INF1003 Tutorial 9

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Topic: Functions
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1. Let $f_1(x) = x^2 + 1$ and $f_2(x) = (x+2)^2$. Find

$$\text{(a) } f_1 + f_2, \quad \text{(b) } f_1 f_2, \quad \text{(c) } f_1 \circ f_2, \quad \text{(d) } f_2 \circ f_1.$$

(a) The sum $(f_1 + f_2)(x)$:

$$\begin{aligned}(f_1 + f_2)(x) &= f_1(x) + f_2(x)\\ &= (x^2+1) + (x+2)^2\\ &= x^2 + 1 + (x^2 + 4x + 4)\\ &= 2x^2 + 4x + 5.\end{aligned}$$

(b) The product $(f_1 f_2)(x)$:

$$\begin{aligned}(f_1 f_2)(x) &= f_1(x)\, f_2(x)\\ &= (x^2+1)(x+2)^2\\ &= (x^2+1)(x^2+4x+4)\\ &= x^2(x^2+4x+4) + 1\cdot(x^2+4x+4)\\ &= x^4 + 4x^3 + 4x^2 + x^2 + 4x + 4\\ &= x^4 + 4x^3 + 5x^2 + 4x + 4.\end{aligned}$$

(c) The composition $(f_1 \circ f_2)(x)$:

$$\begin{aligned}(f_1 \circ f_2)(x) &= f_1(f_2(x))\\ &= f_1\big((x+2)^2\big) = \big((x+2)^2\big)^2 + 1\\ &= (x+2)^4 + 1.\end{aligned}$$

Expanding:

$$\begin{aligned}(x+2)^4 &= (x^2+4x+4)^2\\ &= x^4 + 8x^3 + 24x^2 + 32x + 16,\end{aligned}$$

so

$$(f_1 \circ f_2)(x) = x^4 + 8x^3 + 24x^2 + 32x + 17.$$

(d) The composition $(f_2 \circ f_1)(x)$:

$$\begin{aligned}(f_2 \circ f_1)(x) &= f_2(f_1(x))\\ &= f_2(x^2+1) = (x^2+1+2)^2\\ &= (x^2+3)^2\\ &= x^4 + 6x^2 + 9.\end{aligned}$$

2. Determine whether each of these functions from the set $\{a, b, c, d\}$ to itself is one-to-one and onto.

(a) $f(a) = b,\ f(b) = a,\ f(c) = c,\ f(d) = d.$

Solution. Every element of the codomain $\{a, b, c, d\}$ appears exactly once as an image:

$$f(a) = b,\ f(b) = a,\ f(c) = c,\ f(d) = d.$$

Hence f is one-to-one (no two inputs share the same output) and onto (every element of the codomain is hit). So f is **bijective**.

(b) $f(a) = b,\ f(b) = b,\ f(c) = d,\ f(d) = c.$

Solution. The images are $\{b, b, d, c\} = \{b, c, d\}$.

- Not one-to-one: $f(a) = b$ and $f(b) = b$.
- Not onto: a is not an image of any element.

So f is **neither** one-to-one nor onto.

(c) $f(a) = d,\ f(b) = b,\ f(c) = d,\ f(d) = d.$

Solution. The images are $\{d, b, d, d\} = \{b, d\}$.

- Not one-to-one: $f(a) = d$ and $f(c) = d$.
- Not onto: a and c are not images of any element.

So f is **neither** one-to-one nor onto.

3. Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$ in each case:

(a) $f(n) = \pm n.$

Solution. For a fixed integer n, the notation $\pm n$ represents *two* possible values, n and $-n$. Thus $f(n)$ is not uniquely determined: the rule does not assign exactly one real number to each integer n. Therefore f is **not** a function from $\mathbb{Z}$ to $\mathbb{R}$.

(b) $f(n) = \sqrt{n^2 - 1}.$

Solution. For $n = 0$ we have $n^2 - 1 = -1$ and $\sqrt{-1}$ is not a real number. So $f(0)$ is not defined in $\mathbb{R}$, which means f fails to assign a real value to every integer input. Hence f is **not** a function from $\mathbb{Z}$ to $\mathbb{R}$.

(c) $f(n) = \dfrac{1}{n^2 - 4}.$

Solution. If $n = 2$ or $n = -2$, then $n^2 - 4 = 0$ and $1/(n^2 - 4)$ is undefined. Thus the rule does not give a real value for every integer n. Hence f is **not** a function from $\mathbb{Z}$ to $\mathbb{R}$.

4. Consider the following three assignments from the set of students in our class.

(a) A mapping from the set of all SIT students to the set of Poly Diplomas, mapping each student to his/her polytechnic diploma.

(b) A mapping from the set of all SIT students to a set of strings, mapping each student to his/her student ID number.

(c) A mapping from the set of all SIT students to the set of all first names of female citizens and residents in Singapore, mapping each student to his/her mother's first name.

For each of these, answer:

(I) Under what conditions does the assignment *fail* to be a function?

(II) Under what conditions can it be one-to-one?

(III) Under what conditions can it be onto?

Solution.

(a) **Student $\mapsto$ poly diploma.**

Let the codomain be the set of all possible polytechnic diplomas.

(I) Not a function if:

- Some SIT students did not come from a polytechnic route (for instance, A-level or IB students), so they have no poly diploma to map to.
- A student has more than one poly diploma but the rule does not specify which one to choose, making the output ambiguous.

To have a well-defined function, we could restrict the domain to "SIT students with exactly one poly diploma".

(II) One-to-one if:

- No two distinct students share the same poly diploma, i.e. at most one student in SIT holds any given poly diploma.

This is logically possible but unrealistic in practice.

(III) Onto if:

- Every poly diploma in the codomain is held by at least one SIT student.

Equivalently, the codomain is chosen to be exactly the set of poly diplomas actually possessed by SIT students.

(b) **Student $\mapsto$ student ID.**

(I) Not a function if:

- Some students have no assigned student ID.
- A student has two different student IDs and the rule does not specify a unique choice.

With the usual assumption that every student has exactly one ID, the assignment is a function.

(II) One-to-one if:

- Each student ID is unique to one student (no two students share the same ID).

This is typically true by design, so under standard assumptions the function is one-to-one.

(III) Onto if:

- The codomain is chosen to be exactly the set of student IDs in use at SIT (so every element of the codomain is used by some student).

If the codomain were “all strings” or “all 8-digit numbers”, it would not be onto.

(c) **Student $\mapsto$ mother's first name.**

(I) Not a function if:

- The information about the mother's first name is missing for some students, so no output can be given.
- There is ambiguity about which person counts as “mother” (e.g. more than one recorded mother) and the rule does not specify how to choose.

If we assume every student has exactly one recorded mother with a well-defined first name, the assignment is a function.

(II) One-to-one if:

- No two students' mothers share the same first name.

This is possible for a very small class but unlikely in reality.

(III) Onto if:

- Every female first name in the codomain is used as the mother's first name of at least one student in the class.

This would require choosing a smaller codomain, such as “the set of mothers' first names actually appearing in our class”, rather than all possible female names in Singapore.

5. Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is one-to-one and onto.

(a) $f(n) = n - 1$.

Solution. For injectivity, assume $f(n_1) = f(n_2)$:

$$n_1 - 1 = n_2 - 1 \Rightarrow n_1 = n_2.$$

Hence f is one-to-one.

For surjectivity, let $m \in \mathbb{Z}$ be arbitrary. Choose $n = m + 1$. Then $f(n) = (m + 1) - 1 = m$. So every integer has a preimage. Hence f is onto.

Therefore f is **bijective**.

(b) $f(n) = n^2 + 1$.

Solution.

- Not one-to-one: $f(1) = 1^2 + 1 = 2$ and $f(-1) = (-1)^2 + 1 = 2$.
- Not onto: 0 is not in the range, since $n^2 + 1 \geq 1$ for all $n \in \mathbb{Z}$.

So f is **neither** one-to-one nor onto.

(c) $f(n) = n^3$.

Solution. For injectivity, suppose $f(n_1) = f(n_2)$:

$$n_1^3 = n_2^3 \Rightarrow n_1 = n_2$$

(over integers, the cube function is strictly increasing).

For surjectivity, take any integer m. Setting $n = \sqrt[3]{m}$ gives $n \in \mathbb{Z}$ and $f(n) = m$. For integers this just says: for each m there is a unique integer n with $n^3 = m$.

Thus f is **bijective**.

(d) $f(n) = \left\lceil \frac{n}{2} \right\rceil$.

Solution.

- Not one-to-one: for example

$$f(1) = \left\lceil \frac{1}{2} \right\rceil = 1, \quad f(2) = \left\lceil \frac{2}{2} \right\rceil = 1,$$

  so $f(1) = f(2)$ with $1 \neq 2$.
- Onto: let $m \in \mathbb{Z}$ be arbitrary and choose $n = 2m$. Then

$$f(n) = \left\lceil \frac{2m}{2} \right\rceil = \lceil m \rceil = m.$$

  So every integer has a preimage.

Therefore f is **onto but not one-to-one**.

(e) $f(n) = \lfloor n \rfloor$.

Solution. For any integer n, $\lfloor n \rfloor = n$, so f is the identity function on $\mathbb{Z}$. Hence it is clearly one-to-one and onto.

Therefore f is **bijective**.

6. Determine whether each of these functions from $\mathbb{R}$ to $\mathbb{R}$ is bijective.

(a) $f(x) = 2x + 1$.

Solution. For injectivity, assume $f(x_1) = f(x_2)$:

$$2x_1 + 1 = 2x_2 + 1 \Rightarrow x_1 = x_2.$$

For surjectivity, let $y \in \mathbb{R}$ be arbitrary. Then

$$y = 2x + 1 \Rightarrow x = \frac{y-1}{2} \in \mathbb{R},$$

so there is an x such that $f(x) = y$. Thus f is **bijective**.

(b) $f(x) = x^2 + 1$.

Solution.

- Not one-to-one: $f(1) = 2$ and $f(-1) = 2$.
- Not onto: for example $y = 0$ is not in the image, because $x^2 + 1 \geq 1$ for all x.

So f is **not bijective**.

(c) $f(x) = x^3$.

Solution. The function x^3 is strictly increasing over $\mathbb{R}$, so it is one-to-one.

For any $y \in \mathbb{R}$, take $x = \sqrt[3]{y}$; then $f(x) = y$, so f is onto.

Therefore f is **bijective**.

(d) $f(x) = \dfrac{x^2+1}{x^2+2}$.

Solution. Note that $x^2 \geq 0$, so

$$\frac{x^2+1}{x^2+2} = 1 - \frac{1}{x^2+2}.$$

Since $x^2 + 2 > 2$, we have $0 < \dfrac{1}{x^2+2} < \dfrac{1}{2}$ and hence

$$\frac{1}{2} < f(x) < 1 \quad \text{for all } x \in \mathbb{R}.$$

Therefore f is not onto $\mathbb{R}$, because values such as 0 or 2 are never obtained.

Moreover $f(x) = f(-x)$, so f is not one-to-one.

Hence f is **not bijective**.

7. Suppose $g\colon A \to B$ and $f\colon B \to C$.

(a) If f is one-to-one and $f \circ g$ is one-to-one, must g also be one-to-one?

Solution. Yes.

Proof. Assume f is one-to-one and $f \circ g$ is one-to-one. Let $x_1, x_2 \in A$ and suppose $g(x_1) = g(x_2)$. Then

$$(f \circ g)(x_1) = f(g(x_1)) = f(g(x_2)) = (f \circ g)(x_2).$$

Because $f \circ g$ is one-to-one, we must have $x_1 = x_2$. Hence, whenever $g(x_1) = g(x_2)$ we get $x_1 = x_2$, so g is one-to-one.

(b) If g is one-to-one and $f \circ g$ is one-to-one, must f also be one-to-one?

Solution. No. We give a counterexample.

Let

$$A = \{1, 2\}, \quad B = \{a, b, c\}, \quad C = \{0, 1\}.$$

Define $g\colon A \to B$ by

$$g(1) = a, \quad g(2) = b.$$

This g is one-to-one.

Define $f\colon B \to C$ by

$$f(a) = 0, \quad f(b) = 1, \quad f(c) = 1.$$

Here f is *not* one-to-one, since $f(b) = f(c)$ but $b \neq c$.

However, the composition $f \circ g\colon A \to C$ is

$$(f \circ g)(1) = f(a) = 0, \quad (f \circ g)(2) = f(b) = 1,$$

which is one-to-one on A. Thus g and $f \circ g$ can both be one-to-one even when f is not.

Therefore, we cannot conclude that f is one-to-one.