Roughly a function is continuous at a number a if the graph of the function f on a some open interval (c, d) containing a can be drawn without lifting a pencil. More precisely,

> A function f is **continuous at a number** a if
> $$\lim_{x \to a} f(x) = f(a)$$

In words, if the limit of f at $x = a$ equals the function value of f at $x = a$, then f is continuous at $x = a$. We can rephrase the definition in 3 steps: The function f is continuous at a if

i) $f(a)$ is defined,
ii) $\lim_{x \to a} f(x)$ exists (as a real number, so no infinite limit), and
iii) $\lim_{x \to a} f(x) = f(a)$

On the other, f is **discontinuous at a number** a if any one of three steps fails. Hence,

α) If $f(a)$ is not defined, then f is discontinuous at a.
β) If $\lim_{x \to a} f(x) = \text{DNE}$ or $\pm\infty$, then f is discontinuous at a.
γ) If $\lim_{x \to a} f(x) \neq f(a)$, then f is discontinuous at a.

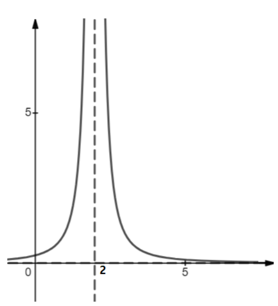
Example 1 The function $f(x) = \frac{1}{(x-2)^2}$ is discontinuous at 2 because $f(2)$ is undefined. It is also discontinuous at 2 because $\lim_{x \to 2} f(x) = \infty$ is not a real number. At everywhere else, however, the function f is continuous.

Example 2 The function

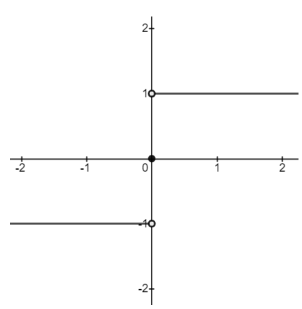
$$g(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is discontinuous at 0 as $\lim_{x \to 0} g(x) = \text{DNE}$. The left limit is -1, but the right limit is 1. At everywhere else the function g is continuous.

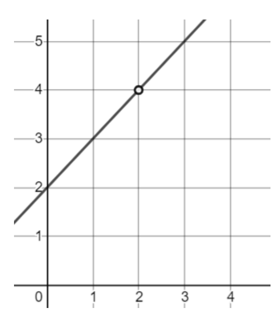
Example 3 The function $h(x) = \frac{x^2-4}{x-2}$ is discontinuous at 2 because $h(2)$ is undefined even $\lim_{x \to 2} h(x) = 4$ exists as a real number. At everywhere else the function h is continuous.



Example 1



Example 2



Example 3

Three Types of Discontinuities:

(1) If f has a vertical asymptote at a, then f is discontinuous at a. It is called an **infinite discontinuity**.
(2) If both one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist as real numbers but do not agree, then f is said to have a **jump discontinuity**.
(3) If $\lim_{x\to a} f(x) = L$ exists as a real number but discontinuous at a, then f is said to have a **removable discontinuity**.

In the last case, we can remove the discontinuity by redefining the function as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ L & \text{if } x = a \end{cases}$$

In **Example 1**, the function f has an infinite discontinuity at 2. The function g in **Example 2** has a jump discontinuity at 0. None of these discontinuities can be removed. The function h in **Example 3** has a removable discontinuity at 2. We can remove the discontinuity by redefining a function as follows:

$$r(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$$

Recall that $\lim_{x\to 2} h(x) = 4$. The graph of the function r is basically the graph of the function h with the hole at $(2, 4)$ is filled up with a solid point $(2, 4)$.

A function f is **continuous from the right at a number** a if

$$\lim_{x\to a^+} f(x) = f(a)$$

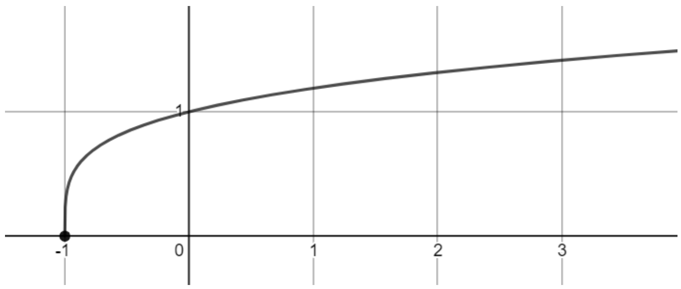
In words, if the right-hand limit of f at $x = a$ equals the function value of f at $x = a$, then f is continuous from right at $x = a$.

A function f is **continuous from the left at a number** a if

$$\lim_{x\to a^-} f(x) = f(a)$$

In words, if the left-hand limit of f at $x = a$ equals the function value of f at $x = a$, then f is continuous from left at $x = a$.

Example 4 Consider a function $f(x) = \sqrt[4]{x+1}$ with the domain $D(f) = [-1, \infty)$, the range $R(f) = [0, \infty)$, the x-intercept $(-1, 0)$, and the y-intercept $(0, 1)$.



The function f is continuous at every number in the domain except at -1.

Since $\lim_{x\to -1^+} f(x) = 0 = f(-1)$, the function f is continuous from the right at -1. However, $\lim_{x\to -1} f(x) = \text{DNE}$. So f is discontinuous at -1.

A function f is **continuous on an interval** if it is continuous at every number in the interval. If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right or continuous from the left.*

According to the definition, we could say that the function f in the previous example is continuous on the interval $[-1, \infty)$ (even though f is technically discontinuous at -1). Some very strict textbook will say that the function f is only continuous on $(-1, \infty)$ and is continuous from the right at -1.

Theorem 4 If f and g are continuous at a and c is a constant, then the following function are also continuous at a: $f + g$, $f - g$, cf, $f \cdot g$, and $\frac{f}{g}$ if $g(a) \neq 0$.

Theorem 5
(a) A polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.
(b) A rational function is continuous wherever it is defined; that is, it is continuous on its domain.
(c) $\sin(x)$ and $\cos(x)$ are continuous everywhere; that is, they are continuous on $\mathbb{R} = (-\infty, \infty)$.

Theorem 7 The following types of functions are continuous at every number in their domains: polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions

So most of the functions we deal with in Calculus are continuous on its domain. We only question continuity of a function at a when the evaluating the function at a goes wrong because a may not be in the domain of the function.

Not unless the function is a piecewise defined function, the function is discontinuous at a if $f(a)$ is undefined.

Theorem 8 If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

In words, we can slide the limit inside a continuous function, and this is useful when evaluating the limit of a composite function $f \circ g$ (or $f(g(x))$).

Example 5 We can evaluate the limit

$$\lim_{x\to 0} \cos(\sin(x))$$

as follows. $\lim_{x\to 0} \sin(x) = 0$, and $\cos(x)$ is continuous at 0. Hence, the limit can slide inside the outside function $\cos(x)$.

$$\lim_{x\to 0} \cos(\sin(x)) = \cos\left(\lim_{x\to 0} \sin(x)\right) = \cos(0) = 1$$

Theorem 9 If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Theorem 10 (Intermediate Value Theorem) Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Recall the function g in **Example 2**. Note that $g(-2) = -1$ and $g(2) = 1$, and $\frac{1}{2}$ $(= N)$ is between $g(-2)$ and $g(2)$. However, there is no number c between -2 and 2 such that $g(c) = \frac{1}{2}$. This does not contradict the theorem above as the function g is not continuous on the closed interval $[-2, 2]$. In particular, it is discontinuous at 0 which is inside the interval.

One of the application of the Intermediate Value Theorem (IVT) is to show an existence of a root of an equation on an open interval. A number that satisfies the equation is called a **root** of the equation. For instance, π is a root of the equation $\sin(x) - \cos(x) = 1$. An equation can have more than one root.

Example 6 Note that $\sin(0) - \cos(0) = -1$ and $\sin(\frac{3\pi}{4}) - \cos(\frac{3\pi}{4}) = \sqrt{2}$. Since the function $\sin(x) - cos(x)$ is continuous on the closed interval $[0, \frac{3\pi}{4}]$ and 1 is between -1 and $\sqrt{2}$, there must be a number c in $(0, \frac{3\pi}{4})$ such that $\sin(c) - \cos(c) = 1$. c is a root of the equation.

Assigned Exercises: (p 124) 3, 5, 7, 13, 15, 19, 21, 25, 27, 31, 39 - 45 (odds), 51 - 57 (odds)