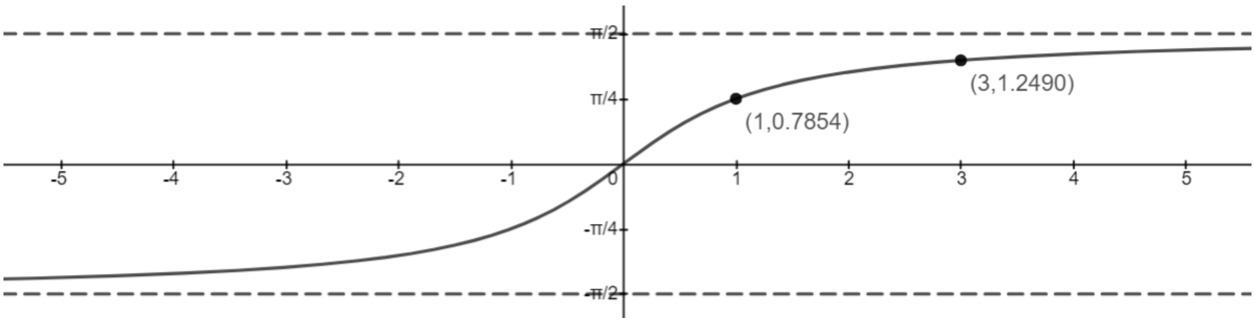
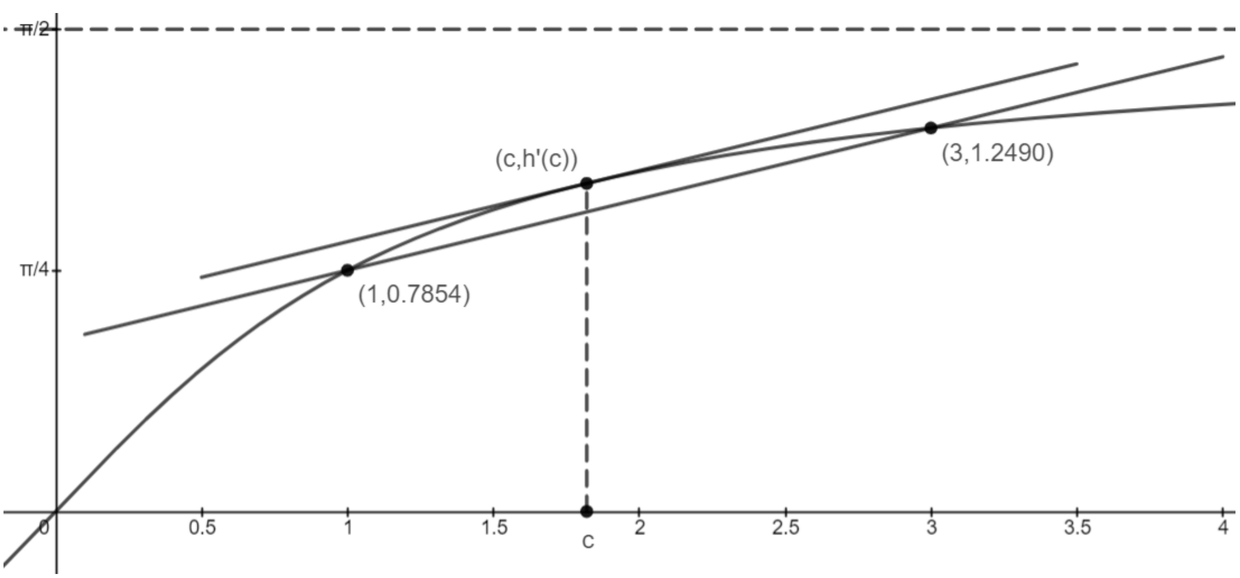
Suppose a height (in feet) of a particle is given by the function $h(t) = \tan^{-1}(t)$ of t (in seconds). Then the particle rises from $h(1) = 0.7854$ feet to $h(3) = 1.2490$ feet in two seconds.



The average velocity of this vertical motion over the interval $1 \leq t \leq 3$ is $\frac{h(3)-h(1)}{3-1} = \frac{\tan^{-1}(3)-\tan^{-1}(1)}{2} = 0.2318$ ft/s. On average, the velocity of the particle is 0.2318 ft/s, but does the particle ever actually reach that velocity at some time between 1 second and 3 seconds? Equivalently, does the particle reach an instantaneous velocity of 0.2318 ft/s at some time between 1 second and 3 seconds? Using math symbols, we can rephrase the question as "Is there a real number c in the interval $[1, 3]$ so that $h'(c) = 0.2318$?" because the instantaneous velocity is the derivative of the height function $h(t)$.

Graphically, the meaning of the average velocity 0.2318 ft/s is the slope of the secant line connecting two points $(1, \tan^{-1}(1))$ and $(3, \tan^{-1}(3))$ on the curve, and the meaning of the instantaneous velocity $h'(c)$ at c (which is unknown at this point) is the slope of the tangent line to the curve at $(c, \tan^{-1}(c))$. If we can find c such that the tangent line is parallel to the secant line, the answer is 'yes'. From a quick eyeballing, we can see that it actually happens for c somewhere between 1.5 seconds and 2 seconds.



Can we actually find the value of c? The slope of the secant line is

$$\frac{h(3)-h(1)}{3-1} = \frac{\tan^{-1}(3)-\tan^{-1}(1)}{2},$$

and the derivative of $h(t)$ is $h'(t) = \frac{1}{1+t^2}$. So we need to solve the equation

$$\frac{1}{1+c^2} = \frac{\tan^{-1}(3)-\tan^{-1}(1)}{2}$$

for the variable c.

$$\begin{aligned} 1+c^2 &= \frac{2}{\tan^{-1}(3)-\tan^{-1}(1)} \\ c^2 &= \frac{2}{\tan^{-1}(3)-\tan^{-1}(1)} - 1 \\ c &= \pm\sqrt{\frac{2}{\tan^{-1}(3)-\tan^{-1}(1)} - 1} \\ &\approx \pm 1.8203 \end{aligned}$$

At $t = 1.8203$ seconds, the particle is ascending with the instantaneous velocity 0.2318 ft/s. We attempted to find c because we were convinced by the graph that there is a point on the curve so that the tangent line is parallel to the secant line. What if it is difficult to draw a graph?

The Mean Value Theorem: Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or, equivalently $f(b) - f(a) = f'(c)(b - a)$.

In the example earlier, the function $h(t)$ is both continuous and differentiable everywhere on its domain $(-\infty, \infty)$. Hence, it definitely satisfies both hypotheses on the intervals $[1, 3]$ and $(1, 3)$. By the theorem, it was guaranteed to find c between 1 and 3 such that $h'(c) = \frac{h(3)-h(1)}{3-1}$.

Note that the theorem is an existential statement. It ensures that 'a' value c exists between a and b, but it does not give a direct way of finding such c or it does tell how many such c exist between a and b.

Example 1 Consider a cubic function $f(x) = x^3 - 3x + 2$. Since it is a polynomial function, it is continuous and differentiable everywhere on its domain $(-\infty, \infty)$, we can find c that satisfy the conclusion of the Mean Value Theorem over the interval $[-2, 2]$. No graphing is necessary! The slope of the secant line is

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{((2)^3 - 3(2) + 2) - ((-2)^3 - 3(-2) + 2)}{2 - (-2)} = 1,$$

and the derivative of $f(x)$ is $f'(x) = 3x^2 - 3$. The Mean Value Theorem guarantees that the equation $3c^2 - 3 = 1$ has at least one solution between -2 and 2. In fact, the solution of the equation are $c = \pm\frac{2}{\sqrt{3}}$ or $\pm\frac{2\sqrt{3}}{3}$. Since $\sqrt{3} \approx 1.7$, both values $-\frac{2\sqrt{3}}{3} \approx -1.1$ and $\frac{2\sqrt{3}}{3} \approx 1.1$ are between -2 and 2. Verify that the graph indeed exhibits such values.

Example 2 For the same function $f(x) = x^3 - 3x + 2$, let us consider the interval $[-\sqrt{3}, 0]$ this time. The function f is still continuous on $[-\sqrt{3}, 0]$ and differentiable on $(-\sqrt{3}, 0)$ as it is a polynomial function. The slope of the secant line is

$$\frac{f(0) - f(-\sqrt{3})}{0 - (-\sqrt{3})} = \frac{((0)^3 - 3(0) + 2) - ((-\sqrt{3})^3 - 3(-\sqrt{3}) + 2)}{\sqrt{3}} = 0.$$

The Mean Value Theorem guarantees that the equation $f'(c) = 0$ has at least one solution between $-\sqrt{3}$ and 0. In Section 4.1, we call this solution a critical number of $f(x)$. This results is a consequence of a special case of the Meant Value Theorem.

Rolle's Theorem: Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

In words, if the function has the same values at each end point of the closed interval, then it has a critical number (or more than one) between the end points.

In the previous example, $f(-\sqrt{3}) = 2$ and $f(0) = 2$, so $f(-\sqrt{3}) = f(0)$. For the completeness's sake, let us find the critical number. The equation is $3c^2 - 3 = 0$, so the solution are $c = \pm 1$. Note that the critical number 1 was not ensured by Rolle's theorem as it is outside the interval $[-\sqrt{3}, 0]$. In fact, $f'(x) < 0$ when $x < -1$ and $f'(x) > 0$ when $x > -1$ (but $x < 1$), so it is not only critical number but $f(x)$ actually has a local maximum at -1.

Rolle's theorem can prove that an equation has only one real root between two values a and b.

Example 3 Consider an equation $x^3 + e^x = 0$. We want to argue the existence of the root of the equation and possibly the uniqueness of the root as well. Define a function $g(x) = x^3 + e^x$. Since both x^3 and e^x are continuous everywhere, their sum is also continuous everywhere. By the Intermediate Value Theorem, the function $g(x)$ has a root (or possibly more) between a and b if $g(a)$ and $g(b)$ are different signs. It is not hard to see that $g(-3) = -26.95$ and $g(0) = 1$ have the different signs. By IVT, there exists at least one real root between -3 and 0.

Now suppose that there are more than one real root between -3 and 0. Then there are at least two real numbers a and b such that $-3 < a < b < 0$, $g(a) = 0$, and $g(b) = 0$. Since the function g is both continuous and differentiable everywhere, by Rolle's Theorem, there exists c between a and b such that $g'(c) = 0$, i.e. $3c^2 + e^c = 0$. Note that $3c^2$ is never negative and e^c is always positive, so it is impossible for $3c^2 + e^c$ to be zero. Thus, the equation $3c^2 + e^c = 0$ has no solution between a and b, which contradicts the conclusion of Rolle's Theorem. Therefore, there cannot be more than one real root between -3 and 0.

Since $g'(x) = 3x^2 + e^x$ is always positive (i.e. $g(x)$ is always increasing), $g(x)$ cannot have any other root outside the interval $[-3, 0]$. So $g(x)$ has a unique real root.

Q: Are all hypotheses of the theorems necessary?

Example 4 Consider a function $h(x) = \tan(x)$. Note that $h(\frac{\pi}{6}) = h(\frac{7\pi}{6}) = \frac{\sqrt{3}}{3}$. However, $h'(x) = \sec^2(x)$ is never zero. Shouldn't it be zero by Rolle's Theorem? This shouldn't be surprising at all as h is neither continuous nor differentiable at $x = \frac{\pi}{2}$ inside the interval $[\frac{\pi}{6}, \frac{7\pi}{6}]$.

Example 5 Consider a function $j(x) = |x|$. Shouldn't the Mean Value Theorem guarantee that there exists c between -2 and 4 so that

$$j'(x) = \frac{j(4) - j(-2)}{4 - (-2)} = \frac{1}{3}?$$

Note that the derivative of $j(x)$ is

$$j'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

So it will never be $\frac{1}{3}$. The function $j(x)$ is continuous everywhere, but it is not differentiable at $x = 0$ inside the interval $[-2, 4]$. Hence, it is not too shocking that the conclusion of the Mean Value Theorem does not follow.

Theorem 5 If $f'(x) = 0$ for all x in an interval (a, b), then f must be constant on (a, b).

Proof. Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. By MVT, there exists c between x_1 and x_2 such that $f'(c) = \frac{f(x_1)-f(x_2)}{x_1-x_2}$, but $f'(c) = 0$. So $f(x_1) - f(x_2) = 0$. Hence, $f(x_1) = f(x_2)$. $\square$

Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b). Consequently, $f(x) = g(x) + c$ for some constant c.

Proof. Apply the theorem above to the function $(f - g)(x)$. $\square$

Example 6 A number a is called a **fixed point** of a function f if $f(a) = a$. For instance, $\sin(x)$ has a fixed point 0 as $\sin(0) = 0$, but no other fixed point. The function $\cos(x)$ also has only one fixed point, called the Dottie number (about 0.739), but it is not obvious to find one. We can exhibit its location between 0 and $\frac{\pi}{2}$ using IVT to the equation $\cos(x) = x$, but the exact value is unknown.

Assume that the function f has a property that $f'(x) \neq 1$ for all real numbers x. Then we want to show that f has at most one fixed point.

Suppose f has two distinct fixed points, say a and b. By the MVT, there exists c between a and b such that $f'(c) = \frac{f(a)-f(b)}{a-b}$. Since a and b are fixed points, $f(a) = a$ and $f(b) = b$. Thus, $f(a) - f(b) = a - b$ and $\frac{f(a)-f(b)}{a-b} = 1$. Thus, $f'(c) = 1$ which contradicts the assumption that $f'(x) \neq 1$ for all real numbers x. Therefore, f has to have at most one fixed point. $\square$

Assigned Exercises: (p 291) 1 - 13 (odds), 17, 19, 21, 25, 27, 37