Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Q: When can a function f change from increasing to decreasing (or from decreasing to increasing)?
A: Possibly when the derivative $f'(x) = 0$. Also potentially when $f'(x)$ is not defined.

Local Extreme Values

Recall that a real number c in the domain of f is a critical number of a function f if $f'(c) = 0$ or $f'(c)$ is undefined.

First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(a) If f' does not change the sign at c, then f has no local extreme value at c.

Practically, here are the steps to follow to find intervals over which a function f is increasing or decreasing and identify local extreme values:

Step 1: Identify the domain of the function f, if possible.

Step 2: Find all of its critical numbers.

Step 3: Draw a number line and partition it using vertical lines at critical numbers into intervals.

Step 4: Pick a convenient number from each interval as a test point.

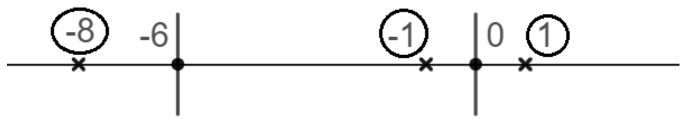
Step 5: If $f'(\text{test point}) > 0$, write $\oplus$ over and $\nearrow$ under the interval.
If $f'(\text{test point}) < 0$, write $\ominus$ over and $\searrow$ under the interval.

Step 6: f has a local maximum at c when the arrows changes from $\nearrow$ to $\searrow$ over c.
f has a local minimum at c when the arrows changes from $\searrow$ to $\nearrow$ over c.
If the arrows do not change the direction over c, then no local extreme value.

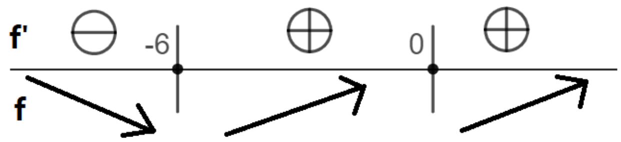
In case there is a local extreme at c, we say that the function f has a local maximum (or minimum) at c and the local maximum (or minimum) value is $f(c)$. Sometimes the coordinate is to be found. Then $(c, f(c))$.

Example 1 Consider a function $f(x) = x^4 + 8x^3 + 20$.

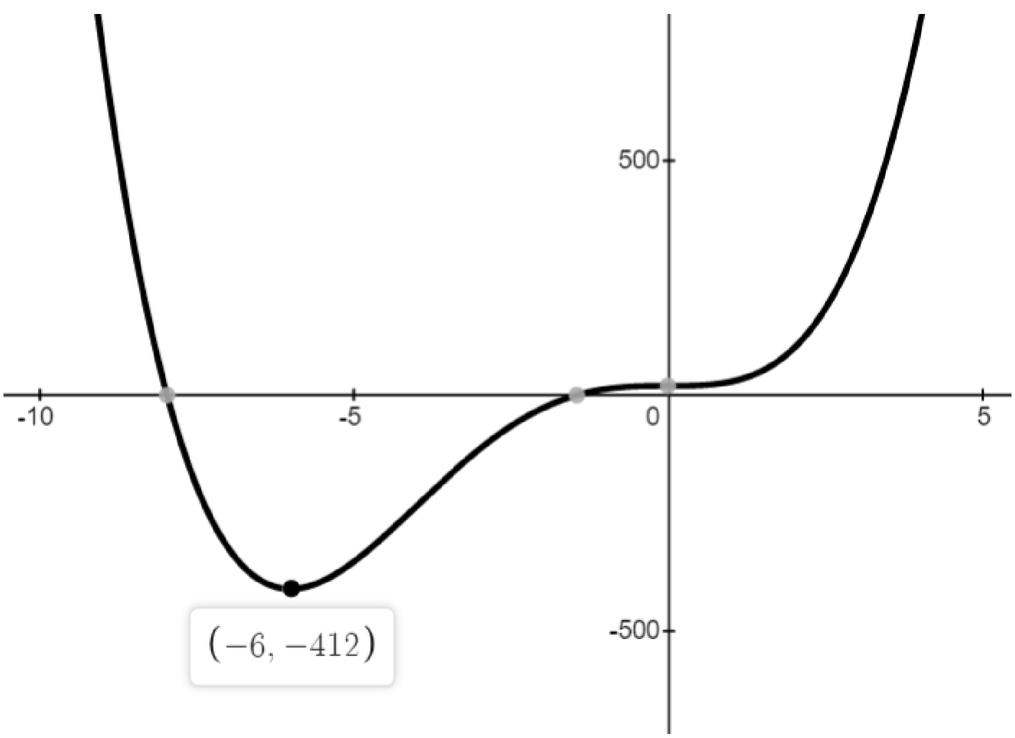
Step 1: Domain of f is $D = (-\infty, \infty)$.
Step 2: $f'(x) = 4x^3 + 24x^2 = 4x^2(x+6)$ set to 0, i.e. $4x^2(x+6) = 0$, i.e. $x = 0$ or -6. Hence, the critical numbers are $c = -6,\ 0$.
Step 3 & **Step 4**: Partition the number line at -6 and 0. Pick -8, -1, and 1 as test points.



Step 5: $f'(-8) = 4(-8)^2(-8+6) < 0$; $f'(-1) = 4(-1)^2(-1+6) > 0$; $f'(1) = 4(1)^2(1+6) > 0$.



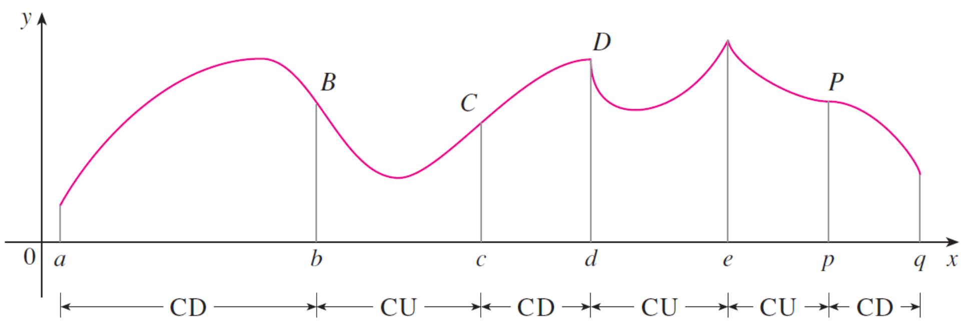
Step 6: The function f has a local minimum value of $f(-6) = (-6)^4 + 8(-6)^3 + 20 = -412$ at $x = -6$. The critical number 0 does not give any extreme value. There is no local maximum value.



What Does f'' Say About f?

If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Alternatively, if the line segment connecting any two points of the graph is below the graph, then concave downard. If the line segment connecting any two points of the graph is above the graph, then concave upward.



Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Practically, here are the steps to follow to find intervals over which a function f is concave upward or downward:

Step 7: Find the second derivative $f''(x)$.

Step 8: Find all values of c in the domain of f so that $f''(c) = 0$ or $f''(c)$ is undefined.

Step 9: Draw a number line and partition it using vertical lines at c into intervals.

Step 10: Pick a convenient number from each interval as a test point.

Step 11: If $f''(\text{test point}) > 0$, write $\oplus$ over and write "CU" under the interval.
If $f''(\text{test point}) < 0$, write $\ominus$ over and write "CD" under the interval.

Step 12: When the concavity changes over c, the point $(c, f(c))$ is an inflection point.

Example 1 (revisited)

Step 7: $f''(x) = 12x^2 + 48x = 12x(x+4)$

Step 8: $f''(x) = 0$, i.e. $12x(x+4) = 0$, i.e. $x = 0$ or $x = -4$.

Step 9 & **Step 10**: Partition the number line at -4 and 0. Pick -5, -1, and 1 as test points.

Step 11: $f''(-5) = 12(-5)(-5+4) > 0$; $f''(-1) = 12(-1)(-1+4) < 0$; $f''(1) = 12(1)(1+4) > 0$. So f has concave upward over the interval $(-\infty, -4)$ and $(0, \infty)$. f has concave downward over the interval $(-4, 0)$.

Step 12: Concavity changes over -4 and 0. Hence, $(-4, f(-4))$ and $(0, f(0))$ are inflection points of the graph. Verify the concavity with the graph above.

The second derivative can be used to determine the local extreme values.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

(c) If $f'(c) = 0$ and $f''(c) = 0$, then it is inconclusive.

Example 1 (revisited again) Recall that the function $f(x) = x^4 + 8x^3 + 20$ has two critical numbers -8 and 0. With the second derivative $f''(x) = 12x^2 + 48x$,

- $f''(-6) = 12(-6)^2 + 48(-6) > 0$, so f has a local minimum at $x = -6$.
- $f''(0) = 12(0)^2 + 48(0) = 0$, so it is inconclusive.

Assigned Exercises: (p 300) 1, 5, 9 - 31 (odds), 35, 37 - 55 (odds)