Recall: The derivative of $f(x)$ at $x = a$ is defined as the limit

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

if it exists. It represents the instantaneous rate of change of f w.r.t. x at $x = a$. It also represents the slope of the tangent line to the graph of f at the point $(a, f(a))$. It's a number!

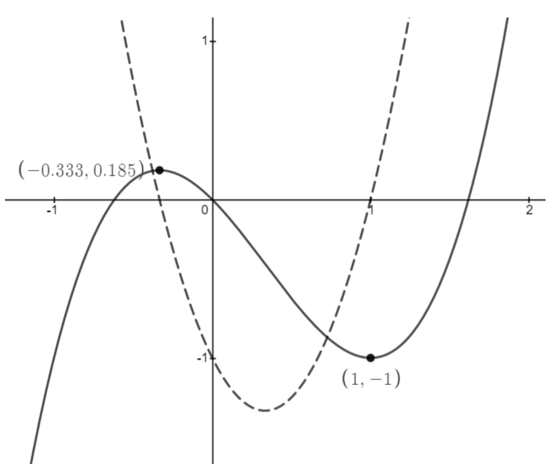
Suppose that the derivative of f exists for all x in some interval I. As there is a slope that is assigned for each x in I, we can consider this assignment as a function $f'(x)$. In the definition above, just replace a with x to get the definition

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

of the **derivative of** $f(x)$. When considering the derivative as a function, there is no need of "at $x =$ specific number." Note that notation causes some challenge when the difference quotient $\frac{f(x)-f(a)}{x-a}$ is used when we replace a by x.

Example 1 Consider $f(x) = x^3 - x^2 - x$. Its derivative is

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - (x+h)^2 - (x+h) - (x^3 - x^2 - x)}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^2 - 2xh - h^2 - x - h - x^3 + x^2 + x}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 2xh - h^2 - h}{h} \quad = \quad \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 - 2x - h - 1)}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 2x - h - 1) \quad = \quad 3x^2 - 2x - 1
\end{aligned}$$



The graphs of $f(x) = x^3 - x^2 - x$ (solid) and $f'(x) = 3x^2 - 2x - 1$ (dotted) are shown.

Note that the graph of f has horizontal slopes at $x \approx -0.333$ and $x = 1$. Since the derivative finds the slope of the tangent line, f' should be zero at these two x-values. Hence, $f'(-0.333) = 0$ and $f'(1) = 0$.

On the interval $(-\infty, -0.333)$, the function f is increasing, which means that the slope of the tangent line at every point on the graph over that interval should be positive. The same is true over the interval $(1, \infty)$. On the other hand, over the interval $(-0.333, 1)$, the function f is decreasing. Hence, the derivative f' should be negative over that interval.

Other Notations for Derivatives

The derivative of the function $y = f(x)$ is so far denoted by y' or $f'(x)$. Here are some other notations.

$$\frac{d}{dx}[y], \quad \frac{dy}{dx}, \quad \frac{d}{dx}[f], \quad \frac{df}{dx}, \quad \frac{d}{dx}[f(x)]$$

The expression $\frac{d}{dx}$ is called a **differentiation operator** with respect to the variable x (introduced by Leibniz). To denote the slope $f'(a)$ using the Leibniz notation, we use

$$\left.\frac{dy}{dx}\right|_{x=a}$$

The vertical bar is read "evaluate at."

> A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** I if it is differentiable at every number in the interval.

Example 2 Consider $g(x) = \sqrt{x}$ with the domain $D(g) = [0, \infty)$. Its derivative is

$$\begin{aligned}
\frac{d}{dx}[\sqrt{x}] &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \quad = \quad \lim_{h\to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \quad = \quad \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \quad = \quad \frac{1}{\sqrt{x} + \sqrt{x}} \quad = \quad \frac{1}{2\sqrt{x}}
\end{aligned}$$

This also means that $g'(x) = \frac{1}{2\sqrt{x}}$. The function g is differentiable at $x > 0$. However, if $x = 0$, the left limit does not make sense. Hence, g is not differentiable at $x = 0$. The slope at $x = 4$ is

$$\left.\frac{dg}{dx}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4} \quad \text{or} \quad g'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

The derivative of the function $y = \sqrt{x}$ is going to appear a lot, so we might as well memorize it.

> $$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} \quad \text{for } x > 0.$$

Example 3 (**Important Example**) Consider an absolute value function $f(x) = |x|$ with the domain $(-\infty, \infty)$. Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Let us find the derivative $f'(a)$ if we 'can'.

$$f'(a) = \lim_{x \to a} \frac{|x| - |a|}{x - a}$$

We have three cases to consider:

Case 1: If $a < 0$, then the limit is

$$f'(a) = \lim_{x \to a} \frac{|x| - |a|}{x - a} = \lim_{x \to a} \frac{-x - (-a)}{x - a} = \lim_{x \to a} \frac{-(x - a)}{x - a} = \lim_{x \to a} -1 = -1$$

Case 2: If $a = 0$, then the limit is

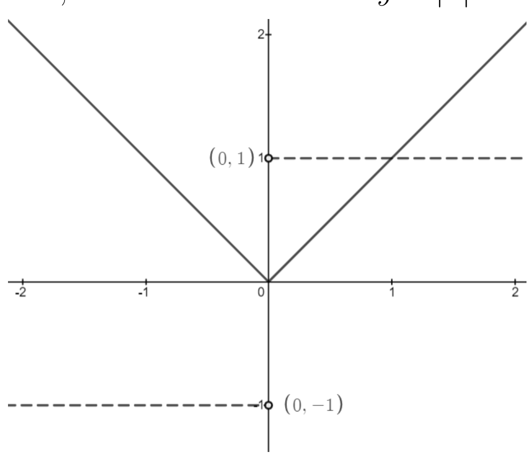
$$f'(a) = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} = \text{DNE}$$

because the left limit -1 and the right limit 1 do not agree.

Case 3: If $a > 0$, then the limit is

$$f'(a) = \lim_{x \to a} \frac{|x| - |a|}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = \lim_{x \to a} 1 = 1$$

Hence, the absolute function $y = |x|$ is differentiable if $x \neq 0$ but not differentiable at $x = 0$.



The solid curve is the graph of the function f, and the dotted curve is the graph of the derivative f'.

The derivative of $|x|$ can be written as

$$(|x|)' = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

with the domain $(-\infty, 0) \cup (0, \infty)$ and the range $\{-1, 1\}$.

$$\boxed{(|x|)' = \frac{|x|}{x} \quad \text{for } x \neq 0.}$$

Theorem 4 If f is differentiable at a, then f is continuous at a.

Proof. Suppose that the function f is differentiable at a. Then the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists. To show that the function f is continuous at a, we need to show the following:
(i) $\lim_{x\to a} f(x)$ exists, (ii) $f(a)$ is defined, and (iii) $\lim_{x\to a} f(x) = f(a)$.

We get (ii) for free as the differentiability at a assumes that $f(a)$ is defined. Note that

$$\begin{aligned}
\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}(f(x)-f(a))\cdot\frac{x-a}{x-a} \\
&= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a) \\
&= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a) \\
&= f'(a)\cdot 0 = 0
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}(f(x)-f(a)+f(a)) \\
&= \lim_{x\to a}(f(x)-f(a)) + \lim_{x\to a} f(a) \\
&= 0 + f(a) = f(a)
\end{aligned}$$

shows that (i) and (iii) are true. Therefore, f is continuous at a. □

The **converse** of a conditional statement "If p, then q" ($p \Rightarrow q$) is the conditional statement "If q, then p" ($q \Rightarrow p$). Even a conditional statement is true, its converse may not be true.

The converse of **Theorem 4** is "If f is continuous at a, then f is differentiable at a." It is FALSE. For instance, the absolute value function $f(x) = |x|$ in **Example 3** is continuous at 0, but it is not differentiable at 0.

The **contrapositive** of a conditional statement "If p, then q" ($p \Rightarrow q$) is the conditional statement "If not q, then not p" ($\neg q \Rightarrow \neg p$). A conditional statement and its contrapositive are logically equivalent. One is true if and only if the other is true.

The contrapositive of **Theorem 4** is

"If f is not continuous at a, then f is not differentiable at a."

How Can a Function Fail To Be Differentiable?

Here are the obvious reasons:

Reason 1: If the function f is <u>not</u> defined at a, then f is <u>not</u> differentiable at a.

$\lim_{x\to a} f(x)$ can exist even if $f(a)$ is not defined as the limit is about when x is near a (not about $x = a$). However, differentiability of f at a requires $f(a)$ to be defined.

Reason 2: If the function f is <u>not</u> continuous at a, then f is <u>not</u> differentiable at a.

Hence, if the function f has a vertical asymptote at $x = a$ (infinite discontinuity) or if the limit $\lim_{x\to a} f(x) = \text{DNE}$ (jump or removable discontinuity), then the function f is not differentiable at a. For instance, $\frac{1}{x}$ is not differentiable at 0 and $\sqrt{x}$ is not differentiable at 0.

Reason 3: If the graph of the function f has an erupt change of the slope of tangent lines at a, then f is not differentiable at a.

The absolute value function $|x|$ has the slope -1 right before $x = 0$ (from the left), then it suddenly changes the slope to 1 right after $x = 0$. In order for a function to have a derivative at a, it is necessary that the slopes should be changing continuously over $x = a$. Graphically, this happens when the graph has a "sharp corner" at $(a, f(a))$.

Reason 4: If the graph has a **vertical tangent line** when $x = a$, then f is not differentiable at a.

Even f is continuous and $f(a)$ is defined, if

$$\lim_{x\to a} f'(x) = \infty \text{ or } -\infty$$

then it is not differentiable there. This is not easy to even imagine.

Example 4 Consider a piecewise function

$$g(x) = \begin{cases} \sqrt{1-(x+1)^2} & \text{if } -2 \leq x \leq 0 \\ \sqrt{1-(x-1)^2} & \text{if } 0 < x \leq 2 \end{cases}$$

with the domain $[-2, 2]$ and the range $[-1, 1]$. It is continuous on its domain. The function g is definitely not differentiable at -2 and 2 as it is not continuous at -2 and 2.

Let us consider differentiability at 0. So we attempt to evaluate the limit

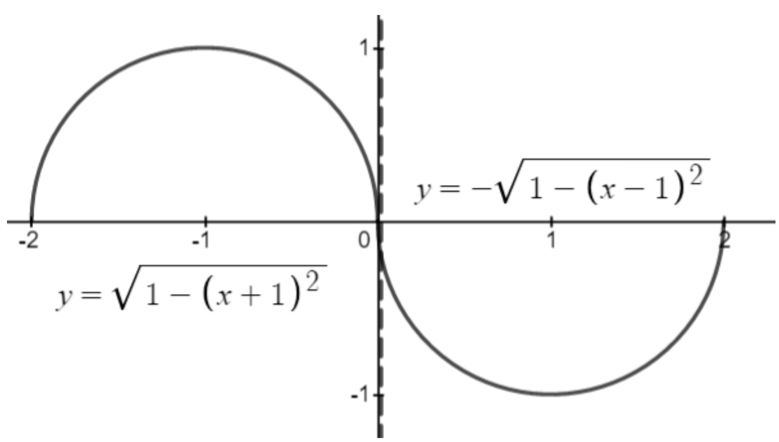
$$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0}$$

Since the function uses different definitions before and after $x = 0$, we should consider one-sided limits. First, the left-hand limit is

$$\begin{aligned}
\lim_{x\to 0^-}\frac{g(x)-g(0)}{x-0} &= \lim_{x\to 0^-}\frac{\sqrt{1-(x+1)^2}-0}{x} = \lim_{x\to 0^-}\frac{\sqrt{1-(x+1)^2}}{x} = \lim_{x\to 0^-}\frac{\sqrt{-x^2-2x}}{x} \\
&= \lim_{t\to -\infty}\frac{\sqrt{-(\frac{1}{t})^2-2(\frac{1}{t})}}{\frac{1}{t}} \qquad \text{Let } t=\frac{1}{x}.\text{ As } x\to 0^-, t\to -\infty. \\
&= \lim_{t\to -\infty} t\sqrt{-\frac{1}{t^2}-\frac{2}{t}} = \lim_{t\to -\infty} -|t|\sqrt{-\frac{1}{t^2}-\frac{2}{t}} \\
&= \lim_{t\to -\infty} -\sqrt{t^2\left(-\frac{1}{t^2}-\frac{2}{t}\right)} = \lim_{t\to -\infty} -\sqrt{-1-2t} = -\infty
\end{aligned}$$

This means that as the point travels along the graph from the left towards $(0,0)$, the tangent line becomes more like a vertical line. Similarly, the right-hand limit is

$$\begin{aligned}
\lim_{x\to 0^+}\frac{g(x)-g(0)}{x-0} &= \lim_{x\to 0^+}\frac{\sqrt{1-(x-1)^2}-0}{x} = \lim_{x\to 0^+}\frac{\sqrt{1-(x-1)^2}}{x} = \lim_{x\to 0^+}\frac{\sqrt{-x^2+2x}}{x} \\
&= \lim_{t\to \infty}\frac{\sqrt{-(\frac{1}{t})^2+2(\frac{1}{t})}}{\frac{1}{t}} \qquad \text{Let } t=\frac{1}{x}.\text{ As } x\to 0^+, t\to \infty. \\
&= \lim_{t\to \infty} t\sqrt{-\frac{1}{t^2}+\frac{2}{t}} = \lim_{t\to \infty} |t|\sqrt{-\frac{1}{t^2}+\frac{2}{t}} \\
&= \lim_{t\to \infty} \sqrt{t^2\left(-\frac{1}{t^2}+\frac{2}{t}\right)} = \lim_{t\to \infty} \sqrt{-1+2t} = \infty
\end{aligned}$$



The tangent line at 0 is drawn with a dotted line. Note that it is literally a vertical line. Hence, its slope is undefined as we confirmed with the limits above.

Higher Derivatives

Since the derivative f' of a function f can be a function, we can consider the derivative of the derivative function f'. It is defined (obviously) as follows with the obvious notation:

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

f'' is the derivative of the derivative of the function f, so it is called the **second derivative** of the original function f. In Leibniz notation,

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}$$

We can do it again. The **third derivative** of the function f is

$$f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h}$$

, i.e. the derivative of the second derivative. In Leibniz notation,

$$\frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d^3y}{dx^3}$$

In general, the n**-th derivative** of the function f is

$$y^{(n)} = f^{(n)}(x) = \lim_{h \to 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}$$

Here, $f^{(n-1)}$ is the $(n-1)$-th derivative of the function f.

If the function $f(t)$ is either height or position function of time t, then its first derivative $f'(t)$ is the (instantaneous) velocity function. So we can denote the velocity function as $v(t) = f'(t)$. The derivative of the velocity function is the acceleration function, i.e. $a(t) = v'(t)$ and $a(t) = f''(t)$. Hence, the acceleration function is the second derivative of the position function.

Assigned Exercises: (p 160) 3, 5, 7, 9, 13, 21, 25, 27, 29, 33, 39, 41, 43, 47, 49, 55, 57, 59, 61