CONSTRUCTIBLE FUNCTION FUNCTOR

$$\mathrm{KAlgSet}/\mathbb{C} \xrightarrow{F} \mathrm{Ab}$$

$$\begin{array}{ccc} X & \longmapsto & F(X) = \{X \to \mathbb{C} \text{ constructible}\} \\ \downarrow f & & \downarrow f_* \quad (\cdots) \\ Y & \longmapsto & F(Y) \end{array}$$

$(\cdots)\quad f_* : F(X) \longrightarrow F(Y)$

$$\mathbb{1}_W \longmapsto (q \mapsto \chi(f^{-1}(q)))$$

HOMOLOGY FUNCTOR

$$\mathrm{KAlgSet}/\mathbb{C} \xrightarrow{H_*} \mathrm{Ab}$$

$$\begin{array}{ccc} X & \longmapsto & H_*(X) \\ \downarrow f & \longmapsto & \downarrow f_* \\ Y & \longmapsto & H_*(Y) \end{array}$$

ALGEBRAIC CYCLE FUNCTOR

$$\mathrm{KAlgSet}/\mathbb{C} \xrightarrow{Z_*} \mathrm{Ab}$$

$$\begin{array}{ccc} X & \longmapsto & Z_*X = \{\sum n_i V_i \mid n_i \in \mathbb{Q}, V_i \underset{\text{subvar}}{\subseteq} X\} \\ \downarrow f & & \downarrow f_* \\ Y & \longmapsto & Z_* Y \end{array}$$

$$f_*(V_i) = f(V_i)$$

THE LOCAL EULER OBSTRUCTION

UNNATURAL TRANSFORMATION $Z_* \overset{T}{\Rightarrow} F$

$$
\begin{tikzcd}
Z_* X \arrow[r, "T_X"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\cancel{\circlearrowleft}"] & FX \arrow[d, "f_*"] \\
Z_* Y \arrow[r, "T_Y"'] & FY
\end{tikzcd}
$$

$$T_X : Z_* X \longrightarrow FX$$

$$V \longmapsto T_X V : (p \mapsto Eu_p(V))$$

Prop: $\forall X \in \mathrm{KAlgVar}/\mathbb{C}$, $T_X : Z_* X \longrightarrow FX$ isomorphism

This gives rise to an inverse unnatural transformation

$$FX \xrightarrow{T^{-1}} Z_* X$$

$T^{-1}(\mathbb{1}_W) = \sum n_i V_i$ so that for all $p \in X$

$$\sum n_i Eu_p(V_i) = \begin{cases} 1 & p \in W \\ 0 & \text{—} \end{cases}$$

EX: Let $C \subseteq \mathbb{P}^2$ be the (projective closure of) the cusp

Since $\operatorname{Sing} C = \{(0:0:1)\}$ and $\operatorname{mult}(C,(0:0:1)) = 2$,

$$T(C)(p) = \begin{cases} 1 & p \in C \setminus \{(0:0:1)\} \\ 2 & p = (0:0:1) \\ 0 & \mathbb{P}^2 \setminus C \end{cases} = \mathbb{1}_C + \mathbb{1}_{\{(0:0:1)\}}$$

$$T^{-1}(\mathbb{1}_C) = 1 \cdot C - 1 \cdot \{(0:0:1)\}$$

THE CHERN-MATHER UNNATURAL TRANSFORMATION

$$Z_* \overset{c_M}{\twoheadrightarrow} H_*$$

$$\begin{tikzcd}
Z_*X \arrow[r, "c_{MX}"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\cancel{\circlearrowleft}"] & H_*X \arrow[d, "f_*"] \\
Z_*Y \arrow[r, "c_{MY}"'] & H_*Y
\end{tikzcd}$$

$$c_{M,X} : Z_*X \longrightarrow H_*X \qquad f_*(f^*c(\tilde{T}_V) \cap [\mathbb{P}^1]) = f_*(3 + [\mathbb{P}^1])$$

$$V \longmapsto incl_* \vartheta_*(c(\tilde{T}_V) \cap [\tilde{V}])$$

$\tilde{V} \subseteq G(M)$ grass. of $(\dim V)$-subspaces of tangent of ambient space M

closure of "obvious section on reg part"

$$\begin{tikzcd}
\tilde{T}_V \arrow[r, hook] \arrow[d] & \text{Tauto of } V \arrow[d] \\
\tilde{V} \arrow[r, hook] \arrow[d, "\vartheta"'] & G(M) \arrow[d] \\
V \arrow[r, hook] & M
\end{tikzcd}$$

$\tilde{T}_V$ = restriction of tautological bundle of $G(M)$

Ex: The Chern-Mather class of the cusp C

$$c_M(C) = 3 + [C] \quad (\text{see } ***)$$

THE CHERN-MACPHERSON NATURAL TRANSFORMATION

$$F \overset{c_*}{\Rightarrow} H_*$$

$$c_* = c_M \circ T^{-1}$$

$$\begin{tikzcd}
FX \arrow[r, "c_{*X}"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowright"] & H_* X \arrow[d, "f_*"] \\
FY \arrow[r, "c_{*Y}"'] & H_*(Y)
\end{tikzcd}$$

(1) Additivity: group homo.

$$c_*\left(\sum n_i \mathbb{1}_{W_i}\right) = \sum n_i c_* \mathbb{1}_{W_i}$$

(2) Naturality:

$$f_*(c_* \mathbb{1}_W) = c_*(f_* \mathbb{1}_W)$$

Moreover,

(3) c_* extends c in the sense that

$$c_* \mathbb{1}_X = c(X) \cap [X] \text{ for } X \text{ smooth.}$$

*** EXAMPLE OF CHERN-MATHER CLASS ***

$$C = V(x^3 - y^2 z) \subseteq \mathbb{P}^2$$

= projectivization of the cusp $\{x^3 - y^2\}$

= im f, where

$$\mathbb{P}^1 \xrightarrow{f} C \subseteq \mathbb{P}^2$$

$$(t,s) \longmapsto (t^2 s, t^3, s^3)$$

We are going to show that

$$c_M(C) = 3 + [C]$$

STEP 1: (Reduction to computation of a Chern class over $\mathbb{P}^1$)

Claim 1: f is a homeomorphism

Pf: One can check that f is bijective. Since it is a continuous bijection between a compact space and a Hausdorff space, it is a homeomorphism.

Claim 2: The Nash blowup $\vartheta : \tilde{C} \longrightarrow C$ is a homeomorphism

Pf: Same as Claim 1. To convince yourself that ϑ is bijective, observe that there is a single "limit tangent plane" on the singular point of C (Obs: C is localy isomorphic to $V(x^3-y^2)$ around each curve's singular point)

We need to compute

$$c_M(C) = \vartheta_*(c^*(\tilde{T}_C) \cap [\tilde{C}])$$

$$\begin{tikzcd}
\tilde{T}_V \arrow[r, hook] \arrow[d] & E \arrow[d] \\
\tilde{C} \arrow[r, hook] \arrow[d, "\vartheta"'] & G(\mathbb{P}^2) \arrow[d] \\
C \arrow[r, hook] & \mathbb{P}^2
\end{tikzcd}$$

The parametrization f and claim 2 allow us to extend this diagram as follows

$$\begin{tikzcd}
\tilde f^*\tilde T_C \arrow[r] \arrow[dd] & \tilde T_C \arrow[r, hook] \arrow[d] & E \arrow[d] \\
 & \tilde C \arrow[r, hook] \arrow[d, shift right, "\nu"'] & G(\mathbb{P}^2) \arrow[d] \\
\mathbb{P}^1 \arrow[r, "f"'] \arrow[ur, "\tilde f"] & C \arrow[r, hook] \arrow[u, shift right] & \mathbb{P}^2
\end{tikzcd}$$

Here (1) $\tilde f^*\tilde T_C$ has the same fibres as $\tilde T_V$ but they lay over $\mathbb{P}^1$, instead of $\tilde C$

(2) $\tilde f = \nu^{-1}\circ f$ is a homeomorphism

Claim 3: $\boxed{c_M(C) = f_*(c^*(\tilde f^*\tilde T_C)\cap[\mathbb{P}^1])}$

Pf:
$$\begin{aligned}
c_M(C) &= \nu_*(c(\tilde T_C)\cap[\tilde C]) \\
&= \nu_*(c(\tilde T_C)\cap \tilde f_*[\mathbb{P}^1]) && \text{(by Claim 1)} \\
&= \nu_*\tilde f_*(c^*(\tilde f^*\tilde T_C)\cap[\mathbb{P}^1]) && \text{(by Proj. Formula)} \\
&= \nu_*\nu_*^{-1} f_*(c^*(\tilde f^*\tilde T_C)\cap[\mathbb{P}^1]) && \text{(by (2))} \\
&= f_*(c^*(\tilde f^*\tilde T_C)\cap[\mathbb{P}^1]).
\end{aligned}$$

STEP 2: Trivializing open cover of $\tilde{f}^*\tilde{T}_C$.

By definition, we have that

$$\tilde{f}^*\tilde{T}_C = \{((t:s),v) \mid (t:s)\in\mathbb{P}^1, v\in T_{f(t:s)}C\}$$

where, for the singular point $(0:1)$ of f, we set

$$T_{(0:1)}C := \lim_{\alpha\mapsto(0:1)} T_{f(\alpha)}C$$

We want to compute the transition functions of a trivializing open cover of $\tilde{f}^*\tilde{T}_C$, making use of the affine open covers

$$\mathbb{P}^1 = \mathcal{U}_1 \cup \mathcal{U}_2 \qquad \mathbb{P}^2 = V_1\cup V_2\cup V_3$$

* Since $f(\mathcal{U}_1)\subseteq V_2$ and $f(\mathcal{U}_2)\subseteq V_3$, we may use

The affine cover of $\mathbb{P}^1$

$$\mathbb{P}^1 = \mathcal{U}_1\cup\mathcal{U}_2 \qquad \mathcal{U}_i = \{(u_1:u_2) \mid u_i\neq 0\}$$

$$\varphi_1:\mathcal{U}_1 \longrightarrow \mathbb{C} \qquad \varphi_2:\mathcal{U}_2\longrightarrow\mathbb{C}$$

$$(u_1:u_2)\longmapsto u_2/u_1 \qquad (u_1:u_2)\longmapsto u_1/u_2$$

$$(1:\ell) \longleftarrow \ell \qquad (k:1)\longleftarrow k$$

The affine cover of $\mathbb{P}^2 = V_1 \cup V_2 \cup V_3$ $V_i = \{(v_1:v_2:v_3) \mid v_i \neq 0\}$

$$\varphi_1 : V_1 \longrightarrow \mathbb{C}^2 \qquad \varphi_2 : V_2 \longrightarrow \mathbb{C}^2 \qquad \varphi_3 : V_3 \longrightarrow \mathbb{C}^2$$

$$(v_1:\_:v_3) \mapsto \left(\frac{v_2}{v_1}, \frac{v_3}{v_1}\right) \qquad (v_1:\_:v_3) \mapsto \left(\frac{v_1}{v_2}, \frac{v_3}{v_2}\right) \qquad (v_1:\_:v_3) \mapsto \left(\frac{v_1}{v_3}, \frac{v_2}{v_3}\right)$$

$$(1:a_1:a_2) \leftarrow\!\!\mapsto (a_1, a_2) \quad (1:b_1:b_2) \leftarrow\!\!\mapsto (b_1, b_2) \quad (1:c_1:c_2) \leftarrow\!\!\mapsto (c_1, c_2)$$

We trivialize $\tilde{f}^*\tilde{T}_C$ by replacing $f|_{\mathcal{U}_1}$ and $f|_{\mathcal{U}_2}$ by

$$\sigma_1 = \varphi_2 \circ f \circ \varphi_1^{-1} : \mathbb{C} \longrightarrow \mathbb{C}^2 \quad \ell \longmapsto (\ell, \ell^3)$$

$$\sigma_2 = \varphi_3 \circ f \circ \varphi_2^{-1} : \mathbb{C} \longrightarrow \mathbb{C}^2 \quad k \longmapsto (k^2, k^3)$$

To be precise, we use the following vector bundle isomorphisms

$$\tilde{f}^*\tilde{T}_C|_{\mathcal{U}_1} = \{((u_1:u_2), v) \mid (u_1:u_2) \in \mathcal{U}_1,\ v \in T_{f(u_1:u_2)} C\}$$

$$\downarrow \varphi_1 \times d\varphi_2$$

$$\cong \{(\ell, \tilde{v}) \mid \ell \in \mathbb{C},\ \tilde{v} \in T_{\sigma_1(\ell)}\{y = x^3\}\}$$

$$= \{(\ell, \lambda(1, 3\ell^2)) \mid \ell, \lambda \in \mathbb{C}\}$$

$$\overset{\alpha_1}{\cong} \{(\ell, \lambda) \mid \ell, \lambda \in \mathbb{C}\} = \mathbb{C} \times \mathbb{C} \text{ as trivial bundle over } \mathbb{C}.$$

$$\boxed{\tilde{f}^*\tilde{T}_C|_{\mathcal{U}_1} \xrightarrow{\alpha_1 \circ (\varphi_1 \times d\varphi_2)} \mathbb{C} \times \mathbb{C}}$$

Analogously,

$$f^*\tilde{T}_{\mathcal{C}}|_{\mathcal{U}_2} = \{((u_1:u_2),v) \mid (u_1:u_2)\in\mathcal{U}_2,\ v\in T_{f(u_1:u_2)}\mathcal{C}\}$$

$$\Big\downarrow \varphi_2\times d\psi_3$$

$$\cong \{(k,\tilde{v}) \mid k\in\mathbb{C},\ \tilde{v}\in T_{\sigma_2(k)}\{x^3-y^2\}\}$$

$$T_{(0,0)}\{x^3-y^2\} := \mathrm{Span}\{(1,0)\}$$

$$= \{(k,\mu(1,\tfrac{3}{2}k)) \mid k,\mu\in\mathbb{C}\}$$

$$\overset{\alpha_2}{=} \{(k,\mu) \mid k,\mu\} = \mathbb{C}\times\mathbb{C} \text{ triv. bndle over } \mathbb{C}.$$

$$\boxed{\tilde{f}^*\tilde{T}_{\mathcal{C}}|_{\mathcal{U}_2} \xrightarrow{\alpha_2\circ(\varphi_2\times d\psi_3)} \mathbb{C}\times\mathbb{C}}$$

STEP 3: The transition functions

We need to compute the expression of

$$t = \alpha_2\circ(\varphi_2\times d\psi_3)\circ(\varphi_1^{-1}\times d\psi_2^{-1})\circ\alpha_1^{-1} =$$

$$= \alpha_2\circ\big((\varphi_2\circ\varphi_1^{-1})\times(d\psi_3\circ d\psi_2^{-1})\big)\circ\alpha_1^{-1}$$

$$= \alpha_2\circ\big((\varphi_2\circ\varphi_1^{-1})\times d(\psi_3\circ\psi_2^{-1})\big)\circ\alpha_1^{-1}$$

* $\varphi_2 \circ \varphi_1^{-1}(\ell) = \varphi_2(1:\ell) = 1/\ell$

* To compute $d(\psi_3 \circ \psi_2^{-1})$, we let

$$V_{23} = V_2 \cap V_3 = \{(v_1 : v_2 : v_3) \in \mathbb{P}^2 \mid v_2 \neq 0 \neq v_3\}$$

$$A_2 := \psi_2(V_{23}) = \{(b_1, b_2) \mid b_2 \neq 0\}$$

$$A_3 := \psi_3(V_{23}) = \{(c_1, c_2) \mid c_2 \neq 0\}$$

$$\begin{tikzcd} & V_{23} \arrow[dl, "\psi_2"'] \arrow[dr, "\psi_3"] & \\ A_2 \arrow[rr] & & A_3 \end{tikzcd}$$

$$(b_1, b_2) \longmapsto \left(\frac{b_1}{b_2}, \frac{1}{b_2}\right)$$

$$\left(\frac{c_1}{c_2}, \frac{1}{c_2}\right) \longleftarrow\!\shortmid (c_1, c_2)$$

Since A_2, A_3 are open subsets of $\mathbb{C}^2$, their tangent spaces are just

$$TA_2 = \{(b_1, b_2, u, v) \mid b_2 \neq 0\}$$

$$TA_3 = \{(c_1, c_2, y_1, y_2) \mid c_2 \neq 0\}$$

The differential matrix of $\psi_3 \circ \psi_2^{-1}$ at (b_1, b_2) is

$$d(\psi_3 \circ \psi_2^{-1})_{(b_1,b_2)} = \begin{pmatrix} \frac{1}{b_2} & -\frac{b_1}{b_2^2} \\ 0 & -\frac{1}{b^2} \end{pmatrix}$$

$$TA_2 \xrightarrow{d(\psi_3 \circ \psi_2^{-1})} TA_3$$

$$((b_1,b_2),(u,v)) \longmapsto \left(\left(\frac{b}{b_2}, \frac{1}{b_2}\right), \left(\frac{b_2 u - b_1 v}{b_2^2}, -\frac{v}{b_2^2}\right)\right)$$

Finally,

$$t(\ell,\lambda) = \left(\alpha_2 \circ \left((\varphi_2 \circ \varphi_1^{-1}) \times d(\psi_3 \circ \psi_2^{-1})\right) \circ \alpha_1^{-1}\right)(\ell,\lambda)$$

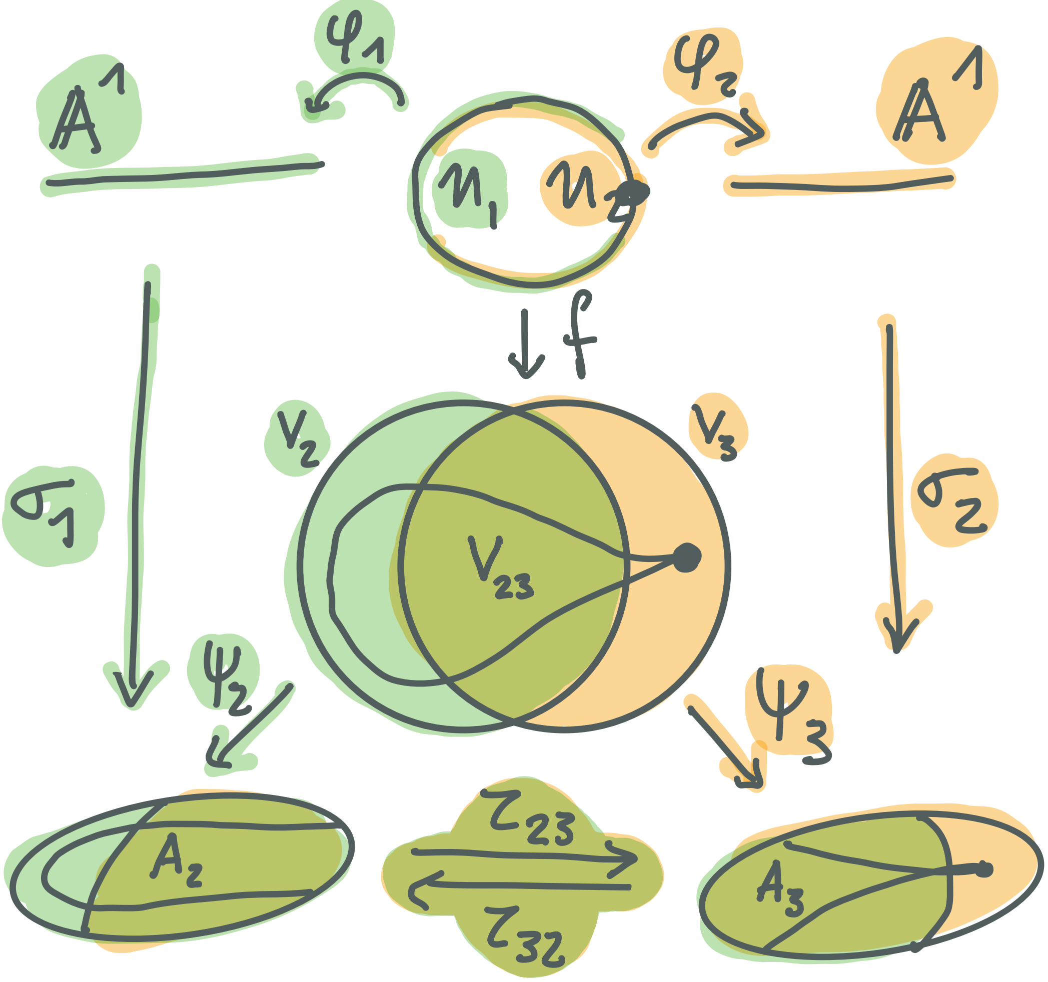
$$= \left(\alpha_2 \circ \left((\varphi_2 \circ \varphi_1^{-1}) \times d(\psi_3 \circ \psi_2^{-1})\right)\right)(\ell,(\lambda, 3\lambda\ell^2))$$

$$\in T_{(\ell,\ell^3)}\mathbb{C}^2$$
$$\begin{array}{c} \| \\ (b_1,b_2) \end{array}$$

$$= \alpha_2\left(\frac{1}{\ell}, \frac{\lambda\ell^3 - 3\lambda\ell^3}{\ell^6}, \frac{-3\lambda\ell^2}{\ell^6}\right)$$

$$= \alpha_2\left(\frac{1}{\ell}, \frac{-2\lambda}{\ell^3}, \frac{-3\lambda}{\ell^4}\right)$$

$$= \left(\frac{1}{\ell}, \frac{-2\lambda}{\ell^3}\right).$$



STEP 4 $c_M(C)$ computation

STEP 1 $\Rightarrow c_M(C) = f_*(c^*(\tilde{f}^*\tilde{T}_C) \cap [\mathbb{P}^1])$

Since the transition functions have degree 3 on the fibres $\left((\ell, \underline{\underline{\lambda}}) \mapsto \left(\frac{1}{\ell}, \underline{\underline{-2\lambda/\ell^3}}\right)\right)$

we conclude that $c^*(\tilde{f}^*\tilde{T}_C) = 1 + 3a$, where a is the generator of $H^2(\mathbb{P}^1)$.

Therefore

$$c^*(\tilde{f}^*\tilde{T}_{C'}) \cap [\mathbb{P}^1] = 3 + [\mathbb{P}^1]$$

and, since f^* is a homeomorphism,

$$c_M(C) = 3 + [C]$$