Chern classes for singular algebraic varieties

By R. D. MACPHERSON*

Our object is to prove the existence of Chern classes for possibly singular complex algebraic varieties, which was conjectured by Deligne and Grothendieck [2]. These are homology classes which for nonsingular varieties are the Poincaré duals of the usual Chern classes; they are characterized by naturality properties described in Section 1 below. As a byproduct of the proof we obtain explicit formulas for these classes which do not involve the resolution of singularities.

My debt to the ideas of Deligne, Grothendieck, and Mather is clear from what follows. I wish to thank Thom for suggesting that I work on this at the appropriate moment, Fulton and Landman for answering my questions on the mysteries of algebraic geometry, and Bott, my thesis advisor, for teaching me how to do mathematics in this spirit.

1. The Deligne-Grothendieck conjecture

A *constructible set* in an algebraic variety is one obtained from the subvarieties by finitely many of the usual set-theoretic operations. A *constructible* function on a variety is one for which the variety has a finite partition into constructible sets such that the function is constant on each set.

PROPOSITION 1. *There is a unique covariant functor* $\mathbf{F}$ *from compact complex algebraic varieties to abelian groups whose value on a variety is the group of constructible functions from that variety to the integers and whose value* f_* *on a map* f *satisfies*

$$f_*(1_W)(p) = \chi(f^{-1}(p) \cap W)\ ,$$

where 1_W *is the function that is identically one on the subvariety* W *and zero elsewhere, and where* χ *denotes the topological Euler characteristic.*

Proof. For any constructible function α on a variety V, $f_*(\alpha)(p)$ is uniquely determined since the functions of the form 1_W form a basis over the integers for the constructible functions. If $\{\mathbb{S}_i\}$ is a stratification ([1], [3]) of V subordinate to both α and f, then

$$f_*(\alpha)(p) = \textstyle\sum_i \alpha(\mathbb{S}_i)\chi_c(\mathbb{S}_i \cap f^{-1}(p))\ ,$$

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where χ_c denotes the Euler characteristic computed using cohomology with compact supports. This may be seen from the fact that χ_c of a stratified object is the sum of the χ_c of the strata. Using this characterization of $f_*(\alpha)$ both the constructibility of $f_*(\alpha)$ and functoriality, $g_*f_*(\alpha) = (gf)_*(\alpha)$, follow using stratification theory and the multiplicativity of χ_c for fiber bundles.

The main result is the following theorem, which was conjectured by Deligne and Grothendieck.

THEOREM 1. *There exists a natural transformation from the functor* **F** *to homology which, on a nonsingular variety V, assigns to the constant function* 1 *the Poincaré dual of the total Chern class of V.*

Explicitly, the theorem asserts that we can assign to any constructible function α on a compact complex algebraic variety V an element $c_*(\alpha)$ of $H_*(V)$ satisfying the following three conditions:

1. $f_*c_*(\alpha) = c_*f_*(\alpha)$
2. $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$
3. $c_*(1) = \text{Dual}\, c(V)$ if V is smooth.

DEFINITION (Deligne). *The total Chern class of any compact variety V is c_* applied to the constant function* 1 *on V.*

The compactness restriction may be dropped with minor modifications of the proof if all maps are taken to be proper and Borel-Moore homology (homology with locally finite supports) is used.

PROPOSITION 2. *There can be at most one natural transformation c_* satisfying the above conditions.*

Proof. For any constructible α on V, $c_*(\alpha)$ may be determined using resolution of singularities as follows. Find integers k_i and maps g_i of smooth varieties X_i to V so that

$$\alpha = \sum k_i g_{i*}(1) .$$

For example, each V_i may be chosen as a resolution of an irreducible subvariety of the support of $\alpha - \sum_1^{i-1} k_i g_{i*}(1)$ of maximal dimension. Then $c_*(\alpha)$ must be $\sum k_i g_{i*}\, \text{Dual}\, c(X_i)$.

Of course, in the above proof the g_i and X_i may be chosen in many ways; our existence theorem implies that $c_*(\alpha)$ so computed is independent of the choices. For example, we obtain the following result about the classical Chern classes:

COROLLARY TO THEOREM 1. *For nonsingular X_i and V, if k_i and*

$g_i\colon X_i \to V$ are chosen so that, for all $p \in V$, $\sum k_i \chi g_i^{-1}(p) = 1$, then

$$\text{Dual}\, c(V) = \sum k_i g_{i*}\, \text{Dual}\, c(X_i)\ .$$

2. The Mather-Chern classes

Suppose V, a v-dimensional variety, is embedded in a smooth variety N. Over the nonsingular part of V there is a section of $G_v(TN)$, the Grassman bundle of v-dimensional subspaces associated to the tangent bundle of N, given by the tangent space to V. The *Nash blowup* $\hat{V}$ of V is the closure in $G_v(TN)$ of the image of this section. It comes equipped with a map $\nu\colon \hat{V} \to V$, the restriction of the projection of $G_v(TN)$, and a vector bundle which we may unambiguously call TV, the restriction of the tautological bundle over $G_v(TN)$. The $\hat{V}$, TV and ν are analytically independent of the embedding, since near each point, V has a unique minimal local analytic embedding; for non-embeddable varieties they can be defined by a patching process.

Mather has defined an extension of Chern classes to singular varieties by the formula

$$c_M(V) = \nu_*\, \text{Dual}\, c(TV)\ ,$$

where Dual denotes the Poincaré duality map defined by capping with the fundamental (orientation) homology class. (Of course Dual is not in general an isomorphism.) We note that if $\nu'\colon \hat{V}' \to V$ is any blowup with a subbundle TV' of $\nu'^* TN$ which, over the smooth locus of V, agrees with the pulled up tangent bundle to V in $\nu'^* TN$, then $c_M(V) = \nu'_*\, \text{Dual}\, c(TV')$.

An algebraic cycle on a variety V is a finite formal linear sum $\sum n_i V_i$ where the n_i are integers and the V_i are irreducible subvarieties of V. We may define c_M on any algebraic cycle of V by

$$c_M\left(\sum n_i V_i\right) = \sum n_i\, \text{incl}_{i*}\, c_M V_i\ ,$$

where incl_i is the inclusion of V_i in V. Our object is to write a formula expressing c_* applied to a constructible function as c_M applied to an associated algebraic cycle.

3. The local Euler obstruction

We define transcendentally an integral character $\text{Eu}_p(V)$ of the singularity type of a variety V at a point $p \in V$. Finding a purely algebraic formula for this is the main problem in extending Theorem 2 below to algebraic varieties in characteristic zero.

As above, choose an embedding of V, a v-dimensional variety, into a smooth variety N and perform the Nash construction. Let $z = (z_1, z_2, \cdots, z_n)$

be local coordinates in N about p such that $z_i(p) = 0$ and let $||z|| = \sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n}$. Since $||z||^2$ is a real-valued function, $d\,||z||^2$ may be considered as a section of TN^* where $*$ denotes the *real* dual bundle retaining only its orientation from the complex structure. $d\,||z||^2$ pulls back and restricts to a section r of TV^*.

LEMMA 1. *For small enough ε, the section r is nonzero over $\nu^{-1}(z)$ where $0 < ||z|| \leqq \varepsilon$.*

Proof. V has a stratification satisfying Whitney's Condition a.

Let B_ε be the ε-ball $\{z \mid ||z|| \leqq \varepsilon\}$ and S_ε be the ε-sphere $\{z \mid ||z|| = \varepsilon\}$. The obstruction to extending r as a nonzero section of TV^* from $\nu^{-1}S_\varepsilon$ to $\nu^{-1}B_\varepsilon$, which we denote by $\mathrm{Eu}(TV^*, r)$, lies in $H^v(\nu^{-1}B_\varepsilon, \nu^{-1}S_\varepsilon; \mathbf{Z})$. If $\mathcal{O}_{(\nu^{-1}B_\varepsilon, \nu^{-1}S_\varepsilon)}$ denotes the orientation class in $H_v(\nu^{-1}B_\varepsilon, \nu^{-1}S_\varepsilon; \mathbf{Z})$, then we define the local Euler obstruction of V at p to be $\mathrm{Eu}(TV^*, r)$ evaluated on $\mathcal{O}_{(\nu^{-1}B_\varepsilon, \nu^{-1}S_\varepsilon)}$ or, symbolically,

$$\mathrm{Eu}_p(V) = \langle \mathrm{Eu}(TV^*, r), \mathcal{O}_{(\nu^{-1}B_\varepsilon, \nu^{-1}S_\varepsilon)} \rangle \,.$$

This is easily seen to be independent of all the choices involved. As with the Mather-Chern classes any blowup $\nu'\colon \hat{V}' \to V$ with a subbundle of ν'^*TN extending the tangent bundle to the smooth part of V could be used in place of $\nu\colon \hat{V} \to V$ with the same result.

The following facts about the local Euler obstruction may be of assistance in getting a feel for it.

1. $\mathrm{Eu}_p(V) = 1$ if V is nonsingular at p.
2. If V is a curve, $\mathrm{Eu}_p(V)$ is the multiplicity of V at p. If V is the cone on a nonsingular plane curve of degree d and p is the vertex, $\mathrm{Eu}_p(V) = 2d - d^2$.
3. $\mathrm{Eu}_{p\times p'}(V \times V') = \mathrm{Eu}_p(V)\cdot\mathrm{Eu}_{p'}(V')$
4. If V is locally reducible at p and V_i are its components, then $\mathrm{Eu}_p(V) = \sum \mathrm{Eu}_p(V_i)$.

Using the local Euler obstruction, we define a mapping T from the algebraic cycles on V to the constructible functions on V by

$$T(\sum n_i V_i)(p) = \sum n_i \mathrm{Eu}_p(V_i) \,.$$

LEMMA 2. *T is a well-defined isomorphism from the group of algebraic cycles to the group of constructible functions.*

Proof. To show that T is well-defined, i.e., that the right hand side is constructible, one works with a stratification of the map ν. To show that it is an isomorphism, one works modulo subvarieties of dimension d and proceeds by descending induction on d.

Now we can state our refinement of Theorem 1.

THEOREM 2. *$c_M T^{-1}$ satisfies the requirements for c_* in the Deligne-Grothendieck conjecture* (i.e., *c_* equals $c_M T^{-1}$*).

The rest of this note will be devoted to the proof of this theorem.

4. First reductions

Trivially $c_M T^{-1}$ satisfies Conditions 2 and 3 (following Theorem 1), so it suffices to show Condition 1 of naturality: $c_M T^{-1} f_* = f_* c_M T^{-1}$.

Next, it is enough to prove this relation for maps $f\colon X \to Y$ where X is a nonsingular variety and for the function identically one on X. For, as in the proof of Proposition 2, any α is $\sum k_i g_{i*} 1$ where the source X_i of g_i is smooth. Then

$$\begin{aligned} f_* c_M T^{-1}(\alpha) &= \sum k_i f_* c_M T^{-1} g_{i*} 1 \\ &= \sum k_i c_M T^{-1} f_* g_{i*} 1 \\ &= c_M T^{-1} f_*(\alpha) \end{aligned}$$

where the middle equality follows from two applications of the result for 1 on X_i and from the functoriality of lower star.

Since for smooth X, $c_M T^{-1}(1) = \mathrm{Dual}\, c(X)$, we are reduced to showing

$$f_* \,\mathrm{Dual}\, c(X) = c_M(T^{-1} f_* 1)$$

or equivalently, for the algebraic cycle $\sum n_i V_i$ such that $T(\sum n_i V_i) = f_*(\alpha)$, the following relation holds:

$$f_* \,\mathrm{Dual}\, c(X) = c_M\left(\sum n_i V_i\right) .$$

Using the definition of $f_*(\alpha)$, this translates into the

Reduction. It is enough to show for a map $f\colon X \to Y$, where X is smooth, that there exists an algebraic cycle $\sum n_i V_i$ on Y such that

$$(1) \qquad f_* \,\mathrm{Dual}\, c(X) = \sum n_i \,\mathrm{incl}_{i*} c_M(V_i)$$

and

$$(2) \qquad \chi f^{-1}(p) = \sum n_i \,\mathrm{Eu}_p(V_i) \qquad \text{for all } p.$$

While the spirit of the above discussion is that (2) determines the algebraic cycle and (1) is the result about it, one can see that the statements of (1) and (2) are somewhat similar. In the next section we will construct the algebraic cycle $\sum n_i V_i$ by totally different means. Then in the two following sections we will give entirely parallel proofs of (1) and (2).

$df: TX \to TN$ $\xrightarrow{\text{smooth}} X \xrightarrow{f} Y \hookrightarrow N$ smooth $f^*TN = TN$

5. The main construction

The heart of the proof is the algebraic cycle Z_∞ constructed below which brings to bear some subtle geometry of Grassmann manifolds.

We start with our map $f: X \to Y$ with X smooth and we assume that Y is embeddable in a smooth variety. (This already covers, for example, the projective case of our theorem. If it is not embeddable, a cumbersome process using local embeddability is required.) We fix an embedding $Y \subseteq N$.

Let d be the dimension of X. Over X we have a natural embedding of $\mathrm{Hom}(TX, TN)$ into the Grassmann bundle $G_d(TX \oplus TN)$ called the *graph construction*: a point in $\mathrm{Hom}(TX, TN)$ is a linear map whose graph is a d-plane in $TX \oplus TN$. For each complex number λ there is a section s_λ of $G_d(TX \oplus TN)$ defined as follows: the vector bundle map $df: TX \to TN$ can be viewed as a section of $\mathrm{Hom}(TX, TN)$ and s_λ is λ times this section followed by the graph construction. Let Z_λ be the algebraic cycle one times the image of s_λ. The Z_λ form an algebraic family of algebraic cycles in $G_d(TX \oplus TN)$ parametrized by $\mathbf{C}$. This family has a unique extension to a rational equivalence, i.e., an algebraic family of algebraic cycles parametrized by the Riemann sphere $\mathbf{P}_1(\mathbf{C})$ considered as $\mathbf{C} \cup \infty$ in the usual way. If W is the closure of the image of the map $X \times \mathbf{C} \to G_d(TX \oplus TN) \times \mathbf{P}_1(\mathbf{C})$ sending (x, λ) to $(s_\lambda(x), \lambda)$, then Z_∞ is the intersection of the cycle $1 \cdot W$ and the cycle $1 \cdot (G_d(TX \oplus TN) \times \infty)$.

As a set, the part of Z_∞ lying over $x \in X$ consists of the points of the form $\lim_{j \to \infty} S_{\lambda_j}(x_j)$ where $\lambda_j \to \infty$ and $x_j \to x$. We note that it is *not* enough to consider such sequences with all $x_j = x$. Intuitively, Z_∞ breaks into irreducible components in a manner reflecting the singularities of the map f.

We now construct the following diagram of spaces, maps and vector bundles. Since the diagram commutes, we may unambiguously follow the convention of calling the pullback of a vector bundle by the same symbol as the bundle itself.

taut. bdl on ξ rel Grassm.

$\hat{V}_i \times_{V_i} V'_i =$

$$\begin{tikzcd}
\xi \subseteq TX \oplus TN \supseteq TV_i \arrow[d] & & & \xi \subseteq TX \oplus TN \arrow[d] & \\
P_i \arrow[rr, "\mu_i"] \arrow[dddd, "\rho_i"'] & & V'_i \arrow[r, hook] \arrow[dddd, "\gamma_i"] & Z \arrow[dd, shift right, "\pi"'] & \\
 & & & & TX \arrow[dl] \\
 & & & X \arrow[uu, shift right, "s_\lambda"'] \arrow[dd, "f"] & TN \arrow[dd] \\
 & TV_i \arrow[dl] & & & \\
\hat{V}_i \arrow[rr, "\nu_i"] & & V_i \arrow[r, hook] & Y \arrow[r, hook] & N
\end{tikzcd}$$

$\supset Z_0 \cong X,\ Z_\infty = H \cup \bigcup m_i V'_i$

Nash bundle

$\hat{V}_i = \mathrm{Nash}(V_i)$ $\qquad V_i = f(\pi(V'_i))$

$$\begin{tikzcd}
Z_\lambda \quad G \times \mathbb{C} \arrow[r, hook] \arrow[d] & G \times \mathbb{P}^1 \arrow[d] & G \times \{\infty\} \arrow[l, hook'] \\
X \times \mathbb{C} \arrow[r, hook] \arrow[d] & X \times \mathbb{P}^1 \arrow[d] & X \times \{\infty\} \arrow[l, hook'] \\
\mathbb{C} \arrow[r, hook] & \mathbb{P}^1 & \{\infty\} \arrow[l, hook']
\end{tikzcd}$$

$Z \subset Gr_d\,(TX \oplus TN)$ Abschluss von s_λ, $\lambda \in \mathbb{C}$

Z_∞ die Faser über $\lambda = \infty$

The space Z is the union of the supports of all the z_λ for λ real and non-negative or infinite. The V'_i are the irreducible components of the support of Z_∞; the integers m_i are their multiplicities (so $Z_\infty = \sum m_i V'_i$). The map π is the restriction of the projection of $G_d(TX \oplus TN)$; s_λ is as defined before. The bundle ξ is the tautological bundle over $G_d(TX \oplus TN)$ restricted to Z; it is a subbundle of (the pullup of) $TX \oplus TN$. V_i is the image in Y of V'_i. $\hat{V}_i$ is the Nash blowup of V_i and the associated bundle TV_i is embedded naturally in TN if we construct the Nash blowup with respect to the embedding of V_i in N. P_i is the join of $\hat{V}_i$ and V'_i over V_i (i.e., the closure in the fiber product $\hat{V}_i \times_{V_i} V'_i$ of the inverse image of the open set of $\hat{V}_i$ projecting isomorphically to V_i). Over P_i are two subbundles of $TX \oplus TN$: ξ pulled back from one factor and TV_i pulled back from the other.

LEMMA 3. *If $\mathcal{O}_{P_i}$ and $\mathcal{O}_X$ are the orientation homology classes of P_i and X, then*

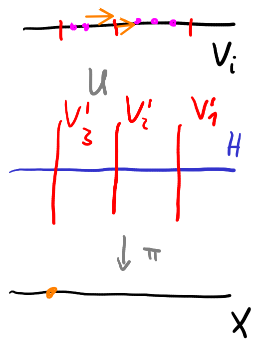
$$s_{0*}\mathcal{O}_X = \sum m_i \mu_{i*} \mathcal{O}_{P_i}\,.$$

Proof. $s_{0*}\mathcal{O}_X$ is the homology class of the algebraic cycle $1 \cdot Z_0$. Since μ_i is birational, $\sum m_i \mu_{i*}\mathcal{O}_{P_i}$ is the homology class of the algebraic cycle $\sum m_i V'_i = Z_\infty$. Now Z_0 and Z_∞ are rationally equivalent and hence homologous in $G_d(TX \oplus TN)$. But the homology may be taken to lie in Z; in fact, if we triangulate the semianalytic set Z and assign ± 1 to each top-dimensional triangle, the boundary of this chain will be $Z_0 - Z_\infty$.

LEMMA 4. *The bundle TV_i over P_i is contained in the bundle ξ over P_i.*

$\leftarrow$ pulled back to P_i

Proof. It suffices to show this inclusion on the complement of a proper subvariety (a dense set). We choose those points p of P_i projecting to non-singular points of $\hat{V}_i$ and V'_i and such that the differential of η_i is surjective at $\mu_i(p)$. Then, since η_i factors through f, TV_i is contained in the image of df: $TX \to TN$ properly pulled back. It will suffice then to show that for each $x \in X$, $Z_\infty \cap \pi^{-1}x$ contains only points representing linear subspaces containing the image of df at z. After locally trivializing TX and TN near x, this reduces to the following. Let X and N be complex vector spaces, F: $X \to N$ a linear map, F_i a sequence of maps converging to F, and λ_j a sequence of complex numbers tending to ∞. Then any vector $(0, F(v))$ in $X \oplus N$ is approximated arbitrarily closely for large enough j by a vector in the graph of $\lambda_j F_j$. But this is clear since $((1/\lambda_j)v,\ F_j(v))$ is in the graph of $\lambda_j F_j$.



Let $\mathcal{O}_{\hat{V}_i}$ denote the orientation class of $\hat{V}_i$ and let $\mathrm{Eu}(\xi/TV_i)$ denote the Euler class of the underlying real vector bundle.

LEMMA 5. ρ_{i*} Dual $c(\xi/TV_i) = \rho_{i*}$ Dual $\mathrm{Eu}(\xi/TV_i)$

$$= p_i \mathcal{O}_{\hat{V}_i}$$

$$S_\lambda(x) = TX \oplus \lambda\cdot df_x(TX) \subset \mathrm{Grass}_d(TX \oplus TN)$$

$x \in X$ regulärer Punkt von f:

$$T_xX \oplus \lambda df_x(T_xX) \xrightarrow{\lambda\to\infty} T \subset 0 \oplus T_{f(x)}N$$ „vertical subspace"

$x \in X$ a non-regular point, $\vec{v} \in T_xX \setminus \{0\}$ with $df_x(\vec{v}) = 0$:

$$\begin{array}{ccccc} T_xX \oplus \lambda df_x(T_xX) & \xrightarrow{\lambda\to\infty} & T & \subset & T_xX \oplus T_{f(x)}N \\ \cup & & \cup & & \cup \\ \langle\vec{v}\rangle \oplus 0 & \xrightarrow{\lambda\to\infty} & \langle\vec{v}\rangle \oplus 0 & & \end{array}$$ has „non-vertical" comp.

„The horizontal part of the limit space T is $T_xV_i \oplus 0$."

take $T/T\cap 0\oplus TN$

Z.Z. Der horizontale Anteil von T ist im tang. Bündel enthalten.

Beweis: Approximiere T durch eine Folge (x_i, λ_i) von Punkten in der guten, regulären Teilmenge $U \subset X$: $T_i = T_{x_i}X \oplus \lambda_i \cdot df_{x_i}(T_{x_i}X)$

OBdA alle x_i in der domain einer Trivialisierung

$$\mathbb{C}^d \oplus \lambda_i F_i(\mathbb{C}^d) = \text{Faser von } \xi \text{ über } s_{\lambda_i}(x_i) \in Gr$$

$$T_{f(x_i)}V = \text{Faser von } \xi \text{ über } p_i \in P \text{ über}$$ $s_{\lambda_i}(x_i)$ (Wähle Folge p_i)

Dann $q_i = \rho(p_i) \in \hat{V}$ mit $T_{q_i}\hat{V} = T_{x_i}V$ wegen Wahl von U.

~~Frage: $T_{x_i}V \subset \mathbb{C}^d \oplus \lambda_i \cdot F_i(\mathbb{C}^d)$~~ ? Quatsch.

$$\begin{tikzcd} P \arrow[r] \arrow[dd] & V' \arrow[r, hook] \arrow[dd] & Gr \arrow[d] \\ & & X \arrow[d, "f"] \\ \hat{V} \arrow[r] & V \arrow[r, hook] & Y \end{tikzcd}$$

$T_{f(x_i)}V$

The space Z is the union of the supports of all the z_λ for λ real and non-negative or infinite. The V_i' are the irreducible components of the support of Z_∞; the integers m_i are their multiplicities (so $Z_\infty = \sum m_i V_i'$). The map π is the restriction of the projection of $G_d(TX \oplus TN)$; s_λ is as defined before. The bundle ξ is the tautological bundle over $G_d(TX \oplus TN)$ restricted to Z; it is a subbundle of (the pullup of) $TX \oplus TN$. V_i is the image in Y of V_i'. $\hat{V}_i$ is the Nash blowup of V_i and the associated bundle TV_i is embedded naturally in TN if we construct the Nash blowup with respect to the embedding of V_i in N. P_i is the join of $\hat{V}_i$ and V_i' over V_i (i.e., the closure in the fiber product $\hat{V}_i \times_{V_i} V_i'$ of the inverse image of the open set of $\hat{V}_i$ projecting isomorphically to V_i). Over P_i are two subbundles of $TX \oplus TN$: ξ pulled back from one factor and TV_i pulled back from the other.

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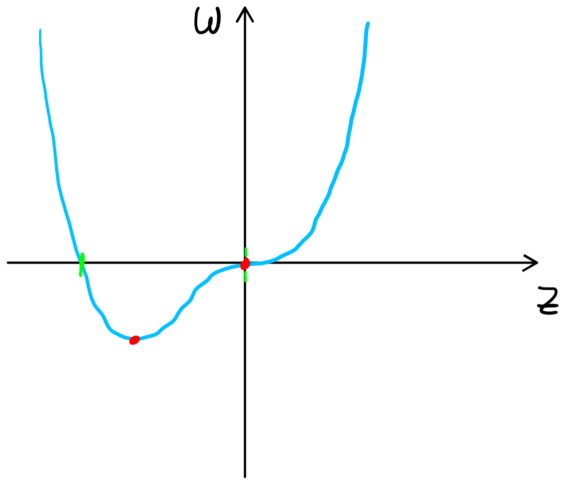
LEMMA 5. ρ_{i*} Dual $c(\xi/TV_i) = \rho_{i*}$ Dual $\mathrm{Eu}(\xi/TV_i)$ $\quad (1 + c_1 + c_2 + \dots + c_d) \cap [P]$

$$= p_i \mathcal{O}_{\hat{V}_i}$$

total CC $\quad$ top CC $\quad \neq c_d \cap [P]$

Warum killt ρ_{i*} alle nieder-dim. Komp. von $c(\xi/TV) \cap [P]$?

Example: $f : \mathbb{C} \to \mathbb{C}, \quad z \mapsto w = z^3(z+a), \quad a \in \mathbb{R}_{<0}$



$f' = 4z^3 + 3az^2 = z^2(4z+3a)$

Set $a = \frac{4}{3}$ and get $c_2 = -1$ as a critical point.

<u>The graph construction:</u> $s_\lambda : z \mapsto (z, (1 : \lambda \cdot f'(z))) \in \mathbb{C} \times \mathbb{P}^1$

This point is the same as $(z, (\frac{1}{\lambda} : f'(z)))$ for $\lambda \neq 0$.

Introduce $\mathbb{P}^1$ with affine coordinate λ and take the closure of

$$\Gamma_s = \left\{ (z, [v], \lambda) : [v] = (\tfrac{1}{\lambda} : f'(z)) \right\} \subset \mathbb{C} \times \mathbb{P}^1 \times \mathbb{P}^1$$

Equations for this: Choose coordinates $(u : v)$ for the middle term.

$$\det \begin{pmatrix} u & \mu \\ v & 4z^3 + 4z^2 \end{pmatrix} = \mu v - \boxed{u \cdot (4z^3 + 4z^2)}$$

$\boxed{\mu = \frac{1}{\lambda}}$ What is the fiber over $\mu = 0$?

$u = 0 : \{ (z, (0 : 1)) \mid z \in \mathbb{C} \}$

$4z^3 + 4z^2 = 0 : \quad \mathrm{Crit}(f) \times \mathbb{P}^1$

$\forall (u:v) \; \exists \; z_n \to c, \mu_n \to 0 : \lim_{n\to\infty} (z_n, (\mu_n : df(z_n))) = (c, (u:v))$

We try this with $(u:v) = (2:3)$ and $c = -1$

$$z_n = -1 + \frac{1}{n}$$

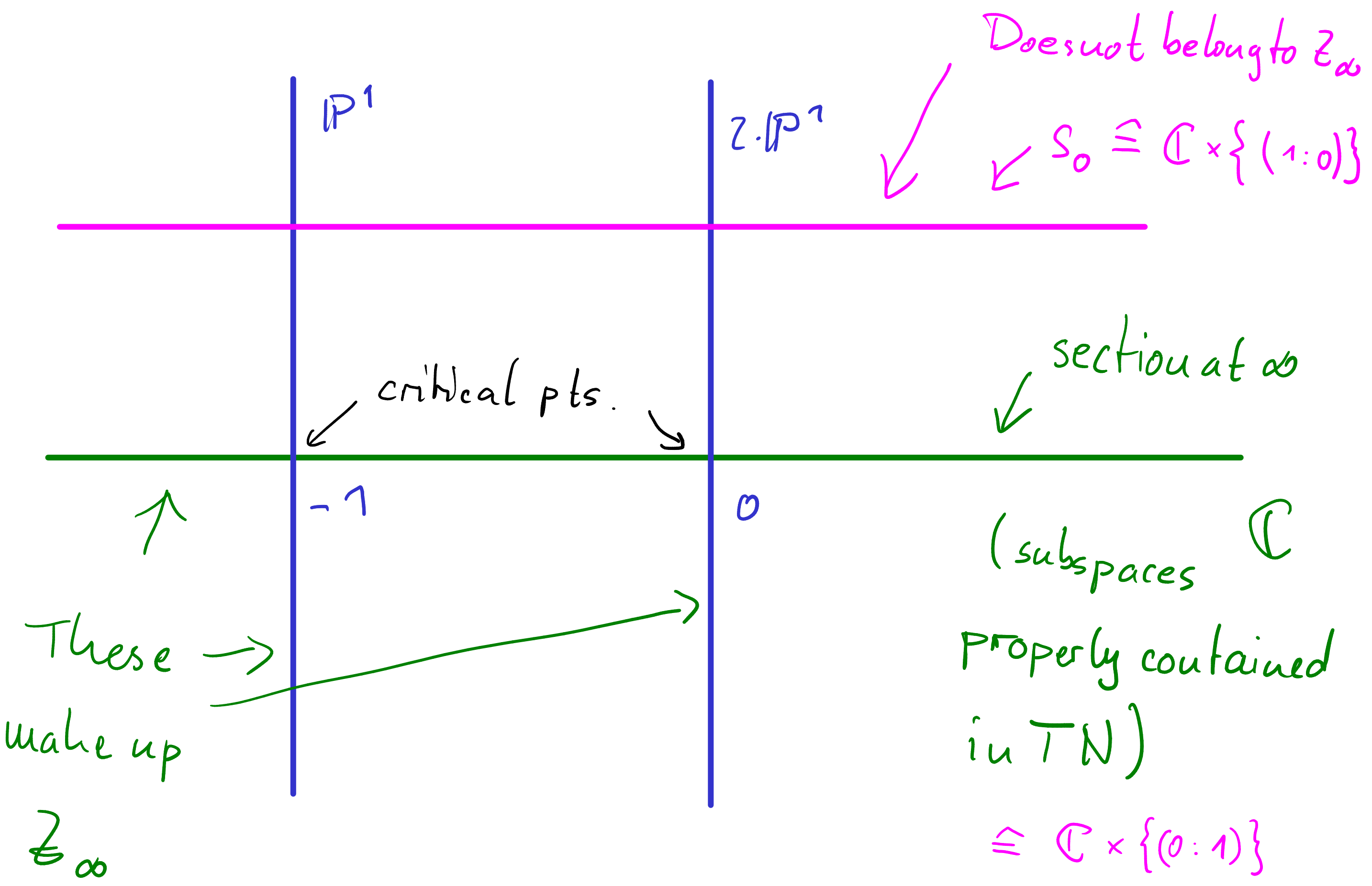
$$f'(z_n) = \underbrace{4\left(\tfrac{1}{n}-1\right)^3 + 4\left(\tfrac{1}{n}-1\right)^2}_{\neq 0} = a_n \longrightarrow 0$$

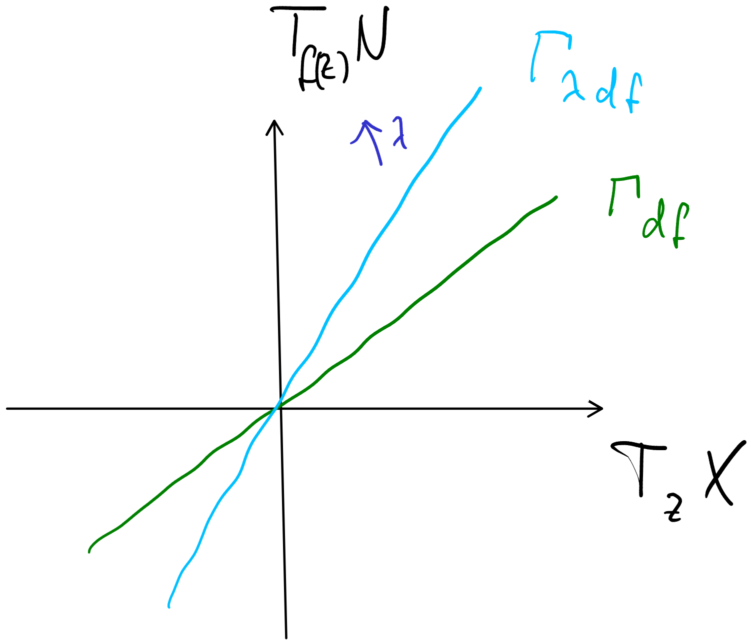
Now choose μ_n: $\quad (2:3) \overset{!}{=} (\mu_n : a_n)$.

Since $a_n \neq 0$, we can choose $\mu_n = \frac{2}{3} a_n$.

This has the right limit <u>and</u> $\mu_n \to 0$, as required.

$$Z_\infty =$$





$s_0 : z \mapsto (z, (1:0))$

Claim: $Z_0 - Z_\infty = (h)$ for some rational function h.

Def: Two cycles $D, E \subset X$ are rationally equivalent, if $\exists\, W \subset X \times \mathbb{P}^1$ irred. s.t. $D = W \cap X \times \{0\}$ and $E = W \cap X \times \{\infty\}$

„Every

$E \sim D \;:\Longleftrightarrow\; E = V[0] \;,\; D = V[\infty]$

for $V \subset X \times \mathbb{P}^1$ <u>irred.</u> and mapping dominantly onto $\mathbb{P}^1$

<u>Example</u>: $X = \mathbb{P}^2$, $E = 1\cdot\{x=0\} + 1\cdot\{y=0\}$

$D = 1\cdot\{x^2 - yz = 0\}$.

with function $h := \dfrac{x\cdot y}{x^2 - yz}$ and/or family

$$V = \left\{ \det\begin{pmatrix} u & xy \\ v & x^2-yz \end{pmatrix} = 0 \right\} \subset \underset{(x:y:z)}{\mathbb{P}^2} \times \underset{(u:v)}{\mathbb{P}^1}$$

But $E' = 1\cdot\{x=0\} \not\sim D$.

In particular, there is no homotopy of the embeddings.

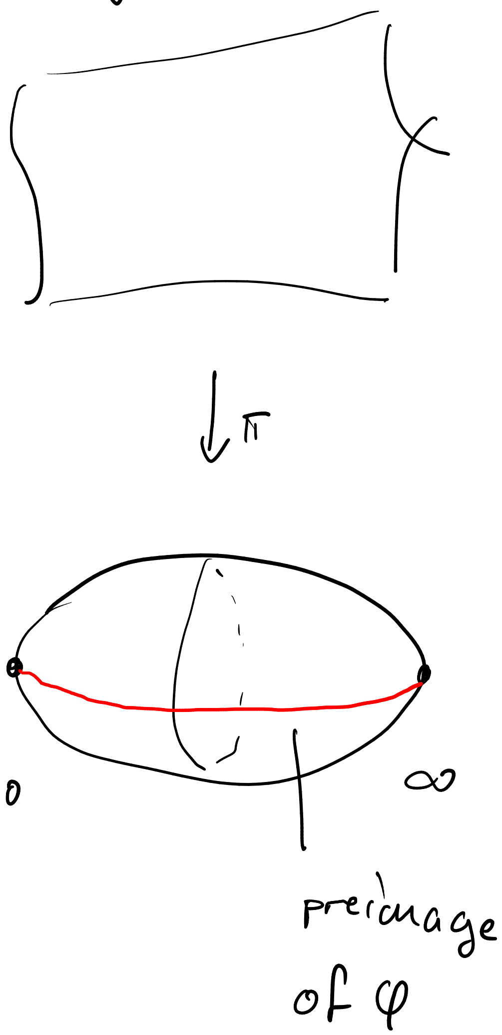
What is the map to homology?

$$\sum_i n_i D_i \longmapsto \sum_i n_i \cdot \mathrm{inc}_*[D_i]$$

fund. class.

w/ coord. λ

$V \subset X \times \mathbb{P}^1$



__Lemma:__ Birational maps preserve fundamental classes.

__Corollary:__ We can replace V by a resolution of its sing's $\rho : \tilde{V} \to V$.

Take this resolution so that $\tilde{V}_\infty$ and $\tilde{V}_0$ is SNC and do the __real blow-up__ of $p \mapsto \arg \lambda$.

Or just consider $\arg \lambda$ on $\tilde{V} \setminus (\tilde{V}_0 \cup \tilde{V}_\infty)$ as a smooth function.

Sard's theorem: $\exists\, \varphi$ regular value of $\arg \lambda$.

$$t \cdot (1 : 0) + (1-t)(0 : 1) \quad , \quad t \in \mathbb{R}$$

Can we do that in our example?

$(1 + c_1 + c_2) \cap [P] = [P] + c_1 \cap [P] + c_2 \cap [P]$

projeziert nach?

$0 \to T\hat{V} \to \xi \to \xi/T\hat{V} \to 0$

$P \downarrow \hat{V}$ i.d.R. mit Dimensionsdefekt

$r = \dim \hat{V} = \operatorname{rank} T\hat{V}$, $m = \dim P$

$\dim [P] = 2m$, $c_{top}(\xi/T\hat{V}) \in H^?(P)$

for some integer p_i. $\boxed{\operatorname{rank} \xi = d}$, $\dim N \geq \dim Y \geq \dim V = \dim \hat{V}$

Proof. A dimension count shows that ρ_{i*} Dual $c(\xi/TV_i) = \rho_{i*}c_n(\xi/TV_i)$ where n is the dimension of ξ/TV_i. The equality with the Euler class then is standard. The expression lives in the same dimension homology as $\mathcal{O}_{\hat{V}_i}$ and, since $\hat{V}_i$ is irreducible, must therefore be a multiple of $\mathcal{O}_{\hat{V}_i}$.

$n = d - r$ = Dimension der generischen Faser von $f^{-1}(V) \to V$

Finally, we are ready to define the algebraic cycle $\sum n_i V_i$ of the reduction. The V_i are the V_i constructed above and n_i is given by $m_i \cdot p_i$.

6. Proof of equation (1)

The constructions of the preceding section were expressly designed to make the following manipultation work. Lemmas 3, 4 and 5 provide the key. The symbol $\mathcal{O}$ with a subscript always denotes the orientation homology class of the (possibly singular) variety named in the subscript.

$$
\begin{aligned}
f_* \operatorname{Dual} c(x) &= f_* \operatorname{Dual} c(TX) \quad ✓\\
&= f_* \operatorname{Dual} c(s_0^*\xi) \quad ✓ \qquad s_0 : X \to X = (TX \oplus 0)\\
&= f_*\mathcal{O}_X \frown s_0^* c(\overset{\pi^*}{\xi}) \quad ✓\\
&= f_*\pi_*s_{0*}(\mathcal{O}_X \frown s_0^*c(\xi)) \quad ✓\\
&= f_*\pi_*(s_{0*}\mathcal{O}_X \frown c(\xi)) \quad \text{projection formula auf Gr}\\
f_*\pi_*\big((\underbrace{\textstyle\sum_i m_i \mu_{i*}\mathcal{O}_{P_i}}_{=\, s_{0*}\mathcal{O}_X}) \cap c(\xi)\big) &= \sum m_i f_*\pi_*(\mu_{i*}\mathcal{O}_{P_i} \frown c(\xi)) \quad ✓\\
&= \sum m_i f_*\pi_*\mu_{i*}(\mathcal{O}_{P_i} \frown \mu_i^* c(\xi)) \quad \text{Rückzug auf } P_i\\
&= \sum m_i \nu_{i*}\rho_{i*}(\mathcal{O}_{P_i} \frown c(\xi)) \quad \text{Kommutativität } \pi\circ\mu_i = \nu_i \circ \rho_i\\
&= \sum m_i \nu_{i*}\rho_{i*}(\mathcal{O}_{P_i} \frown [c(\xi/TV_i) \smile c(TV_i)]) \quad \text{Whitney sum}\\
&= \sum m_i \nu_{i*}\rho_{i*}([\mathcal{O}_{P_i} \frown c(\xi/TV_i)] \frown c(TV_i)) \quad \text{cap-cup-compat.}\\
&= \sum m_i \nu_{i*}\rho_{i*}([\mathcal{O}_{P_i} \overset{\text{typo}}{\frown} c(\xi/TV_i)] \frown \rho_i^* c(TV_i))\\
= \textstyle\sum_i m_i \nu_{i*}\big((\rho_{i*}([\mathcal{O}_{P_i} \cap c(\xi/TV_i)]) \cap c(TV_i)\big) &= \sum m_i \nu_{i*}(\rho_{i*} \operatorname{Dual} c(\xi/TV_i) \frown c(TV_i)) \quad \text{missing bracket}\\
&= \sum m_i \nu_{i*}(p_i\mathcal{O}_{\hat{V}_i}) \frown c(TV_i) \quad \text{Lemma 5}\\
&= \sum m_i p_i \nu_{i*} \operatorname{Dual} c(TV_i) \quad \text{Def. von Dual}\\
&= \sum n_i \operatorname{incl}_{i*} c_M(V_i)\,. \quad \text{Def. Mather-Klasse}
\end{aligned}
$$

We note that this section and the last already contain the proof of the following weakening of the Deligne-Grothendieck conjecture: the information needed to evaluate f_* Dual $c(X)$ may be extracted from *some* locally determined constructible function on Y. ("Locally determined" here means that to evaluate the function on an open set it is enough to know f is restricted to the inverse image of that open set.) Even this appears to be false for other characteristic classes, e.g., the Todd genus. What fails in the proof is the crucial Lemma 5.

7. Proof of equation (2)

This proof will parallel step by step the proof of equation (1) in the last section. The trick is to find an object which manipulates like $c(X)$ but which measures the Euler characteristic of $f^{-1}(p)$ (which, of course, may be singular).

About p on Y in N we construct the ε-ball and the ε-sphere as in the section on the local Euler obstruction. For each space of the diagram on Page 428 we let B, followed by its symbol, denote the pullback of the ε-ball and S, followed by its symbol, denote the pullback of the ε-sphere. We continue the convention that $\mathcal{O}$ subscripted denotes the orientation class of the variety or the pair named in the subscript; $\mathrm{Eu}(\cdot)$ denotes the Euler class of the named vector bundle; and $\mathrm{Eu}(\cdot, r)$ denotes the obstruction to extending the nonvanishing partial section r of the bundle.

Let r be the section of TN^* (real dual) defined by $d\|\ \|^2$. We denote by the same letter r the pullback of this section of TN^* over the other spaces of our diagram and also, as in the section on the local Euler obstruction, the restriction to TV_i^*. Let t be the section of TX^* given by the pullback of r by df. Again we denote by the same letter t the pullback of this section over Z_i and P_i. $t \oplus r$ is a section of $TX^* \oplus TN^*$ over Z; this restricts to a section of ξ^* over Z and over P_i.

LEMMA 6. *For small enough* ε,

(a) t *is a nonvanishing section of* TX^* *over* SX;

(b) $t \oplus r$ *is a nonvanishing section of* ξ^* *over* SZ;

(c) r *is a nonvanishing section of* $T\hat{V}_i^*$ *over* SP_i.

Proof. Statement (a) follows from the Sard-Bertini theorem. Statement (b) for SZ_λ is an exercise in linear algebra and for SZ_∞ follows from Lemma 4 and statement (c). Statement (c) follows directly from Lemma 1.

From now on, we fix ε small enough to satisfy the requirements of Lemma 1 and Lemma 6. We consider all sections as being defined over the preimage of the ε-sphere.

LEMMA 7. $\chi f^{-1}(p) = \langle \mathrm{Eu}(TX^*, t), \mathcal{O}_{(BX,SX)} \rangle$

Proof. BX is a manifold with boundary which deformation retracts onto $f^{-1}(p)$, so $\chi f^{-1}(p) = \chi BX$. The boundary SX of BX is odd-dimensional and hence has Euler characteristic zero, so $\chi BX = \chi(BX, SX)$. That $\chi(BX, SX) = \langle \mathrm{Eu}(TX^*, t), \mathcal{O}_{(BX,SX)} \rangle$ may be seen by generalizing virtually any of the usual proofs relating the Euler class to the Euler characteristic.

The proof of equation (2) now goes as follows:

$$\begin{aligned}
\chi f^{-1}(p) &= \langle \mathrm{Eu}(TX^*, t), \mathcal{O}_{(BX,SX)}\rangle \\
&= \langle s_0^* \,\mathrm{Eu}(\xi^*, t \oplus r), \mathcal{O}_{(BX,SX)}\rangle \\
&= \langle \mathrm{Eu}(\xi^*, t \oplus r), s_{0*}\mathcal{O}_{(BX,SX)}\rangle \\
&= \langle \mathrm{Eu}(\xi^*, t \oplus r), \textstyle\sum m_i \mu_{i*}\mathcal{O}_{(BP_i,SP_i)}\rangle \\
&= \textstyle\sum m_i \langle \mu_i^*(\mathrm{Eu}(\xi^*, t \oplus r)), \mathcal{O}_{(BP_i,SP_i)}\rangle \\
&= \textstyle\sum m_i \langle \mathrm{Eu}(\xi^*/TV_i^* \oplus TV_i^*, 0 \oplus r), \mathcal{O}_{(BP_i,SP_i)}\rangle \\
&= \textstyle\sum m_i \langle \mathrm{Eu}(\xi^*/TV_i^*) \smile \mathrm{Eu}(TV_i^*, r), \mathcal{O}_{(BP_i,SP_i)}\rangle \\
&= \textstyle\sum m_i \langle \rho_i^* \,\mathrm{Eu}(TV_i^*, r), \mathrm{Eu}(\xi^*/TV_i^*) \frown \mathcal{O}_{(BP_i,SP_i)}\rangle \\
&= \textstyle\sum m_i \langle \mathrm{Eu}(TV_i^*, r), \rho_{i*}\,\mathrm{Dual}\,\mathrm{Eu}(\xi^*/TV_i^*)\rangle \\
&= \textstyle\sum m_i \langle \mathrm{Eu}(TV_i^*, r), p_i\mathcal{O}_{(B\hat{V}_i,S\hat{V}_i)}\rangle \\
&= \textstyle\sum m_i p_i \langle \mathrm{Eu}(TV_i^*, r), \mathcal{O}_{(B\hat{V}_i,S\hat{V}_i)}\rangle \\
&= \textstyle\sum n_i \,\mathrm{Eu}_p(V_i)\ .
\end{aligned}$$

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