Recall that $(e^x)' = e^x$ and $e^{\ln(x)} = x$. Let $y = \ln(x)$. Then $e^y = x$. We use implicit differentiation.

$$\begin{aligned} \frac{d}{dx}[e^y] &= \frac{d}{dx}[x] \\ e^y \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x} \end{aligned}$$

$$\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x} \quad \text{or} \quad (\ln(x))' = \frac{1}{x}}$$

Example 1 Consider a function $f(x) = \ln(\ln(x))$. What is the domain of the function? Its derivative is

$$f'(x) = \frac{1}{\ln(x)} \cdot (\ln(x))' = \frac{1}{\ln(x)} \cdot \frac{1}{x} = \frac{1}{x\ln(x)}$$

By Change of Base Theorem,

$$\log_b(x) = \frac{\log_e(x)}{\log_e(b)} = \frac{\ln(x)}{\ln(b)} = \frac{1}{\ln(b)}\ln(x)$$

Hence,

$$\frac{d}{dx}[\log_b(x)] = \frac{d}{dx}\left[\frac{1}{\ln(b)}\ln(x)\right] = \frac{1}{\ln(b)} \cdot \frac{1}{x} = \frac{1}{\ln(b)x}$$

$$\boxed{\frac{d}{dx}[\log_b(x)] = \frac{1}{\ln(b)x} \quad \text{or} \quad (\log_b(x))' = \frac{1}{\ln(b)x}}$$

However, it is little better to write it as $\frac{1}{\ln(b)} \cdot \frac{1}{x}$.

Example 2 Consider a function

$$g(x) = \log\left(\frac{x-1}{x+1}\right)$$

Recall that the base of log is 10. Hence, the derivative is

$$\begin{aligned} g'(x) &= \frac{1}{\ln(10)} \cdot \frac{1}{\frac{x-1}{x+1}} \cdot \left(\frac{x-1}{x+1}\right)' \\ &= \frac{1}{\ln(10)} \cdot \frac{x+1}{x-1} \cdot \frac{(x-1)'(x+1) - (x-1)(x+1)'}{(x+1)^2} \\ &= \frac{1}{\ln(10)} \cdot \frac{x+1}{x-1} \cdot \frac{(x+1) - (x-1)}{(x+1)^2} \\ &= \frac{1}{\ln(10)} \cdot \frac{x+1}{x-1} \cdot \frac{2}{(x+1)^2} = \frac{2}{\ln(10)(x-1)(x+1)} \end{aligned}$$

Here is a different perspective for taking the derivative using a property of logs. Since $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$, the function $g(x) = \log(x-1) - \log(x+1)$. Then

$$\begin{aligned}
g'(x) &= (\log(x-1) - \log(x+1))' \\
&= \frac{1}{\ln(10)} \cdot \frac{1}{x-1} \cdot (x-1)' - \frac{1}{\ln(10)} \cdot \frac{1}{x+1} \cdot (x+1)' \\
&= \frac{1}{\ln(10)} \cdot \frac{1}{x-1} - \frac{1}{\ln(10)} \cdot \frac{1}{x+1} \\
&= \frac{1}{\ln(10)} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) \\
&= \frac{1}{\ln(10)} \cdot \frac{2}{(x-1)(x+1)} \quad = \quad \frac{2}{\ln(10)(x-1)(x+1)}
\end{aligned}$$

We just avoided using the quotient rule!

Logarithmic Differentiation

As seen from the previous example, when taking the derivative of a function that is either product, quotient, or raised to power, it is sometimes easier if the log function is used to break them into small pieces. Divide and Conquer!

Before how this is done let us find the derivative of the expression $\ln(y)$.

$$\frac{d}{dx}[\ln(y)] = \frac{1}{y} \cdot \frac{dy}{dx} = \frac{y'}{y}$$

$$\boxed{\frac{d}{dx}[\ln(y)] = \frac{dy/dx}{y} \quad \text{or} \quad (\ln(y))' = \frac{y'}{y}}$$

Example 3 Consider a function

$$y = \frac{x^2(x^4+1)}{e^x\sqrt{x+e^x}}$$

First close your eyes and imagine what it takes to find the derivative of the function. Quotient rule. Product rule. Instead, we can apply the natural log function first and expand using the properties of log to the equation to get

$$\ln(y) = \ln\left(\frac{x^2(x^4+1)}{e^x\sqrt{x+e^x}}\right) = 2\ln(x) + \ln(x^4+1) - x\ln(e) - \frac{1}{2}\ln(x+e^x)$$

$$\begin{aligned}
(\ln(y))' &= (2\ln(x) + \ln(x^4+1) - x - \frac{1}{2}\ln(x+e^x))' \\
\frac{y'}{y} &= \frac{2}{x} + \frac{1}{x^4+1}(4x^3) - 1 - \frac{1}{2}\cdot\frac{1}{x+e^x}(1+e^x) \\
\frac{y'}{y} &= \frac{2}{x} + \frac{4x^3}{x^4+1} - 1 - \frac{1+e^x}{2(x+e^x)} \\
&= \frac{4(x^4+1)(x+e^x) + 4x^3(2x)(x+e^x) - 2x(x^4+1)(x+e^x) - x(1+e^x)(x^4+1)}{2x(x^4+1)(x+e^x)} \\
&= \frac{(x+e^x)(-2x^5+12x^4-2x+4) - (1+e^x)(x^5+x)}{2x(x^4+1)(x+e^x)} \\
&= \frac{-2x^6+12x^5-2x^2+4x+e^x(-2x^5+12x^4-2x+4) - x^5 - x - e^x(x^5+x)}{2x(x^4+1)(x+e^x)} \\
&= \frac{-2x^6+11x^5-2x^2+3x+e^x(-2x^5+12x^4-2x+4-x^5-x)}{2x(x^4+1)(x+e^x)} \\
&= \frac{-2x^6+11x^5-2x^2+3x+e^x(-3x^5+12x^4-3x+4)}{2x(x^4+1)(x+e^x)} \\
y' &= y\left(\frac{-2x^6+11x^5-2x^2+3x+e^x(-3x^5+12x^4-3x+4)}{2x(x^4+1)(x+e^x)}\right) \\
&= \frac{x^2(x^4+1)}{e^x\sqrt{x+e^x}}\left(\frac{-2x^6+11x^5-2x^2+3x+e^x(-3x^5+12x^4-3x+4)}{2x(x^4+1)(x+e^x)}\right) \\
&= \frac{x(-2x^6+11x^5-2x^2+3x+e^x(-3x^5+12x^4-3x+4))}{2e^x(x+e^x)\sqrt{x+e^x}}
\end{aligned}$$

Example 4 Consider a function $h(x) = x^x$. Then $\ln(h(x)) = \ln(x^x) = x\ln(x)$. Its derivative is

$$\begin{aligned}
(\ln(h(x)))' &= (x\ln(x))' \\
\frac{h'(x)}{h(x)} &= 1\cdot\ln(x) + x\cdot\frac{1}{x} \\
h'(x) &= h(x)(\ln(x)+1) \quad = \quad x^x(\ln(x)+1)
\end{aligned}$$

Natural Base e

Let $f(x) = \ln(x)$ and use the definition of a derivative to find $f'(1)$.

$$f'(1) = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{h\to 0}\frac{1}{h}\ln(1+h) = \lim_{h\to 0}\ln((1+h)^{1/h})$$

Since $f'(1) = 1$, we write $1 = \lim_{h\to 0}\ln((1+h)^{1/h})$. Since an exponential function is continuous, limit can slide through the continuous function.

$$\lim_{h\to 0}(1+h)^{1/h} = \lim_{h\to 0} e^{\ln((1+h)^{1/h})} = e^{\lim_{h\to 0}\ln((1+h)^{1/h})} = e^1 = e$$

Finally, let $n = \frac{1}{h}$. Then $h \to 0$ is equivalent to $n \to \infty$.

$$\lim_{h \to 0} (1+h)^{1/h} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

In fact, the last expression is the first expression used to defined the constant e.

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \quad \text{or} \quad e = \lim_{h \to 0} (1+h)^{\frac{1}{h}}$$

n	$(1+\frac{1}{n})^n$
1	2
1000	2.716923932
10^6	2.718280469
10^{12}	2.718281828

The last value is good to 9 decimal places of the decimal expansion of the constant e. The constant is irrational (cannot be written as a ratio of two integers) and transcendental (cannot be a solution to a polynomial equation with integer coefficients). It is the second most popular constant to π.

Here are some other incarnations of the constant.

$$e = \sum_{n=1}^{\infty} \frac{1}{n!} \qquad e = \lim_{n \to \infty} \frac{n}{\sqrt[n]{n!}}$$

$$e = 1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cfrac{1}{22 + \cfrac{1}{26 + \ddots}}}}}}}$$

Assigned Exercises: (p 223) 3 - 21 (odds), 27, 29, 39 - 49 (odds), 53