Principles of Finance

Risk and Return

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Motivation: First principles

- Firms obtain equity capital either internally by earning money and retaining it or externally by issuing new shares of stocks
- Primary market
  - When a corporation itself issues new shares of stock and sells them to investors, they do so on the primary market
- Secondary market
  - After the initial transaction in the primary market, the shares continue to trade in a secondary market between investors

Risk and Return

- Different securities have different initial prices, pay different dividend amounts, and sell for different future amounts
- To make them comparable, we express their performance in terms of their returns
- The rate of return on an investment can be calculated as

$$R_{t+1} = \frac{Div_{t+1} + P_{t+1}}{P_t} - 1 = \frac{Div_{t+1}}{P_t} + \frac{P_{t+1} - P_t}{P_t}$$

$$= \text{Dividend Yield} + \text{Capital Gain Rate}$$

Calculating Rates of Return

- The return indicates the percentage increase in the value of an investment per dollar initially invested in the security
- Looking **backward**, we can calculate **realized** rates of returns
- Suppose an investor purchases a share of ABC common stock for $100 on January 1, the stock pays $4 in dividends, and it has a market price of $108 on December 31. The realized rate of return on this investment is

$$R_{t+1} = \frac{4}{100} + \frac{108 - 100}{100} = 12\%$$

- Looking **forward**, we can characterize random events using **probability distributions** and calculate **expected** rates of returns

Some facts about returns

- When an investment is risky, it may earn different returns
- Each possible return has some likelihood of occurring
- We summarize this information with a probability distribution, which assigns a probability that each return will occur

Current Stock Price ($)	Stock Price in One Year ($)	Probability Distribution: Return, R	Probability, P_R
	140	0.40	25%
100	110	0.10	50%
	80	−0.20	25%

Some facts about returns

- Given a probability distribution of returns, we can compute the mean or expected return
- Two common measures of risk of a probability distribution are its variance and standard deviation, also called **volatility**
- These quantities can be computed as follows

$$\text{Expected Return} = E[R] = \sum_R p_R \times R$$

$$Var(R) = E[(R - E[R])^2] = \sum_R p_R \times (R - E[R])^2$$

$$SD(R) = \sqrt{Var(R)}$$

Some facts about returns

Calculating the Expected Return and Volatility

Problem
Suppose AMC stock is equally likely to have a 45% return or a −25% return. What are its expected return and volatility?

Solution
First, we calculate the expected return by taking the probability-weighted average of the possible returns:

$$E[R] = \sum_R p_R \times R = 50\% \times 0.45 + 50\% \times (-0.25) = 10.0\%$$

To compute the volatility, we first determine the variance:

$$Var(R) = \sum_R p_R \times (R - E[R])^2 = 50\% \times (0.45 - 0.10)^2 + 50\% \times (-0.25 - 0.10)^2$$
$$= 0.1225$$

Then, the volatility or standard deviation is the square root of the variance:

$$SD(R) = \sqrt{Var(R)} = \sqrt{0.1225} = 35\%$$

Some facts about returns

- The average annual return of an investment over some period is simply the average of the realized returns for each year
- We have

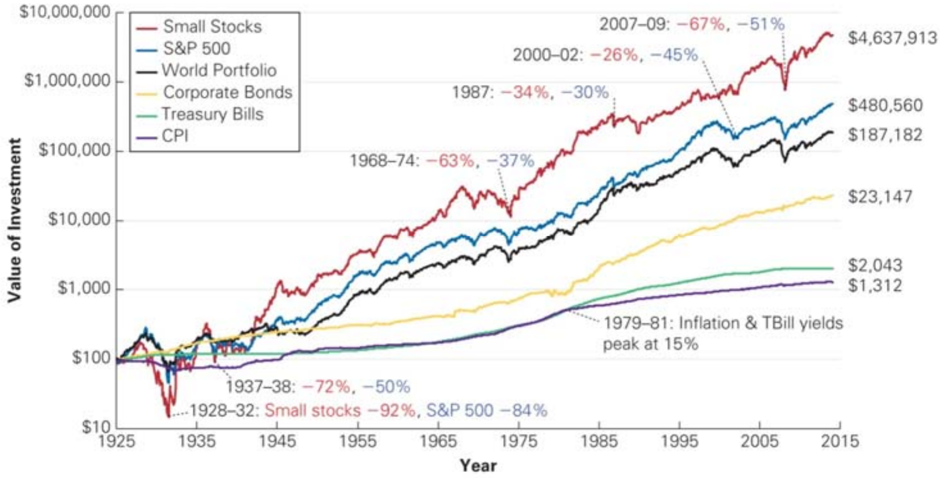
$$\bar{R} = \frac{1}{T}(R_1 + R_2 + \cdots + R_T) = \frac{1}{T}\sum_{t=1}^{T} R_t$$

- Examples (1926-2014)

Investment	Average Annual Return
Small stocks	18.8%
S&P 500	12.0%
Corporate bonds	6.5%
Treasury bills	3.5%

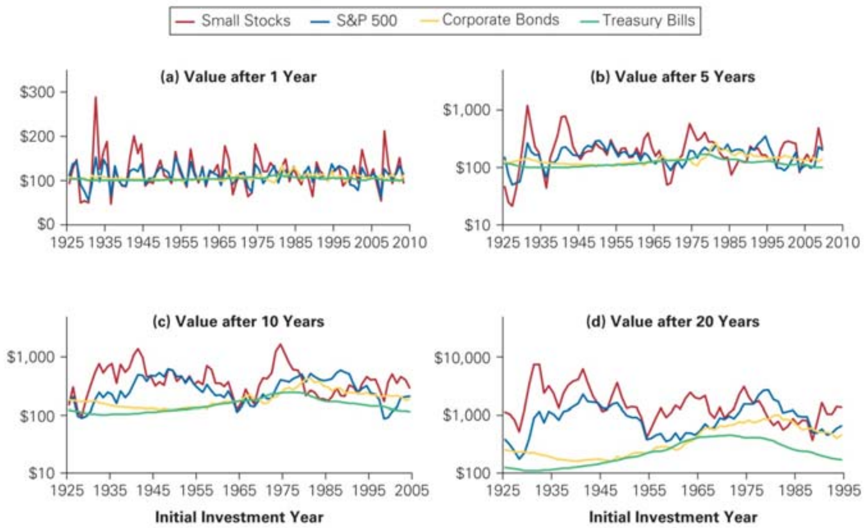
Some facts about returns

- Value in 2015 of $100 invested at the end of 1925



Some facts about returns

- More realistic horizons and different initial investment dates can greatly influence each investment's risk and return



Some facts about returns

Year End	S&P 500 Index	Dividends Paid*	S&P 500 Realized Return	Microsoft Realized Return	1-Month T-Bill Return
2001	1148.08				
2002	879.82	14.53	−22.1%	−22.0%	1.6%
2003	1111.92	20.80	28.7%	6.8%	1.0%
2004	1211.92	20.98	10.9%	8.9%	1.2%
2005	1248.29	23.15	4.9%	−0.9%	3.0%
2006	1418.30	27.16	15.8%	15.8%	4.8%
2007	1468.36	27.86	5.5%	20.8%	4.7%
2008	903.25	21.85	−37.0%	−44.4%	1.5%
2009	1115.10	27.19	26.5%	60.5%	0.1%
2010	1257.64	25.44	15.1%	−6.5%	0.1%
2011	1257.60	26.59	2.1%	−4.5%	0.0%
2012	1426.19	32.67	16.0%	5.8%	0.1%
2013	1848.36	39.75	32.4%	44.2%	0.0%
2014	2058.90	42.47	13.7%	27.5%	0.0%

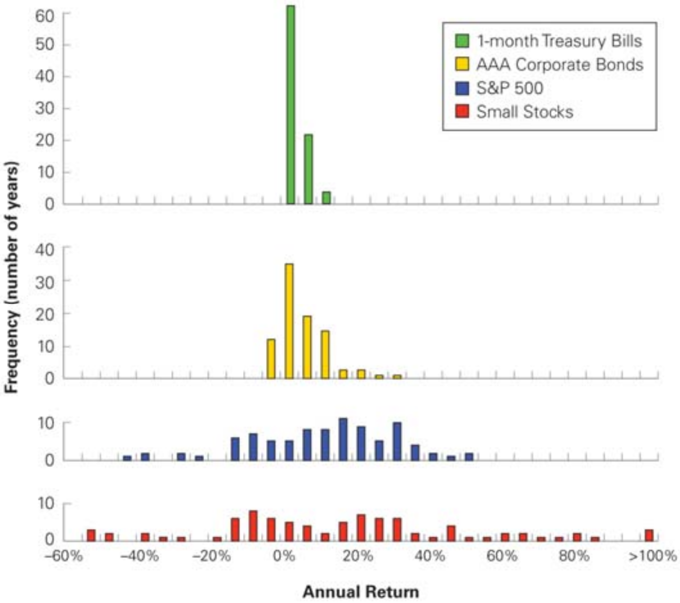
*Total dividends paid by the 500 stocks in the portfolio, based on the number of shares of each stock in the index, adjusted until the end of the year, assuming they were reinvested when paid.

Source: Standard & Poor's, Microsoft and U.S. Treasury Data

- The average annual returns of the S&P500 and Microsoft over this period are respectively 8.7% and 8.6%

Some facts about returns

- The **variability of returns** is very different for each investment



Some facts about returns

□ Using the data in the table, compute the variance and volatility of the S&P500 returns over the period 2001-2014 where

$$Var(R) = \frac{1}{T-1}\sum_{t=1}^{T}(R_t - \bar{R})^2$$

Solution

Earlier, we calculated the average annual return of the S&P 500 during this period to be 8.7%. Therefore,

$$\begin{aligned} Var(R) &= \frac{1}{T-1}\sum_t (R_t - \bar{R})^2 \\ &= \frac{1}{13-1}[(-0.221-0.087)^2 + (0.287-0.087)^2 + \cdots + (0.137-0.087)^2] \\ &= 0.038 \end{aligned}$$

The volatility or standard deviation is therefore $SD(R) = \sqrt{Var(R)} = \sqrt{0.038} = 19.5\%$

Unbiased sample variance

□ In general, the population variance of a finite population of size N with values R_i is given by

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(R_i - \mu)^2$$

where μ is the population mean

$$\mu = \frac{1}{N}\sum_{i=1}^{N} R_i$$

Unbiased sample variance

- Take $n<N$ observations and estimate the variance on the basis of this sample. Directly taking the variance of the sample data gives the average of the squared deviations

$$\sigma_R^2 = \frac{1}{n}\sum_{i=1}^{n}(R_i - \bar{R})^2$$

- Here $\bar{R}$ is the sample mean

$$\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$$

- Since the R_i are selected randomly, both $\bar{R}$ and σ_R^2 are random variables. Their expected values can be evaluated by summing over the ensemble of all possible samples.

- For σ_R^2 this gives

$$E[\sigma_R^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}\left(R_i - \frac{1}{n}\sum_{i=1}^{n} R_i\right)^2\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} E\left[R_i^2 - \frac{2}{n}R_i\sum_{j=1}^{n} R_j + \frac{1}{n^2}\sum_{j=1}^{n} R_j \sum_{k=1}^{n} R_k\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{n-2}{n}E\left[R_i^2\right] - \frac{2}{n}\sum_{i\neq j, j=1}^{n} E\left[R_i R_j\right] + \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k\neq j}^{n} E\left[R_j R_k\right]\right.$$

$$\left. + \frac{1}{n^2}\sum_{j=1}^{n} E\left[R_j^2\right]\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{n-2}{n}(\sigma^2 + \mu^2) - \frac{2}{n}(n-1)\mu^2 + \frac{n(n-1)}{n^2}\mu^2 + \frac{\sigma^2 + \mu^2}{n}\right]$$

$$= \frac{n-1}{n}\sigma^2$$

Unbiased sample variance

- This shows that σ_R^2 gives an estimate of the population variance that is biased by a factor of $\frac{n-1}{n}$
- For this reason, σ_R^2 is referred to as the biased sample variance
- Correcting for this bias yields the *unbiased sample variance*

$$Var(R) = \frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R})^2$$

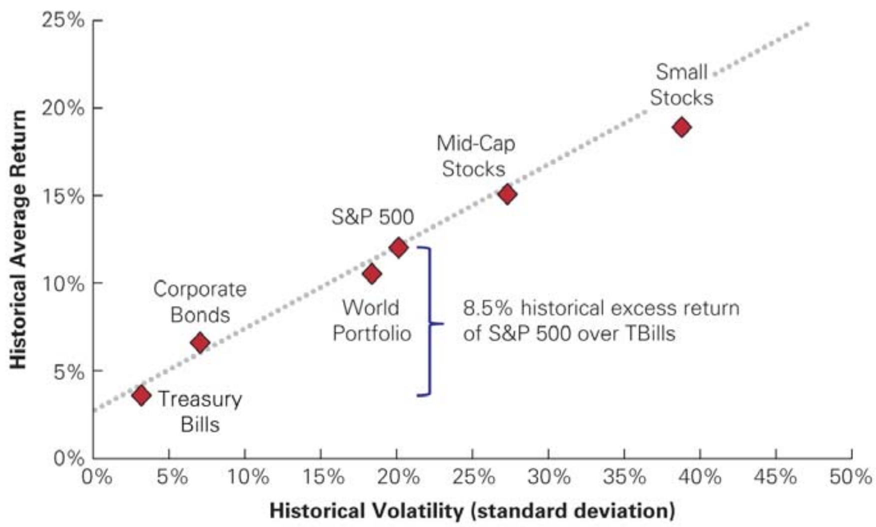
Some facts about returns

- Using the data on various securities, we can compute the variance and volatility of the returns on each of the securities over the period 1926-2014
- We get

Investment	Return Volatility (Standard Deviation)	Excess Return (Average Return in Excess of Treasury Bills)
Small stocks	38.8%	15.3%
S&P 500	20.1%	8.5%
Corporate bonds	7.0%	3.0%
Treasury bills (30-day)	3.1%	0.0%

Some facts about returns

□ Historical tradeoff between risk and return in large portfolios



Some facts about returns

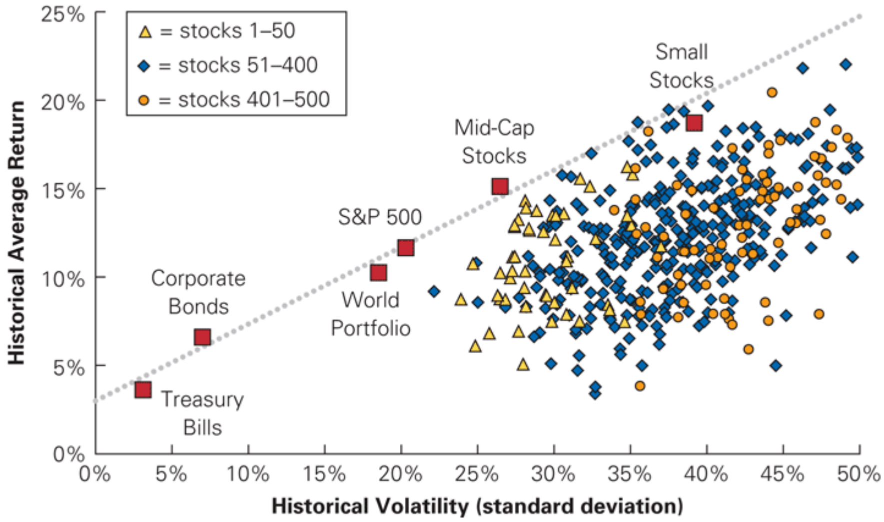
□ Is there a positive relationship between volatility and average returns for **individual** stocks?

□ As shown on the next slide, there is no precise relationship between volatility and average return for individual stocks

- Larger stocks tend to have lower volatility than smaller stocks
- All stocks tend to have higher risk and lower returns than large portfolios

Some facts about returns

□ Historical volatility and return for 500 individual stocks, ranked annually by size



Portfolio returns

□ A portfolio is uniquely defined by (a vector of) **portfolio weights**. Suppose there are N assets, i=1,2,...,N. Define the portfolio weight x_i as

$$x_i = \frac{\text{Value of investment } i}{\text{Total value of portfolio}}$$

□ The **portfolio return** is equal to the weighted average of the returns on the individual assets in the portfolio, where the weights are given by the portfolio weights

□ I.e. denote by x the vector of portfolio weights and by R the vector of portfolio returns. We have $R_P = x'R$ or

$$R_P = x_1 R_1 + x_2 R_2 + \cdots + x_n R_n = \sum_i x_i R_i$$

Portfolio returns

Problem

Suppose you invest \$10,000 by buying 200 shares of the Walt Disney Company at \$30 per share and 100 shares of Coca-Cola stock at \$40 per share. If Disney's share price goes up to \$36 and Coca-Cola's share price falls to \$38, what is the new value of the portfolio, and what return did the portfolio earn? Show that Eq. 11.2 holds. If you don't buy or sell any shares after the price change, what are the new portfolio weights?

Portfolio returns

Solution

The new value of the portfolio is $200 \times \$36 + 100 \times \$38 = \$11{,}000$, for a gain of \$1000 or a 10% return on your initial \$10,000 investment. The return on Disney stock was $36 / 30 - 1 = 20\%$, and the return on Coca-Cola stock was $38 / 40 - 1 = -5\%$. Given the initial portfolio weights of 60% Disney and 40% Coca-Cola, we can also compute the return of the portfolio from Eq. 11.2:

$$R_P = x_D R_D + x_C R_C = 60\% \times 0.2 + 40\% \times (-0.05) = 10\%$$

After the price change, the new portfolio weights are

$$x_D = \frac{200 \times \$36}{11{,}000} = 65.45\%$$

$$x_C = \frac{100 \times \$38}{11{,}000} = 34.55\%$$

Without trading, the portfolio weights will increase for the stocks in the portfolio whose returns are above the overall portfolio return.

Expected portfolio returns

- The expected rate of return on a portfolio can be calculated as follows

$$E[R_P] = E\left[\sum_i x_i R_i\right] = \sum_i E[x_i R_i] = \sum_i x_i E[R_i]$$

- Example

Portfolio Expected Return

Problem
Suppose you invest $10,000 in Ford stock, and $30,000 in Tyco International stock. You expect a return of 10% for Ford and 16% for Tyco. What is your portfolio's expected return?

Expected portfolio returns

Solution
You invested $40,000 in total, so your portfolio weights are 10,000/40,000 = 0.25 in Ford and 30,000/40,000 = 0.75 in Tyco. Therefore, your portfolio's expected return is

$$E[R_P] = x_F E[R_F] + x_T E[R_T] = 0.25 \times 10\% + 0.75 \times 16\% = 14.5\%$$

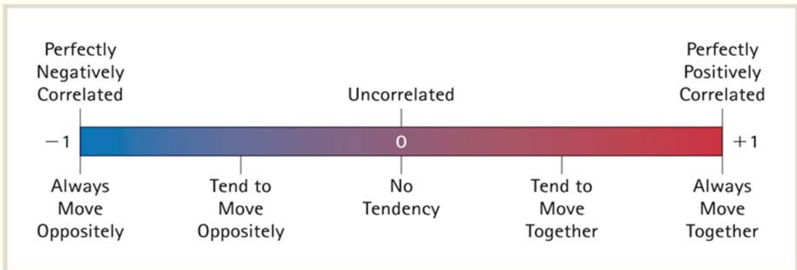
Diversification

- When we combine stocks in a portfolio, **some of their risk is eliminated** through diversification
- Example

Year	Stock Returns: North Air	Stock Returns: West Air	Stock Returns: Tex Oil	Portfolio Returns (1) $1/2R_N + 1/2R_W$	Portfolio Returns (2) $1/2R_W + 1/2R_T$
1998	21%	9%	−2%	15.0%	3.5%
1999	30%	21%	−5%	25.5%	8.0%
2000	7%	7%	9%	7.0%	8.0%
2001	−5%	−2%	21%	−3.5%	9.5%
2002	−2%	−5%	30%	−3.5%	12.5%
2003	9%	30%	7%	19.5%	18.5%
Average Return	10.0%	10.0%	10.0%	10.0%	10.0%
Volatility	13.4%	13.4%	13.4%	12.1%	5.1%

Diversification

- This example demonstrates two important phenomena
  - By combining stocks into a portfolio, we reduce risk
  - The amount of risk that is eliminated in a portfolio depends on the degree to which the stocks face common risks and their prices move together
- The idea of **diversification** is related to that of **correlation**



Diversification

- Covariance between returns R_i and R_j

$$Cov(R_i, R_j) = E[(R_i - E(R_i))(R_j - E(R_j))]$$

- Estimate of the covariance from historical data

$$Cov(R_i, R_j) = \frac{1}{T-1}\sum_{t=1}^{T}(R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j)$$

- Estimate of the correlation

$$Corr(R_i, R_j) = \frac{Cov(R_i, R_j)}{SD(R_i)SD(R_j)}$$

A measure of the common risk shared by stocks that does not depend on their volatility. **The correlation between two stocks will always be between –1 and +1**

Diversification

- Returning to our example

	Deviation from Mean			North Air and West Air	West Air and Tex Oil
Year	$(R_N - \bar{R}_N)$	$(R_W - \bar{R}_W)$	$(R_T - \bar{R}_T)$	$(R_N - \bar{R}_N)(R_W - \bar{R}_W)$	$(R_W - \bar{R}_W)(R_T - \bar{R}_T)$
1998	11%	−1%	−12%	−0.0011	0.0012
1999	20%	11%	−15%	0.0220	−0.0165
2000	−3%	−3%	−1%	0.0009	0.0003
2001	−15%	−12%	11%	0.0180	−0.0132
2002	−12%	−15%	20%	0.0180	−0.0300
2003	−1%	20%	−3%	−0.0020	−0.0060
	Sum $= \sum_t (R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j) =$			0.0558	−0.0642
Covariance:	$Cov(R_i, R_j) = \frac{1}{T-1}$ Sum $=$			0.0112	−0.0128
Correlation:	$Corr(R_i, R_j) = \frac{Cov(R_i, R_j)}{SD(R_i)\,SD(R_j)} =$			0.624	−0.713

Diversification

□ Stock returns move together if they are affected similarly by economic events. Thus, stocks in the same industry tend to have more highly correlated returns than stocks in different industries

□ Some real world examples

	Microsoft	Dell	Delta Air Lines	American Airlines	General Motors	Ford Motor	Anheuser-Busch
Volatility (Standard Deviation)	42%	54%	50%	72%	33%	37%	18%
Correlation with							
Microsoft	1.00	0.65	0.27	0.19	0.22	0.26	−0.07
Dell	0.65	1.00	0.19	0.18	0.32	0.32	0.10
Delta Air Lines	0.27	0.19	1.00	0.69	0.31	0.38	0.19
American Airlines	0.19	0.18	0.69	1.00	0.35	0.58	0.11
General Motors	0.22	0.32	0.31	0.35	1.00	0.64	0.11
Ford Motor	0.26	0.32	0.38	0.58	0.64	1.00	0.10
Anheuser-Busch	−0.07	0.10	0.19	0.11	0.11	0.10	1.00

Diversification

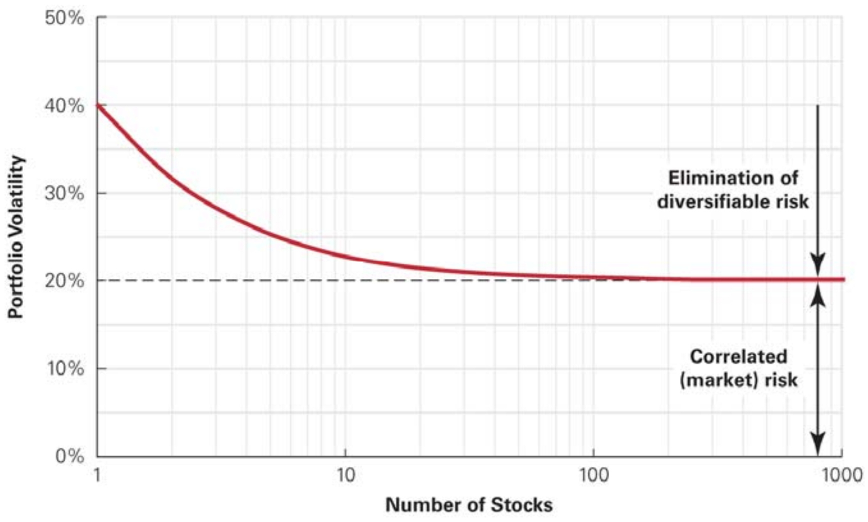
□ In general, the risk (variance) on any individual investment can be broken down into two (orthogonal) sources

- Firm-specific risk ☠: Good or bad news about a company
- Market-wide risk ☺: News that affect all stocks, such as news about the economy

□ When many stocks are combined in a large portfolio, the firm-specific risks for each stock will average out and be diversified. **Market-wide risk cannot be eliminated by diversification**

- Investors are only paid for bearing market-wide risk

Diversification



Diversification

- In general, the independent risks due to firm-specific news are known as:
  - Firm-specific risk
  - Idiosyncratic risk
  - Unsystematic Risk
- The common risks due to market news are also known as
  - Market risk
  - Undiversifiable risk
  - Systematic risk

Diversification

Diversifiable Versus Systematic Risk

Problem

Which of the following risks of a stock are likely to be firm-specific, diversifiable risks, and which are likely to be systematic risks? Which risks will affect the risk premium that investors will demand?

a. The risk that the founder and CEO retires
b. The risk that oil prices rise, increasing production costs
c. The risk that a product design is faulty and the product must be recalled
d. The risk that the economy slows, reducing demand for the firm's products

Solution

Because oil prices and the health of the economy affect all stocks, risks (b) and (d) are systematic risks. These risks are not diversified in a large portfolio, and so will affect the risk premium that investors require to invest in a stock. Risks (a) and (c) are firm-specific risks, and so are diversifiable. While these risks should be considered when estimating a firm's future cash flows, they will not affect the risk premium that investors will require and, therefore, will not affect a firm's cost of capital.

Diversification

- To illustrate the effects of diversification, consider two types of firms
  - *Type S* firms are affected only by *systematic risk*. There is a 50% chance the economy will be strong and *type S* stocks will earn a return of 40%; There is a 50% change the economy will be weak and their return will be –20%. **Because all these firms face the same systematic risk, holding a large portfolio of *type S* firms will not diversify the risk**
  - *Type I* firms are affected only by firm-specific risks. Their returns are equally likely to be 35% or –25%, based on factors specific to each firm's local market. **Because these risks are firm specific, if we hold a portfolio of the stocks of many *type I* firms, the risk is diversified**

Diversification

Volatility When Risks Are Independent

Problem
What is the volatility of an equally weighted average of n independent, identical risks?

□ If risks are independent, they are uncorrelated and their covariance is zero. Therefore

$$SD(R_P) = \sqrt{Var(R_P)} = \sqrt{\frac{1}{n} Var(Indiv.risk)} = \frac{SD(Indiv.risk)}{\sqrt{n}}$$

□ Very large portfolios of assets with independent risks have no risk

Diversification

Portfolio Volatility

Problem
What is the volatility of the average return of ten type S firms? What is the volatility of the average return of ten type I firms?

Solution
Type S firms have equally likely returns of 40% or −20%. Their expected return is $\frac{1}{2}(40\%) + \frac{1}{2}(-20\%) = 10\%$, so

$$SD(R_S) = \sqrt{\tfrac{1}{2}(0.40 - 0.10)^2 + \tfrac{1}{2}(-0.20 - 0.10)^2} = 30\%$$

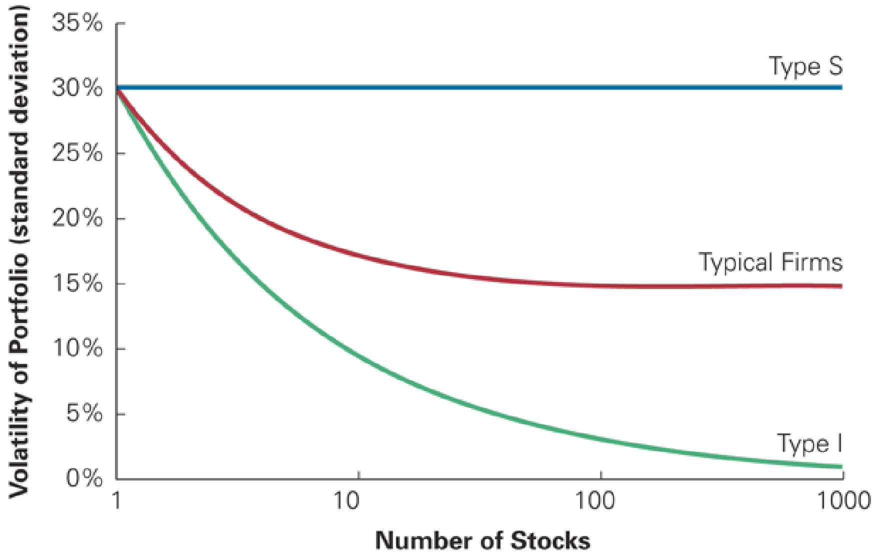
Because all type S firms have high or low returns at the same time, the average return of ten type S firms is also 40% or −20%. Thus, it has the same volatility of 30%, as shown in Figure 10.8.

Type I firms have equally likely returns of 35% or −25%. Their expected return is $\frac{1}{2}(35\%) + \frac{1}{2}(-25\%) = 5\%$, so

$$SD(R_I) = \sqrt{\tfrac{1}{2}(0.35 - 0.05)^2 + \tfrac{1}{2}(-0.25 - 0.05)^2} = 30\%$$

Because the returns of type I firms are independent, using Eq. 10.8, the average return of 10 type I firms has volatility of 30% ÷ $\sqrt{10} = 9.5\%$, as shown in Figure 10.8.

□ *Actual firms* are affected by both market-wide risks and firm-specific risks. **In a large portfolio** of such firms, only the unsystematic risk is diversified. The volatility therefore declines until **only the systematic risk remains**



Diversification

□ The **risk premium for diversifiable risk is zero**, so investors are not compensated for holding firm-specific risk

- If the diversifiable risk of stocks was compensated with a risk premium, investors could buy the stocks, earn this premium, and simultaneously diversify and eliminate the risk

□ **The risk premium of a security is determined by its systematic risk** and does not depend on its diversifiable risk

- This implies that a stock's volatility, which is a measure of total risk (that is, systematic risk plus diversifiable risk), is not especially useful in determining the risk premium that investors will earn

Diversification: Implication

- If you hold a portfolio, *the appropriate measure of risk for any given asset is not the standard deviation*. The portfolio variance satisfies

$$Var(R_P) = Cov(\textstyle\sum_i x_i R_i, R_P)$$

$$= \textstyle\sum_i x_i\, Cov(R_i, R_P) = \sum_i x_i SD(R_i)\, SD(R_P)\, Corr(R_i, R_P)$$

- Dividing both sides by $SD(R_P)$ we get

$$SD(R_P) = \sum_i \overbrace{\underset{\substack{\uparrow \\ \text{Amount} \\ \text{of } i \text{ held}}}{x_i} \times \underset{\substack{\uparrow \\ \text{Total} \\ \text{Risk of } i}}{SD(R_i)} \times \underset{\substack{\uparrow \\ \text{Fraction of } i\text{'s} \\ \text{risk that is} \\ \text{common to } P}}{Corr(R_i, R_P)}}^{\text{Security } i\text{'s contribution to the volatility of the portfolio}}$$

Securities contribute to the volatility of the portfolio according to their volatility scaled by their correlation with the portfolio

Diversification: Implication

- Dividing both sides by $SD(R_P)$ we get

$$SD(R_P) = \sum_i \overbrace{\underset{\substack{\uparrow \\ \text{Amount} \\ \text{of } i \text{ held}}}{x_i} \times \underset{\substack{\uparrow \\ \text{Total} \\ \text{Risk of } i}}{SD(R_i)} \times \underset{\substack{\uparrow \\ \text{Fraction of } i\text{'s} \\ \text{risk that is} \\ \text{common to } P}}{Corr(R_i, R_P)}}^{\text{Security } i\text{'s contribution to the volatility of the portfolio}}$$

- Unless all of the stocks in a portfolio have a perfect positive correlation of +1 with one another, the risk of the portfolio will be lower than the weighted average volatility of the individual stocks

$$SD(R_P) = \sum_i x_i SD(R_i) Corr(R_i, R_P) \le \sum_i x_i SD(R_i)$$

Efficient Portfolio with Two Assets

- The notion of **efficient portolios** is extremely important; it is related to the **risk-return trade-off**
- Consider an investor who can invest in two stocks. **How should the investor choose a portfolio of these two stocks?**
- With 2 assets, the portfolio expected return is

$$E[R_P] = E\left[\sum_i x_i R_i\right] = \sum_i E[x_i R_i] = \sum_i x_i E[R_i]$$

and the portfolio variance is

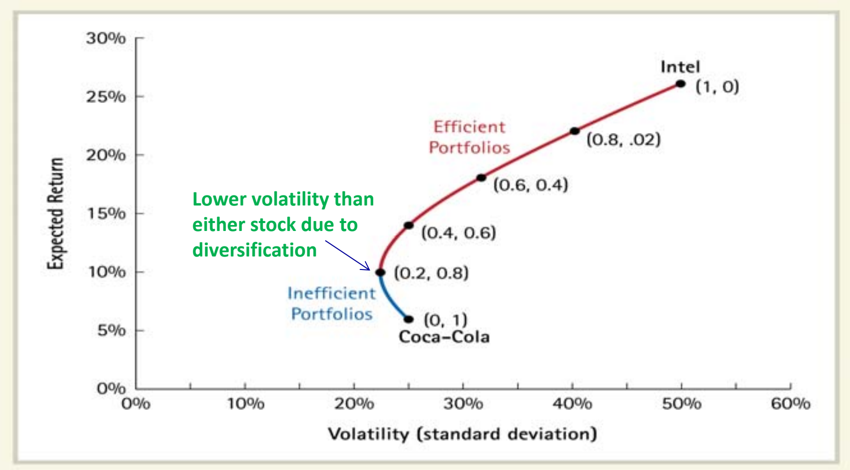
$$Var(R_P) = x_1^2 SD(R_1)^2 + x_2^2 SD(R_2)^2 + 2x_1 x_2 Corr(R_1, R_2) SD(R_1) SD(R_2)$$

Efficient Portfolio with Two Assets

- Assume that the returns on Coca-Cola (C) and Intel (I) are not correlated and have the following characteristics

Portfolio Weights		Expected Return (%)	Volatility (%)
x_I	x_C	$E[R_P]$	$SD[R_P]$
1.00	0.00	26.0	50.0
0.80	0.20	22.0	40.3
0.60	0.40	18.0	31.6
0.40	0.60	14.0	25.0
0.20	0.80	10.0	22.3
0.00	1.00	6.0	25.0

Two Assets



Labels indicate portfolio weights (x_I, x_C) for Intel and Coca-Cola stocks.

Portfolios on the **red** portion of the curve, with at least 20% invested in Intel stock, are **efficient**. **Investors should only invest in efficient portfolios**.

Those on the **blue** portion of the curve, with less than 20% invested in Intel stock, are **inefficient**—an investor can earn a higher expected return with lower risk by choosing an alternative portfolio.

Two Assets

Problem

Sally Ferson has invested 100% of her money in Coca-Cola stock and is seeking investment advice. She would like to earn the highest expected return possible without increasing her volatility. Which portfolio would you recommend?

Solution

In Figure 11.3, we can see that Sally can invest up to 40% in Intel stock without increasing her volatility. Because Intel stock has a higher expected return than Coca-Cola stock, she will earn higher expected returns by putting more money in Intel stock. Therefore, you should recommend that Sally put 40% of her money in Intel stock, leaving 60% in Coca-Cola stock. This portfolio has the same volatility of 25%, but an expected return of 14% rather than the 6% she has now.

The effect of correlation

- If the correlation is 1 then

$$Var(R_P) = x_1^2 Var(R_1) + x_2^2 Var(R_2) + 2x_1 x_2 SD(R_1) SD(R_2)$$

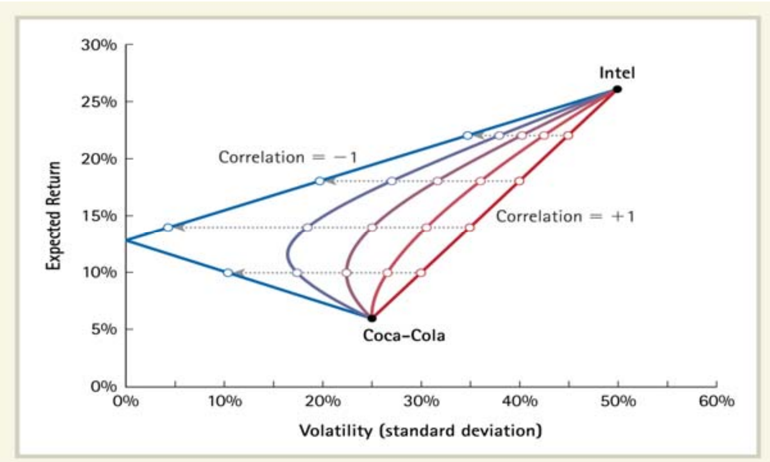
so that

$$SD(R_P) = x_1 SD(R_1) + x_2 SD(R_2)$$

- If the correlation is -1 then

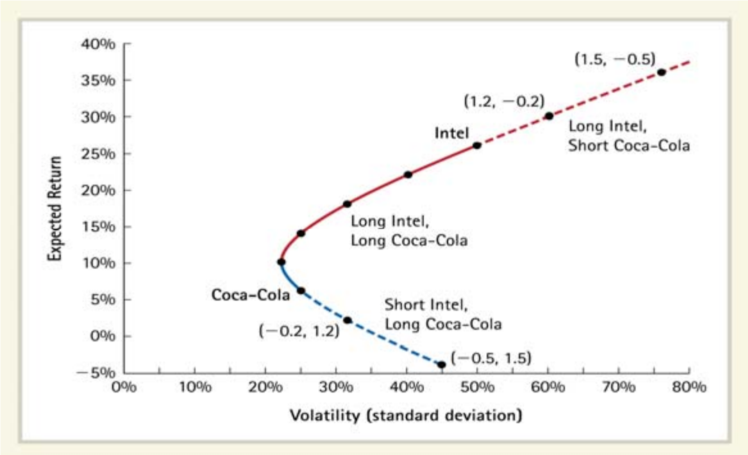
$$SD(R_P) = |x_1 SD(R_1) - x_2 SD(R_2)|$$

The effect of correlation



This figure illustrates correlations of 1, 0.5, 0, -0.5, and -1. The lower the correlation, the lower the risk of the portfolios. Correlation has no effect on the expected return of a portfolio.

Short sales



Labels indicate portfolio weights (x_I, x_C) for Intel and Coca-Cola stocks. Red indicates efficient portfolios, blue indicates inefficient portfolios. The dashed curves indicate positions that require shorting either Coca-Cola (red) or Intel (blue). Shorting Intel to invest in Coca-Cola is inefficient. Shorting Coca-Cola to invest in Intel is efficient.

Three Assets

□ Adding more stocks reduces risk through diversification

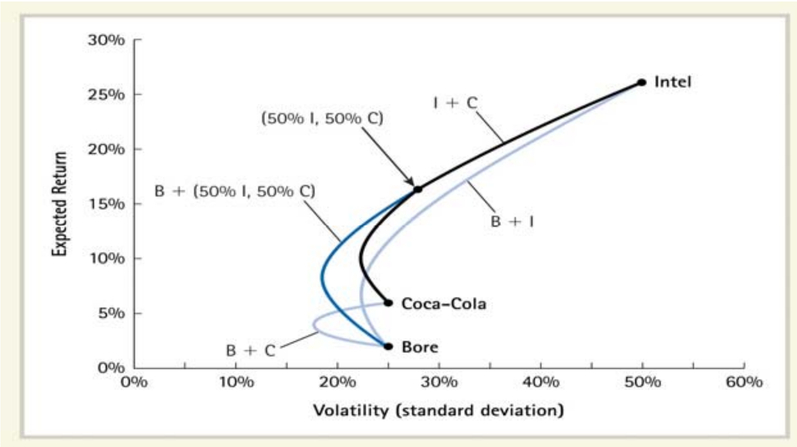
□ Suppose we add a third stock to our portfolio, Bore Industries, which is **uncorrelated with Intel and Coca-Cola** but is expected to have a very low return of 2% with a volatility of 25%

			Correlation with		
Stock	**Expected Return**	**Volatility**	**Intel**	**Coca-Cola**	**Bore Ind.**
Intel	26%	50%	1.0	0.0	0.0
Coca-Cola	6%	25%	0.0	1.0	0.0
Bore Industries	2%	25%	0.0	0.0	1.0

□ Because Bore stock is inferior to Coca-Cola stock, one might guess that no investor would want to hold a long position in Bore

- That conclusion ignores the diversification opportunities that Bore provides

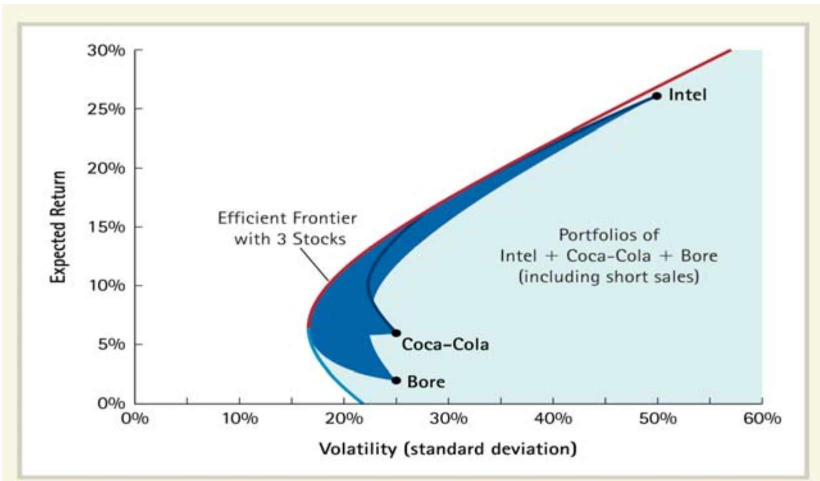
Three Assets



By combining Bore (B) with Intel (I), Coca-Cola (C), and portfolios of Intel and Coca-Cola, we introduce new risk and return possibilities.

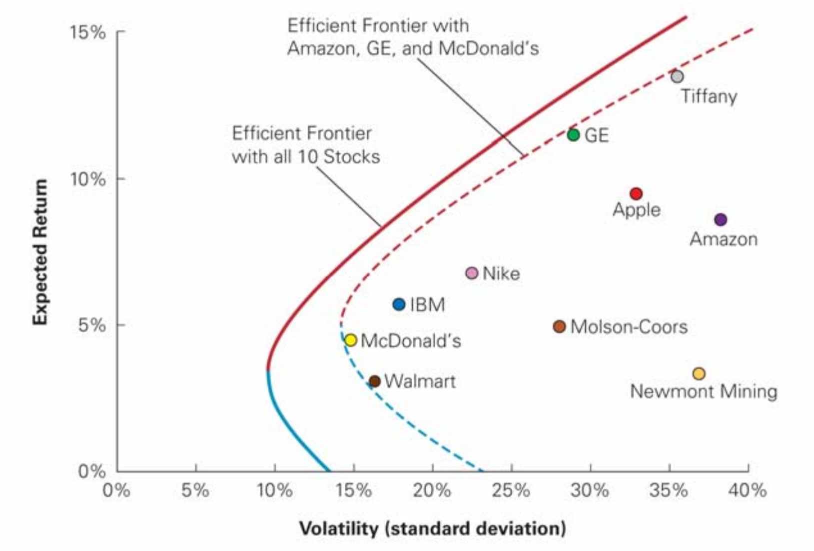
We can do better than with just Coca-Cola and Intel alone (the black curve). Portfolios of Bore and Coca-Cola (B + C) and Bore and Intel (B + I) are shown in light blue in the figure. The dark blue curve is a combination of Bore with a portfolio of Intel and Coca-Cola.

Three Assets



Portfolios of all three stocks are shown, with the dark blue area showing portfolios without short sales, and the light blue area showing portfolios that include short sales. The best risk–return combinations are on the efficient frontier (red curve). **The efficient frontier improves (has a higher return for each level of risk) when we move from two to three stocks**.

Ten Assets



The efficient frontier expands as new investments are added.

Portfolio Choices with Many Risky Assets

- The **minimum variance frontier** is the set of portfolios that minimizes the portfolio standard deviation for a given level of expected return
- The **efficient frontier** is the set of portfolios that maximizes the expected return for a given portfolio standard deviation
- The efficient frontier is **upward sloping portion** of the minimum variance frontier

Mean-Variance Frontier: Some Lagrangian Intuition

□ Start with a vector of asset returns *R*. Denote by *M* the vector of mean returns: *M≡E(R)* and denote by Σ the variance-covariance matrix: Σ=*E[(R-M)(R-M)′]*, where *V′* denotes the transpose of *V*.

□ A portfolio is defined by its weights *x* on the individual securities. The portfolio return is *x′R* where the weights sum to one, *x′1=1*.

□ The problem to choose a portfolio to minimize the variance for a given mean is then

$$min_{\{x\}} x'\Sigma x \quad s.t.\ x'M = \mu; x'1 = 1$$

Mean-Variance Frontier: Some Lagrangian Intuition

□ Introduce the Lagrange multipliers 2λ and 2δ on the constraints. The first order conditions give

$$\Sigma x - \lambda M - \delta 1 = 0 \;\; \Rightarrow \;\; x = \Sigma^{-1}(\lambda M + \delta 1)$$

□ We find the Lagrange multipliers from the constraints

$$M'x = M'\Sigma^{-1}(\lambda M + \delta 1) = \mu \; and \;\; 1'x = 1'\Sigma^{-1}(\lambda M + \delta 1) = 1$$

□ This can be written as

$$\begin{bmatrix} M'\Sigma^{-1}M & M'\Sigma^{-1}1 \\ 1'\Sigma^{-1}M & 1'\Sigma^{-1}1 \end{bmatrix} \begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

□ Let

$$A = M'\Sigma^{-1}M; \quad B = M'\Sigma^{-1}1; \quad C = 1'\Sigma^{-1}1$$

Mean-Variance Frontier: Some Intuition

- Solving this system, we get

$$\lambda = \frac{C\mu - B}{AC - B^2};\ \delta = \frac{A - B\mu}{AC - B^2}$$

- Thus, for a given mean portfolio return μ, the minimum variance portfolio has variance

$$Var(R_{P,\mu}) = \frac{C\mu^2 - 2B\mu + A}{AC - B^2}$$

and is formed by portfolio weights

$$x = \Sigma^{-1}\frac{M(C\mu - B) + 1(A - B\mu)}{AC - B^2}$$

- The variance is thus a quadratic function of the mean. The square root of a parabola is a hyperbola, which is why we draw hyperbolic regions in the expected return-volatility space

Portfolio Choices with Many Risky Assets

- If investors
  - Prefer more to less (other things equal, they prefer a higher expected return)
  - Are risk averse (other things equal, they prefer a lower standard deviation),

then **all investors should choose portfolios from the efficient frontier**

Portfolios with One Risky Asset and One Risk-Free Asset

- Thus far, we have considered the risk and return possibilities that result from combining risky investments into portfolios
- There is another way besides diversification to reduce risk
  - We can keep some of our money in a safe, no-risk investment like Treasury bills or Swiss Government bonds
- Conversely, if we are an aggressive investor, we might decide to borrow money to invest even more in the stock market

Portfolios with One Risky Asset and One Risk-Free Asset

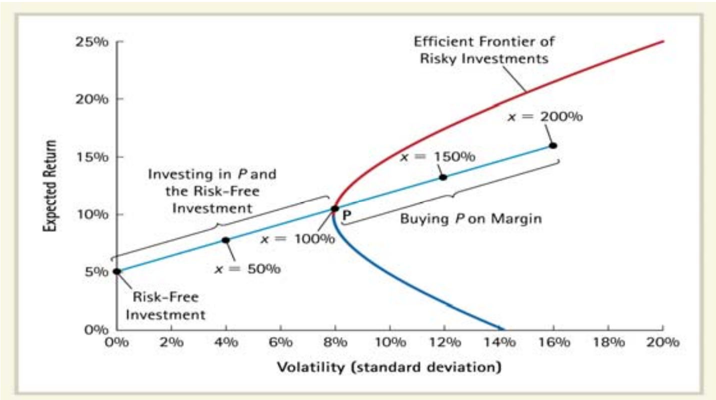
- A risk-free asset has zero variance and zero covariance with any other asset. Thus, investing a fraction (1-x) of the portfolio in the risk-free asset yields an expected return of

$$\begin{aligned} E[R_{xP}] &= (1 - x)r_f + xE[R_p] \\ &= r_f + x(E[R_p] - r_f) \end{aligned}$$

and a standard deviation of

$$\begin{aligned} SD(R_{xP}) &= \sqrt{(1-x)^2 Var(r_f) + x^2 Var(R_p) + 2(1-x)x\, Cov(r_f, R_p)} \\ &= \sqrt{x^2 Var(R_p)} \qquad\qquad\qquad 0 \\ &= x\, SD(R_p) \end{aligned}$$

Portfolios with One Risky Asset and One Risk-Free Asset



Given a risk-free rate of 5%, the risk-free investment is represented in the graph by the point with 0% volatility and an expected return of 5%.

The blue line shows the portfolios obtained by investing x in portfolio P and $(1 - x)$ in the risk-free investment. Investments with weight $x > 100\%$ in portfolio P require borrowing at the risk-free interest rate.

Portfolios with One Risky Asset and One Risk-Free Asset

- Suppose you have \$10,000 in cash, and you decide to borrow another \$10,000 at a 5% interest rate to invest in the stock market. You invest the entire \$20,000 in portfolio Q with a 10% expected return and a 20% volatility
- What is the expected return and volatility of your investment?
- What is your realized return if (1) Q goes up 30% or (2) Q falls by 10% over the course of the year

Portfolios with One Risky Asset and One Risk-Free Asset

Solution

You have doubled your investment in Q by buying stocks on margin, so $x = 200\%$. Using Eq. 11.15 and Eq. 11.16,

$$E[R_{xQ}] = r_f + x(E[R_Q] - r_f) = 5\% + 2 \times (10\% - 5\%) = 15\%$$
$$SD(R_{xQ}) = x\, SD(R_Q) = 2 \times (20\%) = 40\%$$

You have increased both your expected return and your risk relative to the portfolio Q.

If Q goes up 30%, your investment will be worth \$26,000 at year-end. However, you will owe \$10,000 × 1.05 = \$10,500 on your loan. After repaying your loan, you will have \$26,000 − \$10,500 = \$15,500. Because you initially invested \$10,000 of your own money, this is a 55% return.

If Q drops by 10%, you are left with \$18,000 − \$10,500 = \$7500, and your return is −25%.

Note that your returns are more extreme than those of the portfolio: 55% and −25% versus 30% and −10%, respectively. In fact, the range is doubled to 55% − (−25%) = 80% from 30% − (−10%) = 40%. This doubling corresponds to the doubling of the volatility of the portfolio.

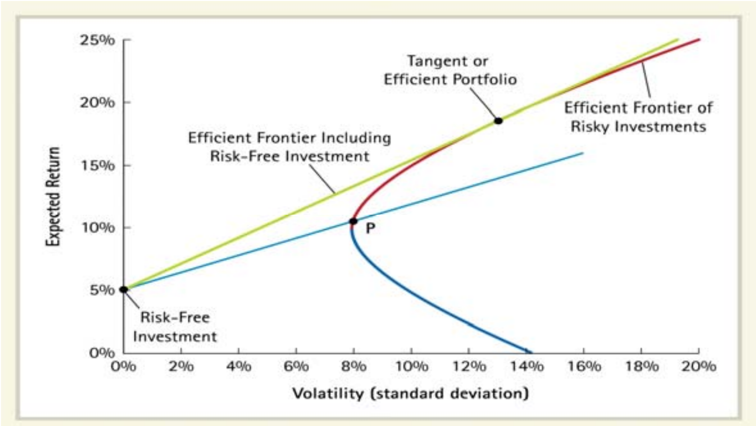
Identifying the Tangent Portfolio

- The slope of the line through a given portfolio P in the expected return-volatility space is often referred to as the **Sharpe Ratio**.
- It is defined as

$$\boxed{\text{Sharpe Ratio} = \frac{\text{Portfolio Excess Return}}{\text{Portfolio Volatility}} = \frac{E[R_P] - r_f}{SD(R_P)}}$$

- The Sharpe ratio measures the ratio of **reward-to-volatility** provided by a portfolio
- The optimal portfolio to combine with the risk-free asset will be the one with the highest Sharpe ratio as it will lead to the steepest possible line. It is called the **tangent portfolio**

Identifying the Tangent Portfolio



The tangent portfolio is the portfolio with the highest Sharpe ratio. Investments on the green line connecting the risk-free investment and the tangent portfolio provide the best risk and return tradeoff available to an investor. **The tangent portfolio is also referred to as the efficient portfolio.**

The efficient portfolio and the cost of capital

- Consider a portfolio of risky securities *P*
- The portfolio *P* is **efficient** if it has the **highest possible Sharpe ratio**, that is if it provides the largest increase in expected return possible for a given increase in volatility
- To determine whether *P* has the highest possible Sharpe ratio, and hence is efficient, let's consider **whether we could raise its Sharpe ratio** by adding more of some investment *i* in the portfolio

The efficient portfolio and the cost of capital

- If we purchase more of investment i, we earn $E[R_i]$
- Each security contributes to the volatility of the portfolio according to its volatility scaled by its correlation with the portfolio
- Thus adding i to our portfolio improves our Sharpe ratio if

$$E[R_i] > r_f + \underbrace{SD(R_i)Corr(R_i, R_P)}_{\text{Incremental volatility from investment i}} \times \underbrace{\frac{E[R_P] - r_f}{SD(R_P)}}_{\text{Return per unit of volatility available from portfolio P}}$$

The efficient portfolio and the cost of capital

- To provide further interpretation of this condition, define the beta of an investment i with portfolio P as

$$\beta_i^P \equiv \frac{SD(R_i) \times Corr(R_i, R_P)}{SD(R_P)} = \frac{Cov(R_i, R_P)}{Var(R_P)}$$

- Increasing the amount invested in i will increase the Sharpe ratio if its expected return $E[R_i]$ exceeds the required return r_i, given by

$$r_i = r_f + \beta_i^P \times (E[R_P] - r_f)$$

- The **required return** is the expected return that is necessary to compensate for the risk investment i contributes to the portfolio

The efficient portfolio and the cost of capital

- Investors add asset *i* in the portfolio until the expected return equals the required return through an increase in the correlation with the portfolio.
- At this point, no trade can possibly improve the risk-reward ratio of the portfolio. **Our portfolio is the optimal, efficient portfolio** and we have

$$\boxed{E[R_i] = r_i \equiv r_f + \beta_i^{eff} \times (E[R_{eff}] - r_f)}$$

where R_{eff} is the return on the efficient portfolio, the portfolio with the highest Sharpe ratio (A portfolio is efficient iff the expected return of every security is equal to its required return).

The efficient portfolio and the cost of capital

The Required Return of a New Investment

Problem

You are currently invested in the Omega Fund, a broad-based fund with an expected return of 15% and a volatility of 20%, as well as in risk-free Treasuries paying 3%. Your broker suggests that you add a real estate fund to your portfolio. The real estate fund has an expected return of 9%, a volatility of 35%, and a correlation of 0.10 with the Omega Fund. Will adding the real estate fund improve your portfolio?

The efficient portfolio and the cost of capital

Solution

Let R_{re} be the return of the real estate fund and R_O be the return of the Omega Fund. From Eq. 11.19, the beta of the real estate fund with the Omega Fund is

$$\beta_{re}^{O} = \frac{SD(R_{re})\,Corr(R_{re}, R_O)}{SD(R_O)} = \frac{35\% \times 0.10}{20\%} = 0.175$$

We can then use Eq. 11.20 to determine the required return that makes the real estate fund an attractive addition to our portfolio:

$$r_{re} = r_f + \beta_{re}^{O}(E[R_O] - r_f) = 3\% + 0.175 \times (15\% - 3\%) = 5.1\%$$

Because its expected return of 9% exceeds the required return of 5.1%, investing some amount in the real estate fund will improve our portfolio's Sharpe ratio.

The efficient portfolio and the cost of capital

Identifying the Efficient Portfolio

Problem

Consider the Omega Fund and real estate fund of Example 11.13. Suppose you have \$100 million invested in the Omega Fund. In addition to this position, how much should you invest in the real estate fund to form an efficient portfolio of these two funds?

Solution

Suppose that for each $1 invested in the Omega Fund, we borrow x_{re} dollars (or sell x_{re} worth of Treasury bills) to invest in the real estate fund. Then our portfolio has a return of $R_P = R_O + x_{re}(R_{re} - r_f)$, where R_O is the return of the Omega Fund and R_{re} is the return of the real estate fund. Table 11.5 shows the change to the expected return and volatility of our portfolio as we increase the investment x_{re} in the real estate fund, using the formulas

$$E[R_P] = E[R_O] + x_{re}(E[R_{re}] - r_f)$$

$$Var(R_P) = Var[R_O + x_{re}(R_{re} - r_f)] = Var(R_O) + x_{re}^2 Var(R_{re}) + 2x_{re}Cov(R_{re}, R_O)$$

Adding the real estate fund initially improves the Sharpe ratio of the portfolio, as defined by Eq. 11.17. As we add more of the real estate fund, however, its correlation with our portfolio rises, computed as

$$Corr(R_{re}, R_P) = \frac{Cov(R_{re}, R_P)}{SD(R_{re})SD(R_P)} = \frac{Cov(R_{re}, R_O + x_{re}(R_{re} - r_f))}{SD(R_{re})SD(R_P)}$$

$$= \frac{x_{re}Var(R_{re}) + Cov(R_{re}, R_O)}{SD(R_{re})SD(R_P)}$$

The beta of the real estate fund—computed from Eq. 11.19—also rises, increasing the required return. The required return equals the 9% expected return of the real estate fund at about $x_{re} = 11\%$, which is the same level of investment that maximizes the Sharpe ratio. Thus, the efficient portfolio of these two funds includes $0.11 in the real estate fund per $1 invested in the Omega Fund.

The efficient portfolio and the cost of capital

x_{re}	$E[R_P]$	$SD(R_P)$	Sharpe Ratio	$Corr(R_{re}, R_P)$	β_{re}^P	Required Return r_{re}
0%	15.00%	20.00%	0.6000	10.0%	0.18	5.10%
4%	15.24%	20.19%	0.6063	16.8%	0.29	6.57%
8%	15.48%	20.47%	0.6097	23.4%	0.40	8.00%
10%	15.60%	20.65%	0.6103	26.6%	0.45	8.69%
11%	15.66%	20.74%	0.6104	28.2%	0.48	9.03%
12%	15.72%	20.84%	0.6103	29.7%	0.50	9.35%
16%	15.96%	21.30%	0.6084	35.7%	0.59	10.60%

The CAPM

- The CAPM (Capital Asset Pricing Model) is a model that relates returns to risk
- There are three main assumptions that underlie the CAPM
  - Investors can buy and sell all securities at competitive market prices and can borrow and lend at the risk-free rate
  - Investors only hold efficient portfolios, that is portfolios that yield the maximum expected return for a given level of volatility
  - Investors have homogeneous expectations regarding volatilities, correlations, and expected rates of returns

Two-fund separation theorem

- Suppose investors have homogeneous expectations
- Then all investors, regardless of their risk preferences, will form optimal portfolios by combining only two assets, the risk-free asset and the tangent portfolio
- Therefore the optimal portfolio of risky investments no longer depends on how conservative or aggressive the investor is; every investor **should invest in the tangent portfolio independent of his taste for risk**
- The preferences of investors will only determine **how much** to invest in the tangent portfolio versus the risk-free investment

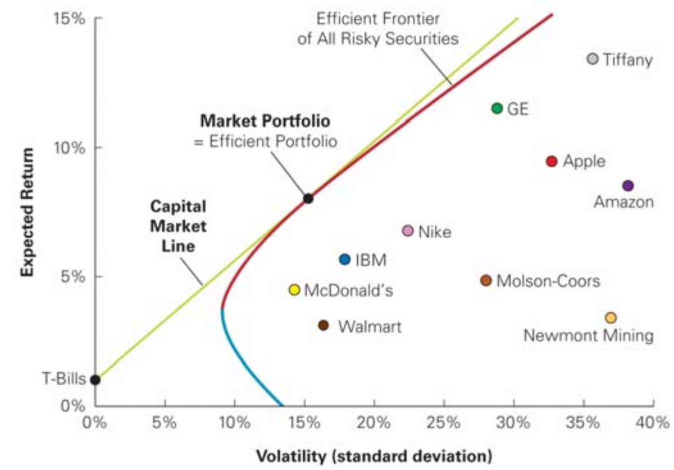
Two-fund separation theorem

□ This two-fund separation theorem implies that all investors will hold the same portfolio of risky assets

□ In equilibrium, supply must equal demand for all assets

□ Thus in equilibrium the tangent or efficient portfolio must be the market portfolio

Two-fund separation theorem



When investors have homogeneous expectations, the market portfolio and the efficient portfolio coincide. Therefore **the capital market line (CML), which is the line from the risk-free investment through the market portfolio, represents the highest expected return available for any level of volatility**.

The CAPM relation

- Under the CAPM assumptions, all investors are holding the market portfolio of risky assets, which is the efficient portfolio
- Using the results derived above we can then write

$$E[R_i] = r_i = r_f + \underbrace{\beta_i^{Mkt}(E[R_{Mkt}] - r_f)}_{\text{Risk premium for security } i}$$

- This implies that the risk premium on a security is equal to the market risk premium multiplied by the amount of market risk present in the security's returns, measured by its beta

The CAPM relation

- In the CAPM relation, the beta of a security with the market portfolio is defined as

$$\beta_i^{Mkt} \equiv \beta_i = \frac{\overbrace{SD(R_i) \times Corr(R_i, R_{Mkt})}^{\text{Volatility of } i \text{ that is common with the market}}}{SD(R_{Mkt})} = \frac{Cov(R_i, R_{Mkt})}{Var(R_{Mkt})}$$

- Statistically, the risk of any asset can be measured by how much an asset moves with the market (the covariance)
- The CAPM specifies that part of the variance can be diversified away, and that only the non-diversifiable portion is rewarded

CAPM intuition

- To summarize, the CAPM
  - Measures the non diversifiable risk with beta, which is standardized around one
  - Translates beta into expected returns

$$E[R_i] = r_i = r_f + \underbrace{\beta_i^{Mkt}(E[R_{Mkt}] - r_f)}_{\text{Risk premium for security } i}$$

CAPM application

Computing the Expected Return for a Stock

Problem

Suppose the risk-free return is 4% and the market portfolio has an expected return of 10% and a volatility of 16%. Campbell Soup stock has a 26% volatility and a correlation with the market of 0.33. What is Campbell Soup's beta with the market? What capital market line portfolio has equivalent market risk, and what is its expected return?

CAPM application

Solution

We can compute beta using Eq. 11.23:

$$\beta_{CPB} = \frac{SD(R_{CPB})\,Corr(R_{CPB}, R_{Mkt})}{SD(R_{Mkt})} = \frac{26\% \times 0.33}{16\%} = 0.54$$

That is, for each 1% move of the market portfolio, Campbell Soup stock tends to move 0.54%. We could obtain the same sensitivity to market risk by investing 54% in the market portfolio, and 46% in the risk-free security. Because it has the same market risk, Campbell's soup stock should have the same expected return as this portfolio, which is (using Eq. 11.15 with $x = 0.54$),

$$\begin{aligned} E[R_{CPB}] &= r_f + x(E[R_{Mkt}] - r_f) = 4\% + 0.54(10\% - 4\%) \\ &= 7.2\% \end{aligned}$$

Because $x = \beta_{CPB}$, this calculation is precisely the CAPM Eq. 11.22. Thus, investors will require an expected return of 7.2% to compensate for the risk associated with Campbell Soup stock.

CAPM application

Computing the Equity Cost of Capital

Problem

Suppose you estimate that eBay's stock has a volatility of 30% and a beta of 1.45. A similar process for UPS yields a volatility of 35% and a beta of 0.79. Which stock carries more total risk? Which has more market risk? If the risk-free interest rate is 3% and you estimate the market's expected return to be 8%, calculate the equity cost of capital for eBay and UPS. Which company has a higher cost of equity capital?

CAPM application

Solution

Total risk is measured by volatility; therefore, UPS stock has more total risk than eBay. Systematic risk is measured by beta. eBay has a higher beta, so it has more market risk than UPS.

Given eBay's estimated beta of 1.45, we expect the price for eBay's stock to move by 1.45% for every 1% move of the market. Therefore, eBay's risk premium will be 1.45 times the risk premium of the market, and eBay's equity cost of capital (from Eq. 12.1) is

$$r_{EBAY} = 3\% + 1.45 \times (8\% - 3\%) = 3\% + 7.25\% = 10.25\%$$

UPS has a lower beta of 0.79. The equity cost of capital for UPS is

$$r_{UPS} = 3\% + 0.79 \times (8\% - 3\%) = 3\% + 3.95\% = 6.95\%$$

Because market risk cannot be diversified, it is market risk that determines the cost of capital; thus eBay has a higher cost of equity capital than UPS, even though it is less volatile.

The market portfolio

- The capital market line (CML), which is the line from the risk-free investment through the market portfolio, represents the highest expected return available for any level of volatility
- Consider a portfolio on the **CML** with a fraction x invested in the market portfolio and the remaining (1-x) invested in the risk free investment.
- We have

$$E[R_{xCML}] = (1 - x)r_f + xE[R_{Mkt}] = r_f + x(E[R_{Mkt}] - r_f)$$

and

$$SD(R_{xCML}) = x\,SD(R_{Mkt})$$

- The portfolio's risk premium and volatility are determined by the fraction x that is invested in the market

The market portfolio

Problem

Your brother-in-law's investment portfolio consists solely of \$10,000 invested in McDonald's stock. Suppose the risk-free rate is 4%, McDonald's stock has an expected return of 9% and a volatility of 27%, and the market portfolio has an expected return of 10% and a volatility of 16%. Under the CAPM assumptions, which portfolio has the lowest possible volatility while having the same expected return as McDonald's stock? Which portfolio has the highest possible expected return while having the same volatility as McDonald's stock?

Solution

The CAPM assumptions imply that the best possible risk–return combinations are combinations of the risk-free investment and the market portfolio—portfolios on the capital market line. First let's find the CML portfolio that has an expected return of 9%, equal to the McDonald's return. From Eq. 12.2, we need to determine the amount x to invest in the market so that

$$9\% = E[R_{xCML}] = r_f + x(E[R_{Mkt}] - r_f) = 4\% + x(10\% - 4\%)$$

Solving for x, we get $x = 0.8333$. That is, your brother-in-law should sell his McDonald's stock and invest \$8333 in the market portfolio and the remaining \$1667 in the risk-free investment. Using Eq. 12.3, this portfolio has a volatility of only

$$SD(R_{xCML}) = 0.8333(16\%) = 13.3\%$$

This volatility is much lower than the volatility of McDonald's stock, and is the lowest possible volatility given an expected return of 9%.

Alternatively, we can choose the CML portfolio that matches McDonald's volatility of 27%. To do so, we use Eq. 12.3 to find x such that

$$27\% = SD(R_{xCML}) = x(16\%)$$

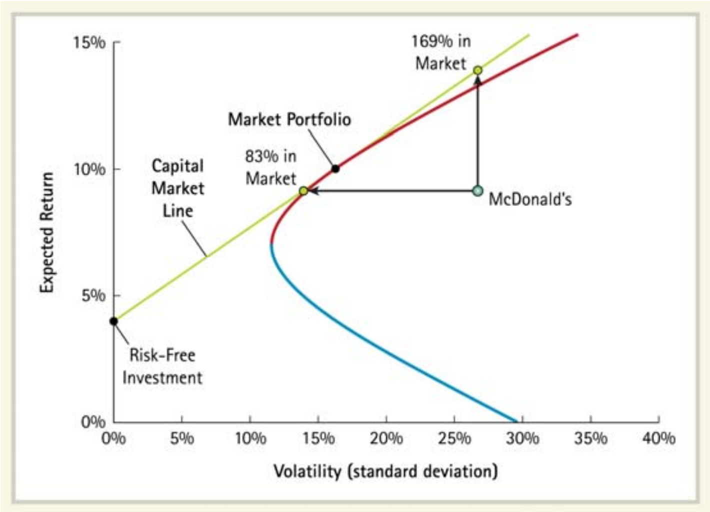
In this case $x = 1.6875$, so the expected return is

$$E[R_{xCML}] = 4\% + 1.6875(10\% - 4\%) = 14.1\%$$

This expected return is much higher than the expected return of McDonald's stock, and the highest possible return we can earn without increasing the volatility. To achieve this portfolio, your brother-in-law needs to sell his McDonald's stock, add (or borrow) an additional \$6875, and invest \$16,875 in the market portfolio.

Figure 12.2 illustrates the two alternatives to investing in McDonald's stock. Any portfolio on the capital market line between these two portfolios (that is, investing between \$8333 and \$16,875 in the market) will have both a higher expected return and a lower volatility than investing in McDonald's stock alone.

The market portfolio



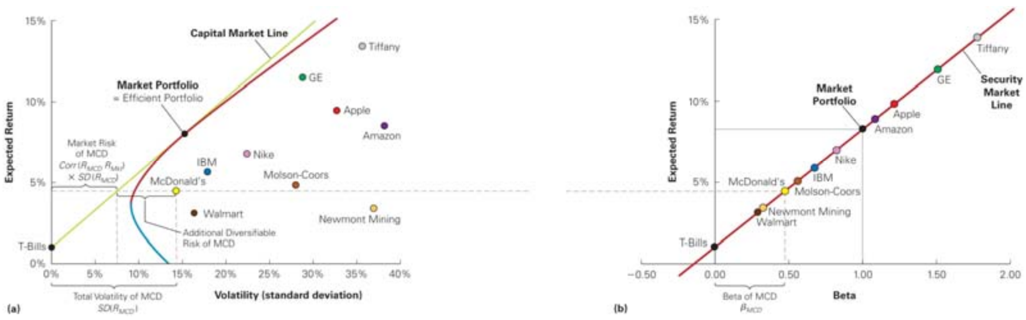
Given the assumptions in the example, portfolios with 83% to 169% invested in the market (and the rest invested or borrowed at the risk-free rate) offer a higher expected return and a lower volatility than investing 100% in McDonald's stock.

The security market line

- The **CML** depicts portfolios combining the risk-free investment and the efficient portfolio, and shows the highest expected return that can be attained for each level of volatility.
  - According to the CAPM, the market portfolio is on the CML and **all other stocks and portfolios contain diversifiable risk and lie to the right of the CML**

- The **SML** shows the required return for each security as a function of its beta with the market.
  - According to the CAPM, the market portfolio is efficient, which is equivalent to the required return equaling the expected return for every security. According to the CAPM, all stocks and portfolios should lie on the SML.

The security market line



(a) The **CML** depicts portfolios combining the risk-free investment and the efficient portfolio, and shows the highest expected return that can be attained for each level of volatility.

(b) The **SML** shows the required return for each security as a function of its beta with the market.

The security market line

- Because the security market line applies to all securities, it applies to portfolios as well
- The expected return on a portfolio should correspond to the portfolio's beta

$$\beta_P = \frac{Cov(R_P, R_{Mkt})}{Var(R_{Mkt})} = \frac{Cov\left(\sum_i x_i R_i, R_{Mkt}\right)}{Var(R_{Mkt})} = \sum_i x_i \frac{Cov(R_i, R_{Mkt})}{Var(R_{Mkt})}$$
$$= \sum_i x_i \beta_i$$

- In other words, the beta of a portfolio is the weighted average beta of the securities in the portfolio

The security market line

The Expected Return of a Portfolio

Problem

Suppose Kraft Foods' stock has a beta of 0.50, whereas Boeing's beta is 1.25. If the risk-free rate is 4%, and the expected return of the market portfolio is 10%, what is the expected return of an equally weighted portfolio of Kraft Foods and Boeing stocks, according to the CAPM?

The security market line

Solution

We can compute the expected return of the portfolio in two ways. First, we can use the SML to compute the expected return of Kraft Foods (KFT) and Boeing (BA) separately:

$$E[R_{KFT}] = r_f + \beta_{KFT}(E[R_{Mkt}] - r_f) = 4\% + 0.50(10\% - 4\%) = 7.0\%$$

$$E[R_{BA}] = r_f + \beta_{BA}(E[R_{Mkt}] - r_f) = 4\% + 1.25(10\% - 4\%) = 11.5\%$$

Then, the expected return of the equally weighted portfolio P is

$$E[R_P] = \tfrac{1}{2}E[R_{KFT}] + \tfrac{1}{2}E[R_{BA}] = \tfrac{1}{2}(7.0\%) + \tfrac{1}{2}(11.5\%) = 9.25\%$$

Alternatively, we can compute the beta of the portfolio using Eq. 11.24:

$$\beta_P = \tfrac{1}{2}\beta_{KFT} + \tfrac{1}{2}\beta_{BA} = \tfrac{1}{2}(0.50) + \tfrac{1}{2}(1.25) = 0.875$$

We can then find the portfolio's expected return from the SML:

$$E[R_P] = r_f + \beta_P(E[R_{Mkt}] - r_f) = 4\% + 0.875(10\% - 4\%) = 9.25\%$$

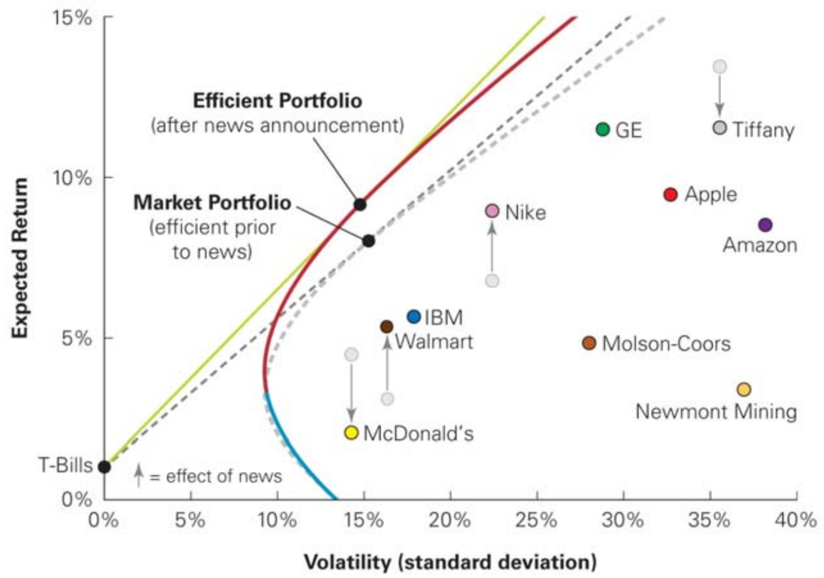
Alpha

- Consider the situation in the above figure, and suppose that new information arrives that increases the expected return of Walmart and Nike and lowers the expected return of Tiffany and McDonald's

- Suppose that if market prices remain unchanged, this news would raise the expected return of Walmart and Nike by 2% and lowers the expected return of Tiffany and McDonald's by 2%

- As we can see on the figure, the market portfolio is no longer efficient as other portfolios offer a better risk-return trade-off

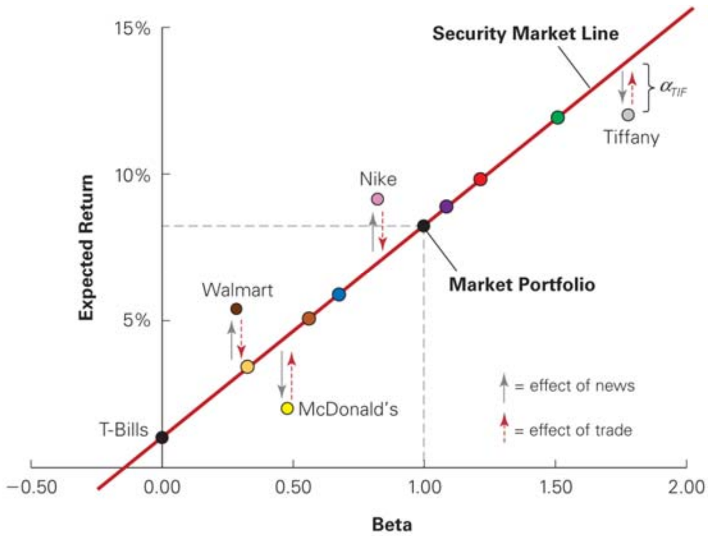
Alpha



Alpha

- To improve the performance of their portfolio, investors who are holding the market portfolio will compare the expected return of each security with its required return from the SML
- The difference between a stock's expected return and its required return is called the stock's **alpha**
- When the market portfolio is efficient, all stocks are on the SML and have an alpha of zero. When a stock's alpha is not zero, investors can improve upon performance of the market portfolio as the Sharpe ratio of a portfolio increases when we buy stocks whose expected return exceeds their required return
- This phenomenon will raise (depress) the price and lower (increase) the returns on positive (negative) alpha stocks until all the stocks are on the SML

Alpha



If the market portfolio is not efficient, then stocks will not all lie on the SML. The distance of a stock above or below the security market line is the stock's alpha. We can improve upon the market portfolio by buying stocks with positive alphas and selling stocks with negative alphas.

The CAPM in practice

- To implement the CAPM, we need the following inputs
  - The current risk-free rate
  - The expected market risk premium
  - The beta of the asset being analyzed

The CAPM in practice

- The risk-free rate is the rate on a risk-free bond
  - In valuation, the time horizon is usually infinite, leading to the conclusion that a long-term riskfree rate is always preferable to a short term rate
  - Using long term government bond on all the cash flows in a long term analysis yields a close approximation
  - Most practitioners use 10 to 30 year risk-free government bonds

- The market risk premium is the rate of return above the risk-free rate (that you just chose) on a portfolio that includes all stocks in the market (market portfolio such as SMI or S&P500)

Estimation set-up

- Decide on an estimation period
  - Traditionally periods range from 2 to 5 years
  - Longer periods provide more data, but firms change
- Decide on the return frequency: Higher frequency yields more observations but suffer from more noise
- Estimate returns (including dividends) on stocks

  $Return = (Price_{end} - Price_{begin} + Div_{period}) / Price_{begin}$

  Include dividends only in ex-dividend month
- Choose a market index and estimate returns on the index

Estimation set-up

Year End	S&P 500 Index	Dividends Paid*	S&P 500 Realized Return	Microsoft Realized Return	1-Month T-Bill Return
2001	1148.08				
2002	879.82	14.53	−22.1%	−22.0%	1.6%
2003	1111.92	20.80	28.7%	6.8%	1.0%
2004	1211.92	20.98	10.9%	8.9%	1.2%
2005	1248.29	23.15	4.9%	−0.9%	3.0%
2006	1418.30	27.16	15.8%	15.8%	4.8%
2007	1468.36	27.86	5.5%	20.8%	4.7%
2008	903.25	21.85	−37.0%	−44.4%	1.5%
2009	1115.10	27.19	26.5%	60.5%	0.1%
2010	1257.64	25.44	15.1%	−6.5%	0.1%
2011	1257.60	26.59	2.1%	−4.5%	0.0%
2012	1426.19	32.67	16.0%	5.8%	0.1%
2013	1848.36	39.75	32.4%	44.2%	0.0%
2014	2058.90	42.47	13.7%	27.5%	0.0%

*Total dividends paid by the 500 stocks in the portfolio, based on the number of shares of each stock in the index, adjusted until the end of the year, assuming they were reinvested when paid.

Source: Standard & Poor's, Microsoft and U.S. Treasury Data

Estimating beta

- The standard procedure for estimating betas is to regress excess stock returns (R_i - r_f) against the market risk premium

$$R_i - r_f = \alpha_i + \beta_i (R_{Mkt} - r_f) + \varepsilon_i$$

- **Interpretation**
  - The *slope* of the regression corresponds to the stock's *beta*. When the market's return increases by 1%, the security's return increases by β_i %.
  - α_i is the intercept term of the regression
  - ε_i is the error term and represents the deviation from the best-fitting line and is zero on average (or else we could improve the fit)

Estimating beta

- Since E[ε_i]=0, we have

$$E[R_i] = r_f + \beta_i (E[R_{Mkt}] - r_f) + \alpha_i$$

- **Interpretation**
  - The *intercept* represents a risk-adjusted performance measure for the historical returns and measures the distance above/below the SML
    - α_i provides a simple measure of performance during the period of regression, relative to the CAPM
    - If $\alpha_i > 0$, then the stock did better than expected during the regression period. This is *Jensen's alpha*

Estimating beta

- Standard variance decomposition implies

$$\boxed{V(R_i) = \beta_i^2 V(R_{Mkt}) + V(\varepsilon_i)}$$

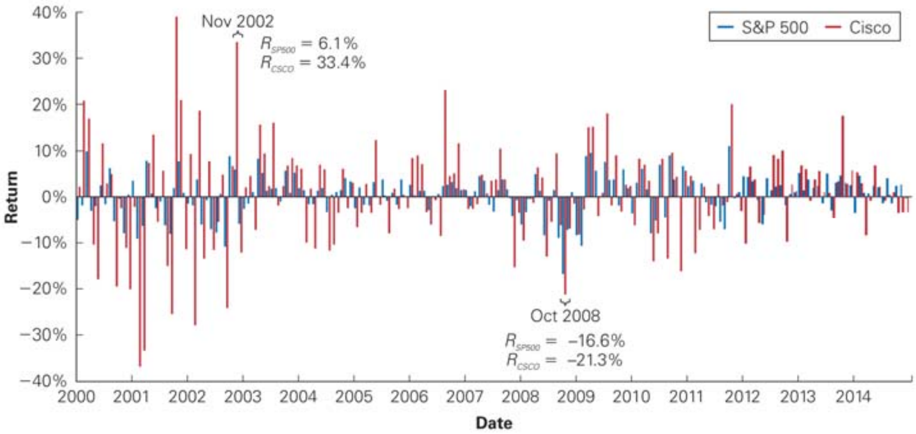
- The R-squared (R^2) of the regression provides an estimate of the proportion of the risk of a firm that can be attributed to market risk

$$\boxed{R^2 = \frac{\beta_i^2 V(R_{Mkt})}{V(R_i)} = 1 - \frac{V(\varepsilon_i)}{V(R_i)}}$$

- The balance can be attributed to firm specific risk

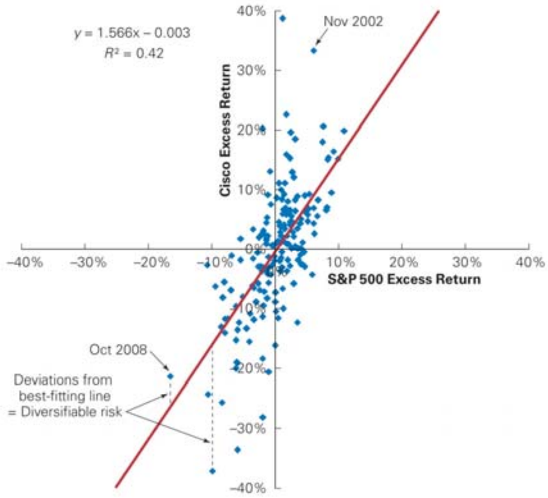
Estimating beta

- Monthly returns for Cisco and the S&P500



Estimating beta

□ Beta corresponds to the slope of the best-fitting line. Beta measures the expected change in Cisco's excess return per 1% change in the market's excess return (monthly T-bill rate 0.1%).



Estimation set-up

□ Estimation for Cisco using monthly data on a 15-year window

$$\boxed{Returns_{Cisco} - r_f = -0.33\% + 1.566\ (Returns_{S\&P500} - r_f) \quad (R^2{=}42\%)}$$

□ The estimated beta is *1.566*

□ *42%* of the risk at Cisco comes from market sources

□ The intercept is *-0.33%*. Cisco did worse than expected. Its monthly return was *0.33%* lower than required by the security market line

Estimating expected returns

- Inputs to the expected returns calculation
  - Cisco's beta = *1.566*
  - Risk free rate = *1.2%* (Long term gvt bond)
  - Market risk premium = *6%* (Historical premium)
- Expected return

$$\boxed{E(R_i) = 1.2\% + 1.566\ [6\%] = 10.6\%}$$

- Managers at Cisco need to make at least *10.6%* (when investing in new projects) as a return for their equity investors to break even

Appendix: Additional resources

- Course on "Investments" given in the MFE program
- Analysis of the investment strategy of the Norwegian Sovereign Wealth Fund:

www0.gsb.columbia.edu/faculty/aang/paper/AngBrandtDenison.pdf

- **Book**: "Asset Management: A Systematic Approach to Factor Investing" by Andrew Ang

Appendix: The APT and multifactor models

□ The arbitrage pricing theory (APT) provides an alternative to the CAPM for the determination of expected returns

□ In the APT, stock returns depend on pervasive factors. The return obeys the following relation

$$Return = a + b_1(r_{factor\ 1}) + b_2(r_{factor\ 2}) + b_3(r_{factor\ 3}) + \ldots + noise$$

□ The theory does not say what the factors are: There could be an oil price factor, an interest rate factor, and so on.

□ Some stocks will be more sensitive to a particular factor than other stocks

→ Exxon Mobil is more sensitive to an oil factor than Google

Appendix: The APT and multifactor models

□ The APT states that the expected risk premium on a stock should depend on the expected risk premium on each factor and the stock's sensitivity to each of the factors

□ The APT will provide a good handle on expected returns only if we can

- Identify a reasonably short list of macroeconomic factors
- Measure the expected risk premium of each of these factors
- Measure the sensitivity of each stock to these factors

Appendix: The three-factor model

- Fama and French show that the stocks of small firms and those with a high book-to-market ratio have provided above average returns
- If investors demand do demand an extra return for taking on exposures to these factors, then we have a measure of the expected return that looks like the APT:

$$E(R_i) - R_f = b_{market}\,(R_{market}) + b_{size}\,(R_{size}) + b_{book\ to\ market}\,(R_{book\ to\ market})$$

- This is known as the Fama-French model. The following table reports the risk premium in the Fama-French model and the CAPM with $R_{market} = 5.2\%$, $R_{size} = 3.2\%$, and $R_{B/M} = 5.4\%$

Appendix: The three-factor model

	Three-factor model				
	Factor sensitivities			Fama-French	CAPM
	b_{market}	b_{size}	$b_{B/M}$	Risk Premium	Risk Premium
Aircraft	1.15	0.51	0.00	7.54	6.43
Banks	1.13	0.13	0.35	8.08	5.55
Chemicals	1.13	-0.03	0.17	6.58	5.57
Computers	0.90	0.17	-0.49	2.49	5.29
Construction	1.21	0.21	-0.09	6.42	6.52
Food	0.88	-0.07	-0.03	4.09	4.44
Petroleum	0.96	-0.35	0.21	4.93	4.32
Pharmaceuticals	0.84	-0.25	-0.63	0.09	4.71
Tobacco	0.86	-0.04	0.24	5.56	4.08