AKLT-States as ZX-Diagrams

Diagrammatic Reasoning for Quantum States

East, R. D., van de Wetering, J., Chancellor, N., & Grushin, A. G. (2022)
PRX quantum, 3(1), 010302.

Presented by Yan Mong Chan

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Overview

AKLT Model

AKLT Model

- Spin-1 Affleck-Kennedy-Lieb-Tasaki (AKLT) model with N sites

$$
\begin{aligned}
H_{\text{AKLT}}^{S=1} &= \frac{1}{24}\sum_{j=1}^{N-1}(\mathbf{S}_j + \mathbf{S}_{j+1})^2\left((\mathbf{S}_j + \mathbf{S}_{j+1})^2 - 2\right) \\
&= \frac{1}{2}\sum_{j=1}^{N-1}\left[\mathbf{S}_j \cdot \mathbf{S}_{j+1} + \frac{1}{3}(\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 + \frac{2}{3}\right]
\end{aligned} \tag{1}
$$

- $\mathbf{S}_i$ are 3x3 spin-1 operators; $\mathcal{H} = \mathbb{C}_2^{\otimes N}$

Why interesting?

- Historically, it was believed that the GS of 1D spin chains of any spins are gapless
- An example that verifies the Haldane conjecture: integer spin antiferromagnetic Heisenberg chains has gapped GS
- Lots of surprising physics: Hidden string order, etc.

Ground state of Spin-1 AKLT

- The Hamiltonian is a projector onto adjacent total spin=2 subspace because

$$(\mathbf{S}_j + \mathbf{S}_{j+1})^2 \left((\mathbf{S}_j + \mathbf{S}_{j+1})^2 - 2\right) |s^{(j,j+1)}_{\text{total}} = 0\rangle = 0$$
$$(\mathbf{S}_j + \mathbf{S}_{j+1})^2 \left((\mathbf{S}_j + \mathbf{S}_{j+1})^2 - 2\right) |s^{(j,j+1)}_{\text{total}} = 1\rangle = 0$$

- Therefore we have

$$\langle s^{(j,j+1)}_{total} = 2 \, |\psi_0\rangle = 0 \quad \forall j = 1, 2, \cdots, N-1$$

- $|\psi_0\rangle$ is the ground state(s) of the Hamiltonian

- To solve GS, replace each spin-1 site with two spin-1/2
- We construct the state so that spins on adjacent sites are in singlet configuration.

$$\frac{1}{\sqrt{2}}\left(|0\rangle_i\,|1\rangle_{i+1} - |1\rangle_i\,|0\rangle_{i+1}\right)$$

- This construction ensures that the total spin on adjacent sites are zero.
- We then do a projection onto the the spin-1 subspace for each lattice site

$$|\uparrow\uparrow\rangle\,,\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right),\ |\downarrow\downarrow\rangle$$

- The ground state wavefunction is unique for close chains and 4-fold degenerate for open-end chains

MPS representation

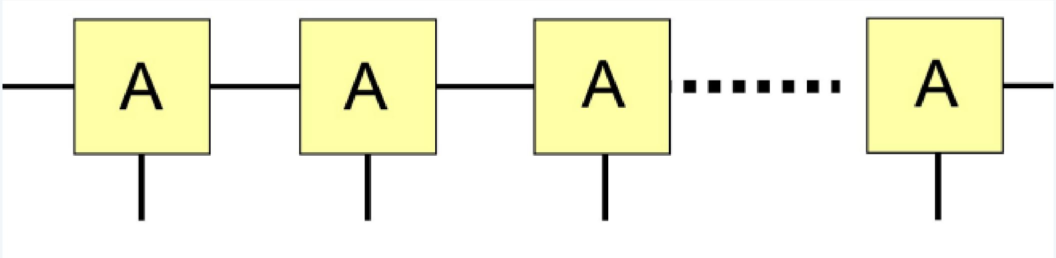

Figure: MPS representation of AKLT ground state taken from Wei et. al. (2022) [2201.09307]

Derivation of the MPS representation

- This derivation is taken from a very useful review by Andreas Haller
- Let $|\mathbf{a}\rangle = |a_1, a_2, \cdots, a_N\rangle$ and $|\mathbf{b}\rangle = |b_1, b2, \cdots, b_N\rangle$ be the auxiliary spin-1/2 states and N be the number of sites. The general wavefunction takes the form

$$|\psi\rangle = \sum_{\mathbf{a},\mathbf{b}} c_{\mathbf{ab}} |\mathbf{a}, \mathbf{b}\rangle$$

- The G.S. adjacent spins are in valence bond, so we have $c_{\mathbf{ab}} = \Sigma_{b_1 a_2} \Sigma_{b_2 a_3} \cdots \Sigma_{b_{N-1} a_N}$, where

$$\Sigma = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

- We the project our subspace onto the spin-1 subspace using the on-site projector

$$P_i = \sum_{a_i, b_i, \sigma_i} M^{\sigma_i}_{a_i b_i} \left|\sigma_i\right\rangle \left\langle a_i, b_i\right|$$

where $\sigma_i = 0, \pm 1$, $a_i, b_i = \pm 1/2$, and the matrices M^{σ} takes the form

$$M^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad M^{(0)} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad M^{(-1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- Therefore, $\left|\psi_0\right\rangle \propto P_1 P_2 \cdots P_N \left|\psi\right\rangle$, which takes the form

$$\begin{aligned} \left|\psi_0\right\rangle &\propto \sum_{\mathbf{a}, \mathbf{b}, \boldsymbol{\sigma}} M^{\sigma_1}_{a_1 b_1} \Sigma_{b_1 a_2} M^{\sigma_2}_{a_2 b_2} \Sigma_{b_2 a_3} \cdots \Sigma_{b_{N-1} a_N} M^{\sigma_N}_{a_N b_N} \left|\boldsymbol{\sigma}\right\rangle \\ &= \sum_{\boldsymbol{\sigma}} A^{\sigma_1} A^{\sigma_2} \cdots A^{\sigma_N} \left|\boldsymbol{\sigma}\right\rangle \end{aligned}$$

where $A^{\sigma_i} = M^{\sigma_i} \Sigma$ and we contract all the As via matrix multiplication

- The tensor-trian at each site is given by

$$A^{(+1)} \propto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A^{(0)} \propto -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad A^{(-1)} \propto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

- Under the convention used by the paper, the MPS matrices are [1]

$$\boxed{\mathcal{M}^{[n]+1} = \sqrt{\frac{2}{3}} \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \quad \mathcal{M}^{[n]0} = \frac{1}{\sqrt{3}} \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \quad \mathcal{M}^{[n]-1} = \sqrt{\frac{2}{3}} \left(\begin{array}{cc} 0 & -1 \\ 0 & 0 \end{array} \right)}$$

[1] MPS matrices are undetermined up to a local change of basis of form $A \mapsto MAM^{-1}$. It is easy to see that the two sets of matrices are related by $\mathcal{M}^{(n)} = \sigma_x A^{(n)} \sigma_x$

Hidden string order

- The AKLT state has a hidden anti-ferromagnetic order

$$\cdots 1, 0, 0, 0, -1, 0, 1, 0, 0, -1 \cdots$$

  i.e. successive non-zero spins must be alternating

- In MPS picture, the matrix element of states of form $|\cdots \pm 1, 0, 0, 0, 0, \pm 1 \cdots\rangle$ is a product of form

$$(\cdots)\sigma_{\pm}\sigma_z^n\sigma_{\pm}(\cdots)$$

- This is obviously zero because $\{\sigma_{\pm}, \sigma_z\} = 0$ and $\sigma_{\pm}^2 = 0$

Quantized Berry phase

- The Berry phase of a **periodic** 1D AKLT state can be calculated by twisting one of the covalent bonds

$$|10\rangle - |01\rangle \to |10\rangle - e^{i\theta}|01\rangle$$

- The Berry phase is then defined as

$$\gamma = -i\int_0^{2\pi} \frac{\langle\psi_\theta|\partial_\theta|\psi_\theta\rangle}{\langle\psi_\theta|\psi_\theta\rangle} d\theta$$

- Thermodynamic limit, $\boxed{\gamma = \pi}$ [**?**] [2]

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[2] According to my understand, this phase is quantized (though I can't follow the exact argument), so does it mean that showing $\gamma = \pi$ in thermodynamic limit is the same as showing it for all finite length?

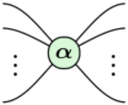
G.S. degeneracies of Spin-1 AKLT

- Replace each spin-1 site with a triplet: $|\uparrow\uparrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), |\downarrow\downarrow\rangle$
- For any adjacent site $\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$, the combined states lies in $\mathbf{1} \oplus \mathbf{3}$ iff there's a valence bond (singlet) between adjacent sites
- The G.S. forms valence bond on adjacent sites
- If the chain is periodic, only one G.S.
- If the chain has open ends, then we have 2 free ends so there are 4 degenerate G.S.
- *Question: I have written down the CG decomposition of the two site case and notice something weird that I don't know how to explain*

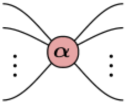
ZXH Calculus

ZXH Calculus

- Like MPS, a diagrammatic representation of tensors
- Equipped with *spiders* with a set of rewriting rules



$$:= \ |0\cdots 0\rangle\langle 0\cdots 0| + e^{i\alpha}\,|1\cdots 1\rangle\langle 1\cdots 1|$$



$$:= \ |+\cdots +\rangle\langle +\cdots +| + e^{i\alpha}\,|-\cdots -\rangle\langle -\cdots -|$$

- There are 3 types of spiders : Z-spider (light/green), X-spider (dark/red), and H-spider (rectangle box)
- Each spider has *n* inputs, *m* output, and a phase α

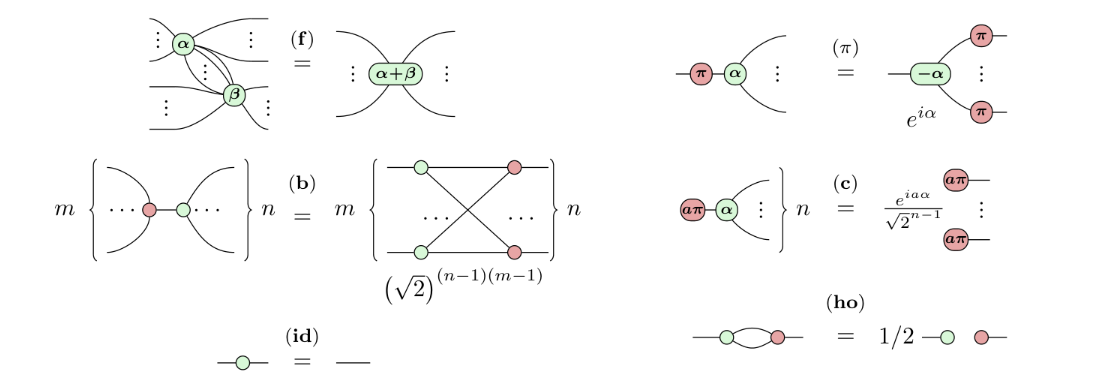
$$[Z(\alpha)]^{j_1 j_2 \cdots j_m}_{i_1 i_2 \cdots i_n} = \begin{cases} 1 & i_1 = i_2 = \cdots = j_1 = j_2 \cdots = j_m = 0 \\ e^{i\alpha} & i_1 = i_2 = \cdots = j_1 = j_2 \cdots = j_m = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$[X(\alpha)]^{j_1 j_2 \cdots j_m}_{i_1 i_2 \cdots i_n} = \frac{1}{2^{(n+m)/2}} \begin{cases} 1 + e^{i\alpha} & (\oplus_\alpha i_\alpha) \oplus (\oplus_\beta j_\beta) = 0 \\ 1 - e^{i\alpha} & (\oplus_\alpha i_\alpha) \oplus (\oplus_\beta j_\beta) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$[H(\alpha)]^{j_1 j_2 \cdots j_m}_{i_1 i_2 \cdots i_n} = \begin{cases} a & i_1 = i_2 = \cdots = j_1 = j_2 \cdots = j_m = 1 \\ 1 & \text{otherwise} \end{cases}$$ [3]

[3] When α is left unspecified, it is understood that $\alpha = 0$ for Z and X, and $\alpha = -1$ for H

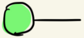
- ZX calculus also comes with a set of rewriting rules



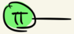
- and etc. etc.
- A comprehensive introduction of these rules can be found in [?]. For reference, I have listed these rules in the appendix of the slides.

Example - States

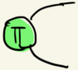
$\frac{1}{\sqrt{2}}$  $= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = |0\rangle$

$\frac{1}{\sqrt{2}}$  $= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

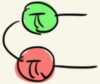
$\frac{1}{\sqrt{2}}$  $= \frac{1}{\sqrt{2}}(|+\rangle + e^{i\pi}|-\rangle) = |1\rangle$

$\frac{1}{\sqrt{2}}$  $= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$

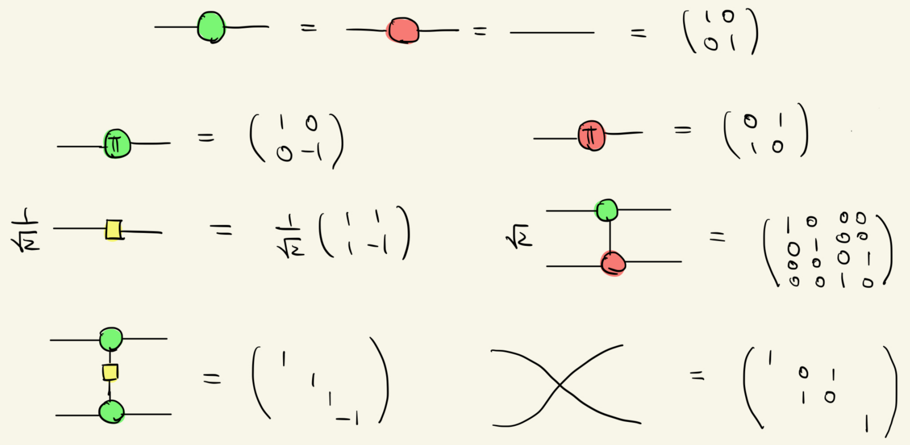
$\frac{1}{\sqrt{2}}$  $= \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

$\frac{1}{\sqrt{2}}$  $= \frac{|00\rangle - |11\rangle}{\sqrt{2}}$

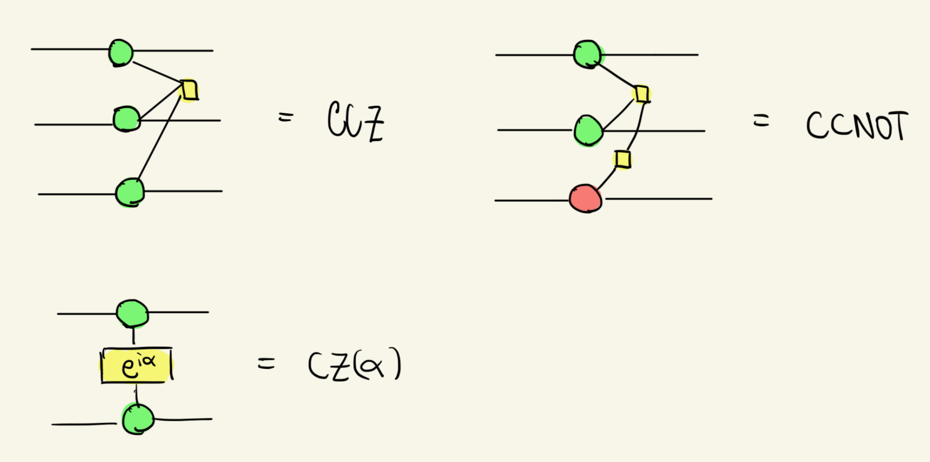
$\frac{1}{\sqrt{2}}$  $= \frac{|01\rangle + |10\rangle}{\sqrt{2}}$

$\frac{1}{\sqrt{2}}$  $= \frac{|01\rangle - |10\rangle}{\sqrt{2}}$

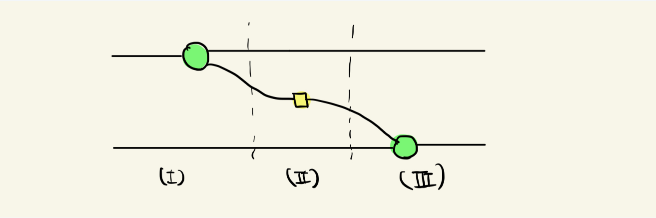
Example - Circuit gates



Example - Control phase gates



Example - Convert diagrams to matrix



$$\underbrace{\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0&0&0\\0&0&0&1\end{pmatrix}\right]}_{(III)}\underbrace{\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&1\\1&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]}_{(II)}\underbrace{\left[\begin{pmatrix}1&0\\0&0\\0&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]}_{(I)}$$

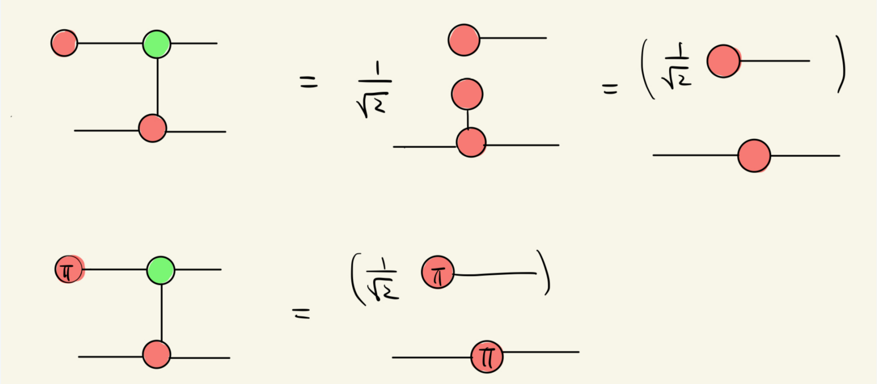
$= \mathrm{diag}(1,1,1,-1) =$ Control-Z

ZX calculus over traditional MPS

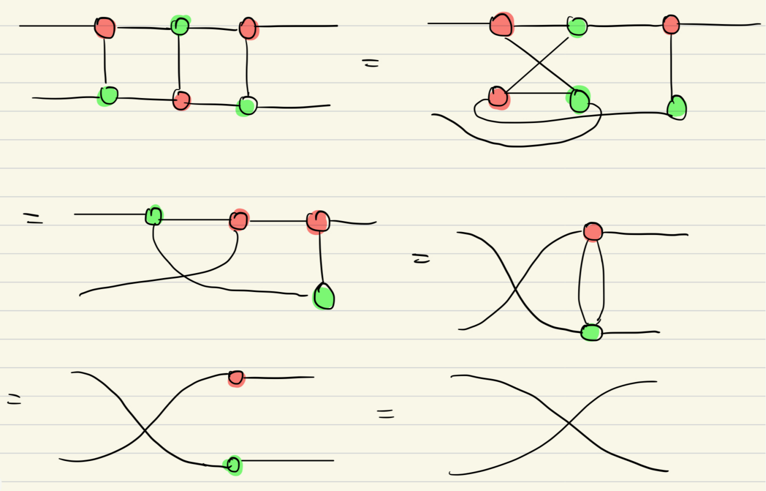
- **Complete**: Any equality of matrices with powers of *n* can be derived purely diagrammatically using the rewritting rules[4]
- **Only connectivity matters**: Two diagrams represent the same circuit if the underlying graphs are the same
- **Programmatic graph reduction**: Calculations can be done <u>automatically</u> and <u>exactly</u> by open source graph reduction packages, e.g. PyZX

[4] For pedagogical introduction to ZX rewritting rules, see [**?**]

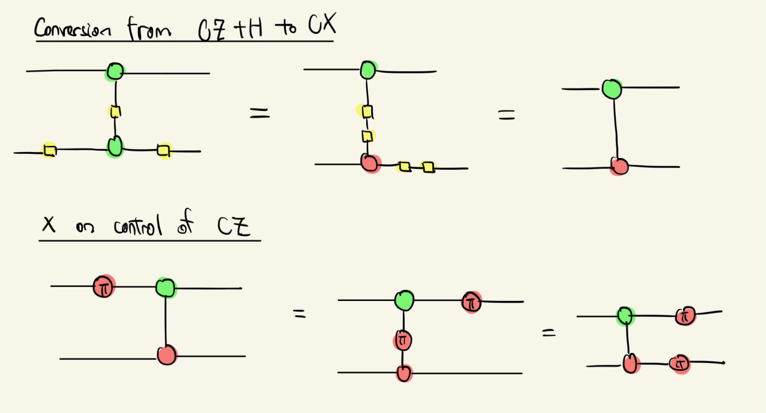
Example - Applying CNOT to states



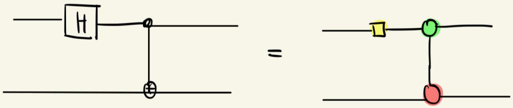
Example - 3 CNOT = Swap



Example - CX and CZ

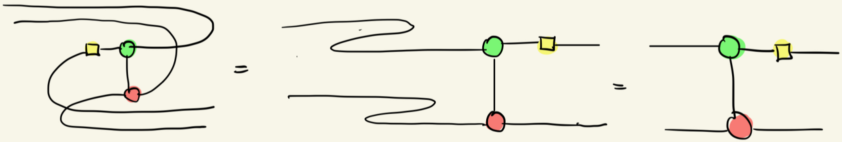


Example - Bell state preparation & Inverse



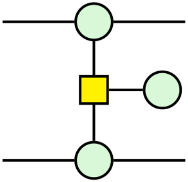
To get the inverse : (1) Turn $\alpha \to -\alpha$ for all phase

(2) Do a transpose



Example - Project away $|11\rangle$

$$\frac{1}{2}$$



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

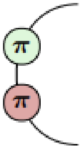
5

[5] Note the control nature of the multi-leg H-box

1D AKLT as ZX diagrams

AKLT GS - Covalent bonds

- Take bell state, apply X_2 and Z_1

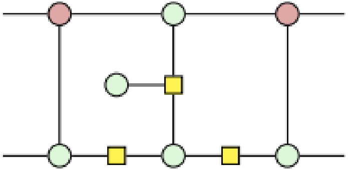


$$= |01\rangle - |10\rangle \,.$$

AKLT GS - Spin-1 projector

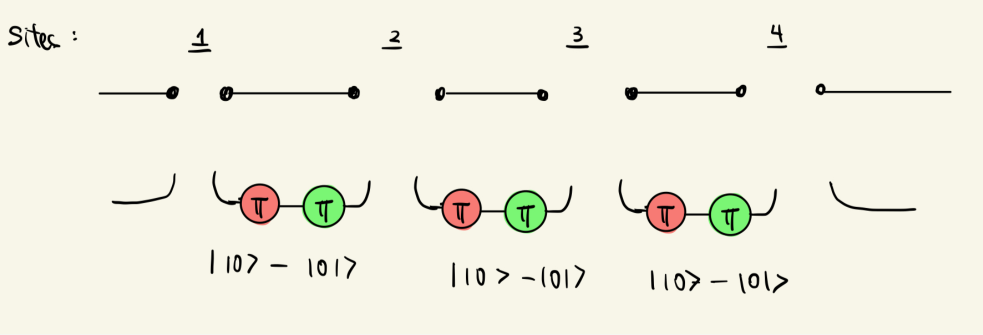
- The Bell state preparation circuit $|11\rangle \rightarrow |\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$
- Strategy: (Bell prep)$^\dagger \circ$ (Project out $|11\rangle$) $\circ$ (Bell prep)†

$\frac{1}{2}$

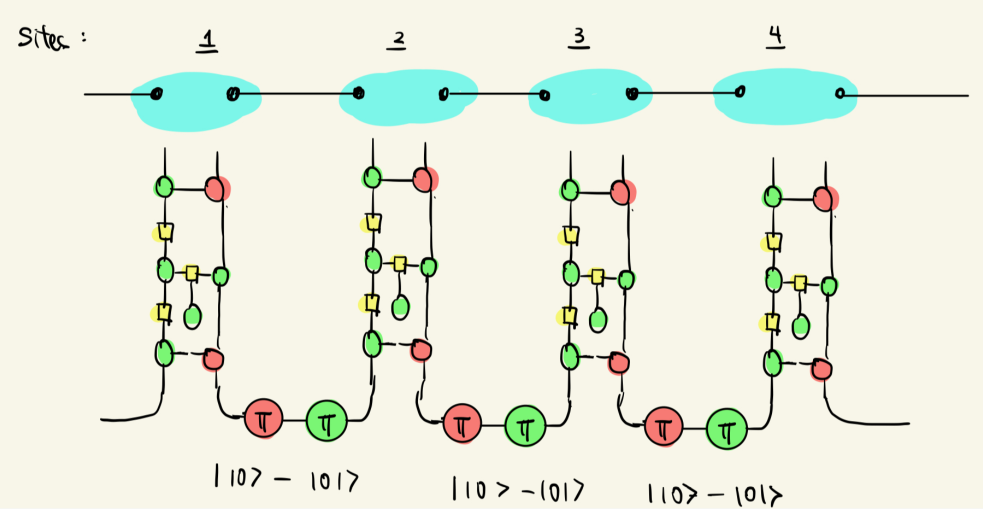


$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To construct the AKLT state, we first write down the valence bond pairs

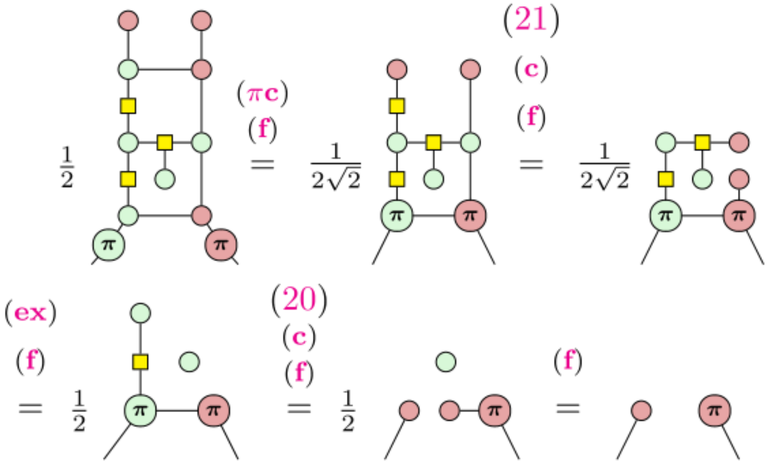


We then apply the porjector to project onto the spin-1 subspace



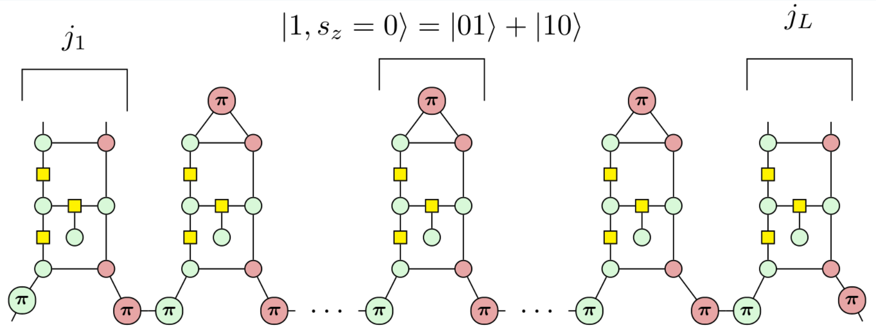
The dangling wires at the edge of the chain signifies the 4-fold degeneracies

Recovering MPL matrices by contraction

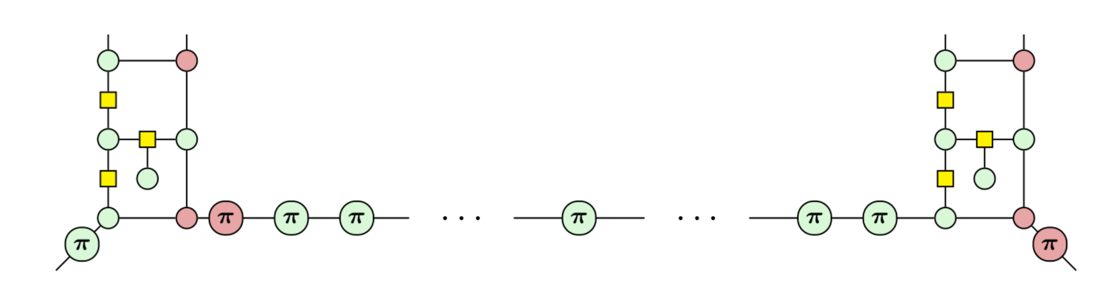


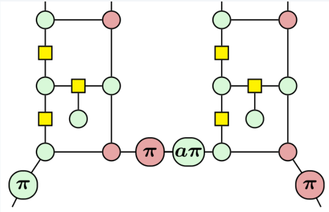
$$= 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{6}} M^{[n]+1}$$

Demonstrating hidden string order



Simplify the repeating block in the middle, we get the following figure





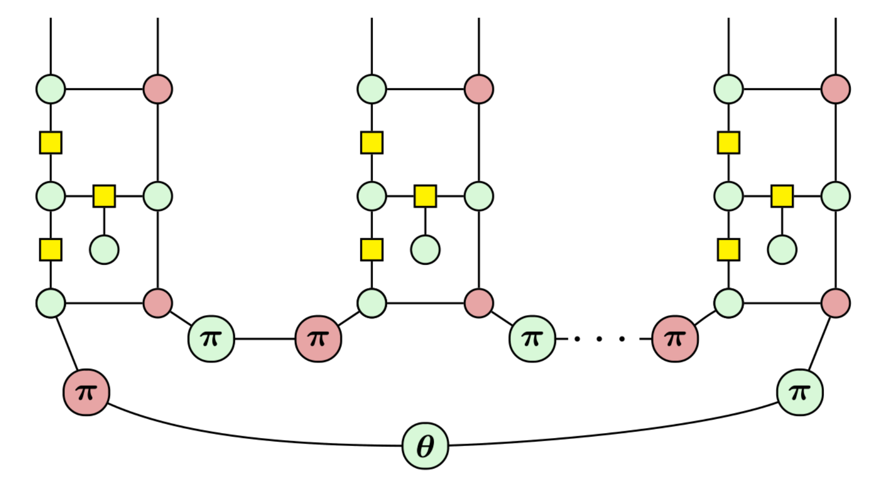
Plugging in ± 1 on both sides shows that the state vanishes if $j_1 = j_L$ does not otherwise (Work out an example that gives 0)

Berry phase computation

- Recall the Berry phase is calculated using

$$\gamma = -i \int_0^{2\pi} \frac{\langle \psi_\theta | \partial_\theta | \psi_\theta \rangle}{\langle \psi_\theta | \psi_\theta \rangle} d\theta$$

  where the chain is twisted at the end
- We calculate the phase directly by representing both $|\psi_\theta\rangle$ and $\langle\psi_\theta|$ as ZX diagrams and doing all the contractions diagrammatically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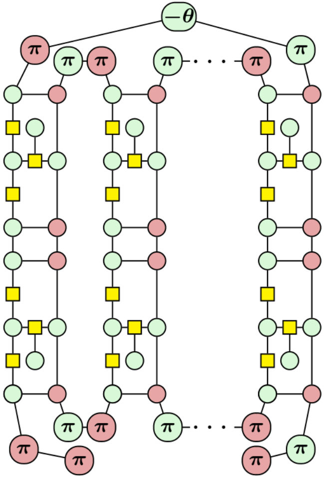
The integral is therefore

$$\gamma = (-i)\int_0^{2\pi} \frac{ie^{i\theta}}{2\langle\psi_\theta|\psi_\theta\rangle}$$



$$d\theta$$

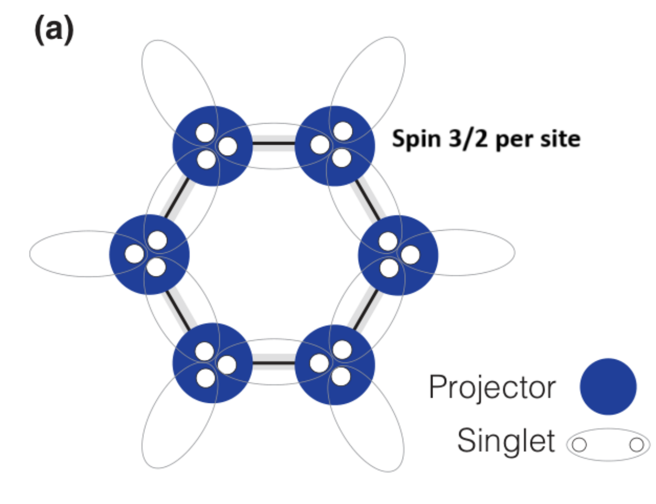
Skipping the simplification steps, the result can be evaluated exactly as

$$\begin{aligned}\gamma &= \frac{1}{2}\int_0^{2\pi} \frac{2(-1)^N e^{i\theta} + 3^N + (-1)^N}{2(-1)^N \cos\theta + 3^N + (-1)^N} d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \frac{g e^{i\theta} + 1}{g\cos\theta + 1} d\theta = \boxed{\pi}\end{aligned}$$

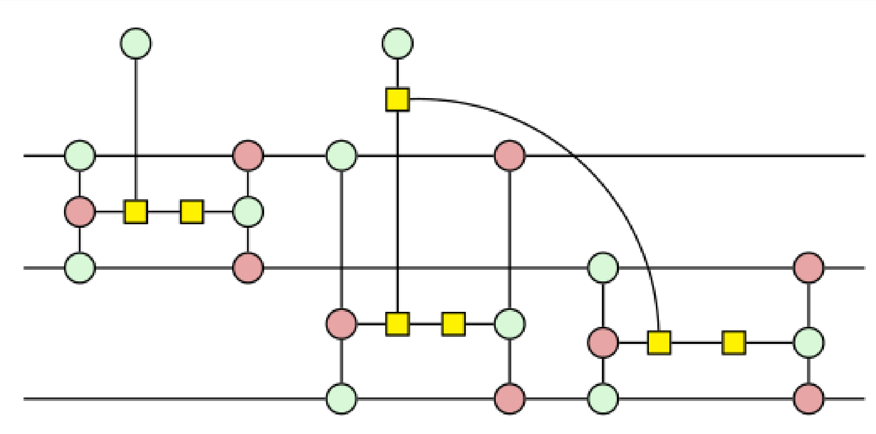
where $g = \frac{2(-1)^N}{3^N + (-1)^N}$, and N is the length of the chain. Thus, the direct computation shows that $\boxed{\gamma = \pi}$ holds for all finite length 1D AKLT chains.

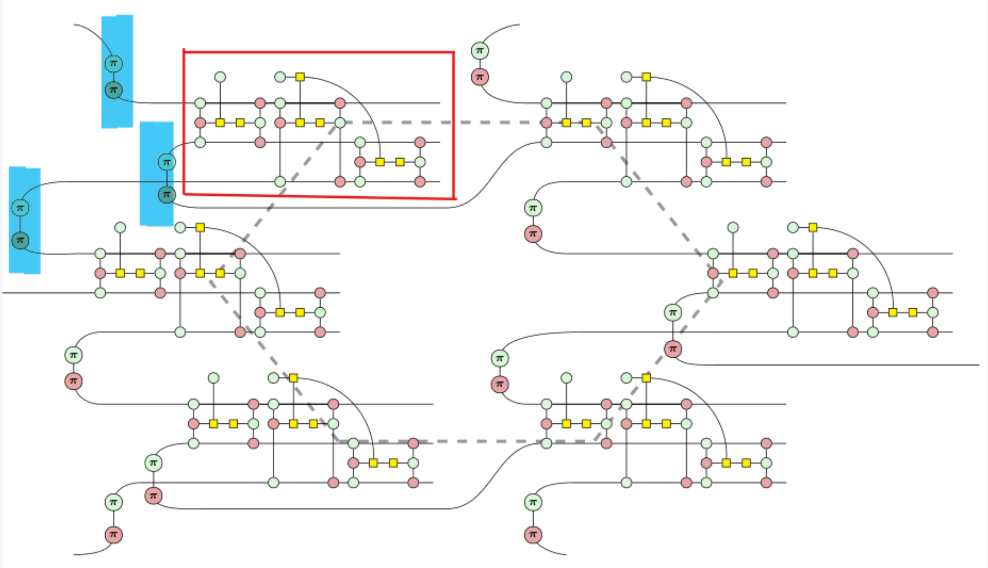
2D AKLT States as ZX diagram

2D AKLT states on honeycomb lattice



Spin-3/2 Projector




π

Previously known facts about this states

- The state can be reduced into a graph state under suitable measurement operations [**?**]
- The measurement is a joint measurement of the 3 qubit on each site

$$
\begin{aligned}
E_z &= \frac{2}{3}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right)\\
E_x &= \frac{2}{3}\left(|+++\rangle\langle +++| + |---\rangle\langle ---|\right)\\
E_y &= \frac{2}{3}\left(|iii\rangle\langle iii| + |-i-i-i\rangle\langle -i-i-i|\right)
\end{aligned}
$$

where $|0\rangle, |+\rangle, |i\rangle$ are the eigenstates of the Z, X, Y operators [6]

[6] Note that $E_x^\dagger E_x + E_y^\dagger E_y + E_z^\dagger E_z = P_{3/2}$

Graph reduction rules:

- After measurement is performed, apply the following graph reduction rules
  (i) Merge neighboring sites with same POVM outcomes
  (ii) Cut pairs of edges that connects two vertices
- The merged sites are called a *domain* and represent a single-logical qubit

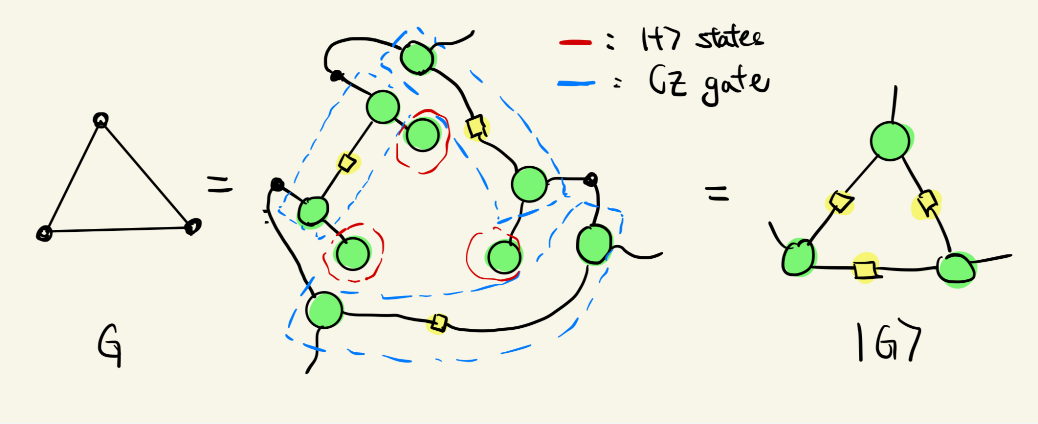
Reasons for the rules to hold:

1. Neighboring sites are antiferromagnetic, so if both of them are $S_a = \pm 3/2$, one can only have combinations of form $|3/2, -3/2, \cdots\rangle_{12}$ or $|-3/2, 3/2, \cdots\rangle_{12}$
2. Let m be the multiplicity of an edge, the inferred graph state stabilizer generators is of form $X_u Z_v^m$.[7] Since $Z^2 = I$, we can remove all but one edge.

Universal Quantum Resource: The construction above gives a cluster state in thermodynamic limit, which is a univeral quantum resource.

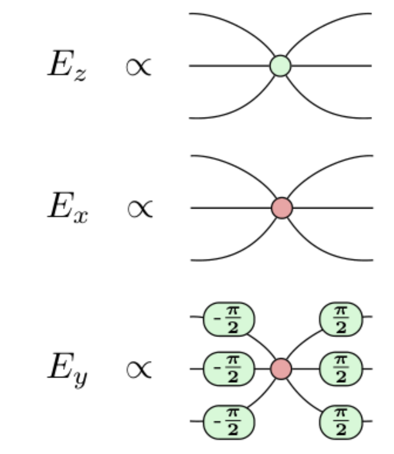
[7] To be honest I can't follow this part of the argument

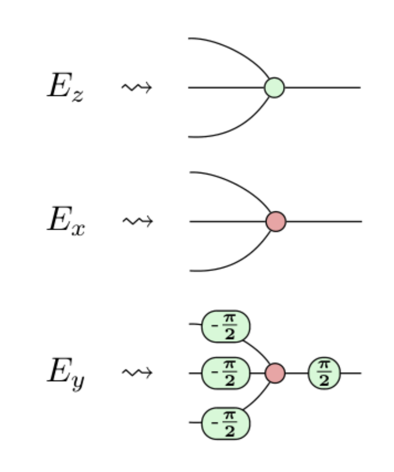
Graph state in ZX calculus



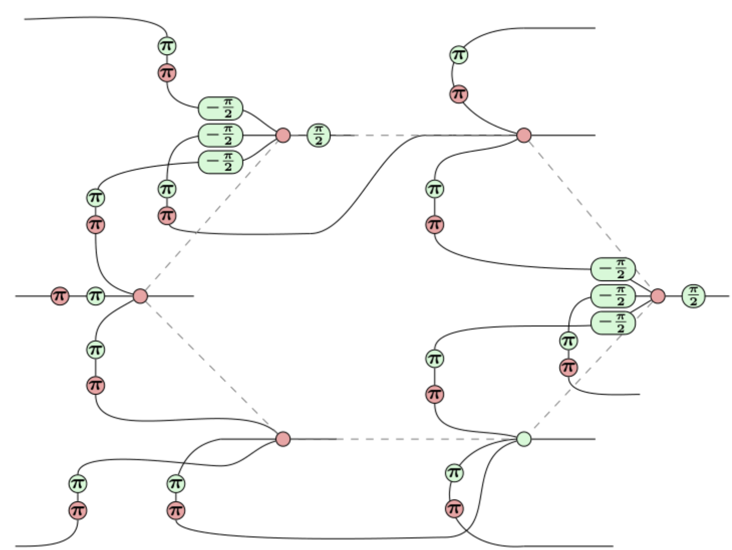
Graph state reduction proof (ZX calculus):

We first write down the POVM operators





All of these operators are symmetric under permutation, so they ”annihilate” the spin-3/2 projector and connects to the singlet directly

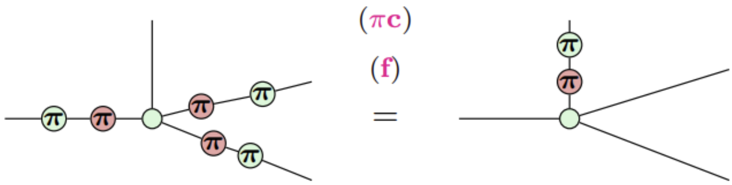


Using known results on Clifford states:

- The diagram only contains contain any higher-arity H-boxes, and the only phases that appear are multiples of $\pi/2$
- It is a so-called Clifford state, which can be presented as a graph state with single-qubit Clifofrd unitaries on its outputs. QED.
- We can also do it explicitly

Explicit graph reduction

- First rewrite the edges so that we have a $(\pi) - (\pi)$ in the output edges

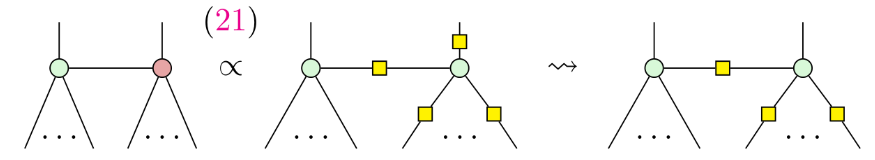


(88)

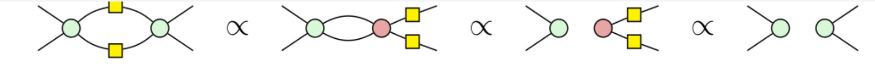
With that we can redefine our basis and eliminate those phases.

- We can then merge all the vertices with same color, this is the same as Rule 1 in [?]

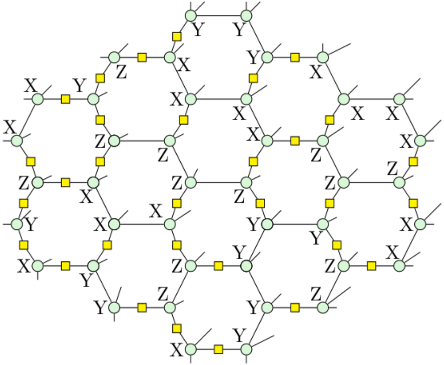
- We can then pull out hadamard gates from red vertices to make our states manifestly "graph-like"



- We can then remove edges with multiplicity $m > 1$ by using the Hopf rule



This is basically Rule 2 in [**?**]



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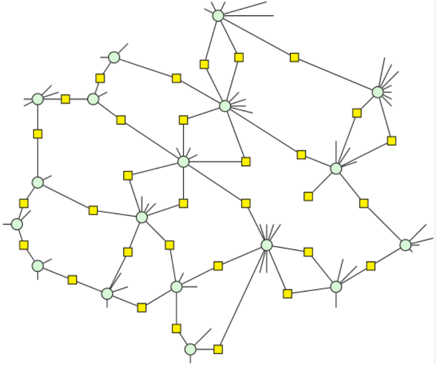

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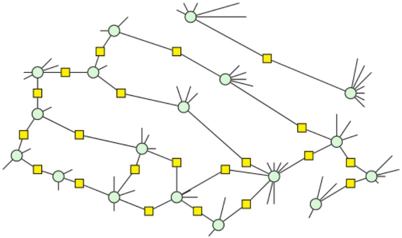

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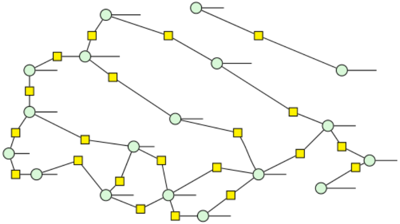


Using PyZX

About PyZX

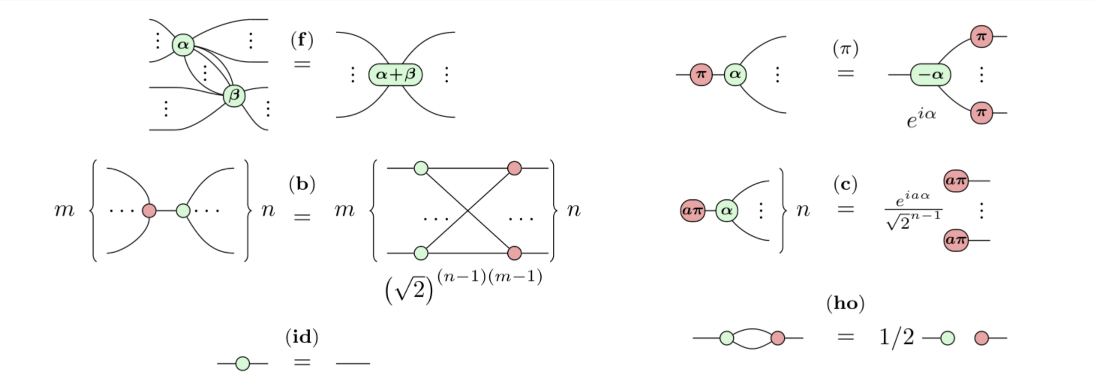
- PyZX is an open-source packages for ZXH calculus
- Allows you to construct circuits and graphs directly
- Also supports automatic graph simplification
- Repo link: https://github.com/Quantomatic/pyzx

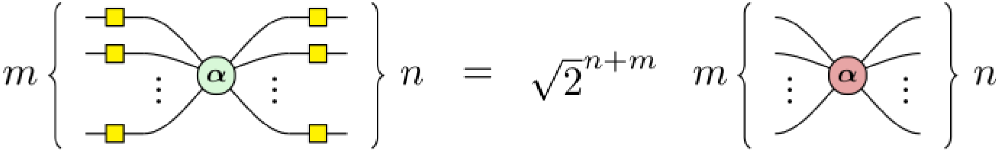
Graphical reduction with PyZX

”Talk is cheap. Show me the code.” - Linus Torvalds

Appendix

ZX calculus rewritting rules





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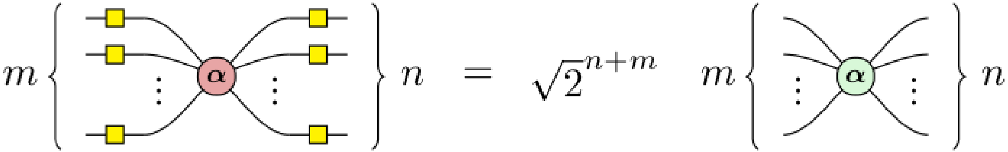

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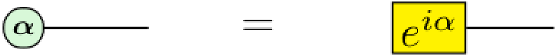

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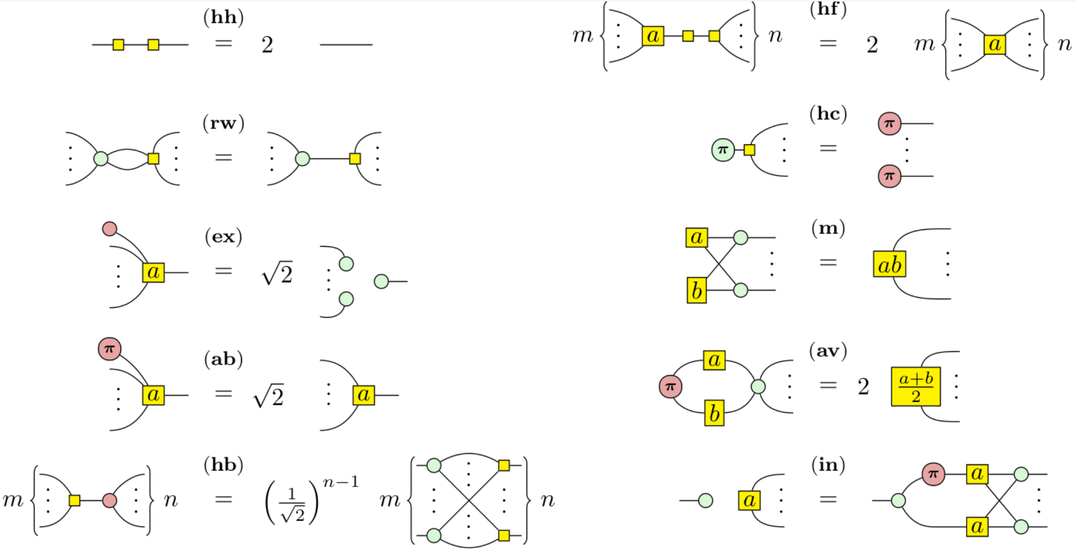
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(ex)
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(ab)
(av)
(hb)
(in)

Constructor of projector of higher spin

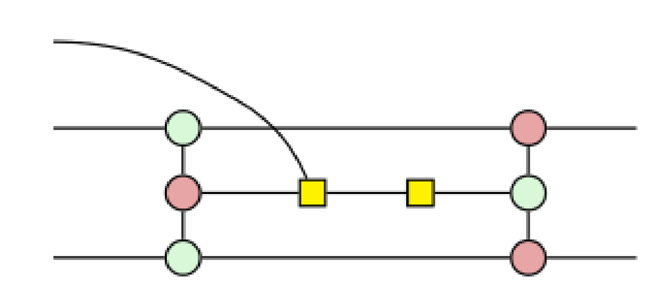
- To represent spins $N/2$ sites, we stack N spin-1/2 and project onto the symmetric subspace[8]
- This is done by the symmetrizing projector

$$P = \frac{1}{n!} \sum_{\sigma \in S_N} U_\sigma$$

where $U_\sigma \left|x_1 x_2 \cdots\right\rangle = \left|x_{\sigma(1)} x_{\sigma(2)} \cdots\right\rangle$

[8] Recall the highest spin subspace of N spin-1/2 is all the states that is symemtric under permutation of spin indices

- Basically we want a coherent superposition of swap wires
- This is done by CSWAP which maps $|0xy\rangle \mapsto |0xy\rangle$ and $|1xy\rangle \mapsto |1yx\rangle$.



- Higher projectors can be constructed inductively by swapping additional qubit with all other qubits[9]

[9] There are some subtleties as to how to make sure only at most one CSWAP is fired each time. Details are presented in Appendix C of the paper.

Higher spin representation of SU(2)

- One way to represent arbitary spins is to stack N spin-1/2

$$\overbrace{\mathbf{2} \otimes \mathbf{2} \cdots \otimes \mathbf{2}}^{N \text{ copies}}$$

- The representation of $\mathfrak{su}(2)$ in this space is basically given by

$$\hat{S}_z = \sum_{i=1}^{N} \hat{S}_z^{(i)} \qquad \hat{S}_\pm = \sum_{i=1}^{N} \hat{S}_x^{(i)} \pm i\hat{S}_y^{(i)}$$

- To construct a spin-N/2 subspace, we take the highest state $|\uparrow \cdots \uparrow\rangle$ and apply $\hat{S}_-$ repeatedly
- The highest state is obviously symmetric under permutation of spin label
- For any state obtained by applying $\hat{S}_-$, we note that

$$\text{Perm}(\hat{S}_- |\psi\rangle, \sigma) = \text{Perm}(\hat{S}_-, \sigma)\text{Perm}(|\psi\rangle, \sigma)$$

  where $\sigma \in S_N$ is a permutation of spin index
- Since both $\hat{S}_-$ and $|\psi\rangle$ are totally symmetric, the state $\hat{S}_- |\psi\rangle$ must also be a totally symmetric state
- Therefore, the subspace spanned by totally symmetric states is the highest spin subspace.

Positive operator valued measure (POVM)

- Let $\{\mathcal{O}_i\}$ be a set of positive semi-definite operators. The set of operator is said to form a positive operator value value measure (POVM) if they satisfy the completeness relations

$$\sum_i \mathcal{O}_i = I$$

- Since it is always possible to write $\mathcal{O}_i$ in terms of Kraus operators $E_i^\dagger E_i$, we can also express our relations in terms of Kraus operators

$$\sum_i E_i^\dagger E_i = I$$

- E_i are not unique because $E_i \mapsto UE_i$ where U is unitary gives you another set of valid Kraus operators

- The post-measurement state is not uniquely determined by $\mathcal{O}_i$
- However, given E_i, the post-measurement state is given by

$$|\psi\rangle \mapsto \frac{E_i |\psi\rangle}{\langle\psi|E_i^\dagger E_i|\psi\rangle}$$

- Note that the post-measurement state is not uniquely defined by $\mathcal{O}_i$ as E_i is undetermined up to a unitary U
- This agree with the notion that we need to supply in addition to $\mathcal{O}_i$ the exact form of E_i to uniquely determine the post-meausrement state

Graph states

- A simple graph $G = (V, E)$ consist of a set of vertices V and a set of edges $E \subseteq \{\{x, y\} \in V \times V, x \neq y\}$
- Let G be a graph, and $|G\rangle_0 = |+\rangle^{\otimes}$ be the empty graph state. The graph state $|G\rangle$ is defined as

$$|G\rangle = \prod_{(i,j)\in E} \mathcal{U}_{ij} |G_0\rangle$$

  where $\mathcal{U}_{ij} = \mathrm{CZ}(i, j)$ with $i, j \in V$
- Equivalently, graph states can be understood using the stabilizer formalism

- A *cluster state* is a graph state where the underlying graph is a lattice
- Any 2D cluster state is a universal resource for measurement-based quantum computation
-

- The spin operator (with $\hbar$ stripped) for spin-1 systems are

$$S_x = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \quad S_y = \begin{pmatrix} 0 & -i\frac{\sqrt{2}}{2} & 0 \\ i\frac{\sqrt{2}}{2} & 0 & -i\frac{\sqrt{2}}{2} \\ 0 & i\frac{\sqrt{2}}{2} & 0 \end{pmatrix} \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Therefore the rotation operator of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is represented as

$$U_x(\pi) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad U_y(\pi) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad U_z(\pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- Note that these are the matrix representation in the spin-1 z basis

- Therefore, if we apply these transformations individually to each site, we have

$$A_i^{\sigma} \mapsto \sum_{\sigma=0,\pm 1} U_{\sigma}^{\sigma'} A_i^{\sigma}$$

  where σ denote the σ-th component of the spin-1 spinor and i is the site index.
- In particular

More examples using ZX rewrite rules

Example: CNOT swapping

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Thank you for your attention

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Presented by Yan Mong Chan

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