirit of the above discussion is that (2) determines the algebraic cycle and (1) is the result about it, one can see that the statements of (1) and (2) are somewhat similar. In the next section we will construct the algebraic cycle $\sum n_i V_i$ by totally different means. Then in the two following sections we will give entirely parallel proofs of (1) and (2).

We believe:

LEOs are hard to compute in general.

Question:

For our example $\left(X = \begin{array}{c} \mathbb{P}^1 \\ \delta \downarrow \\ C \\ \text{cusp} \end{array} \subseteq \mathbb{P}^2\right)$, can we use

Properties of C_* $\quad\mathbb{F}(X) \to H_*(X)$ + knowledge about Mather class $c_M(C) = 3 + [C]$

to compute $Eu_p(C)$ for all $p \in C$?

Step 1: $C_*(\underbrace{f_* 1\!\!1_{\mathbb{P}^1}}_{p \mapsto \chi(f^{-1}(p)) = 1}) = f_* \underbrace{C_*(1\!\!1_{\mathbb{P}^1})}_{2 + [\mathbb{P}^1]}$

$= 2 + [C]$

$\underbrace{C_*(1\!\!1_C)}$ $\qquad \underbrace{\in H_*(C)}$

$C_M \circ \underbrace{T^{-1}(1\!\!1_C)}_{= C - \{0\}}$ $\qquad$ T isom.

$$\left.\begin{array}{l} [\, C_M(C) = 3 + [C] \\ [\, C_M(\{0\}) = 1 \end{array}\right\} \Rightarrow T^{-1}(1\!\!1_C) = C - \{0\}$$

$\Rightarrow T(C) = 1\!\!1_C + T(\{0\})$

$= 1\!\!1_C + 1\!\!1_{\{0\}}$

also $\boxed{Eu_{\{0\}}(C) = 2}$?

$$1 = \chi f^{-1}(0) = Eu_0(C) - \underbrace{Eu_0(\{0\})}_{=1}$$

$\Rightarrow$ <u>Claim</u>

<u>Remark</u> For any resolution $\tilde{f} : \tilde{X} \to X$ of sing. (that is birational) you can use a similar argument, since $\tilde{f}_* [\tilde{X}] = [X]$

$\Sigma \subseteq X$ sing strata

Known $\tilde{f}_* \, c_*(1\!\!1_{\tilde{X}}) \underset{\text{Ax } 3}{=} \tilde{f}_*(c^*(\tilde{X}) \cap [\tilde{X}])$

LHS $c_*(\underbrace{\tilde{f}_* 1\!\!1_{\tilde{X}}}$

$1\!\!1_X + \underbrace{(p \mapsto \chi f^{-1}(p) - 1)}_{=0 \text{ on } X \setminus \Sigma}$

unknown on Σ

Local Euler Obstruction, Dec 5th 2022

Obstruction Theory

CW-complex

Given $p: E \to B$ fibration with fiber F, E n-connected

$G: B_n \to E$ section of p.

n-skeleton of your CW-complex

Q: What is the obstruction to extend G to B_{n+1}?

Let e^{n+1} be an $(n+1)$-cell with attaching map $\varphi: S^n \to B_n$

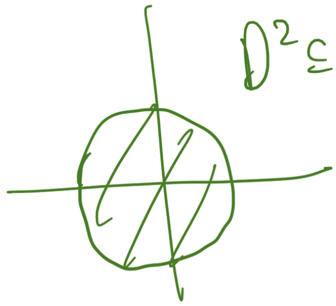
$\Rightarrow \quad [G \circ \varphi] \in \pi_n(p^{-1}(\underbrace{\Delta}_{e^{n+1}})) \cong \pi_n(F)$

$\Rightarrow \quad (e^{n+1} \mapsto [G \circ \varphi])$ defines a cochain $C_G \in C^{n+1}(B; \underbrace{\pi_n(F)}_{\text{local coeff.}})$

This is actually a cocycle and it holds that: G can be extended $\Leftrightarrow C_G = 0$

Note: This can also be done rel A, $A \subseteq B$ subcomplex.

Example:



$D^2 \subseteq \mathbb{R}^2$

$T\mathbb{R}^{2*} \xrightarrow{p|} D^2$ restricted cotangent bundle

$r^2 = x^2 + y^2$

$dr^2 \in$ smooth section of $(T\mathbb{R}^2)^*$ restricted to S^1

CW-decomp.

1-skeleton

Now: $\pi: (T\mathbb{R}^2)^* \setminus \{0\} \to \mathbb{R}^2$ fiber bundle with $\pi^{-1}(b) = \underbrace{T_b\mathbb{R}^{2*} - \{0\}}_{\simeq S^1}$

Still $dr^2|_{S^1}$ is a section of $\pi|_{S^1}$.

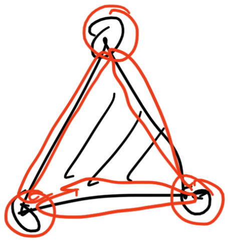
$\mathcal{O} \in H_2(D^2, S^1; \mathbb{Z}) \Rightarrow \underbrace{\langle [C_{dr^2}], \mathcal{O} \rangle}_{\text{local Euler obstruction}} \in \mathbb{Z}$

$[\Delta^2 \xrightarrow{\cong} D^2]$

2nd Criterium

$\exists$ extension of $G|_{B_{n-1}}$ $\iff$ $0 = [C_G] \in H^{n+1}(B; \pi_n(F))$

$G: B_1 \to E$



1. $C_G = 0$ as cocycle

$\Rightarrow$ Can extend $G: S^1 \to E$ to Δ

2. $[C_G] = 0$

$\Rightarrow$ Can extend $G: \overset{\circ}{\circ\ \ \circ} \to E$ to Δ

In $\hat{G}: \Delta \to E$ s.t. $\hat{G}\left[\left(\overset{\circ}{\circ\ \circ}\right)\right] = G$

but $\hat{G}|_{\Delta} \neq G$.

$\boxed{\text{Application:}}$ $\underset{\text{sing}}{V^\nu} \subseteq \underset{\text{smooth}}{N^n}$, $\quad TV \to \hat{V} \xrightarrow{\nu} V$ Nash construction

- - - z local coord. N around $\in V$, $z(p) = 0$.

$z \quad (z_1, \dots, n)$ of p

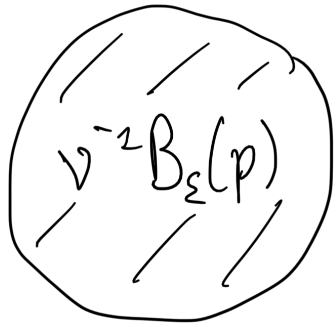
$r^2 = \|z\|^2 = \sum_i z_i \bar{z}_i$

$d\|z\|^2$ = section of TN^* = real dual bundle of TN oriented by complex structure

Prop: $\exists \varepsilon > 0$: The section dr^2 restricts to a non-zero section of $\tilde{T}_V^*$ over $\nu^{-1}(B_\varepsilon(p) \setminus \{p\})$.

Proof: By Whitney conditions.

Idea: $Eu_p(V)$ measures the extendability of this section to p.



$\nu^{-1} S_\varepsilon(p)$

ν = real dim of V

$(\tilde{T}_V \setminus 0)_p \simeq S^{\nu-1}$

Obstruction cocycle:

$$C_{dr^2} \in C^\nu\Big(\nu^{-1}B_\varepsilon(p), \nu^{-1}S_\varepsilon(p); \underbrace{\pi_{\nu-1}(\tilde{T}_V^* \setminus 0)}_{\simeq \mathbb{Z}}\Big)$$

$$H_\nu(\nu^{-1}B_\varepsilon(p), \nu^{-1}S_\varepsilon(p)) = \langle \mathcal{O}_{(\nu^{-1}B_\varepsilon(p), \nu^{-1}S_\varepsilon(p))} \rangle$$

$$Eu_p(V) = \langle [C_{dr^2}], \mathcal{O} \rangle \in \mathbb{Z}$$