Pretty much every variables appear in this section are functions of time variable t.

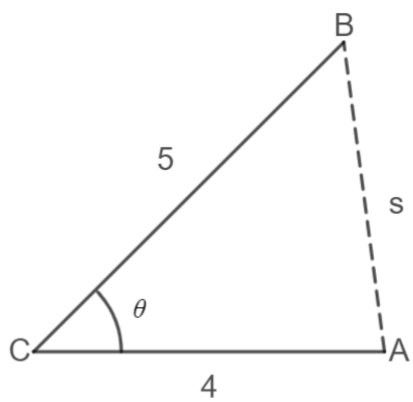
If the variable θ is used to measure an angle (in radian), it will be a function of time t (in seconds), i.e. $\theta(t)$. That means the angle θ most likely changes with time t. We do not know how θ is defined as a function of t and we will never know.

However, we will be given with

$$\frac{d\theta}{dt}$$

It is the (instantaneous) rate of change of the angle θ with respect to time t. Since we will be using this sort of rate of change w.r.t. time a lot, we will just refer as a **rate**. 'Given' rates are usually constants, so $\frac{d\theta}{dt} = k$ for some real number k.

1. If $\frac{d\theta}{dt} > 0$, then the angle is increasing by $\frac{d\theta}{dt}$ every unit of time.
2. If $\frac{d\theta}{dt} < 0$, then the angle is decreasing by $|\frac{d\theta}{dt}|$ every unit of time.
3. If $\frac{d\theta}{dt} = 0$, then the angle is not changing.



For instance, consider two line segments $\overline{AC}$ and $\overline{BC}$ with the same end point C. The segment $\overline{AC}$ has the length $AC = 4$ feet, and the segment $\overline{BC}$ has the length $BC = 5$ feet. Let θ be the angle $\angle ACB$. Let s be the length (in feet) of the third side of the triangle $\triangle ABC$.

As the angle changes its value, if AB and AC are fixed, then only the length s of the third side will change its value.

It does not matter what an initial value of the angle θ is. Suppose that the rate is $\frac{d\theta}{dt} = 0.2$. Then the angle θ is increasing by 0.2 radian every second. If the rate is $\frac{d\theta}{dt} = -\frac{\pi}{12}$. Then the angle θ is decreasing by $|-\frac{\pi}{12}| = \frac{\pi}{12}$ radian every second.

There is another function of time t in the system. The side length s is changing as the angle θ changes. As the angle θ is depending on time t and the side length s is depending on the angle θ, so is the side length s depending on time t as well. Hence, we can consider the side length s as a function of time t, i.e. $s(t)$. Do we know how the side length $s(t)$ is defined as a function of time t? Not really! Why? Because we did not know how $\theta(t)$ is defined from the beginning. We do not need to know how $s(t)$ is defined. At the end of the day, we only need to know the rate $\frac{ds}{dt}$, the (instantaneous) rate of change of the side length w.r.t. time t.

Q: If the rate $\frac{d\theta}{dt}$ is constant, will the rate $\frac{ds}{dt}$ be constant?

Here is one way to find out. We construct table of values of θ and corresponding s to see how the change of the angle θ affects the change of the length s.

Suppose that the rate $\frac{d\theta}{dt} = 0.2$ radians per second. The law of cosine $c^2 = a^2+b^2-2ab\cos(C)$ says that $s = \sqrt{4^2+5^2-2(4)(5)\cos(\theta)} = \sqrt{41-40\cos(\theta)}$.

θ	s	θ	s
0.5	$\sqrt{41-40\cos(0.5)} = 2.4283$	2.5	$\sqrt{41-40\cos(2.5)} = 8.5467$
0.7	$\sqrt{41-40\cos(0.7)} = 3.2259$	2.7	$\sqrt{41-40\cos(2.7)} = 8.7842$

When the angle changes from 0.5 to 0.7 (within 1 second), the average rate of change of the side length s is

$$\frac{\Delta s}{\Delta t} = \frac{3.2259-2.4283}{1} = 0.7979 \text{ ft/sec}$$

When the angle changes from 2.5 to 2.7 (within 1 second), the average rate of change of the side length s is

$$\frac{\Delta s}{\Delta t} = \frac{8.7842-8.5467}{1} = 0.2375 \text{ ft/sec}$$

Although they are average rate (not instantaneous rate), note that they are different even though the rate of angle is constant. Roughly we can conclude that the side length s is not changing with the constant rate. To be more precise, we need to find the (instantaneous) rate $\frac{ds}{dt}$.

The equation

$$s = \sqrt{41-40\cos(\theta)}$$

relates the side length s to the angle θ. Suppose we apply the differential operator

$$\boxed{\frac{d}{dt}}$$

that differentiates the operand with respect to the time variable t to the equation.

$$\begin{aligned}
\frac{d}{dt}[s] &= \frac{d}{dt}[\sqrt{41-40\cos(\theta)}] \\
\frac{ds}{dt} &= \frac{1}{2\sqrt{41-40\cos(\theta)}} \cdot \left(0-40(-\sin(\theta)) \cdot \frac{d\theta}{dt}\right) \\
\frac{ds}{dt} &= \frac{20\sin(\theta)}{\sqrt{41-40\cos(\theta)}} \cdot \frac{d\theta}{dt}
\end{aligned}$$

LHS is just $\frac{ds}{dt}$ as we do not know the definition of s as a function of t. The best we can write is $\frac{ds}{dt}$. On RHS, we used a chain rule. Also we don't know the definition of θ as a function of t. The best we can write is $\frac{d\theta}{dt}$.

We finally have the (instantaneous) rate $\frac{ds}{dt}$. Apparently, it is proportional to the rate $\frac{d\theta}{dt}$ with the proportionality constant changing with the specific value of the angle θ. That is why we had different average rates of the side lengths when we consider the change of the angles from 0.5 to 0.7 and 2.5 to 2.7.

Now we can calculate the precise rate of change of the side length at a specific value of the angle θ. Assuming that $\frac{d\theta}{dt} = 0.2$, for instance, at the very moment the angle is $\theta = 0.5$, the rate of the side length is

$$\frac{ds}{dt} = \frac{20\sin(\theta)}{\sqrt{41 - 40\cos(\theta)}} \cdot \frac{d\theta}{dt} = \frac{20\sin(0.5)}{\sqrt{41 - 40\cos(0.5)}} \cdot 0.2 = 0.7897 \text{ ft/sec}$$

At the very moment the angle is $\theta = 2.5$, the rate of the side length is

$$\frac{ds}{dt} = \frac{20\sin(\theta)}{\sqrt{41 - 40\cos(\theta)}} \cdot \frac{d\theta}{dt} = \frac{20\sin(2.5)}{\sqrt{41 - 40\cos(2.5)}} \cdot 0.2 = 0.2801 \text{ ft/sec}$$

The rate $\frac{ds}{dt}$ was obtained from a relation $s = \sqrt{41 - 40\cos(\theta)}$ of θ and s. The rate $\frac{ds}{dt}$ is related to the rate $\frac{d\theta}{dt}$ by the equation

$$\frac{ds}{dt} = \frac{20\sin(\theta)}{\sqrt{41 - 40\cos(\theta)}} \cdot \frac{d\theta}{dt}$$

Hence, they are called the **related rates**.

Related Rates Problem

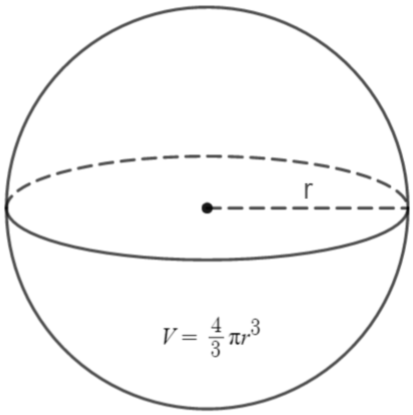
Solving a "Related Rates Problem" is to find the rate of a desired variable using given values of other variables and given rates of other variables. The following is the problem solving strategy suggested by the textbook and modified by me slightly:

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Assign meaningful variables to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives $\frac{d\,\text{VARIABLE}}{dt}$.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6. Use the Implicit Differentiation and the Chain Rule to differentiate both sides of the equation with respect to t. $\frac{d}{dt}[\text{LHS}] = \frac{d}{dt}[\text{RHS}]$
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example 1 (Easy one) The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

1. Do you understand terms used in the problem? Radius, Sphere, Volume, Diameter

2. Volume of Sphere $= \frac{4}{3}\pi r^3$ where r is the Radius. Diameter $= 2r$.



3. r: Radius of Sphere; V: Volume of Sphere; D: Diameter of Sphere. We consider all of them as functions of time t, i.e. $r(t)$, $V(t)$, and $D(t)$. However, we do not care about actually definitions.

4. Rate of radius $\frac{dr}{dt} = 4$ mm/s; Rate of volume $\frac{dV}{dt}$ = desired rate (in mm^3/s).

5. $V = \frac{4}{3}\pi r^3$

6. We apply the differential operator $\frac{d}{dt}$ to both sides of the equation:

$$\begin{aligned}\frac{d}{dt}[V] &= \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]\\ \frac{dV}{dt} &= \frac{4}{3}\pi \cdot \frac{d}{dt}\left[r^3\right]\\ &= \frac{4}{3}\pi\left(3r^2 \cdot \frac{dr}{dt}\right)\\ &= 4\pi r^2 \cdot \frac{dr}{dt}\end{aligned}$$

7. Note that $D = 2r$. Then $r = \frac{D}{2}$.

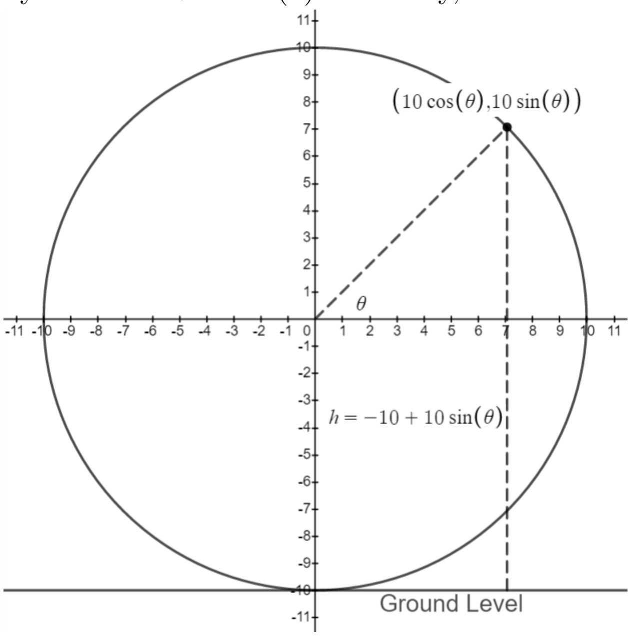
$$\frac{dV}{dt} = 4\pi\left(\frac{D}{2}\right)^2 \cdot \frac{dr}{dt} = 4\pi\left(\frac{80}{2}\right)^2 \cdot 4 = 25600\pi \approx 80424.7719$$

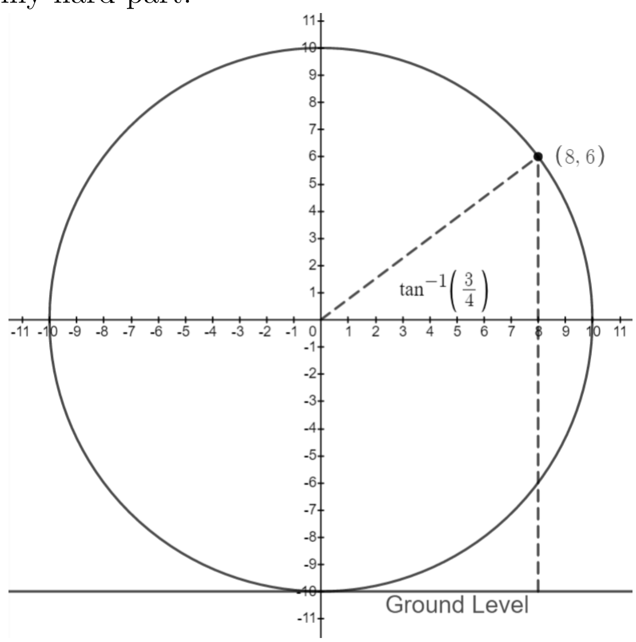
Therefore, the volume increasing at the rate 80,424.7719 mm^3/s when the diameter is 80 mm.

Example 2 (Hard one) A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?

1. Although it is not mentioned (but should have been mentioned), we assume that the Ferris wheel sits right above the ground level. Maybe it does not matter.

2. We model the location of a rider using a scaled unit circle. The coordinate on the circle is $(10\cos(\theta), 10\sin(\theta))$, but the height of the rider from the ground level ($y = -10$) is described by $h = -10 + 10\sin(\theta)$. Actually, this is the only hard part.





3. h: height of a rider (in meters); θ: usual angle from Trigonometry.

4. Since the wheel rotates at a rate of one revolution every 2 minutes, $\frac{d\theta}{dt} = \frac{2\pi}{2} = \pi$ rad/min. $\frac{dh}{dt}$: rate of height (in m/min).

5. Obviously, the relation involving h and θ is $h = -10 + 10\sin(\theta)$.

6. $\frac{d}{dt}[h] = \frac{d}{dt}[-10 + 10\sin(\theta)] \quad \Rightarrow \quad \frac{dh}{dt} = 10\cos(\theta) \cdot \frac{d\theta}{dt}$

7. Note that we do not worry about the height 16 m until now. From the diagram, we can see that $\cos(\theta) = \frac{8}{10}$ when the rider is 16 feet high from the ground level. Then

$$\frac{dh}{dt} = 10\left(\frac{8}{10}\right) \cdot \pi = 8\pi \approx 25.1327$$

Therefore, the rider is rising at the rate 25.1327 m/min when he is 16 m above ground level.

Assigned Exercises: (p 249) 3, 13, 17, 25, 27, 33, 39, 47, 49