Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

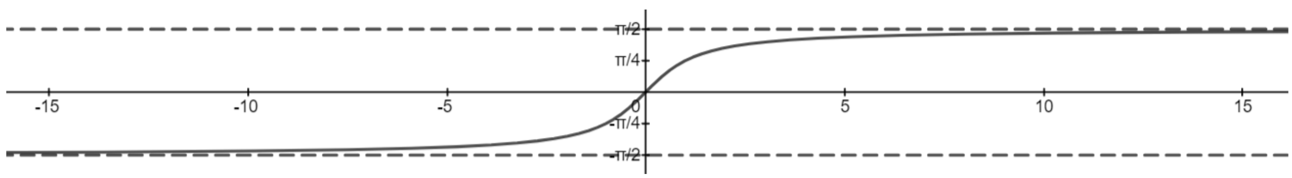
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Similarly, let f be a function defined on some interval $(-\infty, b)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Example 1 Consider $f(x) = \tan^{-1}(x)$. Its domain is $(-\infty, \infty)$.



As seen from the graph, the function values are getting close to $\frac{\pi}{2}$ as x approaches ∞. The function values are getting close to $-\frac{\pi}{2}$ as x approaches $-\infty$. Using the notations in the definition,

$$\boxed{\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}} \quad \text{and} \quad \boxed{\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}}$$

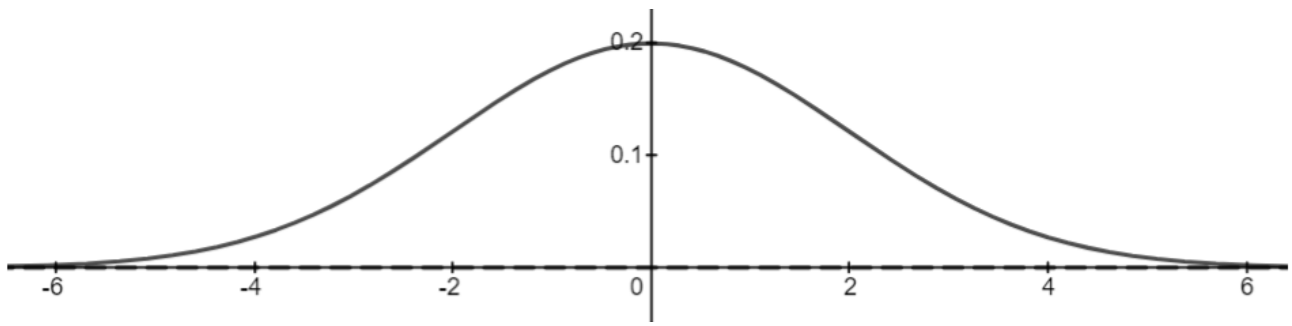
Alternatively, as $x \to \infty$, $f(x) \to \frac{\pi}{2}$. As $x \to -\infty$, $f(x) \to -\frac{\pi}{2}$. These are also called the **end behaviors** (or **asymptotic behaviors**) of the function.

The line $y = L$ is called a **horizontal asymptote** of the curve of $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

In the previous example, both $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes. When graphing a function, it is generally understood that the horizontal asymptotes are graphed together using dotted lines (see above). Sometimes a function has one horizontal asymptote as the end behaviors are the same as $x \to \infty$ and $x \to -\infty$.

Example 2 Consider $N(x) = \frac{1}{\sqrt{2\pi(2)^2}} e^{-\frac{(x-0)^2}{2(2)^2}}$.



It is not obvious even from the graph, but $\lim_{x\to\infty} N(x) = 0$ and $\lim_{x\to-\infty} N(x) = 0$ as well. Hence, the horizontal asymptote of $N(x)$ is the x-axis (or $y = 0$).

In two examples above, the graphs of the functions were provided so that we had some idea about where the functions values are approaching. However, we should be able to find out the end behaviors of many practical functions. Let's start with basics.

Example 3 Let $f(x) = \frac{1}{x}$. Its domain is $(-\infty, 0) \cup (0, \infty)$. It has a vertical asymptote at $x = 0$. What about the end behaviors? As $x \to \infty$, the fraction has a fixed numerator 1 and the denominator growing without a bound to both ends.

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
10	0.1	10^6	$10^{-6} = 0.000001$	-10	-0.1	-10^6	$-10^{-6} = -0.000001$
100	0.01	10^9	$10^{-9} = 0.000000001$	-100	-0.01	-10^9	$-10^{-9} = -0.000000001$
1000	0.001	∞	$\frac{1}{\infty} = ?$	-1000	-0.001	$-\infty$	$-\frac{1}{\infty} = ?$

As can seen, the function values are arbitrarily close to 0. So

$$\boxed{\lim_{x\to\infty} \frac{1}{x} = 0} \quad \text{and} \quad \boxed{\lim_{x\to-\infty} \frac{1}{x} = 0}$$

Theorem 5 If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a reduced rational number with an odd denominator, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

Recall that a **rational number** is a number of the form $\frac{A}{B}$ where A and B are integers with $B \neq 0$. A reduced rational number means A and B has no common factor (as they were canceled). For instance,

$$\lim_{x\to\infty}\frac{1}{x^2}=0 \quad \lim_{x\to\infty}\frac{1}{x^5}=0 \quad \lim_{x\to\infty}\frac{1}{\sqrt{x}}=0 \quad \lim_{x\to\infty}\frac{1}{x^{1/3}}=0 \quad \lim_{x\to\infty}x^{-2/5}=0 \quad \lim_{x\to\infty}\frac{1}{x^{5/2}}=0$$

$$\lim_{x\to-\infty}\frac{1}{x^2}=0 \quad \lim_{x\to-\infty}\frac{1}{x^5}=0 \quad \lim_{x\to-\infty}\frac{1}{\sqrt[3]{x}}=0 \quad \lim_{x\to-\infty}\frac{1}{x^{2/5}}=0 \quad \lim_{x\to-\infty}x^{-4/3}=0$$

Note that if r is a reduced rational number with an even denominator, then x^r is undefined if $x<0$. So none of the following limits would make sense:

$$\lim_{x\to-\infty}\frac{1}{x^{1/2}} \qquad \lim_{x\to-\infty}\frac{1}{\sqrt[6]{x}} \qquad \lim_{x\to-\infty}\frac{1}{x^{3/4}} \qquad \lim_{x\to-\infty}x^{-5/8}$$

Note that these limits are not DNE (does not exist) but do not make sense.

Example 4 Evaluate

$$\lim_{x\to\infty}\frac{-2x^3+6}{x^3-7x^2}$$

Can we use the limit laws from Section 2.3? Let's try.

$$\begin{aligned}
\lim_{x\to\infty}\frac{-2x^3+6}{x^3-7x^2} &= \frac{\lim_{x\to\infty}(-2x^3+6)}{\lim_{x\to\infty}(x^3-7x^2)}\\
&= \frac{\lim_{x\to\infty}-2x^3+\lim_{x\to\infty}6}{\lim_{x\to\infty}x^3-\lim_{x\to\infty}7x^2}\\
&= \frac{-2\lim_{x\to\infty}x^3+6}{\lim_{x\to\infty}x^3-7\lim_{x\to\infty}x^2}
\end{aligned}$$

However, none of $\lim_{x\to\infty}x^3$ and $\lim_{x\to\infty}x^2$ is a real number. So usage of the limit laws cannot be justified. There is a little trick (of course, you need to remember) to deal with this type of question.

Since we are concerned when x is really huge, x is certainly nowhere close to 0. So the expression $\frac{1}{x^3}$ is defined. Let's try again.

$$\begin{aligned}
\lim_{x\to\infty}\frac{-2x^3+6}{x^3-7x^2} &= \lim_{x\to\infty}\frac{-2x^3+6}{x^3-7x^2}\cdot\boxed{\frac{\frac{1}{x^3}}{\frac{1}{x^3}}}\\
&= \lim_{x\to\infty}\frac{\frac{-2x^3+6}{x^3}}{\frac{x^3-7x^2}{x^3}} = \lim_{x\to\infty}\frac{\frac{-2x^3}{x^3}+\frac{6}{x^3}}{\frac{x^3}{x^3}-\frac{7x^2}{x^3}}\\
&= \lim_{x\to\infty}\frac{-2+6\frac{1}{x^3}}{1-7\frac{1}{x}} = \frac{\lim_{x\to\infty}(-2+6\frac{1}{x^3})}{\lim_{x\to\infty}(1-7\frac{1}{x})}\\
&= \frac{\lim_{x\to\infty}-2+6\lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty}1-7\lim_{x\to\infty}\frac{1}{x}} = \frac{-2+6(0)}{1-7(0)} = -2
\end{aligned}$$

With a little trick the limit laws worked so that the limit can be found. There is no difference in argument if $x \to -\infty$. Hence,

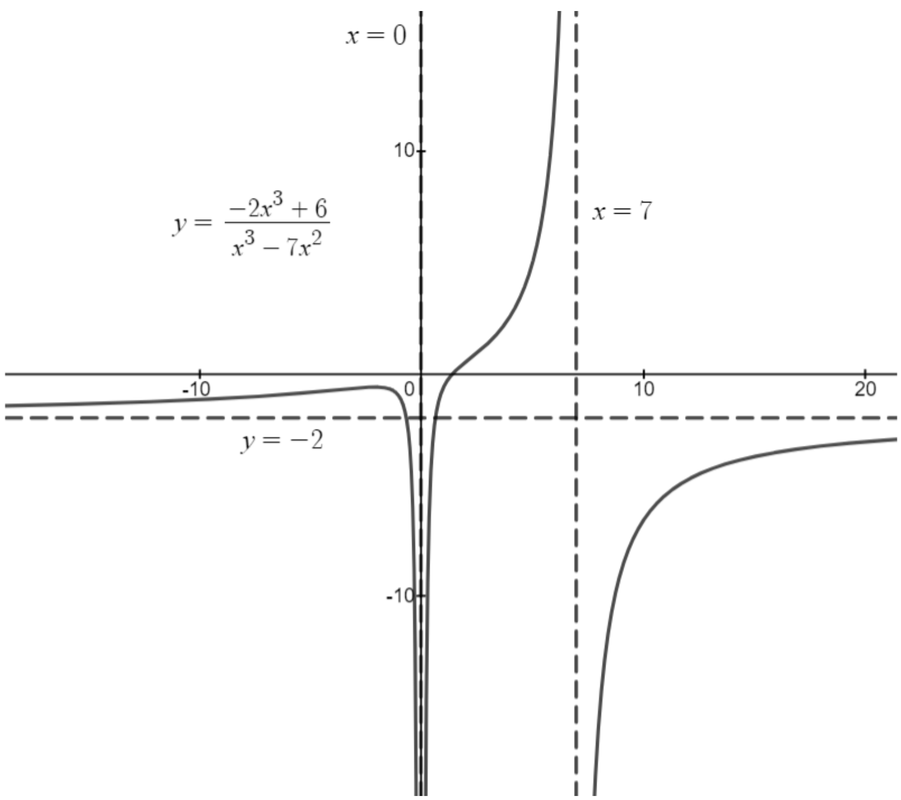
$$\lim_{x \to -\infty} \frac{-2x^3 + 6}{x^3 - 7x^2} = -2$$

Here is an alternative informal way that works better sometime.

For the numerator polynomial $-2x^3 + 6$, how much difference does the constant term 6 make when x is huge like million? Not much. Maybe it is less obvious for the denominator polynomial $x^3 - 7x^2$. The square term $-7x^2$ might have some significance impact because we are subtracting a big chunk of numbers like 7 trillion. But if x is even huger like 10^{1000}, no one will notice 7×10^{2000} is being subtracted from 10^{3000}. When considering the end behavior of a rational expression, what really matters is just the leading terms (the term with the largest exponent power) of the numerator polynomial and the denominator polynomials. The lower terms can be safely ignored. Hence,

$$\lim_{x \to \infty} \frac{-2x^3 + 6}{x^3 - 7x^2} = \lim_{x \to \infty} \frac{-2x^3}{x^3} = \lim_{x \to \infty} -2 = -2$$

For a rational function, this is the quickest way to find the limits at infinity.



Example 5 Find the end behaviors of the function

$$g(x) = \frac{1000x^3 + 6x^2 + 11}{-33x^6 + 7x^2}$$

First, we consider $x \to \infty$.

$$\lim_{x\to\infty} \frac{1000x^3 + 6x^2 + 11}{-33x^6 + 7x^2} = \lim_{x\to\infty} \frac{1000x^3}{-33x^6} = \lim_{x\to\infty} -\frac{1000}{33x^3} = -\frac{1000}{33} \lim_{x\to\infty} \frac{1}{x^3} = -\frac{1000}{33}(0) = 0$$

Now we consider $x \to -\infty$.

$$\lim_{x\to-\infty} \frac{1000x^3 + 6x^2 + 11}{-33x^6 + 7x^2} = \lim_{x\to-\infty} \frac{1000x^3}{-33x^6} = \lim_{x\to-\infty} -\frac{1000}{33x^3} = -\frac{1000}{33} \lim_{x\to-\infty} \frac{1}{x^3} = -\frac{1000}{33}(0) = 0$$

No difference! In fact, for a rational function, the end behaviors will be same either way, if they are numbers. In this case, the horizontal asymptote is $y = 0$, the x-axis.

Example 6 Find the horizontal asymptote(s) of the function

$$h(x) = \frac{4x + 13}{\sqrt{8x^2 + 4x - 7}}$$

Note that this function is not a rational function, but the same argument can be used. However, there could be different end behaviors can happen as it is not a rational function. First, we let $x \to \infty$.

$$\lim_{x\to\infty} \frac{4x + 13}{\sqrt{8x^2 + 4x - 7}} = \lim_{x\to\infty} \frac{4x}{\sqrt{8x^2}} = \lim_{x\to\infty} \frac{4x}{2\sqrt{2}|x|} = \lim_{x\to\infty} \frac{2x}{\sqrt{2}x} = \lim_{x\to\infty} \frac{2}{\sqrt{2}} = \sqrt{2}$$

It is very important that $\sqrt{x^2} = |x|$. Not just x! Since x approaches ∞, the values of x are certainly positive. Recall that the definition of $|x|$. If $x > 0$, then $|x| = x$. Now we let $x \to -\infty$.

$$\lim_{x\to-\infty} \frac{4x + 13}{\sqrt{8x^2 + 4x - 7}} = \lim_{x\to-\infty} \frac{4x}{\sqrt{8x^2}} = \lim_{x\to-\infty} \frac{4x}{2\sqrt{2}|x|} = \lim_{x\to-\infty} \frac{2x}{\sqrt{2}\cdot(-x)} = -\lim_{x\to-\infty} \frac{2}{\sqrt{2}} = -\sqrt{2}$$

This time, we are approaching $-\infty$, so the values of x are negative. Then $|x| = -x$ since $x < 0$.

The horizontal asymptotes of the function h are $y = \sqrt{2}$ and $y = -\sqrt{2}$.

Example 7 Compute $\lim_{x\to\infty}(\sqrt{x^2 + 16} - x)$.

The trick about ignoring the lower terms "only" works if we have a fractional expression. So we <u>cannot</u> do this: $\lim_{x\to\infty}(\sqrt{x^2} - x) = \lim_{x\to\infty}(x - x) = 0$ (incorrect argument). Here is another trick we need to learn.

For an expression $\sqrt{A} + B$ or $\sqrt{A} + \sqrt{B}$, the expression $\sqrt{A} - B$ or $\sqrt{A} - \sqrt{B}$ is called the **conjugate** of the original expression. The expressions $\sqrt{A} + B$ and $\sqrt{A} - B$ are called the

conjugate pairs. The following trick works because when the conjugate pairs are multiplied together, the radical expression disappear. We have seen this trick used in Section 2.3.

$$
\begin{aligned}
\lim_{x\to\infty}(\sqrt{x^2+16}-x) &= \lim_{x\to\infty}\frac{\sqrt{x^2+16}-x}{1}\cdot\frac{\sqrt{x^2+16}+x}{\sqrt{x^2+16}+x} \\
&= \lim_{x\to\infty}\frac{(\sqrt{x^2+16}-x)(\sqrt{x^2+16}+x)}{\sqrt{x^2+16}+x} \\
&= \lim_{x\to\infty}\frac{(\sqrt{x^2+16})^2-x^2}{\sqrt{x^2+16}+x} \\
&= \lim_{x\to\infty}\frac{16}{\sqrt{x^2+16}+x} \qquad \text{Now we have a fractional expression.} \\
&= \lim_{x\to\infty}\frac{16}{\sqrt{x^2}+x} \qquad \text{So we can ignore 16 inside the radical.} \\
&= \lim_{x\to\infty}\frac{16}{|x|+x} = \lim_{x\to\infty}\frac{16}{x+x} = \lim_{x\to\infty}\frac{16}{2x} = 8\lim_{x\to\infty}\frac{1}{x} = 8(0) = 0
\end{aligned}
$$

Example 8 Compute $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$.

$$
\begin{aligned}
\lim_{x\to\infty}(\sqrt{x^2+x}-x) &= \lim_{x\to\infty}\frac{\sqrt{x^2+x}-x}{1}\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty}\frac{(\sqrt{x^2+x})^2-x^2}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x} \qquad \text{Now we have a fractional expression.} \\
&= \lim_{x\to\infty}\frac{x}{\sqrt{x^2}+x} \qquad \text{So we can ignore } x \text{ inside the radical.} \\
&= \lim_{x\to\infty}\frac{x}{|x|+x} = \lim_{x\to\infty}\frac{x}{x+x} = \lim_{x\to\infty}\frac{x}{2x} = \lim_{x\to\infty}\frac{1}{2} = \frac{1}{2}
\end{aligned}
$$

Infinite Limits at Infinity

The end behaviors of a function do not have to approach a real number. As $x\to\infty$ or $-\infty$, the function values can increase or decrease without bound. Here are four possible end behaviors:

If the function values increase without bound, as $x\to\infty$, then $\lim_{x\to\infty} f(x)=\infty$.
If the function values decrease without bound, as $x\to\infty$, then $\lim_{x\to\infty} f(x)=-\infty$.

If the function values increase without bound, as $x \to -\infty$, then $\lim_{x\to-\infty} f(x) = \infty$.
If the function values decrease without bound, as $x \to -\infty$, then $\lim_{x\to-\infty} f(x) = -\infty$.

Not unless we use the precise definition of infinite limits at infinity, we rely on intuition to determine infinite limits at infinity. However, we should be cautious as intuition could lead us to wrong answer. Here are the obvious facts.

1. If n is even positive integer, then $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = \infty$.
2. If n is odd positive integer, then $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$.
3. If n is positive integer, then $\lim_{x\to\infty} x^{1/n} = \infty$ (or $\lim_{x\to\infty} \sqrt[n]{x} = \infty$).
4. If $n > 1$ is positive odd integer, then $\lim_{x\to-\infty} x^{1/n} = -\infty$ (or $\lim_{x\to-\infty} \sqrt[n]{x} = -\infty$).

The following are intuitive: Let $c > 0$ be a positive real number.

$$\begin{array}{llllll} \infty+\infty=\infty; & \infty\cdot\infty=\infty; & \infty+c=\infty; & \infty-c=\infty; & c\cdot\infty=\infty; & \frac{\infty}{c}=\infty \\ \infty\cdot-\infty=-\infty; & c-\infty=-\infty; & -c\cdot\infty=-\infty; & \frac{1}{\pm\infty}=0; & \frac{c}{\pm\infty}=0; & \frac{-c}{\pm\infty}=0 \end{array}$$

Example 9 Find the end behaviors of the function

$$f(x) = \frac{-33x^6 + 7x^2}{1000x^3 + 6x^2 + 11}$$

First, we consider $x \to \infty$.

$$\lim_{x\to\infty} \frac{-33x^6 + 7x^2}{1000x^3 + 6x^2 + 11} = \lim_{x\to\infty} \frac{-33x^6}{1000x^3} = \lim_{x\to\infty} -\frac{33x^3}{1000} = -\frac{33}{1000} \lim_{x\to\infty} x^3 = -\frac{33}{1000} \cdot \infty = -\infty$$

Now we consider $x \to -\infty$.

$$\lim_{x\to-\infty} \frac{-33x^6 + 7x^2}{1000x^3 + 6x^2 + 11} = \lim_{x\to-\infty} \frac{-33x^6}{1000x^3} = \lim_{x\to-\infty} -\frac{33x^3}{1000} = -\frac{33}{1000} \lim_{x\to-\infty} x^3 = -\frac{33}{1000} \cdot -\infty = \infty$$

So $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$.

The following are <u>not</u> intuitively sure: Let $c > 0$ be a positive real number.

$$\infty - \infty = ?; \quad \frac{\infty}{\infty} = ?; \quad 0 \cdot \infty = ?; \quad \frac{\infty}{0} = ? \quad \frac{0}{\infty} = ?; \quad \frac{0}{0} = ?; \quad \frac{\infty}{\infty} = ?; \quad \frac{1}{0} = ?$$

We have to wait till Section 4.4 to evaluate some of these cases.

Assigned Exercises: (p 137) 3 - 41 (odds), 49, 61, 63