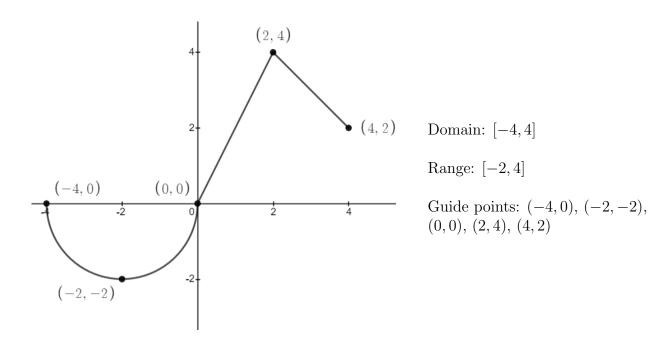
Transformation of Functions

Let y = f(x) be a piecewise defined as follows:

$$f(x) = \begin{cases} -\sqrt{4 - (x+2)^2} & \text{if } -2 \le x < 0 \\ -2|x-2|+4 & \text{if } 0 \le x < 2 \\ -|x-2|+4 & \text{if } 2 \le x \le 4 \end{cases}$$



Vertical and Horizontal Shifts Suppose h > 0 and k > 0. To obtain the graph of

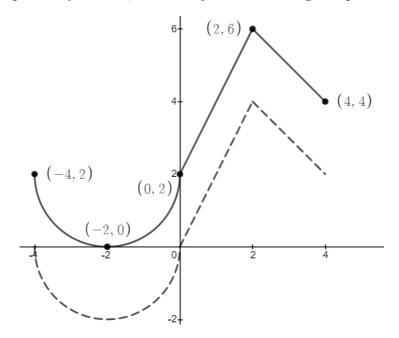
y = f(x) + k, shift the graph of y = f(x) a distant k units upward.

y = f(x) - k, shift the graph of y = f(x) a distant k units downward.

y = f(x - h), shift the graph of y = f(x) a distant h units to the right.

y = f(x + h), shift the graph of y = f(x) a distant h units to the left.

Example 1 Graph the function y = f(x) + 2. Since we are shifting the graph of y = f(x) upward by 2 units, add 2 to y-values of the guide points of the graph of y = f(x).



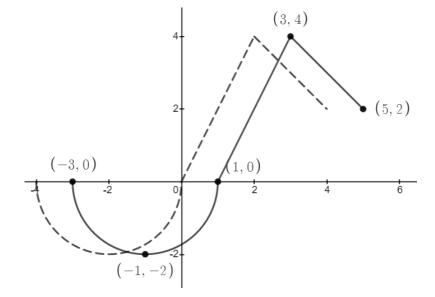
$$(a,b) \mapsto (a,b+2)$$

$$\begin{array}{cccc} (-4,0) & \mapsto & (-4,2) \\ (-2,-2) & \mapsto & (-2,0) \\ (0,0) & \mapsto & (0,2) \\ (2,4) & \mapsto & (2,6) \\ (4,2) & \mapsto & (4,4) \end{array}$$

Domain: [-4, 4] (no change) Range: [-2 + 2, 4 + 2] = [0, 6]

Note that the overall shape (even the size) has not been altered. Such a transformation is called an **isometry** or **rigid motion**.

Example 2 Graph the function y = f(x - 1). Since we are shifting the graph of y = f(x) to the right by 1 unit, add 1 to x-values of the guide points of the graph of y = f(x).



$$(a,b) \mapsto (a+1,b)$$

$$\begin{array}{cccc} (-4,0) & \mapsto & (-3,0) \\ (-2,-2) & \mapsto & (-1,-2) \\ (0,0) & \mapsto & (1,0) \\ (2,4) & \mapsto & (3,4) \\ (4,2) & \mapsto & (5,2) \end{array}$$

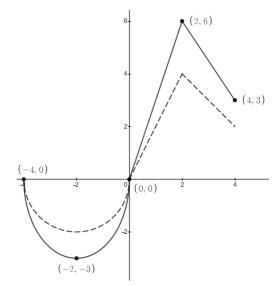
Domain: [-4+1, 4+1] = [-3, 5]Range: [-2, 4] (no change) **Vertical Stretching and Shrinking** The graph of y = Af(x) is obtained as follows:

If A > 1, then the graph of y = Af(x) is obtained by vertically stretching the graph of y = f(x) by a factor of A.

If 0 < A < 1, then the graph of y = Af(x) is obtained by vertically shrinking the graph of y = f(x) by a factor of $\frac{1}{A}$.

That is, (a, b) of the graph of y = f(x) is mapped to (a, Ab) of the graph of y = Af(x).

Example 3 Graph the function y = 1.5f(x). Since we are vertically stretching the graph of y = f(x) by a factor of 1.5, multiply 1.5 to y-values of the guide points of the graph of y = f(x).



$$(a,b) \mapsto (a,1.5b)$$

$$(-4,0) \mapsto (-4,0)$$

 $(-2,-2) \mapsto (-2,-3)$

$$(0,0) \qquad \mapsto \quad (0,0)$$

$$(2,4) \qquad \mapsto \quad (2,6)$$

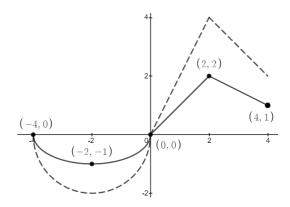
$$(4,2) \qquad \mapsto \quad (4,3)$$

Domain: [-4,4] (no change)

Range: [1.5(-2), 1.5(4)] = [-3, 6]

Note that the points on the x-axis are **stationary** under vertical stretching because multiplying 1.5 does not change the y-value.

Example 4 Graph the function y = 0.5 f(x). Since we are vertically shrinking the graph of y = f(x) by a factor of $\frac{1}{0.5}$, multiply 0.5 to y-values of the guide points of the graph of y = f(x).



$$(a,b) \mapsto (a,0.5b)$$

$$(-4,0) \mapsto (-4,0)$$

 $(-2,-2) \mapsto (-2,-1)$

$$(0,0) \qquad \mapsto \quad (0,0)$$

$$(2,4) \qquad \mapsto \qquad (2,2)$$

$$(4,2) \qquad \mapsto \quad (4,1)$$

Domain: [-4, 4] (no change)

Range: [0.5(-2), 0.5(4)] = [-1, 2]

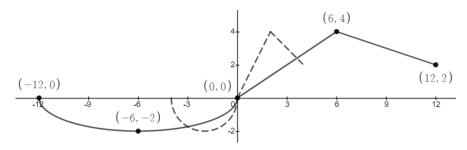
Horizontal Stretching and Shrinking The graph of y = f(Bx) is obtained as follows:

If 0 < B < 1, then the graph of y = f(Bx) is obtained by horizontally stretching the graph of y = f(x) by a factor of $\frac{1}{B}$.

If B > 1, then the graph of y = f(Bx) is obtained by horizontally shrinking the graph of y = f(x) by a factor of B.

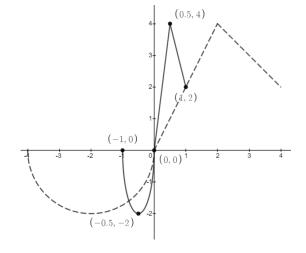
That is, (a,b) of the graph of y=f(x) is mapped to $(\frac{1}{B}a,b)$ of the graph of y=f(Bx).

Example 5 Graph the function $y = f(\frac{1}{3}x)$. Since we are horizontally stretching the graph of y = f(x) by a factor of $\frac{1}{1/3}$, multiply 3 to x-values of the guide points of the graph of y = f(x).



 Domain: [3(-4), 3(4)] = [-12, 12]Range: [-2, 4] (no change)

Example 6 Graph the function y = f(4x). Since we are horizontally shrinking the graph of y = f(x) by a factor of 4, multiply $\frac{1}{4}$ to x-values of the guide points of the graph of y = f(x).



$$(a,b) \qquad \mapsto \quad (\frac{1}{4}a,b)$$

$$\begin{array}{cccc} (-4,0) & \mapsto & (-1,0) \\ (-2,-2) & \mapsto & (-0.5,-2) \\ (0,0) & \mapsto & (0,0) \end{array}$$

$$(0,0) \mapsto (0,0)$$

 $(2,4) \mapsto (0.5,4)$

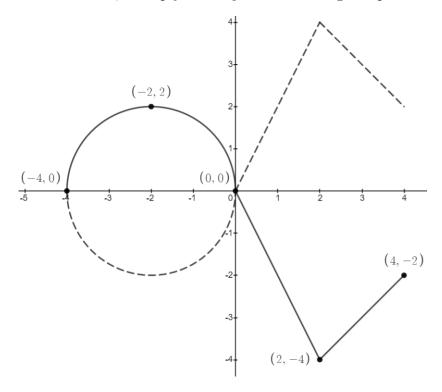
$$(4,2) \qquad \mapsto \quad (1,2)$$

Domain: $\left[\frac{1}{4}(-4), \frac{1}{4}(4)\right] = [-1, 1]$ Range: [-2, 4] (no change) Reflecting about the Axes To obtain the graph of

y = -f(x), reflect the graph of y = f(x) about the x-axis.

y = f(-x), reflect the graph of y = f(x) about the y-axis.

Example 7 Graph the function y = -f(x). Since we are reflecting the graph of y = f(x) about the x-axis, multiply -1 to y-values of the guide points of the graph of y = f(x).



$$(a,b) \mapsto (a,-b)$$

$$(-4,0) \qquad \mapsto \quad (-4,0)$$

$$(-2, -2) \quad \mapsto \quad (-2, 2)$$

$$(0,0) \qquad \mapsto \quad (0,0)$$

$$(2,4) \qquad \mapsto \quad (2,-4)$$

$$(4,2) \qquad \mapsto \quad (4,-2)$$

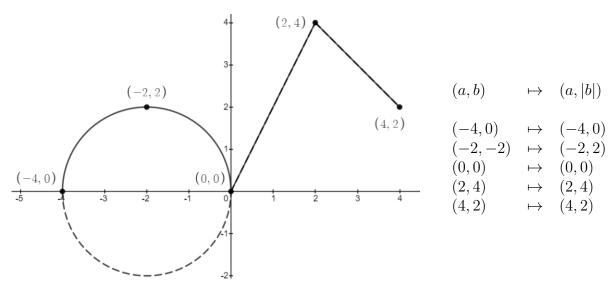
Domain: [-4, 4] (no change) Range: [-4, 2] (note that [2, -4] is an invalid interval notation)

Note that we get a mirror image about the x-axis.

Absolute Value of a Function To obtain the graph of y = |f(x)|, reflect only the portion of the graph of y = f(x) below the x-axis over the x-axis.

Since the y-values of the graph of y = f(x) that are above the x-axis are positive, it is not affected by the absolute value.

Example 8 Graph the function y = |f(x)|. Only the semicircle which is below the x-axis is reflected over the x-axis.



More than one transformation can be applied to a function to achieve more complicated transformation. One popular form is

$$y = k + Af(B(x - h))$$
 or $y = Af(B(x - h)) + k$

In this form, the transformations are carried in the following order:

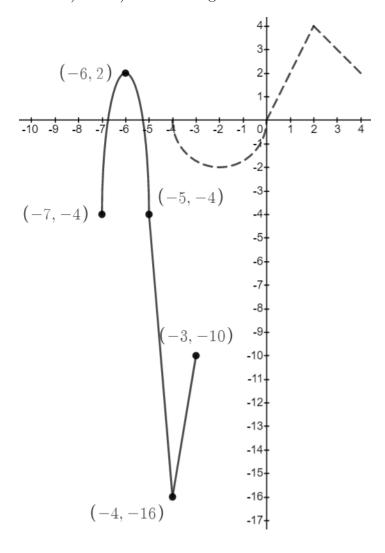
- 1) Horizontal stretching or shrinking using B (reflection about the y-axis if B < 0)
- 2) Horizontal shifting using h
- 3) Vertical stretching or shrinking using A (reflection about the x-axis if A < 0)
- 4) Vertical shifting using k

The argument of the function f may not be written in the form of B(x-h). In that case, the coefficient B must be factored out to write in the form of B(x-h) to figure out the correct horizontal shifting factor h. For instance, the graph of y = f(2x-6) is <u>not</u> obtained by shifting the graph of y = f(x) to the right by 6 units. The argument 2x - 6 must be factored into 2(x-3) to see that the shifting factor is 3 units.

Example 9 Graph the function y = -4 - 3f(2x + 10). Note that the argument 2x + 10 must be factored as 2(x + 5). Hence, the function can be written as y = -4 - 3f(2(x + 5)). The transformations are done in the following order:

- Horizontal shrinking by a factor of 2, i.e. $(a,b) \mapsto (\frac{1}{2}a,b)$ 1)
- Horizontal shifting to the left by 5 units, i.e. $(\frac{1}{2}a,b) \mapsto (\frac{1}{2}a-5,b)$ Vertical stretching by a factor of 3, i.e. $(\frac{1}{2}a-5,b) \mapsto (\frac{1}{2}a-5,3b)$ 2)
- 3)
- Reflect about the x-axis, i.e. $(\frac{1}{2}a 5, 3b) \mapsto (\frac{1}{2}a 5, -3b)$ 4)
- Vertical shifting downward by 4 units, i.e. $(\frac{1}{2}a 5, -3b) \mapsto (\frac{1}{2}a 5, -3b 4)$ 5)

Note: 3) and 4) are done together.



Combination of Functions

Given two functions f(x) and g(x), we can

- i) add them to form the sum function f + g, defined by (f + g)(x) = f(x) + g(x).
- ii) subtract one from the other to form the **difference function** f g, defined by (f g)(x) = f(x) g(x).
- iii) multiply them to form the **product function** $f \cdot g$, defined by $(f \cdot g)(x) = f(x) \cdot g(x)$.
- iv) divide one by the other to form the **quotient function** $\frac{f}{g}$, defined by $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$.

Note 1: The difference of f and g means f-g and the difference of g and f means g-f. Note 2: The quotient of f and g means $\frac{f}{g}$ and the quotient of g and f means $\frac{g}{f}$.

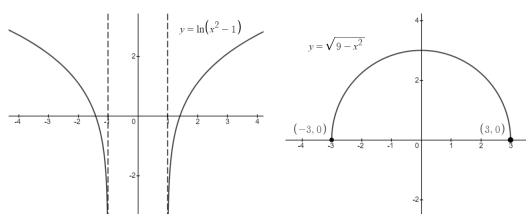
It is <u>not</u> easy to explain how these operations will combine the graphs of f and g. So it might be simply better strategy to create a chart to plot them. However, the domain of f + g, f - g, $f \cdot g$, and $\frac{f}{g}$ can be found without actually plotting them.

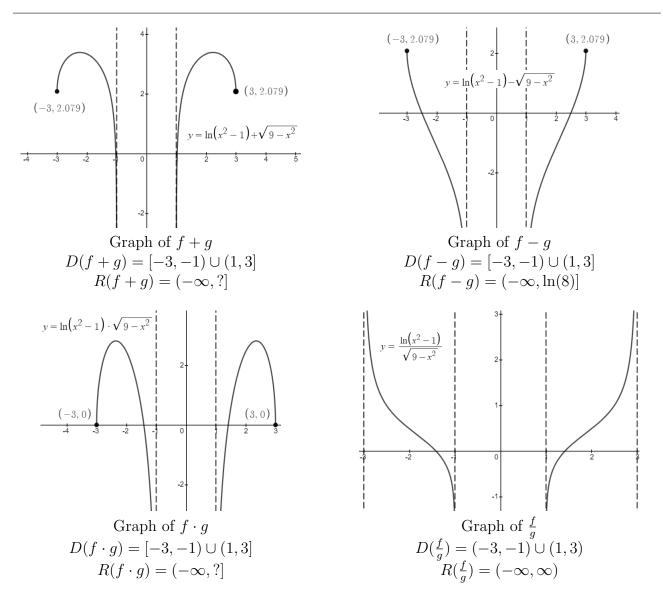
$$D(f+g) = D(f) \cap D(g) \qquad D(f-g) = D(f) \cap D(g) \qquad D(f \cdot g) = D(f) \cap D(g)$$

$$D\left(\frac{f}{g}\right) = D(f) \cap D(g) \cap \{x \in \mathbb{R} : g(x) \neq 0\}$$

Basically, the domain should be the intersection of the domains of f and g. But for the quotient function, we further require that the denominator function g should not be zero.

Example 10 Consider $f(x) = \ln(x^2 - 1)$ and $g(x) = \sqrt{9 - x^2}$. Note that $D(f) = (-\infty, 1) \cup (1, \infty)$ and D(g) = [-3, 3]. Then $D(f) \cap D(g) = [-3, -1) \cup (1, 3]$.





Q: Have you noticed that the domain of the quotient function is slightly different from the rest? Why?

Another way to combine two functions is to "compose" them.

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

Similarly, the composite function $g \circ f$ is defined by

$$(g \circ f)(x) = g(f(x))$$

It is very tricky the find the domain of the composition $f \circ g$. When the composition function is evaluated, note that we first insert the input value into the function g, so the domain of g is the bare bottom line we start with. Then the function f takes the output of g as its input, so that means the output of g must be inside the domain of f. Technically, here is how we can find the domain of the composite function $f \circ g$.

$$D(f \circ g) = \{x \in D(g) : g(x) \in D(f)\}$$

$$D(g \circ f) = \{x \in D(f) : f(x) \in D(g)\}$$

In words, the domain of $f \circ g$ is the set of all real numbers x from the domain of g such that the function values g(x) is in the domain of f. Sometimes, for $f \circ g$, we refer g as an **inside function** and f as an **outside function**.

Example 11 Consider $f(x) = -x^2 - 1$ with the domain $D(f) = (-\infty, \infty)$ and $g(x) = \sqrt{x}$ with the domain $D(g) = [0, \infty)$. The composite function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = -(\sqrt{x})^2 - 1$$

The domain of $f \circ g$ is $[0, \infty)$. Since the domain of f itself is the set of all real numbers, we only have to worry about the domain of g. Note that the final expression is not simplified to -x-1. When finding a domain of a composite function, it is very important not to simplify right after inserting of the inside function into the outside function.

Common Mistake: If you further simplify to write

$$(f \circ g)(x) = -x - 1$$

you will end up making a mistake of thinking the domain of $f \circ g$ as $(-\infty, \infty)$ as -x - 1 is just a linear polynomial.

Once the domain of $f \circ g$ is found, then you may simplify to write

$$(f \circ g)(x) = -x - 1$$

with the domain $D(f \circ g) = [0, \infty)$. So $(f \circ g)(2) = -2 - 1 = -3$, but $(f \circ g)(-2) \neq -(-2) - 1 = 1$ as the composite function $f \circ g$ is not even defined for x = -2.

The composition function $g \circ f$ would not make sense what so ever. All possible output values of f are negative which cannot be inserted into g as we cannot evaluate the square root of negative numbers. Hence, $g \circ f$ is an empty function as its domain is \emptyset .

Assigned Exercises: (p 42) 1, 3, 5, 7, 13, 17, 21, 23, 31, 32, 35, 37, 43 (Find f(x) and g(x) so that $F = f \circ g$), 47 (Find f(x) and g(x) so that $\nu = f \circ g$), 53, 55, 59*