CCN1048 Linear Algebra Final Reminder

Going through from chapter 1 to chapter 10, part of them will not be in detailed. Please be prepared to read this reminder. Good luck.

* is something important

is something out of syllabus.

CCN1048 Linear Algebra Final Reminder

Contents

CCN1048 Linear Algebra Final Reminder

Augmented Matrix

$$\left[\begin{array}{c|c} coefficient & constant \\ matrix & matrix \end{array}\right]$$

System of Linear Equations

Consistent: ≥ 1 solutions

Independent: 1 solution (unique solution)

Dependent: $>$ 1 solution

Inconsistent: no solution

Homogeneous: constant matrix is $(0\ 0\ \dots\ 0)^T$

Consistent:

Trivial solution: e.g. $(x, y, z) = (0,0,0)$

Non-trivial solution: other solutions

Only 2 possibilities:

a. Has only the trivial solution
b. Has infinitely many solutions in addition to the trivial solution.

*A linear system MUST have

$$\left[\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & i & l \end{array}\right]$$

a. Exactly one solution (consistent and independent)

$$\begin{cases} a \neq 0 \\ e \neq 0 \\ i \neq 0 \end{cases}$$

b. No solution (inconsistent)

$$\begin{cases} i = 0 \\ l \neq 0 \end{cases} or\ e = 0$$

c. Infinitely many solutions (consistent and dependent)

$$\begin{cases} i = l = 0 \\ e \neq 0 \end{cases}$$

Elementary Row Operation

$R_i \leftrightarrow R_j$

$$R_1 \leftrightarrow R_2, E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E^{-1}$$

$$\det(E) = -1 = \det(E^{-1})$$

$kR_i \to R_j$

$$2R_3 \to R_3, E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\det(E) = 2, \det(E^{-1}) = 1/2$$

$R_i + kR_j \to R_j$

$$R_2 - 3R_1 \to R_2, E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(E) = 1 = \det(E^{-1})$$

For

$$A \overset{E_1 E_2 E_3}{\to\to\to} B$$

We can write

$$E_3 E_2 E_1 A = B, A = E_1^{-1} E_2^{-1} E_3^{-1} B$$

Matrix Algebra

There are serval types of matrices, here are the most commonly used:

Rows Matrix

Only 1 row.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Columns Matrix

Only 1 column.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Rectangular Matrix

Number of rows is not equal to number of columns

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Square Matrix

Number of rows is equal to number of columns

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Diagonal Matrix

A square matrix that at least one element of principal diagonal is non-zero and all the other elements are zero.

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Note that D^2 is still a diagonal matrix and it is equal to diagonal to the power of 2.

Pre-multiplication

$$D\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 9 \end{bmatrix}$$

Post-multiplication

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}D = \begin{bmatrix} 0 & 10 & 9 \end{bmatrix}$$

Scalar Matrix

A diagonal matrix that all its diagonal elements are the same.

$$\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Identity Matrix

A diagonal matrix that all its diagonal elements are equal to 1.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Upper Triangular Matrix

A square matrix and all its elements below the diagonal are zero.

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

*Note that U^2 is still an upper triangular matrix.

$$\det(U) = adf$$

Lower Triangular Matrix

A square matrix and all its elements above the diagonal are zero.

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

*Note that L^2 is still a lower triangular matrix.

$$\det(L) = acf$$

Null Matrix

All its elements are equal to 0.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Trace of a Matrix

Trace of a square matrix is defined to be the sum of the elements on the main diagonal.

$$tr(A+B) = tr(A) + tr(B)$$

$$tr(cA) = c\ tr(A)$$

$$tr(A) = tr(A^T)$$

$$tr(AB) = tr(BA)$$

$$tr(P^{-1}AP) = tr(A)$$

** In general,

$$tr(ABC) \neq tr(ACB)$$

However, if A, B, C are symmetric matrices, the above equation is true.

Proof:

$tr(ABC)$

$= tr(A^T B^T C^T) = tr(A^T (CB)^T) = tr((CB)^T A^T) = tr((ACB)^T) = tr(ACB)$

Transpose of a Matrix

The matrix resulting from interchanging the rows and columns in the given matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$(A^T)^T = A$$

$$(A \pm B)^T = A^T \pm B^T$$

$$(kA)^T = kA^T$$

$$(AB)^T = B^T A^T$$

Inverse of a matrix

if A and B are square matrices such that $AB = BA = I$.

$$A^{-1} = B \text{ and } B^{-1} = A$$

Invertible = nonsingular = $\det A \neq 0$

Not invertible = singular = $\det A = 0$

A matrix is invertible if and only if determinant is not equal to 0

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$A^{-k} = (A^{-1})^k = (A^k)^{-1}$$

$$(A^{-1})^{-1} = A$$

$$(kA)^{-1} = k^{-1}A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

** Nearly all the prove related to inverse will be based on

$$AB = I \text{ and } BA = I$$

Way to find Inverse

$[A|I] \sim [I|B] \Rightarrow A^{-1} = B$

If the I on r.h.s. cannot be formed, inverse does not exist.

$A^{-1} = \det(A)^{-1}\ adj(A)$

Determinants of matrices

** Always try to find determinants using

$$R_1 + R_2 + \cdots + R_n \to R_1 \; or \; C_1 + C_2 + \cdots + C_n \to C_1$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Using calculator's program to find determinants is suggested.

$$\det(I) = 1$$

$$\det(A^T) = \det(A)$$

$$\det(A^{-1}) = \det(A)^{-1}$$

$$\det(cA) = c^k \det(A) \text{ for a } k \times k \text{ matrix } A$$

$$\det(EB) = \det(E)\det(B)$$

For square matrices A and B of equal size,

$$\det(AB) = \det(A)\det(B)$$

Row interchange or Column interchange

$$\det(B) = -\det(A)$$

Adjoint of Matrix

1x1 generic matrix

The adjoint of the above matrix is I

2x2 generic matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3x3 generic matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}^T$$

** Properties:

for $n \times n$ matrices A and B:

$$adj(I) = I$$

$$adj(A^T) = \left(adj(A)\right)^T$$

$$adj(AB) = adj(B)\,adj(A)$$

Prove:

$$adj(B)adj(A) = \det(B)\,B^{-1}\det(A)\,A^{-1} = \det(AB)\,(AB)^{-1} = adj(AB)$$

$$adj(cA) = c^{n-1}\,adj(A)$$

$$adj(A^k) = adj(A)^k, k \in \mathbb{Z}$$

The prove using $adj(AB) = adj(B)\,adj(A)$ with $B = A^{k-1}$ and performing recursion.

For A is an $n \times n$ matrix with $n \geq 2$,

$$\det\big(adj(A)\big) = \det(A)^{n-1}$$

For A is an invertible $n \times n$ matrix,

$$adj\big(adj(A)\big) = \det(A)^{(n-2)}\, A$$

If A is invertible,

$$A\, adj(A) = \det(A)\, I$$

$$adj(A) = \det(A)\, A^{-1}$$

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Cayley-Hamilton formula

2x2 case

$$adj(A) = I_2\, tr(A) - A$$

$$A^2 - tr(A)A + \det(A)\, I_2 = 0$$

Symmetric Matrix

The square matrix itself is equal to the transpose of itself.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 6 \\ -3 & 6 & 4 \end{bmatrix} = A^T$$

If B has eigenvectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$ and form an orthonormal set of vectors in R^n, there will have an orthogonal matrix Q such that $Q^T B Q = D$ where D is a diagonal matrix.

Orthogonally Diagonalization Algorithm of Symmetric Matrix A

Apply Diagonalization Algorithm

If the collection does not form an orthogonal set, apply projection method

Normalize the orthogonal set

Q be the matrix whose columns are the eigenvectors from the orthonormal set

Skew-symmetric Matrix

The square matrix itself is equal to the negative of its transpose with diagonal elements are equal to 0.

$$A=\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}=-A^T$$

Commutative and anti-commutative matrices

Commute:

$$AB=BA$$

Anti-commute:

$$AB=-BA$$

Periodic matrix

For

$$A^{k+1}=A, k\in\mathbb{Z}^+$$

The least positive integer of k is the period of A.

if $k=1$, so that

$$A^2=A$$

then A is called idempotent.

Nilpotent matrix

For

$$A^p=0. p\in\mathbb{Z}^+$$

The least positive integer of p for $A^p=0$. A is said to be nilpotent of index p.

Orthogonal matrix

A square matrix with columns and rows are orthogonal unit vectors (i.e. orthonormal vectors)

$$A^T A = AA^T = I$$

i.e. $A^{-1} = A^T$

$$1 = \det(I) = \det(A^T A) = \det(A^T)\det(A) = (\det(A))^2$$

The rows and columns of A form an orthonormal set.

Solving System of Linear Equations

Any free variable, e.g. t, remember to write what t is an element to.

For example:

solution set = $\{(250 - 4t, 3t - 100, t) | t \in \mathbb{N}\ AND\ 34 \leq t \leq 62\}$

$(x_1, x_2, x_3) = \{(5 + 3t, 2 + t, t) | t \in \mathbb{R}\}$

For application question, remember the range of the answer, apply the range to the answer you found and carry on elimination.

Row echelon form

It just like the lower triangular part of the coefficient matrix of the augment matrix are 0.

Reduced row echelon form

Just like row echelon form, with all entry above the leading variables are also 0.

Gaussian Elimination

Make it become row echelon form by doing elementary row operation.

Apply back-substitution.

Gauss-Jordan Elimination

Make it become reduced row echelon form by doing elementary row operation.

Using Inverse

Only applicable to independent system.

$$Ax = b \Rightarrow x = A^{-1}b$$

LU-factorization

$$Ax = b$$

First

$$A \overset{E_1 E_2 E_3}{\rightarrow\rightarrow\rightarrow} U$$

$$E_3 E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} U = LU$$

Therefore

$$LUx = b$$

Let

$$y = Ux$$

$$i.e. Ly = b$$

Solve

$$Ly = b$$

Then solve

$$Ux = y$$

** x is the final answer required.

Cramer's Rule

$$Ax = b$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 1 & b & c \\ 2 & e & f \\ 3 & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, x_2 = \frac{\begin{vmatrix} a & 1 & c \\ d & 2 & f \\ g & 3 & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, x_3 = \frac{\begin{vmatrix} a & b & 1 \\ d & e & 2 \\ g & h & 3 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

Vectors in Coordinate System

Given $P_1(x_1, y_1)\ and\ P_2(x_2, y_2)$

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$

For $\vec{v} = (1, -3, 2)$

$$\vec{v} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = (1 \quad -3 \quad 2)^T$$

Norm

For $\vec{u} = (u_1, u_2, \dots, u_n)$ and $\vec{v} = (v_1, v_2, \dots, v_n)$

$$||\vec{u} - \vec{v}|| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

$$||\vec{0}|| = 0$$

$$||k\vec{v}|| = |k|||\vec{v}||$$

Unit Vector

Normalization:

$$\hat{u} = \frac{1}{||\vec{v}||}\vec{v}$$

Dot Product (Euclidean inner product)

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta$$

$$|\vec{u} \cdot \vec{v}| \le ||\vec{u}||||\vec{v}||$$

$$\vec{u} \cdot \vec{v} = (u_1, u_2, \dots, u_n)(v_1, v_2, \dots, v_n)^T = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

$$\vec{u} \cdot \vec{u} = ||\vec{u}||^2$$

The angle θ between $\vec{u}$ and $\vec{v}$ satisfies $0 \le \theta \le \pi$

Parallelogram equation for vectors

$$||\vec{u} + \vec{v}||^2 + ||\vec{u} - \vec{v}||^2 = 2||\vec{u}||^2 + 2||\vec{v}||^2$$

Orthogonality

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||||\vec{v}||}\right)$$

$$\theta = {}^{\pi}/_{2} \text{ if and only if } \vec{u}\cdot\vec{v} = 0$$

** Show they are orthogonal vectors = prove dot product = 0

Normal

$$\vec{n}\cdot\overrightarrow{P_0P} = 0$$

using orthogonal projection, and the vector minus its orthogonal projection is orthogonal to the one projected on it.

For the normal of plane (ABC),

$$\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC}$$

Orthogonal Projection

The orthogonal projection of $\vec{v}$ onto $\vec{w}$

$$(\vec{v}\cdot\widehat{w})\widehat{w} \qquad [(magnitude)direction]$$

i.e. $\vec{v}-(\vec{v}\cdot\widehat{w})\widehat{w}$ is orthogonal onto $\vec{w}$

Orthogonal Projection line onto Plane

The orthogonal projection of $\vec{v}$ onto plane of $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC}$$

The orthogonal projection is

$$\vec{v}-(\vec{v}\cdot\hat{n})\hat{n}$$

Line and Plane

Equation of line:

$$f(x, y) = ax + by + c$$

Equation of plane:

$$f(x, y, z) = ax + by + cz + d$$

Calculate distance between Point P_0 and line or plane

$$D = \frac{|f(P_0)|}{||\vec{n}||}$$

Calculate distance between two lines

* distance only define when two lines are parallel

$$D = \frac{|c_2 - c_1|}{||\vec{n}||}$$

Calculate distance between line and plane

* distance only define when line is parallel to plane

i.e. find a point on the line and let it be P_0

$$D = \frac{|f(P_0)|}{||\vec{n}||}$$

Calculate distance between two planes

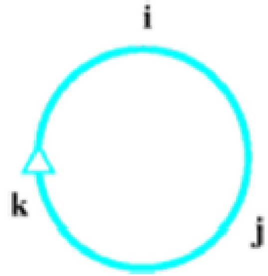
* distance only define when two planes are parallel

$$D = \frac{|d_2 - d_1|}{||\vec{n}||}$$

Cross Product

For $\vec{u} = (u_1 \quad u_2 \quad u_3)^T$ and $\vec{v} = (v_1 \quad v_2 \quad v_3)^T$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$



$$\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = 0$$

$$||\vec{u} \times \vec{v}||^2 = ||\vec{u}||^2 ||\vec{v}||^2 - (\vec{u} \cdot \vec{v})^2$$

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

$$(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{v} \cdot \vec{w})\vec{u}$$

The cross product is orthogonal to both the original vector.

Area of Parallelogram

$$||\vec{u} \times \vec{v}||$$

unit is $unit^2$

The area of the triangle bounded by $\vec{u}$ and $\vec{v}$ are

$$\frac{||\vec{u} \times \vec{v}||}{2}$$

Scalar Triple Product

$$\vec{u} \cdot (\vec{v} \times \vec{w})$$

$|\vec{u} \cdot (\vec{v} \times \vec{w})|$ is the volume of the parallelepiped with unit is $unit^3$

The 3 vectors lie in the same plane if and only if $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$

Linear Independence

Determination

Determination of linearly dependent or linearly independent

Form the augmented matrix by $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots = 0$

Applying row operations, if the solution set contain free variable, it is linear dependent, otherwise it is linear independent.

Orthogonal and Orthonormal Set

* $\vec{0}$ is orthogonal to any vector.

An orthogonal set of vectors means they are mutually orthogonal to each other.

The only way to show a set of vectors is an orthogonal set is to compile dot product between all vectors inside the set.

Projection method to transform non-orthogonal set of vectors into an orthogonal set.

Let $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ be the non-orthogonal set of vectors

$\overrightarrow{v_1}$ be the base vector.

$$\overrightarrow{q_2} = \overrightarrow{v_2} - (\overrightarrow{v_2} \cdot \widehat{v_1})\widehat{v_1}$$

$$\overrightarrow{q_3} = \overrightarrow{v_3} - (\overrightarrow{v_3} \cdot \widehat{v_1})\widehat{v_1} - (\overrightarrow{v_3} \cdot \widehat{q_2})\widehat{q_2}$$

$\{\overrightarrow{v_1}, \overrightarrow{q_2}, \overrightarrow{q_3}\}$ is an orthogonal set of vectors.

Show it is an orthonormal set:

Dot product with each other = 0

Dot product with itself = 1

Eigenvalues and Eigenvectors

Eigenvalues

λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$

Diagonalizability

Square matrix A is diagonalizable if a nonsingular matrix P such that $P^{-1}AP$ is diagonal matrix.

Diagonalization Algorithm

Solve $\det(A - \lambda I)$

Sub λ into $A - \lambda I$ and solve $[A - \lambda I|0]$

The number of eigenvectors need to be the same as the number of columns of matrix A, otherwise it is not diagonalizable.

$$P = \left[\begin{bmatrix} \overrightarrow{v_1} \end{bmatrix} \quad \begin{bmatrix} \overrightarrow{v_2} \end{bmatrix} \quad \begin{bmatrix} \overrightarrow{v_3} \end{bmatrix}\right]$$

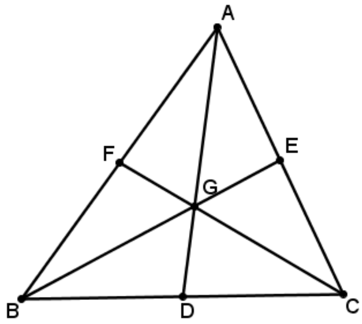
$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

if $P^{-1}AP = D$, then $A = PDP^{-1}$

This implies $A^m = PD^mP^{-1}$

Enrichment

Vector of Centroid



$$\overrightarrow{AG} = \frac{\overrightarrow{AA} + \overrightarrow{AB} + \overrightarrow{AC}}{3} = \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$$

$$\overrightarrow{BG} = \frac{\overrightarrow{BA} + \overrightarrow{BB} + \overrightarrow{BC}}{3} = \frac{1}{3}\overrightarrow{BA} + \frac{1}{3}\overrightarrow{BC}$$

$$\overrightarrow{CG} = \frac{\overrightarrow{CA} + \overrightarrow{CB} + \overrightarrow{CC}}{3} = \frac{1}{3}\overrightarrow{CA} + \frac{1}{3}\overrightarrow{CB}$$

Matrix Binomial Theorem

Suppose $AB = BA$,

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$$

Cayley-Hamilton's Theorem

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We have

$$A^2 - tr(A)A + \det(A)\, I = 0$$

Proof:

$A^2 - tr(A)A + \det(A)\, I$

$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} a^2+bc & ab+bc \\ ac+cd & d^2+bc \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bc \\ ac+cd & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Euler Line and Euler Circle

Let triangle ABC, where point H is the orthocenter.

The lines (AH, BH, CH) perpendicular to (BC, AC, AB) at point (D, F, E) respectively.

Let point G and O be the centroid and circumcenter respectively.

Line GHO is a straight line is it is called as Euler line.

Point D, E, F; Midpoint of AB, AC, BC; Midpoint of HA, HB, HC; all 9 points concyclic. This is called the Euler circle.

CCN1048 Linear Algebra Final Reminder

Quiz

1. Performs Gauss-Jordan elimination to solve the system of linear equations.

$$\begin{cases} 2x & -y & +z & = 0 \\ x & +2y & -2z & = 0 \\ 3x & +y & -z & = 0 \end{cases}$$

2. Use Cramer's rule to solve the system of linear equations.

$$\begin{cases} -x & +3y & & = -72 \\ 3x & +4y & -4z & = -4 \\ -20x & -12y & 5z & = -50 \end{cases}$$

3. Let A be the following matrix

$$A = \begin{bmatrix} 1 & 3 \\ -2 & -8 \end{bmatrix}$$

a. Compute the matrices A^2, AA^T and A^{-1}

b. Find numbers m and n such that $A^2 = mA + nI_2$

c. Write A and A^T as a product of elementary matrices

d. Let $B = A - tI_2$, where t is a scalar. For which values of t is B not invertible?

e. Let $S = X + X^T$, where X is any square matrix. Show that S is symmetric.

4.

$$A = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{bmatrix}$$

a. Find all the eigenvalues and the corresponding eigenvectors of A.

b. Shows A is diagonalizable.

c. Find a non-singular matrix P and a diagonal matrix D such that

$$P^{-1}AP = D$$

d. Find $\det(2(A^{-1})^{1048})$

5. Consider three points $A(0,1,2), B(1,2,0), C(2,0,1)$

a. Prove that the points A, B, C form a triangle

b. Find $\angle ABC$

c. Find the length of the median of AB

d. Find the coordinates of centroid T

e. Find the perimeter of $\triangle ABC$

f. Find the area of $\triangle ABC$

Answer

1.

$$\left[\begin{array}{ccc|c}2 & -1 & 1 & 0\\ 1 & 2 & -2 & 0\\ 3 & 1 & -1 & 0\end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c}1 & 2 & -2 & 0\\ 2 & -1 & 1 & 0\\ 3 & 1 & -1 & 0\end{array}\right]$$

$$\xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c}1 & 2 & -2 & 0\\ 0 & -5 & 5 & 0\\ 3 & 1 & -1 & 0\end{array}\right] \xrightarrow{-3R_1+R_3 \to R_3} \left[\begin{array}{ccc|c}1 & 2 & -2 & 0\\ 0 & -5 & 5 & 0\\ 0 & -5 & 5 & 0\end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2} \left[\begin{array}{ccc|c}1 & 2 & -2 & 0\\ 0 & 1 & -1 & 0\\ 0 & -5 & 5 & 0\end{array}\right] \xrightarrow{5R_2+R_3 \to R_3} \left[\begin{array}{ccc|c}1 & 2 & -2 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$i.e.\ \begin{cases} x = 0 \\ y = t \\ z = t \end{cases}, \forall t \in \mathbb{R}, (x, y, z) = \{(0, t, t) | t \in \mathbb{R}\}$$

2.

$$Let\ A = \begin{bmatrix} -1 & 3 & 0 \\ 3 & 4 & -4 \\ -20 & -12 & 5 \end{bmatrix}$$

$$\det(A) = 223$$

$$x = \det(A)^{-1} \begin{vmatrix} -72 & 3 & 0 \\ -4 & 4 & -4 \\ -50 & -12 & 5 \end{vmatrix} = 12$$

$$y = \det(A)^{-1} \begin{vmatrix} -1 & -72 & 0 \\ 3 & -4 & -4 \\ -20 & -50 & 5 \end{vmatrix} = -20$$

$$z = \det(A)^{-1} \begin{vmatrix} -1 & 3 & -72 \\ 3 & 4 & -4 \\ -20 & -12 & -50 \end{vmatrix} = -10$$

3.

a.

$A^2 = \begin{bmatrix} 1 & 3 \\ -2 & -8 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & -8 \end{bmatrix} = \begin{bmatrix} -5 & -21 \\ 14 & 58 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 3 \\ -2 & -8 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & -8 \end{bmatrix} = \begin{bmatrix} 10 & -26 \\ -26 & 68 \end{bmatrix}$

$A^{-1} = \frac{1}{\left((1)(-8)-(3)(-2)\right)}\begin{bmatrix} -8 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3/2 \\ -1 & -1/2 \end{bmatrix}$

b.

$$\begin{bmatrix} -5 & -21 \\ 14 & 58 \end{bmatrix} = \begin{bmatrix} m & 3m \\ -2m & -8m \end{bmatrix} + \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$$

$by\ 3m = -21\ and -2m = 14,$

$we\ have\ m = -7,$

$i.e. m + n = -5\ and\ n - 8m = 58,$

$we\ have\ n = 2$

$i.e. A^2 = -7A + 2I_2$

c.

$A \xRightarrow{E_1:R_2+2R_1\to R_2} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \xRightarrow{E_2:-\frac{1}{2}R_2\to R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xRightarrow{E_3:R_1-3R_2\to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$where\ E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, E_3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$i.e. E_3E_2E_1A = I \Rightarrow A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$A^T = (E_1^{-1}E_2^{-1}E_3^{-1})^T = (E_3^{-1})^T(E_2^{-1})^T(E_1^{-1})^T$

$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

d.

$\det(B) = \det\begin{pmatrix} 1-t & 3 \\ -2 & -8-t \end{pmatrix} = (1-t)(-8-t) - (3)(-2)$

$= t^2 + 7t - 2$

B is not invertible if and only if $\det(B) = 0$

$i.e. t = -\frac{7}{2} \pm \frac{\sqrt{57}}{2}$

e. $S^T = (X + X^T)^T = X^T + X = S$

i.e. S is symmetric

4.

a.

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & -2 & 1\\ -1 & 3-\lambda & -1\\ 2 & -4 & 3-\lambda\end{vmatrix}$$

$=-\lambda^3+8\lambda^2-13\lambda+6=-(\lambda-1)^2(\lambda-6)$

$For\ \lambda_1=\lambda_2=1,$

$(A-\lambda I)\vec{v}=0$

$$\left[\begin{array}{ccc|c}1 & -2 & 1 & 0\\ -1 & 2 & -1 & 0\\ 2 & -4 & 2 & 0\end{array}\right]\xRightarrow[R_3-2R_1\to R_3]{R_2+R_1\to R_2}\left[\begin{array}{ccc|c}1 & -2 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\vec{v}=s\begin{bmatrix}2\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}\forall s,t\in\mathbb{R}\ and\ s,t\neq 0$$

The two linearly independent eigenvectors corresponding to $\lambda_1=\lambda_2=1$ is $[2\quad 1\quad 0]^T$ and $[-1\quad 0\quad 1]^T$

$For\ \lambda_3=6,$

$(A-\lambda I)\vec{v}=0$

$$\left[\begin{array}{ccc|c}-4 & -2 & 1 & 0\\ -1 & -3 & -1 & 0\\ 2 & -4 & -3 & 0\end{array}\right]\xRightarrow[R_3+\frac{1}{2}R_1\to R_3]{R_2-\frac{1}{4}R_1\to R_2}\left[\begin{array}{ccc|c}-4 & -2 & 1 & 0\\ 0 & -5/2 & -5/4 & 0\\ 0 & -5 & -5/2 & 0\end{array}\right]$$

$$\xRightarrow{R_3-2R_2\to R_3}\left[\begin{array}{ccc|c}-4 & -2 & 1 & 0\\ 0 & -5/2 & -5/4 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\vec{v}=t\begin{bmatrix}1\\-1\\2\end{bmatrix}\forall t\in\mathbb{R}\ and\ t\neq 0$$

The eigenvector corresponding to $\lambda_3=6$ is $[1\quad -1\quad 2]^T$

b.

From eigenvectors, we form matrix P

$$\det(P)=\begin{vmatrix}2 & -1 & 1\\ 1 & 0 & -1\\ 0 & 1 & 2\end{vmatrix}=-5\neq 0$$

i.e. The set of eigenvectors is linearly independent

i.e. A is diagonalizable.

c.

$P^{-1}AP=D$

$$P=\begin{bmatrix}2 & -1 & 1\\ 1 & 0 & -1\\ 0 & 1 & 2\end{bmatrix},D=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 6\end{bmatrix}$$

d.

$P^{-1}AP = D \Rightarrow A = PDP^{-1} \Rightarrow A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$

$\det(2(A^{-1})^{1048}) = 2^3 \det(P(D^{-1})^{1048}P^{-1})$

$= 2^3 \det(P) \det(D^{1048})^{-1} \det(P)^{-1}$

$$= 2^3 \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} \frac{1}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6^{1048} \end{vmatrix}} \frac{1}{\begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix}}$$

$$= \frac{2^3 \times 5}{6^{1048} \times 5} = \frac{1}{3^{1048} \times 2^{1045}}$$

5.

a.

$$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \overrightarrow{AC} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \overrightarrow{BC} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

Because $\overrightarrow{AB} \neq k\overrightarrow{AC}\ \forall\ k \in \mathbb{R}$, i.e. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not collinear.

Points A, B, C do not lie on one straight line.

i.e. points A, B, C form $\triangle\ ABC$

b.

$\overrightarrow{BA} \cdot \overrightarrow{BC} = \left\|\overrightarrow{BA}\right\| \left\|\overrightarrow{BC}\right\| \cos \angle ABC$

$$\angle ABC = \cos^{-1}\left(\frac{(-1)(-1) + (-1)(2) + (2)(-1)}{\sqrt{(-1)^2 + (-1)^2 + 2^2}\sqrt{(-1)^2 + 2^2 + (-1)^2}}\right) = \frac{\pi}{3}$$

c. (General method)

Let the required median be CS, where CS perpendicular bisects AB at S

Required length $= \left\|\overrightarrow{CB} - (\overrightarrow{CB} \cdot \widehat{AB})\widehat{AB}\right\|$

$$= \left\| \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \frac{(1)(1) + (1)(-2) + (-2)(1)}{\sqrt{1^2 + 1^2 + (-2)^2}^2} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\| = \left\| \begin{matrix} \frac{3}{2} \\ -\frac{3}{2} \\ 0 \end{matrix} \right\|$$

$$= \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right)^2} = \sqrt{\frac{9}{2}} = \frac{3}{2}\sqrt{2}\ unit$$

d.

Let $T = (x, y, z)$

$$\overrightarrow{AT} = \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x - 0 \\ y - 1 \\ z - 2 \end{bmatrix}$$

i.e. $x = 1, y = 1, z = 1$

i.e. $T = (1,1,1)$

e.

Perimeter $= \left|\left|\overrightarrow{AB}\right|\right| + \left|\left|\overrightarrow{AC}\right|\right| + \left|\left|\overrightarrow{BC}\right|\right|$

$= \sqrt{1^2 + 1^2 + (-2)^2} + \sqrt{2^2 + (-1)^2 + (-1)^2} + \sqrt{(-1)^2 + 2^2 + (-1)^2}$

$= 3\sqrt{6}\, unit$

f.

$$Area = \frac{1}{2}\left|\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right| = \frac{1}{2}\left\|\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix}\right\|$$

$$= \frac{1}{2}\left\|\begin{vmatrix} +\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \end{vmatrix}\right\| = \frac{1}{2}\left\|\begin{vmatrix} -3 \\ -3 \\ -3 \end{vmatrix}\right\| = \frac{1}{2}\sqrt{3(-3)^2} = \frac{3\sqrt{3}}{2}\, unit^2$$