Lecture 02: Probability Theory

INF1004 Mathematics II

Name: Timothy Chia **Date:** 12/01/2025

1 Types of Probability

Probability is defined as a number in the range $[0, 1]$ that is assigned to an event associated with a random experiment. There are two interpretations of probability theory that guide how such values are assigned:

1.1 Classical Probability

In classical probability, all outcomes in the sample space are assumed to be equally likely. The probability of a random event A occuring is given as

$$P(A) = \frac{\text{Number of different outcomes in A}}{\text{Total number of possible outcomes}}$$

Note: This definition is not very practical for everyday life because it assumes the outcomes are equally likely.

1.2 Empirical Probability

In empirical probability theory, the probability of an event A occurring approximated by the relative frequency of its occurrence over a large number of trials such that

$$P(A) = \frac{\text{Number of times an outcome occurs}}{\text{Total number of observations}}$$

This interpretation relies on collecting data over the years or existing historical data to propose such relative frequencies.

1.3 Definitions

- We define a sample space S as the set of all possible outcomes of a random phenomenon.
- An event A is defined as the set of outcomes under investigation.
- By definition of probability, the following statements must hold to be valid:

$$0 \leq P(A) \leq 1,\ P(S) = 1$$

- If $A_1, A_2, \ldots$ are mutually exclusive events, then

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

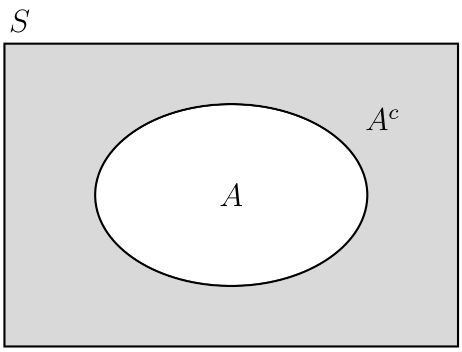
.

2 Complementary Events

The complement of an event A denoted $P(A^c)$ and is given by

$$P(A^c) = P(S) - P(A)$$

Conceptually, the complement of an event can be thought of as the probability that it does not occur. Visually, this can be represented as the area in a venn diagram that is not in A.

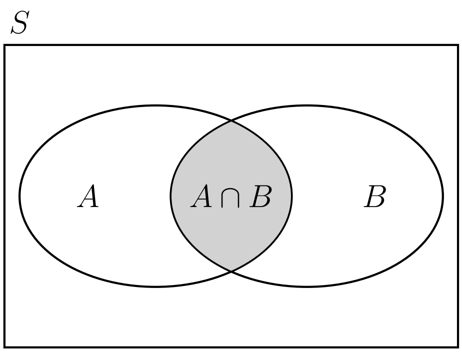


3 Intersection of Events

Given two events A and B, the probability of their intersection denoted

$$P(A \cap B)$$

is the probability that both events occur. In a venn diagram, it is the area where both events overlap.

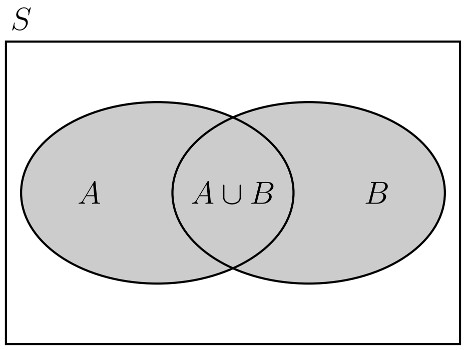


If two events A and B are mutually exclusive, then $P(A \cap B) = 0$ because both events never overlap.

4 Union of Events

Given two events A and B, the *union* of these events, denoted $A \cup B$, is the event that *at least one* of the events occurs. That is, $A \cup B$ occurs if event A occurs, or event B occurs, or both occur.

In terms of probability, $P(A \cup B)$ represents the probability that at least one of the two events occurs. Visually, in a Venn diagram, the union corresponds to the total area covered by both events A and B.



In general, the probability of the union of two events is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

This formula accounts for the fact that the overlapping region $A \cap B$ is included in both $P(A)$ and $P(B)$, and therefore must be subtracted once to avoid double counting.

4.1 Mutually Exclusive Events

If events A and B are *mutually exclusive*, then they cannot occur at the same time. In this case,

$$P(A \cap B) = 0,$$

and the formula simplifies to

$$P(A \cup B) = P(A) + P(B).$$

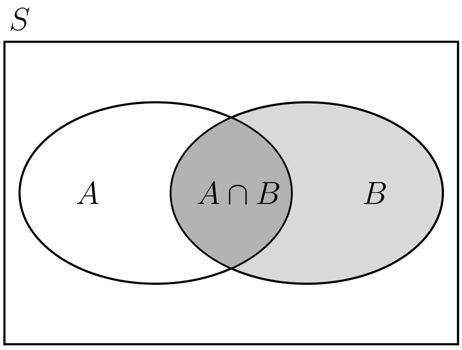
5 Conditional Probabilities

In practice, it happens naturally that probability of an event A can be changed if additional information is available, which leads to the discovery/creation of *conditional probability*.

5.1 Definition

The conditional probability of event A given that event B has occurred, denoted $P(A \mid B)$, is defined (for $P(B) > 0$) as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$



Note: Conditional probability is *not symmetric*. In general,

$$P(A \mid B) \neq P(B \mid A),$$

because the events A and B may have different probabilities and provide different information when conditioned upon.

Additionally, when working with conditional probability $P(A \mid B)$, the sample space is no longer the original sample space S. Instead, the sample space is restricted to event B. All probabilities are therefore measured *relative to* B.

6 Multiplicative Rule of Probability

The definition of conditional probability provides a way to compute the intersection of events A and B. Since,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}.$$

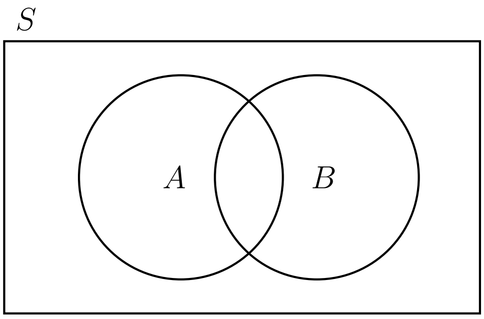
We have that,

$$P(A \cap B) = P(A \mid B) \cdot P(B),$$
$$P(A \cap B) = P(A \mid B) \cdot P(A).$$

7 Independent and Mutually Exclusive Events

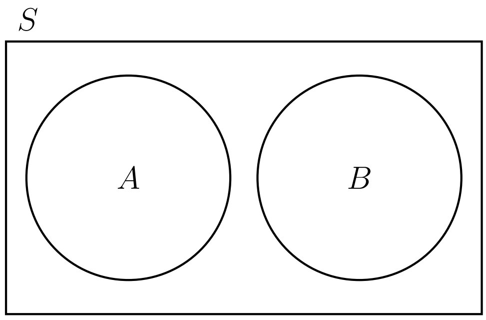
Events may be related in different ways depending on how the occurrence of one affects another. Two important relationships are *independence* and *mutual exclusivity.* These concepts are fundamentally different and should not be confused.

Independent Events



Both events may occur, but the occurrence of one does not affect the probability of the other.

Mutually Exclusive Events



Events cannot occur at the same time, so $A \cap B = \varnothing$. (e.g. If A happens, B cannot—so learning A occurred rules B out.)

Note: Independent events may overlap, whereas mutually exclusive events never do.

7.1 Independent Events

Two events A and B are said to be **independent** if

$$P(A \cap B) = P(A)\,P(B).$$

Equivalently, provided the relevant probabilities are non-zero,

$$P(A \mid B) = P(A) \quad \text{or} \quad P(B \mid A) = P(B).$$

That is, the occurrence of one event does not affect the probability of the other. Using the definition of conditional probability,

$$P(A \cap B) = P(A \mid B)\,P(B).$$

If A and B are independent, then $P(A \mid B) = P(A)$, and hence

$$P(A \cap B) = P(A)\,P(B).$$

This result is known as the **multiplication rule for independent events**.

More generally, if $A_1, A_2, \ldots, A_n$ are mutually independent events, then the probability that all events occur is

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i).$$

8 Law of Total Probability

The **Law of Total Probability** allows us to compute the probability of an event by decomposing the sample space into several mutually exclusive and exhaustive cases.

8.1 Definition

Suppose $S_1, S_2, \ldots, S_n$ form a **partition** of the sample space S, meaning:

- $S_i \cap S_j = \varnothing$ for $i \neq j$ (mutually exclusive),
- $\displaystyle\bigcup_{i=1}^{n} S_i = S$ (exhaustive),
- $P(S_i) > 0$ for all i.

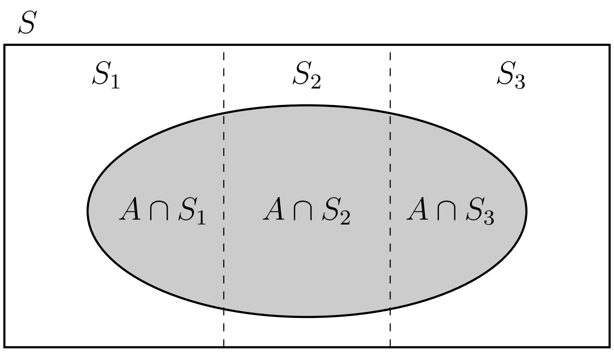
Then for any event $A \subseteq S$, we may write A as the union of its intersections with each S_i:

$$A = (A \cap S_1) \cup (A \cap S_2) \cup \cdots \cup (A \cap S_n).$$

Since these intersections are mutually exclusive, their probabilities add, giving

$$P(A) = P(A \cap S_1) + P(A \cap S_2) + \cdots + P(A \cap S_n).$$

This concept can be visualised with the following Venn diagram.



Using the definition of conditional probability,

$$P(A \cap S_i) = P(A \mid S_i)\, P(S_i).$$

Substituting this into the previous expression yields the **Law of Total Probability**:

$$P(A) = \sum_{i=1}^{n} P(A \mid S_i)\, P(S_i).$$

9 Bayes' Theorem

Bayes' Theorem provides a way to reverse conditional probabilities. In particular, it allows us to compute $P(B \mid A)$ in terms of $P(A \mid B)$, together with prior information about B. From the definition of conditional probability, we have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B \mid A) = \frac{P(A \cap B)}{P(A)}.$$

Rearranging the first expression gives

$$P(A \cap B) = P(A \mid B)\, P(B).$$

Substituting this into the second expression yields **Bayes' Theorem**:

$$P(B \mid A) = \frac{P(A \mid B)\, P(B)}{P(A)}.$$

In practice, the probability $P(A)$ in the denominator is often unknown directly. It is commonly computed using the **Law of Total Probability**.

Suppose B and B^c form a partition of the sample space, meaning they are mutually exclusive and exhaustive. Then

$$P(A) = P(A \mid B)\, P(B) + P(A \mid B^c)\, P(B^c).$$

Substituting this into Bayes' Theorem gives the expanded form:

$$P(B \mid A) = \frac{P(A \mid B)\, P(B)}{P(A \mid B)\, P(B) + P(A \mid B^c)\, P(B^c)}.$$

This formulation highlights how Bayes' Theorem updates the probability of an event using new information, weighted by prior probabilities.