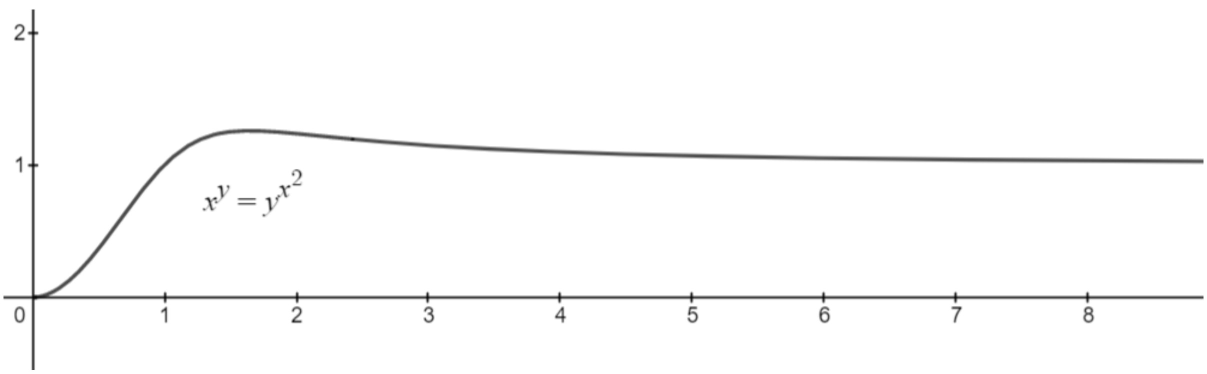
When a function $y = f(x)$ is defined with a definition explicitly in terms of the input variable x, then finding the derivative is not much of challenge. For instance, the derivative of $y = x^2 + e^{x^2}$ is $y' = 2x + e^{x^2} \cdot (2x) = 2x(1 + e^{x^2})$ as, again, the definition of the function $x^2 + e^x$ is 'explicitly' in terms of x.

However, if a function is defined as an equation where the output variable <u>cannot</u> be isolated (or solved for) to be written in the form $y = f(x)$ where the right hand side is only involving the input variable x but not the output variable y. For instance, suppose that the output variable y as a function of the input variable x is defined using the equation

$$x^y = y^{x^2}$$

It may not be obvious if the equation in fact describes y as a function x, i.e. the graph of the equation passes the vertical line test. Here is the graph of the equation.



If the derivative y' is to be found, then first thing is to solve the equation for the output variable y so that it is of the form $y = f(x)$. Then we can find the derivative of the function $f(x)$ using various rules and theorems. However, it is not easy to solve the equation for the variable y. We can try.

$$x^y = y^{x^2} \;\Rightarrow\; \ln(x^y) = \ln(y^{x^2}) \;\Rightarrow\; y\ln(x) = x^2\ln(y) \;\Rightarrow\; \frac{y}{\ln(y)} = \frac{x^2}{\ln(x)}$$

You see? The right hand side is purely in terms of x, but the left hand side is not y.

Implicit Differentiation is used when a variable y is known to be a function of a variable x but no explicit definition of $f(x)$ is known. Here are the steps for Implicit Differentiation: Given an equation LHS (left hand side) = RHS (right hand side),

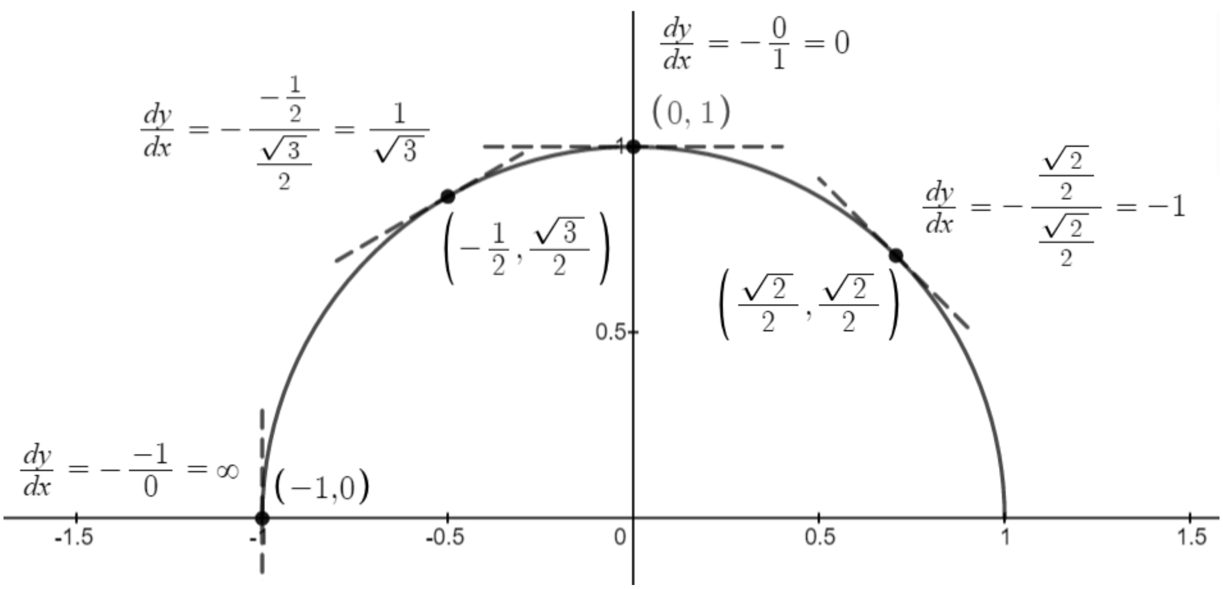
Step 1: Apply the differential operator $\frac{d}{dx}$ to both sides to write $\frac{d}{dx}$[LHS] $= \frac{d}{dx}$[RHS].
Step 2: Use usual differentiation rules to find the derivatives of LHS and RHS. Remember that when the differential operator is applied to the variable y, we write $\frac{dy}{dx}$ or y'.
Step 3: Solve the resulting equation for $\frac{dy}{dx}$ or y'.

Example 1 Consider an equation $x^2 + y^2 = 1$ where $y \geq 0$ which gives us the upper semi-circle centered at the origin with the radius 1. The equation, in fact, can be solved for y easily. However, just as an easy example, let us find the derivative using the implicit differentiation.

Step 1: $\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[1]$
Step 2: $\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = 0$, i.e. $2x + 2y \cdot \frac{dy}{dx} = 0$. Here, $\frac{dy}{dx}$ is the result of using the chain rule to the expression $(y)^2$. The inside function is y, and its derivative is $\frac{dy}{dx}$ or y'.
Step 3: $2y\frac{dy}{dx} = -2x$, i.e. $\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$.
The derivative of the function y of x is

$$\frac{dy}{dx} = -\frac{x}{y}$$

Note that the expression is also not explicitly in terms of x. It contains the variable y in it, which is okay. We can find the slope of the tangent line at various points. Since the expression on the right hand side has the variable y, we somehow need to know the y-coordinate of the point.



In fact, the equation $x^2 + y^2 = 1$ can be solved for y.

$$y^2 = 1 - x^2 \quad \Rightarrow \quad y = \pm\sqrt{1 - x^2} \quad \Rightarrow \quad y = \sqrt{1 - x^2} \quad (y \geq 0 \text{ since upper semi-circle})$$

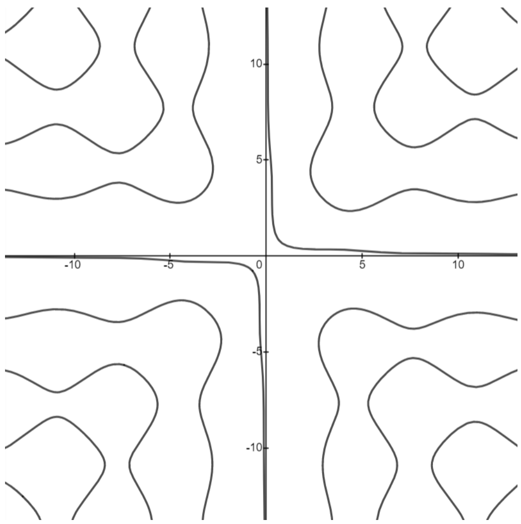
We can check that their derivatives match.

$$y' = \frac{1}{2\sqrt{1 - x^2}} \cdot (-2x) = -\frac{x}{\sqrt{1 - x^2}} = -\frac{x}{y} = \frac{dy}{dx}$$

We can see that the graph has vertical tangent when $y = 0$, i.e. $(-1, 0)$ and $(1, 0)$.

Implicit differentiation can be used to find the slope of the tangent line without y being a function at all.

Example 2 Consider an equation $x\sin(y) + y\sin(x) = 1$. Its graph is clearly not the graph for a function y of x.



$$\begin{aligned}
\tfrac{d}{dx}[x\sin(y) + y\sin(x)] &= \tfrac{d}{dx}[1] \\
\tfrac{d}{dx}[x]\sin(y) + x\tfrac{d}{dx}[\sin(y)] &= 0 \\
+\tfrac{d}{dx}[y]\sin(x) + y\tfrac{d}{dx}[\sin(x)] & \\
1\cdot\sin(y) + x\cdot\cos(y)\cdot\tfrac{dy}{dx} &= 0 \\
+\tfrac{dy}{dx}\cdot\sin(x) + y\cdot\cos(x) & \\
\tfrac{dy}{dx}(x\cos(y) + \sin(x)) &= -\sin(y) - y\cos(x) \\
\tfrac{dy}{dx} &= -\tfrac{\sin(y)+y\cos(x)}{x\cos(y)+\sin(x)}
\end{aligned}$$

It is too bad that it is hard to find the coordinate of even one point on the graph. However, as long as we have the coordinate (x, y) for a point on the graph, we can find the slope of the tangent line to the graph at the point. Now let us find the second derivative $\frac{d^2y}{dx^2}$. Are you ready to handle some mess?

$$\begin{aligned}
\frac{d}{dx}\left[\frac{dy}{dx}\right] &= \frac{d}{dx}\left[-\frac{\sin(y)+y\cos(x)}{x\cos(y)+\sin(x)}\right] \\
\frac{d^2y}{dx^2} &= -\frac{\frac{d}{dx}[\sin(y)+y\cos(x)](x\cos(y)+\sin(x)) - (\sin(y)+y\cos(x))\frac{d}{dx}[x\cos(y)+\sin(x)]}{(x\cos(y)+\sin(x))^2} \\
&= -\frac{(\frac{dy}{dx}(\cos(y)+\cos(x)) - y\sin(x))(x\cos(y)+\sin(x))}{(x\cos(y)+\sin(x))^2} \\
&\quad +\frac{(\sin(y)+y\cos(x))(\cos(y)-x\sin(y)\frac{dy}{dx}+\cos(x))}{(x\cos(y)+\sin(x))^2} \\
&= -\frac{\frac{dy}{dx}(\cos(y)+\cos(x))(x\cos(y)+\sin(x))}{(x\cos(y)+\sin(x))^2} + \frac{y\sin(x)(x\cos(y)+\sin(x))}{(x\cos(y)+\sin(x))^2} \\
&\quad -\frac{x\sin(y)\frac{dy}{dx}(\sin(y)+y\cos(x))}{(x\cos(y)+\sin(x))^2} + \frac{(\sin(y)+y\cos(x))(\cos(y)+\cos(x))}{(x\cos(y)+\sin(x))^2} \\
&= -\frac{dy}{dx}\left(\frac{(\cos(y)+\cos(x))(x\cos(y)+\sin(x)) + x\sin(y)(\sin(y)+y\cos(x))}{(x\cos(y)+\sin(x))^2}\right) \\
&\quad +\frac{y\sin(x)(x\cos(y)+\sin(x)) + (\sin(y)+y\cos(x))(\cos(y)+\cos(x))}{(x\cos(y)+\sin(x))^2}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{dy}{dx}\left(\frac{x\cos^2(y) + x\cos(x)\cos(y) + \sin(x)\cos(y) + \sin(x)\cos(x) + x\sin^2(y) + xy\cos(x)\sin(y)}{(x\cos(y)+\sin(x))^2}\right) \\
&\quad + \frac{xy\sin(x)\cos(y) + y\sin^2(x) + \sin(y)\cos(y) + y\cos(x)\cos(y) + \cos(x)\sin(y) + y\cos^2(x)}{(x\cos(y)+\sin(x))^2} \\
&= -\boxed{\frac{dy}{dx}}\left(\frac{x + x\cos(x)\cos(y) + \sin(x)\cos(y) + \sin(x)\cos(x) + xy\cos(x)\sin(y)}{(x\cos(y)+\sin(x))^2}\right) \\
&\quad + \frac{xy\sin(x)\cos(y) + y + \sin(y)\cos(y) + y\cos(x)\cos(y) + \cos(x)\sin(y)}{(x\cos(y)+\sin(x))^2}
\end{aligned}$$

Finally we put the first derivative back into the expression.

$$\begin{aligned}
&= \frac{\sin(y) + y\cos(x)}{x\cos(y)+\sin(x)}\left(\frac{x + x\cos(x)\cos(y) + \sin(x)\cos(y) + \sin(x)\cos(x) + xy\cos(x)\sin(y)}{(x\cos(y)+\sin(x))^2}\right) \\
&\quad + \frac{xy\sin(x)\cos(y) + y + \sin(y)\cos(y) + y\cos(x)\cos(y) + \cos(x)\sin(y)}{(x\cos(y)+\sin(x))^2} \\
&= \frac{(\sin(y) + y\cos(x))(x + x\cos(x)\cos(y) + \sin(x)\cos(y) + \sin(x)\cos(x) + xy\cos(x)\sin(y))}{(x\cos(y)+\sin(x))^3} \\
&\quad + \frac{xy\sin(x)\cos(y) + y + \sin(y)\cos(y) + y\cos(x)\cos(y) + \cos(x)\sin(y)}{(x\cos(y)+\sin(x))^2}
\end{aligned}$$

Assigned Exercises: (p 215) 5 - 15 (odds), 19, 21, 29, 31, 35, 37, 41, 43, 71