Derivatives of a Constant Function

$$\frac{d}{dx}[c] = 0 \quad \text{or} \quad (c)' = 0$$

Derivative of the Identity Function

$$\frac{d}{dx}[x] = 1 \quad \text{or} \quad (x)' = 1$$

The Power Rule: If n is a positive integer, then

$$\frac{d}{dx}[x^n] = n \cdot x^{n-1} \quad \text{or} \quad (x^n)' = n \cdot x^{n-1}$$

Proof. Binomial Theorem says $(x+h)^n = \sum_{k=0}^{n} {}_nC_k x^{n-k}h^k = x^n + h\sum_{k=1}^{n} {}_nC_k x^{n-k}h^{k-1}$.

$$\begin{aligned}
\frac{(x+h)^n - x^n}{h} &= \frac{x^n + h\sum_{k=1}^{n} {}_nC_k x^{n-k}h^{k-1} - x^n}{h} = \frac{h\sum_{k=1}^{n} {}_nC_k x^{n-k}h^{k-1}}{h} \\
&= \sum_{k=1}^{n} {}_nC_k x^{n-k}h^{k-1} = {}_nC_1 x^{n-1}h^{1-1} + \sum_{k=2}^{n} {}_nC_k x^{n-k}h^{k-1} \\
&= n \cdot x^{n-1} + h\sum_{k=2}^{n} {}_nC_k x^{n-k}h^{k-2}
\end{aligned}$$

The derivative of x^n is

$$(x^n)' = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\left(n \cdot x^{n-1} + h\sum_{k=2}^{n} {}_nC_k x^{n-k}h^{k-2}\right) = n \cdot x^{n-1}$$

Example 1 Let $f(x) = x^{180}$ be a power function. The derivative is $f'(x) = 180 \cdot x^{180-1} = 180x^{179}$.

The General Power Rule: If n is any real number, then

$$\frac{d}{dx}[x^n] = n \cdot x^{n-1} \quad \text{or} \quad (x^n)' = n \cdot x^{n-1}$$

Example 2 Let $g(x) = \sqrt{x}$ be a root function (or a power function with the exponent $\frac{1}{2}$). The derivative is $g'(x) = (\sqrt{x})' = (x^{1/2})' = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2} \cdot x^{-1/2} = \frac{1}{2} \cdot \frac{1}{x^{1/2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$.

Example 3 Let $h(x) = x^{\pi}$. The derivative is $h'(x) = \pi \cdot x^{\pi-1} = \pi x^{\pi-1}$.

Warning: You cannot use the rule if n is not a real number. For instance, the derivative of the function x^x is not $x \cdot x^{x-1}$. You cannot use the rule for an exponential function. The derivative of the function 2^x is not $x \cdot 2^{x-1}$.

The Constant Multiple Rule: If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)] \quad \text{or} \quad (c \cdot f(x))' = c \cdot f'(x)$$

Example 4 Let $q(x) = 4x^6$ be the four times the power function x^6. The derivative is $q'(x) = (4x^6)' = 4(x^6)' = 4(6 \cdot x^{6-1}) = 4(6x^5) = 24x^5$.

The Sum and Difference Rule: If f and g are differentiable functions, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \quad \text{or} \quad (f(x) + g(x))' = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] \quad \text{or} \quad (f(x) - g(x))' = f'(x) - g'(x)$$

Example 5 Let $u(x) = x^4$ and $v(x) = x^6$. Then the derivative of the sum of u and v is

$$(u(x) + v(x))' = u'(x) + v'(x) = 4 \cdot x^{4-1} + 6 \cdot x^{6-1} = 4x^3 + 6x^5$$

Example 6 Let $f(x) = 16$ and $g(x) = \sqrt[4]{x}$. Then the derivative of the difference of f and g is

$$(f(x) - g(x))' = f'(x) - g'(x) = (16)' - (x^{1/4})' = 0 - \frac{1}{4}x^{-3/4} = -\frac{1}{4\sqrt[4]{x^3}}$$

Example 7 Find the derivative of the function $h(x) = -x^2 + 3x - 18 + \frac{\sqrt{3}}{x^2}$.

$$\begin{aligned}
h'(x) &= (-x^2 + 3x - 18 + \sqrt{3}x^{-2})' \\
&= (-x^2)' + (3x)' - (18)' + (\sqrt{3}x^{-2})' \\
&= -(x^2)' + 3(x)' - (18)' + \sqrt{3}(x^{-2})' \\
&= -(2 \cdot x^{2-1}) + 3(1) - 0 + \sqrt{3}(-2 \cdot x^{-2-1}) \\
&= -2x^1 + 3 - 2\sqrt{3} \cdot x^{-3} \\
&= -2x + 3 - \frac{2\sqrt{3}}{x^3}
\end{aligned}$$

Strategy: i) Separate over sums and subtractions; ii) Factor out the constant multiples

Exponential Functions

Let us find the derivative of the exponential function with the natural base e.

$$(e^x)' = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = \lim_{h\to 0} e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x$$

Definition of the Number e

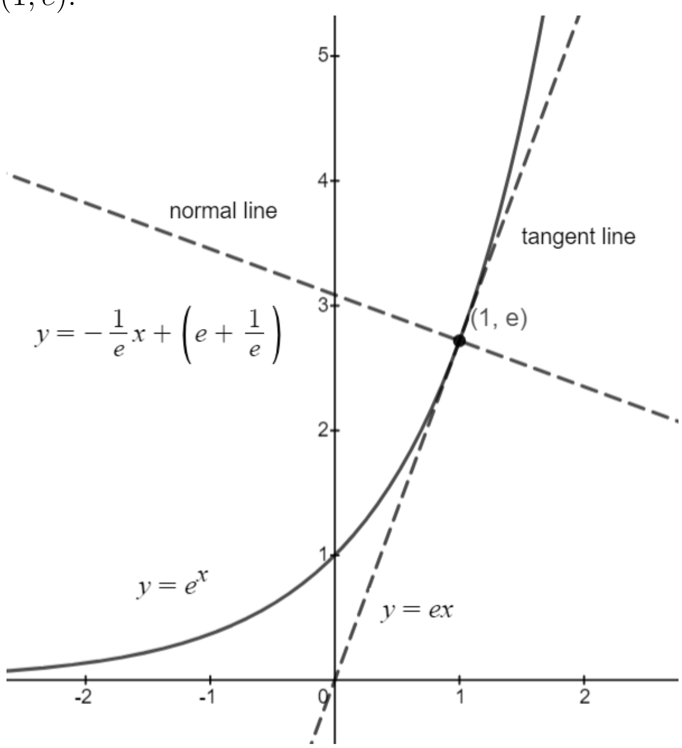
$$e \text{ is the number such that } \lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

Derivative of the Exponential Function

$$\frac{d}{dx}[e^x] = e^x \quad \text{or} \quad (e^x)' = e^x$$

The **normal line** to a curve C at a point P is the line through P and is perpendicular to the tangent line at P. In particular, the slope of the normal line is $-1/f'(a)$.

Example 8 Find the equation of the normal line to a curve of $f(x) = e^x$ at the point $P(1, e)$.



The slope function is $f'(x) = (e^x)' = e^x$. Hence, the slope of the tangent line would be $m = f'(1) = e^1 = e$. The tangent line has the equation $y - e = e(x - 1)$ or $y = ex$.

The normal line, then, has the slope $m_\perp = -1/e$ (perpendicular slope w. r.t. $m = e$). Thus, the equation of the normal line is $y - e = -\frac{1}{e}(x - 1)$ or $y = -\frac{1}{e}x + (e + \frac{1}{e})$.

Assigned Exercises: (p 180) 3 - 31 (odds), 39, 47, 49, 53, 55, 59, 67, 71, 73