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Function

Domain

Range of x such that the function $f(x)$ exists.

Range

Change x to be the subject of $y = f(x)$

Check the range of y such that $x = g(y)$ exists.

Example:

Find domain and range of

$$f(x) = \frac{3x + 4}{x^2 - x}$$

Answer:

Odd

$f(-x) = -f(x)$

Example:

$$f(x) = x|x|$$

Answer:

Even

$f(-x) = f(x)$

Example:

$$f(x) = e^{-x^2}$$

Answer:

Inverse of function

$y = f(x)$

Solving for x in terms of y, we have

$x = g(y)$

$\therefore f^{-1}(x) = g(x)$

Properties

If $f(x)$ is an invertible function with domain A and range B, then

***$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x \ \forall \ x \in X$

***$\left(f^{-1}(x)\right)^{-1} = f(x)$

$(g \circ f)^{-1}(x) = f^{-1}(x) \circ g^{-1}(x)$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{g^{-1}} Y \xrightarrow{f^{-1}} X$$

Graph

If f is invertible, then the graph of the function

$y = f^{-1}(x)$

Is the same as the graph of the equation

$x = f(y)$

Notice that it is identical to $y = f(x)$ except x and y have been reversed. Thus, the graph of $f^{-1}(x)$ can be obtained by reflect the graph of $f(x)$ across the line $y = x$.

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Examples:

Find the inverse function to the given function and determine the domain and range of both functions

$$y = 1 + \log x$$

Answer:

$$^{**}y = \sin 2x + 1$$

Answer:

*****NOTICE:** Domain of a function must be the range of its inverse if and only if it is a one–to–one function. Note that the second example is not a one to one function.

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For the following 6 questions, solve for x. Given that f is an invertible function and that

$f^{-1}(1) = -2$

$f^{-1}(2) = 3$

$f^{-1}(3) = 2$

$f^{-1}(4) = 5$

$f^{-1}(5) = -7$

$f^{-1}(6) = 8$

$f^{-1}(7) = -3$

$f^{-1}(8) = 1$

$f^{-1}(9) = 4$

$f(x+2) = 5$

Answer:

$f(3x-4) = 3$

Answer:

$f(-5x) = 1$

Answer:

$f(-2-x) = 2$

Answer:

$f\left(\frac{1}{x}\right) = 8$

Answer:

$f\left(\frac{5}{x-1}\right) = 3$

Answer:

***Suppose $f(x) = x^n$ for n is a positive integer. For which values of n is f an invertible function? Explain.

Answer:

Period

$f(x+T)=f(x)$

Period of $y=f(x)$, where $f(x)$ is $\sin x$ or $\cos x$

$T=2\pi$

Period of $y=f(x)$, where $f(x)$ is $\tan x$

$T=\pi$

Example:

Find the period of $y=3\tan(3x+5)$

Answer:

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Limit

Properties of limit

Situation of limit $\lim\limits_{x \to a} f(x)$ does not exist

1. The one–sided limits are not equal.
   $\lim\limits_{x \to a^+} f(x) \neq \lim\limits_{x \to a^-} f(x)$
2. The function doesn't approach a finite value.
   (Basic Definition of Limit)
3. The function doesn't approach a particular value.
   (Oscillation)
4. The x–value is approaching the endpoint of a closed interval.
   Either $\lim\limits_{x \to a^+} f(x)$ or $\lim\limits_{x \to a^-} f(x)$ does not exist

Steps handling limits

1. Direct substitution.
2. LH if and only if $\left(\frac{0}{0}\right)$ or $\left(\pm\frac{\infty}{\infty}\right)$
3. Change $\lim\limits_{x \to -\infty} f(x)$ to $\lim\limits_{x \to \infty} f(-x)$
4. Factoring
5. Rationalization
6. Multiplying numerator and denominator by a conjugate.
7. Divide both numerator and denominator by the x to the greatest exponent found in the denominator
8. Solving trigonometry limits

$$\lim_{x \to 0} \frac{\sin x}{x} \left(resp. \frac{x}{\sin x}\right) = 1$$

9. Definition of e

$$\lim_{x \to 0\ (resp.\ \infty)} (1+x)^{\frac{1}{x}} \left(resp. \left(1+\frac{1}{x}\right)^x\right) = e$$

10. Additional limits

$$\lim_{x \to 0} \frac{e^x - 1}{x} \left(resp. \frac{x}{e^x - 1}\right) = 1$$

Examples:

Evaluate the limit if exists.

$$\lim_{x\to0^+}\ln x$$

Answer:

$$\lim_{x\to0}\frac{x^3-2x^2+x}{2x^3+x^2-2x}$$

Answer:

$$\lim_{x\to2}\frac{x^2+5}{x^2-3}$$

Answer:

$$^*\lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}+x\right)^2}{\sqrt[3]{x^6+1}}$$

Answer:

$$^*\lim_{x\to\infty}\frac{(x-1)^{100}(6x+1)^{200}}{(3x+5)^{300}}$$

Answer:

$$^{*}\lim_{x\to\infty}\frac{\sqrt[4]{x^5}+\sqrt[5]{x^3}+\sqrt[6]{x^8}}{\sqrt[3]{x^4+2}}$$

Answer:

$$^{**}\lim_{x\to 0}\frac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$$

Answer:

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$$

Answer:

$$\lim_{x\to-\infty}\left(\sqrt{x^2+x}-x\right)$$

Answer:

$$\lim_{x\to\infty}\left(\frac{2x+1}{2x-3}\right)^{3x}$$

Answer:

$$\lim_{x\to 1}\frac{1-\sqrt[n]{x}}{1-\sqrt[m]{x}}\ \forall\ (m,n)\in\mathbb{R}$$

Answer:

$$\lim_{x\to 0}\frac{x}{\sin 11x}$$

Answer:

$$\lim_{x\to 0}\frac{\tan 6x}{\sin 2x}$$

Answer:

Let $F(x) = \frac{x^2-1}{|x-1|}$, evaluate

$$F(x) = \begin{cases} \dfrac{x^2-1}{1-x} & if\ (x-1) < 0 \Rightarrow x < 1 \\ \dfrac{x^2-1}{x-1} & if\ (x-1) \geq 0 \Rightarrow x \geq 1 \end{cases}$$

$$\lim_{x\to 1^+} F(x)$$

Answer:

$$\lim_{x\to 1^-} F(x)$$

Answer:

Does $\lim_{x\to 1} F(x)$ exist? Explain.

Answer:

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

What is wrong with the above equation?

Answer:

*In view of part 1, explain why the equation $\lim\limits_{x \to 2} \frac{x^2+x-6}{x-2} = \lim\limits_{x \to 2}(x+3)$ is correct.

Answer:

For what value of the constant c is the function

$f(x) = \begin{cases} cx^2 + 2x & if\ x < 2 \\ x^3 - cx & if\ x \geq 2 \end{cases}$ continuous on $(-\infty, \infty)$?

Answer:

**If $\lim_{x\to1}\frac{f(x)-8}{x-1}=10$, find $\lim_{x\to1}f(x)$.

Answer:

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Differentiation

Properties of differentiation

Product rule

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + g(x)f'(x)$$

Quotient rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

***Notice:** The notation of the function

$$f^n(x) = [f(x)]^n$$

$$f^{(n)}(x) = \frac{d^n}{dx^n}f(x)$$

Chain rule

$$\frac{d}{dx}f\left(g\big(h(x)\big)\right) = f'\left(g\big(h(x)\big)\right)g'\big(h(x)\big)h'(x)$$

*****Inverse function theorem**

$$\frac{d}{dx}f^{-1}(x) = (f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}$$

*****LD**

$$y = f\big(g(x)\big)^{h(x)}$$

$$\ln y = h(x)\ln f\big(g(x)\big)$$

$$y' = y\left[\frac{h(x)}{f\big(g(x)\big)}g'(x) + h'(x)\ln f\big(g(x)\big)\right]$$

$$y' = g'(x)h(x)f\big(g(x)\big)^{h(x)-1} + h'(x)f\big(g(x)\big)^{h(x)}\ln f\big(g(x)\big)$$

***Alternative method of LD**

$$\frac{d}{dx}f\big(g(x)\big)^{h(x)}$$

$$= \frac{d}{dx}e^{\ln f(g(x))^{h(x)}}$$

$$= \frac{d}{dx}e^{h(x)\ln f(g(x))}$$

$$= e^{h(x)\ln f(g(x))}\left[\frac{h(x)}{f\big(g(x)\big)}g'(x) + h'(x)\ln f\big(g(x)\big)\right]$$

$$= f\big(g(x)\big)^{h(x)}\left[\frac{h(x)}{f\big(g(x)\big)}g'(x) + h'(x)\ln f\big(g(x)\big)\right]$$

$$= g'(x)h(x)f\big(g(x)\big)^{h(x)-1} + h'(x)f\big(g(x)\big)^{h(x)}\ln f\big(g(x)\big)$$

Examples:

Find $f'(x)$

$$f(x) = e^{e^{2x^2+1}}$$

Answer:

$$f(x) = (6x)^{\cos(2x+1)}$$

Answer:

Find $\frac{dy}{dx}$

$$(2x+y)^4 + 3x^2 + 3y^2 = \frac{x}{y} + 1$$

Answer:

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Application of Differentiation

$f'(x)$ is the **slope** of $f(x)$

$f''(x)$ is the **slope** of $f'(x)$

Asymptote

$$y = mx + c + \frac{f(x)}{g(x)}, \deg\big(f(x)\big) < \deg\big(g(x)\big)$$

Horizontal asymptote

$$y = mx + c$$

Vertical asymptote

Let domain of $g(x)$ is G

$$x = a \;\forall\; a \notin G$$

Relative maximum

$$f'(x) = 0 \; and \; f''(x) < 0$$

Relative minimum

$$f'(x) = 0 \; and \; f''(x) > 0$$

Inflection point

$$f''(x) = 0 \; and \; \begin{cases} f''(x+a) > 0 \; (resp. < 0) \\ f''(x-a) < 0 \; (resp. > 0) \end{cases}$$

Examples:

Let $f(x) = x^4 + x^3 + 1, 0 \leq x \leq 2$

Let $g(x) = f^{-1}(x)$ and $F(x) = f\big(2g(x)\big)$.

***Find an equation for the tangent line to $y = F(x)$ at $x = 3$

Answer:

**Given a right cone with radius r and height h, is circumscribed around a giver sphere with radius R. Find when its volume is a minimum.

Answer:

Integration

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C \ (r \neq 1)$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \ (0 < a \neq 1)$$

Properties of integration

u–substitution

Example:

$$\int x\sqrt{5+x^2}dx$$

Answer:

Trig–substitution

Expression	Substitution
$a^2 - x^2$	$x = a\sin\theta$
$a^2 + x^2$	$x = a\tan\theta$
$x^2 - a^2$	$x = a\sec\theta$

Example:

$$\int \frac{x^2}{\sqrt{4-x^2}} dx$$

Answer:

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Partial Fractions

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$ $(k\in\mathbb{Z}^+)$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$ $(k\in\mathbb{Z}^+)$
$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$	$\dfrac{A_{n-1}x^{n-1}+A_{n-2}x^{n-2}+\cdots+A_0}{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}$

Example:

$$\int \frac{4x+1}{x^2-x-2}\,dx$$

Answer:

Rationalizing Substitution

Examples:

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$$

Answer:

$$\int \frac{1}{1+e^{x}}\,dx$$

Answer:

Integration by Parts

$$\int udv = uv - \int vdu$$

Examples:

$$\int e^x \sin x \, dx$$

Answer:

$$\int \tan^{-1} x \, dx$$

Answer:

Application of integration

Area of graph

Upper (resp. lower) function is $f(x)$ (resp. $g(x)$)

Assuming $f(x)$ and $g(x)$ did not intersect on $[a,b]$

$$\int_a^b [f(x) - g(x)]\, dx$$

Volume

A certain area rotates around $x-$axis to from an object, the corresponding area:

$$\pi \int_a^b [f^2(x) - g^2(x)]\, dx$$

Upper (resp. lower) function is $f(x)$ (resp. $g(x)$)

Assuming $f(x)$ and $g(x)$ did not intersect on $[a,b]$

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Foundation Theorem of Calculus (FTC)

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a) \text{ and } F'(x) = f(x)$$

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f\big(g(x)\big)g'(x)$$

Examples:

***Evaluate the following

$$\int_5^0 (-2x-3)dx$$

Answer:

$$\frac{d}{dx}\int_{a^2+b}^{x} f(t)dt$$

Answer:

$$\frac{d}{dx}\int_{a^2+b}^{x^2} f(t)dt$$

Answer:

$$\frac{d}{dx}\int_{x^2}^{a^2+b} f(t)dt$$

Answer:

$$\frac{d}{dx}\int_{x^2}^{x^3} f(t)dt$$

Answer:

Given $f(0) = 4$, evaluate the following limit

$$\lim_{x\to 0} \frac{\int_0^{x^2} (t+1)f(t)dt}{3x^2}$$

Answer:

Find the area enclosed between $y = x^2$ and $y = x$ between $x = 0$ and $x = 2$

Answer:

A cup−like object is made by rotating the area between $y = 2x^2$ and $y = x + 1$ with $x \geq 0$ around the x−axis. Find the volume of the material needed to make the cup. Units are cm

Answer: