INF1003 Tutorial 8

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1. List the members of the following sets.

(a) $\{x \mid x \text{ is a real number such that } x^2 = 1\}$.

Solution. Solve $x^2 = 1$ over the real numbers:

$$x^2 = 1 \quad \Rightarrow \quad x = \pm 1.$$

So the set is

$$\{-1, 1\}.$$

(b) $\{x \mid x \text{ is the cube of a positive integer such that } x \leq 1728\}$.

Solution. Let $x = k^3$ where k is a positive integer and $k^3 \leq 1728$. Note that $1728 = 12^3$, and $13^3 = 2197 > 1728$. Hence k can be any integer from 1 to 12. Thus the set of cubes is

$$\{1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728\}.$$

(c) $\{x \mid x \text{ is a prime number such that } x < 15\}$.

Solution. The positive integers less than 15 are $2, 3, 4, \ldots, 14$. The primes among them are $2, 3, 5, 7, 11, 13$. So the set is

$$\{2, 3, 5, 7, 11, 13\}.$$

(d) $\{x \mid x \text{ is an integer such that } x^2 = 5\}$.

Solution. If x is an integer and $x^2 = 5$, then $x = \pm\sqrt{5}$, but $\sqrt{5}$ is not an integer. Hence there is no integer solution, so the set is empty:

$$\emptyset.$$

2. Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{6, 4\}$, $D = \{4, 2, 6\}$ and $E = \{4, 8, 6\}$. Determine which of these sets are proper subsets of which other sets. Also list the members of the following.

(a) *Proper subsets.*

Solution.

- $B = \{2, 6\}$ and $A = \{2, 4, 6\}$, so every element of B is in A, but A has 4 as an extra element. Thus $B \subset A$ (proper subset).

- $D = \{4, 2, 6\}$ has exactly the same elements as A. So $A = D$. Therefore $B \subset D$ as well.
- $C = \{6, 4\}$; this is a subset of A (and D) and also of E:

$$C \subset A, \quad C \subset D, \quad C \subset E.$$

- No other non-trivial proper subset relations hold (for instance, E is not a subset of A because $8 \in E$ but $8 \notin A$).

(b) $A \cap C$.

Solution.

$$\begin{aligned} A \cap C &= \{2, 4, 6\} \cap \{6, 4\} \\ &= \{4, 6\}. \end{aligned}$$

(c) $B \cup E$.

Solution.

$$\begin{aligned} B \cup E &= \{2, 6\} \cup \{4, 8, 6\} \\ &= \{2, 4, 6, 8\}. \end{aligned}$$

(d) $D \cap B \cap E$.

Solution. First compute $D \cap B$:

$$\begin{aligned} D \cap B &= \{4, 2, 6\} \cap \{2, 6\} \\ &= \{2, 6\}. \end{aligned}$$

Then intersect with E:

$$\begin{aligned} (D \cap B) \cap E &= \{2, 6\} \cap \{4, 8, 6\} \\ &= \{6\}. \end{aligned}$$

So $D \cap B \cap E = \{6\}$.

3. What is the cardinality of each of these sets?

(a) $\{a\}$

Solution. There is exactly one element, namely a, so

$$|\{a\}| = 1.$$

(b) $\{\{a\}\}$

Solution. Here the single element of the set is the set $\{a\}$. So there is still exactly one element:

$$|\{\{a\}\}| = 1.$$

(c) $\{a, \{a\}\}$

Solution. The elements are a and the set $\{a\}$, which are distinct objects. Hence there are two elements:

$$|\{a, \{a\}\}| = 2.$$

(d) $\{a, \{a\}, \{a, \{a\}\}\}$

Solution. The elements are a, the set $\{a\}$, and the set $\{a, \{a\}\}$, all distinct. So there are three elements:

$$\big|\{a, \{a\}, \{a, \{a\}\}\}\big| = 3.$$

(e) $\emptyset$

Solution. The empty set has no elements, so

$$|\emptyset| = 0.$$

(f) $\{\emptyset\}$

Solution. This set has a single element, namely the empty set. Thus

$$|\{\emptyset\}| = 1.$$

(g) $\{\emptyset, \{\emptyset\}\}$

Solution. The two elements are $\emptyset$ and the set $\{\emptyset\}$, which are distinct. Therefore

$$\big|\{\emptyset, \{\emptyset\}\}\big| = 2.$$

(h) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Solution. The three elements are $\emptyset$, $\{\emptyset\}$, and $\{\emptyset, \{\emptyset\}\}$, all distinct, so

$$\big|\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\big| = 3.$$

4. List the elements of the following sets, where a and b are distinct elements, and $P(\cdot)$ denotes the power set.

(a) $P(\{a, b\})$.

Solution. All subsets of $\{a, b\}$ are

$$\emptyset, \quad \{a\}, \quad \{b\}, \quad \{a, b\}.$$

Hence

$$P(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

(b) $P(\{a, \emptyset\})$.

Solution. Now the two elements are a and $\emptyset$. Subsets:

$$\emptyset, \quad \{a\}, \quad \{\emptyset\}, \quad \{a, \emptyset\}.$$

So

$$P(\{a, \emptyset\}) = \{\emptyset, \{a\}, \{\emptyset\}, \{a, \emptyset\}\}.$$

(c) $P(\{a, \{\emptyset\}\})$.

Solution. The elements are a and the set $\{\emptyset\}$. Subsets:

$$\emptyset, \quad \{a\}, \quad \{\{\emptyset\}\}, \quad \{a, \{\emptyset\}\}.$$

Hence

$$P(\{a, \{\emptyset\}\}) = \{\emptyset, \{a\}, \{\{\emptyset\}\}, \{a, \{\emptyset\}\}\}.$$

(d) $P(\{a, b, \{a, b\}\})$.

Solution. This set has three elements: a, b, and $\{a, b\}$. A 3-element set has $2^3 = 8$ subsets:

$$\emptyset,\ \{a\},\ \{b\},\ \{\{a, b\}\},\ \{a, b\},\ \{a, \{a, b\}\},\ \{b, \{a, b\}\},\ \{a, b, \{a, b\}\}.$$

So

$$P(\{a, b, \{a, b\}\}) = \{\emptyset, \{a\}, \{b\}, \{\{a, b\}\}, \{a, b\}, \{a, \{a, b\}\}, \{b, \{a, b\}\}, \{a, b, \{a, b\}\}\}.$$

(e) $P(P(\emptyset))$.

Solution. First $P(\emptyset) = \{\emptyset\}$. Then $P(\{\emptyset\})$ has subsets

$$\emptyset \quad \text{and} \quad \{\emptyset\}.$$

Hence

$$P(P(\emptyset)) = \{\emptyset, \{\emptyset\}\}.$$

5. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find

(a) $A \times B$;

Solution. By definition,

$$A \times B = \{(x, y) \mid x \in A, y \in B\}.$$

So we pair each element of A with each element of B:

$$A \times B = \{(a, y), (a, z), (b, y), (b, z), (c, y), (c, z), (d, y), (d, z)\}.$$

(b) $B \times A$.

Solution. Now

$$B \times A = \{(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)\}.$$

6. Let the universal set $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and let

$$A = \{0, 2, 4, 6, 8, 10\}, \quad B = \{0, 1, 2, 3, 4, 5, 6\}, \quad C = \{4, 5, 6, 7, 8, 9, 10\}.$$

For each set below, list the members and draw the corresponding Venn diagram.

(a) $A \cap B \cap C$.

Solution. First,

$$A \cap B = \{0, 2, 4, 6\},$$

then

$$\begin{aligned}(A \cap B) \cap C &= \{0, 2, 4, 6\} \cap \{4, 5, 6, 7, 8, 9, 10\} \\ &= \{4, 6\}.\end{aligned}$$

So $A \cap B \cap C = \{4, 6\}$.

(b) $A \cup B \cup C$.

Solution.

$$A \cup B = \{0, 1, 2, 3, 4, 5, 6, 8, 10\},$$

and

$$\begin{aligned}(A \cup B) \cup C &= \{0, 1, 2, 3, 4, 5, 6, 8, 10\} \cup \{4, 5, 6, 7, 8, 9, 10\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ &= U.\end{aligned}$$

So $A \cup B \cup C = U$.

(c) $(A \cup B) \cap C$.

Solution. From above, $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 8, 10\}$. Then

$$\begin{aligned}(A \cup B) \cap C &= \{0, 1, 2, 3, 4, 5, 6, 8, 10\} \cap \{4, 5, 6, 7, 8, 9, 10\} \\ &= \{4, 5, 6, 8, 10\}.\end{aligned}$$

(d) $(A \cap B) \cup C$.

Solution. We have $A \cap B = \{0, 2, 4, 6\}$, hence

$$\begin{aligned}(A \cap B) \cup C &= \{0, 2, 4, 6\} \cup \{4, 5, 6, 7, 8, 9, 10\} \\ &= \{0, 2, 4, 5, 6, 7, 8, 9, 10\}.\end{aligned}$$

(e) $(A \cup (B \cap C))^{\mathrm{c}}$ (complement taken in U).

Solution. First find $B \cap C$:

$$\begin{aligned}B \cap C &= \{0, 1, 2, 3, 4, 5, 6\} \cap \{4, 5, 6, 7, 8, 9, 10\} \\ &= \{4, 5, 6\}.\end{aligned}$$

Then

$$\begin{aligned} A \cup (B \cap C) &= \{0,2,4,6,8,10\} \cup \{4,5,6\} \\ &= \{0,2,4,5,6,8,10\}. \end{aligned}$$

The complement in U is

$$\begin{aligned} (A \cup (B \cap C))^{\mathrm{c}} &= U \setminus \{0,2,4,5,6,8,10\} \\ &= \{1,3,7,9\}. \end{aligned}$$

(f) $(B \cup C) \setminus A^{\mathrm{c}}$.

Solution. First compute $B \cup C$:

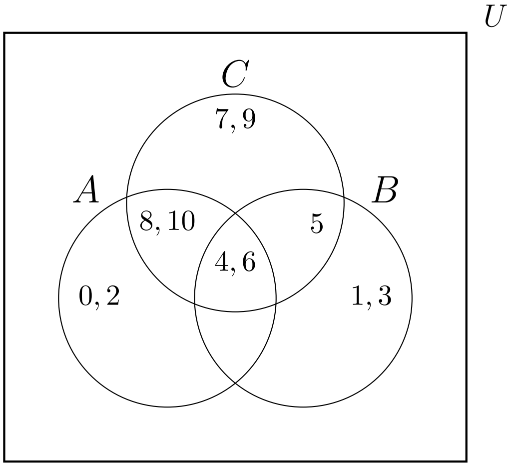
$$\begin{aligned} B \cup C &= \{0,1,2,3,4,5,6,7,8,9,10\} \\ &= U. \end{aligned}$$

Next $A^{\mathrm{c}} = U \setminus A = \{1,3,5,7,9\}$. The set difference $X \setminus Y$ equals $X \cap Y^{\mathrm{c}}$, so

$$\begin{aligned} (B \cup C) \setminus A^{\mathrm{c}} &= (B \cup C) \cap (A^{\mathrm{c}})^{\mathrm{c}} \\ &= U \cap A \\ &= A \\ &= \{0,2,4,6,8,10\}. \end{aligned}$$

Venn diagram for A, B, and C in U.

The diagram below shows all elements of U placed in their corresponding regions with respect to A, B, and C.



7. Suppose that A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

Proof. To show $A \subseteq C$, we must show that every element of A is also an element of C.

Let x be an arbitrary element of A. Since $A \subseteq B$, we have $x \in B$. Since $B \subseteq C$, every element of B is in C, so $x \in C$.

As x was arbitrary in A, this shows $A \subseteq C$. $\square$

8. Find the sets A and B if

$$A \setminus B = \{1, 5, 7, 8\}, \quad B \setminus A = \{2, 10\}, \quad A \cap B = \{3, 6, 9\}.$$

Solution. The set A is the disjoint union of the elements that are only in A and those in $A \cap B$:

$$\begin{aligned} A &= (A \setminus B) \cup (A \cap B) \\ &= \{1, 5, 7, 8\} \cup \{3, 6, 9\} \\ &= \{1, 3, 5, 6, 7, 8, 9\}. \end{aligned}$$

Similarly, B consists of elements only in B and those in $A \cap B$:

$$\begin{aligned} B &= (B \setminus A) \cup (A \cap B) \\ &= \{2, 10\} \cup \{3, 6, 9\} \\ &= \{2, 3, 6, 9, 10\}. \end{aligned}$$

9. Show that if A and B are sets, then $A \setminus B = A \cap B^{\mathrm{c}}$.

Proof. We prove both inclusions.

($\subseteq$) Let $x \in A \setminus B$. By definition of set difference, $x \in A$ and $x \notin B$. The statement $x \notin B$ means $x \in B^{\mathrm{c}}$. Hence $x \in A$ and $x \in B^{\mathrm{c}}$, so $x \in A \cap B^{\mathrm{c}}$.

($\supseteq$) Let $x \in A \cap B^{\mathrm{c}}$. Then $x \in A$ and $x \in B^{\mathrm{c}}$, which means $x \notin B$. Thus $x \in A$ and $x \notin B$, so $x \in A \setminus B$.

Since the two sets are subsets of each other, we conclude

$$A \setminus B = A \cap B^{\mathrm{c}}.$$

$\square$

10. What can you say about the sets A and B if we know that

(a) $A \cup B = A$?

Solution. If $A \cup B = A$, then adding B does not introduce any new elements beyond those in A. Formally, let $x \in B$. Then $x \in A \cup B = A$, so $x \in A$. Therefore $B \subseteq A$.

(b) $A \cap B = A$?

Solution. If $A \cap B = A$, then intersecting with B does not remove any elements from A. Let $x \in A$. Then $x \in A \cap B$, so $x \in B$. Hence $A \subseteq B$.

(c) $A \setminus B = A$?

Solution. Using $A \setminus B = A \cap B^{\mathrm{c}}$, we have

$$A \cap B^{\mathrm{c}} = A.$$

This means that every element of A lies in B^{c}, i.e. no element of A lies in B. So

$$A \cap B = \emptyset,$$

i.e. A and B are disjoint.

(d) $A \cap B = B \cap A$?

Solution. This holds for *all* sets A and B, since set intersection is commutative. Therefore this condition does not give any additional information about A and B.

(e) $A \setminus B = B \setminus A$?

Solution. Suppose $A \setminus B = B \setminus A$.

We first show $A \subseteq B$. Let $x \in A$. If $x \notin B$, then $x \in A \setminus B$. Since $A \setminus B = B \setminus A$, we also have $x \in B \setminus A$, which implies $x \in B$ and $x \notin A$ — a contradiction. Hence our assumption that $x \notin B$ is false, so $x \in B$. Thus $A \subseteq B$.

Similarly, by symmetry, we can show $B \subseteq A$.

Therefore $A = B$.

11. In a group of 343 employees in a company, the following information is known:

- 120 employees are managers.
- 135 employees are engineers.
- 80 employees are in the sales department.
- 50 employees are both managers and engineers.
- 30 employees are both managers and in the sales department.
- 25 employees are both engineers and in the sales department.
- 15 employees are managers, engineers, and in the sales department.

How many employees are neither managers, engineers, nor in the sales department? Draw a Venn diagram to show each of the sets and provide a brief explanation.

Solution. Let M, E, and S be the sets of managers, engineers, and salespeople respectively. We are given:

$$|M| = 120, \quad |E| = 135, \quad |S| = 80,$$
$$|M \cap E| = 50, \quad |M \cap S| = 30, \quad |E \cap S| = 25, \quad |M \cap E \cap S| = 15.$$

We first find the numbers in each of the 7 non-empty regions of the three-set Venn diagram.

- Managers, engineers, and sales: $|M \cap E \cap S| = 15$.
- Managers and engineers only (not sales):

$$\begin{aligned} |M \cap E \text{ only}| &= |M \cap E| - |M \cap E \cap S| \\ &= 50 - 15 \\ &= 35. \end{aligned}$$

- Managers and sales only (not engineers):

$$\begin{aligned} |M \cap S \text{ only}| &= |M \cap S| - |M \cap E \cap S| \\ &= 30 - 15 \\ &= 15. \end{aligned}$$

- Engineers and sales only (not managers):

$$\begin{aligned} |E \cap S \text{ only}| &= |E \cap S| - |M \cap E \cap S| \\ &= 25 - 15 \\ &= 10. \end{aligned}$$

- Managers only (not engineers and not sales):

$$\begin{aligned} |M \text{ only}| &= |M| - \big(|M \cap E \text{ only}| + |M \cap S \text{ only}| + |M \cap E \cap S|\big) \\ &= 120 - (35 + 15 + 15) \\ &= 55. \end{aligned}$$

- Engineers only:

$$\begin{aligned}|E \text{ only}| &= |E| - \big(|M\cap E \text{ only}| + |E\cap S \text{ only}| + |M\cap E\cap S|\big)\\ &= 135 - (35+10+15)\\ &= 75.\end{aligned}$$

- Sales only:

$$\begin{aligned}|S \text{ only}| &= |S| - \big(|M\cap S \text{ only}| + |E\cap S \text{ only}| + |M\cap E\cap S|\big)\\ &= 80 - (15+10+15)\\ &= 40.\end{aligned}$$

The total number of employees in $M\cup E\cup S$ is

$$\begin{aligned}|M\cup E\cup S| &= |M \text{ only}| + |E \text{ only}| + |S \text{ only}|\\ &\quad + |M\cap E \text{ only}| + |M\cap S \text{ only}|\\ &\quad + |E\cap S \text{ only}| + |M\cap E\cap S|\\ &= 55+75+40+35+15+10+15\\ &= 245.\end{aligned}$$

The total number of employees is 343, so the number of employees who are in none of the three sets (neither managers, engineers, nor sales) is

$$343 - 245 = 98.$$

Answer: 98 employees are neither managers, engineers, nor in the sales department.

Venn diagram for M, E, and S.