Tangents

Recall: In Section 2.1, we observed that the limiting value of the slopes

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

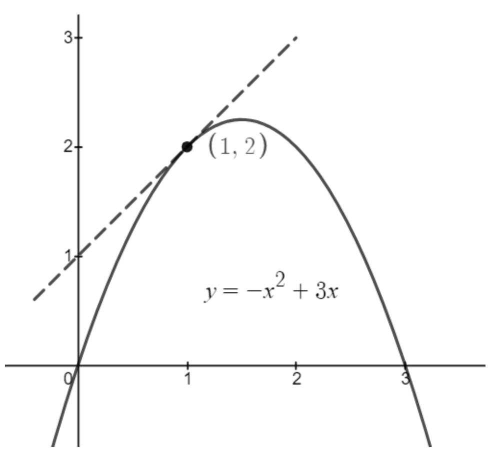
of a secant lines connecting a point $(a, f(a))$ (labeled P) and points Q, as the x-values of Q approach a, is called the **slope of the tangent line** to the graph of f at $(a, f(a))$. In that section, we estimated the slope numerically. In this section, we use the techniques from Sections 2.2 and 2.3 to find the slope of the tangent line by evaluating a limit.

The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1 Consider a function $f(x) = -x^2 + 3x$. The point $(1, 2)$ is on the graph of the function f as $f(1) = 2$. Here the point P is $(1, 2)$ and $a = 1$. Let $(x, f(x))$ be an arbitrary point on the graph with $x \neq a$.



The slope of the tangent line at the point P is

$$\begin{aligned}
m &= \lim_{x\to1} \tfrac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to1} \tfrac{-x^2+3x-(-(1)^2+3(1))}{x-1} \\
&= \lim_{x\to1} \tfrac{-x^2+3x-2}{x-1} \\
&= \lim_{x\to1} \tfrac{-(x^2-3x+2)}{x-1} \\
&= \lim_{x\to1} \tfrac{-(x-1)(x-2)}{x-1} \\
&= \lim_{x\to1} -(x-2) \\
&= 1
\end{aligned}$$

The dotted line shown is the tangent line to the graph at $(1, 2)$. The point-slope form of the tangent line is $y - 2 = 1(x - 1)$, and its slope-intercept form is $y = x + 1$.

The slope of the tangent line at P is often called the **slope of the graph** at P.

If we express an arbitrary x-value using $a + h$, then the difference quotient becomes

$$\frac{f(x) - f(a)}{x - a} = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}$$

If we let $h \to 0$, then the value $a + h$ will approach the value a. Hence, the following is the "alternative" definition of the slope of the tangent line at P:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 1 (revisited) $f(x) = -x^2 + 3x$ with $a = 1$.

$$\begin{aligned}
m &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h \to 0} \frac{-(1+h)^2 + 3(1+h) - 2}{h} \\
&= \lim_{h \to 0} \frac{-(1+2h+h^2) + 3 + 3h - 2}{h} \\
&= \lim_{h \to 0} \frac{-1 - 2h - h^2 + 3 + 3h - 2}{h} \\
&= \lim_{h \to 0} \frac{h - h^2}{h} = \lim_{h \to 0} \frac{h(1-h)}{h} = \lim_{h \to 0} 1 - h = 1
\end{aligned}$$

Velocities

A motion in Calculus 1 is happening on a straight line, called a **rectilinear motion** (rectus + linea). Two possibilities: (1) vertical motion and (2) horizontal motion.

(1) **Vertical Motion**: The height of a moving body is described by the height function $f(t)$ where t is the time elapsed since the beginning of the motion t_0, called the **initial time**. Conventionally, we assume that upward direction is positive and downward is negative. Not unless mentioned, the height is in feet and t is in seconds.

(2) **Horizontal Motion**: The position on a real number line (labeled x) of a moving body is described by the **position function** $f(t)$. Conventionally, moving towards right is positive and left is negative. Imagine a car moving forwards (i.e. moving towards right) and backwards (i.e. moving towards left with the Reverse gear on).

Either way, we have a way to calculate the **average velocity over the time interval** $[a, a+h]$:

$$\textbf{Average Velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{(a+h) - a}$$

Considering this average velocity as the slope of the secant line connecting $(a, f(a))$ and $(a + h, f(a+h))$, the slope of the tangent line is called the **velocity** or **instantaneous velocity** at time $t = a$.

$$v(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Example 2 Suppose that a body (considered as a point) is traveling on a real number line. Its position is described by the function $f(t) = -t^2 + 3t$ in feet with t in seconds. At the beginning, the body is located at the origin since $f(0) = 0$. After 1 second later, its position is at $f(1) = 2$. After 1.5 seconds later (since $t = 0$), its position is at $f(1.5) = -(1.5)^2 + 3(1.5) = 2.25$ ft (away from the origin). After 2 seconds later, its position is at $f(2) = 2$. How is it possible?

According to calculation in **Example 1**, the (instantaneous) velocity of the car at $t = 1$ is 1 ft/sec. Let us find the velocity $v(1.5)$ at $t = 1.5$.

$$\begin{aligned}
v(1.5) &= \lim_{h\to 0} \frac{f(1.5+h) - f(1.5)}{h} \\
&= \lim_{h\to 0} \frac{-(1.5+h)^2 + 3(1.5+h) - 2.25}{h} \\
&= \lim_{h\to 0} \frac{-(2.25 + 3h + h^2) + 4.5 + 3h - 2.25}{h} \\
&= \lim_{h\to 0} \frac{-h^2}{h} = \lim_{h\to 0} -h = 0
\end{aligned}$$

Hence, the velocity of the body at $t = 1.5$ is 0 ft/sec. The zero velocity means that the body was **at rest** at that moment.

Now let us find the velocity at $t = 2$.

$$\begin{aligned}
v(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{-(2+h)^2 + 3(2+h) - 2}{h} \\
&= \lim_{h\to 0} \frac{-(4 + 4h + h^2) + 6 + 3h - 2}{h} \\
&= \lim_{h\to 0} \frac{-h - h^2}{h} = \lim_{h\to 0} \frac{h(-1-h)}{h} = \lim_{h\to 0} -1 - h = -1
\end{aligned}$$

Hence, the velocity of the body at $t = 2$ is -1 ft/sec. A negative velocity means that the body is moving towards left (or backwards). If the body was a car, then the gear is in "Reverse".

In summary, for a horizontal motion

$v(a) > 0$	$\Rightarrow$	The body is moving towards right (forwards).
$v(a) < 0$	$\Rightarrow$	The body is moving towards left (backwards).
$v(a) = 0$	$\Rightarrow$	The body is at rest (at that moment).

Derivatives

The slope of the tangent line to the graph of the function f at the point $P(a, f(a))$ is rather too long. Simply we call it

The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

If the function name is f, then we use the notation f' (read "f prime") to denote the derivative of f. Alternatively,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example 3 Consider a function $g(x) = \frac{1}{x}$. Note that $g(100) = \frac{1}{100}$. Let us find the derivative $g'(100)$.

$$\begin{aligned}
g'(100) &= \lim_{x \to 100} \frac{g(x) - g(100)}{x - 100} \\
&= \lim_{x \to 100} \frac{\frac{1}{x} - \frac{1}{100}}{x - 100} \qquad\qquad \text{LCD} = 100x \\
&= \lim_{x \to 100} \frac{\frac{1}{x} - \frac{1}{100}}{x - 100} \cdot \frac{100x}{100x} \\
&= \lim_{x \to 100} \frac{100 - x}{100x(x - 100)} \\
&= \lim_{x \to 100} \frac{-(x - 100)}{100x(x - 100)} \qquad A - B = -(B - A) \\
&= \lim_{x \to 100} -\frac{1}{100x} = -\frac{1}{10000} = -0.0001
\end{aligned}$$

Imagining the graph of the function g near $x = 100$, is it reasonable that the slope of the tangent line is so small? The equation of the tangent line can be found using the point-slope form:

$$y - \frac{1}{100} = -\frac{1}{10000}(x - 100)$$

or its slope-intercept form $y = -\frac{1}{10000}x + \frac{1}{50}$.

In the previous example, there is actually nothing special about the input value 100. We should be able to find the derivative at any nonzero input value a. Once again for $g(x) = \frac{1}{x}$,

$$\begin{aligned}
g'(a) &= \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= \lim_{x\to a} \frac{\frac{1}{x}-\frac{1}{a}}{x-a} \qquad\qquad \text{LCD} = ax \\
&= \lim_{x\to a} \frac{\frac{1}{x}-\frac{1}{a}}{x-a} \cdot \frac{ax}{ax} \\
&= \lim_{x\to a} \frac{a-x}{ax(x-a)} = \lim_{x\to a} \frac{-(x-a)}{ax(x-a)} = \lim_{x\to a} -\frac{1}{ax} = -\frac{1}{a^2}
\end{aligned}$$

Since we have the derivative formula for the function $g(x) = \frac{1}{x}$, i.e.

$$\boxed{\left(\frac{1}{a}\right)' = -\frac{1}{a^2}}$$

we can just evaluate $g'(a) = -\frac{1}{a^2}$ to find the derivatives at different values. For instance, $g'(1) = -1$ and $g'(0.1) = -100$.

Example 4 Consider $k(x) = \sqrt{x}$ with the domain $[0, \infty)$. For $a \in (0, \infty)$, the derivative is

$$\begin{aligned}
k'(a) &= \lim_{h\to 0} \frac{k(a+h)-k(a)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \cdot \frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}} \\
&= \lim_{h\to 0} \frac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{h(\sqrt{a+h}+\sqrt{a})} \\
&= \lim_{h\to 0} \frac{(\sqrt{a+h})^2-(\sqrt{a})^2}{h(\sqrt{a+h}+\sqrt{a})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{a+h}+\sqrt{a})} = \lim_{h\to 0} \frac{1}{\sqrt{a+h}+\sqrt{a}} = \frac{1}{\sqrt{a+0}+\sqrt{a}} = \frac{1}{2\sqrt{a}}
\end{aligned}$$

This derivative is also worthwhile to memorize:

$$\boxed{(\sqrt{a})' = \frac{1}{2\sqrt{a}}}$$

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a. The point-slope form of the tangent line equation is

$$y - f(a) = f'(a)(x - a)$$

The tangent line equation for $g(x) = \frac{1}{x}$ in **Example 3** is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \quad \text{or} \quad y = -\frac{1}{a^2}x + \frac{2}{a}$$

The tangent line equation for $k(x) = \sqrt{x}$ in **Example 4** is

$$y - \sqrt{a} = \frac{1}{2\sqrt{a}}(x - a) \quad \text{or} \quad y = \frac{1}{2\sqrt{a}}x + \frac{\sqrt{a}}{2}$$

Rates of Change

In application we do not only consider height or position function. The function value of $f(x)$ and the input variable x can describe many interesting things besides positions and time. For instance, T can represent the temperature and the function value $P = f(T)$ can represent the pressure. Then the difference of the temperatures over the interval $[T_1, T_2]$ is denoted by $\Delta T = T_2 - T_1$, and the corresponding difference of the pressure over the interval is denoted by $\Delta P = P_2 - P_1 = f(T_2) - f(T_1)$. The difference quotient

$$\frac{\Delta P}{\Delta T} = \frac{P_2 - P_1}{T_2 - T_1} = \frac{f(T_2) - f(T_1)}{T_2 - T_1}$$

is called the **average rate of change** of P with respect to T over the interval $[T_1, T_2]$. When the difference $\Delta T \to 0$, then the limiting value of the average rate of change $\Delta P/\Delta T$ is called the **(instantaneous) rate of change** of P with respect to T, and it is denoted by

$$\lim_{\Delta T \to 0} \frac{\Delta P}{\Delta T}$$

More specifically, the instantaneous rate of change of P at T_1 is

$$\textbf{Instantaneous Rate of Change} = \lim_{T \to T_1} \frac{f(T) - f(T_1)}{T - T_1} \quad \text{or} \quad \lim_{h \to 0} \frac{f(T_1 + h) - f(T_1)}{h}$$

This is basically the derivative of f at T_1, i.e. $f'(T_1)$.

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Example 5 Consider the volume of sphere function $V = f(R)$ of the radius R. According to a formula in Geometry, $f(R) = \frac{4}{3}\pi R^3$. When the radius changes from $R_1 = 3$ m to $R_2 = 6$ m, the volume of the sphere changes from $V_1 = f(R_1) = f(3) = \frac{4}{3}\pi(3)^3 = 36\pi$ m^3 to $V_2 = f(R_2) = f(6) = \frac{4}{3}\pi(6)^3 = 288\pi$ m^3. The average rate of change of the volume with respect to the radius over the interval $[3, 6]$ is

$$\frac{\Delta V}{\Delta R} = \frac{V_2 - V_1}{R_2 - R_1} = \frac{f(R_2) - f(R_1)}{R_2 - R_1} = \frac{288\pi - 36\pi}{6 - 3} = \frac{252\pi}{3} = 84\pi \approx 263.8938 \ \frac{\text{m}^3}{\text{m}}$$

This average rate of change indicates that the volume of sphere increases by roughly 264 m^3 for every increase of the radius by 1 m <u>on average</u>. Note that, in reality, the volume does not really change by 264 m^3.

Q: When a balloon is inflated using a air pump, what is the rate of change of the volume with respect to the radius at the "very" moment of the radius $R = 3$ m is the instantaneous rate of change

$$\begin{aligned}
f'(3) &= \lim_{R\to 3} \frac{f(R) - f(3)}{R - 3} \quad \text{or} \quad \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\
&= \lim_{R\to 3} \frac{\frac{4}{3}\pi R^3 - \frac{4}{3}\pi(3)^3}{R - 3} \\
&= \frac{4}{3}\pi \lim_{R\to 3} \frac{R^3 - 3^3}{R - 3} \\
&= \frac{4}{3}\pi \lim_{R\to 3} \frac{(R-3)(R^2 + 3R + 9)}{R - 3} \\
&= \frac{4}{3}\pi \lim_{R\to 3} (R^2 + 3R + 9) = \frac{4}{3}\pi(9 + 9 + 9) = 36\pi = 113.0974 \ \frac{\text{m}^3}{\text{m}}
\end{aligned}$$

If we want to find an expression of the derivative for an arbitrary value of a, we can find

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{4}{3}\pi(a+h)^3 - \frac{4}{3}\pi a^3}{h} \\
&= \frac{4}{3}\pi \lim_{h\to 0} \frac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h} \\
&= \frac{4}{3}\pi \lim_{h\to 0} \frac{3a^2h + 3ah^2 + h^3}{h} \\
&= \frac{4}{3}\pi \lim_{h\to 0} \frac{h(3a^2 + 3ah + h^2)}{h} \\
&= \frac{4}{3}\pi \lim_{h\to 0} (3a^2 + 3ah + h^2) = \frac{4}{3}\pi(3a^2) = 4\pi a^2
\end{aligned}$$

Coincidentally, this rate of change is precisely the surface area of the sphere.

Assigned Exercises: (p 148) 1, 3, 5, 7, 9, 11, 15, 23, 25, 29, 33, 37, 41, 47, 57