Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

exist. Then

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$
2. $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$
3. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cL$
4. $\lim_{x\to a} f(x)g(x) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x)) = LM$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ if $M \neq 0$.
6. $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n = L^n$ where n is a positive integer.
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer.
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer. (If n is even, we assume that $a \geq 0$.)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ where n is a positive integer. (If n is even, we assume that $\lim_{x\to a} f(x) \geq 0$.)

Direct Substitution Property If f is one of the polynomial, rational, root, power, exponential, logarithmic, trigonometric functions, or a function obtained by combining them in various ways and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

So most of time, the limit can be found by simply evaluating the function at $x = a$. Here are some uninteresting examples:

$$\lim_{x\to 2} x^2 + x = (2)^2 + 2 = 6 \qquad \lim_{x\to -2} \frac{x^2-4}{x^3-8} = \frac{(-2)^2-4}{(-2)^3-8} = 0 \qquad \lim_{x\to 0.5} \sqrt{1-x^2} = \sqrt{1-(0.5)^2} = \frac{\sqrt{3}}{2}$$

$$\lim_{x\to \pi} e^{\sin(x)} = e^{\sin(\pi)} = 1 \qquad \lim_{x\to 1} \ln(x^2) = \ln((1)^2) = 0 \qquad \lim_{x\to 0} \frac{3^x - x^4}{\cos(x)} = \frac{3^0-(0)^4}{\cos(0)} = 1$$

The interesting cases are when things go not as smoothly as you wish. In that case, there are few tricks we can pull out of a bag. If all these tricks run out, then we just go back to Section 2.2 and find the limit numerically or graphically.

Example 1 Find the limit.

$$\lim_{x\to 2}\frac{x^2-4}{x^3-8}$$

Say you didn't care if 2 is in the domain of the function $\frac{x^2-4}{x^3-8}$ or not. Naively just plug in.

$$\frac{(2)^2-4}{(2)^3-8}=\frac{0}{0}$$

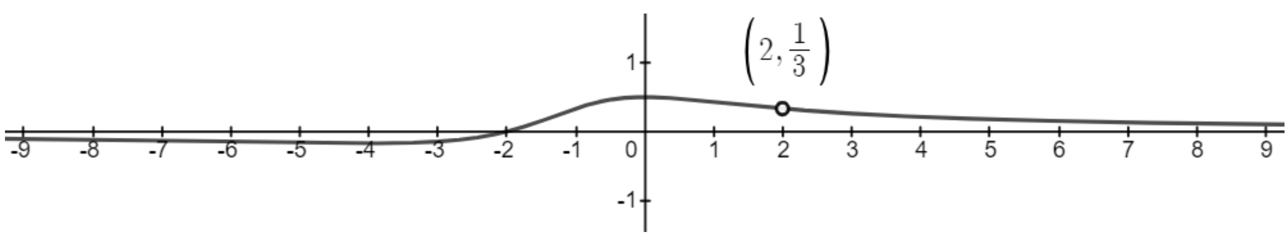
which is undefined. However, this does <u>not</u> mean that the limit does not exist. More efforts needed. Note that the numerator can be factored as $(x-2)(x+2)$ and the denominator as $(x-2)(x^2+2x+4)$. Then the rational expression can be simplified by canceling the common factor $x-2$.

$$\frac{x^2-4}{x^3-8}=\frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)}=\frac{x+2}{x^2+2x+4}$$

Important Remark: We can cancel the factor $x-2$ because the limit is about when x gets close to the value 2 but still different from 2. So $x-2$ is still nonzero. Then

$$\lim_{x\to 2}\frac{x^2-4}{x^3-8}=\lim_{x\to 2}\frac{x+2}{x^2+2x+4}=\frac{2+2}{(2)^2+2(2)+4}=\frac{1}{3}$$

Remember this trick. Review your factoring skill. By factoring and canceling, we removed the factor that causes the denominator to be zero. From a graphical point of view, this means that the graph has a point discontinuity at $(2,\frac{1}{3})$.



Example 2 Find the limit.

$$\lim_{h\to 0}\frac{(2+h)^4-16}{h}$$

Naively if we just plug 0 into h, then $\frac{(2+0)^4-16}{0}=\frac{0}{0}$ undefined! However, this does <u>not</u> mean that the limit does not exist. More efforts needed. This time, we expand the numerator instead of factoring. Using the Binomial Theorem,

$$(2+h)^4=\sum_{k=0}^{4}{}_4C_k(2)^{4-k}(h)^k=2^4+4\cdot 2^3h+6\cdot 2^2h^2+4\cdot 2h^3+h^4=16+32h+24h^2+8h^3+h^4$$

$$\frac{(2+h)^4-16}{h} = \frac{16+32h+24h^2+8h^3+h^4-16}{h} = \frac{h(32+24h+8h^2+h^3)}{h} = 32+24h+8h^2+h^3$$

Important Remark: We can cancel the factor h because the limit is about when h gets close to the value 0 but still different from 0. So h is still nonzero. Then

$$\lim_{h\to 0}\frac{(2+h)^4-16}{h} = \lim_{h\to 0}(32+24h+8h^2+h^3) = 32+24(0)+8(0)^2+(0)^3 = 32$$

Remember this trick. By canceling, we removed the factor that causes the denominator to be zero.

Example 3 Find the limit.

$$\lim_{x\to 2}\frac{\sqrt{8-x^2}-2}{x-2}$$

Naively if we just plug 2 into x, then $\frac{\sqrt{8-(2)^2}-2}{2-2} = \frac{0}{0}$ undefined! However, this does <u>not</u> mean that the limit does not exist. More efforts needed. This time, there is nothing to factor or expand. Observe!

$$\begin{aligned}
\frac{\sqrt{8-x^2}-2}{x-2}\cdot\frac{\sqrt{8-x^2}+2}{\sqrt{8-x^2}+2} &= \frac{(\sqrt{8-x^2}-2)(\sqrt{8-x^2}+2)}{(x-2)(\sqrt{8-x^2}+2)} \\
&= \frac{(\sqrt{8-x^2})^2-2^2}{(x-2)(\sqrt{8-x^2}+2)} \\
&= \frac{-x^2+4}{(x-2)(\sqrt{8-x^2}+2)} \\
&= \frac{-(x-2)(x+2)}{(x-2)(\sqrt{8-x^2}+2)} \\
&= -\frac{x+2}{\sqrt{8-x^2}+2}
\end{aligned}$$

Important Remark: We can cancel the factor $x-2$ because the limit is about when x gets close to the value 2 but still different from 2. So $x-2$ is still nonzero. Then

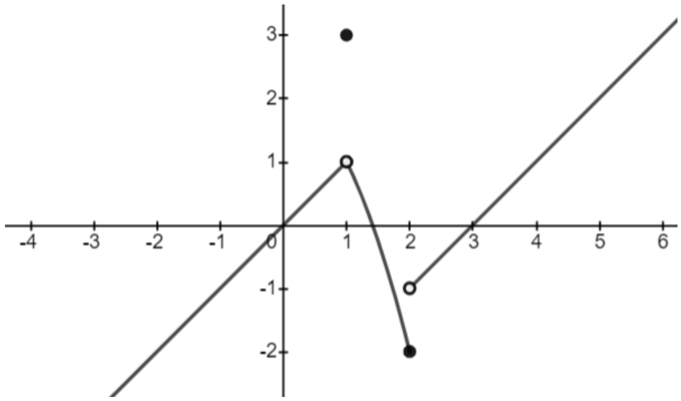
$$\lim_{x\to 2}\frac{\sqrt{8-x^2}-2}{x-2} = \lim_{x\to 2}-\frac{x+2}{\sqrt{8-x^2}+2} = -\frac{2+2}{\sqrt{8-(2)^2}+2} = -1$$

Remember this trick. By canceling, we removed the factor that causes the denominator to be zero.

Theorem 1

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

Example 4 Consider a piecewise defined function $g(x)$.



$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \le 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

Since x, 3, $2 - x^2$, and $x - 3$ are polynomials, the limits can be evaluated as $x \to a$ using the direct substitution property if a is some open interval contained in their domains. For a piecewise defined function, we need to be cautious when we find a limit as x approaches there the pieces are patched. For instance, as $x \to 1$ or as $x \to 2$ in this example. Based on the graph, we can easily see that the limit as $x \to 1$ exits as the left limit and the right limit agree.

$$\lim_{x \to 1} g(x) = 1 \quad \text{because} \quad \lim_{x \to 1^-} g(x) = 1 = \lim_{x \to 1^+} g(x)$$

However, the limit as $x \to 2$ does not exist as the one-sided limits are different.

$$\lim_{x \to 2} g(x) = \text{DNE} \quad \text{because} \quad \lim_{x \to 2^-} g(x) = -2 \neq -1 = \lim_{x \to 2^+} g(x)$$

But we should be able to find the one-sided limits without looking at the graph.

To evaluate the left limit of $g(x)$ as $x \to 1$, we would only use the piece x as the x-values considered are less than 1.

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x = 1$$

To evaluate the right limit of $g(x)$ as $x \to 1$, we would only use the piece $2 - x^2$ as the x-values considered are greater than 1 (but not too great to be greater than 2 to consider using the piece $x - 3$).

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (2 - x^2) = 2 - (1)^2 = 1$$

To evaluate the left limit of $g(x)$ as $x \to 2$, we would only use the piece $2 - x^2$ as the x-values considered are less than 2 (but not too less to be less than 1 to consider using the piece x).

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (2 - x^2) = 2 - (2)^2 = -2$$

To evaluate the right limit of $g(x)$ as $x \to 2$, we would only use the piece $x - 3$ as the x-values considered are greater than 2.

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (x - 3) = 2 - 3 = -1$$

Sandwich Theorem

Last special case is to use a theorem named **Sandwich theorem** or **Squeeze theorem**. We basically sandwich or squeeze the limiting value of the function $g(x)$ using the limits of two functions: one above $g(x)$ and one below $g(x)$.

Theorem 2 If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g are both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

Sandwich Theorem / Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

Recall that the limit of the function $f(x) = \sin(\frac{1}{x})$ as $x \to 0$ did not exist as it has a crazy oscillation near $x = 0$.

Example 5 Consider the function $g(x) = x\sin(\frac{1}{x})$. We want to find the limit

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$$

Note that

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

If $x > 0$, then

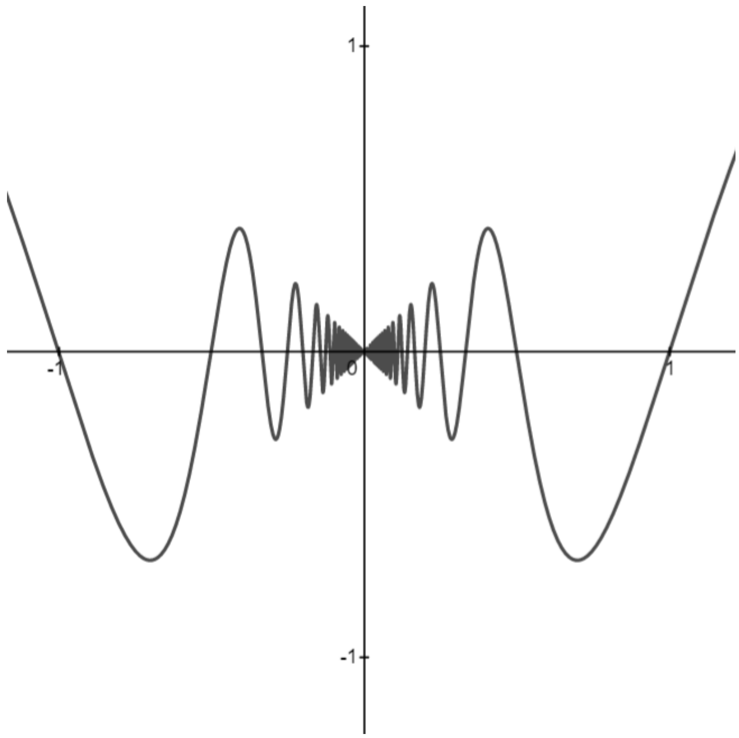
$$-1 \cdot x \leq x \cdot \sin\left(\frac{1}{x}\right) \leq 1 \cdot x \quad \Rightarrow \quad -x \leq x \sin\left(\frac{1}{x}\right) \leq x$$

Here $-x$ is the lower function and x is the upper function. If $x < 0$, then

$$-1 \cdot x \geq x \cdot \sin\left(\frac{1}{x}\right) \geq 1 \cdot x \quad \Rightarrow \quad -x \geq x \sin\left(\frac{1}{x}\right) \geq x$$

Here x is the lower function and $-x$ is the upper function.

Since $\lim_{x\to 0^-} -x = \lim_{x\to 0^-} x = 0$, by Sandwich Theorem, $\lim_{x\to 0^-} g(x) = 0$. Similarly, $\lim_{x\to 0^+} g(x) = 0$ since $\lim_{x\to 0^+} x = \lim_{x\to 0^+} -x = 0$ by the same theorem. So $\lim_{x\to 0} g(x) = 0$ as both one-sided limits agree.



Alternatively, we know that

$$\left| x \cdot \sin\left(\frac{1}{x}\right) \right| = |x| \left| sin\left(\frac{1}{x}\right) \right| \leq |x| \cdot 1 = |x|$$

Hence,

$$-|x| \leq x \cdot \sin\left(\frac{1}{x}\right) \leq |x|$$

In this case, $-|x|$ is the lower function and $|x|$ is the upper function. Since $\lim_{x\to 0} -|x| = 0 = \lim_{x\to 0} |x|$, by Sandwich Theorem, we can conclude that

$$\lim_{x\to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$$

Assigned Exercises: (p 102) 1 - 31 (odds), 35, 37, 39, 41, 43, 45, 49, 51