In order to talk about a function, we first need two sets A and B of numbers (mostly in Calculus).

A **function** f is a rule that assigns to each element x in a set A "exactly one" element, denoted by $f(x)$, in a set B.

Symbolically, we denote it by $f : A \to B$. The set A is called the **domain** of the function f, and the set B is called the **codomain** of the function f. The letter f is the **name of the function**.

An actual assignment (or rule) of the elements can be described in many ways: pictorially, using a table, a set of ordered pairs (called a graph), or an explicit equation (or an implicit equation). For example, let us consider a function f defined on the domain $A = \mathbb{N}$ (the set of natural numbers) and the codomain $B = \mathbb{Q}$ (the set of rational numbers) as follows:

$$\begin{array}{ccc} 1 & \to & \frac{1}{1} \\ 2 & \to & \frac{1}{2} \\ 3 & \to & \frac{1}{3} \\ \vdots & & \vdots \end{array}$$

or

x	$f(x)$
1	$\frac{1}{1}$
2	$\frac{1}{2}$
3	$\frac{1}{3}$
$\vdots$	$\vdots$

or $\{(1, \frac{1}{1}), (2, \frac{1}{2}), (3, \frac{1}{3}), \ldots\}$

Given a set of ordered pairs, we can discern if the set represents a function or not by finding two distinct pairs whose first component is identical. If such pair exists, then the set does not represent a function. For instance, we cannot find such a pair in the example.

For an assignment $3 \to \frac{1}{3}$ or $(3, \frac{1}{3})$, the value 3 (from the domain) is called an **input value**, and the value $\frac{1}{3}$ is the corresponding **output value** or **function value** (from the codomain). If we express an arbitrary element of the domain using a variable x and the corresponding unique element in the codomain can be expressed in terms of x, then we can express the function using the set builder's notation as follows: $f = \left\{\left(x, \frac{1}{x}\right) : x \in \mathbb{N}\right\}$.

If a variable x (called the **input variable**) holds an input value, a variable y (called the **output variable**) holds an output value, and there exists an equation involving x and y that defines an assignment of f uniquely, then we can also represent the function f as follows:

$$f = \{(x, y) : x \in \mathbb{N}, xy = 1\}.$$

Furthermore, if we can solve the equation for the output variable y, then we call the equation an **explicit** equation for the function. We can also represent the function f as follows:

$$f = \left\{(x, y) : x \in \mathbb{N}, y = \frac{1}{x}\right\}.$$

We can finally represent the function by plotting the set of ordered pairs in the Cartesian coordinate plane with the x-axis as the horizontal axis and the y-axis as the vertical axis. We call this **drawing the graph** of the function. In this case, we can discern if the graph represent a function or not by finding a vertical line that passes through more than one point of the graph. If such vertical line exists, then the graph does not represent a function. It is called the **vertical line test** (VLT). For instance, we can not find such a vertical line for the example.

Have you noticed that the output values from the example only form a proper subset of the codomain $B = \mathbb{Q}$ given? If we form a set of all output values from the example, we get the set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ or $\{\frac{1}{x} : x \in \mathbb{N}\}$. The set of all output values is called the **range** of the function. Up to Calculus-level math courses, we actually care more about the range than a codomain. From now on, we will just let $\mathbb{R}$ be the codomain.

Function Notation

Now let us be real! For low-level math courses like Calculus, we discuss functions using the function notation. For the example we have considered so far, we define the function as follows:

$$f(x) = \frac{1}{x}$$

Here, f is the **name** of the function, x is the **input variable** (or **independent variable**), and the right hand side of the equal sign, i.e. $\frac{1}{x}$, is called the **definition of the function** (or **function rule**) of the function f.

We should not read the notation as "$f(x)$ equals $\frac{1}{x}$." but read as "$f(x)$ is defined by $\frac{1}{x}$. Sometimes, we use the explicit equation $y = \frac{1}{x}$ to define the function. In that case, the variable y is called the **output variable** (or **dependent variable**) of the function. When do we use $f(x) = \frac{1}{x}$? When do we use $y = \frac{1}{x}$? If we want to emphasize the name of the function, we use the former. If we want to emphasize the output variable, then we use the latter. Simple as that.

Given a function notation $f(x) = \frac{1}{x}$ and an input value 3, we **evaluate** the function f with the input value 3 by "substituting" every occurrence of the input variable x in the definition of the function $\frac{1}{x}$ with the input value 3. Symbolically, we denote it by $f(3) = \frac{1}{(3)}$. After we simplify the expression after the substitution, we get the the output value $\frac{1}{3}$ that corresponds to the input value 3.

Using the function notation, there is no mention of a domain (which should be given before the rule is mentioned). Once again for low-level math courses, we are first given with the function notation and are expected to find the **(implied) domain**, which is a maximal subset of $\mathbb{R}$ in a sense that it contains all possible and/or valid real numbers that can substitute the input variable in the definition of the function. For $f(x) = \frac{1}{x}$, the (implied) domain is $D(f) = \{x \in \mathbb{R} : x \neq 0\}$, i.e. the set of all nonzero real numbers. We may omit the expression $\in \mathbb{R}$ and just state the domain as $D(f) = \{x : x \neq 0\}$. Then the corresponding range of the function is $R(f) = \{y : y \neq 0\}$. There are few undefined (as real numbers) expressions that should be considered when finding the domain of a function:

i) $\sqrt{x}$ is undefined if $x < 0$.

ii) $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$ is undefined if $x < 0$ and n is an even positive integer.

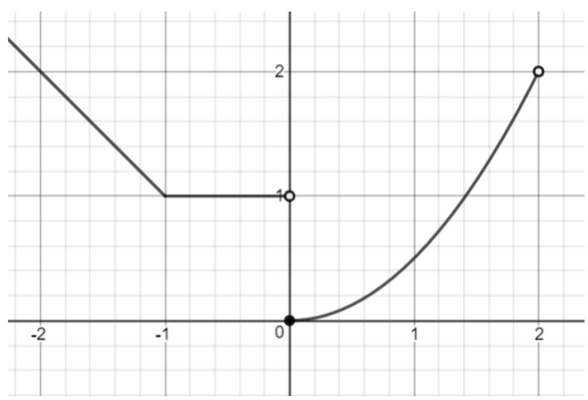
iii) $\frac{1}{x}$ is undefined if $x = 0$.

Piecewise Defined Function

Example 1 The following piecewise defined function is composed of three different functions.

$$f(x) = \begin{cases} -x & \text{if} \quad x < -1 \\ 1 & \text{if} \quad -1 \leq x < 0 \\ \frac{1}{2}x^2 & \text{if} \quad 0 \leq x < 2 \end{cases}$$

Domain $= \{x : x < 2\}$ or $(-\infty, 2)$
Range $= \{y : y \geq 0\}$ or $[0, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$



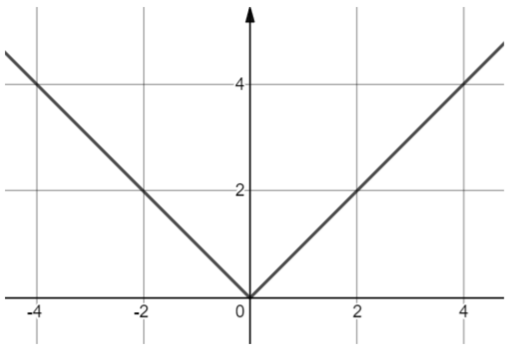
To evaluate $f(x)$, we need to see where x is located first. Check which inequality the x-value given satisfies, then we can choose which piece of functions to use to evaluate. For instance, $f(-2) = -(-2) = 2$ since $-2 < -1$. $f(-1) = 1$ since $-1 \leq -1 < 0$. $f(0) = \frac{1}{2}(0)^2 = 0$ since $0 \leq 0 < 2$. $f(1) = \frac{1}{2}(1)^2 = \frac{1}{2}$ since $0 \leq 1 < 2$. $f(2)$ is undefined.

Absolute Value Function For all real numbers x,

$$|x| = \begin{cases} -x & \text{if} \quad x < 0 \\ x & \text{if} \quad x \geq 0 \end{cases}$$

In words, if what is inside the absolute value symbol $|\cdot|$ is negative, then the output is obtained by negating the inside. If what is inside the absolute value symbol is zero or positive, then the output is obtained by simply removing the absolute value symbol.

For instance, $|-4| = -(-4)$ since -4 is negative. $|0| = 0$ and $|4| = 4$ since 4 is positive.



Here is the graph of the absolute value function $f(x) = |x|$.

When $x < 0$, we use $y = -x$ graph.
When $x \geq 0$, we use $y = x$ graph.

Domain $= (-\infty, \infty)$
Range $= [0, \infty)$.

Here are some important facts about the absolute values: For all real numbers a and b,

i) $|ab| = |a| \cdot |b|$

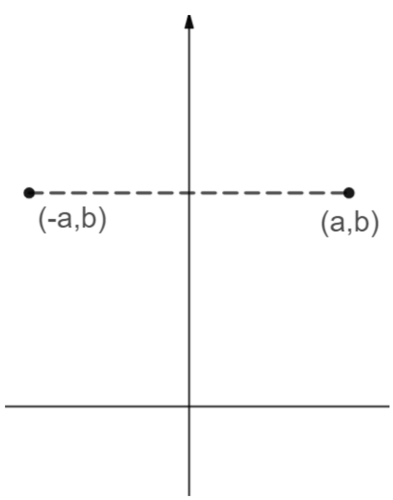
ii) $|\frac{a}{b}| = \frac{|a|}{|b|}$

iii) $|a + b| \leq |a| + |b|$

iv) $\sqrt{x^2} = |x|$ (or $\sqrt[n]{x^n} = |x|$ if n is even.)

Symmetry

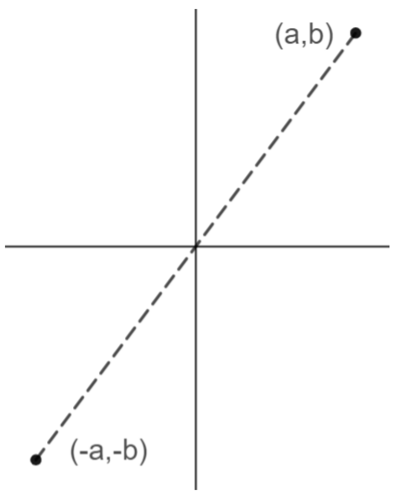
A function $f(x)$ is called an **even function** if $f(-x) = f(x)$ for all $x \in D(f)$.
A function $f(x)$ is called an **odd function** if $f(-x) = -f(x)$ for all $x \in D(f)$.



Suppose that $f(x)$ is an even function and $f(3) = 4$. Then the point $(3, 4)$ is on the graph of f. Since f is even, $f(-3) = 4$. Hence, $(-3, 4)$ must be on the graph as well. Hence, on the graph of an even function, if (a, b) is on the graph, then $(-a, b)$ must be as well. These two points are symmetric about the y-axis.

The graph of an even function is symmetric about the y-axis.

The left hand side of the graph of an even function must fall on top of the right hand side of the graph if it is folded at the y-axis.



Suppose that $f(x)$ is an odd function and $f(3) = 4$. Then the point $(3, 4)$ is on the graph of f. Since f is odd, $f(-3) = -4$. Hence, $(-3, -4)$ must be on the graph as well. Hence, on the graph of an odd function, if (a, b) is on the graph, then $(-a, -b)$ must be as well. These two points are symmetric about the origin.

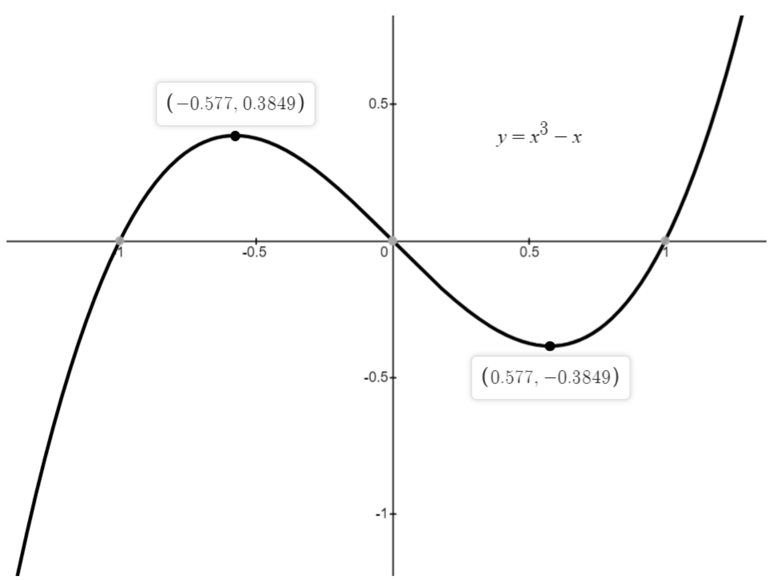
The graph of an odd function is symmetric about the origin.

If you reflect the graph of an odd function about the x-axis and then about the y-axis, you should see no difference between before and after.

Increasing and Decreasing Functions

When determining if a function is increasing or decreasing, we always move from left to right. Imagine you are riding a roller coaster which is traveling along the graph of the function from left end to right end.

Example 2 The following is the graph of the function $f(x) = x^3 - x$ which can be factored as $x(x-1)(x+1)$. Hence, it has three x-intercepts at $(-1, 0)$, $(0, 0)$, and $(1, 0)$. If understand how a generic cubic polynomial graph looks like, it is not hard to draw a graph like below. However, finding two outstanding points $(-0.577, 0.3849)$ (peak) and $(0.577, -0.3849)$ (valley) is not easy. Calculus will teach you in a later chapter how to find these coordinates with exact values.



If you start your roller coaster from at least at the far left end of the graph and moving towards right, then you will experience the ride moving up until you reach $x = -0.577$. That is equivalent to the function being "increasing." Hence, we describe that $f(x)$ is increasing on the interval $(-\infty, -0.577)$. At the very moment at $x = -0.577$, technically the function is neither increasing nor decreasing. Right after $x = -0.577$, the ride is going down until you reach $x = 0.577$. So we say that the function $f(x)$ is "decreasing" on the interval $(-0.577, 0.577)$. Again at the very bottom of the valley, the function is neither increasing nor decreasing. Finally the ride will shoot up to sky there upon. Then the function $f(x)$ is "increasing" on the interval $(0.577, \infty)$. In summary, **the function** $f(x)$ **is increasing on the interval** $(-\infty, -0.577) \cup (0.577, \infty)$ **and decreasing on the interval** $(-0.577, 0.577)$. The symbol $\cup$ (read "union") is used to conjoin two separate intervals. Here is the formal definition.

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \in I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \in I$$

If a function is neither increasing nor decreasing over the entire domain of the function, the function is called a **constant function**. A constant function is always defined as

$$f(x) = k$$

where k is some real number. The graph of a constant function is a horizontal line passing through $(0, k)$. If $k = 0$, then the function $f(x) = 0$ is called the **zero function**, and its graph is basically the x-axis.

Assigned Exercises: (p 19) 3, 7, 25 - 37 (odds), 43, 47, 55 - 63 (odds), 71 - 79 (odds)