Goal: Find the derivative of a function formed by composing two functions.

Consider two functions $f(x) = x^4$ and $g(x) = x^2 + 3x + 5$ and their composition $h(x) = (f \circ g)(x) = f(g(x)) = (x^2 + 3x + 5)^4$. To differentiate $h(x)$, we have two options:

Option 1: Expand the fourth power and differentiate.

Using **Expand**$[(x^2 + 3x + 5)^4]$ on Wolfram Alpha, we get $h(x) = x^8 + 12x^7 + 74x^6 + 288x^5 + 771x^4 + 1440x^3 + 1850x^2 + 1500x + 625$. Then $h'(x) = 8x^7 + 84x^6 + 444x^5 + 1440x^4 + 3084x^3 + 4320x^2 + 3700x + 1500$. If the exponent was 12 instead of 4, then this option seems a bit impractical. Also without Wolfram Alpha, it could have been a challenge to expand the fourth power.

Option 2: Use the Product Rule many times.

Since $h(x) = (x^2 + 3x + 5)^2 \cdot (x^2 + 3x + 5)^2$, we first use the product rule to write $h'(x) = ((x^2 + 3x + 5)^2)'(x^2 + 3x + 5)^2 + (x^2 + 3x + 5)^2((x^2 + 3x + 5)^2)'$. To find $((x^2 + 3x + 5)^2)'$, we use the Product Rule again. $((x^2+3x+5)\cdot(x^2+3x+5))' = (x^2+3x+5)'(x^2+3x+5)+(x^2+3x+5)(x^2+3x+5)' = (2x+3)(x^2+3x+5)+(x^2+3x+5)(2x+3) = 2(2x+3)(x^2+3x+5)$. Hence, $h'(x) = 4(2x + 3)(x^2 + 3x + 5)^3$. This option is not feasible if the exponent is large.

Who would have thought that $8x^7+84x^6+444x^5+1440x^4+3084x^3+4320x^2+3700x+1500$ can be factored as $4(2x + 3)(x^2 + 3x + 5)^3$?

How about the Power Rule? It works for x^4. Like $\frac{d}{dx}[x^4] = 4x^3$, why not $\frac{d}{dx}[(x^2+3x+5)^4] = 4(x^2 + 3x + 5)^3$? Since we know the right answer, we should know that using the Power Rule does not work. However, they look almost the same. It is just missing an extra factor $2x + 3$, which happens to be the derivative of $x^2 + 3x + 5$, the inside function $g(x)$.

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $h = f \circ g$ defined by $h(x) = f(g(x))$ is differentiableat at x and h' is given by the product

$$h'(x) = f'(g(x)) \cdot (g'(x))$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

"Take the derivative of the outside and keep the inside. Multiply by the derivative of the inside."

Example 1 Let $f(x) = x^4$ and $g(x) = x^2+3x+5$. Define $h(x) = f(g(x)) = (x^2+3x+5)^4$. Since both the outside function f and the inside function g are differentiable everywhere, the composite $h = f \circ g$ is also differentiable by the Chain Rule. Note that the derivative of the outside function is $f'(x) = 4x^3$ and the derivative of the inside function is $g'(x) = 2x+3$. The derivative of h is

$$h'(x) = f'(g(x)) \cdot (g'(x)) = 4(x^2+3x+5)^3 \cdot (2x+3) = 4(2x+3)(x^2+3x+5)^3$$

If we write $y = f(u) = u^4$ and $u = x^2+3x+5$, then $\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = 2x+3$. Hence, $\frac{dy}{dx} = 4u^3 \cdot (2x+3) = 4(x^2+3x+5)^3 \cdot (2x+3) = 4(2x+3)(x^2+3x+5)^3$.

As differentiating a power of a function is quite ubiquitous,

Power of Function If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = n \cdot u^{n-1}\frac{du}{dx}$$

Alternatively, $\frac{d}{dx}[(g(x))^n] = n(g(x))^{n-1} \cdot \frac{d}{dx}[g(x)]$ or $((g(x))^n)' = n(g(x))^{n-1} \cdot (g'(x))$.

Example 2 Consider a function $h(x) = \sqrt{x^2+3x+5}$. Since taking a square root is same as raising $\frac{1}{2}$ power, we can rewrite h as $h(x) = (x^2+3x+5)^{1/2}$. Using the rule above, the derivative of h is

$$h'(x) = \frac{1}{2}(x^2+3x+5)^{-1/2} \cdot (2x+3) = \frac{1}{2(x^2+3x+5)^{1/2}} \cdot (2x+3) = \frac{2x+3}{2\sqrt{x^2+3x+5}}$$

Example 3 Consider a function $h(x) = e^{x^2+3x+5}$. The outside function is $f(x) = e^x$, and the inside function is $g(x) = x^2+3x+5$. Note that $f'(x) = e^x$ and $g'(x) = 2x+3$. Hence,

$$h'(x) = f'(g(x)) \cdot (g'(x)) = f'(x^2+3x+5) \cdot (2x+3) = e^{x^2+3x+5} \cdot (2x+3) = (2x+3)e^{x^2+3x+5}$$

Example 4 Consider a function $h(x) = 2^x$. By a property of log ($e^{\ln(b)} = b$), we can rewrite $2^x = e^{\ln(2)x}$. Then the derivative of h is

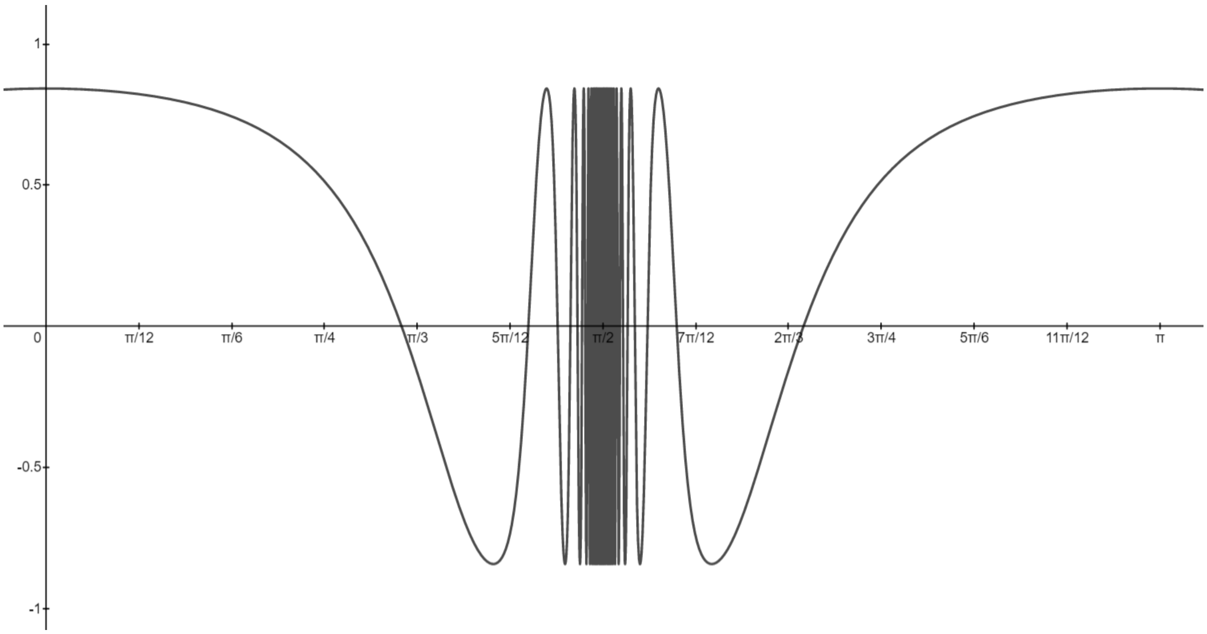
$$h'(x) = (e^{\ln(2)x})' = e^{\ln(2)x} \cdot \ln(2) = \ln(2)e^{\ln(2)x} = \ln(2)2^x$$

We should just remember this example as a rule.

$$\frac{d}{dx}[b^x] = \ln(b) \cdot b^x$$

Q: Why is the rule called the 'Chain' Rule?

Example 3 Consider a hideous function $h(x) = \sin(\cos(\tan(x)))$.



Let us not worry about where the function is differentiable. Here, $f(x) = \sin(x)$ is the outside function and $g(x) = \cos(\tan(x))$ is the inside function. Since $f'(x) = \cos(x)$, the Chain Rule imposes that

$$h'(x) = f'(g(x)) \cdot (g'(x)) = f'(\cos(\tan(x)) \cdot (g'(x)) = \cos(\cos(\tan(x))) \cdot \boxed{(\cos(\tan(x)))'}$$

As you can see inside the box, we need to find the derivative of the inside function $\cos(\tan(x))$, which is again a composite function. Hence, we need to use the Chain Rule again. For the function $\cos(\tan(x))$, the outside function is $\cos(x)$ and the inside function is $\tan(x)$. Using the Chain Rule again like a chain reaction, we have

$$(\cos(\tan(x)))' = -\sin(\tan(x)) \cdot (\tan(x))' = -\sin(\tan(x)) \cdot \sec^2(x)$$

Therefore, the derivative of h is

$$h'(x) = \cos(\cos(\tan(x))) \cdot (-\sin(\tan(x)) \cdot \sec^2(x)) = -\sec^2(x)\sin(\tan(x))\cos(\cos(\tan(x)))$$

Assigned Exercises: (p 204) 1, 5, 7 - 31 (odds), 35 - 41 (odds), 59, 61, 63, 65, 77, 83