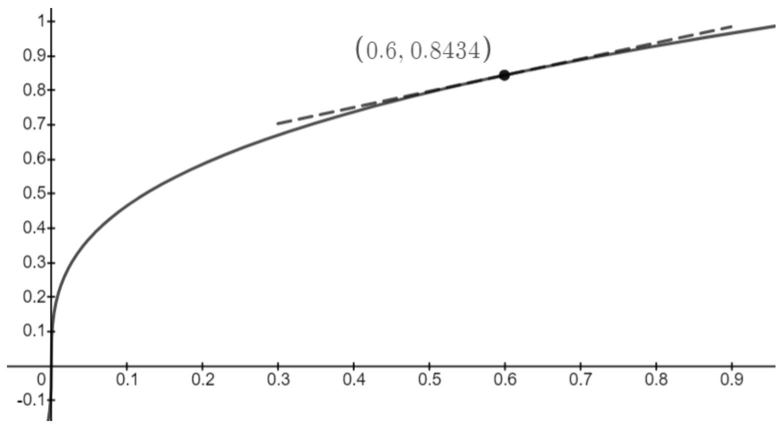
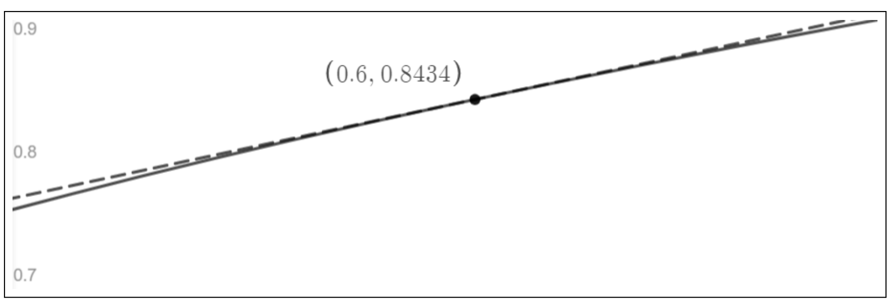
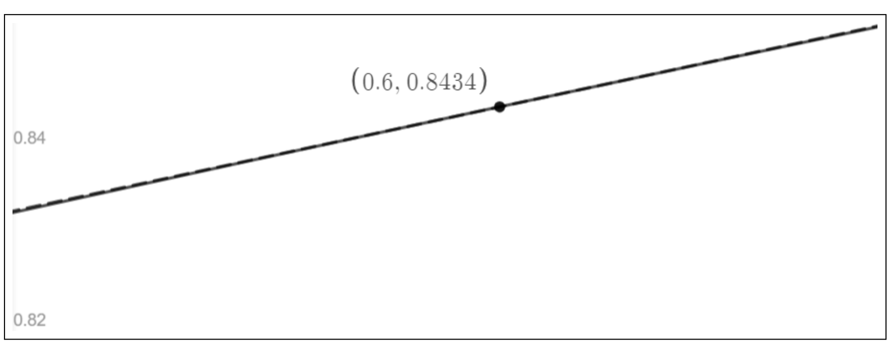
Suppose you look at the earth from ISS, you agree that the earth is somewhat sphere looking, but in the middle of the ocean you don't really feel that the surface of water is curved.

If a function $f(x)$ is "differentiable" at a, i.e. $f'(a)$ exists, then there is not much of difference between the small portion of the graph near $(a, f(a))$ and the small portion of the tangent line to the graph at the point $(a, f(a))$.



There seems to be small gap between the graph of the function $f(x) = \sqrt[3]{x}$ and its graph of the tangent line at $P(0.6, \sqrt[3]{0.6})$. The gap becomes even smaller when we are closer to the point P. We can see this phenomenon even better if we zoom in about the point P. The following graphs are zoomed-in images. In the second image, we cannot even see much difference between the graph of the function and the tangent line to the naked eyes.





This is a feature of differentiability. If you imagine how a graph fails to be differentiable at a number, this phenomenon is impossible. Recall that the equation of the tangent line to the graph of a function $y = f(x)$ at the point $P(a, f(a))$ is $y - f(a) = f'(a)(x - a)$, i.e.

$$y = f(a) + f'(a)(x - a)$$

Since the tangent line is pretty close to the graph, we can approximate the output value $f(x)$ using the output value of the tangent line equation.

In the previous example, the tangent line equation is $y - \sqrt[3]{0.6} = \frac{1}{3}(0.6)^{-2/3}(x - 0.6)$, i.e. $y = 0.4686x + 0.5623$. So we can approximate the output value of $f(x) = \sqrt[3]{x}$ using the output value of $y = 0.4686x + 0.5623$. Here is the table of output values.

x	approximated value	true output value $f(x)$	error = |approximate - true|
0.4000	0.7497	0.7368	0.0129
0.5000	0.7966	0.7937	0.0029
0.5900	0.8388	0.8387	0.0001
0.5999	0.8434	0.8434	0.0000
0.6000	0.8435	0.8434	0.0000
0.6001	0.8455	0.8435	0.0000
0.6100	0.8481	0.8481	0.0001
0.7000	0.8903	0.8879	0.0024
0.8000	0.9372	0.9283	0.0089

When $f(x)$ is approximated by the tangent line, we call it a **tangent line approximation** of f at a or a **linear approximation** of f at a. Here, the number a is important as the approximating is not good when the input value x is further away from a. In the example, the approximation is only good over the interval $(0.6 - \varepsilon, 0.6 + \varepsilon)$ for small ε. Note that $\varepsilon = 0.2$ is too large to approximate the output value within two decimal places as $|L(0.4) - f(0.4)| = 0.0129 \not< 0.01$. We need $\varepsilon = 0.1$ to approximate the output values within two decimal places.

The linear function

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

is also called a **linearization** of f at a.

Differentials

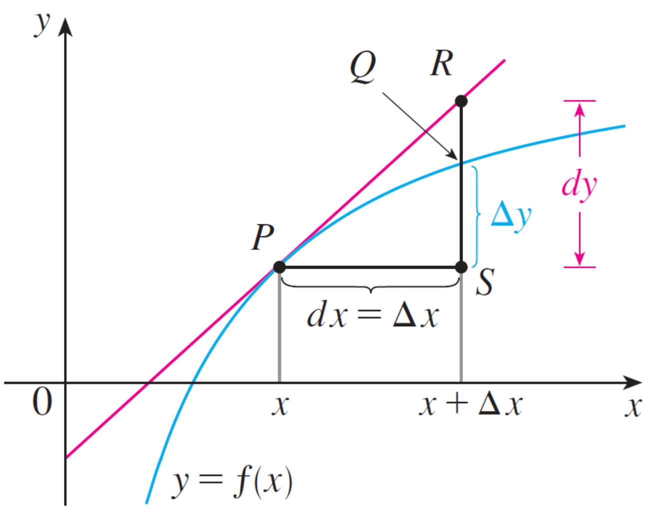
If $y = f(x)$ is a differentiable function, then the expression dx (non-separable notation) is an independent variable called a **differential**. There is a corresponding dependent variable

dy which is also called a differential. The differential dy is defined as follows:

$$\boxed{dy = f'(x)dx}$$

So dy depends on the values of x and dx. The differential dx usually represents a very very small change of the x-values, and the differential dy is then corresponding small change of the y-value when the very very small change of the x-values is made to the specific value of x.

Here is the geometric interpretation of the differentials dx and dy:



x: specific input value

Δx: actual change of x-values

Δy: actual change of y-values
$\Delta y = f(x + \Delta x) - f(x)$
called the **true error**.

dx: usually taken to be Δx

dy: $L(x + \Delta x) - L(x)$,
where $L(x)$ is the linearization at x, called **approximated error**.

P is $(x, f(x))$, Q is $(x+\Delta x, f(x+\Delta x))$, Q, R is $(x+\Delta x, L(x+\Delta x))$, and S is $(x+\Delta x, f(x))$. Then dy is the distance between R and S, i.e. if $dx = \Delta x$, then

$$\begin{aligned} dy &= L(x + \Delta x) - f(x) \\ &= f(x) + f'(x)((x + \Delta x) - x) - f(x) \\ &= f'(x)((x + \Delta x) - x) \\ &= f'(x)\Delta x \\ &= f'(x)dx \end{aligned}$$

We would like to believe that

$$\boxed{\Delta y \approx dy}$$

when $dx = \Delta x$ is very very small. Basically, we are saying that the linear approximation at x is pretty good way to estimate the function.

Example 1 Let $y = e^{x+1/x}$ be a function. It is differentiable everywhere on its domain $(-\infty, 0) \cup (0, \infty)$. The derivative is

$$f'(x) = e^{x+1/x} \cdot \left(1 - \frac{1}{x^2}\right) = \frac{e^{x+1/x}(x^2 - 1)}{x^2}$$

At $x = 2$, let $dx = \Delta x = 0.01$. So we go from $x = 2$ to $x = 2.01$ (small change). Without actually computing, we want to estimate the difference in y-values, i.e. Δy, using the differential dy. Since $dy = f'(x)dx$,

$$dy = \frac{e^{2+1/2}((2)^2-1)}{(2)^2} \cdot 0.01 = \frac{3e^{5/2}}{400} = 0.0914$$

Let us now compute the actual difference in the y-values to see how this estimate is good.

$$\Delta y = f(2+0.01) - f(2) = e^{2.01+1/2.01} - e^{2+1/2} = 0.0919$$

There is little difference, but still dy is a pretty good estimate of Δy.

The ratio

$$\boxed{\frac{\Delta y}{y}}$$

is called the **relative error** in y. Of course, $\frac{dx}{x}$ is the relative error in x. When a relative error in a variable is expressed as a percent form, it is called the **percentage error** of the variable.

Example 2 Use a linear approximation (or differentials) to estimate $\sqrt{100.5}$.

Since 100.5 is close to 100 and $\sqrt{100}$ can be easily computed, we use the linear approximation at 100. Let $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$. The linear approximation at 100 is $L(x) = f(100) + f'(100)(x-100) = \sqrt{100} + \frac{1}{2\sqrt{100}}(x-100) = 10 + \frac{1}{20}(x-100)$, i.e.

$$L(x) = \frac{1}{20}x + 5$$

Therefore, $\sqrt{100.5} \approx L(100.5) = \frac{1}{20}(100.5) + 5 = 10.025$.

<u>In differential notation</u>: $100.5 = 100 + \Delta x = 100 + dx$ where $\Delta x = dx = 0.5$. Then

$$\Delta y \approx dy = f'(x)dx = f'(100) \cdot 0.5 = \frac{1}{2\sqrt{100}} \cdot 0.5 = 0.025$$

Hence, $\sqrt{100.5} = f(100) + \Delta y \approx \sqrt{100} + dy = 10 + 0.025 = 10.025$. We still do not know the value of Δy, and we do not want to find out. The relative error $\frac{\Delta y}{y}$ can be also approximated by $\frac{dy}{y} = \frac{0.025}{10} = 0.0025$ in y, and its percent error is 0.25%. The relative error in x is $\frac{\Delta x}{x} = \frac{0.5}{100} = 0.005$, and its percent error is 0.5%.

Assigned Exercises: (p 256) 1, 3, 5, 11 - 31 (odds), 33, 35