Notes on the deformation of thick-walled vessels affected by transmural pressure changes

Timo Ströhle [*] **Nicolas Petit** [*]

[*] *Centre Automatique et Systèmes (CAS), Mines Paris, Université PSL, 75006 Paris, France (e-mail: timo.strohle@minesparis.psl.eu and nicolas.petit@minesparis.psl.eu).*

Abstract: These are supplementary notes to (Ströhle and Petit (2025)), where the flow of incompressible fluids within complaint vessels and its control is investigated. To model this physical system, a law for the fluid-structure interaction needs to be established. Therefore, in these notes, the deformation of thick-walled vessels affected by a change in transmural pressure is investigated. We aim to give more details on this issue.

A thick-walled vessel with cylindrical symmetry is investigated. The vessel has an inner (luminal) radius a and an outer radius b such that the vessels wall has thickness $h = b - a$. Furthermore, the vessel is affected by the luminal and extraluminal pressure P_a and P_b, resp. (see Fig. 1). We aim to find a law that relates the transmural pressure $P = P_a - P_b$ to the luminal cross-sectional area $A = \pi a^2$. For the sake of generality, a thick-walled vessel is considered, but the formula is also valid for thin walls, considered in Ströhle and Petit (2025).

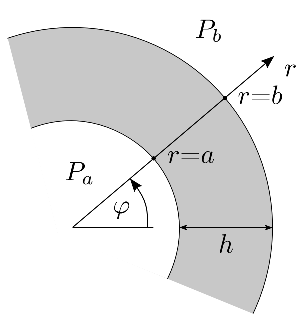

Fig. 1. Illustration of a segment of a circular cross-sectional area of an elastic thick-walled vessel with wall-thickness h, affected by transmural pressure $P = P_a - P_b$.

The static equilibrium conditions in radial direction of an infinitesimal small piece of the vessels wall can be established in polar coordinates r and φ as

$$(\sigma_r + d\sigma_r)\ (r + dr)\ d\varphi = d\varphi\ \sigma_\varphi\ dr + \sigma_r r\ d\varphi\ .$$

When neglecting higher order terms, this condition reduces to the following ordinary differential equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} = 0\ . \tag{1}$$

Herein, the radial and circumferential stresses σ_r and σ_φ, resp. has been introduced (see Fig. 2). Due to symmetry, the static equilibrium condition in circumferential direction proves the invariance of σ_φ w.r.t. changes in φ.

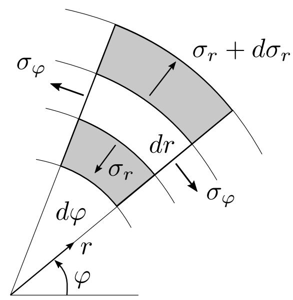

Fig. 2. Infinitesimal piece of a thick-walled vessel affected by radial and circumferential stresses σ_r and σ_φ, respectively.

By assuming material linearity, both stresses σ_r and σ_φ can be related to the deformation of the vessels wall, i.e. to the radial and circumferential strains ε_r and ε_φ, respectively. For the plane stress case the following stress-strain relation can be established

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\varphi)\ , \qquad \varepsilon_\varphi = \frac{1}{E}(\sigma_\varphi - \nu\sigma_r)\ . \tag{2}$$

Herein the Young's modulus E and the Poisson's ratio ν have been introduced. The radial displacement x is a function of the radial variable r. To link $x(r)$ to the radial and circumferential strains, small deformations are assumed s.t. the following kinematical relations for the strains

$$\varepsilon_r = \frac{x + dx - x}{dr} = \frac{dx}{dr} \qquad \varepsilon_\varphi = \frac{(r + x)\ d\varphi - r\ d\varphi}{r\ d\varphi} = \frac{x}{r} \tag{3}$$

can be established (cf. Fig. 3). The combination of (1) - (3) gives rise to an ordinary differential equation of Cauchy-Euler type, governing the displacement of the vessels wall, i.e.

$$x''(r) + \frac{1}{r}x'(r) - \frac{x}{r^2} = 0\ , \tag{4}$$

that is solved by the general solution $x = C_1 r + C_2 \frac{1}{r}$.

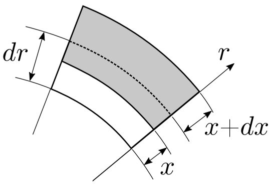

Fig. 3. Radial displacement $x(r)$ of an infinitesimal piece of a thick-walled vessel.

By noticing that $\sigma_r(a) = P_a$ and $\sigma_r(b) = P_b$, the coefficients C_1 and C_2 are determined by static boundary conditions

$$\frac{dx}{dr}(a) = C_1 - C_2\frac{1}{a^2} = \frac{1}{E}(P_a - \nu\,\sigma_\varphi(a)) \tag{5}$$

$$\frac{dx}{dr}(b) = C_1 - C_2\frac{1}{b^2} = \frac{1}{E}(P_b - \nu\,\sigma_\varphi(b)) \tag{6}$$

as well as geometric boundary conditions

$$x(a) = C_1 a + C_2\frac{1}{a} = \frac{a}{E}(\sigma_\varphi(a) - \nu\,P_a) \tag{7}$$

$$x(b) = C_1 b + C_2\frac{1}{b} = \frac{b}{E}(\sigma_\varphi(b) - \nu\,P_b) \tag{8}$$

such that

$$C_1 = \frac{(1+\nu)(1-2\nu)P_a a^2 - P_b b^2}{E(b^2-a^2)} \tag{9}$$

$$C_2 = \frac{(1+\nu)a^2b^2(P_a - P_b)}{E(b^2-a^2)} \tag{10}$$

can be established. By assuming incompressible material for the tube, i.e. $\nu = \frac{1}{2}$ hence $C_1 = 0$, the displacement of the tube at $r = a$ can be established as

$$x(a) = \frac{3a^2b^2}{2E(b^2-a^2)}\frac{P_a - P_b}{a}$$

Introducing the deformed luminal cross-sectional area as $A = \pi(a + x(a))^2$ as well as the initial luminal cross-sectional area $A_0 = a^2\pi$, the (luminal) pressure P_a can be related to A and A_0 as

$$P_a = \frac{2E}{3}\frac{b^2-a^2}{b^2}\left(\left(\frac{A}{A_0}\right)^{\frac{1}{2}} - 1\right) + P_b \tag{11}$$

It is convenient to describe this relation in terms of the ratio $\frac{h}{a}$, where $h = b - a$ indicates the wall thickness. Thus

$$\frac{b^2-a^2}{b^2} = 2\frac{h}{a}\phi \quad \text{with} \quad \phi = \frac{1 + \frac{1}{2}\frac{h}{a}}{1 + 2\frac{h}{a} + \frac{h^2}{a^2}} \tag{12}$$

and (11) can be written as

$$P_a = \phi\frac{4\sqrt{\pi}(Eh)(s)}{3\sqrt{A_0}}\left(\left(\frac{A}{A_0}\right)^{\frac{1}{2}} - 1\right) + P_b \tag{13}$$

This relation can be rewritten as

$$P_a = \phi\beta(s)f(A) + P_b \tag{14}$$

by introducing

$$f(A) = A^{\frac{1}{2}} - A_0^{\frac{1}{2}}\ , \tag{15}$$

and

$$\beta(s) = \frac{4\sqrt{\pi}(Eh)(s)}{3A_0}\ , \tag{16}$$

For the special case of thin-walled vessels $\phi \equiv 1$. For more details on the derivation of the tube-law see e.g. Formaggia et al. (2003), Mynard and Nithiarasu (2008), Parker (2021) and references therein.

REFERENCES

Formaggia, L., Lamponi, D., and Quarteroni, A. (2003). One-dimensional models for blood flow in arteries. *Journal of Engineering Mathematics*, 47, 251–276.

Mynard, J.P. and Nithiarasu, P. (2008). A 1d arterial blood flow model incorporating ventricular pressure, aortic valve and regional coronary flow using the locally conservative galerkin (lcg) method. *Communications in Numerical Methods in Engineering*, 24(5), 367–417. doi: https://doi.org/10.1002/cnm.1117.

Parker, K.H. (2021). Arterial tube laws and wave speeds. URL `https://arxiv.org/abs/2106.10061`.

Ströhle, T. and Petit, N. (2025). Dynamics and motion-planning for microfluidic flow within elastic walls. Submitted for publication.