Understanding MacPherson - Lemma 5 — Proof of Naturality

19.12.2022

- Remember: → Orientation / Fundamental classes always exist, since we use Borel-Moore homology.

  Notation: $\mathcal{O}_{\hat{V}_i} \in H_{2\cdot\dim_{\mathbb{C}}(\hat{V}_i)}(\hat{V}_i)$

  → $\mu \in H_k(M)$, $\phi \in H^r(M) \Rightarrow \mu \cap \phi \in H_{k-r}(M)$

- (Complex) dimension count

$$\begin{tikzcd}
\xi \subseteq TX \oplus TN \supseteq TV_i \arrow[d] & & \xi \subseteq TX \oplus TN \arrow[d] & \\
P_i \arrow[r, "\mu_i"] \arrow[dd, "\rho_i"'] & V_i'^{\,d} \arrow[r, hook] \arrow[dd, "\eta_i"'] & Z \arrow[d, "\pi"'] & \\
 & & X^{d} \arrow[u, bend right, "s_\lambda"'] \arrow[d, "f"'] & TX \arrow[l] \quad TN \arrow[d] \\
\hat{V}_i^{\,m} \arrow[r, "\nu_i"] & V_i^{\,m} \arrow[r, hook] & Y \arrow[r, hook] & N
\end{tikzcd}$$

$TV_i \to \hat{V}_i$

Q:

- $\mathcal{O}_{P_i} \in H_{2d}(P_i)$
- $\dim(\xi / TV_i) = \underbrace{\dim(\mu_i^* \xi)}_{d} - \underbrace{\dim(TV_i)}_{m} = d-m$

  top Chern class $c^{2d-2m}(\xi/TV_i)$
- $\dim(\hat{V}_i) = m$

Facts about P_i:

- $P_i \subseteq \hat{V}_i \times_{V_i} V_i' \xrightarrow{\pi_2} V_i'$, $\quad \downarrow \pi_1$, $\quad \downarrow \eta_i$, $\quad \hat{V}_i \xrightarrow{\nu_i} V_i$

  $P_i = \text{closure}\left(\rho_i^{-1}(\nu_i^{-1}(V_i)_{reg})\right) \subseteq \hat{V}_i \times_{V_i} V_i'$

  $\dim(\eta_i^{-1}(v_i)) = d-m$

  $\Rightarrow \dim_{\mathbb{C}} P_i = \underbrace{d-m}_{\text{Fibers}} + \underbrace{m}_{\text{base}} = d$

$\rho_i^*(TV_i) \leq \mu_i^*(\xi)$ (P_i)

Lemma 5. $\rho_{i*} \operatorname{Dual} c(\overline{\xi/TV_i}) = \rho_{i*} \operatorname{Dual} \operatorname{Eu}(\xi/TV_i)$ ✓

$= p_i \mathcal{O}_{\hat{V}_i}$ for some $p_i \in \mathbb{Z}$.

Proof: $c^*(\cdot) = 1 + c^1(\cdot) + \ldots + \underbrace{c^{\min(rk, \dim Base)}(\cdot)}_{= \operatorname{Eu}(\cdot)}$ $\quad rk \leq \dim_{Base}$

$\operatorname{Dual} c^* = \mathcal{O}_{P_i} \cap 1 + \mathcal{O}_{P_i} \cap c^1(\cdot) + \ldots + \underbrace{\mathcal{O}_{P_i} \cap \operatorname{Eu}}_{\in H_{2d-(2d-2m)}(P_i) = H_{2m}(P_i)} \in H_*(P_i)$

$\rho_{i*}(\operatorname{Dual} c^*) = \underbrace{\rho_{i*}(\mathcal{O}_{P_i} \cap 1) + \rho_{i*}(\mathcal{O}_{P_i} \cap c^1(\cdot)) + \ldots}_{\overset{!}{=} 0 \text{ by dimension.}} + \underbrace{\rho_{i*}(\mathcal{O}_{P_i} \cap \operatorname{Eu})}_{H_{2m}(\hat{V}_i)} \in H_*(\hat{V}_i)$

$H_k(\hat{V}_i) = 0$ for $k > 2m$

Topology: $H_{2m}(\hat{V}_i) = \langle \mathcal{O}_{\hat{V}_i} \rangle$

BUILDING BLOCKS

- Naturality: $c^*(f^*V) = f^*c^*(V)$
- Whitney-Sum: $0 \to V \to W \to U \to 0 \Rightarrow c^*(W) = c^*(U) \cup c^*(V)$
- Pushforward-Pullback-Relation: $f_*(\mu \cap f^*\phi) = (f_*\mu) \cap \phi$
- Distributivity of $\cup$ & $\cap$: $\mu \cap (\phi \cup \psi) = (\mu \cap \phi) \cap \psi$
- Definition: $\mathrm{Dual}(\phi) = \mathcal{O}_? \cap \phi$

  $s_0 : X \xrightarrow{\xi} X = TX \oplus 0 \subseteq Gr_d(TX \oplus TN)$
- Lemma 3: $s_{0*}\mathcal{O}_X = \sum m_i \mu_{i*} \mathcal{O}_{P_i}$
- Lemma 5: $\rho_{i*}\,\mathrm{Dual}\, c(\xi/TV_i) = \rho_{i*}\,\mathrm{Dual}\,\mathrm{Eu}(\xi/TV_i)$
  $= p_i \mathcal{O}_{\hat{V}_i}$

$$
\begin{aligned}
f_*\,\mathrm{Dual}\, c(x) &= f_*\,\mathrm{Dual}\, c(TX) && X \text{ smooth}\\
&= f_*\,\mathrm{Dual}\, c(s_0^*\xi) && \text{def of } s_0 \ \& \ \xi\\
&= f_*(\mathcal{O}_X \frown s_0^* c(\xi)) && \text{nat. of } c \ \& \text{ def dual}\\
&= f_*\pi_* s_{0*}(\mathcal{O}_X \frown s_0^* c(\xi)) && \pi \circ s_0 = \mathrm{id}_X\\
&= f_*\pi_*(\underbrace{s_{0*}\mathcal{O}_X}_{=\sum m_i \mu_{i*}\mathcal{O}_{P_i}} \frown c(\xi)) && \text{Pushforward-Pullback-Relation} \quad \text{Lemma 3}\\
&= \sum m_i f_*\pi_*(\mu_{i*}\mathcal{O}_{P_i} \frown c(\xi)) && \text{Linearity}\\
&= \sum m_i f_*\pi_*\mu_{i*}(\mathcal{O}_{P_i} \frown \mu_i^* c(\xi)) && \text{Pushforward-Pullback-Relation}\\
&= \sum m_i \nu_{i*}\rho_{i*}(\mathcal{O}_{P_i} \frown c(\xi)) && \text{"} \nu_i \circ \xi_i = \eta_i \circ \mu_i = (f \circ \pi) \circ \mu_i \text{"}\\
&= \sum m_i \nu_{i*}\rho_{i*}(\mathcal{O}_{P_i} \frown [c(\xi/TV_i) \smile c(TV_i)]) && 0 \to TV_i \to \xi \to \xi/TV_i \to 0\\
&= \sum m_i \nu_{i*}\rho_{i*}([\mathcal{O}_{P_i} \frown c(\xi/TV_i)] \frown c(TV_i)) && \cup - \cap - \text{Relation}\\
&= \sum m_i \nu_{i*}\rho_{i*}([\mathcal{O}_{P_i} \frown c(\xi/TV_i)] \frown \rho_i^* c(TV_i)) && \xi_i^* TV_i \text{ over } P_i \quad \text{Naturality of } c\\
&= \sum m_i \nu_{i*}(\underbrace{\rho_{i*}\,\mathrm{Dual}\, c(\xi/TV_i)}_{\text{Lemma 5 } = p_i \mathcal{O}_{\hat{V}_i}}) \frown c(TV_i)) && \text{Pushforward-Pullback-Relation}\\
&= \sum m_i \nu_{i*}(p_i \mathcal{O}_{\hat{V}_i}) \frown c(TV_i)\\
&= \sum m_i p_i \nu_{i*}\,\mathrm{Dual}\, c(TV_i) && \text{def Dual}\\
&= \sum n_i\, \mathrm{incl}_{i*}\, c_M(V_i)\,. && \text{Def Mather class}
\end{aligned}
$$

Next time: $(f_* \mathbb{1}_X)(p) = \chi\, f^{-1}(p)$

$\overset{!}{=} \sum n_i \cdot \mathrm{Eu}_p(V_i)$

Then: $f_* \underbrace{\mathrm{Dual}\, c(X)}_{c(\mathbb{1}_X)} = c(f_* \mathbb{1}_X)$