PSP Coursework

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Solutions

Q1: Random Variables

a) **We place uniformly at random $n = 200$ points in the unit interval $[0, 1]$. Denote by the random variable X the distance between 0 and the first random point on the left.**

i) **Find the probability distribution function $F_X(x)$.**

Note: we use $F_X(x)$ to refer to the PDF and not the CDF in this question only. Since n points are placed uniformly on $[0, 1]$, the probability that no point lies within the interval $[0, x]$ with $x \in [0, 1]$ is given by:

$$\begin{aligned} F_X(x) &= \text{Probability of no point in } [0, x] \\ &= (1 - x)^n \\ &= (1 - x)^{200} \end{aligned}$$

Where $0 \le x \le 1$. Here, points are placed such that they are i.i.d..

ii) **Derive the limit as $n \to \infty$ and comment on your expression.**

We revise our notation to parametrise the number of samples drawn.

$$F_X(x; n) = (1 - x)^n$$

For large n, we take the limit as $n \to \infty$, observe that for any fixed $x \in (0, 1]$:

$$F_X(x; n) \to 0$$

However, for $x = 0$:

$$\lim_{n\to\infty} F_X(0; n) = \lim_{n\to\infty} (1 - 0)^n = 1$$

This is because for the uniform distribution, the probability of selecting a single point ($x = 0$) is always zero. So we are sure with absolute certainty that there will be no points directly on 0. In conclusion, when $n \to \infty$:

$$F_X(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \in (0, 1] \\ 0 & \text{otherwise} \end{cases}$$

b) **The random variable X is uniform in the interval $(0, 1)$. Find the density function of the random variable $Y = -\ln X$.**

Since X is uniform on $(0, 1)$, the probability density function (PDF) of X is:

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

The transformation $Y = -\ln X$ implies that $X = e^{-Y}$. Taking the derivative:

$$\frac{dX}{dY} = -e^{-Y}.$$

The support of Y is derived as follows:

$$X \in (0,1) \implies -\ln X \in (0,\infty) \implies Y \in (0,\infty)$$

Using the change of variables formula:

$$f_Y(y) = f_X(x)\left|\frac{dX}{dY}\right|, \quad x = e^{-y}, \quad \left|\frac{dX}{dY}\right| = e^{-y}$$

Substituting $f_X(x) = 1$ for $x \in (0,1)$, we get:

$$f_Y(y) = e^{-y}, \quad y > 0$$

Therefore, the PDF of Y is:

$$f_Y(y) = \begin{cases} e^{-y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

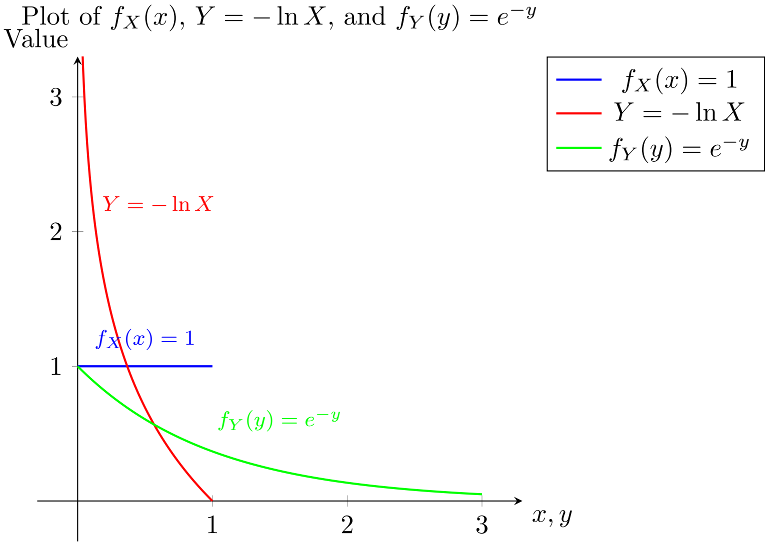

Figure 1: Combined plot of $f_X(x)$, $Y = -\ln X$, and $f_Y(y) = e^{-y}$. Note that $f_Y(y)$ is the reflection of Y over the line $y = x$

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c) **X and Y are independent, identically distributed (i.i.d.) random variables with common PDF $f_X(x) = e^{-x}$, $x > 0$. Find the PDFs of the following random variables:**

i) $Z = X \cdot Y$**.**

Because X and Y are independent, the PDF of $Z = X \cdot Y$ is just the product of both PDFs. The joint distribution of X and Y is:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = e^{-x}e^{-y}, \quad x,y > 0.$$

For $Z = X \cdot Y$, we transform the PDF using the Jacobian Determinant for 1D transformation:

$$f_Z(z) = \int_0^\infty f_{X,Y}\left(x, \frac{z}{x}\right)\left|\frac{\partial y}{\partial z}\right| dx,$$

where $y = \frac{z}{x}$. The Jacobian determinant accounts for the change of variables:

$$\frac{\partial y}{\partial z} = \frac{1}{x}.$$

Thus, the absolute value of the Jacobian determinant is:

$$\left|\frac{\partial y}{\partial z}\right| = \frac{1}{x}.$$

Substituting this into the integral, we have:

$$f_Z(z) = \int_0^\infty f_{X,Y}\left(x, \frac{z}{x}\right)\frac{1}{x}dx,$$

where $f_{X,Y}(x,y) = f_X(x)f_Y(y) = e^{-x}e^{-y}$, $x, y > 0$.

$$f_Z(z) = \int_0^\infty f_{X,Y}(x, z/x)\frac{1}{x}\,dx.$$

Substituting $f_{X,Y}(x, z/x) = e^{-x}e^{-z/x}/x$, the integral becomes:

$$f_Z(z) = \int_0^\infty e^{-x}e^{-z/x}\frac{1}{x}\,dx, \quad z > 0.$$

Solving this integral (using substitution or standard tables):

$$f_Z(z) = 2K_0(2\sqrt{z}), \quad z > 0,$$

where K_0 is the modified Bessel function of the second kind.

ii) $Z = \frac{X}{Y}$**.** The CDF of Z is defined as:

$$F_Z(z) = P\left(\frac{X}{Y} \leq z\right) = P(X \leq zY).$$

Since X and Y are independent, we can express this probability as an integral over the PDF of Y:

$$F_Z(z) = \int_0^\infty P(X \leq zy)f_Y(y)\,dy.$$

Given that X is exponential, the probability $P(X \leq zy)$ becomes:

$$P(X \leq zy) = \int_0^{zy} e^{-x}\,dx = 1 - e^{-zy}.$$

Substituting back into the expression for $F_Z(z)$:

$$F_Z(z) = \int_0^\infty \left(1 - e^{-zy}\right)e^{-y}\,dy.$$

Split the integral into two parts:

$$F_Z(z) = \int_0^\infty e^{-y}\,dy - \int_0^\infty e^{-y}e^{-zy}\,dy = I_1 - I_2.$$

Compute I_1 and I_2:

(a) **First Integral I_1:**

$$I_1 = \int_0^\infty e^{-y}\,dy = \left[-e^{-y}\right]_0^\infty = 0 - (-1) = 1.$$

(b) **Second Integral I_2:**

$$I_2 = \int_0^\infty e^{-y(1+z)}\,dy = \frac{1}{1+z}.$$

This follows from the standard integral of the exponential function:

$$\int_0^\infty e^{-ky}\,dy = \frac{1}{k}, \quad \text{for } k > 0.$$

Therefore, the CDF $F_Z(z)$ becomes:

$$F_Z(z) = 1 - \frac{1}{1+z} = \frac{z}{1+z}, \quad z \geq 0.$$

iii) **Differentiate to Find the PDF $f_Z(z)$:**
The PDF is the derivative of the CDF:

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz}\left(\frac{z}{1+z}\right).$$

Compute the derivative using the quotient rule:

$$f_Z(z) = \frac{(1+z)(1) - z(1)}{(1+z)^2} = \frac{1}{(1+z)^2}, \quad z \geq 0.$$

Thus, the PDF of Z is:

$$f_Z(z) = \frac{1}{(1+z)^2}, \quad z \geq 0.$$

iv) $Z = \max(X, Y)$**.**
The CDF of Z is:

$$F_Z(z) = P(\max(X,Y) \leq z) = P(X \leq z \text{ and } Y \leq z) = P(X \leq z)P(Y \leq z),$$

since X and Y are independent. Substituting the CDFs of X and Y:

$$F_Z(z) = (1 - e^{-z})^2, \quad z > 0.$$

Differentiating to get the PDF:

$$f_Z(z) = \frac{d}{dz} F_Z(z) = 2(1 - e^{-z})e^{-z}, \quad z > 0.$$