Characteristic classes on singular spaces – continued
Advertisement for the Seminar

After our introductory talks during the past semester, we would like to continue our seminar on Characteristic classes on singular varieties. To start the semester, we want to understand MacPherson's solution of the Grothendieck-Deligne conjecture. Afterwards, we move our attention to Riemann-Roch type theorems on singular spaces, their connection with characteristic classes and applications in singularity theory. Let us emphasize that suggestions for further topics are welcome and will be discussed.

Organization

- **Contact:** essig@math.uni-kiel.de and zach@mathematik.uni-kl.de
- Seminar time: **Monday, 2:15pm CET**
- We use Zoom to host the seminar.
  Zoom link, Meeting ID: 678 5844 3356, Passcode: 67424
- **First meeting** on November 7th. The first meeting consists of an introduction and a discussion about the date and time of the seminar. Please come to the first meeting, if you are interested.
- **First Talk**: The first talk will be on November 21st.

The tentative schedule is the following.

Part 1: Characteristic classes for singular varieties

We want to get a better feeling of the local Euler obstruction and MacPherson's graph conjecture. The talks are more or less organized as follows.

1st talk *(Constr. Functions, Nash-blowup, Mather class, local Euler obstruction)*

Explain the "Grothendieck-Deligne conjecture" and compare it to char. classes on manifolds. Introduce the Nash blowup and Nash bundle. Define the local Euler obstruction. In the best case, show a good example of a sing. variety embedded in some smooth variety. (E.g. Whitney umbrella) (Mostly 1. – 4. of [6]).

2nd talk *(MacPherson's Graph Construction)*

Explain the graph construction and how it helps to prove the MacPherson-Chern class theorem. (Chapters 5. – 7. of [6]).

Part 2: Riemann-Roch type theorems in singularity theory

We continue with the list of talks and outline the motivation afterwards.

3rd talk *(Gonzalez-Sprinberg)*

Explain the work in [3] and how the local Euler obstruction can be computed algebraically using Chow rings. Introduce the necessary objects and compute at least one non-trivial concrete example. One might want to look into [7].

4th talk *(Classical Riemann-Roch theorems)*
Discuss the classical Riemann-Roch theorem for curves. Assume Serre duality and proof the modified version with the holomorphic Euler characteristic on the left hand side. Discuss one example for a divisor on a plane curve or on a family of plane curves in detail: Compute the cohomology of the associated line bundles, the degrees, etc.

5th talk *(Riemann-Roch for surfaces and in general)*
Discuss the Hirzebruch-Riemann-Roch theorem and how it specializes to its versions for curves and surfaces. Compute and verify one non-trivial example for a projective surface.

6th talk *(Excursion: K-groups and derived categories)*
Introduce the K-groups of vector bundles and coherent sheaves as needed for [1]. Introduce the higher direct images of a coherent sheaf along a proper map and explain how $K_0^{\mathrm{alg}}(X)$ becomes a covariant functor. Exhibit at least one non-trivial example of a proper morphism, say, with projective fibers and a derived pushforward of a coherent sheaf or a complex of coherent sheaves.

7th talk *(Riemann-Roch for singular varieties)*
Go through the paper by Baum, Fulton, and MacPherson [1].

8th talk *(Local Riemann-Roch theorems)*
Explain how the Milnor's formula can be interpreted as a local "Riemann-Roch-type theorem". Compute the local Euler obstruction of the affine cone of a smooth projective variety with various different methods.

Background on Part 2

Consider the classical Riemann-Roch theorem for a line bundle $\mathcal{L}$ on a smooth projective curve C:

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^\vee \otimes \omega_C) = \deg(\mathcal{L}) + 1 - g. \tag{1}$$

Using Serre duality, one can rewrite the left hand side as

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^\vee \otimes \omega_C) = h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \chi(\mathcal{L})$$

the *holomorphic Euler characteristic* of $\mathcal{L}$. On the right hand side, we find a purely *topological* invariant given by the degree

$$\deg(\mathcal{L}) = \int_C c_1(\mathcal{L}) = \int_C \mathrm{Eu}(\mathcal{L}),$$

which is nothing but the integral over the Euler class, and a correction term given by the *genus* g of the curve. In this spirit, whenever we speak of a "Riemann-Roch type theorem", we mean a theorem that does exactly this: Express a topological invariant as a holomorphic Euler characteristic.

Where does this occur in Singularity theory?

– One example is Milnor's formula for an isolated hypersurface singularity $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$:

$$\mu(f) = \dim_{\mathbb{C}} \mathcal{O}_{n+1}/\mathrm{Jac}(f).$$

The left hand side is the topological obstruction to extending the 1-form $\mathrm{d}f$ from a small sphere $S_\varepsilon^{2n+1} \subset \mathbb{C}^{n+1}$ around the origin as a nowhere vanishing section to its interior. The right hand side can be interpreted as the holomorphic Euler characteristic of the *Koszul-complex* $(\Omega^*_{\mathbb{C}^{n+1},0}, \mathrm{d}f \wedge -)$ induced by the differential of f.

Why is this interesting?

– Topological invariants are notoriously hard to compute. For algebraic invariants, such as holomorphic Euler characteristics, we do have a chance using computer algebra systems. Thus, whenever we can express a topological invariant in terms of the latter, we have a chance to actually get our hands on it. As a tentative goal for this seminar I propose that we reach a position in which we can effectively solve the following problem:
"Given $f \in \mathbb{Q}[x_0, \dots, x_n]$, compute a Whitney stratification for the variety $X = V(f)$ and the constructible function $\mathrm{E}u$ given by the local Euler obstruction along the strata. "

As a starting point for the reading we propose [1] to first look into the following questions:

- What is the K-group $K^0(X)$ of a topological space X?
- What is the Grothendieck group of coherent sheaves $K_0^{\mathrm{alg}}(X)$? How is it covariant? How does this relate to the derived category of X?
- What is its topological analogue $K_0^{\mathrm{top}}(X)$?
- How are the Chern characters defined in either theory? Why do the obvious diagrams commute?
- How do the classical Riemann-Roch theorems follow from this?

After this, we turn towards the local theory. Gonzalez-Sprinberg has given an algebraic formula for the local Euler obstruction via an involved blowup procedure [3] that we could start with. I propose to go through the proof and see how things can be rephrased so that one could possibly obtain another algebraic formula without the blowups – possibly through some projection formula. (Note, that there is a (partial) English translation of the Gonzales-Sprinberg paper by Jiang in [5].)

As a tentative direction, I would like to go towards algebraic residues as described in [4, Chapter 5]. To explain this, recall from [4] that the multidimensional residue of a rational $n+1$-form

$$\omega = \frac{h(z)\mathrm{d}z_0 \wedge \cdots \wedge \mathrm{d}z_n}{f_0 \cdots f_n}$$

for some regular sequence $f_0, \dots, f_n \in \mathcal{O}_{n+1}$ can be transformed into a sphere integral:

$$\frac{1}{(2\pi\sqrt{-1})^{n+1}} \int_{|f_0|=r_0} \cdots \int_{|f_n|=r_n} \omega = \int_{S_\varepsilon^{2n+1}} \eta_\omega$$

where η_ω is the *dolbeault representative* of ω. While the left hand side can be evaluated algebraically, at least for algebraic input, the right hand side has the form of a local topological obstruction: It could arise as the degree of a map of spheres. In fact, this leads to a residue formula for the Milnor number, see [2].

In the case where $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ is the affine cone of a projective variety $Y \subset \mathbb{P}^n$, we can in fact compute the local Euler obstruction of $(X, 0)$ at the origin via an algebraic residue calculus. The hope is to find an "easy" algebraic formula for the local Euler obstruction which extends this result in general.

References

[1] P. Baum, W. Fulton, and R. MacPherson. "Riemann-Roch and topological K-theory for singular varieties". In: *Acta mathematica* (1979).

[2] J. P. Brasselet et al. "Milnor numbers and classes of local complete intersections". In: *Proceedings of the Japan Academy of Sciences.* A 75.10 (1999).

[3] Gerardo González-Sprinberg. "L'obstruction locale d'Euler et le théorème de MacPherson". In: *The Euler-Poincaré characteristic (French).* Vol. 83. Astérisque. Soc. Math. France, Paris, 1981, pp. 7–32.

[4] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978, pp. xii+813. ISBN: 0-471-32792-1.

[5] Yunfeng Jiang. "Note on MacPherson's Local Euler Obstruction". In: *Michigan Mathematical Journal* 68.2 (2019), pp. 227 –250. DOI: 10.1307/mmj/1548817530. URL: https://doi.org/10.1307/mmj/1548817530.

[6] R. D. MacPherson. "Chern classes for singular algebraic varieties". In: *Ann. of Math. (2)* 100 (1974), pp. 423–432. ISSN: 0003-486X. DOI: 10.2307/1971080. URL: https://doi.org/10.2307/1971080.

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