Answers to questions in
Lab 1: Filtering operations

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Instructions: Complete the lab according to the instructions in the notes and respond to the questions stated below. Keep the answers short and focus on what is essential. Illustrate with figures only when explicitly requested.

Good luck!

Question 1: Repeat this exercise with the coordinates p and q set to (5, 9), (9, 5), (17, 9), (17, 121), (5, 1) and (125, 1) respectively. What do you observe?

Answers:
(5,9): Wavefunction with diagonally from top left to bottom right.
(9,5): Wavefunction with same size but rotated to the right compared to (5,9).
(17,9): Wavefunction with smaller wavelength compared to the first two values.
(17, 121): After centering the v-value 121 is projected to -7. Further reduced wavelength compared to (17,9). Wavefunction with direction from bottom left to top right.
(5,1): Higher wavelength compared to the prior values. Nearly vertical wavefunction slightly shifted rotated to the left.
(125,1): After centering the u-value 125 is projected to -3. High wavelength with waves going from bottom left to upper right.
For all the values imag(F) appears to be a shifted version of real(F).

Question 2: Explain how a position (p, q) in the Fourier domain will be projected as a sine wave in the spatial domain. Illustrate with a Matlab figure.

Answers:

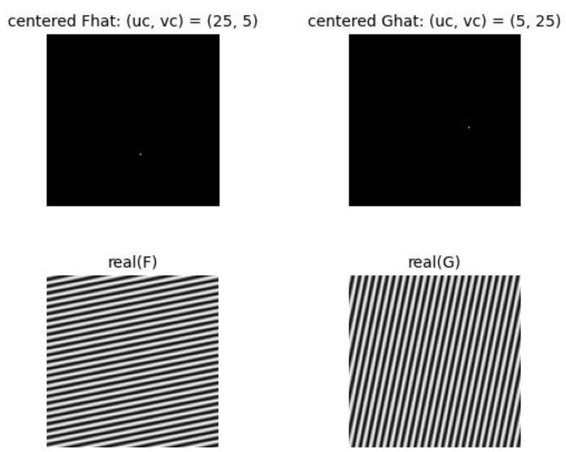

Fig. 1

Each possible point (p,q) is projected onto a unique complex wave function in the spatial domain. As can be seen from the examples in Fig. 1, the direction of the real part of the wave functions always is parallel to the vector (p,q) originating from the origin (if the vector is interpreted in spatial-domain actually (q,p) as p is corresponding to the y-axis). The same can be said for the imaginary part, as it is merely a shifted version of the real waves.

$$F(x,y) = e^{i\frac{2\pi}{N}(py+qx)} = \frac{1}{N}\cos\left(\frac{2\pi(py+qx)}{N}\right) + i\frac{1}{N}\sin\left(\frac{2\pi(py+qx)}{N}\right)$$

Question 3: How large is the amplitude? Write down the expression derived from Equation (4) in the notes. Complement the code (variable amplitude) accordingly.

Answers:

As there is only one point in the Fourier-domain, which is not zero, the function for the spatial domain simplifies to a single complex exponential wavefunction.

$$A = |F(x,y)| = \left|\frac{1}{N} e^{i\frac{2\pi}{N}(px+qy)}\right| = \frac{1}{N}\left|e^{i\theta}\right| = \frac{1}{N}1 = \frac{1}{N}$$

Question 4: How does the direction and length of the sine wave depend on p and q? Write down the explicit expression that can be found in the lecture notes. Complement the code (variable wavelength) accordingly.

Answers:
In the script the following formula for the wavelength is given:

$$\lambda = \frac{1}{\sqrt{u^2+v^2}}$$

However, this is for the continuous case. To adjust this for the discrete case we have to scale with the size of the image and therefore use:

$$\lambda = \frac{N}{\sqrt{ps^2+qs^2}}$$

Here (ps,qs) are the centered values of the original (p,q) vector. The wavelength gives us only the length of the waves. The direction of the waves in the spatial-domain is parallel to the position (p,q) in the Fourier-domain (see Question 2).

Question 5: What happens when we pass the point in the center and either p or q exceeds half the image size? Explain and illustrate graphically with Matlab!

Answers:

The Fourier domain is periodic with a period of *N* (= sz =128 in our case). After applying the shift, we no longer consider the interval {0, …, N-1}, but rather the interval {-N/2, …, N/2-1}. Due to the periodic properties

$$(p,q) = (p,q-N) and (p,q) = (p-N,q)$$

points where p or q exceed $N/2 - 1$ are projected onto their negative equivalents. Consequently, the frequency components beyond half the image size wrap around to the opposite side of the spectrum, demonstrating the periodic nature of the Fourier transform.

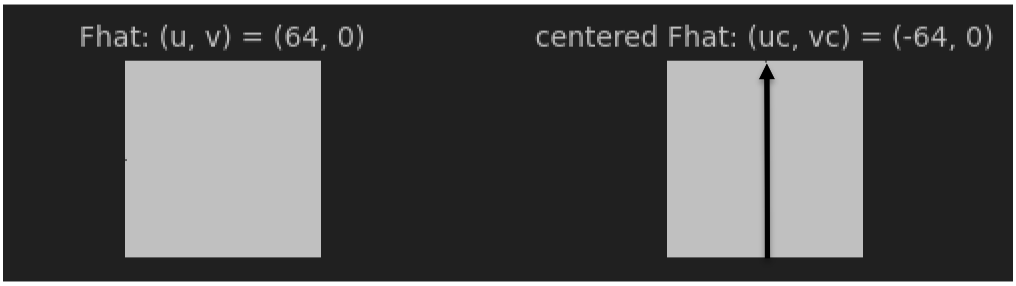

Fig. 2

This is shown in Fig. 2: (64,0) can be depicted in $\{0, \ldots, 127\}^2$ but not in $\{-64,63\}^2$ after centering. Therefore the value is mapped into the new range with (uc,vc) = (64-128,0) = (-64,0) with the movement indicated by the arrow.

Question 6: What is the purpose of the instructions following the question *What is done by these instructions?* in the code?

Answers:

The instructions perform the shift discussed in Question 5. For *u,v* new values *uc,vc* are computed: If *u,v* are exceeding *N/2 - 1*, they are mapped into {-N/2, ..., N/2-1} again, by subtracting N.

Question 7: Why are these Fourier spectra concentrated to the borders of the images? Can you give a mathematical interpretation? Hint: think of the frequencies in the source image and consider the resulting image as a Fourier transform applied to a 2D function. It might be easier to analyze each dimension separately!

Answers:

First, consider the image F and its corresponding Fourier transform $\hat{F}$. The generated picture is constant in horizontal direction. Therefore, all wavefunctions are zero frequency in horizontal direction, which is indicated in the Fourier-domain by all points lying on a straight line on the left boarder (v = 0). Because the image has a rectangular structure with sharp vertical edges, the Fourier spectrum displays a sinc-shaped distribution in horizontal direction. For G and its Fourier transform $\hat{G}$, the result is analogous: since G is a 90° rotation of F, its Fourier transform $\hat{G}$ is likewise a 90° rotation of $\hat{F}$. Consequently, the Fourier spectrum is concentrated along the upper border of the image.

Question 8: Why is the logarithm function applied?

Answers:

The logarithm function performs a grey-level transformation. It is “useful for compressing large dynamic range and make details visible” (slides). Due to the high dynamic range in the

Fourier spectra, a lot of details could not be captured in the image without the logarithmic transformation.

Question 9: What conclusions can be drawn regarding linearity? From your observations can you derive a mathematical expression in the general case?

Answers:

H is defined as a linear combination of F and G with H = F + 2G. From the Fourier Spectra, we can observe, that $\hat{H}$ is also a linear combination of $\hat{F}$ and $\hat{G}$ with $\hat{H} = \hat{F} + 2\hat{G}$.
This observation can be generalized mathematically:

$$F(aF + bG) = aF(F) + bF(G)$$

This means, the Fourier transform is a linear operator.

Question 10: Are there any other ways to compute the last image? Remember what multiplication in Fourier domain equals to in the spatial domain! Perform these alternative computations in practice.

Answers:

The multiplication of F and G results in a cubic shape centered around the origin. The Fourier spectrum therefore contains the corresponding sinc-waves, similar to question 7, but in this case oriented in both the horizontal and vertical directions.

The multiplication in the spatial domain is corresponding to the convolution in the Fourier domain and the other way round. Therefore, the same spectrum could be achieved by computing the Fourier transformations $\hat{F}$, $\hat{G}$ and compute the convolution of these two.

Question 11: What conclusions can be drawn from comparing the results with those in the previous exercise? See how the source images have changed and analyze the effects of scaling.

Answers:

Compared to the square from Exercise 1.5, the vertical rectangle is compressed along the vertical axis and stretched along the horizontal axis. When examining the corresponding Fourier spectra, it becomes evident that these are vertically stretched and horizontally compressed. Thus, the changes in the Fourier domain are inverse to those in the spatial domain. This observation confirms the result discussed in the lecture:
"Compression (scale down) in the spatial domain is the same as expansion (scale up) in the Fourier domain (and vice versa)."

Question 12: What can be said about possible similarities and differences? Hint: think of the frequencies and how they are affected by the rotation.

Answers:

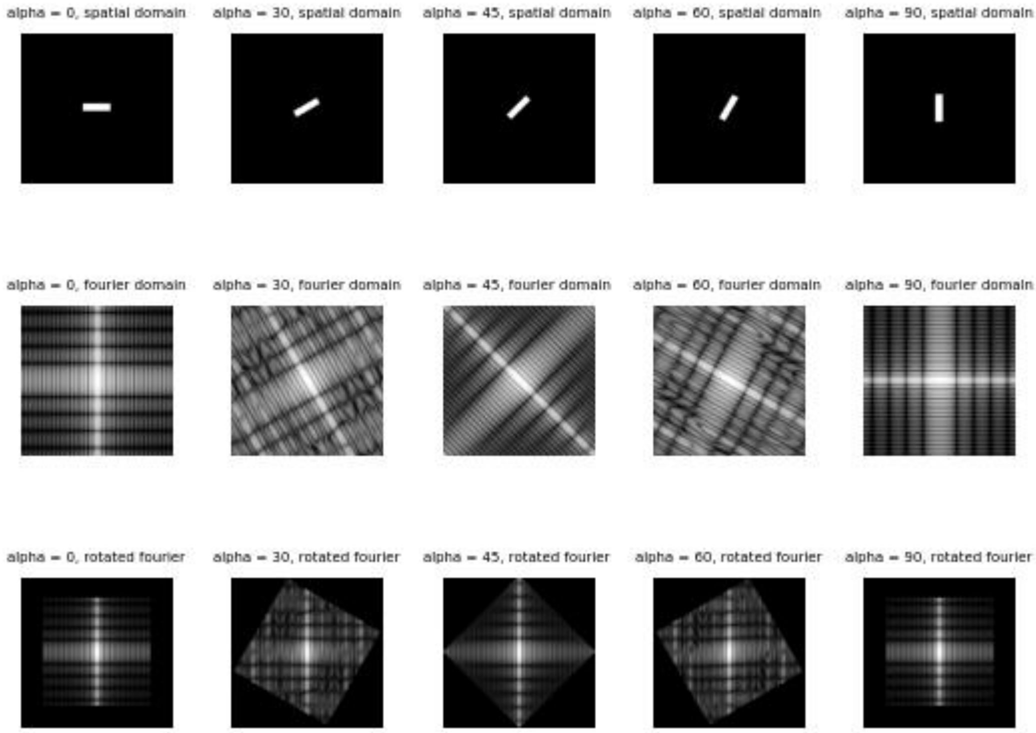

Fig. 3

As can be seen in Fig. 3, a rotation in the spatial domain corresponds to a rotation by the same angle in the Fourier domain, and vice versa. This can be explained by the fact that representing a rotated shape requires the same wave functions but rotated accordingly. For certain rotation angles (e.g., 30 degrees, 60 degrees), the shape changes visibly because diagonal edges cannot be represented perfectly due to the discrete pixel grid (and the very low resolution used here). As a result, the spectrum also shows irregularities in these cases, meaning that $\hat{H}$ does not exactly correspond to $\hat{F}$ for alpha 0.

Question 13: What information is contained in the phase and in the magnitude of the Fourier transform?

Answers:

There is a strong difference between the images with manipulated magnitude and those with manipulated phase. The images processed with "pow2image()" appear as heavily noise-distorted versions of the original image, but all important structures remain recognizable. In contrast, when using "randphaseimage()", the entire image appears noisy, and no similarities with the original image can be identified.

This difference arises from the distinct information contained in the magnitude and phase components: the phase determines the relative shifts of the various wave functions and thus defines the positions of edges in the image. The magnitude, on the other hand, indicates the strength of each wave function and is therefore responsible for the different shades of grey on both sides of an edge.

Question 14: Show the impulse response and variance for the above-mentioned t-values. What are the variances of your discretized Gaussian kernel for t = 0.1, 0.3, 1.0, 10.0 and 100.0?

Answers:

0.1	$\begin{bmatrix} 0.013 & 0.000 \\ 0.000 & 0.013 \end{bmatrix}$
0.3	$\begin{bmatrix} 0.281 & 0.000 \\ 0.000 & 0.281 \end{bmatrix}$
1.0	$\begin{bmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \end{bmatrix}$
10.0	$\begin{bmatrix} 10.000 & 0.000 \\ 0.000 & 10.000 \end{bmatrix}$
100.0	$\begin{bmatrix} 100.000 & 0.000 \\ 0.000 & 100.000 \end{bmatrix}$

Fig. 4

Fig. 4 shows the variance matrices rounded to three decimal places for the corresponding t-values.

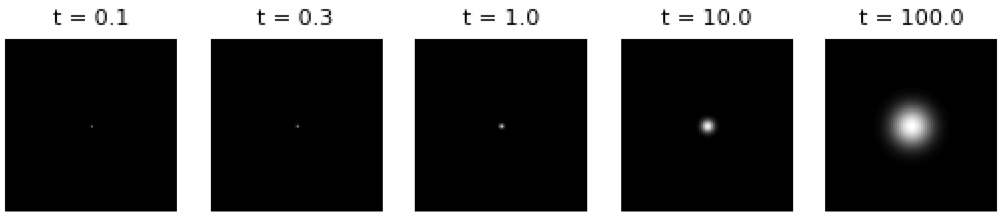

Fig. 5

Fig. 5 shows the corresponding filters, i.e., the impulse responses for the respective standard deviations.

Question 15: Are the results different from or similar to the estimated variance? How does the result correspond to the ideal continuous case? Lead: think of the relation between spatial and Fourier domains for different values of t.

Answers:

For relatively large values (>1.0 in this example), the variance matches that of the ideal continuous filter up to three decimal places. For smaller values (0.1;0.3), larger deviations can be observed. In these cases, the Gaussians in the spatial domain exhibit a much steeper slope, which leads to a greater sampling error of the continuous function compared to the more gradually decaying Gaussians with larger t-values.

Question 16: Convolve a couple of images with Gaussian functions of different variances (like t = 1.0, 4.0, 16.0, 64.0 and 256.0) and present your results. What effects can you observe?

Answers:

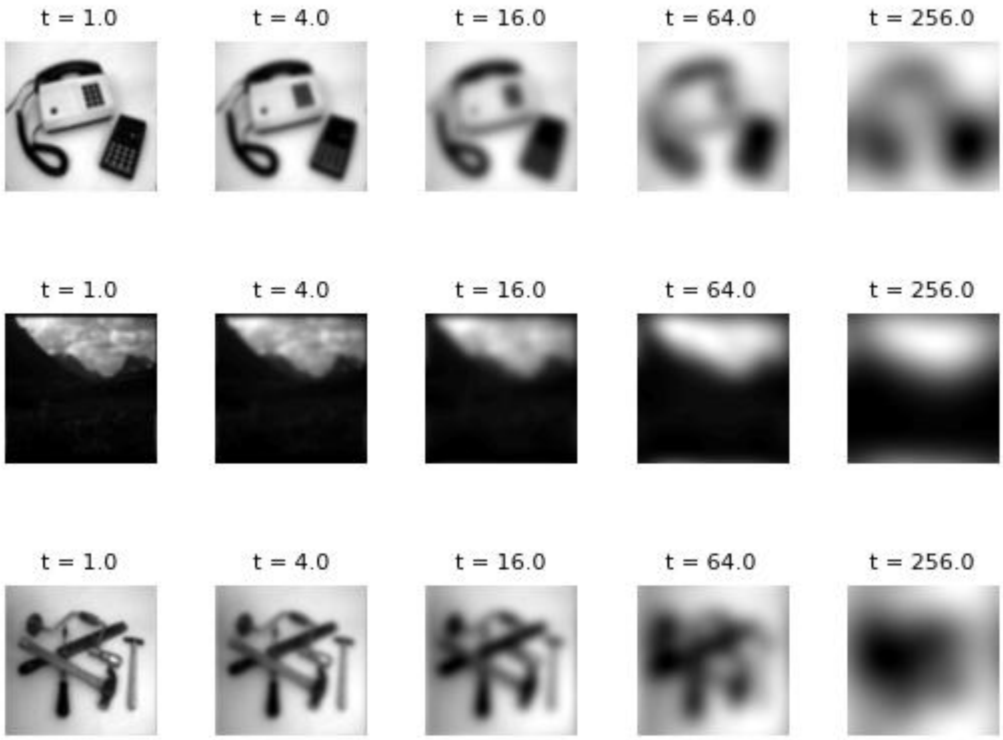

Fig. 6

In Fig. 6, it can be observed that larger t-values lead to increasingly stronger blurring. As t increases, fewer edges - which correspond to the high frequencies in the Fourier domain - are visible. This relationship can be clearly illustrated using the Fourier domain: the larger the t-value, the wider the Gaussian becomes in the spatial domain. At the same time, the Gaussian becomes narrower in the Fourier domain, meaning that fewer high frequencies are transmitted, and the edges become progressively more blurred.

Question 17: What are the positive and negative effects for each type of filter? Describe what you observe and name the effects that you recognize. How do the results depend on the filter parameters? Illustrate with Matlab figure(s).

Answers:

Fig. 7 shows the application of various filters and parameters for Gaussian noise. Fig. 8 presents the corresponding results for salt-and-pepper noise.

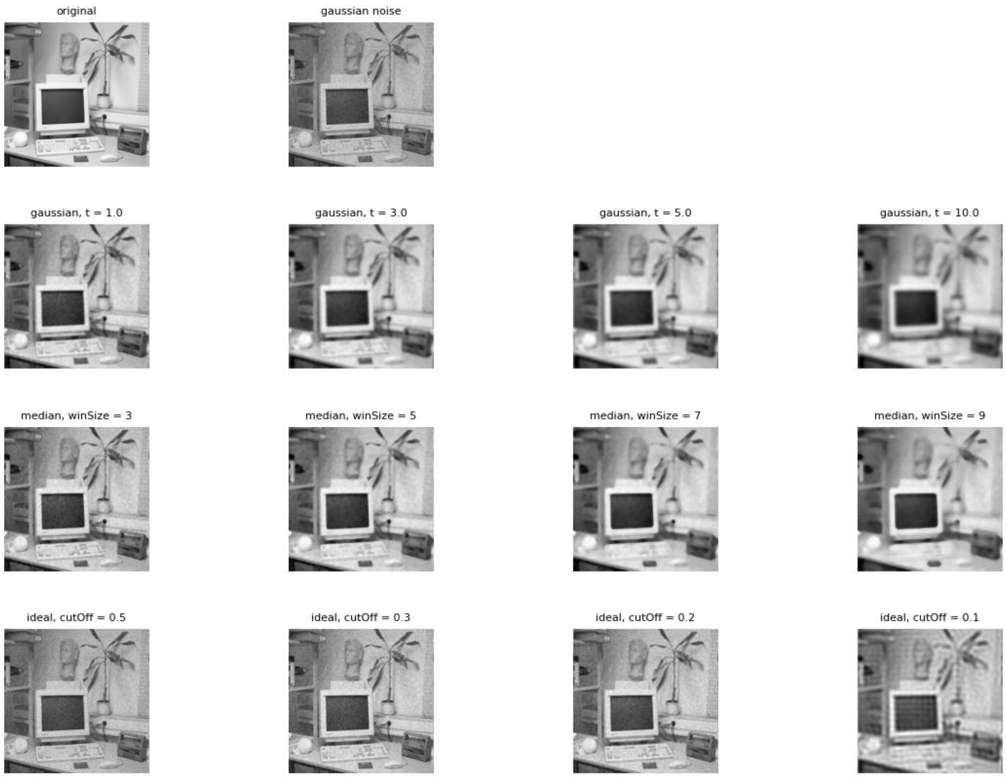
original
gaussian noise
gaussian, t = 1.0
gaussian, t = 3.0
gaussian, t = 5.0
gaussian, t = 10.0
median, winSize = 3
median, winSize = 5
median, winSize = 7
median, winSize = 9
ideal, cutOff = 0.5
ideal, cutOff = 0.3
ideal, cutOff = 0.2
ideal, cutOff = 0.1

Fig. 7

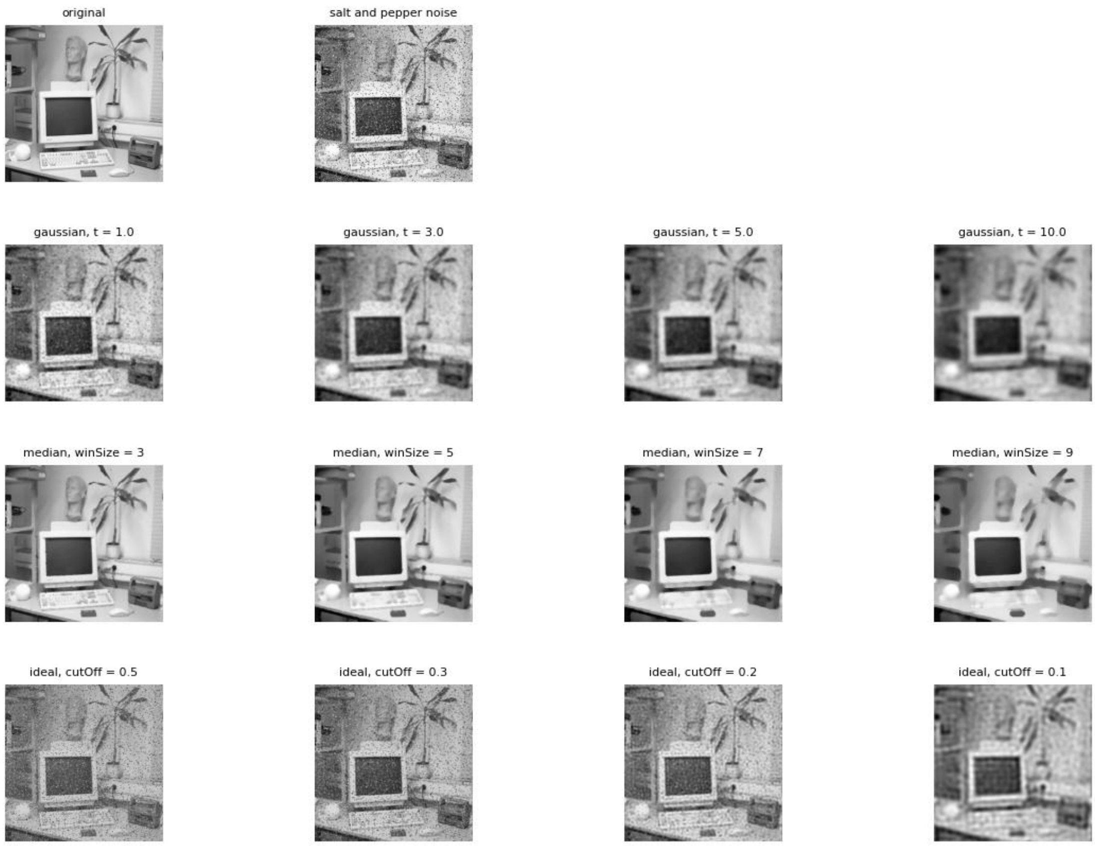
original
salt and pepper noise
gaussian, t = 1.0
gaussian, t = 3.0
gaussian, t = 5.0
gaussian, t = 10.0
median, winSize = 3
median, winSize = 5
median, winSize = 7
median, winSize = 9
ideal, cutOff = 0.5
ideal, cutOff = 0.3
ideal, cutOff = 0.2
ideal, cutOff = 0.1

Fig. 8

Gaussian

Increasing the parameter t leads to stronger blurring, as described in Question 16. The method performs well in removing Gaussian noise, particularly for larger t values. However, it is less effective against salt-and-pepper noise since the extreme pixel values are not removed but rather spread across neighboring regions during the blurring process.

Median

Larger winH,winW values result in more pixels being considered when computing the median. As a consequence, the blurring effect becomes stronger; for higher values, the processed image exhibits a paint-like appearance. The median filter performs slightly worse than the Gaussian filter in the presence of Gaussian noise, but it is effective against salt-and-pepper noise since the median is robust to outliers.

Ideal low-pass

Parameter Cut-off frequency determines the maximum frequency that is allowed to pass. The lower the cut-off frequency, the stronger the resulting blurring effect. The main drawback becomes particularly evident at low cut-off frequencies: distinct ring-like structures can be observed. This occurs because the ideal low-pass filter in the Fourier domain corresponds to a sinc-function in the spatial domain, which leads to undesired artifacts in the image. For this reason, the ideal low-pass filter is rarely used in practice.

Question 18: What conclusions can you draw from comparing the results of the respective methods?

Answers:

The use of a median filter is appropriate for removing salt-and-pepper noise, while a Gaussian filter is effective for reducing Gaussian noise. The use of an ideal low-pass filter, however, is generally not recommended.

Question 19: What effects do you observe when subsampling the original image and the smoothed variants? Illustrate both filters with the best results found for iteration i = 4.

Answers:

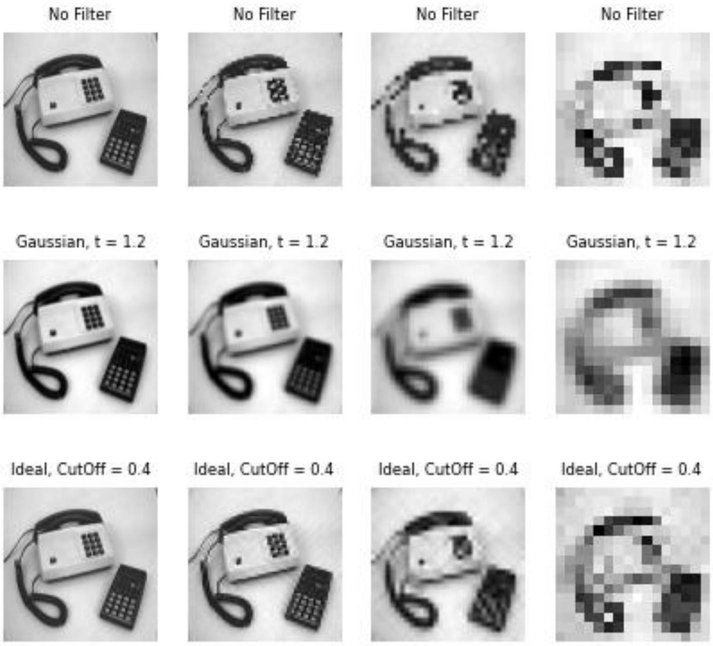

Fig. 9

In Fig. 9, the process of subsampling is shown, once without filtering, once with a Gaussian filter (a value of 1.2 produced the subjectively best results), and once with an ideal low-pass filter (a value of 0.4 yielded the subjectively best results). Using these filters, a significantly improved image quality can be observed after several iterations, as aliasing is effectively prevented.

Question 20: What conclusions can you draw regarding the effects of smoothing when combined with subsampling? Hint: think in terms of frequencies and side effects.

Answers:

The low-pass filters used here clearly have a positive effect on image quality. This is because, as low-pass filters, they suppress high frequencies that would otherwise lead to aliasing artifacts during subsampling by being mapped onto lower frequencies. However, an overly strong low-pass filter (for example, gaussian filter with high t-values) can result in a blurred image and thereby degrade the image quality.
