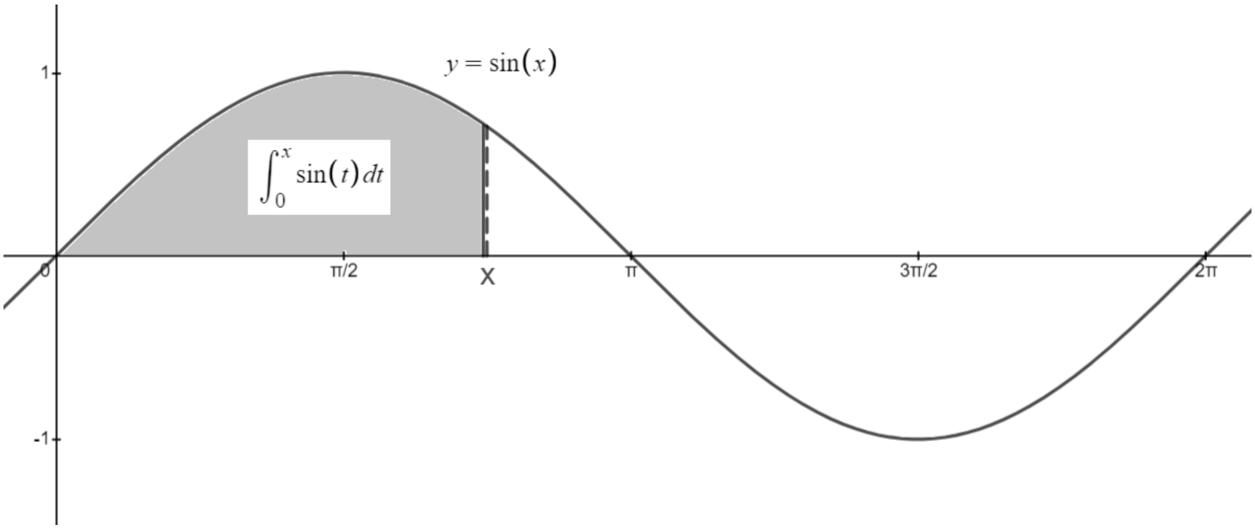
Let $f(x) = \sin(x)$. We can define a function using the definite integral of $f(x)$ from 0 to x (independent variable) as follows:

$$g(x) = \int_0^x \sin(t)\,dt$$

It is the area function as it finds the area between the curve of $\sin(x)$ and the x-axis from 0 to x. Of course, we consider the area below the x-axis as negative area.



$$g(0) = \int_0^0 \sin(t)\,dt = 0$$

$$g\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \sin(t)\,dt = 1$$

$$g(\pi) = \int_0^{\pi} \sin(t)\,dt = 2$$

$$g\left(\frac{3\pi}{2}\right) = \int_0^{3\pi/2} \sin(t)\,dt = 1$$

$$g(2\pi) = \int_0^{2\pi} \sin(t)\,dt = 0$$

Q: Is the function $g(x)$ continuous? Differentiable?

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Example 1 Find the derivative of $g(x) = \int_0^x \sin(t)\,dt$.

Since $f(x) = \sin(x)$ is continuous on $[0, \infty)$, it is continuous on $[0, b]$ for any $b > 0$. Then $g(x)$ is continuous on $[0, b]$ for any $b > 0$ and differentiable on $(0, b)$. Also $g'(x) = f(x) = \sin(x)$.

> If f is continuous on $[a, b]$, $h(x)$ is differentiable on (a, b), and
>
> $$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$
>
> then the composite function $(g \circ h)(x)$
>
> $$(g \circ h)(x) = g(h(x)) = \int_a^{h(x)} f(t)\,dt \qquad a \leq x \leq b$$
>
> is continuous on $[a, b]$ and differentiable on (a, b), and $(g \circ h)'(x) = f(h(x))h'(x)$.

Example 2 Find the derivative of $p(x) = \int_0^{2x} \sin(t)\,dt$.

Here, $f(x) = \sin(x)$ and $h(x) = 2x$. Then $p'(x) = f(h(x))h'(x) = \sin(2x) \cdot (2x)' = 2\sin(2x)$.

> **The Fundamental Theorem of Calculus, Part 2** If f is continuous on $[a, b]$, then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a)$$
>
> where F is any antiderivative of f, that is, a function such that $F' = f$.

Example 3 Find the definite integral

$$\int_0^{\pi} \sin(x)\,dx$$

The function $f(x) = \sin(x)$ is continuous on $[0, \pi]$ and its antiderivative is $F(x) = -\cos(x) + C$. By FTC2,

$$\int_0^{\pi} \sin(x)\,dx = F(\pi) - F(0) = -\cos(\pi) + C - (-\cos(0) + C) = -(-1) + C - (-1 + C) = 2$$

When using FTC2, we do not need to add C for an antiderivative. As seen from the example, $+C$ will be canceled when computing $F(b) - F(a)$.

The following popular notation should be used.

$$F(x)\Big|_{x=a}^{x=b} = F(b) - \boxed{(F(a))} \quad \text{or} \quad F(x)\Big|_{a}^{b} = F(b) - \boxed{(F(a))}$$

The a pair of parentheses around $F(a)$ is 'must' if the antiderivative has more than one term.

Example 4 Find the definite integral

$$\int_2^5 x^2\,dx$$

The function $f(x) = x^2$ is continuous on $[2,5]$ and its antiderivative is $F(x) = \frac{1}{3}x^3$. By FTC2,

$$\int_2^5 x^2\,dx = \frac{1}{3}x^3\Big|_2^5 = \frac{1}{3}(5)^3 - \frac{1}{3}(2)^3 = \frac{117}{3} = 39$$

Example 5 Find the definite integral

$$\int_{-3}^4 (3x-2)\,dx$$

The function $f(x) = 3x-2$ is continuous on $[-3,4]$ and its antiderivative is $F(x) = \frac{3}{2}x^2 - 2x$. By FTC2,

$$\int_{-3}^4 (3x-2)\,dx = \frac{3}{2}x^2 - 2x\Big|_{-3}^4 = \frac{3}{2}(4)^2 - 2(4) - \left(\frac{3}{2}(-3)^2 - 2(-3)\right) = -\frac{7}{2} = -3.5$$

Example 6 The following computation

$$\int_{-1}^1 \frac{1}{x}\,dx = \ln(|x|)\Big|_{-1}^1 = \ln(|1|) - \ln(|-1|) = 0 \quad \text{IS WRONG!}$$

We should have checked whether $f(x) = \frac{1}{x}$ is continuous on $[-1,1]$ or not.

The Fundamental Theorem of Calculus Suppose f is continuous on $[a,b]$.

1. If $g(x) = \displaystyle\int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Assigned Exercises: (p 399) 7 - 43 (odds), 45, 47, 55 - 65 (odds), 69, 73