INF1003 Tutorial 1

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Student ID: 2501530 **Due Date:** 7th September 2025, 10:00 PM

1. **What are the values of these sums?**

(a) $\sum_{k=1}^{5}(k+1)$.

Solution.

$$\sum_{k=1}^{5}(k+1) = \sum_{k=1}^{5} k \ + \ \sum_{k=1}^{5} 1.$$

First,

$$\sum_{k=1}^{5} k = 1+2+3+4+5 = 15.$$

Next,

$$\sum_{k=1}^{5} 1 = 1+1+1+1+1 = 5.$$

Therefore,

$$\sum_{k=1}^{5}(k+1) = 15+5 = 20.$$

(b) $\sum_{j=0}^{4}(-2)^j$.

Solution. Write out the terms:

$$\sum_{j=0}^{4}(-2)^j = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4.$$

Compute each power:

$$(-2)^0 = 1, \quad (-2)^1 = -2, \quad (-2)^2 = 4, \quad (-2)^3 = -8, \quad (-2)^4 = 16.$$

Hence

$$\sum_{j=0}^{4}(-2)^j = 1-2+4-8+16.$$

Add step by step:

$$1-2=-1, \quad -1+4=3, \quad 3-8=-5, \quad -5+16=11.$$

So

$$\sum_{j=0}^{4}(-2)^j = 11.$$

(c) $\displaystyle\sum_{i=1}^{10} 3.$

Solution. Each term in the sum is 3, and there are 10 terms, so

$$\sum_{i=1}^{10} 3 = 3 + 3 + \cdots + 3 \quad (10 \text{ terms}) = 10 \times 3 = 30.$$

(d) $\displaystyle\sum_{j=0}^{8}\big(2^{j+1} - 2^j\big).$

Solution. Factor each term:

$$2^{j+1} - 2^j = 2^j(2-1) = 2^j.$$

Therefore the sum simplifies to

$$\sum_{j=0}^{8}\big(2^{j+1} - 2^j\big) = \sum_{j=0}^{8} 2^j.$$

This is a geometric series with first term 1 and common ratio 2. With 9 terms ($j = 0, 1, \ldots, 8$), the sum is

$$\sum_{j=0}^{8} 2^j = \frac{2^9 - 1}{2 - 1} = 2^9 - 1 = 512 - 1 = 511.$$

2. Find the next four terms of each of the following sequences after the initial conditions, defined by each of these recurrence relations.

(a) $a_n = a_{n-1}^2, \quad a_1 = 2.$

Solution. We repeatedly square the previous term.

For $n = 2$:

$$a_2 = a_1^2 = 2^2 = 4.$$

For $n = 3$:

$$a_3 = a_2^2 = 4^2 = 16.$$

For $n = 4$:

$$a_4 = a_3^2 = 16^2 = 256.$$

For $n = 5$:

$$a_5 = a_4^2 = 256^2 = 65{,}536.$$

Thus the next four terms are

$$a_2 = 4, \quad a_3 = 16, \quad a_4 = 256, \quad a_5 = 65{,}536.$$

(b) $a_n = na_{n-1} + n^2 a_{n-2}, \quad a_1 = 1,\ a_2 = 1.$

Solution. We compute successively for $n = 3, 4, 5, 6$.

For $n = 3$:

$$a_3 = 3a_2 + 3^2 a_1 = 3(1) + 9(1) = 3 + 9 = 12.$$

For $n = 4$:

$$a_4 = 4a_3 + 4^2 a_2 = 4(12) + 16(1) = 48 + 16 = 64.$$

For $n = 5$:

$$a_5 = 5a_4 + 5^2 a_3 = 5(64) + 25(12) = 320 + 300 = 620.$$

For $n = 6$:

$$a_6 = 6a_5 + 6^2 a_4 = 6(620) + 36(64) = 3720 + 2304 = 6024.$$

Hence the next four terms are

$$a_3 = 12, \quad a_4 = 64, \quad a_5 = 620, \quad a_6 = 6024.$$

(c) $a_n = a_{n-1} + a_{n-3}, \quad a_1 = 1,\ a_2 = 2,\ a_3 = 0.$

Solution. For $n = 4$:

$$a_4 = a_3 + a_1 = 0 + 1 = 1.$$

For $n = 5$:

$$a_5 = a_4 + a_2 = 1 + 2 = 3.$$

For $n = 6$:

$$a_6 = a_5 + a_3 = 3 + 0 = 3.$$

For $n = 7$:

$$a_7 = a_6 + a_4 = 3 + 1 = 4.$$

Thus the next four terms are

$$a_4 = 1, \quad a_5 = 3, \quad a_6 = 3, \quad a_7 = 4.$$

3. Suppose that \$1,000 is invested in an account that pays compound interest at a fixed rate of 7% annually. How much is there in the account after 4 years?

Solution. With annual compounding, the amount after n years at interest rate r is

$$A = P(1+r)^n,$$

where P is the principal.

Here $P = 1000$, $r = 0.07$, $n = 4$:

$$A = 1000(1.07)^4.$$

Compute step by step:

$$(1.07)^2 = 1.1449, \quad (1.07)^4 = (1.07)^2 \cdot (1.07)^2 \approx 1.1449^2 \approx 1.310796\ldots$$

So

$$A \approx 1000 \times 1.310796 \approx 1310.80.$$

Therefore, after 4 years there is approximately

$$\$1{,}310.80$$

in the account.

4. Find the sum of the first n terms of an arithmetic series whose first term is 1 and whose common difference is 5.

Solution. An arithmetic series with first term a_1 and common difference d has k-th term

$$a_k = a_1 + (k-1)d.$$

Here $a_1 = 1$ and $d = 5$, so

$$a_k = 1 + 5(k-1).$$

The sum of the first n terms is

$$S_n = \frac{n}{2}\big(a_1 + a_n\big).$$

We first find a_n:

$$a_n = 1 + 5(n-1) = 1 + 5n - 5 = 5n - 4.$$

Thus

$$S_n = \frac{n}{2}\big(1 + (5n-4)\big) = \frac{n}{2}(5n-3).$$

So the sum of the first n terms is

$$S_n = \frac{n(5n-3)}{2}.$$

5. An arithmetic progression has its seventh term equal to 7 and the sum of its first 10 terms is 60. Find the first term and the common difference.

Solution. Let the first term be a and the common difference be d.

The n-th term is

$$T_n = a + (n-1)d.$$

Given that $T_7 = 7$:

$$a + 6d = 7. \tag{1}$$

The sum of the first 10 terms is

$$S_{10} = \frac{10}{2}\big(2a + 9d\big) = 5(2a + 9d).$$

We are told $S_{10} = 60$, so

$$5(2a + 9d) = 60 \quad \Rightarrow \quad 2a + 9d = 12. \tag{2}$$

Now solve the simultaneous equations (1) and (2).

From (1):

$$a = 7 - 6d.$$

Substitute into (2):

$$2(7 - 6d) + 9d = 12 \quad \Rightarrow \quad 14 - 12d + 9d = 12 \quad \Rightarrow \quad 14 - 3d = 12.$$

Hence

$$-3d = 12 - 14 = -2 \quad \Rightarrow \quad d = \frac{2}{3}.$$

Then

$$a = 7 - 6d = 7 - 6 \cdot \frac{2}{3} = 7 - 4 = 3.$$

So the first term and common difference are

$$a = 3, \qquad d = \frac{2}{3}.$$

6. Find an expression for

$$2 + 2(3) + 2(3^2) + \cdots + 2(3^n).$$

Solution. Factor out 2:

$$2 + 2(3) + 2(3^2) + \cdots + 2(3^n) = 2\big(1 + 3 + 3^2 + \cdots + 3^n\big).$$

The sum inside the brackets is a geometric series with first term 1, common ratio 3, and $(n+1)$ terms.

The sum of the first $(n+1)$ terms of a geometric series is

$$1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{3 - 1} = \frac{3^{n+1} - 1}{2}.$$

Therefore

$$2\big(1 + 3 + 3^2 + \cdots + 3^n\big) = 2 \cdot \frac{3^{n+1} - 1}{2} = 3^{n+1} - 1.$$

So the required expression is

$$3^{n+1} - 1.$$

7. What is the value of each of the following sums?

(a) $\displaystyle\sum_{j=0}^{8} 3 \cdot 2^j.$

Solution. Factor out the constant 3:

$$\sum_{j=0}^{8} 3 \cdot 2^j = 3 \sum_{j=0}^{8} 2^j.$$

The inner sum is geometric with first term 1, common ratio 2, and 9 terms:

$$\sum_{j=0}^{8} 2^j = \frac{2^9 - 1}{2 - 1} = 512 - 1 = 511.$$

Hence

$$\sum_{j=0}^{8} 3 \cdot 2^j = 3 \cdot 511 = 1533.$$

(b) $\displaystyle\sum_{j=2}^{8} (-3)^j.$

Solution. This is a geometric series with first term

$$a_1 = (-3)^2 = 9,$$

common ratio

$$r = -3,$$

and the indices run from $j = 2$ to $j = 8$, so there are $8 - 2 + 1 = 7$ terms.

The sum of m terms of a geometric series is

$$S_m = a_1 \frac{1 - r^m}{1 - r}.$$

Here $m = 7$, so

$$\sum_{j=2}^{8} (-3)^j = 9 \cdot \frac{1 - (-3)^7}{1 - (-3)}.$$

Compute:

$$(-3)^7 = -2187, \quad 1 - (-3)^7 = 1 - (-2187) = 1 + 2187 = 2188,$$

and

$$1 - (-3) = 1 + 3 = 4.$$

Thus

$$\sum_{j=2}^{8} (-3)^j = 9 \cdot \frac{2188}{4} = \frac{9 \cdot 2188}{4} = \frac{19692}{4} = 4923.$$

(c) $\displaystyle\sum_{j=4}^{10} (2 + 3j)$.

Solution. Split the sum:

$$\sum_{j=4}^{10} (2 + 3j) = \sum_{j=4}^{10} 2 + \sum_{j=4}^{10} 3j.$$

There are $10 - 4 + 1 = 7$ terms, so

$$\sum_{j=4}^{10} 2 = 2 \times 7 = 14.$$

For the second part,

$$\sum_{j=4}^{10} 3j = 3 \sum_{j=4}^{10} j.$$

Now

$$\sum_{j=1}^{10} j = \frac{10 \cdot 11}{2} = 55, \quad \sum_{j=1}^{3} j = \frac{3 \cdot 4}{2} = 6,$$

so

$$\sum_{j=4}^{10} j = \sum_{j=1}^{10} j - \sum_{j=1}^{3} j = 55 - 6 = 49.$$

Hence

$$\sum_{j=4}^{10} 3j = 3 \cdot 49 = 147.$$

Putting the parts together:

$$\sum_{j=4}^{10} (2 + 3j) = 14 + 147 = 161.$$

8. Suppose $a_n = \dfrac{1}{2^{2n}}$. Find the limit, as $n \to \infty$, of

$$S_n = a_0 + a_1 + \cdots + a_{n-1}.$$

Solution. First write down the first few terms:

$$a_0 = \frac{1}{2^0} = 1, \quad a_1 = \frac{1}{2^2} = \frac{1}{4}, \quad a_2 = \frac{1}{2^4} = \frac{1}{16}, \ \ldots$$

So

$$S_n = 1 + \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{2^{2(n-1)}}.$$

This is a geometric series with

$$\text{first term } a = 1, \qquad \text{common ratio } r = \frac{1}{4}.$$

There are n terms in S_n (from a_0 to a_{n-1}). The sum of the first n terms is

$$S_n = a\,\frac{1 - r^n}{1 - r} = \frac{1 - \left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}} = \frac{1 - \left(\frac{1}{4}\right)^n}{\frac{3}{4}} = \frac{4}{3}\left(1 - \left(\frac{1}{4}\right)^n\right).$$

As $n \to \infty$, $\left(\frac{1}{4}\right)^n \to 0$, so

$$\lim_{n\to\infty} S_n = \frac{4}{3}(1 - 0) = \frac{4}{3}.$$

Therefore

$$\boxed{\lim_{n\to\infty} S_n = \frac{4}{3}.}$$