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In[55]:= (* The following Calculations are done by
          using Jaakko's d and e2 vectors parametrizations ,
          which was proven is lower energy in equilibrium state.
          The parametrization is  $e_\alpha^2 = -\cos\theta \hat{z} -\sin\theta \hat{y}$ 
           $d_\alpha = \cos\theta \hat{y} -\sin\theta \hat{z}$  *)
(***)  $\Xi_{33}, \alpha=3, \lambda=3$  (***)
Clear["context`*"];
Clear[ $\Xi_{33}, c1, c2, gD, \gamma, \chi v, \alpha, \lambda, V, V1, V2, V3, Kij, K1, K1ij, K2, K2ij,$ 
       $K3, K3ij, Q\beta j, Q1\beta j, Q2\beta j, \Delta p, \Delta v1, \Delta v2, X, \theta, x1, x2, x3, b, a, d, Va, Vb,$ 
       $Vd, Vj, V\beta, Xa, Xb, Xd, \Lambda_{ij33}, \Lambda_{ij331}, \Lambda_{ij332}, \Lambda_{ij333}, \Lambda_{ij334}, \Lambda_{ij335}, Pai1,$ 
       $Pai2, Pai3, Paj1, Paj2, Paj3, Pai, Paj, PaiVb, PajVb, Kij\Lambda_{ij\alpha\lambda} Ka\Lambda_{\alpha\lambda},$ 
       $Kba\Lambda_{ba\alpha\lambda}, Rj\lambda\alpha\beta Q\beta j, VdVbej\lambda\xi\epsilon\alpha\beta\gamma, VdVbej\lambda\xi\epsilon\alpha\beta\gamma Q\beta j, LeviCivita\epsilon_{ijk}$ ];
 $\alpha =$ 
3;
 $\lambda = 3$ ;
 $Ka\Lambda_{\alpha\alpha\lambda} = 0$ ;
 $Kba\Lambda_{ba\alpha\lambda} = 0$ ;
 $Rb\lambda\alpha Qb = 0$ ;
 $Rdb\lambda\alpha Qdb = 0$ ;
 $VVbe\lambda\epsilon\alpha Qb = 0$ ;
 $VdVbe\lambda\epsilon\alpha Qbd = 0$ ;
 $V1 = \{1, 0, 0\}$ ;
 $V2 = \{0, -\sin[\theta[x1, x2, x3]], -\cos[\theta[x1, x2, x3]]\}$ ;
 $V3 = \{0, \cos[\theta[x1, x2, x3]], -\sin[\theta[x1, x2, x3]]\}$ ;
(*****)
(*V tensor for d e1 e2*)
V = {V1, V2, V3};
(*TreeForm [V]*) (*Level[V, {2}][[1]]*)
(*****)
(*X tensor for  $\bar{X}^1 \bar{X}^2 \bar{X}^3$ *)
X = {{ $\Delta v1, 0, 0$ }, {0,  $\Delta v2, 0$ }, {0, 0,  $\Delta p$ }};
(* $\bar{X}_i^a$  tensor symbol Row is a index and colum is spatial index *)
(*****)
(** Sumover ba **)
(*****)
(*K tensor  $K_{ij}^{ba}$ *)
Do[Do[
  Xb = Level[X, {1}][[b]]; Xa = Level[X, {1}][[a]];
  K1ij = K1Array[KroneckerDelta, {3, 3}][Tr[{Xb}].{Xa}];
  K2ij = K2[{Xb}].{Xa};
  K3ij = K3[{Xa}].{Xb};
  Kij = K1ij + K2ij + K3ij;
  Print[Style["Kij=", Red, 12], Kij//MatrixForm,
    ", " Style["b=", Red, 12], b, ", ", Style["a=", Red, 12], a];
  (*****)
  (*  $\Lambda_{ij33}^{ba}$  *)
  Vb = Level[V, {1}][[b]]; Va = Level[V, {1}][[a]];

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Pai=Grad[δs[x1,x2,x3],{x1,x2,x3}];
Paj=Grad[δs[x1,x2,x3],{x1,x2,x3}];
PaiPaj=Grad[Grad[δs[x1,x2,x3],{x1,x2,x3}],{x1,x2,x3}];
(***** the symbol Δijxx(1..5) means the 1st to 5th terms of Δijxx,
        where xx is αλ *****)
Δij331=(Grad[Grad[Vb[[λ]],{x1,x2,x3}],{x1,x2,x3}])Va[[α]];
Δij331=Δij111δs[x1,x2,x3];
(*****)
(*Δij112=
  (FullSimplify[Vb.Va,Assumptions→{{x1,x2,x3,θ}∈Reals}]-Vb[[1]]Va[[1]])
  (({Pai})T.{Paj});*)
Δij332=(FullSimplify[Vb.Va,Assumptions→{{x1,x2,x3,θ}∈Reals}]
  KroneckerDelta[α,λ]-Vb[[α]]Va[[λ]]) (PaiPaj);
(*Δij332/MatrixForm*)
(*****)
PaiVb=(Grad[Vb,{x1,x2,x3}])T;
Δij333={ (FullSimplify[(PaiVb).(Va),Assumptions→{{x1,x2,x3,θ}∈Reals}]
  KroneckerDelta[α,λ]-
  ((Grad[Vb[[α]],{x1,x2,x3}]) (Va[[λ]])))T.{Paj};
(*****)
PajVb=(Grad[Vb,{x1,x2,x3}])T;
Δij334=
  ({Pai})T.{ (FullSimplify[(PajVb).(Va),Assumptions→{{x1,x2,x3,θ}∈Reals}]
  KroneckerDelta[α,λ]-((Grad[Vb[[α]],{x1,x2,x3}]) (Va[[λ]])))};
(*****)
(*b=1;a=1;Vb=Level[V,{2}][[b]];Va=Level[V,{2}][[a]];*)
Δij335=-(Grad[Grad[Vb[[α]],{x1,x2,x3}],{x1,x2,x3}])Va[[λ]];
Δij335=Δij115δs[x1,x2,x3];
(*****)
Δij33=Δij331+Δij332+Δij333+Δij334+Δij335;
FullSimplify[Δij33,Assumptions→{{x1,x2,x3,θ,K1,K2,K3}∈Reals}]];
(*****)
KijΔijαλ=FullSimplify[Tr[Kij.(Δij33T)],
  Assumptions→{{x1,x2,x3,θ,K1,K2,K3}∈Reals}]];
Print[Style["KijΔijαλFull",Red,12],KijΔijαλ/TraditionalForm,
  ", "Style["b=",Red,12],b," ",Style["a=",Red,12],a];
Print[Style["KijΔijαλ",Red,12],
  (KijΔijαλ. {δs(0,0,1)[x1,x2,x3]→0,θ(0,0,1)[x1,x2,x3]→0,
    δs(0,0,2)[x1,x2,x3]→0,δs(1,0,1)[x1,x2,x3]→0}) // TraditionalForm,
  ", "Style["b=",Red,12],b," ",Style["a=",Red,12],a];
KaΛaαλ=KaΛaαλ+FullSimplify[KijΔijαλ. {δs(0,0,1)[x1,x2,x3]→0,
  θ(0,0,1)[x1,x2,x3]→0,δs(0,0,2)[x1,x2,x3]→0,δs(1,0,1)[x1,x2,x3]→0},
  Assumptions→{{x1,x2,x3,θ}∈Reals}]];
, {a,3}];
KbaΛbaαλ=KbaΛbaαλ+KaΛaαλ; KaΛaαλ=0; , {b,3}];

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KbaAbaaλ =
FullSimplify [KbaAbaaλ, Assumptions → {{x1, x2, x3, θ} ∈ Reals}] // TraditionalForm
(*****
(*****
LeviCivita[i_, j_, k_] := Module[{test1 = (i == j || j == k || i == k), test2 =
  ((i == 1 && j == 2 && k == 3) || (i == 2 && j == 3 && k == 1) || (i == 3 && j == 1 && k == 2))},
  If[test1, eijk = 0; , If[test2, eijk = 1; , eijk = -1; ];]; eijk];
Do[Do[(* Q tensor Qβjbd*)
  Xb = Level[X, {1}][[b]]; Xd = Level[X, {1}][[d]];
  Q1βj = ({Xb})†.{Xd};
  Q2βj = ({Xd})†.{Xb};
  Qβj = (Q1βj + Q2βj);
  Print[Style["Qβj=", Red, 12], Qβj // MatrixForm ,
    " ", Style["d=", Red, 12], d, " ", Style["b=", Red, 12], b];
  (*****
  (* Rdbjλαβ *)
  Vd = Level[V, {1}][[d]]; Vb = Level[V, {1}][[b]];
  Rjλαβ = (({Vd})†.{Vb}) KroneckerDelta[λ, α] -
    (({Vd})†.{Vb}[α]) . {Table[KroneckerDelta[λ, β], {β, 1, 3, 1}]}];
  (*Print["Rjλαβ=", Rjλαβ // MatrixForm, " ", "d=", d, " ", "b=", b]; *)
  RjλαβQβj = FullSimplify [Tr[Rjλαβ.Qβj], Assumptions →
    {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
  Print[Style["RjλαβQβj=", Red, 12], RjλαβQβj // TraditionalForm ,
    " ", Style["d=", Red, 12], d, " ", Style["b=", Red, 12], b];
  RbλαQb = RbλαQb + RjλαβQβj;
  (*****
  (* VdζVbγεjλεαβγQbdβj *)
  (*****
  Vj = Table[Sum [Vd[[ξ]] LeviCivita[j, λ, ξ], {ξ, 3}], {j, 3}];
  Vβ = Table[Sum [Vb[[γ]] LeviCivita[α, β, γ], {γ, 3}], {β, 3}];
  VdVbejλξεαβγ = ({Vj})†.{Vβ};
  VdVbejλξεαβγQβj = FullSimplify [Tr[VdVbejλξεαβγ.Qβj], Assumptions →
    {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
  Print[Style["VdVbejλξεαβγQβj=", Red, 12], VdVbejλξεαβγQβj // TraditionalForm ,
    " ", Style["d=", Red, 12], d, " ", Style["b=", Red, 12], b];
  VVbeλεαQb = VVbeλεαQb + VdVbejλξεαβγQβj; , {b, 3}];
  RdbλαQdb = RdbλαQdb + RbλαQb;
  RbλαQb = 0;
  VdVbeλεαQbd = VdVbeλεαQbd + VVbeλεαQb;
  VVbeλεαQb = 0; , {d, 3}];
  (***** Show results *****)
  RdbλαQdb = FullSimplify [RdbλαQdb,
    Assumptions → {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
  VdVbeλεαQbd = FullSimplify [VdVbeλεαQbd,
    Assumptions → {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
  Print[Style["RdbλαQdb=", Red, 12], RdbλαQdb // TraditionalForm, " ",
    Style["α=", Red, 12], α, " ", Style["λ=", Red, 12], λ];

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Print[Style["VdVbeλeαQbd=", Red, 12], VdVbeλeαQbd // TraditionalForm,
      ", ", Style["α=", Red, 12], α, ", ", Style["λ=", Red, 12], λ];
(*****)
(*****  $\Xi_{11} = \frac{v^2}{\chi^v} Kba\Lambda ba\alpha\lambda + \frac{6}{5} \frac{qD}{\chi^v} v^2 Rdb\lambda\alpha Qdb + \frac{6}{5} \frac{qD}{\chi^v} v^2 VdVbe\lambda e\alpha Qbd$  *****)
(** c1= $\frac{v^2}{\chi^v}$  c2= $\frac{6}{5} \frac{qD}{\chi^v} v^2$  **)
Print[Style[" $\Xi_{33}$  in Jaakko's parametrization : ", Red, 18]]
 $\Xi_{33} = (c1 Kba\Lambda ba\alpha\lambda + c2 Rdb\lambda\alpha Qdb + c2 VdVbe\lambda e\alpha Qbd);$ 
 $\Xi_{33} // TraditionalForm$ 

$$K_{ij} = \begin{pmatrix} K1 \Delta v^2 + K2 \Delta v^2 + K3 \Delta v^2 & 0 & 0 \\ 0 & K1 \Delta v^2 & 0 \\ 0 & 0 & K1 \Delta v^2 \end{pmatrix}, b=1, a=1$$

 $K_{ij} \Delta_{ij} \alpha \lambda Full = \Delta v^2$ 

$$\left( (K1 + K2 + K3) \delta s^{(2,0,0)}(x1, x2, x3) + K1 (\delta s^{(0,0,2)}(x1, x2, x3) + \delta s^{(0,2,0)}(x1, x2, x3)) \right), b=1, a=1$$

 $K_{ij} \Delta_{ij} \alpha \lambda = \Delta v^2 \left( (K1 + K2 + K3) \delta s^{(2,0,0)}(x1, x2, x3) + K1 \delta s^{(0,2,0)}(x1, x2, x3) \right), b=1, a=1$ 

$$K_{ij} = \begin{pmatrix} 0 & K2 \Delta v^2 & 0 \\ K3 \Delta v^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b=1, a=2$$

 $K_{ij} \Delta_{ij} \alpha \lambda Full = 0, b=1, a=2$ 
 $K_{ij} \Delta_{ij} \alpha \lambda = 0, b=1, a=2$ 

$$K_{ij} = \begin{pmatrix} 0 & 0 & K2 \Delta p \Delta v^2 \\ 0 & 0 & 0 \\ K3 \Delta p \Delta v^2 & 0 & 0 \end{pmatrix}, b=1, a=3$$

 $K_{ij} \Delta_{ij} \alpha \lambda Full = 0, b=1, a=3$ 
 $K_{ij} \Delta_{ij} \alpha \lambda = 0, b=1, a=3$ 

$$K_{ij} = \begin{pmatrix} 0 & K3 \Delta v^2 & 0 \\ K2 \Delta v^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b=2, a=1$$

 $K_{ij} \Delta_{ij} \alpha \lambda Full = 0, b=2, a=1$ 
 $K_{ij} \Delta_{ij} \alpha \lambda = 0, b=2, a=1$ 

$$K_{ij} = \begin{pmatrix} K1 \Delta v^2 & 0 & 0 \\ 0 & K1 \Delta v^2 + K2 \Delta v^2 + K3 \Delta v^2 & 0 \\ 0 & 0 & K1 \Delta v^2 \end{pmatrix}, b=2, a=2$$

 $K_{ij} \Delta_{ij} \alpha \lambda Full = \Delta v^2 \sin(\theta(x1, x2, x3))$ 

$$\left( 2 \cos(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + K1 \delta s^{(0,0,1)}(x1, x2, x3) \theta^{(0,0,1)}(x1, x2, x3) + K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) + \sin(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(0,0,2)}(x1, x2, x3) + K1 \delta s^{(2,0,0)}(x1, x2, x3) \right) \right), b=2, a=2$$

 $K_{ij} \Delta_{ij} \alpha \lambda = \Delta v^2 \sin(\theta(x1, x2, x3))$ 

$$\left( 2 \cos(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) + \sin(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(2,0,0)}(x1, x2, x3) \right) \right), b=2, a=2$$


$$K_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K2 \Delta p \Delta v^2 \\ 0 & K3 \Delta p \Delta v^2 & 0 \end{pmatrix}, b=2, a=3$$


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$$K_{ij}A_{ij\alpha\lambda}Full = \Delta p \Delta v^2 (-(K_2 + K_3)) \cos(\theta(x_1, x_2, x_3))$$

$$\left( (\delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,0,1)}(x_1, x_2, x_3) + \delta s^{(0,0,1)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3)) \right.$$

$$\left. \cos(\theta(x_1, x_2, x_3)) + \delta s^{(0,1,1)}(x_1, x_2, x_3) \sin(\theta(x_1, x_2, x_3)) \right), b=2, a=3$$

$$K_{ij}A_{ij\alpha\lambda} = \Delta p \Delta v^2 (-(K_2 + K_3)) \delta s^{(0,1,1)}(x_1, x_2, x_3) \sin(\theta(x_1, x_2, x_3)) \cos(\theta(x_1, x_2, x_3)), b=2, a=3$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & K_3 \Delta p \Delta v^1 \\ 0 & 0 & 0 \\ K_2 \Delta p \Delta v^1 & 0 & 0 \end{pmatrix}, b=3, a=1$$

$$K_{ij}A_{ij\alpha\lambda}Full = 0, b=3, a=1$$

$$K_{ij}A_{ij\alpha\lambda} = 0, b=3, a=1$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K_3 \Delta p \Delta v^2 \\ 0 & K_2 \Delta p \Delta v^2 & 0 \end{pmatrix}, b=3, a=2$$

$$K_{ij}A_{ij\alpha\lambda}Full = \Delta p \Delta v^2 (K_2 + K_3) \sin(\theta(x_1, x_2, x_3))$$

$$\left( (\delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,0,1)}(x_1, x_2, x_3) + \delta s^{(0,0,1)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3)) \right.$$

$$\left. \sin(\theta(x_1, x_2, x_3)) - \delta s^{(0,1,1)}(x_1, x_2, x_3) \cos(\theta(x_1, x_2, x_3)) \right), b=3, a=2$$

$$K_{ij}A_{ij\alpha\lambda} = \Delta p \Delta v^2 (-(K_2 + K_3)) \delta s^{(0,1,1)}(x_1, x_2, x_3) \sin(\theta(x_1, x_2, x_3)) \cos(\theta(x_1, x_2, x_3)), b=3, a=2$$

$$K_{ij} = \begin{pmatrix} K_1 \Delta p^2 & 0 & 0 \\ 0 & K_1 \Delta p^2 & 0 \\ 0 & 0 & K_1 \Delta p^2 + K_2 \Delta p^2 + K_3 \Delta p^2 \end{pmatrix}, b=3, a=3$$

$$K_{ij}A_{ij\alpha\lambda}Full = \Delta p^2 \cos(\theta(x_1, x_2, x_3)) \left( \cos(\theta(x_1, x_2, x_3)) \right.$$

$$\left( (K_1 + K_2 + K_3) \delta s^{(0,0,2)}(x_1, x_2, x_3) + K_1 (\delta s^{(0,2,0)}(x_1, x_2, x_3) + \delta s^{(2,0,0)}(x_1, x_2, x_3)) \right) -$$

$$2 \sin(\theta(x_1, x_2, x_3)) \left( (K_1 + K_2 + K_3) \delta s^{(0,0,1)}(x_1, x_2, x_3) \theta^{(0,0,1)}(x_1, x_2, x_3) + \right.$$

$$K_1 (\delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) +$$

$$\left. \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3) \right) \left. \right), b=3, a=3$$

$$K_{ij}A_{ij\alpha\lambda} =$$

$$\Delta p^2 \cos(\theta(x_1, x_2, x_3)) \left( K_1 (\delta s^{(0,2,0)}(x_1, x_2, x_3) + \delta s^{(2,0,0)}(x_1, x_2, x_3)) \cos(\theta(x_1, x_2, x_3)) - \right.$$

$$2 K_1 (\delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) + \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3))$$

$$\left. \sin(\theta(x_1, x_2, x_3)) \right), b=3, a=3$$

Out[63]/TraditionalForm=

$$\Delta v^2 \left( (K_1 + K_2 + K_3) \delta s^{(2,0,0)}(x_1, x_2, x_3) + K_1 \delta s^{(0,2,0)}(x_1, x_2, x_3) \right) + \Delta v^2 \sin(\theta(x_1, x_2, x_3)) \left( 2 \cos(\theta(x_1, x_2, x_3)) \right.$$

$$\left( (K_1 + K_2 + K_3) \delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) + K_1 \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3) \right) +$$

$$\left. \sin(\theta(x_1, x_2, x_3)) \left( (K_1 + K_2 + K_3) \delta s^{(0,2,0)}(x_1, x_2, x_3) + K_1 \delta s^{(2,0,0)}(x_1, x_2, x_3) \right) \right) +$$

$$\Delta p^2 \cos(\theta(x_1, x_2, x_3)) \left( K_1 (\delta s^{(0,2,0)}(x_1, x_2, x_3) + \delta s^{(2,0,0)}(x_1, x_2, x_3)) \cos(\theta(x_1, x_2, x_3)) - \right.$$

$$2 K_1 (\delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) + \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3)) \sin(\theta(x_1, x_2, x_3)) \left. \right) -$$

$$\Delta p \Delta v^2 (K_2 + K_3) \delta s^{(0,1,1)}(x_1, x_2, x_3) \sin(2 \theta(x_1, x_2, x_3))$$

$$Q\beta j = \begin{pmatrix} 2 \Delta v^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d=1, b=1$$

$$Rj\lambda\alpha\beta Q\beta j = 2 \Delta v^2, d=1, b=1$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=1, b=1$$

$$Q\beta j = \begin{pmatrix} 0 & \Delta v^1 \Delta v^2 & 0 \\ \Delta v^1 \Delta v^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d=1, b=2$$

$$Rj\lambda\alpha\beta Q\beta j = -\Delta v^1 \Delta v^2 \sin(\theta(x_1, x_2, x_3)), d=1, b=2$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = -\Delta v^1 \Delta v^2 \sin(\theta(x_1, x_2, x_3)), d=1, b=2$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & \Delta p \Delta v1 \\ 0 & 0 & 0 \\ \Delta p \Delta v1 & 0 & 0 \end{pmatrix}, d=1, b=3$$

$$Rj\lambda\alpha\beta Q\beta j = 0, d=1, b=3$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=1, b=3$$

$$Q\beta j = \begin{pmatrix} 0 & \Delta v1 \Delta v2 & 0 \\ \Delta v1 \Delta v2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d=2, b=1$$

$$Rj\lambda\alpha\beta Q\beta j = -\Delta v1 \Delta v2 \sin(\theta(x1, x2, x3)), d=2, b=1$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = -\Delta v1 \Delta v2 \sin(\theta(x1, x2, x3)), d=2, b=1$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\Delta v2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d=2, b=2$$

$$Rj\lambda\alpha\beta Q\beta j = 2\Delta v2^2 \sin^2(\theta(x1, x2, x3)), d=2, b=2$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=2, b=2$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta p \Delta v2 \\ 0 & \Delta p \Delta v2 & 0 \end{pmatrix}, d=2, b=3$$

$$Rj\lambda\alpha\beta Q\beta j = -\Delta p \Delta v2 \cos^2(\theta(x1, x2, x3)), d=2, b=3$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=2, b=3$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & \Delta p \Delta v1 \\ 0 & 0 & 0 \\ \Delta p \Delta v1 & 0 & 0 \end{pmatrix}, d=3, b=1$$

$$Rj\lambda\alpha\beta Q\beta j = -\Delta p \Delta v1 \sin(\theta(x1, x2, x3)), d=3, b=1$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=3, b=1$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta p \Delta v2 \\ 0 & \Delta p \Delta v2 & 0 \end{pmatrix}, d=3, b=2$$

$$Rj\lambda\alpha\beta Q\beta j = \Delta p \Delta v2 \sin^2(\theta(x1, x2, x3)), d=3, b=2$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=3, b=2$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\Delta p^2 \end{pmatrix}, d=3, b=3$$

$$Rj\lambda\alpha\beta Q\beta j = 0, d=3, b=3$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=3, b=3$$

$$Rdb\lambda\alpha Qdb =$$

$$-\Delta v1 (\Delta p + 2\Delta v2) \sin(\theta(x1, x2, x3)) - \Delta v2 (\Delta p + \Delta v2) \cos(2\theta(x1, x2, x3)) + 2\Delta v1^2 + \Delta v2^2, \alpha=3, \lambda=3$$

$$VdVbe\lambda\epsilon\alpha Qbd = -2\Delta v1 \Delta v2 \sin(\theta(x1, x2, x3)), \alpha=3, \lambda=3$$

$E_{33}$  in Jaakko's parametrization :

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$$\begin{aligned}
& c1 \left( \cos(\theta(x1, x2, x3)) \left( K1 \cos(\theta(x1, x2, x3)) \left( \delta s^{(0,2,0)}(x1, x2, x3) + \delta s^{(2,0,0)}(x1, x2, x3) \right) - 2 K1 \sin(\theta(x1, x2, x3)) \right. \right. \\
& \quad \left. \left( \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) \right) \Delta p^2 - \\
& \quad (K2 + K3) \Delta v2 \sin(2 \theta(x1, x2, x3)) \delta s^{(0,1,1)}(x1, x2, x3) \Delta p + \\
& \quad \Delta v1^2 \left( K1 \delta s^{(0,2,0)}(x1, x2, x3) + (K1 + K2 + K3) \delta s^{(2,0,0)}(x1, x2, x3) \right) + \\
& \quad \Delta v2^2 \sin(\theta(x1, x2, x3)) \left( 2 \cos(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \right. \right. \\
& \quad \left. \left. K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) + \right. \\
& \quad \left. \sin(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(2,0,0)}(x1, x2, x3) \right) \right) \right) + \\
& c2 \left( -\Delta v1 (\Delta p + 2 \Delta v2) \sin(\theta(x1, x2, x3)) - \Delta v2 (\Delta p + \Delta v2) \cos(2 \theta(x1, x2, x3)) + 2 \Delta v1^2 + \Delta v2^2 \right) - \\
& 2 \\
& c2 \\
& \Delta v1 \\
& \Delta v2 \\
& \sin(\theta(x1, x2, x3))
\end{aligned}$$

```

In[72]:= (* The following Calculations are done by
          using Jaakko's d and e^2 vectors parametrizations,
          which was proven is lower energy in equilibrium state.
          The parametrization is e_alpha^2 = -cos theta z-hat -sin theta y-hat
          d_alpha = cos theta y-hat -sin theta z-hat *)
(***) E22, alpha=2, lambda=2 (***)
Clear["context`*"];
Clear[E22, c1, c2, gD, gamma, xV, alpha, lambda, V, V1, V2, V3, Kij, K1, K1ij, K2, K2ij,
      K3, K3ij, Qbetaj, Q1betaj, Q2betaj, Delta p, Delta v1, Delta v2, X, theta, x1, x2, x3, b, a, d, Va, Vb,
      Vd, Vj, Vbeta, Xa, Xb, Xd, Aij33, Aij331, Aij332, Aij333, Aij334, Aij335, Pai1,
      Pai2, Pai3, Paj1, Paj2, Paj3, Pai, Paj, PaiVb, PajVb, KijAijalpha KaLambda alpha lambda,
      KbaLambda ba alpha lambda, Rj lambda beta Qbetaj, VdVbeta lambda epsilon alpha beta gamma, VdVbeta lambda epsilon alpha beta gamma Qbetaj, LeviCivita epsilon ijk];
alpha =
  2;
lambda = 2;
KaLambda alpha alpha lambda = 0;
KbaLambda ba alpha lambda = 0;
Rb lambda alpha Qb = 0;
Rdb lambda alpha Qdb = 0;
VVbeta epsilon alpha Qb = 0;
VdVbeta epsilon alpha Qbd = 0;
V1 = {1, 0, 0};
V2 = {0, -Sin[theta[x1, x2, x3]], -Cos[theta[x1, x2, x3]]};
V3 = {0, Cos[theta[x1, x2, x3]], -Sin[theta[x1, x2, x3]]};
(*****)
(*V tensor for d e^1 e^2*)
V = {V1, V2, V3};
(*TreeForm [V] *) (*Level[V, {2}][[1]] *)
(*****)
(*X tensor for X^-1 X^-2 X^-3 *)

```

```

X = {{Δv1, 0, 0}, {0, Δv2, 0}, {0, 0, Δp}};
(*Xia tensor symbol Row is a index and column is spatial index *)
(*****)
(** Sumover ba **)
(*****)
(*K tensor Kijba*)
Do[Do[
  Xb = Level[X, {1}][[b]]; Xa = Level[X, {1}][[a]];
  K1ij = K1Array[KroneckerDelta, {3, 3}][Tr[{Xb}].{Xa}];
  K2ij = K2[{Xb}].{Xa};
  K3ij = K3[{Xa}].{Xb};
  Kij = K1ij + K2ij + K3ij;
  Print[Style["Kij=", Red, 12], Kij//MatrixForm,
    ", " Style["b=", Red, 12], b, ", ", Style["a=", Red, 12], a];
  (*****)
  (* Δij22ba *)
  Vb = Level[V, {1}][[b]]; Va = Level[V, {1}][[a]];
  Pai = Grad[δs[x1, x2, x3], {x1, x2, x3}];
  Paj = Grad[δs[x1, x2, x3], {x1, x2, x3}];
  PaiPaj = Grad[Grad[δs[x1, x2, x3], {x1, x2, x3}], {x1, x2, x3}];
  (***** the symbol Δijxx(1..5) means the 1st to 5th terms of Δijxx,
    where xx is αλ *****)
  Δij331 = (Grad[Grad[Vb[[λ]], {x1, x2, x3}], {x1, x2, x3}]) Va[[α]];
  Δij331 = Δij111δs[x1, x2, x3];
  (*****)
  (*Δij112=
    (FullSimplify[Vb.Va, Assumptions → {x1, x2, x3, θ} ∈ Reals]) - Vb[[1]] Va[[1]]
    ({Pai}).{Paj});*)
  Δij332 = (FullSimplify[Vb.Va, Assumptions → {x1, x2, x3, θ} ∈ Reals])
    KroneckerDelta[α, λ] - Vb[[α]] Va[[λ]] (PaiPaj);
  (*Δij332//MatrixForm *)
  (*****)
  PaiVb = (Grad[Vb, {x1, x2, x3}])T;
  Δij333 = {FullSimplify[(PaiVb).(Va), Assumptions → {x1, x2, x3, θ} ∈ Reals]}
    KroneckerDelta[α, λ] -
    ((Grad[Vb[[α]], {x1, x2, x3}]) (Va[[λ]]))T.{Paj};
  (*****)
  PajVb = (Grad[Vb, {x1, x2, x3}])T;
  Δij334 =
    ({Pai}).{FullSimplify[(PajVb).(Va), Assumptions → {x1, x2, x3, θ} ∈ Reals]}
    KroneckerDelta[α, λ] - ((Grad[Vb[[α]], {x1, x2, x3}]) (Va[[λ]]))T;
  (*****)
  (*b=1; a=1; Vb=Level[V, {2}][[b]]; Va=Level[V, {2}][[a]];*)
  Δij335 = -(Grad[Grad[Vb[[α]], {x1, x2, x3}], {x1, x2, x3}]) Va[[λ]];
  Δij335 = Δij115δs[x1, x2, x3];
  (*****)
  Δij33 = Δij331 + Δij332 + Δij333 + Δij334 + Δij335;

```



```

FullSimplify[Aij33, Assumptions → {{x1, x2, x3, θ, K1, K2, K3} ∈ Reals}]];
(*****)
KijAijα = FullSimplify[Tr[Kij.(Aij33T)],
  Assumptions → {{x1, x2, x3, θ, K1, K2, K3} ∈ Reals}]];
Print[Style["KijAijαFull=", Red, 12], KijAijα//TraditionalForm,
  ", ", Style["b=", Red, 12], b, ", ", Style["a=", Red, 12], a];
Print[Style["KijAijα", Red, 12],
  (KijAijα. {δs(0,0,1)[x1, x2, x3] → 0, θ(0,0,1)[x1, x2, x3] → 0,
    δs(0,0,2)[x1, x2, x3] → 0, δs(1,0,1)[x1, x2, x3] → 0}) // TraditionalForm,
  ", ", Style["b=", Red, 12], b, ", ", Style["a=", Red, 12], a];
KaAαα = KaAαα + FullSimplify[KijAijα. {δs(0,0,1)[x1, x2, x3] → 0,
  θ(0,0,1)[x1, x2, x3] → 0, δs(0,0,2)[x1, x2, x3] → 0, δs(1,0,1)[x1, x2, x3] → 0},
  Assumptions → {{x1, x2, x3, θ} ∈ Reals}]];
, {a, 3}];
KbaAbaα = KbaAbaα + KaAαα; KaAαα = 0; , {b, 3}];
KbaAbaα =
  FullSimplify[KbaAbaα, Assumptions → {{x1, x2, x3, θ} ∈ Reals}] // TraditionalForm
(*****)
(*****)
LeviCivita[i_, j_, k_] := Module[{test1 = (i == j || j == k || i == k), test2 =
  ((i == 1 && j == 2 && k == 3) || (i == 2 && j == 3 && k == 1) || (i == 3 && j == 1 && k == 2))},
  If[test1, eijk = 0; , If[test2, eijk = 1; , eijk = -1;];]; eijk];
Do[Do[(* Q tensor Qβjbd*)
  Xb = Level[X, {1}][[b]]; Xd = Level[X, {1}][[d]];
  Q1βj = ({Xb})T.{Xd};
  Q2βj = ({Xd})T.{Xb};
  Qβj = (Q1βj + Q2βj);
  Print[Style["Qβj=", Red, 12], Qβj // MatrixForm,
    ", ", Style["d=", Red, 12], d, ", ", Style["b=", Red, 12], b];
  (*****)
  (* Rdbjλαβ *)
  Vd = Level[V, {1}][[d]]; Vb = Level[V, {1}][[b]];
  Rjλαβ = (({Vd})T.{Vb}) KroneckerDelta[λ, α] -
    (({Vd})T.{Vb}[[α]]).{Table[KroneckerDelta[λ, β], {β, 1, 3, 1}]}];
  (*Print["Rjλαβ=", Rjλαβ//MatrixForm, ", ", "d=", "d", ", ", "b=", "b"];*)
  RjλαβQβj = FullSimplify[Tr[Rjλαβ.Qβj], Assumptions →
    {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
  Print[Style["RjλαβQβj=", Red, 12], RjλαβQβj // TraditionalForm,
    ", ", Style["d=", Red, 12], d, ", ", Style["b=", Red, 12], b];
  RbλαQb = RbλαQb + RjλαβQβj;
  (*****)
  (* VdζVbγ ∈jλζ ∈αβγ Qbdβj *)
  (*****)
  Vj = Table[Sum[Vd[[ξ]] LeviCivita[j, λ, ξ], {ξ, 3}], {j, 3}];
  Vβ = Table[Sum[Vb[[γ]] LeviCivita[α, β, γ], {γ, 3}], {β, 3}];

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VdVbejλζεαβγ = ({Vj})T. {Vβ};
VdVbejλζεαβγQβj = FullSimplify [Tr[VdVbejλζεαβγ.Qβj], Assumptions →
  {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
Print[Style["VdVbejλζεαβγQβj=", Red, 12], VdVbejλζεαβγQβj // TraditionalForm,
  ",", Style["d=", Red, 12], d, ",", Style["b=", Red, 12], b];
VVbeλεαQb = VVbeλεαQb + VdVbejλζεαβγQβj; , {b, 3}];
RdbλαQdb = RdbλαQdb + RbλαQb;
RbλαQb = 0;
VdVbeλεαQbd = VdVbeλεαQbd + VVbeλεαQb;
VVbeλεαQb = 0; , {d, 3}];

(***** Show results *****)
RdbλαQdb = FullSimplify [RdbλαQdb,
  Assumptions → {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
VdVbeλεαQbd = FullSimplify [VdVbeλεαQbd,
  Assumptions → {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
Print[Style["RdbλαQdb=", Red, 12], RdbλαQdb // TraditionalForm, ",",
  Style["α=", Red, 12], α, ",", Style["λ=", Red, 12], λ];
Print[Style["VdVbeλεαQbd=", Red, 12], VdVbeλεαQbd // TraditionalForm,
  ",", Style["α=", Red, 12], α, ",", Style["λ=", Red, 12], λ];

(*****
(*****  $\Xi_{11} = \frac{v^2}{x^v} Kba\lambda\alpha\alpha + \frac{6}{5} \frac{qD}{x^v} \frac{v^2}{x^v} Rdb\lambda\alpha Qdb + \frac{6}{5} \frac{qD}{x^v} \frac{v^2}{x^v} VdVbe\lambda\alpha Qbd$  *****)
(** c1= $\frac{v^2}{x^v}$  c2= $\frac{6}{5} \frac{qD}{x^v}$  **)
Print[Style["  $\Xi_{22}$  in Jaakko's parametrization : ", Red, 18]]
 $\Xi_{22} = (c1 Kba\lambda\alpha\alpha + c2 Rdb\lambda\alpha Qdb + c2 VdVbe\lambda\alpha Qbd)$ ;
 $\Xi_{22}$  // TraditionalForm


$$K_{ij} = \begin{pmatrix} K1 \Delta v1^2 + K2 \Delta v1^2 + K3 \Delta v1^2 & 0 & 0 \\ 0 & K1 \Delta v1^2 & 0 \\ 0 & 0 & K1 \Delta v1^2 \end{pmatrix}, b=1, a=1$$


$$K_{ij} \Delta_{ij} \alpha \lambda Full = \Delta v1^2$$


$$\left( (K1 + K2 + K3) \delta_S^{(2,0,0)}(x1, x2, x3) + K1 \left( \delta_S^{(0,0,2)}(x1, x2, x3) + \delta_S^{(0,2,0)}(x1, x2, x3) \right) \right), b=1, a=1$$


$$K_{ij} \Delta_{ij} \alpha \lambda = \Delta v1^2 \left( (K1 + K2 + K3) \delta_S^{(2,0,0)}(x1, x2, x3) + K1 \delta_S^{(0,2,0)}(x1, x2, x3) \right), b=1, a=1$$


$$K_{ij} = \begin{pmatrix} 0 & K2 \Delta v1 \Delta v2 & 0 \\ K3 \Delta v1 \Delta v2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b=1, a=2$$


$$K_{ij} \Delta_{ij} \alpha \lambda Full = 0, b=1, a=2$$


$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=1, a=2$$


$$K_{ij} = \begin{pmatrix} 0 & 0 & K2 \Delta p \Delta v1 \\ 0 & 0 & 0 \\ K3 \Delta p \Delta v1 & 0 & 0 \end{pmatrix}, b=1, a=3$$


$$K_{ij} \Delta_{ij} \alpha \lambda Full = 0, b=1, a=3$$


$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=1, a=3$$


$$K_{ij} = \begin{pmatrix} 0 & K3 \Delta v1 \Delta v2 & 0 \\ K2 \Delta v1 \Delta v2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b=2, a=1$$


$$K_{ij} \Delta_{ij} \alpha \lambda Full = 0, b=2, a=1$$


$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=2, a=1$$


```

$$K_{ij} = \begin{pmatrix} K1 \Delta v^2 & 0 & 0 \\ 0 & K1 \Delta v^2 + K2 \Delta v^2 + K3 \Delta v^2 & 0 \\ 0 & 0 & K1 \Delta v^2 \end{pmatrix}, \text{b}=2, \text{a}=2$$

$$K_{ij} A_{ij} \alpha \lambda \text{Full} = \Delta v^2 \cos(\theta(x1, x2, x3)) \\ (\cos(\theta(x1, x2, x3)) ((K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(0,0,2)}(x1, x2, x3) + \\ K1 \delta s^{(2,0,0)}(x1, x2, x3)) - 2 \sin(\theta(x1, x2, x3)) \\ ((K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + K1 \delta s^{(0,0,1)}(x1, x2, x3) \\ \theta^{(0,0,1)}(x1, x2, x3) + K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3))) , \text{b}=2, \text{a}=2$$

$$K_{ij} A_{ij} \alpha \lambda = \Delta v^2 \cos(\theta(x1, x2, x3)) \\ (\cos(\theta(x1, x2, x3)) ((K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(2,0,0)}(x1, x2, x3)) - \\ 2 \sin(\theta(x1, x2, x3)) ((K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \\ K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3))) , \text{b}=2, \text{a}=2$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K2 \Delta p \Delta v^2 \\ 0 & K3 \Delta p \Delta v^2 & 0 \end{pmatrix}, \text{b}=2, \text{a}=3$$

$$K_{ij} A_{ij} \alpha \lambda \text{Full} = \Delta p \Delta v^2 (- (K2 + K3) \sin(\theta(x1, x2, x3)) \\ ((\delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,0,1)}(x1, x2, x3) + \delta s^{(0,0,1)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3)) \\ \sin(\theta(x1, x2, x3)) - \delta s^{(0,1,1)}(x1, x2, x3) \cos(\theta(x1, x2, x3))) , \text{b}=2, \text{a}=3$$

$$K_{ij} A_{ij} \alpha \lambda = \Delta p \Delta v^2 (K2 + K3) \delta s^{(0,1,1)}(x1, x2, x3) \sin(\theta(x1, x2, x3)) \cos(\theta(x1, x2, x3)) , \text{b}=2, \text{a}=3$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & K3 \Delta p \Delta v^1 \\ 0 & 0 & 0 \\ K2 \Delta p \Delta v^1 & 0 & 0 \end{pmatrix}, \text{b}=3, \text{a}=1$$

$$K_{ij} A_{ij} \alpha \lambda \text{Full} = 0, \text{b}=3, \text{a}=1$$

$$K_{ij} A_{ij} \alpha \lambda = 0, \text{b}=3, \text{a}=1$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K3 \Delta p \Delta v^2 \\ 0 & K2 \Delta p \Delta v^2 & 0 \end{pmatrix}, \text{b}=3, \text{a}=2$$

$$K_{ij} A_{ij} \alpha \lambda \text{Full} = \Delta p \Delta v^2 (K2 + K3) \cos(\theta(x1, x2, x3)) \\ ((\delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,0,1)}(x1, x2, x3) + \delta s^{(0,0,1)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3)) \\ \cos(\theta(x1, x2, x3)) + \delta s^{(0,1,1)}(x1, x2, x3) \sin(\theta(x1, x2, x3))) , \text{b}=3, \text{a}=2$$

$$K_{ij} A_{ij} \alpha \lambda = \Delta p \Delta v^2 (K2 + K3) \delta s^{(0,1,1)}(x1, x2, x3) \sin(\theta(x1, x2, x3)) \cos(\theta(x1, x2, x3)) , \text{b}=3, \text{a}=2$$

$$K_{ij} = \begin{pmatrix} K1 \Delta p^2 & 0 & 0 \\ 0 & K1 \Delta p^2 & 0 \\ 0 & 0 & K1 \Delta p^2 + K2 \Delta p^2 + K3 \Delta p^2 \end{pmatrix}, \text{b}=3, \text{a}=3$$

$$K_{ij} A_{ij} \alpha \lambda \text{Full} = \Delta p^2 \sin(\theta(x1, x2, x3)) \\ (2 \cos(\theta(x1, x2, x3)) ((K1 + K2 + K3) \delta s^{(0,0,1)}(x1, x2, x3) \theta^{(0,0,1)}(x1, x2, x3) + \\ K1 (\delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \\ \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3))) + \\ \sin(\theta(x1, x2, x3)) ((K1 + K2 + K3) \delta s^{(0,0,2)}(x1, x2, x3) + \\ K1 (\delta s^{(0,2,0)}(x1, x2, x3) + \delta s^{(2,0,0)}(x1, x2, x3)))) , \text{b}=3, \text{a}=3$$

$$K_{ij} A_{ij} \alpha \lambda = \Delta p^2 \sin(\theta(x1, x2, x3)) \\ (2 K1 (\delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3)) \\ \cos(\theta(x1, x2, x3)) + \\ K1 (\delta s^{(0,2,0)}(x1, x2, x3) + \delta s^{(2,0,0)}(x1, x2, x3)) \sin(\theta(x1, x2, x3))) , \text{b}=3, \text{a}=3$$

Out[80]/TraditionalForm=

$$\begin{aligned} & \Delta v_1^2 \left( (K_1 + K_2 + K_3) \delta_s^{(2,0,0)}(x_1, x_2, x_3) + K_1 \delta_s^{(0,2,0)}(x_1, x_2, x_3) \right) + \Delta v_2^2 \cos(\theta(x_1, x_2, x_3)) \\ & \left( \cos(\theta(x_1, x_2, x_3)) \left( (K_1 + K_2 + K_3) \delta_s^{(0,2,0)}(x_1, x_2, x_3) + K_1 \delta_s^{(2,0,0)}(x_1, x_2, x_3) \right) - 2 \sin(\theta(x_1, x_2, x_3)) \right. \\ & \left. \left( (K_1 + K_2 + K_3) \delta_s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) + K_1 \delta_s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3) \right) \right) + \\ & \Delta p^2 \sin(\theta(x_1, x_2, x_3)) \left( 2 K_1 \left( \delta_s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) + \delta_s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3) \right) \right. \\ & \left. \cos(\theta(x_1, x_2, x_3)) + K_1 \left( \delta_s^{(0,2,0)}(x_1, x_2, x_3) + \delta_s^{(2,0,0)}(x_1, x_2, x_3) \right) \sin(\theta(x_1, x_2, x_3)) \right) + \\ & \Delta p \Delta v_2 (K_2 + K_3) \delta_s^{(0,1,1)}(x_1, x_2, x_3) \sin(2 \theta(x_1, x_2, x_3)) \end{aligned}$$

$$Q\beta j = \begin{pmatrix} 2 \Delta v_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d=1, b=1$$

$$Rj \lambda \alpha \beta Q \beta j = 2 \Delta v_1^2, d=1, b=1$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q \beta j = 0, d=1, b=1$$

$$Q\beta j = \begin{pmatrix} 0 & \Delta v_1 \Delta v_2 & 0 \\ \Delta v_1 \Delta v_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d=1, b=2$$

$$Rj \lambda \alpha \beta Q \beta j = 0, d=1, b=2$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q \beta j = 0, d=1, b=2$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & \Delta p \Delta v_1 \\ 0 & 0 & 0 \\ \Delta p \Delta v_1 & 0 & 0 \end{pmatrix}, d=1, b=3$$

$$Rj \lambda \alpha \beta Q \beta j = -\Delta p \Delta v_1 \sin(\theta(x_1, x_2, x_3)), d=1, b=3$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q \beta j = -\Delta p \Delta v_1 \sin(\theta(x_1, x_2, x_3)), d=1, b=3$$

$$Q\beta j = \begin{pmatrix} 0 & \Delta v_1 \Delta v_2 & 0 \\ \Delta v_1 \Delta v_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d=2, b=1$$

$$Rj \lambda \alpha \beta Q \beta j = -\Delta v_1 \Delta v_2 \sin(\theta(x_1, x_2, x_3)), d=2, b=1$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q \beta j = 0, d=2, b=1$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 \Delta v_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d=2, b=2$$

$$Rj \lambda \alpha \beta Q \beta j = 0, d=2, b=2$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q \beta j = 0, d=2, b=2$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta p \Delta v_2 \\ 0 & \Delta p \Delta v_2 & 0 \end{pmatrix}, d=2, b=3$$

$$Rj \lambda \alpha \beta Q \beta j = \Delta p \Delta v_2 \sin^2(\theta(x_1, x_2, x_3)), d=2, b=3$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q \beta j = 0, d=2, b=3$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & \Delta p \Delta v_1 \\ 0 & 0 & 0 \\ \Delta p \Delta v_1 & 0 & 0 \end{pmatrix}, d=3, b=1$$

$$Rj \lambda \alpha \beta Q \beta j = -\Delta p \Delta v_1 \sin(\theta(x_1, x_2, x_3)), d=3, b=1$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q \beta j = -\Delta p \Delta v_1 \sin(\theta(x_1, x_2, x_3)), d=3, b=1$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta p \Delta v_2 \\ 0 & \Delta p \Delta v_2 & 0 \end{pmatrix}, d=3, b=2$$

$$Rj\lambda\alpha\beta Q\beta j = -\Delta p \Delta v^2 \cos^2(\theta(x1, x2, x3)), d=3, b=2$$

$$VdVbej\lambda\xi\epsilon\alpha\beta\gamma Q\beta j = 0, d=3, b=2$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\Delta p^2 \end{pmatrix}, d=3, b=3$$

$$Rj\lambda\alpha\beta Q\beta j = 2\Delta p^2 \sin^2(\theta(x1, x2, x3)), d=3, b=3$$

$$VdVbej\lambda\xi\epsilon\alpha\beta\gamma Q\beta j = 0, d=3, b=3$$

$$Rdb\lambda\alpha Qdb =$$

$$-\Delta v1 (2\Delta p + \Delta v2) \sin(\theta(x1, x2, x3)) - \Delta p (\Delta p + \Delta v2) \cos(2\theta(x1, x2, x3)) + \Delta p^2 + 2\Delta v1^2, \alpha=2, \lambda=2$$

$$VdVbe\lambda\epsilon\alpha Qbd = -2\Delta p \Delta v1 \sin(\theta(x1, x2, x3)), \alpha=2, \lambda=2$$

$E_{22}$  in Jaakko's parametrization :

Out[88]/TraditionalForm=

$$\begin{aligned} & c1 \left( \sin(\theta(x1, x2, x3)) \left( 2 K1 \cos(\theta(x1, x2, x3)) \right. \right. \\ & \quad \left( \delta_s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \delta_s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) + \\ & \quad \left. K1 \sin(\theta(x1, x2, x3)) \left( \delta_s^{(0,2,0)}(x1, x2, x3) + \delta_s^{(2,0,0)}(x1, x2, x3) \right) \right) \Delta p^2 + \\ & \quad (K2 + K3) \Delta v2 \sin(2\theta(x1, x2, x3)) \delta_s^{(0,1,1)}(x1, x2, x3) \Delta p + \\ & \quad \Delta v1^2 \left( K1 \delta_s^{(0,2,0)}(x1, x2, x3) + (K1 + K2 + K3) \delta_s^{(2,0,0)}(x1, x2, x3) \right) + \\ & \quad \Delta v2^2 \cos(\theta(x1, x2, x3)) \left( \cos(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta_s^{(0,2,0)}(x1, x2, x3) + K1 \delta_s^{(2,0,0)}(x1, x2, x3) \right) - \right. \\ & \quad \left. 2 \sin(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta_s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \right. \right. \\ & \quad \left. \left. K1 \delta_s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) \right) \left. \right) + \\ & c2 \left( -\Delta v1 (2\Delta p + \Delta v2) \sin(\theta(x1, x2, x3)) - \Delta p (\Delta p + \Delta v2) \cos(2\theta(x1, x2, x3)) + \Delta p^2 + 2\Delta v1^2 \right) - \\ & 2 \\ & c2 \\ & \Delta p \\ & \Delta v1 \\ & \sin(\theta(x1, x2, x3)) \end{aligned}$$

```
In[106]:= (* The following Calculations are done by
using Jaakko's d and e^2 vectors parametrizations,
which was proven is lower energy in equilibrium state.
The parametrization is e_alpha^2 = -cos(theta) z-hat -sin(theta) y-hat
d_alpha = cos(theta) y-hat -sin(theta) z-hat *)
(***) E32, alpha=3, lambda=2 (***)
Clear["context`*"];
Clear[E32, c1, c2, gD, gamma, xV, alpha, lambda, V, V1, V2, V3, Kij, K1, K1ij, K2, K2ij,
K3, K3ij, Qbeta j, Q1beta j, Q2beta j, Delta p, Delta v1, Delta v2, X, theta, x1, x2, x3, b, a, d, Va, Vb,
Vd, Vj, Vbeta, Xa, Xb, Xd, Aij33, Aij331, Aij332, Aij333, Aij334, Aij335, Pai1,
Pai2, Pai3, Paj1, Paj2, Paj3, Pai, Paj, PaiVb, PajVb, KijAijalpha, KaLambda alpha,
KbaLambda ba alpha, Rj lambda alpha beta Qbeta j, VdVbej lambda xi epsilon alpha beta gamma, VdVbej lambda xi epsilon alpha beta gamma Qbeta j, LeviCivita epsilon ijk];
alpha =
3;
lambda = 2;
KaLambda alpha = 0;
KbaLambda ba alpha = 0;
```

```

Rbλ $\alpha$ Qb = 0;
Rdbλ $\alpha$ Qdb = 0;
VVb $\epsilon$  $\lambda$  $\epsilon$  $\alpha$ Qb = 0;
VdVb $\epsilon$  $\lambda$  $\epsilon$  $\alpha$ Qbd = 0;
V1 = {1, 0, 0};
V2 = {0, -Sin[ $\theta$ [x1, x2, x3]], -Cos[ $\theta$ [x1, x2, x3]]};
V3 = {0, Cos[ $\theta$ [x1, x2, x3]], -Sin[ $\theta$ [x1, x2, x3]]};
(*****
(*V tensor for d e1 e2*)
V = {V1, V2, V3};
(*TreeForm [V]*) (*Level[V, {2}][[1]]*)
(*****
(*X tensor for  $\bar{X}^1 \bar{X}^2 \bar{X}^3$ *)
X = {{ $\Delta$ v1, 0, 0}, {0,  $\Delta$ v2, 0}, {0, 0,  $\Delta$ p}};
(*Xia tensor symbol Row is a index and colum is spatial index *)
(*****
(** Sumover ba **)
(*****
(*K tensor Kijba*)
Do[Do[
  Xb = Level[X, {1}][[b]]; Xa = Level[X, {1}][[a]];
  K1ij = K1Array[KroneckerDelta, {3, 3}] (Tr[{Xb}]T.{Xa}]);
  K2ij = K2 ({Xb})T.{Xa};
  K3ij = K3 ({Xa})T.{Xb});
  Kij = K1ij + K2ij + K3ij;
  Print[Style["Kij=", Red, 12], Kij//MatrixForm,
    " ", Style["b=", Red, 12], b, " ", Style["a=", Red, 12], a];
  (*****
  (*  $\Lambda_{ij33}^{ba}$  *)
  Vb = Level[V, {1}][[b]]; Va = Level[V, {1}][[a]];
  Pai = Grad[ $\delta$ s[x1, x2, x3], {x1, x2, x3}];
  Paj = Grad[ $\delta$ s[x1, x2, x3], {x1, x2, x3}];
  PaiPaj = Grad[Grad[ $\delta$ s[x1, x2, x3], {x1, x2, x3}], {x1, x2, x3}];
  (***** the symbol  $\Lambda_{ijxx}(1..5)$  means the 1st to 5th terms of  $\Lambda_{ijxx}$ ,
  where xx is  $\alpha\lambda$  *****)
   $\Lambda_{ij331}$  = (Grad[Grad[Vb[[ $\lambda$ ]], {x1, x2, x3}], {x1, x2, x3}]) Va[[ $\alpha$ ]];
   $\Lambda_{ij331}$  =  $\Lambda_{ij111}\delta$ s[x1, x2, x3];
  (*****
  (* $\Lambda_{ij112}$ =
    (FullSimplify [Vb.Va, Assumptions  $\rightarrow$  {x1, x2, x3,  $\theta$ } $\in$ Reals}] - Vb[[1]] Va[[1]])
    (({Pai})T.{Paj});*)
   $\Lambda_{ij332}$  = (FullSimplify [Vb.Va, Assumptions  $\rightarrow$  {x1, x2, x3,  $\theta$ } $\in$ Reals}]
    KroneckerDelta[ $\alpha$ ,  $\lambda$ ] - Vb[[ $\alpha$ ]] Va[[ $\lambda$ ]]) (PaiPaj);
  (* $\Lambda_{ij332}$ //MatrixForm *)
  (*****
  PaiVb = (Grad[Vb, {x1, x2, x3}])T;
   $\Lambda_{ij333}$  = { (FullSimplify [(PaiVb).(Va), Assumptions  $\rightarrow$  {x1, x2, x3,  $\theta$ } $\in$ Reals}]

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KroneckerDelta[α, λ] -
((Grad[Vb[[α]], {x1, x2, x3}]) (Va[[λ]]))T.{Paj};
(*****)
PajVb = (Grad[Vb, {x1, x2, x3}])T;
Aij334 =
({Pai})T.{(FullSimplify[(PajVb).(Va), Assumptions → {{x1, x2, x3, θ} ∈ Reals}])
KroneckerDelta[α, λ] - ((Grad[Vb[[α]], {x1, x2, x3}]) (Va[[λ]]))T};
(*****)
(*b=1;a=1;Vb=Level[V,{2}][[b]];Va=Level[V,{2}][[a]];*)
Aij335 = - (Grad[Grad[Vb[[α]], {x1, x2, x3}], {x1, x2, x3}]) Va[[λ]];
Aij335 = Aij115δs[x1, x2, x3];
(*****)
Aij33 = Aij331+Aij332+Aij333+Aij334+Aij335;
FullSimplify[Aij33, Assumptions → {{x1, x2, x3, θ, K1, K2, K3} ∈ Reals}]];
(*****)
KijAijαλ = FullSimplify[Tr[Kij.(Aij33T)],
Assumptions → {{x1, x2, x3, θ, K1, K2, K3} ∈ Reals}]];
Print[Style["KijAijαλFull", Red, 12], KijAijαλ/TraditionalForm,
", " Style["b=", Red, 12], b, ", ", Style["a=", Red, 12], a];
Print[Style["KijAijαλ", Red, 12],
(KijAijαλ. {δs(0,0,1)[x1, x2, x3] → 0, θ(0,0,1)[x1, x2, x3] → 0,
δs(0,0,2)[x1, x2, x3] → 0, δs(1,0,1)[x1, x2, x3] → 0}) // TraditionalForm,
", " Style["b=", Red, 12], b, ", ", Style["a=", Red, 12], a];
KaAaαλ = KaAaαλ+FullSimplify[KijAijαλ. {δs(0,0,1)[x1, x2, x3] → 0,
θ(0,0,1)[x1, x2, x3] → 0, δs(0,0,2)[x1, x2, x3] → 0, δs(1,0,1)[x1, x2, x3] → 0},
Assumptions → {{x1, x2, x3, θ} ∈ Reals}]];
, {a, 3}];
KbaAbaαλ = KbaAbaαλ+KaAaαλ; KaAaαλ = 0; , {b, 3}];
KbaAbaαλ =
FullSimplify[KbaAbaαλ, Assumptions → {{x1, x2, x3, θ} ∈ Reals}] // TraditionalForm
(*****)
(*****)
LeviCivita[i_, j_, k_] := Module[{test1 = (i == j || j == k || i == k), test2 =
((i == 1 && j == 2 && k == 3) || (i == 2 && j == 3 && k == 1) || (i == 3 && j == 1 && k == 2))},
If[test1, eijk = 0; , If[test2, eijk = 1; , eijk = -1;];]; eijk];
Do[Do[(* Q tensor Qβjbd*)
Xb = Level[X, {1}][[b]]; Xd = Level[X, {1}][[d]];
Q1βj = ({Xb})T.{Xd};
Q2βj = ({Xd})T.{Xb};
Qβj = (Q1βj+Q2βj);
Print[Style["Qβj=", Red, 12], Qβj // MatrixForm,
", " , Style["d=", Red, 12], d, ", ", Style["b=", Red, 12], b];
(*****)
(* Rdbjλαβ *)
Vd = Level[V, {1}][[d]]; Vb = Level[V, {1}][[b]];

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Rjλαβ = (({Vd})†.{Vb}) KroneckerDelta[λ, α] -
  (({Vd})†(Vb[[α]])).{Table[KroneckerDelta[λ, β], {β, 1, 3, 1}]}];
(*Print["Rjλαβ=", Rjλαβ//MatrixForm, "", "d=", "d", "", "b=", "b"];*)
RjλαβQβj = FullSimplify[Tr[Rjλαβ.Qβj], Assumptions →
  {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
Print[Style["RjλαβQβj=", Red, 12], RjλαβQβj // TraditionalForm,
  "", Style["d=", Red, 12], d, "", Style["b=", Red, 12], b];
RbλαQb = RbλαQb + RjλαβQβj;

(*****
(* VdζVbγεjλζεαβγQbdβj *)
(*****
Vj = Table[Sum[Vd[[ξ]] LeviCivita[j, λ, ξ], {ξ, 3}], {j, 3}];
Vβ = Table[Sum[Vb[[γ]] LeviCivita[α, β, γ], {γ, 3}], {β, 3}];
VdVbejλξεαβγ = ({Vj})†.{Vβ};
VdVbejλξεαβγQβj = FullSimplify[Tr[VdVbejλξεαβγ.Qβj], Assumptions →
  {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
Print[Style["VdVbejλξεαβγQβj=", Red, 12], VdVbejλξεαβγQβj // TraditionalForm,
  "", Style["d=", Red, 12], d, "", Style["b=", Red, 12], b];
VVbeλεαQb = VVbeλεαQb + VdVbejλξεαβγQβj; , {b, 3}];
RdbλαQdb = RdbλαQdb + RbλαQb;
RbλαQb = 0;
VdVbeλεαQbd = VdVbeλεαQbd + VVbeλεαQb;
VVbeλεαQb = 0; , {d, 3}];

(***** Show results *****)
RdbλαQdb = FullSimplify[RdbλαQdb,
  Assumptions → {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
VdVbeλεαQbd = FullSimplify[VdVbeλεαQbd,
  Assumptions → {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
Print[Style["RdbλαQdb=", Red, 12], RdbλαQdb // TraditionalForm, "",
  Style["α=", Red, 12], α, "", Style["λ=", Red, 12], λ];
Print[Style["VdVbeλεαQbd=", Red, 12], VdVbeλεαQbd // TraditionalForm,
  "", Style["α=", Red, 12], α, "", Style["λ=", Red, 12], λ];

(*****
(***** E11=  $\frac{V^2}{\chi^v} K_{ba\lambda ba\alpha\lambda} + \frac{6}{5} \frac{qD}{\chi^v} \frac{V^2}{\chi^v} R_{db\lambda\alpha Qdb} + \frac{6}{5} \frac{qD}{\chi^v} \frac{V^2}{\chi^v} V_{dVbe\lambda\alpha Qbd}$  *****)
(** c1= $\frac{V^2}{\chi^v}$  c2= $\frac{6}{5} \frac{qD}{\chi^v} \frac{V^2}{\chi^v}$  **)
Print[Style[" E32 in Jaakko's parametrization : ", Red, 18]]
E32 = (c1 Kbaλbaαλ + c2 RdbλαQdb + c2 VdVbeλεαQbd);
E32 // TraditionalForm

Kij =  $\begin{pmatrix} K1 \Delta v1^2 + K2 \Delta v1^2 + K3 \Delta v1^2 & 0 & 0 \\ 0 & K1 \Delta v1^2 & 0 \\ 0 & 0 & K1 \Delta v1^2 \end{pmatrix}, b=1, a=1$ 

KijAijαλFull=0, b=1, a=1
KijAijαλ=0, b=1, a=1

Kij =  $\begin{pmatrix} 0 & K2 \Delta v1 \Delta v2 & 0 \\ K3 \Delta v1 \Delta v2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b=1, a=2$ 

```



$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = 0, b=1, a=2$$

$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=1, a=2$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & K2 \Delta p \Delta v1 \\ 0 & 0 & 0 \\ K3 \Delta p \Delta v1 & 0 & 0 \end{pmatrix}, b=1, a=3$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = 0, b=1, a=3$$

$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=1, a=3$$

$$K_{ij} = \begin{pmatrix} 0 & K3 \Delta v1 \Delta v2 & 0 \\ K2 \Delta v1 \Delta v2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b=2, a=1$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = 0, b=2, a=1$$

$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=2, a=1$$

$$K_{ij} = \begin{pmatrix} K1 \Delta v2^2 & 0 & 0 \\ 0 & K1 \Delta v2^2 + K2 \Delta v2^2 + K3 \Delta v2^2 & 0 \\ 0 & 0 & K1 \Delta v2^2 \end{pmatrix}, b=2, a=2$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = \Delta v2^2 \sin(\theta(x1, x2, x3)) \left( 2 \sin(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + K1 \delta s^{(0,0,1)}(x1, x2, x3) \theta^{(0,0,1)}(x1, x2, x3) + K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) - \cos(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(0,0,2)}(x1, x2, x3) + K1 \delta s^{(2,0,0)}(x1, x2, x3) \right) \right), b=2, a=2$$

$$K_{ij} \Delta_{ij} \alpha \lambda = \Delta v2^2 \sin(\theta(x1, x2, x3)) \left( 2 \sin(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) - \cos(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(2,0,0)}(x1, x2, x3) \right) \right), b=2, a=2$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K2 \Delta p \Delta v2 \\ 0 & K3 \Delta p \Delta v2 & 0 \end{pmatrix}, b=2, a=3$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = \Delta p \Delta v2 \left( -(K2 + K3) \cos(\theta(x1, x2, x3)) \left( \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,0,1)}(x1, x2, x3) + \delta s^{(0,0,1)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) \right) \sin(\theta(x1, x2, x3)) - \delta s^{(0,1,1)}(x1, x2, x3) \cos(\theta(x1, x2, x3)) \right), b=2, a=3$$

$$K_{ij} \Delta_{ij} \alpha \lambda = \Delta p \Delta v2 (K2 + K3) \delta s^{(0,1,1)}(x1, x2, x3) \cos^2(\theta(x1, x2, x3)), b=2, a=3$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & K3 \Delta p \Delta v1 \\ 0 & 0 & 0 \\ K2 \Delta p \Delta v1 & 0 & 0 \end{pmatrix}, b=3, a=1$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = 0, b=3, a=1$$

$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=3, a=1$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K3 \Delta p \Delta v2 \\ 0 & K2 \Delta p \Delta v2 & 0 \end{pmatrix}, b=3, a=2$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = \Delta p \Delta v2 \left( -(K2 + K3) \sin(\theta(x1, x2, x3)) \left( \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,0,1)}(x1, x2, x3) + \delta s^{(0,0,1)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) \right) \cos(\theta(x1, x2, x3)) + \delta s^{(0,1,1)}(x1, x2, x3) \sin(\theta(x1, x2, x3)) \right), b=3, a=2$$

$$K_{ij} \Delta_{ij} \alpha \lambda = \Delta p \Delta v2 \left( -(K2 + K3) \delta s^{(0,1,1)}(x1, x2, x3) \sin^2(\theta(x1, x2, x3)) \right), b=3, a=2$$

$$K_{ij} = \begin{pmatrix} K1 \Delta p^2 & 0 & 0 \\ 0 & K1 \Delta p^2 & 0 \\ 0 & 0 & K1 \Delta p^2 + K2 \Delta p^2 + K3 \Delta p^2 \end{pmatrix}, b=3, a=3$$

$$\begin{aligned} K_{ij} \Delta_{ij} \alpha \lambda \text{Full} = & \Delta p^2 \cos(\theta(x_1, x_2, x_3)) \\ & \left( 2 \cos(\theta(x_1, x_2, x_3)) \left( (K_1 + K_2 + K_3) \delta s^{(0,0,1)}(x_1, x_2, x_3) \theta^{(0,0,1)}(x_1, x_2, x_3) + \right. \right. \\ & K_1 \left( \delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) + \right. \\ & \left. \left. \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3) \right) \right) + \\ & \sin(\theta(x_1, x_2, x_3)) \left( (K_1 + K_2 + K_3) \delta s^{(0,0,2)}(x_1, x_2, x_3) + \right. \\ & \left. K_1 \left( \delta s^{(0,2,0)}(x_1, x_2, x_3) + \delta s^{(2,0,0)}(x_1, x_2, x_3) \right) \right) \Big), \mathbf{b}=3, \mathbf{a}=3 \end{aligned}$$

$$\begin{aligned} K_{ij} \Delta_{ij} \alpha \lambda = & \Delta p^2 \cos(\theta(x_1, x_2, x_3)) \\ & \left( 2 K_1 \left( \delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) + \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3) \right) \right. \\ & \left. \cos(\theta(x_1, x_2, x_3)) + \right. \\ & \left. K_1 \left( \delta s^{(0,2,0)}(x_1, x_2, x_3) + \delta s^{(2,0,0)}(x_1, x_2, x_3) \right) \sin(\theta(x_1, x_2, x_3)) \right) \Big), \mathbf{b}=3, \mathbf{a}=3 \end{aligned}$$

Out[114]/TraditionalForm=

$$\begin{aligned} & 2 \delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) \left( \Delta v^2 (K_1 + K_2 + K_3) \sin^2(\theta(x_1, x_2, x_3)) + \Delta p^2 K_1 \cos^2(\theta(x_1, x_2, x_3)) \right) + \\ & \frac{1}{2} \sin(2 \theta(x_1, x_2, x_3)) \left( \delta s^{(0,2,0)}(x_1, x_2, x_3) \left( \Delta p^2 K_1 - \Delta v^2 (K_1 + K_2 + K_3) \right) + \right. \\ & \left. K_1 (\Delta p - \Delta v) (\Delta p + \Delta v) \delta s^{(2,0,0)}(x_1, x_2, x_3) \right) + \cos(2 \theta(x_1, x_2, x_3)) \\ & \left( K_1 (\Delta p - \Delta v) (\Delta p + \Delta v) \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3) + \Delta p \Delta v (K_2 + K_3) \delta s^{(0,1,1)}(x_1, x_2, x_3) \right) + \\ & K_1 \left( \Delta p^2 + \Delta v^2 \right) \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3) \end{aligned}$$

$$Q\beta j = \begin{pmatrix} 2 \Delta v^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{d}=1, \mathbf{b}=1$$

$$Rj \lambda \alpha \beta Q\beta j = 0, \mathbf{d}=1, \mathbf{b}=1$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q\beta j = 0, \mathbf{d}=1, \mathbf{b}=1$$

$$Q\beta j = \begin{pmatrix} 0 & \Delta v^2 \Delta v^2 & 0 \\ \Delta v^2 \Delta v^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{d}=1, \mathbf{b}=2$$

$$Rj \lambda \alpha \beta Q\beta j = \Delta v^2 \Delta v^2 \cos(\theta(x_1, x_2, x_3)), \mathbf{d}=1, \mathbf{b}=2$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q\beta j = 0, \mathbf{d}=1, \mathbf{b}=2$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & \Delta p \Delta v^2 \\ 0 & 0 & 0 \\ \Delta p \Delta v^2 & 0 & 0 \end{pmatrix}, \mathbf{d}=1, \mathbf{b}=3$$

$$Rj \lambda \alpha \beta Q\beta j = 0, \mathbf{d}=1, \mathbf{b}=3$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q\beta j = -\Delta p \Delta v^2 \cos(\theta(x_1, x_2, x_3)), \mathbf{d}=1, \mathbf{b}=3$$

$$Q\beta j = \begin{pmatrix} 0 & \Delta v^2 \Delta v^2 & 0 \\ \Delta v^2 \Delta v^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{d}=2, \mathbf{b}=1$$

$$Rj \lambda \alpha \beta Q\beta j = 0, \mathbf{d}=2, \mathbf{b}=1$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q\beta j = \Delta v^2 \Delta v^2 \cos(\theta(x_1, x_2, x_3)), \mathbf{d}=2, \mathbf{b}=1$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 \Delta v^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{d}=2, \mathbf{b}=2$$

$$Rj \lambda \alpha \beta Q\beta j = -\Delta v^2 \sin(2 \theta(x_1, x_2, x_3)), \mathbf{d}=2, \mathbf{b}=2$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q\beta j = 0, \mathbf{d}=2, \mathbf{b}=2$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta p \Delta v^2 \\ 0 & \Delta p \Delta v^2 & 0 \end{pmatrix}, \mathbf{d}=2, \mathbf{b}=3$$

$$Rj\lambda\alpha\beta Q\beta j = -\frac{1}{2}\Delta p\Delta v2 \sin(2\theta(x1, x2, x3)), d=2, b=3$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=2, b=3$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & \Delta p\Delta v1 \\ 0 & 0 & 0 \\ \Delta p\Delta v1 & 0 & 0 \end{pmatrix}, d=3, b=1$$

$$Rj\lambda\alpha\beta Q\beta j = 0, d=3, b=1$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=3, b=1$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta p\Delta v2 \\ 0 & \Delta p\Delta v2 & 0 \end{pmatrix}, d=3, b=2$$

$$Rj\lambda\alpha\beta Q\beta j = -\frac{1}{2}\Delta p\Delta v2 \sin(2\theta(x1, x2, x3)), d=3, b=2$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=3, b=2$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\Delta p^2 \end{pmatrix}, d=3, b=3$$

$$Rj\lambda\alpha\beta Q\beta j = 0, d=3, b=3$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=3, b=3$$

$$Rdb\lambda\alpha Qdb = \Delta v1\Delta v2 \cos(\theta(x1, x2, x3)) - \Delta v2(\Delta p + \Delta v2) \sin(2\theta(x1, x2, x3)), \alpha=3, \lambda=2$$

$$VdVbe\lambda\epsilon\alpha Qbd = \Delta v1(\Delta v2 - \Delta p) \cos(\theta(x1, x2, x3)), \alpha=3, \lambda=2$$

$\Xi_{32}$  in Jaakko's parametrization :

Out[122]/TraditionalForm=

$$\begin{aligned} & c1 \left( 2 \left( K1 \Delta p^2 \cos^2(\theta(x1, x2, x3)) + (K1 + K2 + K3) \Delta v2^2 \sin^2(\theta(x1, x2, x3)) \right) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \right. \\ & \quad K1 (\Delta p^2 + \Delta v2^2) \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) + \\ & \quad \cos(2\theta(x1, x2, x3)) \left( (K2 + K3) \Delta p \Delta v2 \delta s^{(0,1,1)}(x1, x2, x3) + \right. \\ & \quad \quad K1 (\Delta p - \Delta v2) (\Delta p + \Delta v2) \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \Big) + \\ & \quad \frac{1}{2} \sin(2\theta(x1, x2, x3)) \left( (K1 \Delta p^2 - (K1 + K2 + K3) \Delta v2^2) \delta s^{(0,2,0)}(x1, x2, x3) + \right. \\ & \quad \quad \left. K1 (\Delta p - \Delta v2) (\Delta p + \Delta v2) \delta s^{(2,0,0)}(x1, x2, x3) \right) \Big) + \\ & \quad c2 \Delta v1 (\Delta v2 - \Delta p) \cos(\theta(x1, x2, x3)) + c2 (\Delta v1 \Delta v2 \cos(\theta(x1, x2, x3)) - \Delta v2 (\Delta p + \Delta v2) \sin(2\theta(x1, x2, x3))) \end{aligned}$$

(\* The following Calculations are done by  
using Jaakko's d and e<sup>2</sup> vectors parametrizations,  
which was proven is lower energy in equilibrium state.

The parametrization is  $e_\alpha^2 = -\cos\theta \hat{z} -\sin\theta \hat{y}$

$$d_\alpha = \cos\theta \hat{y} -\sin\theta \hat{z} *)$$

(\*\*\*  $\Xi_{23}, \alpha=2, \lambda=3$  \*\*\*)

Clear["context`\*"];

Clear[ $\Xi_{23}, c1, c2, gD, \gamma, \chi v, \alpha, \lambda, v, v1, v2, v3, Kij, K1, K1ij, K2, K2ij,$   
 $K3, K3ij, Q\beta j, Q1\beta j, Q2\beta j, \Delta p, \Delta v1, \Delta v2, X, \theta, x1, x2, x3, b, a, d, Va, Vb,$   
 $Vd, Vj, V\beta, Xa, Xb, Xd, \Lambda ij33, \Lambda ij331, \Lambda ij332, \Lambda ij333, \Lambda ij334, \Lambda ij335, Pai1,$   
 $Pai2, Pai3, Paj1, Paj2, Paj3, Pai, Paj, PaiVb, PajVb, Kij\Lambda ij\alpha, Ka\Lambda a\alpha\lambda,$   
 $Kba\Lambda ba\alpha\lambda, Rj\lambda\alpha\beta Q\beta j, VdVbej\lambda\zeta\epsilon\alpha\beta\gamma, VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j, LeviCivita\epsilon ijk]$ ;

```

α =
  2;
λ = 3;
KaΛaαλ = 0;
KbaΛbaαλ = 0;
RbλaQb = 0;
RdbλaQdb = 0;
VbVbeλaQb = 0;
VdVbeλaQbd = 0;
V1 = {1, 0, 0};
V2 = {0, -Sin[θ[x1, x2, x3]], -Cos[θ[x1, x2, x3]]};
V3 = {0, Cos[θ[x1, x2, x3]], -Sin[θ[x1, x2, x3]]};
(*****
(*V tensor for d e1 e2*)
V = {V1, V2, V3};
(*TreeForm [V] *) (*Level[V, {2}] [[1]] *)
(*****
(*X tensor for  $\bar{X}^1 \bar{X}^2 \bar{X}^3$  *)
X = {{Δv1, 0, 0}, {0, Δv2, 0}, {0, 0, Δp}};
(*Xia tensor symbol Row is a index and colum is spatial index *)
(*****
(** Sumover ba **)
(*****
(*K tensor Kijba*)
Do[Do[
  Xb = Level[X, {1}] [[b]]; Xa = Level[X, {1}] [[a]];
  K1ij = K1Array[KroneckerDelta, {3, 3}] (Tr[{Xb}]T.{Xa});
  K2ij = K2 ({Xb})T.{Xa};
  K3ij = K3 ({Xa})T.{Xb};
  Kij = K1ij + K2ij + K3ij;
  Print[Style["Kij=", Red, 12], Kij//MatrixForm,
    ", " Style["b=", Red, 12], b, ", ", Style["a=", Red, 12], a];
  (*****
  (*  $\Lambda_{ij33}^{ba}$  *)
  Vb = Level[V, {1}] [[b]]; Va = Level[V, {1}] [[a]];
  Pai = Grad[δs[x1, x2, x3], {x1, x2, x3}];
  Paj = Grad[δs[x1, x2, x3], {x1, x2, x3}];
  PaiPaj = Grad[Grad[δs[x1, x2, x3], {x1, x2, x3}], {x1, x2, x3}];
  (***** the symbol  $\Lambda_{ijxx}(1..5)$  means the 1st to 5th terms of  $\Lambda_{ijxx}$ ,
  where xx is αλ *****)
  Λij331 = (Grad[Grad[Vb[[λ]], {x1, x2, x3}], {x1, x2, x3}]) Va[[a]];
  Λij331 = Λij111δs[x1, x2, x3];
  (*****
  (*Λij112=
  (FullSimplify [Vb.Va, Assumptions → {x1, x2, x3, θ} ∈ Reals]) - Vb[[1]] Va[[1]]
  (({Pai})T.{Paj}); *)
  Λij332 = (FullSimplify [Vb.Va, Assumptions → {x1, x2, x3, θ} ∈ Reals])

```

```

KroneckerDelta[α, λ] - Vb[[α]] Va[[λ]]) (PaiPaj);
(*Δij332/MatrixForm *)
(*****)
PaiVb = (Grad[Vb, {x1, x2, x3}])T;
Δij333 = (FullSimplify[(PaiVb).(Va), Assumptions → {{x1, x2, x3, θ} ∈ Reals}])
KroneckerDelta[α, λ] -
((Grad[Vb[[α]], {x1, x2, x3}]) (Va[[λ]]))T.{Paj};
(*****)
PajVb = (Grad[Vb, {x1, x2, x3}])T;
Δij334 =
({Pai})T.{(FullSimplify[(PajVb).(Va), Assumptions → {{x1, x2, x3, θ} ∈ Reals}])
KroneckerDelta[α, λ] - ((Grad[Vb[[α]], {x1, x2, x3}]) (Va[[λ]]))T};
(*****)
(*b=1;a=1;Vb=Level[V,{2}][[b]];Va=Level[V,{2}][[a]];*)
Δij335 = -(Grad[Grad[Vb[[α]], {x1, x2, x3}], {x1, x2, x3}]) Va[[λ]];
Δij335 = Δij115δs[x1, x2, x3];
(*****)
Δij33 = Δij331+Δij332+Δij333+Δij334+Δij335;
FullSimplify[Δij33, Assumptions → {{x1, x2, x3, θ, K1, K2, K3} ∈ Reals}]];
(*****)
KijΔijαλ = FullSimplify[Tr[Kij.(Δij33T)],
Assumptions → {{x1, x2, x3, θ, K1, K2, K3} ∈ Reals}]];
Print[Style["KijΔijαλFull", Red, 12], KijΔijαλ/TraditionalForm,
", " Style["b=", Red, 12], b, ", ", Style["a=", Red, 12], a];
Print[Style["KijΔijαλ", Red, 12],
(KijΔijαλ. {δs(0,0,1)[x1, x2, x3] → 0, θ(0,0,1)[x1, x2, x3] → 0,
δs(0,0,2)[x1, x2, x3] → 0, δs(1,0,1)[x1, x2, x3] → 0}) // TraditionalForm,
", " Style["b=", Red, 12], b, ", ", Style["a=", Red, 12], a];
KaΛααλ = KaΛααλ+FullSimplify[KijΔijαλ. {δs(0,0,1)[x1, x2, x3] → 0,
θ(0,0,1)[x1, x2, x3] → 0, δs(0,0,2)[x1, x2, x3] → 0, δs(1,0,1)[x1, x2, x3] → 0},
Assumptions → {{x1, x2, x3, θ} ∈ Reals}]];
, {a, 3}];
KbaΛbaαλ = KbaΛbaαλ+KaΛααλ; KaΛααλ = 0; , {b, 3}];
KbaΛbaαλ =
FullSimplify[KbaΛbaαλ, Assumptions → {{x1, x2, x3, θ} ∈ Reals}] // TraditionalForm
(*****)
(*****)
LeviCivita[i_, j_, k_] := Module[{test1 = (i == j || j == k || i == k), test2 =
((i == 1 && j == 2 && k == 3) || (i == 2 && j == 3 && k == 1) || (i == 3 && j == 1 && k == 2))},
If[test1, eijk = 0; , If[test2, eijk = 1; , eijk = -1; ];]; eijk];
Do[Do[(* Q tensor Qβjbd*)
Xb = Level[X, {1}][[b]]; Xd = Level[X, {1}][[d]];
Q1βj = ({Xb})T.{Xd};
Q2βj = ({Xd})T.{Xb};
Qβj = (Q1βj+Q2βj);

```

```

Print[Style["Qβj=", Red, 12], Qβj // MatrixForm ,
  ", ", Style["d=", Red, 12], d, ", ", Style["b=", Red, 12], b];
(*****
(* Rdbjλαβ *)
Vd = Level[V, {1}][[d]]; Vb = Level[V, {1}][[b]];
Rjλαβ = (({Vd})T.{Vb}) KroneckerDelta[λ, α] -
  (({Vd})T(Vb[[α]])) . {Table[KroneckerDelta[λ, β], {β, 1, 3, 1}]}];
(*Print["Rjλαβ=", Rjλαβ // MatrixForm, ", ", "d=", "d", ", ", "b=", b]; *)
RjλαβQβj = FullSimplify[Tr[Rjλαβ.Qβj], Assumptions →
  {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
Print[Style["RjλαβQβj=", Red, 12], RjλαβQβj // TraditionalForm,
  ", ", Style["d=", Red, 12], d, ", ", Style["b=", Red, 12], b];
RbλαQb = RbλαQb + RjλαβQβj;
(*****
(* VdζVbγ ∈jλζ ∈αβγ Qbdβj *)
(*****
Vj = Table[Sum[Vd[[ζ]] LeviCivita[j, λ, ζ], {ζ, 3}], {j, 3}];
Vβ = Table[Sum[Vb[[γ]] LeviCivita[α, β, γ], {γ, 3}], {β, 3}];
VdVbejλζ ∈αβγ = ({Vj})T.{Vβ};
VdVbejλζ ∈αβγQβj = FullSimplify[Tr[VdVbejλζ ∈αβγ.Qβj], Assumptions →
  {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
Print[Style["VdVbejλζ ∈αβγQβj=", Red, 12], VdVbejλζ ∈αβγQβj // TraditionalForm,
  ", ", Style["d=", Red, 12], d, ", ", Style["b=", Red, 12], b];
VVbeλεαQb = VVbeλεαQb + VdVbejλζ ∈αβγQβj; , {b, 3}];
RdbλαQdb = RdbλαQdb + RbλαQb;
RbλαQb = 0;
VdVbeλεαQbd = VdVbeλεαQbd + VVbeλεαQb;
VVbeλεαQb = 0; , {d, 3}];
(***** Show results *****)
RdbλαQdb = FullSimplify[RdbλαQdb,
  Assumptions → {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
VdVbeλεαQbd = FullSimplify[VdVbeλεαQbd,
  Assumptions → {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals && Δp > 0 && Δv1 > 0 && Δv2 > 0}];
Print[Style["RdbλαQdb=", Red, 12], RdbλαQdb // TraditionalForm, ", ",
  Style["α=", Red, 12], α, ", ", Style["λ=", Red, 12], λ];
Print[Style["VdVbeλεαQbd=", Red, 12], VdVbeλεαQbd // TraditionalForm,
  ", ", Style["α=", Red, 12], α, ", ", Style["λ=", Red, 12], λ];
(*****
(***** E11 =  $\frac{V^2}{X^V} K_{ba\lambda ba\alpha\lambda} + \frac{6}{5} \frac{gD}{X^V} \frac{V^2}{X^V} R_{db\lambda\alpha Qdb} + \frac{6}{5} \frac{gD}{X^V} \frac{V^2}{X^V} VdVbe\lambda\epsilon\alpha Qbd$  *****)
(** c1 =  $\frac{V^2}{X^V}$  c2 =  $\frac{6}{5} \frac{gD}{X^V} \frac{V^2}{X^V}$  **)
Print[Style[" E23 in Jaakko's parametrization : ", Red, 18]]
E23 = (c1 Kbaλbaαλ + c2 RdbλαQdb + c2 VdVbeλεαQbd);
E23 // TraditionalForm

```

$$K_{ij} = \begin{pmatrix} K_1 \Delta v_1^2 + K_2 \Delta v_1^2 + K_3 \Delta v_1^2 & 0 & 0 \\ 0 & K_1 \Delta v_1^2 & 0 \\ 0 & 0 & K_1 \Delta v_1^2 \end{pmatrix}, b=1, a=1$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = 0, b=1, a=1$$

$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=1, a=1$$

$$K_{ij} = \begin{pmatrix} 0 & K2 \Delta v1 \Delta v2 & 0 \\ K3 \Delta v1 \Delta v2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b=1, a=2$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = 0, b=1, a=2$$

$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=1, a=2$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & K2 \Delta p \Delta v1 \\ 0 & 0 & 0 \\ K3 \Delta p \Delta v1 & 0 & 0 \end{pmatrix}, b=1, a=3$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = 0, b=1, a=3$$

$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=1, a=3$$

$$K_{ij} = \begin{pmatrix} 0 & K3 \Delta v1 \Delta v2 & 0 \\ K2 \Delta v1 \Delta v2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b=2, a=1$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = 0, b=2, a=1$$

$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=2, a=1$$

$$K_{ij} = \begin{pmatrix} K1 \Delta v2^2 & 0 & 0 \\ 0 & K1 \Delta v2^2 + K2 \Delta v2^2 + K3 \Delta v2^2 & 0 \\ 0 & 0 & K1 \Delta v2^2 \end{pmatrix}, b=2, a=2$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = -\Delta v2^2 \cos(\theta(x1, x2, x3)) \\ (2 \cos(\theta(x1, x2, x3)) ((K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + K1 \delta s^{(0,0,1)}(x1, \\ x2, x3) \theta^{(0,0,1)}(x1, x2, x3) + K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3)) + \\ \sin(\theta(x1, x2, x3)) ((K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(0,0,2)}(x1, x2, x3) + \\ K1 \delta s^{(2,0,0)}(x1, x2, x3))) , b=2, a=2$$

$$K_{ij} \Delta_{ij} \alpha \lambda = -\Delta v2^2 \cos(\theta(x1, x2, x3)) \\ (2 \cos(\theta(x1, x2, x3)) ((K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \\ K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3)) + \sin(\theta(x1, x2, x3)) \\ ((K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(2,0,0)}(x1, x2, x3))) , b=2, a=2$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K2 \Delta p \Delta v2 \\ 0 & K3 \Delta p \Delta v2 & 0 \end{pmatrix}, b=2, a=3$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = \Delta p \Delta v2 (-(K2 + K3)) \sin(\theta(x1, x2, x3)) \\ ((\delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,0,1)}(x1, x2, x3) + \delta s^{(0,0,1)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3)) \\ \cos(\theta(x1, x2, x3)) + \delta s^{(0,1,1)}(x1, x2, x3) \sin(\theta(x1, x2, x3))) , b=2, a=3$$

$$K_{ij} \Delta_{ij} \alpha \lambda = \Delta p \Delta v2 (-(K2 + K3)) \delta s^{(0,1,1)}(x1, x2, x3) \sin^2(\theta(x1, x2, x3)) , b=2, a=3$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & K3 \Delta p \Delta v1 \\ 0 & 0 & 0 \\ K2 \Delta p \Delta v1 & 0 & 0 \end{pmatrix}, b=3, a=1$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = 0, b=3, a=1$$

$$K_{ij} \Delta_{ij} \alpha \lambda = 0, b=3, a=1$$

$$K_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K3 \Delta p \Delta v2 \\ 0 & K2 \Delta p \Delta v2 & 0 \end{pmatrix}, b=3, a=2$$

$$K_{ij} \Delta_{ij} \alpha \lambda_{Full} = \Delta p \Delta v2 (-(K2 + K3)) \cos(\theta(x1, x2, x3)) \\ ((\delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,0,1)}(x1, x2, x3) + \delta s^{(0,0,1)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3)) \\ \sin(\theta(x1, x2, x3)) - \delta s^{(0,1,1)}(x1, x2, x3) \cos(\theta(x1, x2, x3))) , b=3, a=2$$

$$K_{ij} \Delta_{ij} \alpha \lambda = \Delta p \Delta v^2 (K_2 + K_3) \delta s^{(0,1,1)}(x_1, x_2, x_3) \cos^2(\theta(x_1, x_2, x_3)), \mathbf{b}=3, \mathbf{a}=2$$

$$K_{ij} = \begin{pmatrix} K_1 \Delta p^2 & 0 & 0 \\ 0 & K_1 \Delta p^2 & 0 \\ 0 & 0 & K_1 \Delta p^2 + K_2 \Delta p^2 + K_3 \Delta p^2 \end{pmatrix}, \mathbf{b}=3, \mathbf{a}=3$$

$$K_{ij} \Delta_{ij} \alpha \lambda \text{Full} = \Delta p^2 \sin(\theta(x_1, x_2, x_3)) \left( \cos(\theta(x_1, x_2, x_3)) \right. \\ \left( (K_1 + K_2 + K_3) \delta s^{(0,0,2)}(x_1, x_2, x_3) + K_1 \left( \delta s^{(0,2,0)}(x_1, x_2, x_3) + \delta s^{(2,0,0)}(x_1, x_2, x_3) \right) \right) - \\ 2 \sin(\theta(x_1, x_2, x_3)) \left( (K_1 + K_2 + K_3) \delta s^{(0,0,1)}(x_1, x_2, x_3) \theta^{(0,0,1)}(x_1, x_2, x_3) + \right. \\ \left. K_1 \left( \delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) + \right. \right. \\ \left. \left. \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3) \right) \right) \left. \right), \mathbf{b}=3, \mathbf{a}=3$$

$$K_{ij} \Delta_{ij} \alpha \lambda = \\ \Delta p^2 \sin(\theta(x_1, x_2, x_3)) \left( K_1 \left( \delta s^{(0,2,0)}(x_1, x_2, x_3) + \delta s^{(2,0,0)}(x_1, x_2, x_3) \right) \cos(\theta(x_1, x_2, x_3)) - \right. \\ \left. 2 K_1 \left( \delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) + \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3) \right) \right. \\ \left. \sin(\theta(x_1, x_2, x_3)) \right), \mathbf{b}=3, \mathbf{a}=3$$

Out[131]/TraditionalForm=

$$-2 \delta s^{(0,1,0)}(x_1, x_2, x_3) \theta^{(0,1,0)}(x_1, x_2, x_3) \left( \Delta v^2 (K_1 + K_2 + K_3) \cos^2(\theta(x_1, x_2, x_3)) + \Delta p^2 K_1 \sin^2(\theta(x_1, x_2, x_3)) \right) + \\ \frac{1}{2} \sin(2 \theta(x_1, x_2, x_3)) \left( \delta s^{(0,2,0)}(x_1, x_2, x_3) \left( \Delta p^2 K_1 - \Delta v^2 (K_1 + K_2 + K_3) \right) + \right. \\ \left. K_1 (\Delta p - \Delta v^2) (\Delta p + \Delta v^2) \delta s^{(2,0,0)}(x_1, x_2, x_3) \right) + \cos(2 \theta(x_1, x_2, x_3)) \\ \left( K_1 (\Delta p - \Delta v^2) (\Delta p + \Delta v^2) \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3) + \Delta p \Delta v^2 (K_2 + K_3) \delta s^{(0,1,1)}(x_1, x_2, x_3) \right) - \\ K_1 (\Delta p^2 + \Delta v^2) \delta s^{(1,0,0)}(x_1, x_2, x_3) \theta^{(1,0,0)}(x_1, x_2, x_3)$$

$$Q\beta j = \begin{pmatrix} 2 \Delta v^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{d}=1, \mathbf{b}=1$$

$$Rj \lambda \alpha \beta Q\beta j = 0, \mathbf{d}=1, \mathbf{b}=1$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q\beta j = 0, \mathbf{d}=1, \mathbf{b}=1$$

$$Q\beta j = \begin{pmatrix} 0 & \Delta v^1 \Delta v^2 & 0 \\ \Delta v^1 \Delta v^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{d}=1, \mathbf{b}=2$$

$$Rj \lambda \alpha \beta Q\beta j = 0, \mathbf{d}=1, \mathbf{b}=2$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q\beta j = \Delta v^1 \Delta v^2 \cos(\theta(x_1, x_2, x_3)), \mathbf{d}=1, \mathbf{b}=2$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & \Delta p \Delta v^1 \\ 0 & 0 & 0 \\ \Delta p \Delta v^1 & 0 & 0 \end{pmatrix}, \mathbf{d}=1, \mathbf{b}=3$$

$$Rj \lambda \alpha \beta Q\beta j = -\Delta p \Delta v^1 \cos(\theta(x_1, x_2, x_3)), \mathbf{d}=1, \mathbf{b}=3$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q\beta j = 0, \mathbf{d}=1, \mathbf{b}=3$$

$$Q\beta j = \begin{pmatrix} 0 & \Delta v^1 \Delta v^2 & 0 \\ \Delta v^1 \Delta v^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{d}=2, \mathbf{b}=1$$

$$Rj \lambda \alpha \beta Q\beta j = 0, \mathbf{d}=2, \mathbf{b}=1$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q\beta j = 0, \mathbf{d}=2, \mathbf{b}=1$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 \Delta v^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{d}=2, \mathbf{b}=2$$

$$Rj \lambda \alpha \beta Q\beta j = 0, \mathbf{d}=2, \mathbf{b}=2$$

$$VdVbej \lambda \zeta \in \alpha \beta \gamma Q\beta j = 0, \mathbf{d}=2, \mathbf{b}=2$$



$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta p \Delta v 2 \\ 0 & \Delta p \Delta v 2 & 0 \end{pmatrix}, d=2, b=3$$

$$Rj\lambda\alpha\beta Q\beta j = \Delta p \Delta v 2 \sin(\theta(x1, x2, x3)) \cos(\theta(x1, x2, x3)), d=2, b=3$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=2, b=3$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & \Delta p \Delta v 1 \\ 0 & 0 & 0 \\ \Delta p \Delta v 1 & 0 & 0 \end{pmatrix}, d=3, b=1$$

$$Rj\lambda\alpha\beta Q\beta j = 0, d=3, b=1$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = -\Delta p \Delta v 1 \cos(\theta(x1, x2, x3)), d=3, b=1$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta p \Delta v 2 \\ 0 & \Delta p \Delta v 2 & 0 \end{pmatrix}, d=3, b=2$$

$$Rj\lambda\alpha\beta Q\beta j = \Delta p \Delta v 2 \sin(\theta(x1, x2, x3)) \cos(\theta(x1, x2, x3)), d=3, b=2$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=3, b=2$$

$$Q\beta j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \Delta p^2 \end{pmatrix}, d=3, b=3$$

$$Rj\lambda\alpha\beta Q\beta j = \Delta p^2 \sin(2\theta(x1, x2, x3)), d=3, b=3$$

$$VdVbej\lambda\zeta\epsilon\alpha\beta\gamma Q\beta j = 0, d=3, b=3$$

$$Rdb\lambda\alpha Qdb = \Delta p \cos(\theta(x1, x2, x3)) (2(\Delta p + \Delta v 2) \sin(\theta(x1, x2, x3)) - \Delta v 1), \alpha=2, \lambda=3$$

$$VdVbe\lambda\epsilon\alpha Qbd = \Delta v 1 (\Delta v 2 - \Delta p) \cos(\theta(x1, x2, x3)), \alpha=2, \lambda=3$$

$E_{23}$  in Jaakko's parametrization :

Out[139]/TraditionalForm=

$$\begin{aligned} & c1 \left( -2 \left( (K1 + K2 + K3) \Delta v 2^2 \cos^2(\theta(x1, x2, x3)) + K1 \Delta p^2 \sin^2(\theta(x1, x2, x3)) \right) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) - \right. \\ & \quad K1 (\Delta p^2 + \Delta v 2^2) \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) + \\ & \quad \cos(2\theta(x1, x2, x3)) \left( (K2 + K3) \Delta p \Delta v 2 \delta s^{(0,1,1)}(x1, x2, x3) + \right. \\ & \quad \quad K1 (\Delta p - \Delta v 2) (\Delta p + \Delta v 2) \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \Big) + \\ & \quad \frac{1}{2} \sin(2\theta(x1, x2, x3)) \left( (K1 \Delta p^2 - (K1 + K2 + K3) \Delta v 2^2) \delta s^{(0,2,0)}(x1, x2, x3) + \right. \\ & \quad \quad \left. K1 (\Delta p - \Delta v 2) (\Delta p + \Delta v 2) \delta s^{(2,0,0)}(x1, x2, x3) \right) \Big) + \\ & \quad c2 \Delta v 1 (\Delta v 2 - \Delta p) \cos(\theta(x1, x2, x3)) + c2 \Delta p \cos(\theta(x1, x2, x3)) \\ & \quad (2(\Delta p + \Delta v 2) \sin(\theta(x1, x2, x3)) - \Delta v 1) \end{aligned}$$

```
In[144]:= Print[Style["E33+E22=", Red, 18]]
FullSimplify[E33+E22,
  Assumptions → Assumptions → {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals &&
    Δp > 0 && Δv1 > 0 && Δv2 > 0}] // TraditionalForm
```

**E33+E22=**

Out[145]/TraditionalForm=

$$\begin{aligned}
& c1 \left( \left( \sin(\theta(x1, x2, x3)) \left( 2 K1 \cos(\theta(x1, x2, x3)) \right. \right. \right. \\
& \quad \left( \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) + \\
& \quad \left. K1 \sin(\theta(x1, x2, x3)) \left( \delta s^{(0,2,0)}(x1, x2, x3) + \delta s^{(2,0,0)}(x1, x2, x3) \right) \right) \Delta p^2 + \\
& \quad (K2 + K3) \Delta v2 \sin(2 \theta(x1, x2, x3)) \delta s^{(0,1,1)}(x1, x2, x3) \Delta p + \\
& \quad \Delta v1^2 \left( K1 \delta s^{(0,2,0)}(x1, x2, x3) + (K1 + K2 + K3) \delta s^{(2,0,0)}(x1, x2, x3) \right) + \Delta v2^2 \cos(\theta(x1, x2, x3)) \\
& \quad \left( \cos(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(2,0,0)}(x1, x2, x3) \right) - \right. \\
& \quad \left. 2 \sin(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \right. \right. \\
& \quad \left. \left. K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) \right) \right) + \\
& \quad \left( \cos(\theta(x1, x2, x3)) \left( K1 \cos(\theta(x1, x2, x3)) \left( \delta s^{(0,2,0)}(x1, x2, x3) + \delta s^{(2,0,0)}(x1, x2, x3) \right) - \right. \right. \\
& \quad \left. \left. 2 K1 \sin(\theta(x1, x2, x3)) \right. \right. \\
& \quad \left( \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) \left. \right) \Delta p^2 - \\
& \quad (K2 + K3) \Delta v2 \sin(2 \theta(x1, x2, x3)) \delta s^{(0,1,1)}(x1, x2, x3) \Delta p + \\
& \quad \Delta v1^2 \left( K1 \delta s^{(0,2,0)}(x1, x2, x3) + (K1 + K2 + K3) \delta s^{(2,0,0)}(x1, x2, x3) \right) + \\
& \quad \Delta v2^2 \sin(\theta(x1, x2, x3)) \left( 2 \cos(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \right. \right. \\
& \quad \left. \left. \theta^{(0,1,0)}(x1, x2, x3) + K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) + \right. \\
& \quad \left. \sin(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(2,0,0)}(x1, x2, x3) \right) \right) \right) \right) + \\
& c2 \left( -5 \Delta v1 (\Delta p + \Delta v2) \sin(\theta(x1, x2, x3)) - (\Delta p + \Delta v2)^2 \cos(2 \theta(x1, x2, x3)) + \right. \\
& \quad \Delta p^2 + \\
& \quad 4 \Delta v1^2 + \\
& \quad \left. \Delta v2^2 \right)
\end{aligned}$$

```
In[142]:= Print[Style["E32-E23=", Red, 18]]
FullSimplify[E33+E22,
  Assumptions → Assumptions → {{x1, x2, x3, θ, Δp, Δv1, Δv2} ∈ Reals &&
    Δp > 0 && Δv1 > 0 && Δv2 > 0}}] // TraditionalForm
```

**E32-E23=**

Out[143]//TraditionalForm=

$$\begin{aligned}
& c1 \left( \left( \sin(\theta(x1, x2, x3)) \left( 2 K1 \cos(\theta(x1, x2, x3)) \right. \right. \right. \\
& \quad \left( \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) + \\
& \quad \left. K1 \sin(\theta(x1, x2, x3)) \left( \delta s^{(0,2,0)}(x1, x2, x3) + \delta s^{(2,0,0)}(x1, x2, x3) \right) \right) \Delta p^2 + \\
& \quad (K2 + K3) \Delta v2 \sin(2 \theta(x1, x2, x3)) \delta s^{(0,1,1)}(x1, x2, x3) \Delta p + \\
& \quad \Delta v1^2 \left( K1 \delta s^{(0,2,0)}(x1, x2, x3) + (K1 + K2 + K3) \delta s^{(2,0,0)}(x1, x2, x3) \right) + \Delta v2^2 \cos(\theta(x1, x2, x3)) \\
& \quad \left( \cos(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(2,0,0)}(x1, x2, x3) \right) - \right. \\
& \quad \left. 2 \sin(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \right. \right. \\
& \quad \left. \left. K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) \right) \right) + \\
& \quad \left( \cos(\theta(x1, x2, x3)) \left( K1 \cos(\theta(x1, x2, x3)) \left( \delta s^{(0,2,0)}(x1, x2, x3) + \delta s^{(2,0,0)}(x1, x2, x3) \right) - \right. \right. \\
& \quad \left. \left. 2 K1 \sin(\theta(x1, x2, x3)) \right. \right. \\
& \quad \left. \left( \delta s^{(0,1,0)}(x1, x2, x3) \theta^{(0,1,0)}(x1, x2, x3) + \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) \right) \Delta p^2 - \\
& \quad (K2 + K3) \Delta v2 \sin(2 \theta(x1, x2, x3)) \delta s^{(0,1,1)}(x1, x2, x3) \Delta p + \\
& \quad \Delta v1^2 \left( K1 \delta s^{(0,2,0)}(x1, x2, x3) + (K1 + K2 + K3) \delta s^{(2,0,0)}(x1, x2, x3) \right) + \\
& \quad \Delta v2^2 \sin(\theta(x1, x2, x3)) \left( 2 \cos(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,1,0)}(x1, x2, x3) \right. \right. \\
& \quad \left. \left. \theta^{(0,1,0)}(x1, x2, x3) + K1 \delta s^{(1,0,0)}(x1, x2, x3) \theta^{(1,0,0)}(x1, x2, x3) \right) + \right. \\
& \quad \left. \sin(\theta(x1, x2, x3)) \left( (K1 + K2 + K3) \delta s^{(0,2,0)}(x1, x2, x3) + K1 \delta s^{(2,0,0)}(x1, x2, x3) \right) \right) \right) \right) + \\
& c2 \left( -5 \Delta v1 (\Delta p + \Delta v2) \sin(\theta(x1, x2, x3)) - (\Delta p + \Delta v2)^2 \cos(2 \theta(x1, x2, x3)) + \right. \\
& \quad \Delta p^2 + \\
& \quad 4 \Delta v1^2 + \\
& \quad \left. \Delta v2^2 \right)
\end{aligned}$$