Given a function defined on an interval I, a differentiable function F is called an **antiderivative** of f on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

If $F(x) = 4x^3$, then $F'(x) = 12x^2$. Hence, the function $F(x)$ is an antiderivative of the function $f(x) = 12x^2$. An antiderivative means "ante-derivative," that is a function "before" taking the derivative. Note that $4x^3 + 5$ is also an antiderivative of $f(x) = 12x^2$. This is a consequence of the corollary from the Mean Value Theorem section. If two functions have the same derivative, two functions are only differed by a constant. If $F(x)$ is an antiderivative of a function $f(x)$, then so is $F(x) + C$ for some constant C.

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

After observing that the derivative of the function $F(x) = x\ln(x) - x$ is $F'(x) = 1 \cdot \ln(x) + x \cdot \frac{1}{x} - 1 = \ln(x)$, it is easy to say that the most general antiderivative of the function $f(x) = \ln(x)$ is $F(x) = x\ln(x) - x + C$. But finding an antiderivative of a so-and-so function can be quite challenging. Finding an antiderivative of e^x requires almost no thinking. It takes a little recollection to see that the function $F(x) = \sin(x)$ is an antiderivative of the function $f(x) = \cos(x)$. Or maybe even little more thinking to see that the function $F(x) = -\cos(x)$ is an antiderivative of the function $f(x) = \sin(x)$.

Example 1 Find an antiderivative and the most general antiderivative of the function

$$f(x) = 5x^4 + \frac{3}{x^5}$$

Goal: Find a function $F(x)$ so that $F'(x) = 5x^4 + 3x^{-5}$. When taking the derivative of a sum of two functions, one rule allows us to take the derivative of each function and add the result, that is $(g(x)+h(x))' = g'(x)+h'(x)$. If we let $g'(x)+h'(x) = 5x^4+3x^{-5}$, then we can see that an antiderivative can be the sum $g(x)+h(x)$ so that $g'(x) = 5x^4$ and $h'(x) = 3x^{-5}$. Here, $g(x)$ is an antiderivative of $5x^4$ and $h(x)$ is an antiderivative of $3x^{-5}$.

Recall the power rule: $(x^n)' = nx^{n-1}$. The function $5x^4$ looks like it could have come from the rule. If $n = 5$, then $(x^5)' = 5x^4$. So the function x^5 is an antiderivative of the function $5x^4$. Now onto $3x^{-5}$. This one does not seem to come from the power rule. If $n = -4$, then

$(x^{-4})' = -4x^{-5}$ matches the exponent but not the coefficient. Note that

$$3x^{-5} = 3(x^{-5}) = 3\left(\frac{-4}{-4}x^{-5}\right) = \boxed{\frac{3}{-4}} \cdot -4x^{-5}$$

There is also a rule: $(c \cdot p(x))' = c \cdot p'(x)$. Using the rule,

$$\left(\frac{3}{-4} \cdot x^{-4}\right)' = \frac{3}{-4}(x^{-4})' = \frac{3}{-4}(-4x^{-5}) = 3x^{-5}$$

So the function $-\frac{3}{4}x^{-4}$ is an antiderivative of $3x^{-5}$. Next time, just use our first two antidifferentiation formulas.

Antidifferentiation Formula 1: If $F(x)$ is an antiderivative of the function $f(x)$,

an antiderivative of $c \cdot f(x)$ is $c \cdot F(x)$.

Antidifferentiation Formula 2: If $n \neq -1$,

an antiderivative of x^n is $\dfrac{1}{n+1}x^{n+1}$.

If you combine two formulas together, we have

If $n \neq -1$,

an antiderivative of ax^n is $\dfrac{a}{n+1}x^{n+1}$.

Finally, the function $F(x) = x^5 + \frac{3}{-4}x^{-4}$ is an antiderivative of the function $f(x) = 5x^4 + 3x^{-5}$, and the most general antiderivative is $F(x) = x^5 - \frac{3}{4x^4} + C$.

Example 2 If a man is free-falling from a height 855 feet, how long would he take to reach the ground?

Since it is free-falling, the initial velocity is 0 ft/s. The acceleration due to gravity is -32 ft/s/s, so the velocity reduces by 32 ft/s every second. Thus, the velocity function is

$$v(t) = -32t$$

Since the derivative of the height function $h(t)$ is the velocity function, we have $h'(t) = v(t)$. So $h(t)$ is an antiderivative of $v(t)$. By the rule we found above,

$$h(t) = \frac{-32}{1+1}t^{1+1} = -16t^2$$

But this function does not work for us as $h(0) = -16(0)^2 = 0$ ft when the initial height of the man is, in fact, 855 ft. That's because we did not obtain the most general antiderivative, that is, $h(t) = -16t^2 + C$. In order to have $h(0) = 855$, we figure that $C = 855$. Hence, the height function is

$$h(t) = -16t^2 + 855$$

Reaching ground means that the height is zero. $h(t) = 0$, i.e. $-16t^2 + 855 = 0$, i.e. $t^2 = \frac{855}{16}$, i.e. $t = \pm\frac{\sqrt{855}}{4} = \pm\frac{3\sqrt{95}}{4} \approx 7.31$ seconds.

What we did in the previous example is called **solving a differential equation with an initial value**. An equation that involves the derivatives of a function is called a **differential equation**. The **general solution** of a differential equation is often times the most general antiderivative of some function. If a condition (sometimes called an **initial value**) is given, then we may use them to determine the value of the constant in the general solution. In that case, we call such a solution the **particular solution**.

In the example above, the differential equation we formulated based on gravity was $h' = -32t$. It is called a differential equation as it involves the derivative h'. The general solution we obtained was $h(t) = -16t^2 + C$. Because of an arbitrary constant C, it is called the "general" solution. The initial value was $h(0) = 855$, and we used it to find the value of $C = 855$. The solution $h(t) = -16t^2 + 855$ is, then, called the particular solution as the value of the constant C is determined.

Example 3 Find f if $f'(x) = \frac{1}{x}$ and $f(1) = -1$.

First, we need to find an antiderivative of the function $\frac{1}{x} = x^{-1}$. Using the antidifferentiation formula 2, the antiderivative of x^{-1} should be

$$\frac{1}{-1+1}x^{-1+1} = \frac{1}{0}x^0 \quad \text{THIS IS WRONG!}$$

Note that the formula has the condition $n \neq -1$, so we cannot use the formula for this function. We should recall that

$$(\ln(x))' = \frac{1}{x}$$

for $x > 0$. Hence, an antiderivative of $\frac{1}{x}$ should be $\ln(x)$ for $x > 0$. So the general solution is $f(x) = \ln(x) + C$ for some constant C. The initial value says that $f(1) = -1$, i.e. $f(1) = \ln(1) + C = 0 + C = C$ which should be -1. Hence, $C = -1$. Therefore, the particular solution is $f(x) = \ln(x) - 1$.

Here are antidifferentiation formulas:

Function	An antiderivative	Most general antiderivative
$c \cdot f(x)$	$c \cdot F(x)$	$c \cdot F(x) + C$
$f(x) + g(x)$	$F(x) + G(x)$	$F(x) + G(x) + C$
$f(x) - g(x)$	$F(x) - G(x)$	$F(x) - G(x) + C$
x^n (if $n \neq 1$)	$\frac{1}{n+1}x^{n+1}$	$\frac{1}{n+1}x^{n+1} + C$
ax^n (if $n \neq 1$)	$\frac{a}{n+1}x^{n+1}$	$\frac{a}{n+1}x^{n+1} + C$
$\frac{1}{x}$	$\ln(\lvert x\rvert)$	$\ln(\lvert x\rvert) + C$
e^x	e^x	$e^x + C$
b^x	$\frac{1}{\ln(b)}b^x$	$\frac{1}{\ln(b)}b^x + C$
$\cos(x)$	$\sin(x)$	$\sin(x) + C$
$\sin(x)$	$-\cos(x)$	$-\cos(x) + C$
$\sec^2(x)$	$\tan(x)$	$\tan(x) + C$
$\sec(x)\tan(x)$	$\sec(x)$	$\sec(x) + C$
$\csc^2(x)$	$-\cot(x)$	$-\cot(x) + C$
$\csc(x)\cot(x)$	$-\csc(x)$	$-\csc(x) + C$
$\cosh(x)$	$\sinh(x)$	$\sinh(x) + C$
$\sinh(x)$	$\cosh(x)$	$\cosh(x) + C$

The list can go on with the rest of hyperbolic functions. What about antiderivatives of $\sec(x)$, $\csc(x)$, $\tan(x)$, and $\cot(x)$? They are rather difficult to obtain at this moment. Later, we will discuss their antiderivatives.

In the table, you find that an antiderivative of $\frac{1}{x}$ is $\ln(|x|)$. Not just $\ln(x)$. It is not wrong to consider $\ln(x)$ as an antiderivative of $\frac{1}{x}$ as long as we assume that $x > 0$ (which is also the domain of $\ln(x)$). The formula tries to find an antiderivative of $\frac{1}{x}$ even for the case $x < 0$. Note that

$$\frac{d}{dx}[\ln(|x|)] = \frac{1}{|x|} \cdot \frac{d}{dx}[|x|] = \frac{1}{|x|} \cdot \frac{|x|}{x} = \frac{1}{x}$$

So $\ln(|x|)$ is indeed an antiderivative of $\frac{1}{x}$ on its domain $(-\infty, 0) \cup (0, \infty)$.

Warning: From now on, when asked to find an antiderivative of $\frac{1}{x}$, ALWAYS answer $\ln(|x|)$ regardless what the domain of $\frac{1}{x}$ is set.

Assigned Exercises: (p 355) 1 - 21 (odds), 23, 25 - 47 (odds), 49, 51, 61, 63, 65, 69