INF1003 Tutorial 5

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1. Let Prime(n) be the predicate "n is a prime number" and In(n, a, b) be the predicate $a \leq n \leq b$, where n, a, b range over all integers. For each statement below, determine whether it is *True*, *False*, or *Neither*, and give a brief explanation.

(a) $\neg\text{Prime}(10) \vee \text{In}(10, 5, 20)$.

Solution.

- 10 is not prime, so Prime(10) is false and $\neg$Prime(10) is true.
- 10 lies between 5 and 20, so In(10, 5, 20) is also true.
- The whole statement is $T \vee T$, which is true.

Hence the statement is **True**.

(b) $\exists n\, \text{Prime}(n)$.

Solution. There are many prime integers (for example, 2, 3, 5, …). Choosing $n = 2$ gives Prime(2) true, so the existential statement is satisfied. Therefore this statement is **True**.

(c) $\exists n\, \neg\text{Prime}(n)$.

Solution. There are integers that are not prime, for example 1, 4, 6, …. Taking $n = 4$ gives $\neg$Prime(4) true. So the existential statement holds and is **True**.

(d) $\forall n\, \text{Prime}(n)$.

Solution. This says every integer is prime. But 4 (or 1, or 10, etc.) is not prime, so there is a counterexample. Because at least one n makes Prime(n) false, the universal statement is false. Hence it is **False**.

(e) $\forall n\, \neg\text{Prime}(n)$.

Solution. This says no integer is prime. However 2 is prime, so for $n = 2$ we have $\neg$Prime(2) false. Thus the universal statement fails and is **False**.

(f) $\neg\forall n\, \text{Prime}(n)$.

Solution. Using a standard equivalence:

$$\neg\forall n\, \text{Prime}(n) \equiv \exists n\, \neg\text{Prime}(n).$$

In part (c) we saw that $\exists n\, \neg\text{Prime}(n)$ is true (e.g. $n = 4$). Therefore this statement is **True**.

(g) $\forall n\,(\text{In}(n,1,3) \to \text{Prime}(n))$.

Solution. The antecedent $\text{In}(n,1,3)$ is true exactly when $n = 1, 2$, or 3. We check these cases:

- $n = 1$: $\text{In}(1,1,3)$ is true, but 1 is not prime, so $\text{Prime}(1)$ is false. Hence the implication is $T \to F$, which is false.

Because the universal quantifier requires the implication to hold for *every* n, the single counterexample $n = 1$ makes the whole statement **False**.

(h) $\forall n\,(\text{In}(n,8,10) \to \text{Prime}(n))$.

Solution. Here $\text{In}(n,8,10)$ is true for $n = 8, 9, 10$.

- $n = 9$: $\text{In}(9,8,10)$ is true, but 9 is not prime $(9 = 3 \cdot 3)$, so $\text{Prime}(9)$ is false. The implication is therefore $T \to F$, which is false.

Hence the universal statement is **False**.

(i) $\forall n\,(\text{In}(n,a,b) \to \neg\text{Prime}(n))$, where a and b are integers smaller than 10.

Solution. Here a and b are fixed integers less than 10, but we are *not* told which ones.

- If the interval $[a,b]$ happens to contain a prime (for example $[2,4]$), then the statement is false (because some n in $[a,b]$ will have $\text{Prime}(n)$ true).
- If $[a,b]$ is chosen to contain no primes (for example $[1,1]$ or $[8,9]$), then the statement is true.

Since the truth value depends on the particular (unspecified) values of a and b, we cannot classify it as always true or always false. Therefore the answer is **Neither**.

(j) $\exists n\,\big(\text{Prime}(n) \to \text{In}(n,30,40)\big)$.

Solution. First interpret the formula carefully:

$$\exists n\,(\text{Prime}(n) \to \text{In}(n,30,40)).$$

For a *fixed* integer n, the implication $\text{Prime}(n) \to \text{In}(n,30,40)$ is false only when $\text{Prime}(n)$ is true and $\text{In}(n,30,40)$ is false; in all other cases it is true.

- Take $n = 1$. Then $\text{Prime}(1)$ is false and so the implication is $F \to \text{In}(1,30,40)$, which is true regardless of the consequent.

Thus there exists at least one integer n (for example $n = 1$) making the implication true, so the existential statement holds. Therefore it is **True**.

2. Let $\text{BB}(x)$ be the statement "x plays basketball every week", where the domain of x is all ICT students. Express each logical formula in English.

(a) $\exists x\,\text{BB}(x)$.

Solution. There is at least one ICT student who plays basketball every week.

(b) $\forall x\, \mathrm{BB}(x)$.

Solution. Every ICT student plays basketball every week.

(c) $\neg\forall x\, \mathrm{BB}(x)$.

Solution. First note that

$$\neg\forall x\, \mathrm{BB}(x) \equiv \exists x\, \neg\mathrm{BB}(x).$$

So in English: Not every ICT student plays basketball every week.

(d) $\exists x\, \neg\mathrm{BB}(x)$.

Solution. There is an ICT student who does not play basketball every week.

3. For each statement, do the following:

(I) Express it using quantified logical expressions.

(II) Form the negation, pushing all negations directly onto predicates (not onto quantifiers).

(III) Express the negated statement in English.

(a) No rabbit knows calculus.

Solution. Let the domain be all rabbits, and let $K(x)$ mean "x knows calculus".

(I) Original statement:

$$\forall x\, \neg K(x)$$

(equivalently $\neg\exists x\, K(x)$).

(II) Negation:

$$\neg\forall x\, \neg K(x) \equiv \exists x\, \neg\neg K(x) \equiv \exists x\, K(x).$$

(III) English: There is a rabbit that knows calculus.

(b) There is a bird that can talk.

Solution. Let the domain be all birds and $T(x)$ mean "x can talk".

(I) Original statement:

$$\exists x\, T(x).$$

(II) Negation:

$$\neg\exists x\, T(x) \equiv \forall x\, \neg T(x).$$

(III) English: No bird can talk (equivalently: all birds cannot talk).

(c) There is no one in this class who knows French and Russian.

Solution. Let the domain be all people in this class. Let $F(x)$ mean "x knows French" and $R(x)$ mean "x knows Russian".

(I) Original statement:

$$\neg\exists x\,(F(x) \land R(x)) \quad \text{or equivalently} \quad \forall x\,\neg(F(x) \land R(x)).$$

(II) Negation:

$$\neg\neg\exists x\,(F(x) \land R(x)) \equiv \exists x\,(F(x) \land R(x)).$$

(III) English: There is someone in this class who knows both French and Russian.

(d) Everyone in this class is a Marvel fan.

Solution. Let the domain be all people in this class, and let $M(x)$ mean "x is a Marvel fan".

(I) Original statement:

$$\forall x\, M(x).$$

(II) Negation:

$$\neg\forall x\, M(x) \equiv \exists x\,\neg M(x).$$

(III) English: There is someone in this class who is not a Marvel fan.

4. Express each of the following statements using predicates and quantifiers. Clearly define all predicates and variables.

(a) There is a student who has taken more than 21 credit hours in a semester and received all A's.

Solution. Let the domain of x be all students, and the domain of y be all modules.

- $\text{Credits}(x, y)$: student x has taken more than y credit hours in a semester.
- $A(x, y)$: student x received grade A in module y.

The statement "received all A's" can be modelled as "for every module y, student x received A in y".

$$\exists x\,\big(\text{Credits}(x, 21) \land \forall y\, A(x, y)\big).$$

(b) A passenger on an airline qualifies as an "Elite Flyer" if the passenger flies more than 25000 miles in a year or takes more than 25 flights during that year.

Solution. Let the domain of x be all airline passengers.

- $\text{Miles}(x, y)$: passenger x flies more than y miles in a year.
- $\text{Flights}(x, y)$: passenger x takes more than y flights in a year.
- $\text{Elite}(x)$: passenger x qualifies as an Elite Flyer.

The rule "qualifies as an Elite Flyer if …" becomes an implication for all passengers:

$$\forall x \left((\text{Miles}(x, 25000) \vee \text{Flights}(x, 25)) \rightarrow \text{Elite}(x) \right).$$

(c) A man qualifies for the marathon if his best previous time is less than 3 hours and a woman qualifies for the marathon if her best previous time is less than 3.5 hours.

Solution. Let the domain of x be all people.

- $M(x)$: x is a man.
- $W(x)$: x is a woman.
- $Q(x)$: x qualifies for the marathon.
- $\text{Best}(x, t)$: the best previous time of x is less than t hours.

We can encode both conditions in a single formula:

$$\forall x \left((M(x) \wedge \text{Best}(x, 3)) \vee (W(x) \wedge \text{Best}(x, 3.5)) \rightarrow Q(x) \right).$$

(Equivalently, we could have written two separate implications, one for men and one for women.)

5. Use the following predicates:

$$L(x) : x \text{ has a laptop}, \quad D(x) : x \text{ has a desktop computer}, \quad M(x) : x \text{ uses macOS}, \quad W(x) : x \text{ u}$$

Domain 1 is "students in this class". Domain 2 is "all people in the world". You may also use an additional predicate

$$S(x) : x \text{ is a student in this class}$$

when working in Domain 2.

For each statement, give an expression in both domains.

(a) Some students have both a laptop and a desktop computer.

Solution.

- **Domain 1 (students in this class):**

$$\exists x \, (L(x) \wedge D(x)).$$

  Here the domain already consists of students, so we do not need $S(x)$.
- **Domain 2 (all people in the world):** We must restrict to those who are students in this class:

$$\exists x \, (S(x) \wedge L(x) \wedge D(x)).$$

(b) All students who use macOS have a laptop.

Solution.

- **Domain 1:**
$$\forall x\,(M(x) \to L(x)).$$
Every (class) student using macOS has a laptop.
- **Domain 2:** We again restrict to students in this class:
$$\forall x\,((S(x) \land M(x)) \to L(x)).$$

(c) Every student uses either macOS, Windows, or both.

Solution.

- **Domain 1:**
$$\forall x\,(M(x) \lor W(x)).$$
(All students in this class use macOS, Windows, or both.)
- **Domain 2:** We restrict the universal quantifier to students in this class:
$$\forall x\,(S(x) \to (M(x) \lor W(x))).$$

6. [**OPTIONAL**] Translate each logical statement into a clear English sentence. The domain of every variable is the set of all real numbers.

(a) $\forall x\,\exists y\,(x < y)$.

Solution. For every real number x, there is a real number y that is larger than x. (Every real number has a larger real number.)

(b) $\forall x\,\forall y\,(((x \geq 0) \land (y \geq 0)) \to (xy \geq 0))$.

Solution. For all real numbers x and y, if both x and y are non-negative, then their product xy is also non-negative.

(c) $\forall x\,\forall y\,\exists z\,(x + y = z)$.

Solution. For any two real numbers x and y, there exists a real number z equal to their sum. (The sum of any two real numbers is a real number.)

7. [**OPTIONAL**] Let $F(x, y)$ be the predicate "x and y are friends", where x and y range over all students in SIT. Translate each statement into clear English (avoid using the symbols x and y in your English sentences).

(a) $\forall x\,\exists y\,\big(F(x, y) \land \forall z\,((y \neq z) \to \neg F(x, z))\big)$.

Solution. For every student, there is some student who is their friend, and they have no other friends. In words: *Every student in SIT has exactly one friend.*

(b) $\exists x\,\forall y\,\forall z\,\big((F(x, y) \land F(x, z) \land (y \neq z)) \to \neg F(y, z)\big)$.

Solution. There is at least one student such that whenever two different students are both friends with that person, those two students are not friends with each other. In words: *There is a student whose friends are not friends with one another.*