l'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Example 1 Find the limit

$$\lim_{x\to 1} \frac{\ln(x)}{x^2-1}$$

If we naively evaluate the expression for $x = 1$, we obtain $\frac{0}{0}$, which is one of the indeterminate forms. So l'Hospital's rule!

$$\lim_{x\to 1} \frac{\ln(x)}{x^2-1} = \lim_{x\to 1} \frac{(\ln(x))'}{(x^2-1)'} = \lim_{x\to 1} \frac{\frac{1}{x}}{2x} = \frac{1}{2}$$

Make sure you do <u>use</u> the quotient rule.

l'Hospital's rule works for limits at infinity, i.e. $x \to \infty$ or $x \to -\infty$.

Example 2 Find the limit

$$\lim_{xto\infty} \frac{\ln(x)}{\sqrt{x}}$$

As both $\ln(x)$ and $\sqrt{x}$ approach ∞ as $x \to \infty$, we have $\frac{\infty}{\infty}$, which is one of the indeterminate forms. So l'Hospital's rule!

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{(\ln(x))'}{(\sqrt{x})'} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

Sometimes you need to apply l'Hospital's rule more than once.

Example 3 Find the limit

$$\lim_{x\to 0^+} \frac{\sin^2(x)}{x^3}$$

If we naively evaluate the expression for $x = 0$, we obtain $\frac{0}{0}$, which is one of the indeterminate forms. So l'Hospital's rule!

$$\lim_{x\to 0^+} \frac{\sin^2(x)}{x^3} = \lim_{x\to 0^+} \frac{\frac{d}{dx}[\sin^2(x)]}{\frac{d}{dx}[x^3]} = \lim_{x\to 0^+} \frac{2\sin(x)\cos(x)}{3x^2} = \lim_{x\to 0^+} \frac{\sin(2x)}{3x^2}$$

If we naively evaluate the last expression for $x = 0$, we again obtain $\frac{0}{0}$, so we apply l'Hospital's rule again.

$$\lim_{x\to 0^+} \frac{\sin^2(x)}{x^3} = \cdots = \lim_{x\to 0^+} \frac{\sin(2x)}{3x^2} = \lim_{x\to 0^+} \frac{(\sin(2x))'}{(3x^2)'} = \lim_{x\to 0^+} \frac{2\cos(2x)}{6x} = \infty$$

Do not use l'Hospital's rule mindlessly.

Example 4 Find the limit

$$\lim_{x\to 1} \frac{\ln(x)}{x^2+1}$$

Let's use l'Hospital's rule!

$$\lim_{x\to 1} \frac{\ln(x)}{x^2+1} = \lim_{x\to 1} \frac{(\ln(x))'}{(x^2+1)'} = \lim_{x\to 1} \frac{\frac{1}{x}}{2x} = \lim_{x\to 1} \frac{1}{2x^2} = \frac{1}{2} \quad \text{THIS IS WRONG!!!}$$

In fact, when naively evaluate the original expression for $x = 1$, we just get a very nice limit.

$$\lim_{x\to 1} \frac{\ln(x)}{x^2+1} = 0$$

Here is the indeterminate product form.

$$0 \cdot \infty \quad \text{or} \quad \infty \cdot 0$$

If a naive evaluation of $f(x) \cdot g(x)$ gives the indeterminate product form $0 \cdot \infty$, then rewriting the expression as $\frac{f(x)}{1/g(x)}$ will give us the indeterminate form $\frac{0}{0}$. So l'Hospital's rule! Also rewriting as $\frac{g(x)}{1/f(x)}$ will use the indeterminate for $\frac{\infty}{\infty}$.

Example 5 Find the limit

$$\lim_{x\to\infty} e^{-x} \ln(x)$$

Since $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to\infty} \ln(x) = \infty$, we have the indeterminate product form $0 \cdot \infty$.

$$\lim_{x\to\infty} e^{-x}\ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{\frac{1}{e^{-x}}} = \lim_{x\to\infty} \frac{\ln(x)}{e^x} = \lim_{x\to\infty} \frac{(\ln(x))'}{(e^x)'} = \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0$$

We could have divided by the reciprocal of $\ln(x)$, but that is not a smart move as the derivative of $\frac{1}{\ln(x)}$ isn't nicer than the derivative of $\ln(x)$.

Here is the indeterminate difference form.

$$\infty - \infty$$

Example 6 Find the limit

$$\lim_{x\to\infty} (x - \ln(x))$$

Naively we have $\infty - \infty$, so l'Hospital's rule. We do a little trick.

$$\lim_{x\to\infty} (x-\ln(x)) = \lim_{x\to\infty} x\left(1 - \frac{\ln(x)}{x}\right) = \lim_{x\to\infty} x\cdot \lim_{x\to\infty}\left(1 - \frac{\ln(x)}{x}\right) = \lim_{x\to\infty} x\cdot\left(1 - \lim_{x\to\infty}\frac{\ln(x)}{x}\right)$$

Now we can evaluate the last limit using l'Hospital's rule (because naively $\frac{\infty}{\infty}$).

$$\lim_{x\to\infty}\frac{\ln(x)}{x} = \lim_{x\to\infty}\frac{(\ln(x))'}{(x)'} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0$$

Then

$$\lim_{x\to\infty} (x - \ln(x)) = \cdots = \lim_{x\to\infty} x\cdot\left(1 - \lim_{x\to\infty}\frac{\ln(x)}{x}\right) = \infty(1-0) = \infty$$

Here is the indeterminate power form.

$$0^0 \qquad \infty^0 \qquad 1^\infty$$

Example 7 Find the limit

$$\lim_{x\to\infty} x^{1/x}$$

Naively we see ∞^0, which is an indeterminate power form. So l'Hospital's rule! But how? Suppose that the limit does exist and let it be L, i.e.

$$L = \lim_{x\to\infty} x^{1/x}$$

We apply the natural log function ln to both sides. Since $\ln(x)$ is continuous, we have

$$\ln(L) = \ln\left(\lim_{x\to\infty} x^{1/x}\right) \quad\Rightarrow\quad \ln(L) = \lim_{x\to\infty}\ln(x^{1/x}) \quad\Rightarrow\quad \ln(L) = \lim_{x\to\infty}\frac{1}{x}\cdot\ln(x)$$

Last example yielded that $\lim_{x\to\infty}\frac{\ln(x)}{x} = 0$. Hence, $\ln(L) = 0$. Then $L = e^0 = 1$. Therefore,

$$\lim_{x\to\infty} x^{1/x} = 1$$

Assigned Exercises: (p 311) 9 - 29 (odds), 33, 35, 41 - 53 (odds), 57 - 67, 85, 89