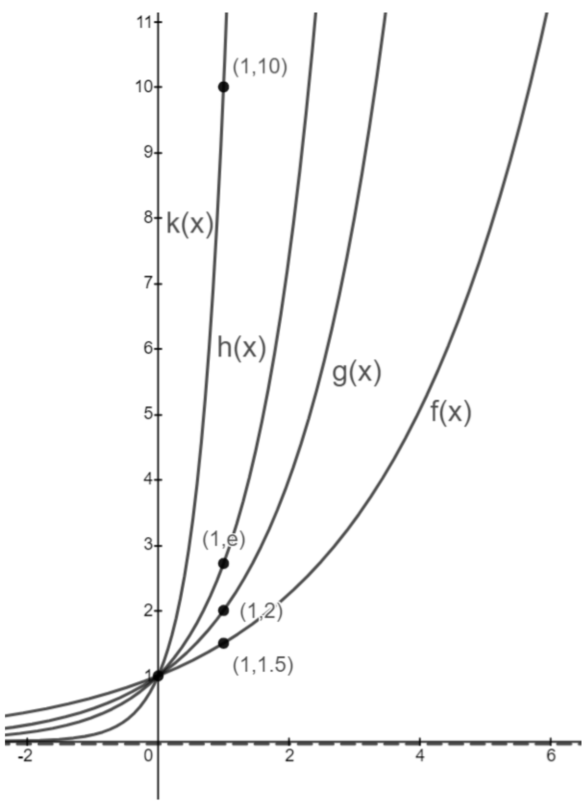
The function

$$f(x) = b^x,$$

where $b > 0$ and $b \neq 1$, is an **exponential function** with the base b.

1. If $b > 1$, it is a growth (increasing) model. If $0 < b < 1$, it is a decay (decreasing) model.
2. Its domain is $(-\infty, \infty)$ and range is $(0, \infty)$.
3. Since $f(0) = b^0 = 1$, the y-intercept is $(0, 1)$.
4. There is no value for x so that $b^x = 0$, so it has no x-intercept.
5. Three points $(-1, \frac{1}{b})$, $(0, 1)$, and $(1, b)$ are three guide points for graphing.



Growth models $(b > 1)$

$f(x) = 1.5^x$

$g(x) = 2^x$; double-life growth

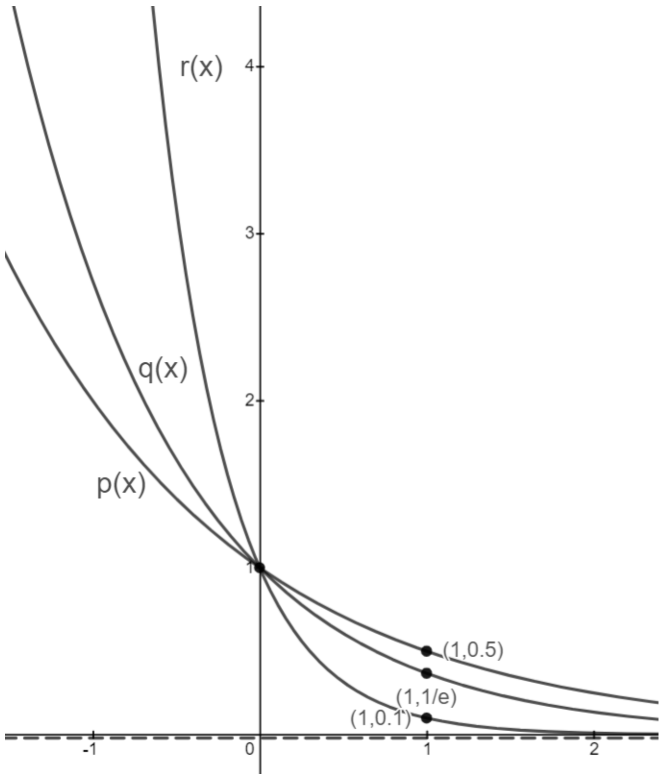
$h(x) = e^x$; e is called the **natural base**.
$e = 2.7182818284\ldots$ sine fine

$k(x) = 10^x$; 10 is a common base.

x	$f(x)$	$g(x)$	$h(x)$	$k(x)$
-10	0.017	0.001	0.00005	ten billionth
-1	0.67	0.5	0.37	0.1
0	1	1	1	1
1	1.5	2	e	10
10	57.7	1024	22026.5	ten billion

These functions grow really fast. If we fold a paper (0.005″) about 42 times, then the thickness of the folded paper can reach the moon.

Observe that if $x > 0$, then $f(x) < g(x) < h(x) < k(x)$. On the other hand, if $x < 0$, then $f(x) > g(x) > h(x) > k(x)$.



Decay models ($0 < b > 1$)

$p(x) = (\frac{1}{2})^x$ or 0.5^x; half-life decay

$q(x) = (\frac{1}{e})^x$ or e^{-x}

$r(x) = (\frac{1}{10})^x$

x	$p(x)$	$q(x)$	$r(x)$
-10	1024	22026.5	ten billion
-1	2	e	10
0	1	1	1
1	0.5	0.37	0.1
10	0.001	0.00005	ten billionth

These functions decay really fast.

No matter what the base is, the function values are getting close to the x-axis but not actually touching the x-axis. In this case, we call the x-axis a **horizontal asymptote** of the graph. When a horizontal asymptote is present, it is always recommended to draw it with dotted line. Its equation is $y = 0$ (the x-axis).

Summary of Exponential Function: For the exponential function $f(x) = b^x$,

· Domain = $(-\infty, \infty)$
· Range = $(0, \infty)$
· Horizontal Asymptote: $y = 0$

Exponent Rules: If a and b are positive numbers and x and y are any real numbers, then

$$b^{x+y} = b^x b^y \qquad b^{x-y} = \frac{b^x}{b^y} \qquad (b^x)^y = b^{xy} \qquad (ab)^x = a^x b^x$$

When the exponential functions are used for a mathematical modeling, it usually takes the form

$$f(t) = Ae^{kt}$$

where A is considered an initial value and k is the growth/decay constant. Note that $f(0) = Ae^0 = A$. Hence, the initial value. If $k > 0$, then growth. If $k < 0$, then decay.

Example 1 A bacteria culture starts with 500 bacteria and doubles in size every half hour.

(a) How many bacteria are there after 3 hours?

Time elapsed t (in hour)	0	0.5	1.0	1.5	2.0	2.5	3.0
Population P	500	$500 \cdot 2$	$500 \cdot 2^2$	$500 \cdot 2^3$	$500 \cdot 2^4$	$500 \cdot 2^5$	$500 \cdot 2^6$

There are $500 \cdot 2^6 = 32{,}000$ bacteria.

(b) How many bacteria are there after t hours?
It is reasonable to assume that $P(t) = 500 \cdot 2^{2t}$ or $500 \cdot 4^t$ where t is measured in hours.

(c) How many bacteria are there after 40 minutes?
Since 40 minutes is $\frac{40}{60}$ hours, $P(\frac{40}{60}) = 500 \cdot 2^{2(40/60)} = 1259.9211$ or 1260 bacteria.

Note 1: The time it takes for an amount of substance or a population to double is called the **double-life**. In the example, a half an hour is the double-life for the bacteria population.

Note 2: We could have modeled the population growth with t measured in minutes instead of hours. Then the model would be $Q(t) = 500 \cdot 2^{t/30}$. The answer for part (a) can be obtained using $Q(180)$, and the answer for part (c) would be obtained using $Q(40)$.

Example 2 Starting with the graph of $y = e^x$, write the equation of the graph that results from the following sequence of transformations: (a) reflecting about the x-axis, (b) shifting 4 units to the left, (c) stretching vertically by a factor of $\frac{3}{2}$, and (d) shifting 2 units upward.

$$y = e^x \quad \overset{\text{(a)}}{\Longrightarrow} \quad y = -e^x \quad \overset{\text{(b)}}{\Longrightarrow} \quad y = -e^{x+4} \quad \overset{\text{(c)}}{\Longrightarrow} \quad y = \frac{3}{2}(-e^{x+4}) \quad \overset{\text{(d)}}{\Longrightarrow} \quad y = -\frac{3}{2}e^{x+4} + 2$$

The final equation still has the domain $(-\infty, \infty)$, but the range is $(-\infty, 2)$ with the horizontal asymptote $y = 2$.

Example 3 Starting with the graph of $y = e^x$, write the equation of the graph that results from the following sequence of transformations: (a) stretching horizontally by a factor 10, (b) reflecting about the x-axis, (c) shifting 5 units downward, and (d) compressing vertically by a factor of 2.

$$y = e^x \quad \overset{\text{(a)}}{\Longrightarrow} \quad y = e^{\frac{1}{10}x} \quad \overset{\text{(b)}}{\Longrightarrow} \quad y = -e^{x/10} \quad \overset{\text{(c)}}{\Longrightarrow} \quad y = -e^{x/10} - 5 \quad \overset{\text{(d)}}{\Longrightarrow} \quad y = \frac{1}{2}(-e^{x/10} - 5)$$

That can be simplified to $y = -\frac{1}{2}e^{x/10} - \frac{5}{2}$. The final equation still has the domain $(-\infty, \infty)$, but the range is $(-\infty, -\frac{5}{2})$ with the horizontal asymptote $y = -\frac{5}{2}$.

Assigned Exercises: (p 53) 1, 3, 11, 13, 15, 17, 21, 23*, 31