INF1003 Tutorial 10

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1. Let M_R be the matrix of a relation R (with respect to the ordering $1, 2, 3, 4$ of the underlying set $A = \{1, 2, 3, 4\}$):

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Determine whether R is reflexive, symmetric, antisymmetric, or transitive.

Reflexive. R is reflexive on A iff all diagonal entries are 1: $M_{11} = M_{22} = M_{33} = M_{44} = 1$. From the matrix this holds, so R is *reflexive*.

Symmetric. R is symmetric iff $M_{ij} = M_{ji}$ for all i, j. Here $M_{12} = 0$ but $M_{21} = 1$, so $M_R \neq M_R^{\mathsf{T}}$. Thus R is *not symmetric*.

Antisymmetric. R is antisymmetric iff for all $i \neq j$, if $M_{ij} = 1$ and $M_{ji} = 1$ then $i = j$ (contradiction). But $M_{13} = 1$ and $M_{31} = 1$ with $1 \neq 3$. Hence R is *not antisymmetric*.

Transitive. R is transitive iff whenever (i, j) and (j, k) are in R, then (i, k) is also in R. From the matrix:

$$(1, 3) \in R \text{ (since } M_{13} = 1), \quad (3, 2) \in R \text{ (since } M_{32} = 1),$$

but

$$(1, 2) \notin R \text{ (since } M_{12} = 0).$$

So the transitivity condition fails. Therefore R is *not transitive*.

2. Let R be the relation on $A = \{1, 2, 3, 4\}$ defined by

$$R = \{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}.$$

Decide whether R is reflexive, symmetric, antisymmetric, and/or transitive.

Reflexive? On A, reflexivity requires $(1,1), (2,2), (3,3), (4,4) \in R$. Here only $(2,2)$ and $(3,3)$ are present; $(1,1)$ and $(4,4)$ are missing. So R is *not reflexive*.

Symmetric? Symmetry requires: whenever $(a,b) \in R$, then $(b,a) \in R$. We have $(2,4) \in R$ but $(4,2) \notin R$, so R is *not symmetric*.

Antisymmetric? Antisymmetry requires: if $(a,b) \in R$ and $(b,a) \in R$ with $a \neq b$, this must never happen. But $(2,3) \in R$ and $(3,2) \in R$ with $2 \neq 3$, so R is *not antisymmetric*.

Transitive? Transitivity: if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

We check all possible chains:

- From $(2,2)$:

$$(2,2), (2,2) \Rightarrow (2,2) \in R; \quad (2,2), (2,3) \Rightarrow (2,3) \in R; \quad (2,2), (2,4) \Rightarrow (2,4) \in R.$$

- From $(2,3)$:

$$(2,3), (3,2) \Rightarrow (2,2) \in R; \quad (2,3), (3,3) \Rightarrow (2,3) \in R; \quad (2,3), (3,4) \Rightarrow (2,4) \in R.$$

- From $(3,2)$:

$$(3,2), (2,2) \Rightarrow (3,2) \in R; \quad (3,2), (2,3) \Rightarrow (3,3) \in R; \quad (3,2), (2,4) \Rightarrow (3,4) \in R.$$

- From $(3,3)$:

$$(3,3), (3,2) \Rightarrow (3,2) \in R; \quad (3,3), (3,3) \Rightarrow (3,3) \in R; \quad (3,3), (3,4) \Rightarrow (3,4) \in R.$$

- From $(2,4)$ or $(3,4)$ as first pair: there is no pair in R whose first component is 4, so no further chain to check.

In every case where (a,b) and (b,c) belong to R, (a,c) is also in R. Hence R is *transitive*.

3. For each of the relations in Question 3 (given as directed graphs on the worksheet), determine whether the relation is reflexive, symmetric, antisymmetric, and/or transitive. For any property that does not hold, give a brief explanation.

Denote the three relations by R_a, R_b and R_c.

(a) For R_a:

- **Reflexive:** Yes. Each element of the underlying set has a loop (x, x), so every element is related to itself.
- **Symmetric:** Yes. For every arrow $x \to y$ in the digraph, there is also an arrow $y \to x$, so if $(x, y) \in R_a$ then $(y, x) \in R_a$.
- **Antisymmetric:** No. Since the relation is symmetric and there are distinct elements joined by two-way arrows, we have distinct $x \neq y$ with both (x, y) and (y, x) in R_a, contradicting antisymmetry.
- **Transitive:** No. There is at least one path $x \to y \to z$ where there is no direct edge $x \to z$, so (x, y) and (y, z) are in R_a but (x, z) is not.

(b) For R_b:

- **Reflexive:** No. At least one element has no loop (x, x).
- **Symmetric:** No. There is at least one arrow $x \to y$ with no matching arrow $y \to x$.
- **Antisymmetric:** No. There exists a pair of distinct vertices $x \neq y$ with arrows in both directions, (x, y) and (y, x).
- **Transitive:** No. There is a two-step path $x \to y \to z$ with no direct arrow $x \to z$.

(c) For R_c:

- **Reflexive:** No. Some vertex misses its loop (x, x).
- **Symmetric:** No. There is an arrow $x \to y$ without a reverse arrow $y \to x$.
- **Antisymmetric:** No. There is at least one pair of distinct vertices $x \neq y$ with arrows both ways.
- **Transitive:** No. Again there is a path $x \to y \to z$ with no edge $x \to z$.

4. Which of the following relations on the set of SIT students are equivalence relations? For those that are, describe the equivalence classes. For each relation, briefly justify your answer.

(a) $\{(a, b) \mid a \text{ and } b \text{ are taking a module together}\}$.

Two students are related if they share at least one common module.

Reflexive: If every student is enrolled in at least one module, then each student trivially shares a module with themselves, so a is related to a. **Symmetric:** If a and b share a module, then b and a do as well. **Transitive:** Not necessarily. It can happen that a and b share module X, and b and c share module $Y \neq X$, but a and c do not share any module. Hence the relation is *not transitive* and so *not an equivalence relation.*

(b) $\{(a, b) \mid a \text{ and } b \text{ study in the same programme}\}$.

Reflexive: Every student is in the same programme as themselves. **Symmetric:** If a is in the same programme as b, then b is in the same programme as a. **Transitive:** If a and b are in the same programme and b and c are in the same programme, then a and c are in that same programme.

Thus this is an *equivalence relation.* The equivalence classes are the sets of students in each programme (e.g. all AAI students, all SFT students, etc.).

(c) $\{(a, b) \mid \text{GPA of } a \text{ is greater than or equal to GPA of } b\}$.

Reflexive: $\text{GPA}(a) \geq \text{GPA}(a)$, so (a, a) is always in the relation. **Symmetric:** Fails in general: if $\text{GPA}(a) > \text{GPA}(b)$ then (a, b) is in the relation but (b, a) is not. **Transitive:** If $\text{GPA}(a) \geq \text{GPA}(b)$ and $\text{GPA}(b) \geq \text{GPA}(c)$, then $\text{GPA}(a) \geq \text{GPA}(c)$.

Since it is not symmetric, this is *not* an equivalence relation.

(d) $\{(a, b) \mid -0.5 \leq \text{GPA}(a) - \text{GPA}(b) \leq 0.5\}$.

Reflexive: $\text{GPA}(a) - \text{GPA}(a) = 0$, which lies in $[-0.5, 0.5]$, so reflexive.

Symmetric: If $-0.5 \leq \text{GPA}(a) - \text{GPA}(b) \leq 0.5$, then $\text{GPA}(b) - \text{GPA}(a) = -(\text{GPA}(a) - \text{GPA}(b))$ also lies in $[-0.5, 0.5]$. So the relation is symmetric.

Transitive: Not always. Example: let

$$\text{GPA}(a) = 3.0, \quad \text{GPA}(b) = 3.4, \quad \text{GPA}(c) = 3.8.$$

Then $\text{GPA}(a) - \text{GPA}(b) = -0.4$ and $\text{GPA}(b) - \text{GPA}(c) = -0.4$ are both in $[-0.5, 0.5]$, so (a, b) and (b, c) are related. But $\text{GPA}(a) - \text{GPA}(c) = -0.8$, which lies outside $[-0.5, 0.5]$, so (a, c) is not related. Hence the relation is *not transitive* and so not an equivalence relation.

(e) $\{(a, b) \mid a \text{ is taking the same number of academic credits as } b\}$.

Reflexive: Each student takes the same number of credits as themselves. **Symmetric:** If a takes the same number of credits as b, then b takes the same number of credits as a. **Transitive:** If a and b take the same number of credits and b and c also take the same number, then a and c take that same number.

Thus this is an *equivalence relation*. The equivalence classes are the sets of students grouped by credit load (e.g. all students taking 20 credits, all taking 15 credits, etc.).

5. Let R be the relation on the set of all strings of English letters such that aRb iff $l(a) = l(b)$, where $l(x)$ is the length of the string x.

(a) Is R reflexive?

Yes. For any string a, we have $l(a) = l(a)$, so $(a, a) \in R$.

(b) Is R symmetric?

Yes. If $l(a) = l(b)$, then $l(b) = l(a)$, so $(a, b) \in R$ implies $(b, a) \in R$.

(c) Is R transitive?

Yes. If $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so $(a, c) \in R$.

(d) Is R an equivalence relation? If so, what are the equivalence classes?

Since R is reflexive, symmetric, and transitive, it is an *equivalence relation*. The equivalence classes are sets of strings of the same length. For each integer $n \geq 0$, there is one equivalence class consisting of all strings of length n.

6. Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Let the relation be aRb iff a divides b (written $a \mid b$).

Reflexive: For every positive integer a, we have $a \mid a$, since $a = a \cdot 1$. So the relation is reflexive.

Transitive: If $a \mid b$ and $b \mid c$, then there exist integers m, n such that $b = am$ and $c = bn = (am)n = a(mn)$, so $a \mid c$. Thus the relation is transitive.

Symmetric: Not in general. For example $2 \mid 4$ (since $4 = 2 \cdot 2$), but $4 \nmid 2$. Therefore the relation is *not symmetric*.

Since it fails symmetry, the divides relation is *not* an equivalence relation.

7. Define a relation R from $\mathbb{Z}$ to $\mathbb{Z}$ by

$$(m, n) \in R \iff (m - n) \text{ is odd.}$$

(a) Which of the following ordered pairs are in R?

(i) $(8, 2)$: $8 - 2 = 6$, which is even, so $(8, 2) \notin R$.

(ii) $(1, 4)$: $1 - 4 = -3$, which is odd, so $(1, 4) \in R$.

(iii) $(5, -3)$: $5 - (-3) = 8$, which is even, so $(5, -3) \notin R$.

(iv) $(3, 2)$: $3 - 2 = 1$, which is odd, so $(3, 2) \in R$.

(b) List five integers that are related by R to 5.

We need integers k such that $(k, 5) \in R$, i.e. $k - 5$ is odd. Because 5 is odd, $k - 5$ is odd exactly when k is even. So all even integers are related to 5.

Examples of five such integers:

$$0,\ 2,\ 4,\ 6,\ 8.$$

(c) Prove that if n is any even integer, then $(n, 3) \in R$.

Proof. Let n be an even integer. Then there exists an integer k such that $n = 2k$. Consider $n - 3$:

$$n - 3 = 2k - 3 = 2k - 4 + 1 = 2(k - 2) + 1.$$

Since $k - 2$ is an integer, $n - 3$ is of the form 2(integer) + 1, so it is odd. Therefore $(n, 3) \in R$ by definition of R. $\square$

(d) Is R an equivalence relation? If so, how many equivalence classes does it have?

We examine the usual properties on $\mathbb{Z}$.

Reflexive? For reflexivity we would need $(n, n) \in R$ for all integers n. But $n - n = 0$, which is even, so $(n, n) \notin R$ for every n. Thus R is *not reflexive.*

Symmetric? If $(m, n) \in R$ then $m - n$ is odd. Then $n - m = -(m - n)$ is also odd, so $(n, m) \in R$. Hence R is symmetric.

Transitive? Not always. Take $1, 2, 3$:

$$1 - 2 = -1 \text{ (odd)} \Rightarrow (1, 2) \in R, \quad 2 - 3 = -1 \text{ (odd)} \Rightarrow (2, 3) \in R,$$

but

$$1 - 3 = -2 \text{ (even)} \Rightarrow (1, 3) \notin R.$$

So R is *not transitive.*

Since R is not reflexive and not transitive, it is *not* an equivalence relation. (So the question of equivalence classes does not apply.)