Definition of a Definite Integral If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0(= a), x_1, x_2, \ldots, x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_k^* lies in the ith subinterval $[x_{k-1}, x_k]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

The expression on the left hand side is read "integral of $f(x)$ from a to b (with respect to x)." It is <u>mostly</u> the area under the curve we discussed in the previous section.

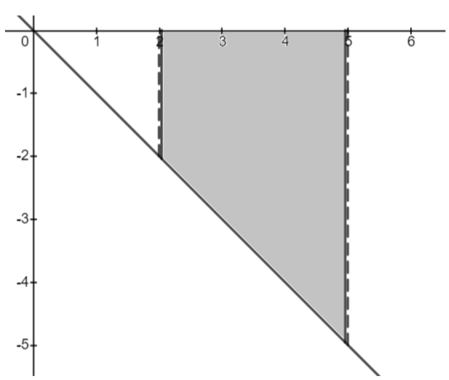
$f(x)$:	integrand
a and b:	limits of integration
a:	lower limit (of integration)
b:	upper limit (of integration)
dx:	does not have a name. It is always there, and x must be the independent variable of the integrand $f(x)$.

Finding the limit on the right hand side is called the **integration**. Nothing special about the variable x, so

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(\phi)\,d\phi$$

Example 1 Integrate the function $f(x) = -x$ from 2 to 5.

$$\int_2^5 -x\,dx$$



Since the shape of the region is a trapezoid, we can easily find the area without much trouble. The area shaded is $\frac{1}{2}(2+5)(3) = 10.5$. However, the integration is <u>not</u> 10.5. Why?

In fact, the integration should be -10.5. Why? Note that when we calculate the area of a rectangle, we used $f(x_k^*)$ as the height.

If the curve of the function $f(x)$ is below the x-axis, then the area would come out as negative. Hence, we obtain a **negative area**. Although it does not make sense that an area is negative, we will not try to fix this. Hence,

$$\int_2^5 -x\,dx = -10.5$$

Example 2 Integrate the function $g(t) = \cos(t)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$$\int_{-\pi/2}^{\pi/2} \cos(t)\,dt = 0$$

Example 3 Integrate the function $h(w) = \sin(w)$ from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

$$\int_{\pi/2}^{3\pi/2} \sin(w)\,dw = 0$$

Theorem 3 If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

Theorem 4 If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i \cdot \Delta x$.

The theorem tells that the limit of the RER estimate is good enough to find the limit. In fact, the theorem could be stated with the LER estimate on the right hand side. So let us actually calculate the area under some polynomial functions.

Example 4 Integrate $f(x) = 3x - 2$ from -3 to 4.

$$\int_{-3}^{4} (3x - 2)\,dx$$

It is basically the difference of areas of two right triangles. However, we will use the theorem to integrate.

Width: $\Delta x = \frac{4-(-3)}{n} = \frac{7}{n}$

Sample points: $x_k = -3 + k \cdot \frac{7}{n} = -3 + \frac{7k}{n}$

Riemann sum: $\sum_{k=1}^{n} f(-3+\frac{7k}{n})\cdot\Delta x = \sum_{k=1}^{n}(3(-3+\frac{7k}{n})-2)(\frac{7}{n})$

To take the limit, we need to simplify the Riemann sum.

$$\sum_{k=1}^{n}\left(3\left(-3+\frac{7k}{n}\right)-2\right)\left(\frac{7}{n}\right)=\sum_{k=1}^{n}\left(\frac{147k}{n^2}-\frac{77}{n}\right)$$

Here are some properties of the sigma notation we should recall:

1. $\displaystyle\sum_{i=m}^{n}(A_i+B_i)=\sum_{i=m}^{n}A_i+\sum_{i=m}^{n}B_i$
2. $\displaystyle\sum_{i=m}^{n}(A_i-B_i)=\sum_{i=m}^{n}A_i-\sum_{i=m}^{n}B_i$
3. $\displaystyle\sum_{i=m}^{n}C\cdot A_i=C\sum_{i=m}^{n}A_i$ where C is an expression that is not depending on the index i.

The Riemann sum becomes

$$\sum_{k=1}^{n}\left(\frac{147k}{n^2}-\frac{77}{n}\right)=\sum_{k=1}^{n}\frac{147k}{n^2}-\sum_{k=1}^{n}\frac{77}{n}=\frac{147}{n^2}\boxed{\sum_{k=1}^{n}k}-\frac{77}{n}\boxed{\sum_{k=1}^{n}1}$$

Summation Formula to be memorized:

$$\sum_{k=1}^{n}1=n$$

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)}{2}\cdot\frac{2n+1}{3}=\frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n}k^3=\left(\frac{n(n+1)}{2}\right)^2=\frac{n^2(n+1)^2}{4}$$

$$\frac{147}{n^2}\boxed{\sum_{k=1}^{n}k}-\frac{77}{n}\boxed{\sum_{k=1}^{n}1}=\frac{147}{n^2}\cdot\frac{n(n+1)}{2}-\frac{77}{n}\cdot n=\frac{147n(n+1)}{2n^2}-77$$

Then

$$\int_{-3}^{4}(3x-2)\,dx=\lim_{n\to\infty}\left(\frac{147n(n+1)}{2n^2}-77\right)=\frac{147}{2}-77=-3.5$$

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

where $\Delta x = \frac{b-a}{n}$.

Reversing the Limits of Integration:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

Properties of the Integral

1. $\displaystyle\int_a^b c\,dx = c(b-a),$ where c is any constant.
2. $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\displaystyle\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx,$ where c is any constant.
4. $\displaystyle\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
5. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
6. If $f(x)\geq 0$ for $a\leq x\leq b,$ then $\displaystyle\int_a^b f(x)\,dx \geq 0.$
7. If $f(x)\geq g(x)$ for $a\leq x\leq b,$ then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$
8. If $m\leq f(x)\leq M$ for $a\leq x\leq b,$ then

$$m(b-a)\leq \int_a^b f(x)\,dx \leq M(b-a).$$

Assigned Exercises: (p 388) 3 - 9 (odds), 17 - 25 (odds), 33 - 43 (odds), 47, 49, 55 - 63 (odds)