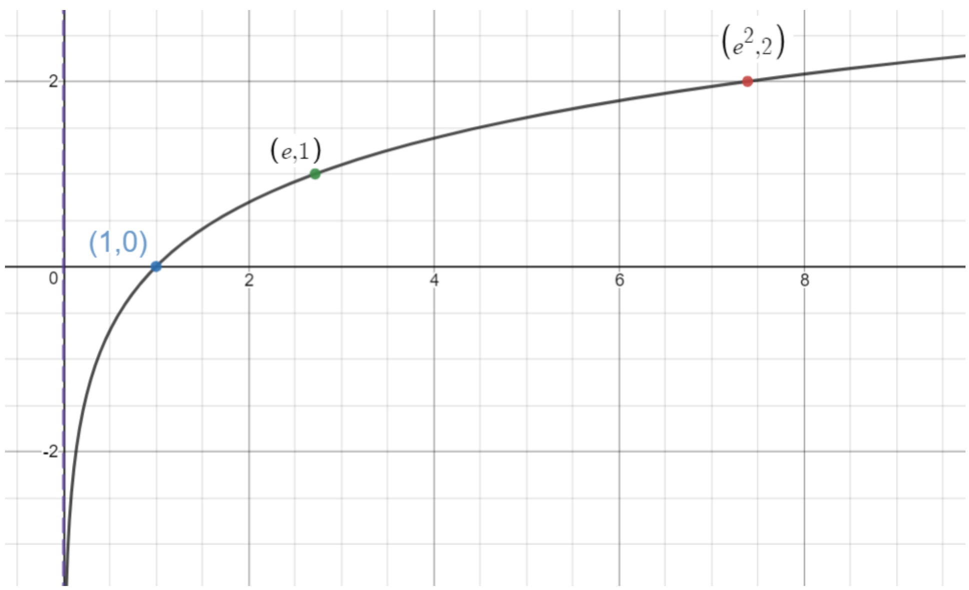
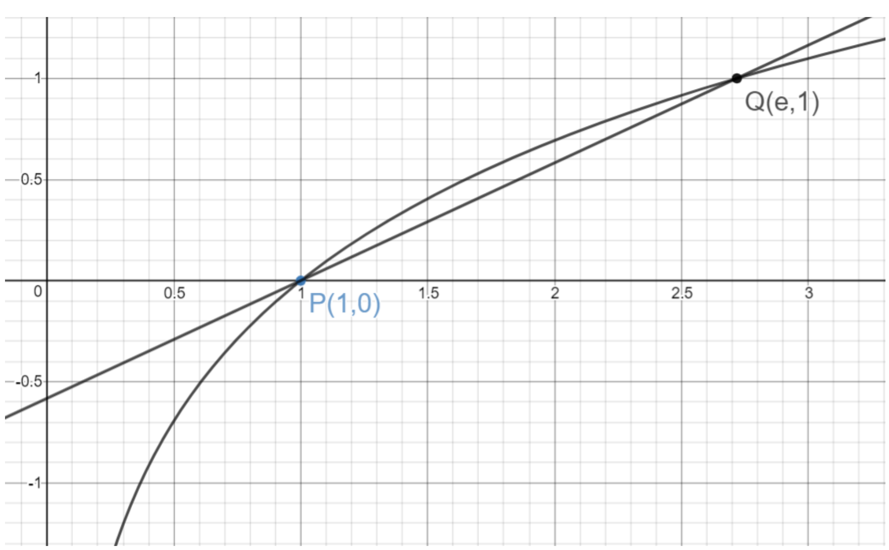
The Tangent Problem

Recall: $f(x) = \ln(x)$ is an increasing (but slowly) function that has the domain $(0, \infty)$ and the range $(-\infty, \infty)$. The graph has a vertical asymptote at $x = 0$ and only one x-intercept at $(1, 0)$ (as $\ln(1) = 0$).



Given two points P and Q on the graph of $f(x)$, the line $\overleftrightarrow{PQ}$ connecting P and Q is called a **secant line** to the graph of $f(x)$. The following graph shows a secant line connecting $P(1, 0)$ and $Q(e, 1)$.



The slope of the secant line can be found using a usual slope formula between two points P and Q.

$$m_{PQ} = \frac{1-0}{e-1} = \frac{1}{e-1} \approx 0.582 > 0$$

which indicates that the function value increases when x value jumps from 1 to e.

Goal: We want to scrutinize what happens to the slope of secant lines as the point Q (moving) approaches to the point P (fixed).

One way for a point Q on the graph to approach the point $P(1,0)$ is to shortening the distance between the x value of Q to the x value of P, that is 1. The following table shows the coordinates of Q and the slope of the secant lines connecting moving point Q to the fixed point $P(1,0)$.

x value of Q	$f(x)$	Q	m_{PQ}
$e \approx 2.718$	$\ln(e) = 1$	$(e, 1)$	$\frac{1}{e-1} \approx 0.582$
2.5	$\ln(2.5)$	$(2.5, \ln(2.5))$	$\frac{\ln(2.5)-0}{2.5-1} = \frac{2}{3}\ln(2.5) \approx 0.611$
2	$\ln(2)$	$(2, \ln(2))$	$\frac{\ln(2)-0}{2-1} = \ln(2) \approx 0.693$
1.5	$\ln(1.5)$	$(1.5, \ln(1.5))$	$\frac{\ln(1.5)-0}{1.5-1} = 2\ln(1.5) \approx 0.811$

It seems like the slope is increasing. Let us approach to the point P even closer.

x value of Q	$f(x)$	Q	m_{PQ}
1.1	$\ln(1.1)$	$(1.1, \ln(1.1))$	$\frac{\ln(1.1)-0}{1.1-1} = 10\ln(1.1) \approx 0.953$
1.01	$\ln(1.01)$	$(1.01, \ln(1.01))$	$\frac{\ln(1.01)-0}{1.01-1} = 100\ln(1.01) \approx 0.995$
1.001	$\ln(1.001)$	$(1.001, \ln(1.001))$	$\frac{\ln(1.001)-0}{1.001-1} = 1000\ln(1.001) \approx 0.9995$
1.0001	$\ln(1.0001)$	$(1.0001, \ln(1.0001))$	$\frac{\ln(1.0001)-0}{1.0001-1} = 10000\ln(1.0001) \approx 0.99995$

It seems like the slope is not only increasing but also approaching 1. No matter how close we get, the slope will never be 1. By the way, we cannot let Q to be actually the same point as P. If Q is P, then we can neither draw a secant line nor compute the slope of the secant line.

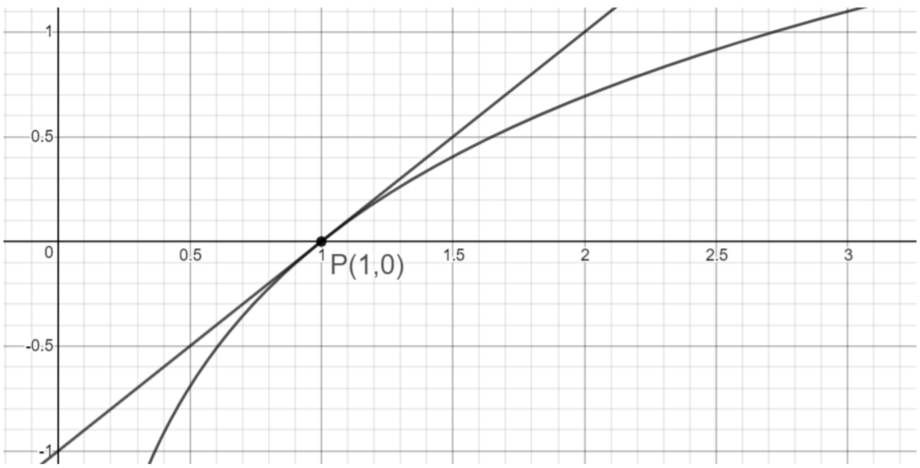
Now let us approach to the point $P(1,0)$ from the left hand side of P. It actually does not matter where we begin the approach.

x value of Q	$f(x)$	Q	m_{PQ}
0.9	$\ln(0.9)$	$(0.9, \ln(0.9))$	$\frac{\ln(0.9)-0}{0.9-1} = -10\ln(0.9) \approx 1.054$
0.99	$\ln(0.99)$	$(0.99, \ln(0.99))$	$\frac{\ln(0.99)-0}{0.99-1} = -100\ln(0.99) \approx 1.005$
0.999	$\ln(0.999)$	$(0.999, \ln(0.999))$	$\frac{\ln(0.999)-0}{0.999-1} = -1000\ln(0.999) \approx 1.0005$
0.9999	$\ln(0.9999)$	$(0.9999, \ln(0.9999))$	$\frac{\ln(0.9999)-0}{0.9999-1} = -10000\ln(0.9999) \approx 1.00005$

Again it seems like the slope is approaching 1. If we "believe" that the slopes of the secant lines are actually approaching to a single value, then we call such a value the **limiting value** or **limit** of the slopes of the secant lines as approaching to the point P. Symbolically, we write as

$$\lim_{Q \to P} m_{PQ} \quad \text{or} \quad \lim_{x \to 1} \frac{\ln(x) - 0}{x - 1}$$

At the very moment when Q is passing by P, we do not have a secant line $\overleftrightarrow{PQ}$. However, we can visualize a line passing through the point P with the slope equal to the limiting value of the slopes of the secant lines. For the current example, it would be the line passing through $P(1,0)$ with the slope 1. The equation for such a line is $y = x - 1$.



We can consider this line as the limiting line of the secant lines connecting P and Q as Q approaches to P.

We call this line the **tangent line** to the graph of $f(x)$ at P (or $x = 1$).

In this sense, we can consider the limiting value of the slopes of the secant lines as the **slope of the tangent line** to the graph of $f(x)$ as approaching to the point P.

The Velocity Problem

A bit of physics: There is this thing called gravity at the very center of the earth, and this gravity pulls everything within its reach towards itself. It is known that the pull accelerates the object being pulled by 32 ft/s every second. If we consider moving away from the center of the earth as positive direction, then this acceleration due to gravity is -32 ft/s^2. It is negative as it accelerates towards the center of the earth. If we assume that an object in motion is in vacuum, then the height of the object can be described by the equation

$$h(t) = -16t^2 + v_0 t + h_0$$

where v_0 is the initial velocity (at $t = 0$) of the object and h_0 is the initial height (at $t = 0$) of the object. Here, the height h is measured in feet and the time t is measured in seconds. Of course, v_0 would be in feet per second.

If $v_0 > 0$, then the object is being shot up at $t = 0$. If a cannonball is shot up with the initial velocity of 400 ft/s from the ground, the height of the cannonball would be given by $h(t) = -16t^2 + 400t + 0$. If $v_0 = 0$, then it is called a free fall. If a diver simply walks off from the 64-feet-high diving board instead of being sprung up, then the height of the diver is given by $h(t) = -16t^2 + 64$. If $v_0 < 0$, then the object is being shot down at $t = 0$. If a sniper is shooting a subject on the ground from the roof top of a 1,024-feet-high building with the muzzle velocity of 3,600 ft/s, the height of the bullet is given by $h(t) = -16t^2 - 3600t + 1024$.

If a function $f(t)$ measures a position of an object at time t, the slope of the secant line connecting two points $(a, f(a))$ and $(b, f(b))$ is called the **average velocity** of the position function $f(t)$ over the interval $a \leq t \leq b$ or $[a, b]$, that is

$$\frac{f(b) - f(a)}{b - a}$$

Consider the last example of a bullet whose height is given by $h(t) = -16t^2 - 3600t + 1024$. The average velocity of the bullet for the first 0.05 seconds is

$$\frac{h(0.05) - h(0)}{0.05 - 0} = \frac{843.96 - 1024}{0.05} = -3600.8 \text{ ft/s}$$

Note that this velocity is not the velocity of the bullet at the very moment at $t = 0.05$ seconds after the bullet left the gun. In order to find the velocity of the bullet at the very moment of $t = 0.05$ seconds, we need to find the slope of the tangent line to the graph of $f(t)$ at $t = 0.05$. Let's estimate the slope of the tangent line using the slopes of secant lines as t approaches 0.05 seconds from both left and right hand sides. Approaching from the left of $t = 0.05$, the slope of the secant line would be

$$\frac{h(t) - h(0.05)}{t - 0.05} = \frac{(-16t^2 - 3600t + 1024) - 843.96}{t - 0.05}$$

t	$h(t)$	Slope of secant line
0.049	$-16(0.049)^2 - 3600(0.049) + 1024 = 847.561584$	-3601.584
0.0499	$-16(0.0499)^2 - 3600(0.0499) + 1024 = 844.3201598$	-3601.5984
0.04999	$-16(0.04999)^2 - 3600(0.04999) + 1024 = 843.996016$	-3601.59984

Approaching from the right of $t = 0.05$,

t	$h(t)$	Slope of secant line
0.051	$-16(0.051)^2 - 3600(0.051) + 1024 = 840.358384$	-3601.616
0.0501	$-16(0.0501)^2 - 3600(0.0501) + 1024 = 843.59983984$	-3601.6016
0.05001	$-16(0.05001)^2 - 3600(0.05001) + 1024 = 843.9239839984$	-3601.60016

Q: Any guess on what the slope of the tangent line at $(0.05, h(0.05))$ should be?

This limiting value of the average velocities at $t = 0.05$ is called the **instantaneous velocity** at $t = 0.05$. If you measure the speed of the bullet using a speed gun at the very moment of $t = 0.05$, the speed gun will read -3601.6 ft/s.

Assigned Exercises: (p 82) 1, 3, 5, 7, 9