We just need to know the derivatives of $\sin(x)$ and $\cos(x)$. Since four other trig functions can be written as quotients using $\sin(x)$ and/or $\cos(x)$, we can find their derivatives using the quotient rule.

$$\frac{d}{dx}[\sin(x)] = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

For this limit, we need to know two not-so-obvious limits:

$$\boxed{\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1 \quad \text{and} \quad \lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} = 0}$$

Using the sine summation formula $\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$,

$$\begin{aligned}
\frac{\sin(x+h)-\sin(x)}{h} &= \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \\
&= \frac{\sin(x)\cos(h)-\sin(x)+\sin(h)\cos(x)}{h} \\
&= \frac{\sin(x)\cos(h)-\sin(x)}{h} + \frac{\sin(h)\cos(x)}{h} \\
&= \sin(x)\cdot\frac{\cos(h)-1}{h} + \cos(x)\cdot\frac{\sin(h)}{h}
\end{aligned}$$

Then the derivative of the sine function is

$$\begin{aligned}
\frac{d}{dx}[\sin(x)] &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\left(\sin(x)\cdot\frac{\cos(h)-1}{h} + \cos(x)\cdot\frac{\sin(h)}{h}\right) \\
&= \lim_{h\to 0}\sin(x)\cdot\frac{\cos(h)-1}{h} + \lim_{h\to 0}\cos(x)\cdot\frac{\sin(h)}{h} \\
&= \sin(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \sin(x)\cdot 0 + \cos(x)\cdot 1 \\
&= \cos(x)
\end{aligned}$$

$$\boxed{\frac{d}{dx}[\sin(x)] = \cos(x) \quad \text{or} \quad (\sin(x))' = \cos(x)}$$

We can use the cosine sum formula $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ to show that

$$\boxed{\frac{d}{dx}[\cos(x)] = -\sin(x) \quad \text{or} \quad (\cos(x))' = -\sin(x)}$$

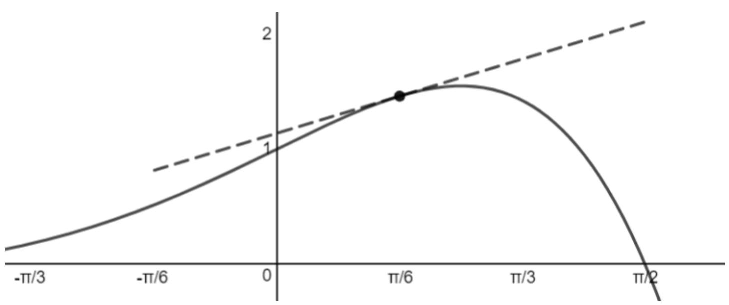
Knowing that $\frac{d}{dx}[\sin(x)] = \cos(x)$ and $\frac{d}{dx}[\cos(x)] = -\sin(x)$, for instance, we can use the quotient rule to find the derivatives of $\tan(x)$, $\cot(x)$, $\csc(x)$, and $\sec(x)$.

$$\begin{aligned}
(\tan(x))' &= \left(\frac{\sin(x)}{\cos(x)}\right)' \\
&= \frac{(\sin(x))'\cos(x) - \sin(x)(\cos(x))'}{(\cos(x))^2} = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)
\end{aligned}$$

Derivatives of Trigonometric Functions

$\frac{d}{dx}[\sin(x)] = \cos(x)$	$\frac{d}{dx}[\cos(x)] = -\sin(x)$
$\frac{d}{dx}[\tan(x)] = \sec^2(x)$	$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$
$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$	$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$

Example 1 Find the equation of the tangent line to the curve of $f(x) = e^x\cos(x)$ at $(\frac{\pi}{6}, \frac{\sqrt{3}}{2}e^{\pi/6})$. The derivative is $f'(x) = e^x\cos(x) - e^x\sin(x) = e^x(\cos(x) - \sin(x))$.



Slope at $x = \frac{\pi}{6}$: $m = \frac{\sqrt{3}-1}{2}e^{\pi/6}$

$$\begin{aligned}
y - \tfrac{\sqrt{3}}{2}e^{\pi/6} &= \tfrac{\sqrt{3}-1}{2}e^{\pi/6}(x - \tfrac{\pi}{6}) \\
y - 1.46 &= 0.62(x - 0.52) \\
y &= 0.62x + 1.14
\end{aligned}$$

Example 2 Find the 2019$^{\text{th}}$ derivative of $y = \sin(x)$, that is $y^{(2019)}$.

So we take the derivatives 2019 times. Let's do this! $y' = \cos(x)$, $y'' = -\sin(x)$, $y''' = -\cos(x)$, $y^{(4)} = -(-\sin(x)) = \sin(x)$. Note that we are back to the original function $y = \sin(x)$, so $y^{(5)}$ would be the same as $y' = \cos(x)$. We say that the derivatives of $\sin(x)$ is periodic with period 4. For instance, $y^{(6)} = y^{(6-4)} = y^{(2)} = y'' = -\sin(x)$. If m is divided by 4 and leaves the remainder d, then $y^{(m)} = y^{(d)}$. When 2019 is divided by 4, the remainder is 3. Hence, $y^{(2019)} = y^{(3)} = y''' = -\cos(x)$.

Assigned Exercises: (p 196) 1 - 15 (odds), 23, 31, 33, 37, 51, 53