Guidelines for Sketching a Curve

A. **Domain**: Of course!

B. **Intercepts**: (i) y-intercept: $(0, f(0))$; (ii) x-intercept: Solve the equation $f(x) = 0$. Only the real solutions. Be aware that in most of cases you will not be able to solve this equation.

C. **Symmetry**: If f is even, then we get the portion of the graph on the left hand side of y-axis for free. The same is true for an odd function. We just need to flip the portion of the graph over the x-axis and then y-axis to get the portion on the left hand side. But this feature rarely occurs.

D. **Asymptotes**: (i) Horizontal Asymptote (HA) is found by finding the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$; (ii) Vertical Asymptote (VA) happens at $x = a$ when $\lim_{x\to a^\pm} f(x) = \pm\infty$; (iii) Slant Asymptote (SA): The function of the form $f(x)/g(x)$ has a slant asymptote $y = mx+b$ if the quotient and the remainder obtained upon dividing $f(x)$ by $g(x)$ using the long division method is mx and b respectively.

E. **Intervals of Increase or Decrease**: Use the Increasing/Decreasing test.

F. **Local Maximum and Minimum Values**: Use the first derivative test.

G. **Concavity and Points of Inflection**: Use the Concavity test.

H. **Sketch the Curve**.

Example 1 Use the guidelines to sketch the curve of $y = \sqrt[3]{x^3+1}$.

A. $D = (-\infty, \infty)$

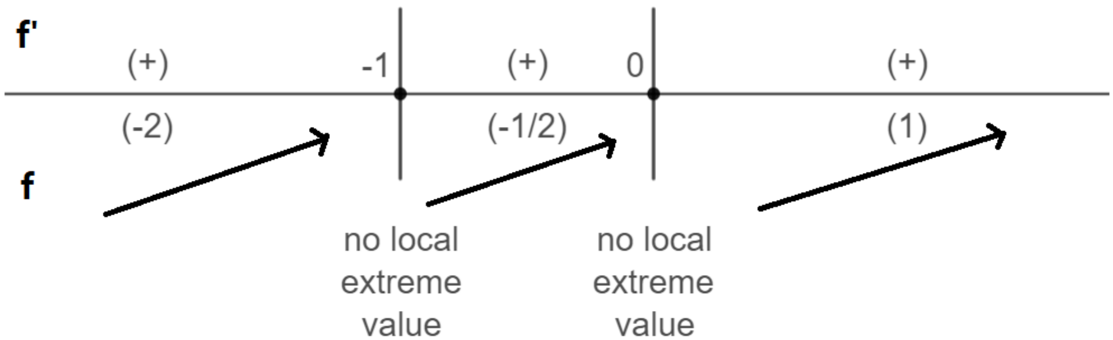
B. y-intercept: $(0, \sqrt[3]{0^3+1}) = (0, 1)$.
x-intercept: $\sqrt[3]{x^3+1} = 0$, i.e. $x^3 + 1 = 0^3$, i.e. $x = \sqrt[3]{-1} = -1$, so $(-1, 0)$.

C. The graph, in fact, has a symmetry, but not the kind that we should be concerned of.

D. As the domain is the set of all real numbers, we have no hole or vertical asymptote. For horizontal asymptotes, we evaluate

$$\lim_{x\to\infty} \sqrt[3]{x^3+1} = \sqrt[3]{\lim_{x\to\infty}(x^3+1)} = \infty \quad \text{and} \quad \lim_{x\to-\infty} \sqrt[3]{x^3+1} = \sqrt[3]{\lim_{x\to-\infty}(x^3+1)} = -\infty$$

E. $y' = \frac{1}{3} \cdot (x^3+1)^{-2/3} \cdot (3x^2) = \frac{x^2}{\sqrt[3]{(x^3+1)^2}}$
$y' = 0$, i.e. $x^2 = 0$, i.e. $x = 0 \in D$.
y' is undefined if $\sqrt[3]{(x^3+1)^2} = 0$, i.e. $x = -1 \in D$.
Critical numbers: -1 and 0. We partition the real number line at -1 and 0.
Test numbers: -2 on $(-\infty, -1)$; $-\frac{1}{2}$ on $(-1, 0)$; 1 on $(0.\infty)$.
$f'(-2) = \frac{(-2)^2}{\sqrt[3]{(-2)^3+1)^2}} > 0$; $f'(-\frac{1}{2}) = \frac{(-1/2)^2}{\sqrt[3]{(-1/2)^3+1)^2}} > 0$; $f'(1) = \frac{(1)^2}{\sqrt[3]{(1)^3+1)^2}} > 0$.
In fact, the derivative can be written $y' = (\frac{x}{\sqrt[3]{x^3+1}})^2$, so it is always positive.



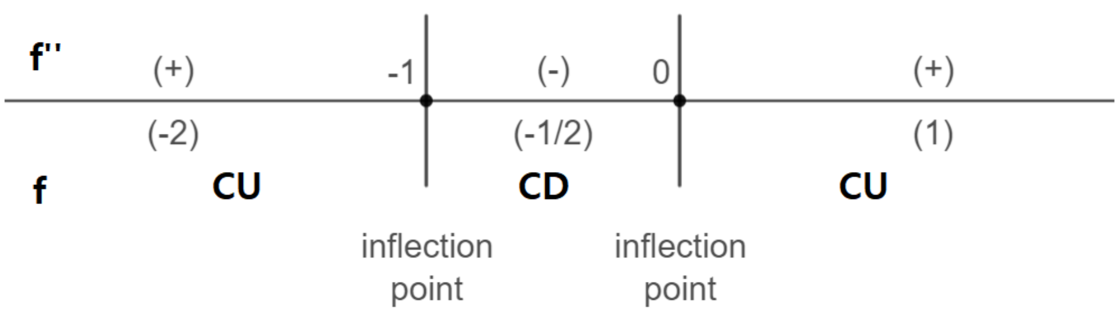
f is increasing on $(-\infty, -1) \cup (-1, 0) \cup (0.\infty)$; f is nowhere decreasing.

F. Since the derivative does not change its sign, no local extreme value exist.

G. Write $y' = x^2(x^3+1)^{-2/3}$. Using the product rule,

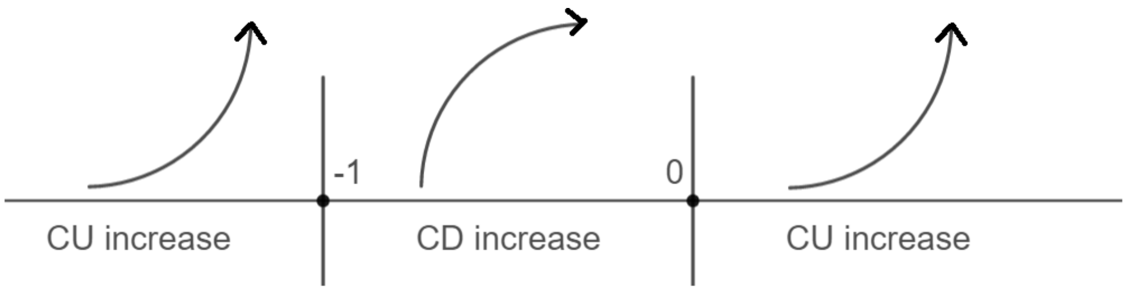
$$y'' = 2x \cdot (x^3+1)^{-2/3} + x^2 \cdot -\frac{2}{3}(x^3+1)^{-5/3}(3x^2) = \cdots = \frac{2x}{\sqrt[3]{(x^3+1)^5}}$$

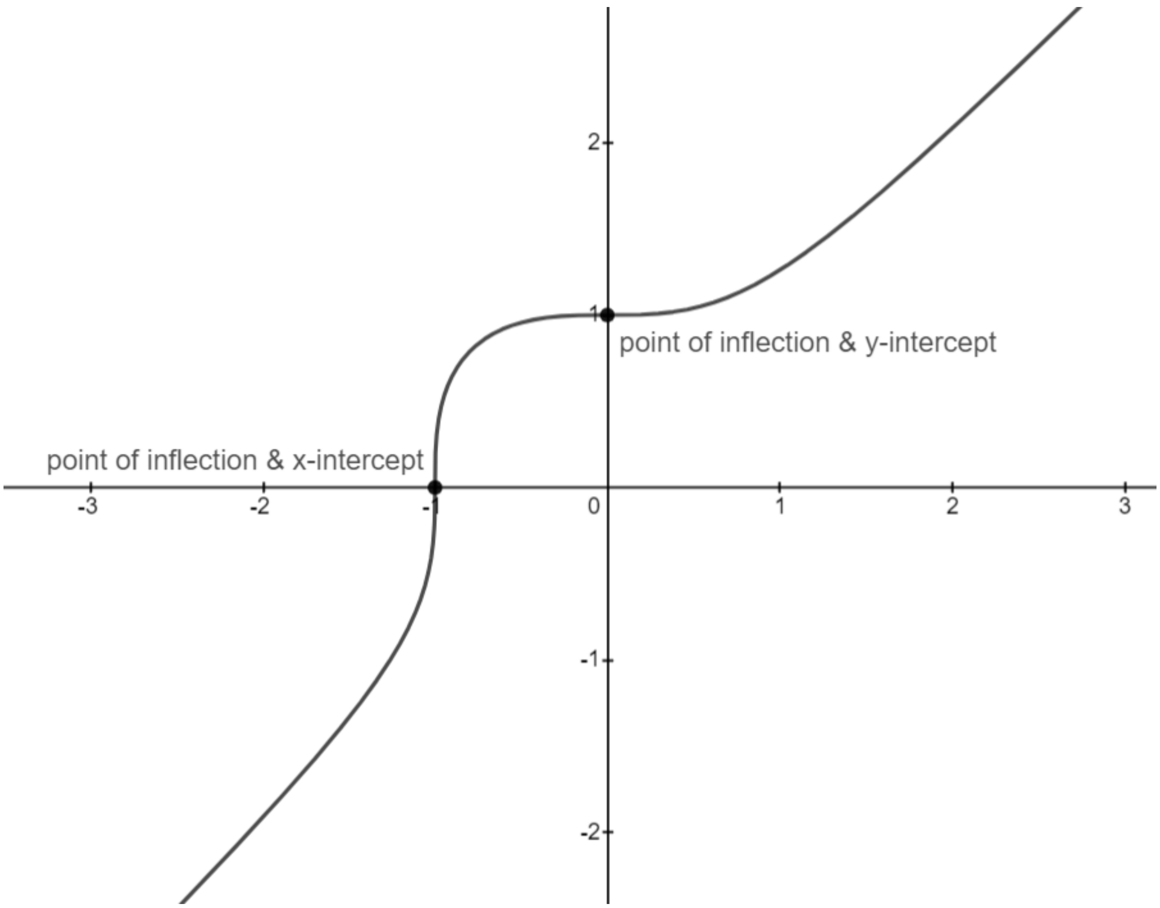
$y'' = 0$, i.e. $2x = 0$, i.e. $x = 0 \in D$.
y'' is undefined if $\sqrt[3]{(x^3+1)^5} = 0$, i.e. $x = -1 \in D$.
Test numbers: -2 on $(-\infty, -1)$; $-\frac{1}{2}$ on $(-1, 0)$; 1 on $(0.\infty)$.
$f''(-2) = \frac{2(-2)}{\sqrt[3]{(-2)^3+1)^5}} > 0$; $f''(-\frac{1}{2}) = \frac{2(-1/2)}{\sqrt[3]{(-1/2)^3+1)^5}} < 0$; $f''(1) = \frac{2(1)}{\sqrt[3]{(1)^3+1)^5}} > 0$.



f is concave up on $(-\infty, -1) \cup (0, \infty)$; f is concave down on $(-1, 0)$.
$f(-1) = \sqrt[3]{(-1)^3+1} = 0$; $f(0) = \sqrt[3]{0^3+1} = 1$.
Points of inflection: $(-1, 0)$ and $(0, 1)$.

H. Using results from E and G,





Make sure that everything we found from A through G is reflected.

Extra: Note that the limits of the derivative at ∞ and $-\infty$ are

$$\lim_{x\to\infty} \frac{x^2}{\sqrt[3]{(x^3+1)^2}} = 1 \quad \text{and} \quad \lim_{x\to-\infty} \frac{x^2}{\sqrt[3]{(x^3+1)^2}} = 1$$

which are evident from the shape of the graph as it becomes like the line with the slope 1. The graph does not curve much at ∞ and $-\infty$ as the limits of the second derivative at ∞ and $-\infty$ are

$$\lim_{x\to\infty} \frac{2x}{\sqrt[3]{(x^3+1)^5}} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{2x}{\sqrt[3]{(x^3+1)^5}} = 0$$

Example 2 Use the guidelines to sketch the curve of

$$y = \frac{-2x^2 + 5x + 1}{x - 2}$$

A. $D = (-\infty, 2) \cup (2, \infty)$

B. y-intercept: $(0, \frac{-2(0)^2+5(0)+1}{0-2}) = (0, -\frac{1}{2})$.
x-intercept: $-2x^2 + 5x + 1 = 0$, i.e. $x = \frac{-5\pm\sqrt{5^2-4(-2)(1)}}{2(-2)}$, i.e. $x = \frac{-5\pm\sqrt{33}}{-4} = \frac{5\pm\sqrt{33}}{4} = \frac{5}{4} \pm \frac{\sqrt{33}}{4}$, so $(\frac{5}{4} - \frac{\sqrt{33}}{4}, 0) \approx (-0.2, 0)$ and $(\frac{5}{4} + \frac{\sqrt{33}}{4}, 0) \approx (2.7, 0)$.

C. $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd function.

D. We could not factor the numerator, so the simplified form of the function is the same as the original rational function. Hence, the value $x = 2$ still makes the denominator zero even after simplifying. Hence, we have a vertical asymptote at $x = 2$.

$$\lim_{x\to\infty} \frac{-2x^2 + 5x + 1}{x - 2} = \lim_{x\to\infty} \frac{-2x^2}{x} = \lim_{x\to\infty} -2x = -\infty$$

$$\lim_{x\to-\infty} \frac{-2x^2 + 5x + 1}{x - 2} = \lim_{x\to-\infty} \frac{-2x^2}{x} = \lim_{x\to-\infty} -2x = \infty$$

So f has no horizontal asymptote. However, the degree of the numerator is one more than the degree of the denominator. Then we have a slant asymptote, and we can find its equation by dividing the numerator by the denominator (using long-division or synthetic division). The quotient is $-2x + 1$, and the remainder is "we couldn't care less." The equation of the slant asymptote is $y = -2x + 1$.

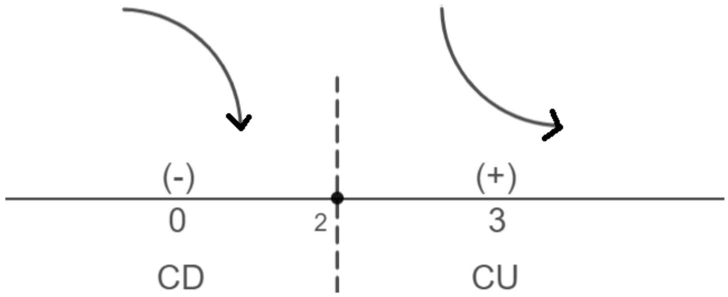
E. $y' = \frac{(-4x+5)(x-2)-(-2x^2+5x+1)(1)}{(x-2)^2} = \frac{-4x^2+5x+8x-10+2x^2-5x-1}{(x-2)^2} == \frac{-2x^2+8x-11}{(x-2)^2}$
$y' = 0$, i.e. $-2x^2 + 8x - 11 = 0$, but its discriminant $8^2 - 4(-2)(-11) < 0$, so no real solution.
y' is undefined if $(x - 2)^2 = 0$, i.e. $x = 2 \notin D$.
So f has no critical number. Note that we still need to choose a test number where on the domain D so that we can determine if f is increasing or decreasing.
Test numbers: 0; $f'(0) = \frac{-2(0)^2+8(0)-11}{(0-2)^2} < 0$. f is decreasing everywhere.

F. Since the derivative does not change its sign, no local extreme value exist.

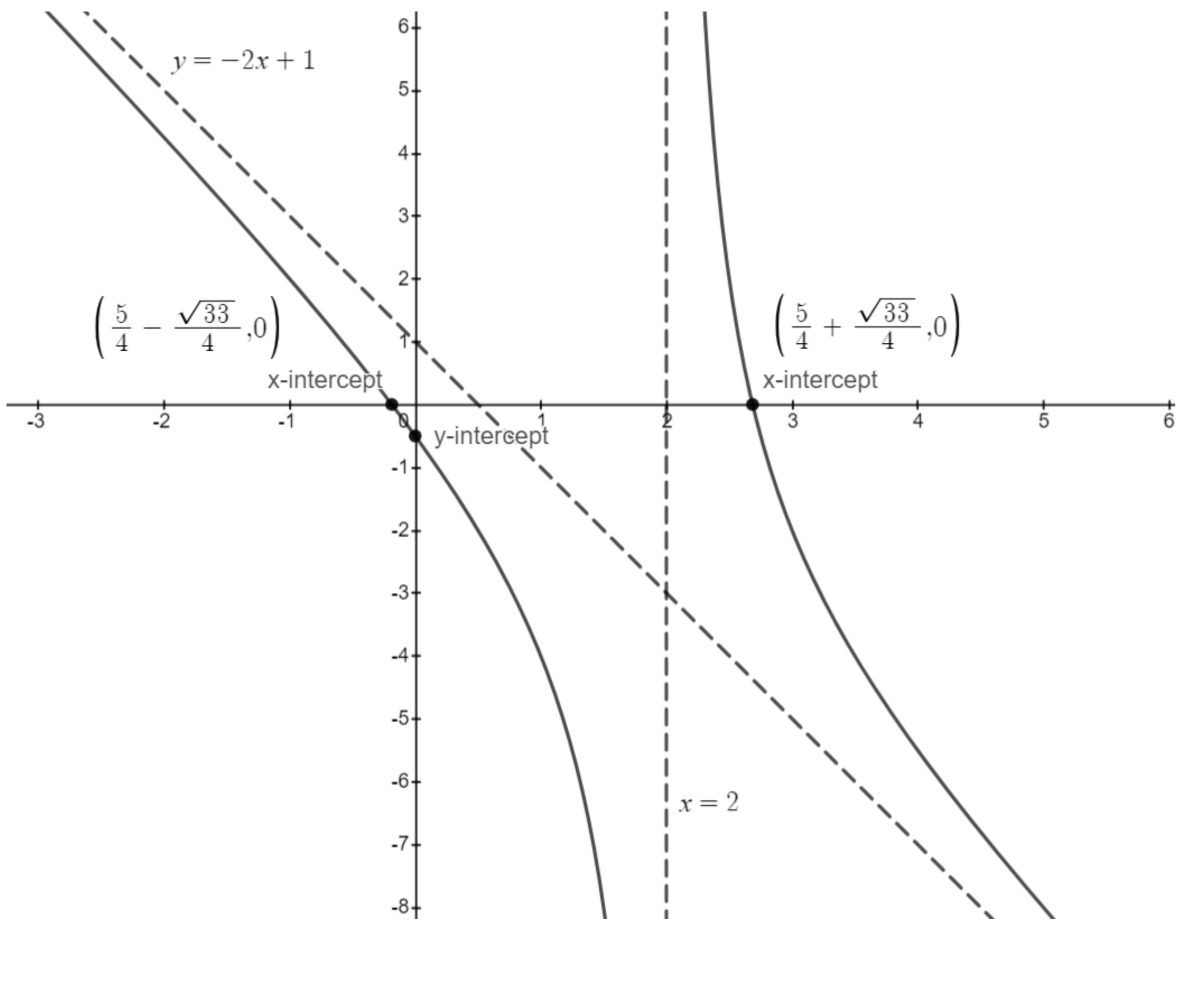
G. Using the quotient rule,

$$y'' = \frac{(-4x + 8)(x - 2)^2 - (-2x^2 + 8x - 11)(2(x - 2))}{((x - 2)^2)^2} = \cdots = \frac{6}{(x - 2)^3}$$

$y'' = 0$, i.e. $6 = 0$, which is not happening.
y'' is undefined if $(x-2)^3 = 0$, i.e. $x = 2 \notin D$.
Test numbers: 0 and 3; $f''(0) = \frac{6}{(0-2)^3} < 0$; $f''(3) = \frac{6}{(3-2)^3} > 0$.
f is concave down on $(-\infty, 2)$. f is concave up on $(2, \infty)$.
Although f is changing concavity over $x = 2$, we have a vertical asymptote there. Hence, no point of inflection.



H. Start with drawing two dotted lines $y = -2x + 1$ and $x = 2$.



Assigned Exercises: (p 321) 5, 11, 17, 21, 25, 29, 33, 67