The Product Rule

$$\begin{aligned}
\frac{f(x+h)g(x+h)-f(x)g(x)}{h} &= \frac{f(x+h)g(x+h)\boxed{-f(x+h)g(x)+f(x+h)g(x)}-f(x)g(x)}{h} \\
&= \frac{f(x+h)(g(x+h)-g(x))+(f(x+h)-f(x))g(x)}{h} \\
&= \frac{f(x+h)(g(x+h)-g(x))}{h}+\frac{(f(x+h)-f(x))g(x)}{h} \\
&= f(x+h)\frac{g(x+h)-g(x)}{h}+\frac{f(x+h)-f(x)}{h}g(x)
\end{aligned}$$

Then the derivative of the product function of f and g is

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h}+\frac{f(x+h)-f(x)}{h}g(x) \\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x) \\
&= f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot g(x) \\
&= f(x)\cdot\frac{d}{dx}[g(x)]+\frac{d}{dx}[f(x)]\cdot g(x) \quad = \quad \frac{d}{dx}[f(x)]\cdot g(x)+f(x)\cdot\frac{d}{dx}[g(x)]
\end{aligned}$$

The Product Rule: If f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)]\cdot g(x)+f(x)\cdot\frac{d}{dx}[g(x)]$$

In simpler notation, $(fg)' = f'\cdot g + f\cdot g' = f'g+fg'$.

Example 1 The function $h(x) = x^5\cdot e^x$ is the product of two functions $f(x) = x^5$ and $g(x) = e^x$. Then the derivative of h is

$$h'(x) = (x^5)'\cdot e^x + x^5\cdot(e^x)' = 5x^4\cdot e^x + x^5\cdot e^x = (5x^4+x^5)e^x = x^4(x+5)e^x$$

Also the second derivative of h is

$$\begin{aligned}
h''(x) &= ((x^5+5x^4)e^x)' \\
&= (x^5+5x^4)'\cdot e^x + (x^5+5x^4)\cdot(e^x)' \\
&= (5x^4+20x^3)e^x + (x^5+5x^4)e^x \\
&= (x^5+10x^4+20x^3)e^x \\
&= x^3(x^2+10x+20)e^x
\end{aligned}$$

The Quotient Rule

$$
\begin{aligned}
\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} &= \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \\
&= \frac{f(x+h)g(x)\boxed{-f(x)g(x) + f(x)g(x)} - f(x)g(x+h)}{g(x+h)g(x)} \\
&= \frac{(f(x+h) - f(x))g(x) + f(x)(g(x) - g(x+h))}{g(x+h)g(x)} \\
&= \frac{(f(x+h) - f(x))g(x) - f(x)(g(x+h) - g(x))}{g(x+h)g(x)}
\end{aligned}
$$

Then the derivative of the quotient function of f and g is

$$
\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(f(x+h) - f(x))g(x) - f(x)(g(x+h) - g(x))}{g(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(f(x+h) - f(x))g(x)}{g(x+h)g(x)} - \lim_{h\to 0} \frac{1}{h} \cdot \frac{f(x)(g(x+h) - g(x))}{g(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \frac{g(x)}{g(x+h)g(x)} - \lim_{h\to 0} \frac{f(x)}{g(x+h)g(x)} \cdot \frac{g(x+h) - g(x)}{h} \\
&= \frac{d}{dx}[f(x)] \cdot \frac{g(x)}{g(x)g(x)} - \frac{f(x)}{g(x)g(x)} \cdot \frac{d}{dx}[g(x)] \\
&= \frac{\frac{d}{dx}[f(x)] \cdot g(x)}{g(x)g(x)} - \frac{f(x) \cdot \frac{d}{dx}[g(x)]}{g(x)g(x)} \quad = \quad \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{(g(x))^2}
\end{aligned}
$$

The Quotient Rule: If f and g are differentiable functions and $g(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{(g(x))^2}$$

Please note the "minus" sign in the numerator. In simpler notation,

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

Example 2 The function $k(x) = \frac{x}{e^x - 1}$ is the quotient of two functions $f(x) = x$ and $g(x) = e^x - 1$. Then the derivative of the quotient k is

$$k'(x) = \frac{(x)'(e^x - 1) - x(e^x - 1)'}{(e^x - 1)^2} = \frac{1 \cdot (e^x - 1) - x(e^x - 0)}{(e^x - 1)^2} = \frac{e^x - 1 - xe^x}{(e^x - 1)^2} = \frac{(1 - x)e^x - 1}{(e^x - 1)^2}$$

Assigned Exercises: (p 188) 3 - 25 (odds), 35, 41, 43, 47, 49, 61