INF1003 Tutorial 7

Name: Timothy Chia **Topic:** Proof Methods and Strategies
Student ID: 2501530 **Due Date:** 28/10/2025, 10:00PM

1. **(Direct proof)** Use a direct proof to show that the sum of two odd integers is even.

   Proof. Let a and b be odd integers. By definition of an odd integer, there exist integers i and j such that

   $$a = 2i + 1 \quad \text{and} \quad b = 2j + 1.$$

   Then

   $$\begin{aligned} a + b &= (2i + 1) + (2j + 1) \\ &= 2i + 2j + 2 \\ &= 2(i + j + 1). \end{aligned}$$

   Since $i + j + 1$ is an integer, $a + b$ is of the form $2k$ for some integer k.
   Hence $a + b$ is even. □

2. Show that for any positive integer n, n is odd if and only if $5n + 6$ is odd.

   We must show, for all positive integers n,

   $$n \text{ is odd} \leftrightarrow 5n + 6 \text{ is odd}.$$

   By the hint, it is enough to prove both directions:

   $$(\Rightarrow) \quad n \text{ odd} \rightarrow 5n + 6 \text{ odd}, \qquad (\Leftarrow) \quad 5n + 6 \text{ odd} \rightarrow n \text{ odd}.$$

   (a) **(n odd $\Rightarrow$ $5n + 6$ odd; direct proof)**
   Assume n is odd. Then there exists an integer k such that

   $$n = 2k + 1.$$

   Compute $5n + 6$:

   $$\begin{aligned} 5n + 6 &= 5(2k + 1) + 6 \\ &= 10k + 5 + 6 \\ &= 10k + 11 \\ &= 2(5k + 5) + 1. \end{aligned}$$

   Since $5k + 5$ is an integer, $5n + 6$ is of the form $2m + 1$, so $5n + 6$ is odd. □

(b) **($5n+6$ odd $\Rightarrow$ n odd; proof by contraposition)**

The statement

$$5n + 6 \text{ odd} \to n \text{ odd}$$

has contrapositive

$$n \text{ even} \to 5n + 6 \text{ even}.$$

We prove this contrapositive.

Assume n is even. Then there exists an integer k such that

$$n = 2k.$$

Compute $5n + 6$:

$$\begin{aligned} 5n + 6 &= 5(2k) + 6 \\ &= 10k + 6 \\ &= 2(5k + 3). \end{aligned}$$

As $5k + 3$ is an integer, $5n + 6$ is of the form $2m$, hence is even. Therefore the contrapositive is true, so the original implication "$5n + 6$ odd $\to$ n odd" is also true.

Combining both directions, we conclude that for every positive integer n, n is odd if and only if $5n + 6$ is odd. $\square$

3. Show that for any integer n, if $n^3 + 5$ is odd, then n is even. Provide two different proofs.

(a) **Proof by contraposition.**

The statement is

$$\forall n \in \mathbb{Z} \left((n^3 + 5 \text{ is odd}) \to (n \text{ is even}) \right).$$

Its contrapositive is

$$\forall n \in \mathbb{Z} \left((n \text{ is odd}) \to (n^3 + 5 \text{ is even}) \right).$$

Assume n is odd. Then there exists an integer k such that $n = 2k + 1$. Compute $n^3 + 5$:

$$\begin{aligned} n^3 + 5 &= (2k + 1)^3 + 5 \\ &= 8k^3 + 12k^2 + 6k + 1 + 5 \\ &= 8k^3 + 12k^2 + 6k + 6 \\ &= 2\left(4k^3 + 6k^2 + 3k + 3\right). \end{aligned}$$

The expression in brackets is an integer, so $n^3 + 5$ is even. Hence the contrapositive holds, and therefore the original statement is true. $\square$

(b) **Proof by contradiction.**

Suppose, for the sake of contradiction, that there exists an integer n such that $n^3 + 5$ is odd *and* n is odd.

Since n is odd, let $n = 2k + 1$ for some integer k. As in part (a),

$$\begin{aligned} n^3 + 5 &= (2k+1)^3 + 5 \\ &= 8k^3 + 12k^2 + 6k + 6 \\ &= 2\big(4k^3 + 6k^2 + 3k + 3\big), \end{aligned}$$

which is even. This contradicts the assumption that $n^3 + 5$ is odd.

Therefore our assumption that n could be odd when $n^3 + 5$ is odd must be false. Hence, if $n^3 + 5$ is odd, n must be even. $\square$

4. Prove that if $m + n$ and $n + p$ are even integers, where m, n, p are integers, then $m + p$ is even. State the strategies used in the proof.

Proof (direct proof using algebra and parity). Assume $m + n$ and $n + p$ are even.

Then there exist integers a and b such that

$$m + n = 2a \quad \text{and} \quad n + p = 2b.$$

Add these equations:

$$(m + n) + (n + p) = 2a + 2b.$$

So

$$m + 2n + p = 2(a + b).$$

Rearrange to isolate $m + p$:

$$\begin{aligned} m + p &= 2(a + b) - 2n \\ &= 2(a + b - n). \end{aligned}$$

Since $a + b - n$ is an integer, $m + p$ is of the form $2k$ and hence is even.

Strategy used: a *direct proof* with the definition of even integers and simple algebraic manipulation. $\square$

5. Prove that $(n^2+1) \geq 2n$ where n is a positive integer with $1 \leq n \leq 4$.

Proof (exhaustive check). We must check the inequality for each integer $n = 1, 2, 3, 4$.

- $n = 1$:
$$n^2+1 = 1^2+1 = 2, \quad 2n = 2(1) = 2,$$
and $2 \geq 2$ is true.
- $n = 2$:
$$n^2+1 = 2^2+1 = 5, \quad 2n = 2(2) = 4,$$
and $5 \geq 4$ is true.
- $n = 3$:
$$n^2+1 = 3^2+1 = 10, \quad 2n = 2(3) = 6,$$
and $10 \geq 6$ is true.
- $n = 4$:
$$n^2+1 = 4^2+1 = 17, \quad 2n = 2(4) = 8,$$
and $17 \geq 8$ is true.

Since the inequality holds for all n in the finite set $\{1, 2, 3, 4\}$, it is true for every positive integer n with $1 \leq n \leq 4$. □

6. Prove or disprove the following statements, where n ranges over all integers.

(a) $\forall n\,(6n \text{ is even} \to n \text{ is even})$.

Disproof (counterexample). For any integer n, $6n = 2(3n)$ is always even. So the implication "$6n$ is even $\to$ n is even" reduces to requiring that n itself be even.

Take $n = 1$. Then $6n = 6$ is even, but 1 is not even. The conditional
$$6n \text{ even} \to n \text{ even}$$
becomes
$$\text{True} \to \text{False},$$
which is false. Hence the universal statement is **false**.

(b) $\forall n\,(6n \text{ is even} \to n \text{ is odd})$.

Disproof (counterexample). Again $6n$ is even for every integer n.

Take $n = 2$. Then $6n = 12$ is even but 2 is not odd. The implication is True $\to$ False, hence false. Therefore the universal statement is **false**.

(c) $\exists n\,(6n \text{ is even} \to n \text{ is even})$.

Proof. Take $n = 2$. Then $6n = 12$ is even and n is even. So the implication
$$6n \text{ even} \to n \text{ even}$$
is True $\to$ True, which is true. Hence there exists an integer n making the implication true, and the existential statement is **true**.

(d) $\exists n\,(6n \text{ is even} \to n \text{ is odd})$.

Proof. Take $n = 1$. Then $6n = 6$ is even and n is odd. So the implication

$$6n \text{ even} \to n \text{ odd}$$

is True $\to$ True, which is true. Hence the statement is **true**.

(e) $\forall n\,(n \text{ is even} \to 6n \text{ is even})$.

Proof. Assume n is even. Then there exists an integer k such that

$$n = 2k.$$

Then

$$\begin{aligned} 6n &= 6(2k) \\ &= 12k \\ &= 2(6k), \end{aligned}$$

which is even. Since this works for every even integer n, the universal statement is **true**.

(f) $\exists n\,(n \text{ is even} \to 6n \text{ is odd})$.

Proof. We analyse the implication for different n.

If n is even, write $n = 2k$. Then

$$6n = 6(2k) = 12k = 2(6k),$$

which is even, so "$6n$ is odd" is false. Hence for every even n, the implication

$$n \text{ even} \to 6n \text{ odd}$$

is True $\to$ False, which is false.

If n is odd, the antecedent "n is even" is false, so the entire implication is vacuously true.

Thus there *do* exist integers n (for example $n = 1$) for which the implication is true, so formally the existential statement is **true**. This illustrates how implications with false antecedent are automatically true (vacuous truth).

$\square$

7. Prove or disprove the statement: *"The product of two irrational numbers is always irrational."* State the strategies used in the proof.

Disproof (counterexample). Consider the irrational number $\sqrt{2}$. The product of $\sqrt{2}$ with itself is

$$\sqrt{2} \cdot \sqrt{2} = 2,$$

and 2 is rational. Thus we have two irrational numbers whose product is rational.

Therefore the statement "The product of two irrational numbers is always irrational" is **false**.

Strategy used: counterexample to disprove a universal claim. □

8. A ceiling function maps x to the least integer greater than or equal to x, denoted $\lceil x \rceil$. For example, $\lceil 2.1 \rceil = 3$ and $\lceil -2.1 \rceil = -2$. Prove that for any odd integer n,

$$\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2+3}{4}.$$

Proof (direct proof). Let n be any odd integer. Then there exists an integer k such that

$$n = 2k+1.$$

Compute $n^2/4$:

$$\begin{aligned} \frac{n^2}{4} &= \frac{(2k+1)^2}{4} \\ &= \frac{4k^2+4k+1}{4} \\ &= k^2+k+\frac{1}{4}. \end{aligned}$$

The integer part of this expression is k^2+k, and the fractional part is $\frac{1}{4}$, which is strictly between 0 and 1. Therefore the least integer greater than or equal to $k^2+k+\frac{1}{4}$ is k^2+k+1:

$$\left\lceil \frac{n^2}{4} \right\rceil = \left\lceil k^2+k+\frac{1}{4} \right\rceil = k^2+k+1.$$

Now compute the right-hand side:

$$\begin{aligned} \frac{n^2+3}{4} &= \frac{(2k+1)^2+3}{4} \\ &= \frac{4k^2+4k+1+3}{4} \\ &= \frac{4k^2+4k+4}{4} \\ &= k^2+k+1. \end{aligned}$$

Hence

$$\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2+3}{4}$$

for every odd integer n. $\square$

9. Prove that if x is irrational, then $1/x$ is irrational.

Proof (by contraposition).
We work with real numbers x with $x \neq 0$ (since $1/x$ is undefined for $x = 0$).

The statement to prove is

$$x \text{ irrational} \rightarrow 1/x \text{ irrational.}$$

Its contrapositive is

$$1/x \text{ rational} \rightarrow x \text{ rational.}$$

Assume $1/x$ is rational. Then we can write

$$\frac{1}{x} = \frac{a}{b}$$

for some integers a and b with $b \neq 0$ and the fraction in lowest terms. Since $1/x = a/b \neq 0$, we also have $a \neq 0$.

Rearranging,

$$x = \frac{b}{a}.$$

Because a and b are integers with $a \neq 0$, the number $x = b/a$ is rational.

Thus the contrapositive is true, so the original statement "If x is irrational, then $1/x$ is irrational" holds for all real $x \neq 0$. $\square$