Lecture 02: Probability Theory

INF1004 Mathematics II

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1 Types of Probability

Probability is defined as a number in the range $[0, 1]$ that is assigned to an event associated with a random experiment. There are two interpretations of probability theory that guide how such values are assigned:

1.1 Classical Probability

In classical probability, all outcomes in the sample space are assumed to be equally likely. The probability of a random event A occuring is given as

$$P(A) = \frac{\text{Number of different outcomes in A}}{\text{Total number of possible outcomes}}$$

Note: This definition is not very practical for everyday life because it assumes the outcomes are equally likely.

1.2 Empirical Probability

In empirical probability theory, the probability of an event A occurring approximated by the relative frequency of its occurrence over a large number of trials such that

$$P(A) = \frac{\text{Number of times an outcome occurs}}{\text{Total number of observations}}$$

This interpretation relies on collecting data over the years or existing historical data to propose such relative frequencies.

1.3 Definitions

- We define a sample space S as the set of all possible outcomes of a random phenomenon.
- An event A is defined as the set of outcomes under investigation.
- By definition of probability, the following statements must hold to be valid:

$$0 \leq P(A) \leq 1, \ P(S) = 1$$

- If $A_1, A_2, \ldots$ are mutually exclusive events, then

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

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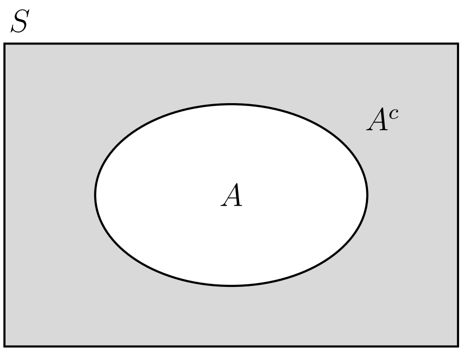
2 Complementary Events

The complement of an event A denoted $P(A^c)$ and is given by

$$P(A^c) = P(S) - P(A)$$

Conceptually, the complement of an event can be thought of as the probability that it does not occur. Visually, this can be represented as the area in a venn diagram that is not in A.

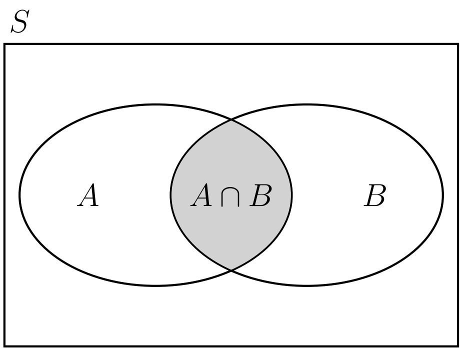


3 Intersection of Events

Given two events A and B, the probability of their intersection denoted

$$P(A \cap B)$$

is the probability that both events occur. In a venn diagram, it is the area where both events overlap.

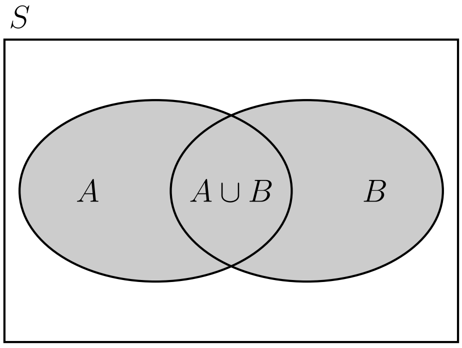


If two events A and B are mutually exclusive, then $P(A \cap B) = 0$ because both events never overlap.

4 Union of Events

Given two events A and B, the *union* of these events, denoted $A \cup B$, is the event that *at least one* of the events occurs. That is, $A \cup B$ occurs if event A occurs, or event B occurs, or both occur.

In terms of probability, $P(A \cup B)$ represents the probability that at least one of the two events occurs. Visually, in a Venn diagram, the union corresponds to the total area covered by both events A and B.



In general, the probability of the union of two events is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

This formula accounts for the fact that the overlapping region $A \cap B$ is included in both $P(A)$ and $P(B)$, and therefore must be subtracted once to avoid double counting.

4.1 Mutually Exclusive Events

If events A and B are *mutually exclusive*, then they cannot occur at the same time. In this case,

$$P(A \cap B) = 0,$$

and the formula simplifies to

$$P(A \cup B) = P(A) + P(B).$$

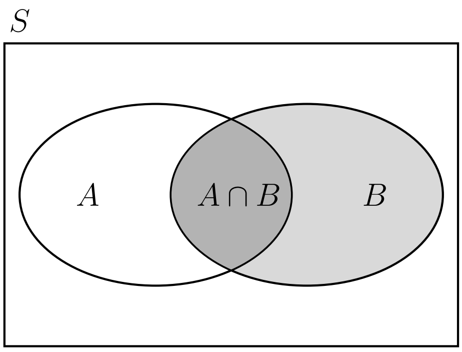
5 Conditional Probabilities

In practice, it happens naturally that probability of an event A can be changed if additional information is available, which leads to the discovery/creation of *conditional probability*.

5.1 Definition

The conditional probability of event A given that event B has occurred, denoted $P(A \mid B)$, is defined (for $P(B) > 0$) as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$



Note: Conditional probability is *not symmetric*. In general,

$$P(A \mid B) \neq P(B \mid A),$$

because the events A and B may have different probabilities and provide different information when conditioned upon.

Additionally, when working with conditional probability $P(A \mid B)$, the sample space is no longer the original sample space S. Instead, the sample space is restricted to event B. All probabilities are therefore measured *relative to* B.

6 Multiplicative Rule of Probability

The definition of conditional probability provides a way to compute the intersection of events A and B. Since,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}.$$

We have that,

$$P(A \cap B) = P(A \mid B) \cdot P(B),$$
$$P(A \cap B) = P(A \mid B) \cdot P(A).$$

7 Independent and Mutually Exclusive Events

7.1 Independent Events

Two events A and B are said to be **independent** if

$$P(A \cap B) = P(A)\,P(B).$$

Equivalently, provided the relevant probabilities are non-zero,

$$P(A \mid B) = P(A) \quad \text{or} \quad P(B \mid A) = P(B).$$

That is, the occurrence of one event does not affect the probability of the other. Using the definition of conditional probability,

$$P(A \cap B) = P(A \mid B)\,P(B).$$

If A and B are independent, then $P(A \mid B) = P(A)$, and hence

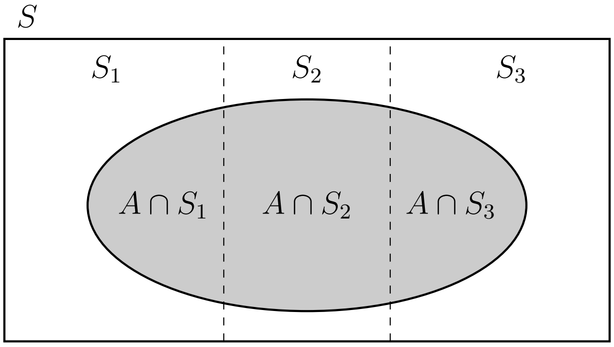
$$P(A \cap B) = P(A)\,P(B).$$

This result is known as the **multiplication rule for independent events**.

More generally, if $A_1, A_2, \ldots, A_n$ are mutually independent events, then the probability that all events occur is

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i).$$

8 Law of Total Probability



9 Bayes' Theorem