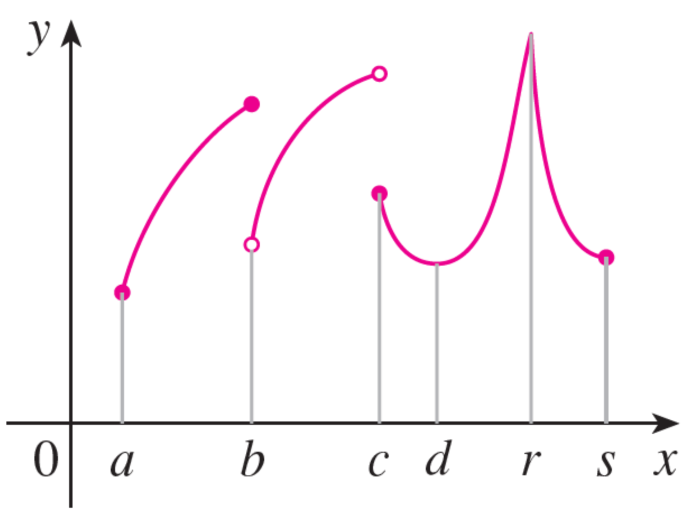
99.99% of mathematical applications to business, economics, science, and engineering is **optimization problem**. Find maximum or minimum values of a function possibly with some restrictions.

Let c be a number in the domain D of a function f. Then $f(c)$ is the
· **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
· **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

Instead of the adjective "absolute," the word "global" can be used, so global maximum and global minimum values. Note that these are output values (or function values) of f. They are collectively called the **extreme value(s)** or **extremum (pl. extrema)** of f.

Example 1 Consider the graph of a function $y = f(x)$.



Domain of f is $D = [a, s]$.

It seems that $f(r) \geq f(x)$ for all $x \in D$, so $f(r)$ is the absolute maximum value of f.

It seems that $f(a) \leq f(x)$ for all $x \in D$, so $f(a)$ is the absolute minimum value of f.

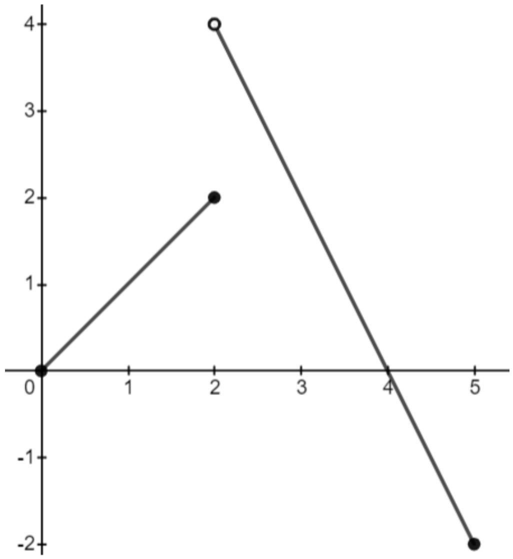
Let c be a number in an open interval $(c - \delta, c + \delta)$ which is contained in the domain D of a function f. Then $f(c)$ is a
· **local maximum** value of f if $f(c) \geq f(x)$ when x is near c.
· **local minimum** value of f if $f(c) \leq f(x)$ when x is near c.

Instead of the adjective "local," the word "relative" can be used, so relative maximum and relative minimum values. Note that these are output values (or function values) of f. They are collectively called the **local extreme value(s)** or **local extremum (pl. extrema)** of f.

Example 1 (cont.) $f(d)$ is a local minimum value as $f(d) \leq f(x)$ for all $x \in (d-\delta, d+\delta)$ where δ can be taken as the lesser value between $|c - d|$ and $|d - r|$. $f(b)$ and $f(r)$ are local maximum values. What about $f(a)$, $f(c)$, or $f(s)$? Note that the function f must be defined on a small open interval containing a or s, but f is not $x < a$ and $x > s$, so $f(a)$ and $f(s)$ are out of consideration. At $x = c$, $f(c) \geq f(x)$ for $x > c$ (but not too large) but $f(c) \leq f(x)$ for $x < c$ (but not too small). Hence, $f(c)$ is neither local minimum value nor local maximum value.

The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Example 2 Consider the graph of a piece-wise defined function $g(x)$.

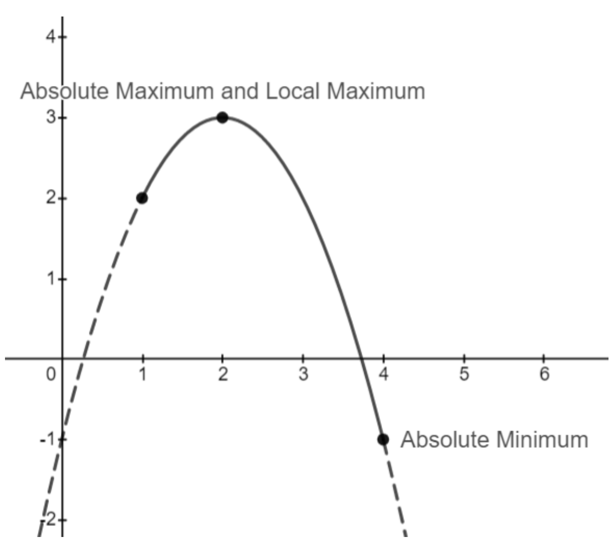


Over the closed interval $[0, 2]$, the function has both absolute maximum value of 2 and absolute minimum value of 0. And it was guaranteed as f is continuous on the that interval.

On the closed interval $[0, 5]$, the function g is not continuous (at 2 specifically). Hence, we should not be too surprised that the function does not attain an absolute maximum value.

The same is true over the closed interval $[2, 5]$.

Example 3 Consider a function $h(x) = -(x - 2)^2 + 3$.



Over the closed interval $[1, 4]$, the function h has the absolute maximum value of $f(2) = 3$. It is also a local maximum.

The function has the absolute minimum value of $f(4) = -1$. However, no local minimum value exist.

Remember that only absolute extrema are guaranteed by the continuity of the function. Not local extrema.

The converse of the theorem is certainly not true. Consider the function $h(x) = \frac{|x|}{x}$. Over the closed interval $[-1, 1]$, the function h attains the absolute maximum value of 1 and the absolute minimum value of -1. However, it is not continuous at 0 which is in the interval.

> **Fermat's Theorem** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof. Suppose that f has a local minimum at c. By the definition, $f(c) \leq f(x)$ if x is sufficiently close to c. Since $f'(c)$ exists, we know that the limit

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. If $h > 0$ with $h = x - c$,

$$\begin{aligned}
f(c) &\leq f(x) \\
f(c) &\leq f(c+h) \\
f(c) - f(c) &\leq f(c+h) - f(c) \\
0 &\leq \frac{f(c+h) - f(c)}{h} \\
\lim_{h \to 0^+} 0 &\leq \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \\
0 &\leq f'(c)
\end{aligned}$$

If $h < 0$ with $h = x - c$, similarly

$$\begin{aligned}
f(c) - f(c) &\leq f(c+h) - f(c) \\
0 &\geq \frac{f(c+h) - f(c)}{h} \\
\lim_{h \to 0^-} 0 &\geq \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \\
0 &\geq f'(c)
\end{aligned}$$

The inequality reverses the direction as h is negative. Since $f'(c) \geq 0$ and $f'(c) \leq 0$, we conclude that $f'(c) = 0$. $\square$

The converse of the theorem is the statement is "If $f'(c) = 0$, then f has a local maximum or minimum at c." It is false! Consider a function $f(x) = x^3$. Its derivative is $f'(x) = 3x^2$, and $f'(0) = 0$. However, the graph of the function has no local extremum.

> A **critical number** of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

To find critical number(s) of a differentiable function $f(c)$ with the domain D,

1. Find the derivative $f'(x)$. Simplify as much as possible. Especially, try to write as a simple fraction if the derivative involves fractions.
2. Find the value(s) c such that $f'(c) = 0$ or $f'(c)$ is undefined.
3. If c is in the domain D, then it is a critical number of f. If not, throw away.

Example 4 Consider a function $f(x) = \sqrt[3]{4 - x^2}$. Note that the domain of f is $D = (-\infty, \infty)$. Find the critical number(s) of f.

1. The derivative is $f'(x) = \frac{-2x}{3\sqrt[3]{(4-x^2)^2}}$.
2. $f'(c) = 0 \quad \Rightarrow \quad \frac{-2c}{3\sqrt[3]{(4-c^2)^2}} = 0 \quad \Rightarrow \quad -2c = 0$, so $c = 0$.
   $f'(c) = \frac{-2c}{3\sqrt[3]{(4-c^2)^2}}$ is undefined if $3\sqrt[3]{(4-c^2)^2} = 0 \quad \Rightarrow \quad 4 - c^2 = 0$, i.e. so $c = \pm 2$.
3. Since $c = -2$, 0, and 2 are in the domain D, they are all critical numbers of f.

Example 5 Consider a function $g(x) = \frac{1}{\sqrt{4-x^2}}$. Note that the domain of g is $D = (-2, 2)$. Find the critical number(s) of g.

1. The derivative is $g'(x) = \frac{x}{\sqrt{(4-x^2)^3}}$.
2. $g'(c) = 0 \quad \Rightarrow \quad \frac{c}{\sqrt{(4-c^2)^3}} = 0 \quad \Rightarrow \quad c = 0$, so $c = 0$.
   $g'(c) = \frac{c}{\sqrt{(4-c^2)^3}}$ is undefined if $\sqrt{(4-c^2)^3} = 0 \quad \Rightarrow \quad 4 - c^2 = 0$, i.e. so $c = \pm 2$.
3. The value 0 is in the the domain D, but $c = \pm 2$ are not in the domain D. Hence, 0 is the only critical number.

If f has a local maximum or minimum at c, then c is a critical number of f.

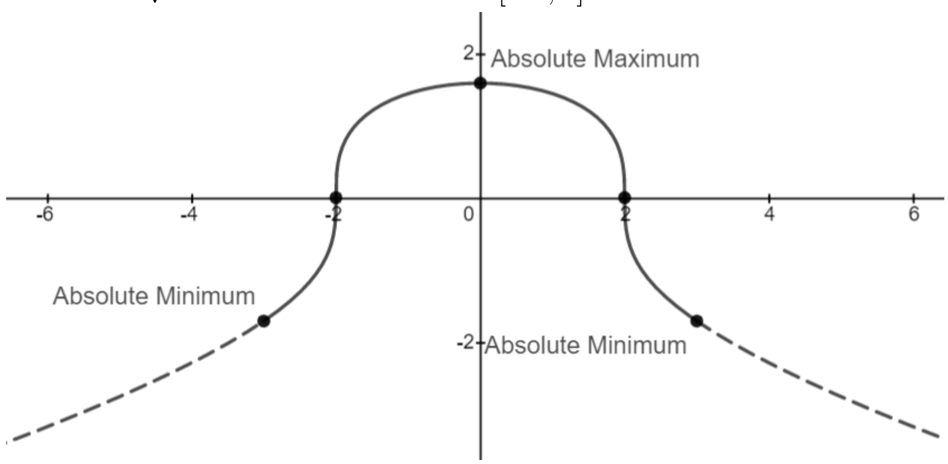
The Closed Interval Method To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers, if exist, of f in (a, b).
2. Find the values $f(a)$ and $f(b)$ of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 6 Consider a function $f(x) = \sqrt[3]{4 - x^2}$ again with the domain $D = (-\infty, \infty)$. Recall that its critical numbers are -2, 0, and 2. Find the absolute maximum and the absolute minimum values on the given closed interval.

(a) $[-3, 3]$

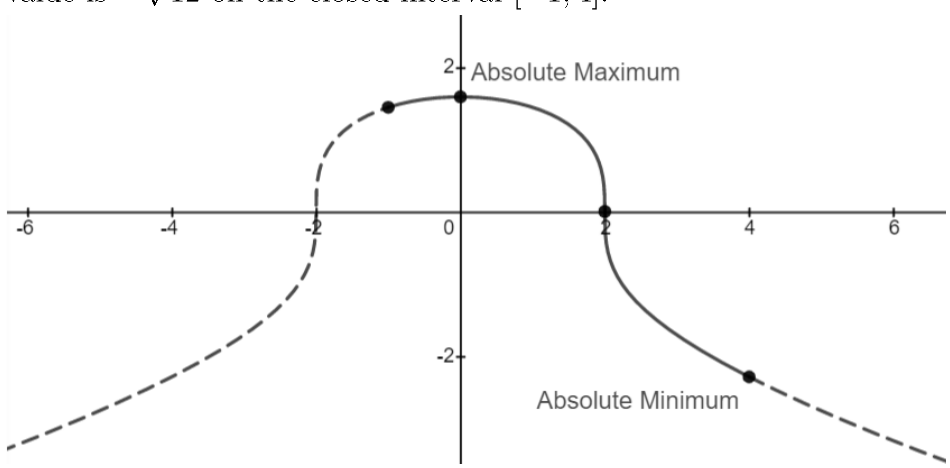
1. All of its critical numbers are in $(-3, 3)$.
   $f(-2) = \sqrt[3]{4-(-2)^2} = 0$; $f(0) = \sqrt[3]{4-0^2} = \sqrt[3]{4}$; $f(2) = \sqrt[3]{4-2^2} = 0$.
2. $f(-3) = \sqrt[3]{4-(-3)^2} = \sqrt[3]{-5} = -\sqrt[3]{5}$; $f(3) = \sqrt[3]{4-3^2} = \sqrt[3]{-5} = -\sqrt[3]{5}$.
3. Since $-\sqrt[3]{5} < 0 < \sqrt[3]{4}$, the absolute maximum value is $\sqrt[3]{4}$ and the absolute minimum value is $-\sqrt[3]{5}$ on the closed interval $[-3, 3]$.



A function can attain the absolute minimum value at multiple places, but the absolute minimum 'value' itself is unique.

(b) $[-1, 4]$

1. Only the critical numbers 0 and 2 are in $(-1, 4)$.
   $f(0) = \sqrt[3]{4-0^2} = \sqrt[3]{4}$; $f(2) = \sqrt[3]{4-2^2} = 0$.
2. $f(-1) = \sqrt[3]{4-(-1)^2} = \sqrt[3]{3}$; $f(4) = \sqrt[3]{4-4^2} = \sqrt[3]{-12} = -\sqrt[3]{12}$.
3. Since $-\sqrt[3]{12} < 0 < \sqrt[3]{4}$, the absolute maximum value is $\sqrt[3]{4}$ and the absolute minimum value is $-\sqrt[3]{12}$ on the closed interval $[-1, 4]$.



Assigned Exercises: (p 283) 3 - 13 (odds), 27 - 43 (odds), 49, 53, 55, 57, 61, 65, 67, 75, 77