$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

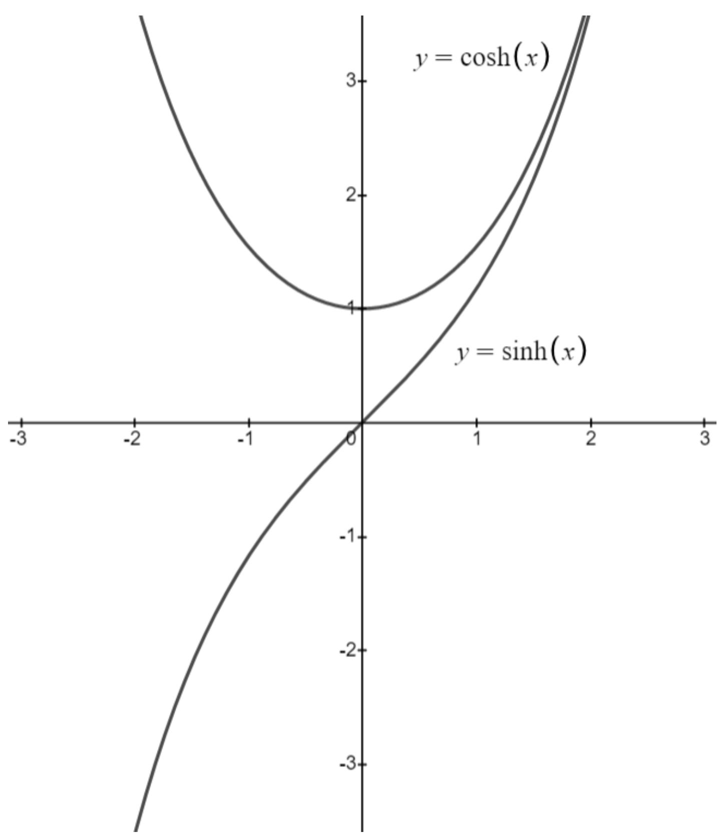
They are called the **hyperbolic functions**. Hyperbolic sine, hyperbolic cosine, and etc.

Pronunciation Guide: sinh "seen-sh", cosh "ko-sh", and for the rest just add "sh" sound at the end of the usual pronunciation of the function names.

Here are some obvious observations:

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{and} \quad \coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The hyperbolic functions have nothing to do with trigonometric functions, but some of identities to be mentioned later resemble the identities of the trigonometric functions. And these functions were precisely defined so that those identities would hold.



Just like $\sin(x)$, the hyperbolic sine $\sinh(x)$ is an odd function, i.e. $\sinh(-x) = -\sinh(x)$. Hence, the graph is symmetric about the origin. Its domain and range is $(-\infty, \infty)$.

The hyperbolic cosine is even function, i.e. $\cosh(-x) = \cosh(x)$. Hence, the graph is symmetric about the y-axis. It looks like a parabola, but no parabola can imitate the hyperbolic cosine function. The domain is $(-\infty, \infty)$, and the range is $[1, \infty)$.

Also note that these functions are not periodic functions like trigonometric sine or cosine.

Example 1 Find the derivatives of $\sinh(x)$ and $\cosh(x)$.

$$\frac{d}{dx}[\sinh(x)] = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh(x)$$
$$\frac{d}{dx}[\cosh(x)] = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

Here are the derivatives of the hyperbolic functions.

$$\frac{d}{dx}[\sinh(x)] = \cosh(x) \qquad \frac{d}{dx}[\operatorname{csch}(x)] = -\operatorname{csch}(x)\coth(x)$$
$$\frac{d}{dx}[\cosh(x)] = \sinh(x) \qquad \frac{d}{dx}[\operatorname{sech}(x)] = -\operatorname{sech}(x)\tanh(x)$$
$$\frac{d}{dx}[\tanh(x)] = \operatorname{sech}^2(x) \qquad \frac{d}{dx}[\coth(x)] = -\operatorname{csch}^2(x)$$

Hyperbolic Identities

$\sinh(-x) = -\sinh(x)$ $\qquad$ $\cosh(-x) = \cosh(x)$

$\cosh^2(x) - \sinh^2(x) = 1$ $\qquad$ $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

$\sinh(x + y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x)$

$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

We can deduce more.

$\sinh(2x) = 2\sinh(x)\cosh(x)$
$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1$

Using the identity $\cosh^2(t) - \sinh^2(t) = 1$ can be used to graph the parametric equation

$$\begin{cases} x(t) = \cosh(t) \\ y(t) = \sinh(t) \end{cases}$$

By the identity, $x^2 - y^2 = (\cosh(t))^2 - (\sinh(t))^2 = 1$, and its graph is a hyperbola. Hence, the name "hyperbolic".

Example 2 Simplify $(\cosh(x) + \sinh(x))^2$.

$$\begin{aligned} (\cosh(x) + \sinh(x))^2 &= \cosh^2(x) + 2\cosh(x)\sinh(x) + \sinh^2(x) \\ &= \cosh^2(x) + \sinh^2(x) + \cosh(x)\sinh(x) \\ &= \cosh(2x) + \sinh(2x) \end{aligned}$$

Assigned Exercises: (p 264) 9, 11, 21, 31 - 39 (odds), 47, 55