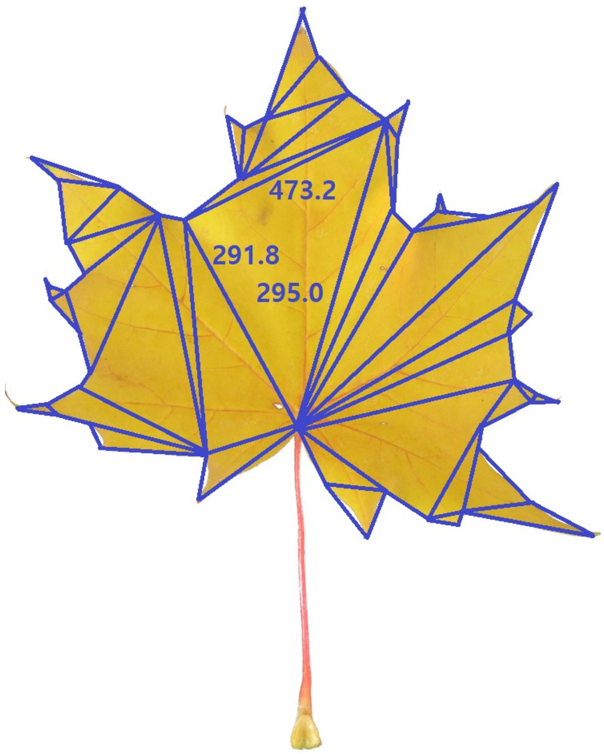
Q: How do we find the area of irregular shape?



A: Triangularization! The area of the leaf can be found by adding the areas of smaller triangles. For a smaller triangle, we can find the area if we can measure three sides.

One of the triangles is measured to have 291.8, 295.0, and 473.2. Using Heron's formula, we can find the area.

Heron's Formula for a triangle $\triangle ABC$ with the sides lengths a, b, and c:

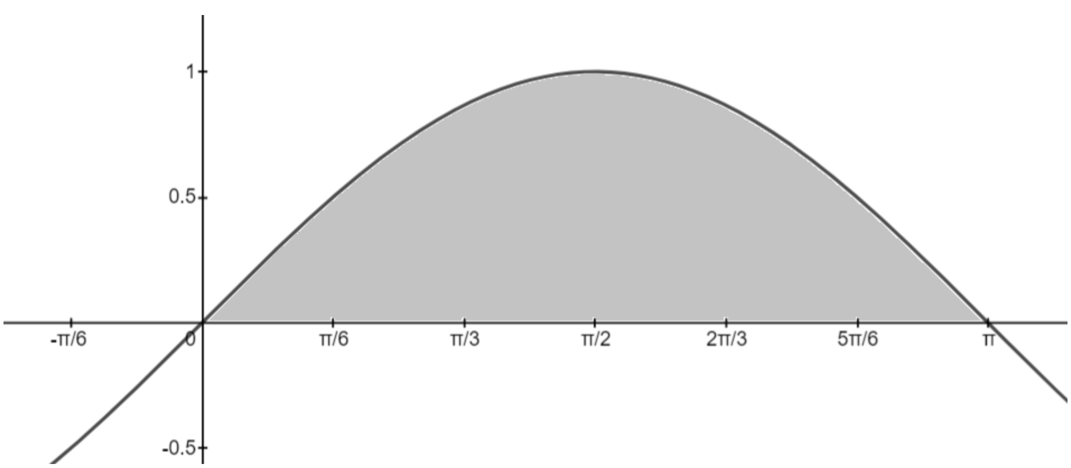
Semi-perimeter: $s = (a+b+c)/2$
Area: $A = \sqrt{s(s-a)(s-b)(s-c)}$

$s = (291.8 + 295.0 + 473.2)/2 = 530$

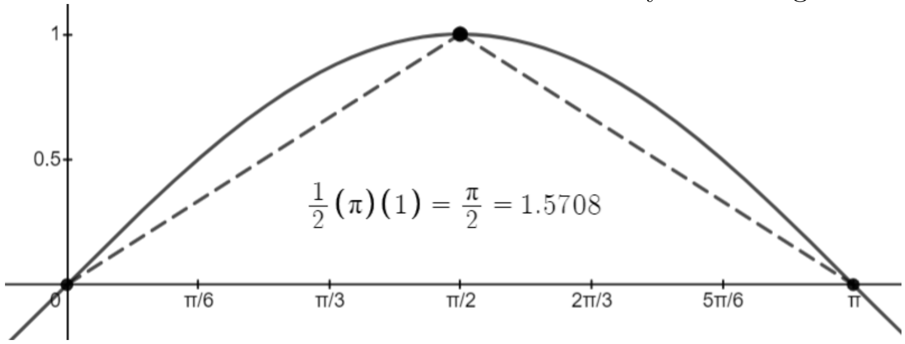
$$A = \sqrt{530(530-291.8)(530-295.0)(530-473.2)} = \sqrt{1685131608} = 41050.4$$

Goal: We want to estimate the area between a curve of a function and x-axis.

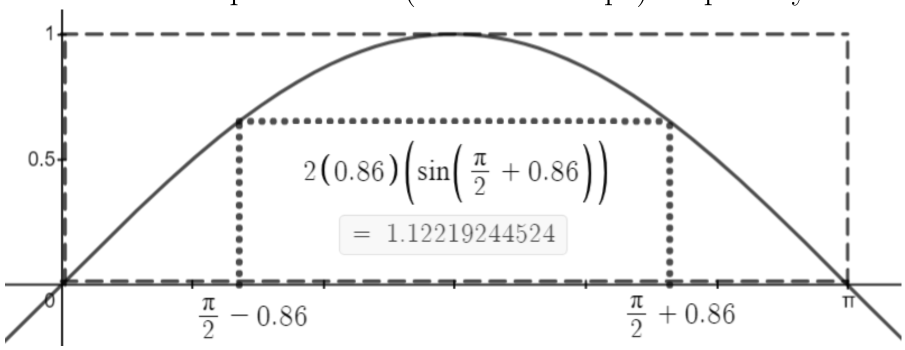
For instance, what is the area underneath the portion of the sine graph from $x = 0$ to $x = \pi$.



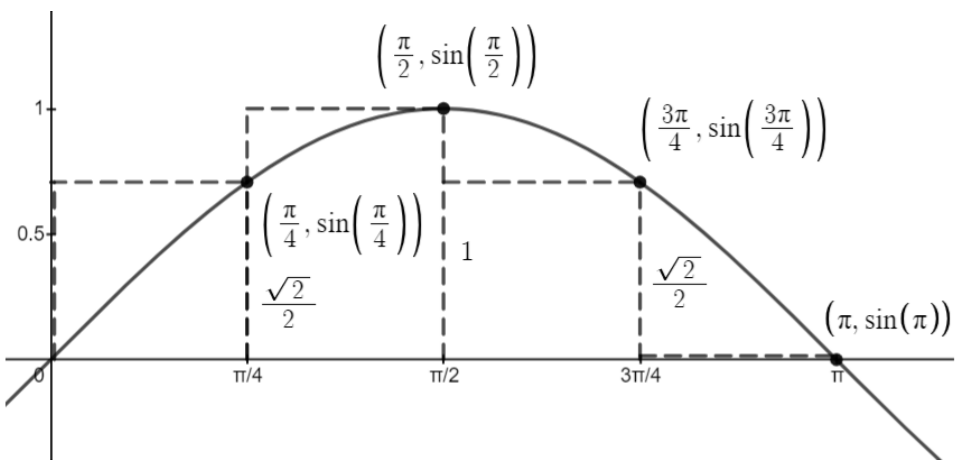
We can get the estimate of the area by using one really large triangle connecting $(0,0)$, $(\frac{\pi}{2}, 1)$, and $(\pi, 0)$, which yields the area $\approx \frac{1}{2}(\pi)(1) = 1.5708$. This estimate is clearly underestimation as there are some leftover area not covered by the triangle. We can make a better estimate if we can further triangularize the leftover areas. More triangles are used, better the estimate will be.



Although there are many algorithms for triangularizing a shape out there, it is still far from being simple enough to be formulated as a formula. The next shape we can consider to estimate the area is a quadrilateral (four-sided shape). Especially a rectangle. Its area is simple. We can consider two rectangles. The most obvious rectangle is the one that has the region inscribed, and its area is $(\pi)(1) = \pi$. The other one is the rectangle with the maximum area that is inscribed in the region, and its area is $2(0.86)(\sin(\frac{\pi}{2}+0.86)) = 1.1222$. One is an overestimate, and the other is a underestimate.



Last Suggestion: How about using more than one rectangle? We start with four rectangles.



We first partition the real number line from $x = 0$ to $x = \pi$ into four equal lengths, so the markers of the partition are $x = 0$, $x = \frac{\pi}{4}$, $x = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$, $x = 3 \cdot \frac{\pi}{4} = \frac{3\pi}{4}$, and $x = 4 \cdot \frac{\pi}{4} = \pi$. Above the closed intervals $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{4}]$, and $[\frac{3\pi}{4}, \pi]$, imagine four rectangles drawn so that the portion underneath the curve of $\sin(x)$ over the closed interval $[0, \pi]$ can be estimated.

Q: How do we determine the height of a rectangle?

There could be infinitely many ways to determine the height of each rectangle, but for the sake of "consistency," we use the right-end number of each closed interval to determine the height of the rectangle. For instance, for the closed interval $[0, \frac{\pi}{4}]$, the right-end number is $\frac{\pi}{4}$. Then we draw a rectangle whose top is drawn passing through the point $(\frac{\pi}{4}, \sin(\frac{\pi}{4}))$ over the closed interval $[0, \frac{\pi}{4}]$. The area of such a rectangle is $(\frac{\pi}{4} - 0)\sin(\frac{\pi}{4}) = \frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}\pi}{8}$. For the rest of the intervals, if we use the right-end number to determine the height, then

$$\text{Area} \approx \left(\frac{\pi}{4} - 0\right)\sin\left(\frac{\pi}{4}\right) + \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right) + \left(\frac{3\pi}{4} - \frac{\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right) + \left(\pi - \frac{3\pi}{4}\right)\sin(\pi)$$

Since we divide the closed interval $[0, \pi]$ into four equal lengths, the base of the rectangle should be all $\frac{\pi}{4}$. Then

$$\begin{aligned} \text{Area} &\approx \frac{\pi}{4} \cdot \sin\left(\frac{\pi}{4}\right) + \frac{\pi}{4} \cdot \sin\left(\frac{\pi}{2}\right) + \frac{\pi}{4} \cdot \sin\left(\frac{3\pi}{4}\right) + \frac{\pi}{4} \cdot \sin(\pi) \\ &= \frac{\pi}{4}\left(\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{4}\right) + \sin(\pi)\right) \\ &= \frac{\pi}{4}\left(\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0\right) \\ &= \frac{\pi(1+\sqrt{2})}{4} \quad (\approx 1.8961) \end{aligned}$$

The first two rectangles overestimated the region, but last two rectangles underestimated. Just from an inspection of the graph, it seems that we underestimated the region overall.

Q: How can we improve the estimate?

One can imagine that the portion being overestimated and underestimated can be reduced if we use more rectangles. So let us use n rectangles.

Markers: First, we need to partition the closed interval $[0, \pi]$ into n equal closed intervals. The first marker will be 0, and let us use the notation $x_0 = 0$ (initial number). Then the second marker will be $x_1 = \frac{\pi}{n}$. The next one is $x_2 = \frac{\pi}{n} + \frac{\pi}{n} = 2 \cdot \frac{\pi}{n}$. The next one is $x_3 = \frac{2\pi}{n} + \frac{\pi}{n} = 3 \cdot \frac{\pi}{n}$. In this way, the very last one would be $x_n = n \cdot \frac{\pi}{n} = \pi$. For instance, if we use $n = 16$ rectangles, the markers are $x_0 = 0$, $x_1 = \frac{\pi}{16}$, $x_2 = \frac{2\pi}{16}$, $x_3 = \frac{3\pi}{16}$, ..., and $x_{16} = \frac{16\pi}{16} = \pi$.

Intervals: There are n closed intervals starting from $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, ..., $[x_{n-1}, x_n]$ or $[0, \frac{\pi}{n}]$, $[\frac{\pi}{n}, \frac{2\pi}{n}]$, $[\frac{2\pi}{n}, \frac{3\pi}{n}]$, ..., $[\frac{(n-1)\pi}{n}, \frac{n\pi}{n}]$.

Base: The base of the rectangle should be all equal to $\frac{\pi}{n}$.

Heights: If we use the right-end number from each closed interval, the heights are given by $\sin(x_1)$, $\sin(x_2)$, $\sin(x_3)$, …, $\sin(x_n)$ or

$$\sin\left(\frac{\pi}{n}\right), \sin\left(\frac{2\pi}{n}\right), \sin\left(\frac{3\pi}{n}\right), \ldots, \sin\left(\frac{n\pi}{n}\right)$$

Area: Finally the area would be

$$R_n = \frac{\pi}{n}\left(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{3\pi}{n}\right) + \cdots + \sin\left(\frac{n\pi}{n}\right)\right)$$

The letter R stands for the "right-end," and n represents the number of the rectangles. Before, we found out that

$$R_4 = \frac{\pi(1+\sqrt{2})}{4} \approx 1.8961 \quad \text{and} \quad R_6 = \frac{\pi(2+\sqrt{3})}{6} \approx 1.9541$$

Here are some more of estimates for different n values with a help from WolframAlpha:

n	R_n	WolframAlpha command	Symbol from WolframAlpha
10	1.98352	(Pi/10)*Sum[Sin[k*(Pi/10)],k,1,10]	$\frac{\pi}{10}\sum_{k=1}^{10}\sin(k\cdot\frac{\pi}{10})$
100	1.99984	(Pi/100)*Sum[Sin[k*(Pi/100)],k,1,100]	$\frac{\pi}{100}\sum_{k=1}^{100}\sin(k\cdot\frac{\pi}{100})$
500	1.99999	(Pi/500)*Sum[Sin[k*(Pi/500)],k,1,500]	$\frac{\pi}{500}\sum_{k=1}^{500}\sin(k\cdot\frac{\pi}{500})$

Q: Any guess on what this estimate will be close to as $n \to \infty$?

Sigma Notation

Instead of writing

$$R_{10} = \frac{\pi}{10}\left(\sin\left(\frac{\pi}{10}\right) + \sin\left(2\cdot\frac{\pi}{10}\right) + \sin\left(3\cdot\frac{\pi}{10}\right) + \cdots + \sin\left(10\cdot\frac{\pi}{10}\right)\right)$$

we can write using Σ (capital letter 'sigma' in Greek alphabet) notation.

$$R_{10} = \frac{\pi}{10}\boxed{\sum_{k=1}^{10}\sin\left(k\cdot\frac{\pi}{10}\right)}$$

It is a succinct way of writing a sum of many expressions that has some patterns that can be described in a nice expression. The sigma notation is written as

$$\sum_{\text{running index} = \text{beginning index value}}^{\text{ending index value}} \quad \text{expression usually involving the running index}$$

Then the notation represents the sum of the expression by evaluating the expression with the running index starting from the beginning index value to the ending index value "incrementing by 1." For instance,

$$\sum_{k=1}^{10} k = 1+2+3+4+5+6+7+8+9+10$$

$$\sum_{k=1}^{10}(2k+1) = (2\cdot 1+1)+(2\cdot 2+1)+(2\cdot 3+1)+\cdots+(2\cdot 10+1)$$

$$\sum_{k=6}^{10}\pi = \pi+\pi+\pi+\pi+\pi$$

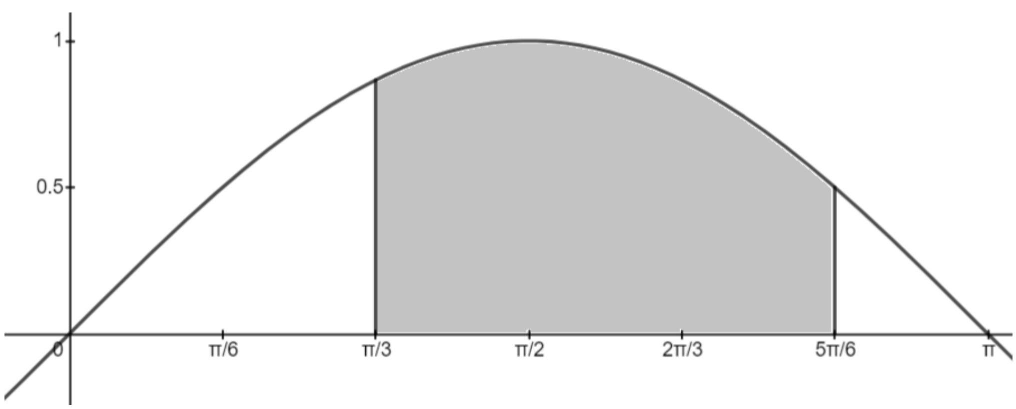
Then

$$R_n = \frac{\pi}{n}\sum_{k=1}^{n}\sin\left(k\cdot\frac{\pi}{n}\right)$$

The area under the curve of $y=\sin(x)$ over the closed interval $[0,\pi]$ is best estimated by

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{\pi}{n}\sum_{k=1}^{n}\sin\left(k\cdot\frac{\pi}{n}\right) \text{ which we suspected to be 2}$$

Example 1 Find the area under the curve of $y=\sin(x)$ over the closed interval $[\frac{\pi}{3},\frac{5\pi}{6}]$.



Let us try $n=10$ rectangles. The equal width of the interval is obtained by

$$\Delta x = \frac{\text{length of the closed interval}}{n} = \frac{\frac{5\pi}{6}-\frac{\pi}{3}}{10} = \frac{\pi}{20}$$

Markers: The first marker is $x_0=\frac{\pi}{3}$ (not zero any longer). The second marker is $x_1=\frac{\pi}{3}+\frac{\pi}{20}$. The next marker is $x_2=\frac{\pi}{3}+2\cdot\frac{\pi}{20}$. The rest are

$$x_3=\frac{\pi}{3}+3\cdot\frac{\pi}{20},\quad x_4=\frac{\pi}{3}+4\cdot\frac{\pi}{20},\quad \ldots,\quad x_9=\frac{\pi}{3}+9\cdot\frac{\pi}{20},\quad x_{10}=\frac{\pi}{3}+10\cdot\frac{\pi}{20}$$

Intervals: The closed intervals are $[x_{k-1}, x_k]$ where $k = 1, 2, \ldots, 10$ or $[\frac{\pi}{3} + (k-1) \cdot \frac{\pi}{20}, \frac{\pi}{3} + k \cdot \frac{\pi}{20}]$ where $k = 1, 2, \ldots, 10$.

Base: $\Delta x = \frac{\pi}{20}$.

Heights: Using the right-end number, the height is $\sin(\frac{\pi}{3} + k \cdot \frac{\pi}{20})$ where $k = 1, 2, \ldots, 10$.

Area: The area is estimated using

$$R_{10} = \frac{\pi}{20} \sum_{k=1}^{10} \sin\left(\frac{\pi}{3} + k \cdot \frac{\pi}{20}\right) \approx 1.33447$$

Right-End Rectangle (RER)

There is nothing special about $\frac{\pi}{3}$, $\frac{5\pi}{6}$, $\sin(x)$, or $n = 10$. Suppose that we want to estimate the area under the curve of $y = f(x)$ over the closed interval $[a, b]$ using n rectangles.

Base: The equal length of the closed intervals is

$$\boxed{\Delta x = \frac{b-a}{n}}$$

Markers: $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2 \cdot \Delta x$, $\ldots$, $x_n = a + n \cdot \Delta x = b$. Or

$$\boxed{x_k = a + k \cdot \Delta x, \text{ where } k = 1, 2, \ldots, n}$$

Heights: $f(x_1)$, $f(x_2)$, $f(x_3)$, $\ldots$, $f(x_n)$. Or

$$\boxed{f(x_k) = f(a + k \cdot \Delta x)}$$

Area: The area is estimated using

$$\boxed{R_n = \Delta x \sum_{k=1}^{n} f(x_k) = \Delta x \sum_{k=1}^{n} f(a + k \cdot \Delta x)}$$

This is called the **right-end rectangle** estimate R_n.

Example 2 Find the area under the curve $f(x) = x^2$ over the closed interval $[2, 5]$. Use $n = 6$ rectangles.

Base: The equal length of the closed intervals is $\Delta x = \frac{5-2}{6} = \frac{1}{2}$

Markers: $x_0 = 2$, $x_1 = 2 + \frac{1}{2}$, $\ldots$, $x_6 = 2 + 6 \cdot \frac{1}{2} = 5$.
Or $x_k = 2 + k \cdot \frac{1}{2} = 2 + \frac{k}{2}$ where $k = 1, 2, 3, 4, 5, 6$.

Heights: $f(x_k) = (x_k)^2 = (2 + \frac{k}{2})^2$

Area: The area is estimated using the right-end rectangle estimate

$$\begin{aligned} R_6 &= \frac{1}{2}\sum_{k=1}^{6}\left(2+\frac{k}{2}\right)^2 \\ &= \frac{1}{2}\left(\left(2+\frac{1}{2}\right)^2+\left(2+\frac{2}{2}\right)^2+\left(2+\frac{3}{2}\right)^2+\left(2+\frac{4}{2}\right)^2+\left(2+\frac{5}{2}\right)^2+\left(2+\frac{6}{2}\right)^2\right) \\ &= \frac{355}{8} \quad = \quad 44.375 \end{aligned}$$

Assigned Exercises: