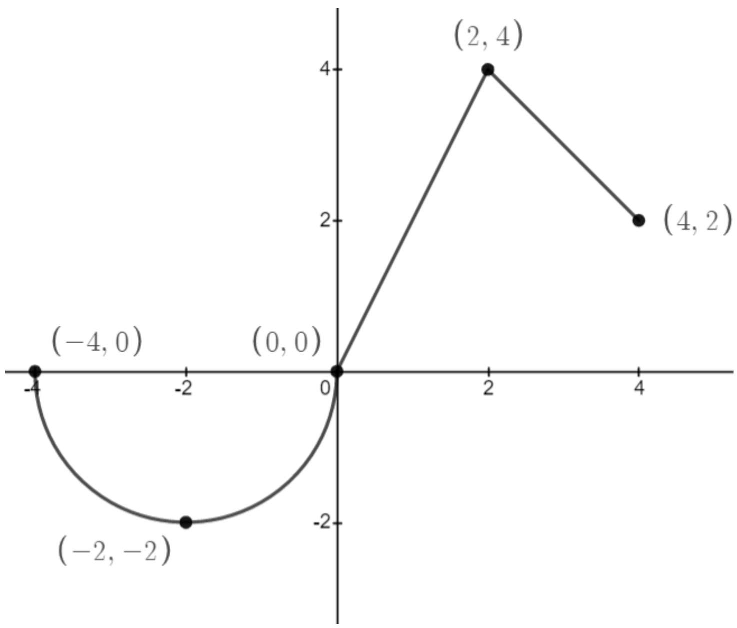
Transformation of Functions

Let $y = f(x)$ be a piecewise defined as follows:

$$f(x) = \begin{cases} -\sqrt{4-(x+2)^2} & \text{if} \quad -2 \leq x < 0 \\ -2|x-2|+4 & \text{if} \quad 0 \leq x < 2 \\ -|x-2|+4 & \text{if} \quad 2 \leq x \leq 4 \end{cases}$$



Domain: $[-4, 4]$

Range: $[-2, 4]$

Guide points: $(-4, 0)$, $(-2, -2)$, $(0, 0)$, $(2, 4)$, $(4, 2)$

Vertical and Horizontal Shifts Suppose $h > 0$ and $k > 0$. To obtain the graph of

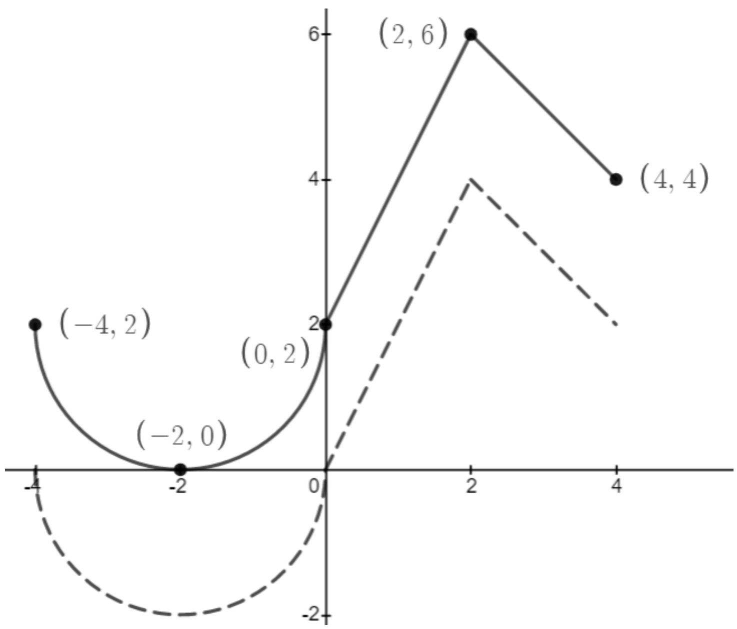
$y = f(x) + k$, shift the graph of $y = f(x)$ a distant k units upward.

$y = f(x) - k$, shift the graph of $y = f(x)$ a distant k units downward.

$y = f(x - h)$, shift the graph of $y = f(x)$ a distant h units to the right.

$y = f(x + h)$, shift the graph of $y = f(x)$ a distant h units to the left.

Example 1 Graph the function $y = f(x) + 2$. Since we are shifting the graph of $y = f(x)$ upward by 2 units, add 2 to y-values of the guide points of the graph of $y = f(x)$.

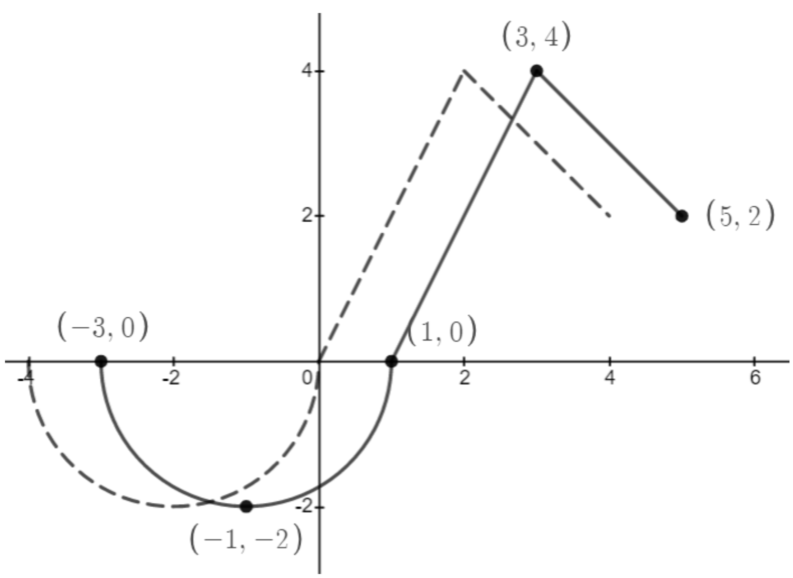


(a, b)	$\mapsto$	$(a, b+2)$
$(-4, 0)$	$\mapsto$	$(-4, 2)$
$(-2, -2)$	$\mapsto$	$(-2, 0)$
$(0, 0)$	$\mapsto$	$(0, 2)$
$(2, 4)$	$\mapsto$	$(2, 6)$
$(4, 2)$	$\mapsto$	$(4, 4)$

Domain: $[-4, 4]$ (no change)
Range: $[-2 + 2, 4 + 2] = [0, 6]$

Note that the overall shape (even the size) has not been altered. Such a transformation is called an **isometry** or **rigid motion**.

Example 2 Graph the function $y = f(x - 1)$. Since we are shifting the graph of $y = f(x)$ to the right by 1 unit, add 1 to x-values of the guide points of the graph of $y = f(x)$.



(a, b)	$\mapsto$	$(a+1, b)$
$(-4, 0)$	$\mapsto$	$(-3, 0)$
$(-2, -2)$	$\mapsto$	$(-1, -2)$
$(0, 0)$	$\mapsto$	$(1, 0)$
$(2, 4)$	$\mapsto$	$(3, 4)$
$(4, 2)$	$\mapsto$	$(5, 2)$

Domain: $[-4 + 1, 4 + 1] = [-3, 5]$
Range: $[-2, 4]$ (no change)

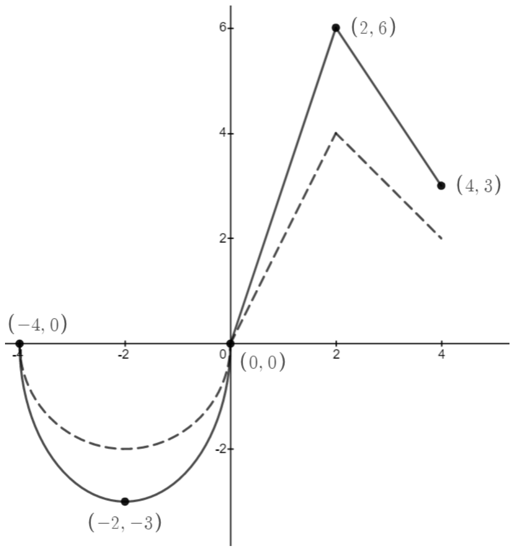
Vertical Stretching and Shrinking The graph of $y = Af(x)$ is obtained as follows:

If $A > 1$, then the graph of $y = Af(x)$ is obtained by vertically stretching the graph of $y = f(x)$ by a factor of A.

If $0 < A < 1$, then the graph of $y = Af(x)$ is obtained by vertically shrinking the graph of $y = f(x)$ by a factor of $\frac{1}{A}$.

That is, (a, b) of the graph of $y = f(x)$ is mapped to (a, Ab) of the graph of $y = Af(x)$.

Example 3 Graph the function $y = 1.5f(x)$. Since we are vertically stretching the graph of $y = f(x)$ by a factor of 1.5, multiply 1.5 to y-values of the guide points of the graph of $y = f(x)$.

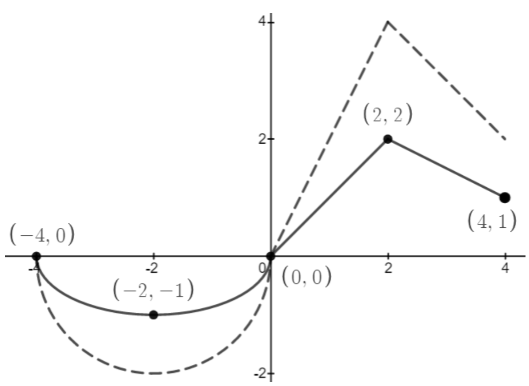


$$\begin{array}{lcl} (a,b) & \mapsto & (a,1.5b) \\ \\ (-4,0) & \mapsto & (-4,0) \\ (-2,-2) & \mapsto & (-2,-3) \\ (0,0) & \mapsto & (0,0) \\ (2,4) & \mapsto & (2,6) \\ (4,2) & \mapsto & (4,3) \end{array}$$

Domain: $[-4, 4]$ (no change)
Range: $[1.5(-2), 1.5(4)] = [-3, 6]$

Note that the points on the x-axis are **stationary** under vertical stretching because multiplying 1.5 does not change the y-value.

Example 4 Graph the function $y = 0.5f(x)$. Since we are vertically shrinking the graph of $y = f(x)$ by a factor of $\frac{1}{0.5}$, multiply 0.5 to y-values of the guide points of the graph of $y = f(x)$.

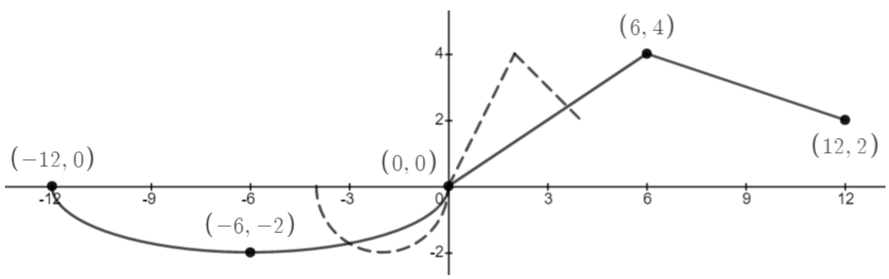


$$\begin{array}{lcl} (a,b) & \mapsto & (a,0.5b) \\ \\ (-4,0) & \mapsto & (-4,0) \\ (-2,-2) & \mapsto & (-2,-1) \\ (0,0) & \mapsto & (0,0) \\ (2,4) & \mapsto & (2,2) \\ (4,2) & \mapsto & (4,1) \end{array}$$

Domain: $[-4, 4]$ (no change)
Range: $[0.5(-2), 0.5(4)] = [-1, 2]$

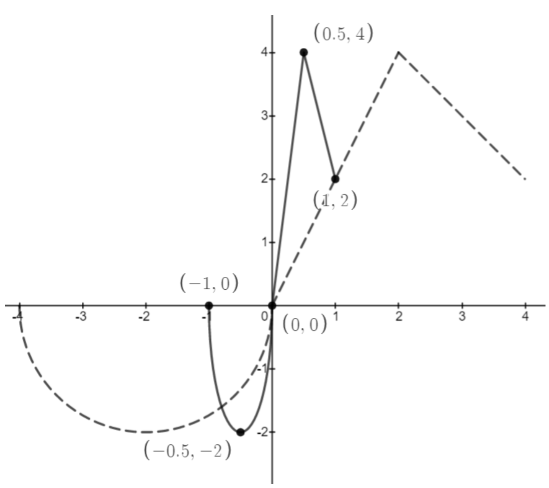
> **Horizontal Stretching and Shrinking** The graph of $y = f(Bx)$ is obtained as follows:
>
> If $0 < B < 1$, then the graph of $y = f(Bx)$ is obtained by horizontally stretching the graph of $y = f(x)$ by a factor of $\frac{1}{B}$.
>
> If $B > 1$, then the graph of $y = f(Bx)$ is obtained by horizontally shrinking the graph of $y = f(x)$ by a factor of B.
>
> That is, (a, b) of the graph of $y = f(x)$ is mapped to $(\frac{1}{B}a, b)$ of the graph of $y = f(Bx)$.

Example 5 Graph the function $y = f(\frac{1}{3}x)$. Since we are horizontally stretching the graph of $y = f(x)$ by a factor of $\frac{1}{1/3}$, multiply 3 to x-values of the guide points of the graph of $y = f(x)$.



(a,b)	$\mapsto$	$(3a,b)$	$(0,0)$	$\mapsto$	$(0,0)$	Domain: $[3(-4), 3(4)] = [-12, 12]$
$(-4,0)$	$\mapsto$	$(-12,0)$	$(2,4)$	$\mapsto$	$(6,4)$	Range: $[-2, 4]$ (no change)
$(-2,-2)$	$\mapsto$	$(-6,-2)$	$(4,2)$	$\mapsto$	$(12,2)$	

Example 6 Graph the function $y = f(4x)$. Since we are horizontally shrinking the graph of $y = f(x)$ by a factor of 4, multiply $\frac{1}{4}$ to x-values of the guide points of the graph of $y = f(x)$.



$$(a,b) \mapsto (\tfrac{1}{4}a, b)$$

$$\begin{aligned}(-4,0) &\mapsto (-1,0)\\ (-2,-2) &\mapsto (-0.5,-2)\\ (0,0) &\mapsto (0,0)\\ (2,4) &\mapsto (0.5,4)\\ (4,2) &\mapsto (1,2)\end{aligned}$$

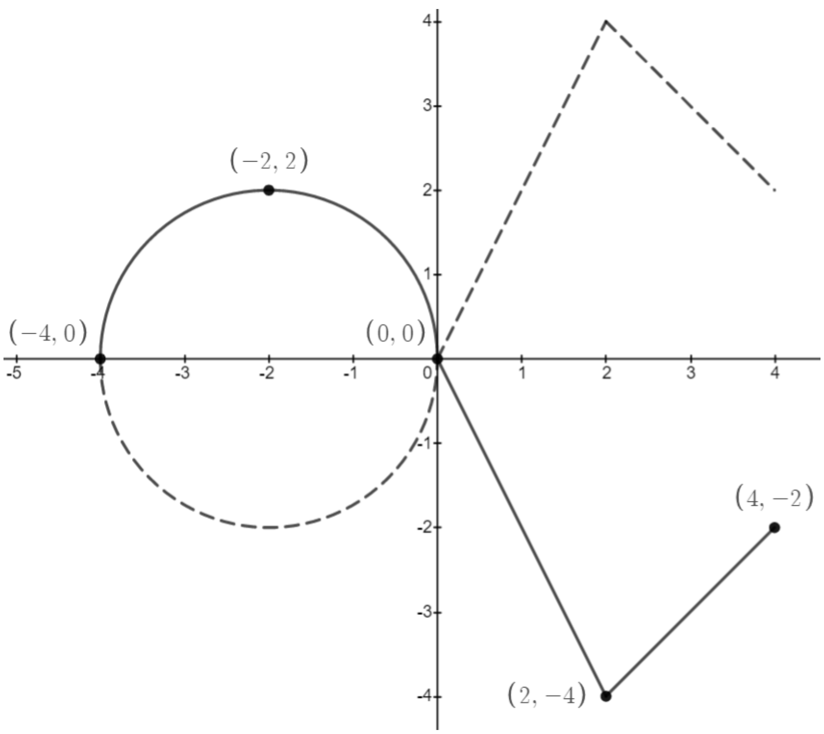
Domain: $[\frac{1}{4}(-4), \frac{1}{4}(4)] = [-1, 1]$
Range: $[-2, 4]$ (no change)

Reflecting about the Axes To obtain the graph of

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.

Example 7 Graph the function $y = -f(x)$. Since we are reflecting the graph of $y = f(x)$ about the x-axis, multiply -1 to y-values of the guide points of the graph of $y = f(x)$.



(a, b)	$\mapsto$	$(a, -b)$
$(-4, 0)$	$\mapsto$	$(-4, 0)$
$(-2, -2)$	$\mapsto$	$(-2, 2)$
$(0, 0)$	$\mapsto$	$(0, 0)$
$(2, 4)$	$\mapsto$	$(2, -4)$
$(4, 2)$	$\mapsto$	$(4, -2)$

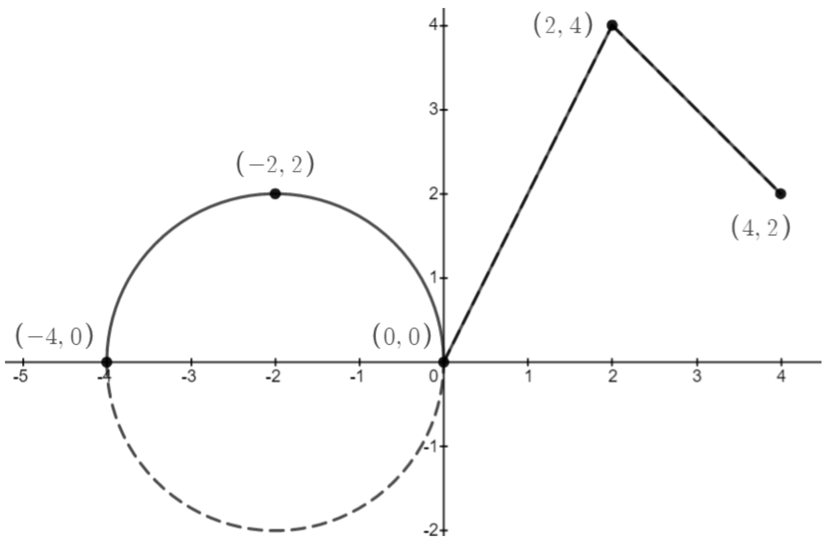
Domain: $[-4, 4]$ (no change)
Range: $[-4, 2]$ (note that $[2, -4]$ is an invalid interval notation)

Note that we get a mirror image about the x-axis.

Absolute Value of a Function To obtain the graph of $y = |f(x)|$, reflect only the portion of the graph of $y = f(x)$ below the x-axis over the x-axis.

Since the y-values of the graph of $y = f(x)$ that are above the x-axis are positive, it is not affected by the absolute value.

Example 8 Graph the function $y = |f(x)|$. Only the semicircle which is below the x-axis is reflected over the x-axis.



(a, b)	$\mapsto$	$(a, \lvert b \rvert)$
$(-4, 0)$	$\mapsto$	$(-4, 0)$
$(-2, -2)$	$\mapsto$	$(-2, 2)$
$(0, 0)$	$\mapsto$	$(0, 0)$
$(2, 4)$	$\mapsto$	$(2, 4)$
$(4, 2)$	$\mapsto$	$(4, 2)$

More than one transformation can be applied to a function to achieve more complicated transformation. One popular form is

$$y = k + Af(B(x - h)) \quad \text{or} \quad y = Af(B(x - h)) + k$$

In this form, the transformations are carried in the following order:

1) Horizontal stretching or shrinking using B (reflection about the y-axis if $B < 0$)
2) Horizontal shifting using h
3) Vertical stretching or shrinking using A (reflection about the x-axis if $A < 0$)
4) Vertical shifting using k

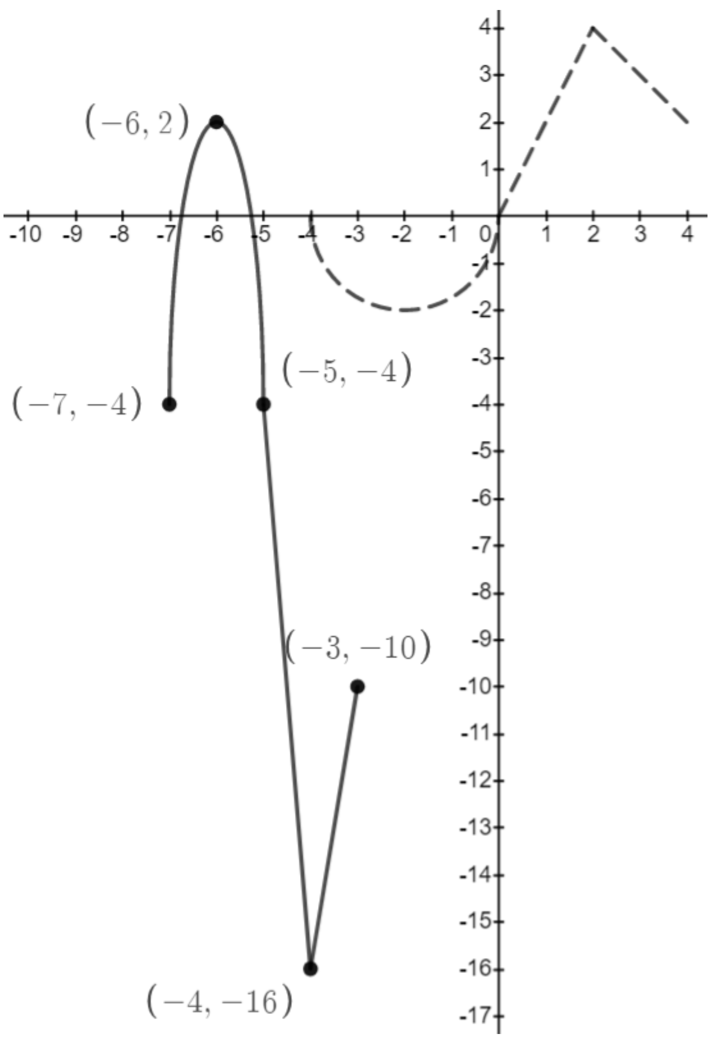
The argument of the function f may not be written in the form of $B(x - h)$. In that case, the coefficient B must be factored out to write in the form of $B(x - h)$ to figure out the correct horizontal shifting factor h. For instance, the graph of $y = f(2x - 6)$ is not obtained by shifting the graph of $y = f(x)$ to the right by 6 units. The argument $2x - 6$ must be factored into $2(x - 3)$ to see that the shifting factor is 3 units.

Example 9 Graph the function $y = -4 - 3f(2x + 10)$. Note that the argument $2x + 10$ must be factored as $2(x + 5)$. Hence, the function can be written as $y = -4 - 3f(2(x + 5))$. The transformations are done in the following order:

1) Horizontal shrinking by a factor of 2, i.e. $(a, b) \mapsto (\frac{1}{2}a, b)$
2) Horizontal shifting to the left by 5 units, i.e. $(\frac{1}{2}a, b) \mapsto (\frac{1}{2}a - 5, b)$
3) Vertical stretching by a factor of 3, i.e. $(\frac{1}{2}a - 5, b) \mapsto (\frac{1}{2}a - 5, 3b)$
4) Reflect about the x-axis, i.e. $(\frac{1}{2}a - 5, 3b) \mapsto (\frac{1}{2}a - 5, -3b)$
5) Vertical shifting downward by 4 units, i.e. $(\frac{1}{2}a - 5, -3b) \mapsto (\frac{1}{2}a - 5, -3b - 4)$

(a,b)	$\mapsto$	$(\frac{1}{2}a, b)$	$\mapsto$	$(\frac{1}{2}a-5, b)$	$\mapsto$	$(\frac{1}{2}a-5, -3b)$	$\mapsto$	$(\frac{1}{2}a-5, -3b-4)$
$(-4,0)$	$\mapsto$	$(-2,0)$	$\mapsto$	$(-7,0)$	$\mapsto$	$(-7,0)$	$\mapsto$	$(-7,-4)$
$(-2,-2)$	$\mapsto$	$(-1,-2)$	$\mapsto$	$(-6,-2)$	$\mapsto$	$(-6,6)$	$\mapsto$	$(-6,2)$
$(0,0)$	$\mapsto$	$(0,0)$	$\mapsto$	$(-5,0)$	$\mapsto$	$(-5,0)$	$\mapsto$	$(-5,-4)$
$(2,4)$	$\mapsto$	$(1,4)$	$\mapsto$	$(-4,4)$	$\mapsto$	$(-4,-12)$	$\mapsto$	$(-4,-16)$
$(4,2)$	$\mapsto$	$(2,2)$	$\mapsto$	$(-3,2)$	$\mapsto$	$(-3,-6)$	$\mapsto$	$(-3,-10)$

Note: 3) and 4) are done together.



Combination of Functions

Given two functions $f(x)$ and $g(x)$, we can

i) add them to form the **sum function** $f+g$, defined by $(f+g)(x) = f(x) + g(x)$.
ii) subtract one from the other to form the **difference function** $f-g$, defined by $(f-g)(x) = f(x) - g(x)$.
iii) multiply them to form the **product function** $f \cdot g$, defined by $(f \cdot g)(x) = f(x) \cdot g(x)$.
iv) divide one by the other to form the **quotient function** $\frac{f}{g}$, defined by $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$.

Note 1: The difference of f and g means $f-g$ and the difference of g and f means $g-f$.
Note 2: The quotient of f and g means $\frac{f}{g}$ and the quotient of g and f means $\frac{g}{f}$.

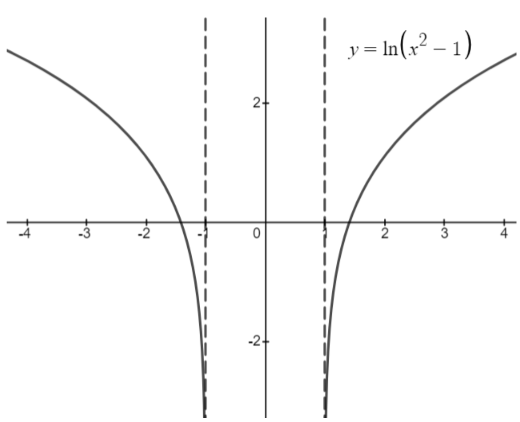
It is <u>not</u> easy to explain how these operations will combine the graphs of f and g. So it might be simply better strategy to create a chart to plot them. However, the domain of $f+g$, $f-g$, $f \cdot g$, and $\frac{f}{g}$ can be found without actually plotting them.

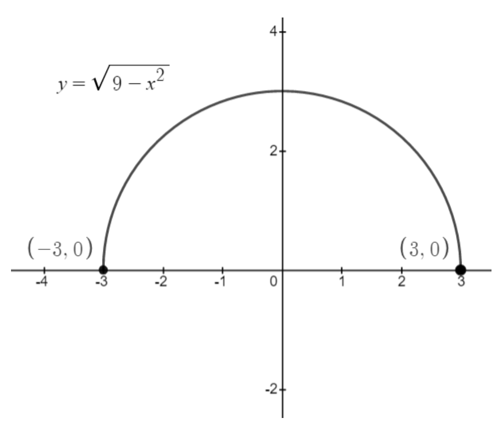
$$D(f+g) = D(f) \cap D(g) \qquad D(f-g) = D(f) \cap D(g) \qquad D(f \cdot g) = D(f) \cap D(g)$$

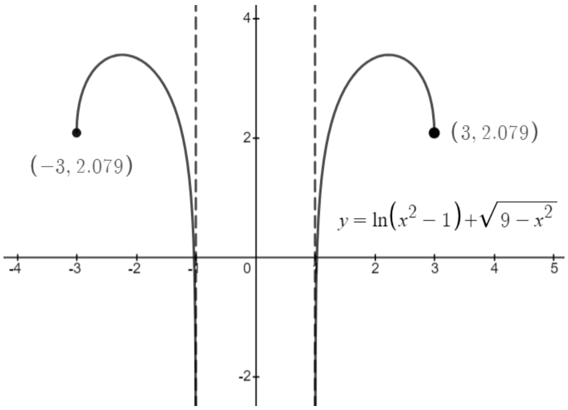
$$D\left(\frac{f}{g}\right) = D(f) \cap D(g) \cap \{x \in \mathbb{R} : g(x) \neq 0\}$$

Basically, the domain should be the intersection of the domains of f and g. But for the quotient function, we further require that the denominator function g should not be zero.

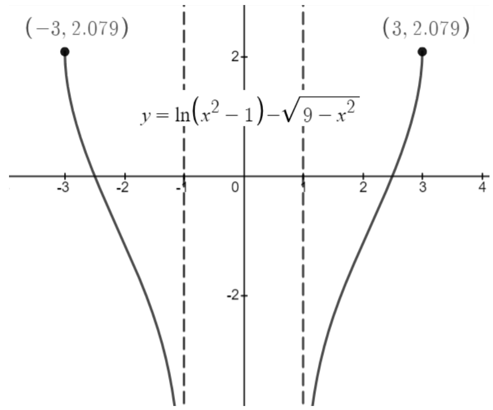
Example 10 Consider $f(x) = \ln(x^2 - 1)$ and $g(x) = \sqrt{9 - x^2}$. Note that $D(f) = (-\infty, 1) \cup (1, \infty)$ and $D(g) = [-3, 3]$. Then $D(f) \cap D(g) = [-3, -1) \cup (1, 3]$.



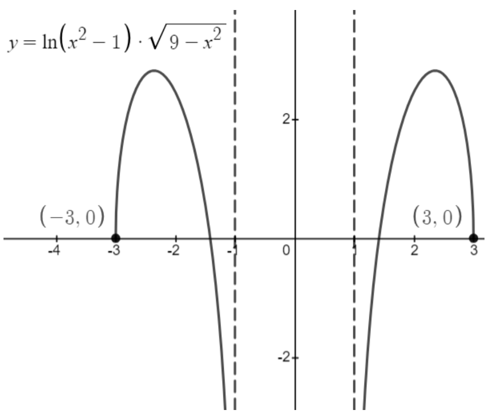




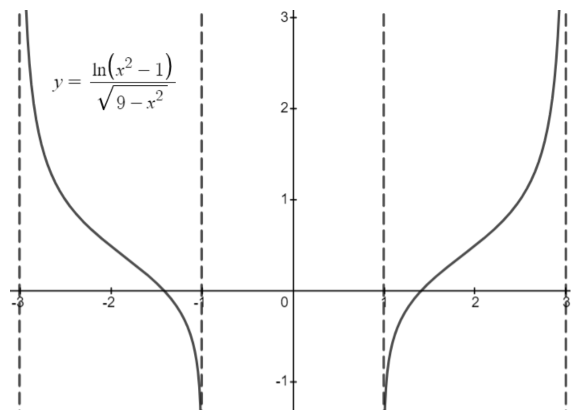
Graph of $f + g$
$D(f + g) = [-3, -1) \cup (1, 3]$
$R(f + g) = (-\infty, ?]$



Graph of $f - g$
$D(f - g) = [-3, -1) \cup (1, 3]$
$R(f - g) = (-\infty, \ln(8)]$



Graph of $f \cdot g$
$D(f \cdot g) = [-3, -1) \cup (1, 3]$
$R(f \cdot g) = (-\infty, ?]$



Graph of $\frac{f}{g}$
$D(\frac{f}{g}) = (-3, -1) \cup (1, 3)$
$R(\frac{f}{g}) = (-\infty, \infty)$

Q: Have you noticed that the domain of the quotient function is slightly different from the rest? Why?

Another way to combine two functions is to "compose" them.

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition of f and g**) is defined by

$$(f \circ g)(x) = f(g(x))$$

Similarly, the composite function $g \circ f$ is defined by

$$(g \circ f)(x) = g(f(x))$$

It is very tricky the find the domain of the composition $f \circ g$. When the composition function is evaluated, note that we first insert the input value into the function g, so the domain of g is the bare bottom line we start with. Then the function f takes the output of g as its input, so that means the output of g must be inside the domain of f. Technically, here is how we can find the domain of the composite function $f \circ g$.

$$D(f \circ g) = \{x \in D(g) : g(x) \in D(f)\}$$
$$D(g \circ f) = \{x \in D(f) : f(x) \in D(g)\}$$

In words, the domain of $f \circ g$ is the set of all real numbers x from the domain of g such that the function values $g(x)$ is in the domain of f. Sometimes, for $f \circ g$, we refer g as an **inside function** and f as an **outside function**.

Example 11 Consider $f(x) = -x^2 - 1$ with the domain $D(f) = (-\infty, \infty)$ and $g(x) = \sqrt{x}$ with the domain $D(g) = [0, \infty)$. The composite function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = -(\sqrt{x})^2 - 1$$

The domain of $f \circ g$ is $[0, \infty)$. Since the domain of f itself is the set of all real numbers, we only have to worry about the domain of g. Note that the final expression is not simplified to $-x - 1$. **When finding a domain of a composite function, it is very important not to simplify right after inserting of the inside function into the outside function.**

Common Mistake: If you further simplify to write

$$(f \circ g)(x) = -x - 1$$

you will end up making a mistake of thinking the domain of $f \circ g$ as $(-\infty, \infty)$ as $-x - 1$ is just a linear polynomial.

Once the domain of $f \circ g$ is found, then you may simplify to write

$$(f \circ g)(x) = -x - 1$$

with the domain $D(f \circ g) = [0, \infty)$. So $(f \circ g)(2) = -2 - 1 = -3$, but $(f \circ g)(-2) \neq -(-2) - 1 = 1$ as the composite function $f \circ g$ is not even defined for $x = -2$.

The composition function $g \circ f$ would not make sense what so ever. All possible output values of f are negative which cannot be inserted into g as we cannot evaluate the square root of negative numbers. Hence, $g \circ f$ is an empty function as its domain is $\emptyset$.

Assigned Exercises: (p 42) 1, 3, 5, 7, 13, 17, 21, 23, 31, 32, 35, 37, 43 (Find $f(x)$ and $g(x)$ so that $F = f \circ g$), 47 (Find $f(x)$ and $g(x)$ so that $\nu = f \circ g$), 53, 55, 59*