Here is an 'informal' definition of a limit of a function.

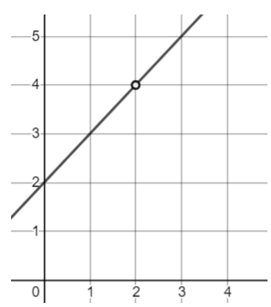
> Suppose $f(x)$ is defined when x is near the number a, that is, f is defined on some open interval that contains a except possibly at a itself. Then we write
>
> $$\lim_{x\to a} f(x) = L \qquad (\text{or } f(x) \to L \text{ as } x \to a)$$
>
> (read "the limit of $f(x)$, as x approaches a, equals L") if we can make the values $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

What does it mean by "x approaches a" or "x to be sufficiently close to a"? Here, a is an actual real number on a real number line (specifically the x-axis). Approaching a means to consider a sequence of real numbers x_1, x_2, x_3, ... such that x_2 is closer to a than x_1 is, and x_3 is closer to a than x_2 is, and etc. There are numerous ways of approaching a. However, for our practical purpose, we will use the following sequence to approach a throughout the course: $x_n^+ = a + \frac{1}{10^n}$ where $n \in \mathbb{N}$. For instance, if $a = 2$, then the sequence is $(2.1, 2.01, 2.001, 2.0001, \ldots)$. Note that each term in the sequence is greater than a, so the sequence is said to approach to a from the right (hand side). We would also like to approach a from the left (hand side), so we also consider the sequence $x_n^- = a - \frac{1}{10^n}$. For $a = 2$, the sequence is $(1.9, 1.99, 1.999, 1.9999, \ldots)$.

Example 1 Consider the function $f(x) = \frac{x^2-4}{x-2}$. Note that the domain is the set of all reals but 2, i.e. $D(f) = (-\infty, 2) \cup (2, \infty)$. The function value is undefined for $x = 2$ as $f(2) = \frac{0}{0}$ is undefined. We want to see if the function has a limiting value L as x approaches 2. To see numerically, we make the table of values using the sequences x_n^+ and x_n^-:

x	$f(x)$
2.1	$\frac{(2.1)^2-4}{2.1-2} = 4.1$
2.01	$\frac{(2.01)^2-4}{2.01-2} = 4.01$
2.001	$\frac{(2.001)^2-4}{2.001-2} = 4.001$
2.0001	$\frac{(2.0001)^2-4}{2.0001-2} = 4.0001$

x	$f(x)$
1.9	$\frac{(1.9)^2-4}{1.9-2} = 3.9$
1.99	$\frac{(1.99)^2-4}{1.99-2} = 3.99$
1.999	$\frac{(1.999)^2-4}{1.999-2} = 3.999$
1.9999	$\frac{(1.9999)^2-4}{1.9999-2} = 3.9999$



If we have to guess a limiting value of $f(x)$ as x approaches 2, it would certainly be $L = 4$. Hence, we write

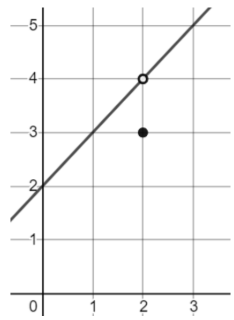
$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$$

We can verify that the limiting value is indeed 4 by observing the graph of the function near $x = 2$.

Example 2 Consider $g(x)$ defined piece-wisely as follows.

$$g(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$

The function $g(x)$ is basically $f(x)$ except that $g(2)$ is defined to be 3. The domain of $g(x)$ is $D(g) = (-\infty, \infty)$.



Suppose we want to find the limiting value of $g(x)$ as x approaches 2. A common mistake is to think that the limiting value of $g(x)$, as x approaches 2, is 3. The definition of the limit only concerns approaching 2, so the function value 3 of $g(x)$ at $x = 2$ is completely out of consideration.

Graphically, we can see that the limiting value of $g(x)$, as x approaches 2, is still 4. Thus,

$$\lim_{x \to 2} g(x) = 4$$

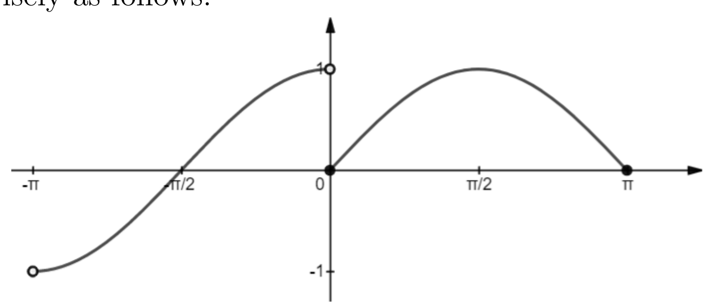
The limit of the function $g(x)$ exists as x approaches any other value. For instance, graphically, we can see that

$$\lim_{x \to 0} g(x) = 2, \quad \lim_{x \to 1} g(x) = 3, \quad \text{and} \quad \lim_{x \to 3} g(x) = 5$$

Example 3 Consider $h(x)$ defined piece-wisely as follows.

$$h(x) = \begin{cases} \cos(x) & \text{if } -\pi < x < 0 \\ \sin(x) & \text{if } 0 \leq x \leq \pi \end{cases}$$

The domain of $h(x)$ is $D(h) = (-\pi, \pi]$.



We want to find the limiting value of $h(x)$ as x approaches 0. As x approaches 0 from the left using the sequence $x_n^- = (-0.1, -0.01, -0.001, \ldots)$, we can see from the graph of $h(x)$ that the function values get close to 1. Although $h(0) = 0$, it is utterly irrelevant when finding the limit. However, when x approaches 0 from the right using the sequence $x_n^+ = (0.1, 0.01, 0.001, \ldots)$, the limiting value of $h(x)$ is 0. It is not obvious from the definition of the limit, but we require the **limit of a function to be unique single value**. Hence, if there are two different limiting values (one when approaching from the left and the other when approaching from the right), we say the limit **Does Not Exist** or $\lim_{x \to 0} h(x) = \text{DNE}$. From the graph, one can easily see that the following limits exist.

$$\lim_{x \to -\pi/2} h(x) = 0 \quad \text{and} \quad \lim_{x \to \pi/2} h(x) = 1$$

Does $h(x)$ have a limit as x approaches π? Not really. As the function $h(x)$ is not even defined when $x > \pi$, there are no function values to be inspected for the sequence $x_n^+ = (\pi + 0.1, \pi + 0.01, \pi + 0.001, \ldots)$. Hence, the limit does not exist as x approaches π.

One-Sided Limits

We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of** $f(x)$ **as** x **approaches** a (or the **limit of** $f(x)$ **as** x **approaches** a **from the left**) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *less than* a.

In the previous example, we note that the limit of the function $h(x)$ does not exist as x approaches 0. However, it has the left-hand limit. The graph suggests that the left-hand limit is

$$\lim_{x \to 0^-} h(x) = 1$$

We write

$$\lim_{x \to a^+} f(x) = L$$

and say the **right-hand limit of** $f(x)$ **as** x **approaches** a (or the **limit of** $f(x)$ **as** x **approaches** a **from the right**) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *greater than* a.

Again, in the previous example, although the limit of the function $h(x)$ as x approaches 0 does not exist, it has the right-hand limit. The graph suggests that the left-hand limit is

$$\lim_{x \to 0^+} h(x) = 0$$

Similarly, the limit does not exist as x approaches π and $-\pi$. However, the one-sided limits do exist.

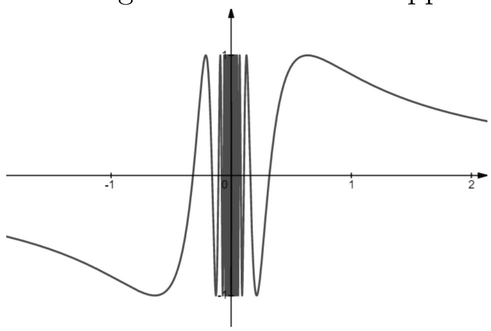
$$\lim_{x \to \pi^-} h(x) = 0 \quad \text{and} \quad \lim_{x \to -\pi^+} h(x) = -1$$

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \text{“} \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L\text{”}$$

In words, for the limit of the function $f(x)$ to exist as x approaches a, (i) the left-hand limit exists, (ii) the right-hand limit exists, and (iii) they agree. This also means that the limit of the function $f(x)$ does not exist if any of three criteria is not satisfied. In the previous example, the limit of the function $h(x)$ as x approaches 0 does not exist because it fails the condition (iii).

Q: How can a function fail to have a one-sided limit?
A: Crazy oscillation!

Example 4 Consider $c(x) = \sin(\frac{1}{x})$ with the domain $D(c) = (-\infty, 0) \cup (0, \infty)$. Does $c(x)$ have the right-hand limit as x approaches 0?



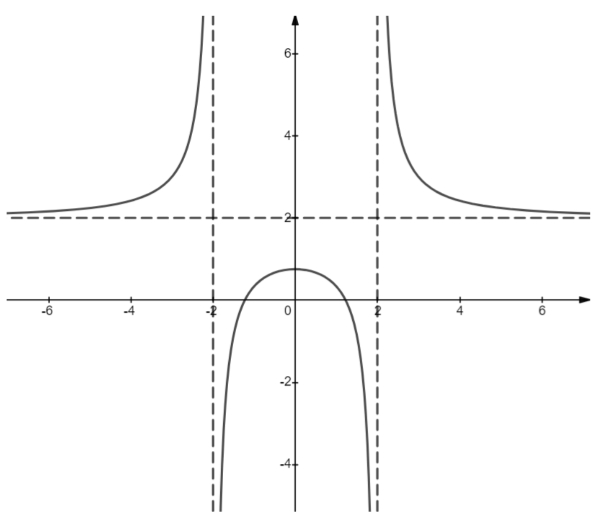
From the graph it is rather difficult to decide if the limiting value exist or not. You might consider zooming in to see, but it would not work as it gets even worse. Crazy oscillation! We should consider numerical computations.
Since we consider the right-hand limit, we use the sequence $x_n^+ = (0.1, 0.01, 0.001, 0.0001, \ldots)$.
Note that $c(\frac{1}{10^n}) = \sin(10^n)$.

x	$c(x)$
0.1	$\sin(10) \approx -0.5440$
0.01	$\sin(100) \approx -0.5064$
0.001	$\sin(1000) \approx 0.8269$
0.0001	$\sin(10000) \approx -0.3056$
0.00001	$\sin(100000) \approx 0.0357$
0.000001	$\sin(100000) \approx -0.3500$

Apparently the function values seem random values between -1 and 1. Hence, there is no particular real number that the function values get closer to as x approaches 0 from the right. So the right-hand limit of the function $c(x)$ as x approaches 0 does not exist. Similarly, we can also conclude that the left-hand limit does not exist.

Q: How else can a function fail to have a one-sided limit?
A: Increasing or decreasing without bound.

Example 5 Consider a rational function $r(x) = \frac{2x^2-3}{x^2-4}$. Its domain is $D(r) = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. It has two vertical asymptotes $x = -2$ and $x = 2$.



Note that the function values increase or decrease without bound when x approaches where the vertical asymptote is located. The function $r(x)$ has no limiting value as x approaches 2 from the right, for instance, because the function values increase without bound.

x	$r(x)$
2.1	14.1951
2.01	236.6883
2.001	1251.6876
2.0001	12501.6875
2.00001	125001.6875

Technically, the function $r(x)$ has no one-sided limit as x approaches -2 or 2.

Infinite Limits

Q: What is ∞?

First, it is not a number. In Calculus, it would be sufficient to consider ∞ as something that is larger than any real number. No matter how large real number you can think of, ∞ can beat it. Although ∞ is not a number, we get to use it to compare with real numbers. For instance, the U.S. national debt \$22,027,894,379,236 $< \infty$ is a fair expression in Calculus. A sequence of real numbers that increase without bound is said to **converge** to ∞. Similarly, $-\infty$ is something that is smaller than any real number. For instance, $-\infty < -123,456,789$. In the previous example, technically, we concluded that the function $r(x)$ has no one-sided limits as x approaches -2 or 2. If we take an advantage of the symbols ∞ or $-\infty$, we can let ∞ or $-\infty$ be a limiting value of the function values that are increasing or decreasing without bound. So

$$\lim_{x\to-2^-} r(x)=\infty,\quad \lim_{x\to-2^+} r(x)=-\infty,\quad \lim_{x\to2^-} r(x)=-\infty,\quad \text{and}\quad \lim_{x\to2^+} r(x)=\infty$$

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x)=\infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large we please) by taking x sufficiently close to a, but not equal to a.

Just like before,

$$\lim_{x\to a} f(x)=\infty \quad \text{if and only if} \quad \lim_{x\to a^-} f(x)=\infty \quad \text{and} \quad \lim_{x\to a^+} f(x)=\infty$$

Similarly,

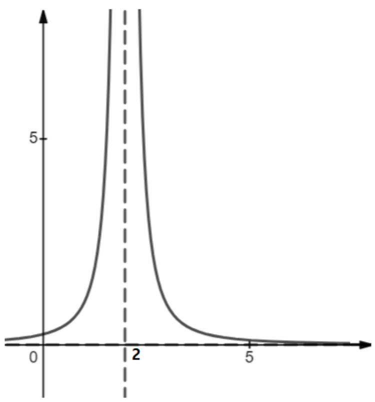
$$\lim_{x\to a} f(x)=-\infty \quad \text{if and only if} \quad \lim_{x\to a^-} f(x)=-\infty \quad \text{and} \quad \lim_{x\to a^+} f(x)=-\infty$$

The vertical line $x=a$ is called a **vertical asymptote** of the curve $y=f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a^-} f(x)=\infty \qquad \lim_{x\to a^+} f(x)=\infty \qquad \lim_{x\to a^-} f(x)=\infty$$

$$\lim_{x\to a^-} f(x)=-\infty \qquad \lim_{x\to a^+} f(x)=-\infty \qquad \lim_{x\to a^-} f(x)=-\infty$$

Example 6 Consider a rational function $t(x) = \frac{1}{(x-2)^2}$. Its domain is $D(t) = (-\infty, 2) \cup (2, \infty)$. It has one vertical asymptote $x = -2$.



x	$t(x)$
1.9	100
1.99	10000
1.999	1000000
1.9999	100000000

x	$t(x)$
2.1	100
2.01	10000
2.001	1000000
2.0001	100000000

From the graph, we can see that the left-hand limit of the function $t(x)$ as x approaches 2 is ∞, and the right-hand limit of the function $t(x)$ as x approaches 2 is also ∞. That is, $\lim_{x \to 2^-} t(x) = \infty$ and $\lim_{x \to 2^+} t(x) = \infty$. Hence, the limit of the function $t(x)$ is ∞ as x approaches 2, i.e. $\lim_{x \to 2} t(x) = \infty$.

Assigned Exercises: (p 92) 5 - 11 (odds), 15, 17, 19, 31 - 43 (odds), 47, 55*