A **mathematical model** is a mathematical description of a real-world phenomenon by means of a function or an equation.

Linear Model

Slope-intercept form: $f(x) = mx + b$ or $y = mx + b$; m: slope and $(0, b)$: y-intercept.

Interpretation of slope m: For every increase of x value by 1 unit, y value increases by m (or decreases by $|m|$ if $m < 0$). If $m = 0$, then no value of y changes.

Point-slope form: $y - y_1 = m(x - x_1)$ or $f(x) = y_1 + m(x - x_1)$; The line passes through the point (x_1, y_1).

Polynomial Model

A linear model is a just an example of polynomial models. A polynomial function is of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where real numbers a_i's are called the coefficients of the polynomial. Especially, a_n is called the **leading coefficient** and a_0 is called the **constant term** of the polynomial. The largest exponent n is called the **degree** of the polynomial. For instance, a linear function $f(x) = 3x+4$ has the leading coefficient $a_1 = 3$ (slope) and the constant term $a_0 = 4$ (y-intercept), and it has the degree $n = 1$. A polynomial is named according to the degree.

Degree	Name	Degree	Name
0	constant	3	cubic
1	linear	4	quartic
2	quadratic	5	quintic

Exception: The **zero polynomial** $p(x) = 0$ has degree $-\infty$.

Quadratic Polynomial Function: $f(x) = ax^2 + bx + c$ or $f(x) = a(x - h)^2 + k$ (vertex form)

The graph of the quadratic function is called a **parabola** ($\pi\alpha\rho\alpha + \beta\alpha\lambda\lambda\omega$) with the vertex $(h, k) = (-\frac{b}{2a}, c - \frac{b^2}{4a})$. The **quadratic formula**

$$x = \frac{-(b) - \sqrt{(b)^2 - 4(a)(c)}}{2(a)} \text{ or } \frac{-(b) + \sqrt{(b)^2 - 4(a)(c)}}{2(a)}$$

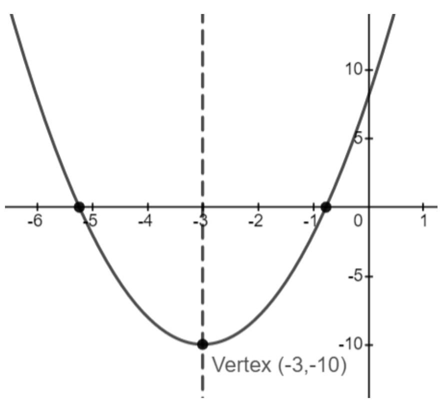
finds the x-intercept(s) if the **discriminant** $b^2 - 4ac \geq 0$. It has no x-intercept if $b^2 - 4ac < 0$.

To get the vertex form, we **complete the square** using $A^2 + 2AB + B^2 = (A+B)^2$:

$$\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c && \text{Factor out } a \text{ from first two terms.}\\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 - \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2\right) + c && \text{Add and subtract } \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 \text{ inside.}\\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - \frac{b^2}{4a} && \text{Multiply } -\left(\frac{b}{2a}\right)^2 \text{ out.}\\
&= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}\\
&= a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + c - \frac{b^2}{4a}
\end{aligned}$$

Example 1 Consider $g(x) = 2x^2 + 12x + 8 = 2(x^2 + 6x) + 8 = 2(x^2 + 6x + (\frac{6}{2})^2 - (\frac{6}{2})^2) + 8 = 2(x^2 + 2\cdot 3x + 3^2) + 8 - 2(3)^2 = 2(x+3)^2 - 10 = 2(x-(-3))^2 - 10$.

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4(2)(8)}}{2(2)} = \frac{-12 \pm 4\sqrt{5}}{4} = \frac{4(-3 \pm \sqrt{5})}{4} = -3 \pm \sqrt{5}$$



Vertex: $(-3, 10)$

x-intercepts:
$(-3-\sqrt{5}, 0) \approx (-5.2, 0)$ and $(-3+\sqrt{5}, 0) \approx (-0.8, 0)$

Axis of symmetry: $x = -3$; The graph is symmetric about this line. In general, the axis of symmetry is $x = -\frac{b}{2a}$, which is the average of the x-intercepts.

Since the discriminant $80 > 0$, it has two x-intercepts.

Rational Functions

A function of the form

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0},$$

where the numerator $P(x)$ is a polynomial of degree m and the denominator $Q(x)$ is a non-zero polynomial of degree n, is called a **rational function**. The domain of the function is $D(f) = \{x : Q(x) \neq 0\}$, and it is very important that you find the domain before simplifying the rational function into its reduced form, that is, completely factoring $P(x)$ and $Q(x)$ then canceling any common factor between the numerator and the denominator.

Example 2 Consider a function

$$h(x) = \frac{x}{x^2 + 2x}$$

The denominator is $x^2 + 2x$ which is zero if and only if $x(x+2) = 0$, i.e. $x = 0$ or $x = -2$. Hence, the domain of h is

$$D(h) = \{x : x \neq -2 \text{ and } x \neq 0\} = (-\infty, -2) \cup (-2, 0) \cup (0, \infty)$$

If the function happens to be simplified as

$$h(x) = \frac{x}{x(x+2)} \Rightarrow \frac{1}{x+2}$$

first, then one would make a mistake of finding the domain as $\{x : x \neq -2\}$ which is a wrong domain.

Algebraic Functions

A function f is called an **algebraic function** if it can be obtained using algebraic operations (addition, subtraction, multiplication, division, and taking roots) starting with polynomials.

Of course, polynomial, rational, and root functions are algebraic. Their combinations are as well. For example,

$$f(x) = \frac{\sqrt{x^2-1}}{3x^3 + 2x^2 + x}; \quad g(x) = \sqrt[3]{3x-5} - \sqrt{7 - x^2}; \quad h(x) = (x^2 + x - 1)\sqrt{x^4 - 2}$$

The functions that are not algebraic are pretty much **transcendental functions**. For instance, exponential functions, logarithmic functions, and trigonometric functions are transcendental.

Trigonometric Functions

We will review using Appendix D.

Assigned Exercises: (p 33) 3, 5, 11, 17