Example 1 (Easy) A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into a circle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) a minimum?

Let x be the length for the portion to be made into a circle. Then the rest of the length is $10-x$ which will be used for a square.

Circle: Since the circumference of the circle is x, its radius can be found by the formula $x = 2\pi r$. Hence, $r = \frac{x}{2\pi}$. Then the area of the circle is $A_C = \pi(\frac{x}{2\pi})^2 = \frac{x^2}{4\pi}$.

Square: Since the perimeter of the square is $10-x$, its side length s would be $s = \frac{10-x}{4}$. Then its area would be $A_S = s^2 = (\frac{10-x}{4})^2 = \frac{(10-x)^2}{16}$.

The total area enclosed is $A(x) = A_C + A_S = \frac{x^2}{4\pi} + \frac{(10-x)^2}{16}$ with the domain of discourse is $[0, 10]$. Technically, the domain should be $(0, 10)$, but the closed interval makes things little easier.

Optimization: i) The derivative is

$$A'(x) = \frac{2x}{4\pi} + \frac{2(10-x)(-1)}{16} = \frac{x}{2\pi} + \frac{x-10}{8} = \frac{4x+\pi(x-10)}{8\pi} = \frac{(4+\pi)x - 10\pi}{8}$$

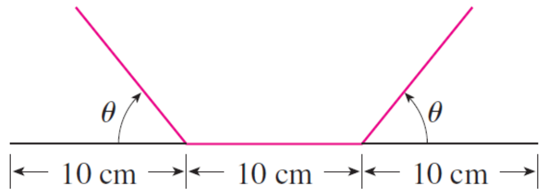
ii) $A'(x) = 0 \quad \Rightarrow \quad \frac{(4+\pi)x-10\pi}{8} = 0 \quad \Rightarrow \quad x = \frac{10\pi}{4+\pi}$ is the only critical number.

iii) $A(0) = \frac{100}{16} = 6.25$, $A(\frac{10\pi}{4+\pi}) = 3.5006$, and $A(10) = \frac{100}{4\pi} = 7.9577$.

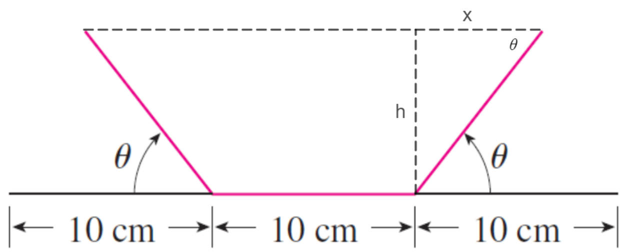
Note that the total area enclosed is a continuous function on the closed interval $[0, 10]$. By the Extreme Value Theorem, if we are allowed to not make a square, then the maximum total area enclosed is 7.9577 m^2 (by just bending the wire to make a circle without cutting). The minimum total area enclosed is 3.5006 m^2.

<u>To answer the original question</u>, we should not cut the wire and just make it into a circle to obtain the maximum total area enclosed. Or if we cut the wire into two pieces with lengths $\frac{10\pi}{4+\pi} = 4.3990$ m (for circle) and $10 - \frac{10\pi}{4+\pi} = 5.6001$ m (for square) to obtain the minimum total area enclosed.

Example 2 (Normal) A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ. How should θ be chosen so that the gutter will carry the maximum amount of water?



Goal: Maximize the area of the trapezoid. To find the area of the trapezoid, we need to find the height h and the top base length $2x + 10$. Since we are concerned about the angle θ, we need to express them as functions of θ.



The most obvious relation is $\sin(\theta) = \frac{h}{10}$ and $\cos(\theta) = \frac{x}{10}$. Then the area of the trapezoid is

$$A(\theta) = \frac{1}{2}(10+(10+2x))h = (10+x)h = (10+10\cos(\theta))10\sin(\theta) = 100(\sin(\theta)+\sin(\theta)\cos(\theta))$$

with the domain of discourse is $[0, \frac{\pi}{2}]$.

Optimization: i) The derivative is

$$A'(\theta) = 100(\cos(\theta) + \cos(\theta)\cos(\theta) + \sin(\theta)(-\sin(\theta))) = 100(\cos(\theta) + \cos^2(\theta) - \sin^2(\theta))$$

ii) $A'(\theta) = 0 \quad \Rightarrow \quad 100(\cos(\theta) + \cos^2(\theta) - \sin^2(\theta)) = 0 \quad \Rightarrow \quad \cos(\theta) + \cos^2(\theta) - \sin^2(\theta) = 0$
Using the Pythagorean identity, $\cos(\theta) + \cos^2(\theta) - (1 - \cos^2(\theta)) = 0$
$2\cos^2(\theta) + \cos(\theta) - 1 = 0$ which can be factored as $(2\cos(\theta) - 1)(\cos(\theta) + 1) = 0$.
Hence, $\cos(\theta) = \frac{1}{2}$, which yields the solutions $\theta = \pm\frac{\pi}{3} + 2\pi k$. The other possibility is $\cos(\theta) = -1$, which yields the solutions $\theta = \pi + 2\pi k$. The only solution in the domain $[0, \frac{\pi}{2}]$ is $\frac{\pi}{3}$, so $\frac{\pi}{3}$ is the only critical number.

iii) $A(0) = 100(0 + 0) = 0$, $A(\frac{\pi}{3}) = 100(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{2}) = 75\sqrt{3} = 129.9038$, and $A(\frac{\pi}{2}) = 100(1 + 1 \cdot 0) = 100$.

Since the area function $A(\theta)$ is a continuous function on the closed interval $[0, \frac{\pi}{2}]$, the absolute maximum value is 129.9038 cm^2 by the Extreme Value Theorem.

To answer the original question, we should bend the metal sheet by 60° so that the gutter will carry the maximum amount of water.

Assigned Exercises: (p 336) 13, 21, 23, 31, 45, 51, 73, 77