Q: How to convince ourselves that a function $f(x)$ has a limit L as x approaches a?

We looked at many examples of finding limits of functions. Now it is time to make the definition of a limit more precisely so that even a mathematician will be satisfied.

The key is to make the verb 'approach' more rigorously. We say that the limit is the real number L that the function values approach as x approaches the real number a. There are two values that are approaching to two different values. How close do the functions values need to get to L? How close do the x values need to get to a? Since the limit is not about the actual function value $f(a)$, we will never let x to be actually a.

On the x-axis the distance between x (variable) and a (fixed value) is measured by an absolute value $|x-a|$. So saying that x is close to a means that $|x-a|$ is small.

On the y-axis the distance between $f(x)$ (varying function values depending on x) and L (proposed limiting value) is measured by an absolute value $|f(x)-L|$. Saying that $f(x)$ is close to L means that $|f(x)-L|$ is small.

Here is the logic behind the definition of a limit. We want to convince that $|f(x)-L|$ can be as small as we wish. We denote this small distance by a Greek letter ε (read 'epsilon' and probably chosen for εrror). So we write $|f(x)-L|<\varepsilon$. If $\varepsilon=0.001$, then $|f(x)-L|<0.001$, which means that the distance between the function values and the proposed limiting value is no more than 0.001 (within thousandths). If we solve this absolute value inequality, we obtain $-0.001<f(x)-L<0.001$, i.e. $L-0.001<f(x)<L+0.001$. If $\varepsilon=10^{-6}$, then $L-0.000001<f(x)<L+0.000001$. If we want them to be nano-scale close, then $\varepsilon=10^{-9}$. If we want them to be pico-scale close, then $\varepsilon=10^{-12}$. It depends on the unit, but 10^{-12} is very small, so very close. But we are not satisfied with it. We will not specify how small, so we do not say what ε is. We call that "arbitrarily close" because the value of ε is arbitrary (unspecified). Just keep in mind that $|f(x)-L|<\varepsilon$ is equivalent to $L-\varepsilon<f(x)<L+\varepsilon$ and ε is wickedly small.

The same is true for $|x-a|$. Except that we use a Greek letter δ (read 'delta' and probably chose for δifference or δistance) to denote a small distance. So we write $|x-a|<\delta$. The distance δ, however, is not arbitrary as we need to make sure that δ is small 'enough' so that when x is close to a within δ distance, the function values $f(x)$ will be close to L within ε distance. This is correct way because $f(x)$ is depending on the value of x (independent variable).

Example 1 It is obvious that

$$\lim_{x\to 2}(2x+1)=5$$

Here, $f(x)=2x+1$, $a=2$, and $L=5$. For this example, we will actually choose a specific value for ε and want to explore how δ should be chosen carefully so that $|f(x)-L|<\varepsilon$, i.e. $|(2x+1)-5|<\varepsilon$. Suppose we want the function values to be close to 5 within tenths, that is,

$\varepsilon = 0.1$ or $\frac{1}{10}$. Let us see what that meant in terms of mathematical symbols:

$$|f(x) - L| < \varepsilon \quad \Rightarrow \quad |(2x+1) - 5| < \frac{1}{10} \quad \Rightarrow \quad |2x - 4| < \frac{1}{10}$$

It would be a good time to remind you that $|ab| = |a||b|$ for any real numbers a and b.

$$|2(x-2)| < \frac{1}{10} \quad \Rightarrow \quad 2|x-2| < \frac{1}{10} \quad \Rightarrow \quad |x-2| < \frac{1}{20}$$

So if we let $\delta = \frac{1}{20}$, then the condition $|x-2| < \frac{1}{20}$ will guarantee the condition $|(2x+1)-5| < \frac{1}{10}$ as all of the implications above can be reversed. Let us check.

$$\begin{aligned}
|x-2| < \frac{1}{20} \quad &\Rightarrow \quad -\frac{1}{20} < x - 2 < \frac{1}{20} \\
&\Rightarrow \quad 2 - \frac{1}{20} < x < 2 + \frac{1}{20} \\
&\Rightarrow \quad \frac{39}{20} < x < \frac{41}{20} \\
&\Rightarrow \quad \frac{39}{10} < 2x < \frac{41}{10} \\
&\Rightarrow \quad \frac{39}{10} + 1 < 2x + 1 < \frac{41}{10} + 1 \\
&\Rightarrow \quad \frac{49}{10} < 2x + 1 < \frac{51}{10} \\
&\Rightarrow \quad \frac{49}{10} - 5 < (2x+1) - 5 < \frac{51}{10} - 5 \\
&\Rightarrow \quad -\frac{1}{10} < (2x+1) - 5 < \frac{1}{10} \\
&\Rightarrow \quad |(2x+1) - 5| < \frac{1}{10}
\end{aligned}$$

Precise Definition of a Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of** $f(x)$ **as** x **approaches** a **is** L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

The example above just verified that the inequalities hold for $\varepsilon = \frac{1}{10}$ by carefully choosing $\delta = \frac{1}{20}$. The definition requires "for every number ε (arbitrary choice of ε)." Note that there is nothing special about $\varepsilon = \frac{1}{10}$. We could find the expression for δ for any ε.

$$|(2x+1)-5| < \varepsilon \quad \Rightarrow \quad |2x-4| < \varepsilon \quad \Rightarrow \quad |2(x-2)| < \varepsilon \quad \Rightarrow \quad 2|x-2| < \varepsilon \quad \Rightarrow \quad |x-2| < \frac{\varepsilon}{2}$$

So $\delta = \frac{\varepsilon}{2}$ would work. Again we can verify that

$$\begin{aligned}
|x-2| < \frac{\varepsilon}{2} \quad &\Rightarrow \quad -\frac{\varepsilon}{2} < x - 2 < \frac{\varepsilon}{2} \\
&\Rightarrow \quad 2 - \frac{\varepsilon}{2} < x < 2 + \frac{\varepsilon}{2} \\
&\Rightarrow \quad 4 - \varepsilon < 2x < 4 + \varepsilon \\
&\Rightarrow \quad 5 - \varepsilon < 2x + 1 < 5 + \varepsilon \\
&\Rightarrow \quad -\varepsilon < (2x+1) - 5 < \varepsilon \\
&\Rightarrow \quad |(2x+1) - 5| < \varepsilon
\end{aligned}$$

The argument here to find δ in terms of ε is rather easy because the function is linear. If the function is anything other than a linear function, then finding δ is <u>not</u> easy.

However, in this class, we only deal with the precise definition of a limit for linear functions only. So learn the way by observing the examples in the notes.

If asked to prove that $\lim_{x\to a} f(x) = L$ using the precise definition of a limit, follow the steps:

Step 1: Write the inequality $|f(x) - L| < \varepsilon$ as $-\varepsilon < f(x) - L < \varepsilon$.
Step 2: Put the inequality as $-\delta < x - a < \delta$. Here, δ would be an expression in terms of ε.
Step 3: Start the proof with "Let $\varepsilon > 0$ be given. Let δ be <u>insert the expression from Step 2</u>."
Step 4: Start from $|x - a| < \delta$ and keep deriving expressions until $|f(x) - L| < \varepsilon$ is obtained.

Example 2 Prove that

$$\lim_{x\to\frac{1}{2}} \left(\frac{2}{3}x - 3\right) = -\frac{8}{3}$$

Preparation: Here, $f(x) = \frac{2}{3}x - 3$, $a = \frac{1}{2}$, and $L = -\frac{8}{3}$.

Step 1: $|f(x) - L| < \varepsilon \Rightarrow |(\frac{2}{3}x - 3) - (-\frac{8}{3})| < \varepsilon \Rightarrow -\varepsilon < (\frac{2}{3}x - 3) - (-\frac{8}{3}) < \varepsilon$

Step 2: $-\varepsilon < \frac{2}{3}x - \frac{1}{3} < \varepsilon \Rightarrow -3\varepsilon < 2x - 1 < 3\varepsilon \Rightarrow -3\varepsilon + 1 < 2x < 3\varepsilon + 1 \Rightarrow -\frac{3}{2}\varepsilon + \frac{1}{2} < x < \frac{3}{2}\varepsilon + \frac{1}{2}$.
Finally, $-\frac{3}{2}\varepsilon < x - \frac{1}{2} < \frac{3}{2}\varepsilon$. So $\delta = \frac{3}{2}\varepsilon$.

Now the actual proof begins here with Step 3 and Step 4.

Proof: Let $\varepsilon > 0$ be given. Let $\delta = \frac{3}{2}\varepsilon$.

$$\begin{aligned}
\left|x - \frac{1}{2}\right| < \frac{3}{2}\varepsilon \quad &\Rightarrow \quad -\frac{3}{2}\varepsilon < x - \frac{1}{2} < \frac{3}{2}\varepsilon \\
&\Rightarrow \quad -\frac{3}{2}\varepsilon + \frac{1}{2} < x < \frac{3}{2}\varepsilon + \frac{1}{2} \\
&\Rightarrow \quad \frac{2}{3}\left(-\frac{3}{2}\varepsilon + \frac{1}{2}\right) < \frac{2}{3}x < \frac{2}{3}\left(\frac{3}{2}\varepsilon + \frac{1}{2}\right) \\
&\Rightarrow \quad -\varepsilon + \frac{1}{3} < \frac{2}{3}x < \varepsilon + \frac{1}{3} \\
&\Rightarrow \quad -\varepsilon + \frac{1}{3} - 3 < \frac{2}{3}x - 3 < \varepsilon + \frac{1}{3} - 3 \\
&\Rightarrow \quad -\varepsilon - \frac{8}{3} < \frac{2}{3}x - 3 < \varepsilon - \frac{8}{3} \\
&\Rightarrow \quad -\varepsilon < \left(\frac{2}{3}x - 3\right) - \left(-\frac{8}{3}\right) < \varepsilon
\end{aligned}$$

Hence, $|(\frac{2}{3}x - 3) - (-\frac{8}{3})| < \varepsilon$. Therefore, $\lim_{x\to\frac{1}{2}} \left(\frac{2}{3}x - 3\right) = -\frac{8}{3}$. QED

Assigned Exercises: (p 113) 13, 15, 17, 19