National University of Singapore
Department of Mathematics

2020/2021 Semester 2 MA3110 Mathematical Analysis II Tutorial 1

1. Suppose that the function f is defined in a deleted neighborhood of a point c and $L \in \mathbb{R}$. Prove that
$$\lim_{x\to c} f(x) = L \quad \text{if and only if} \quad \lim_{x\to c^+} f(x) = L = \lim_{x\to c^-} f(x).$$

2. Let $f : (0,1) \to \mathbb{R}$ and $L \in \mathbb{R}$. Prove that $\lim_{x\to 0^+} f(x) = L$ if and only if $\lim_{y\to\infty} f(1/y) = L$.

3. Let r be a positive rational number, and let $f(x) = x^r$ for $x > 0$. Prove that
$$f'(x) = rx^{r-1} \quad \text{for all } x > 0.$$

4. *What happens if $f'(c) = 0$ in Theorem 6.2.4?*
Let I be an interval, and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J := f(I)$ and let $g : J \to \mathbb{R}$ be the inverse function of f. Prove that if f is differentiable at $c \in I$ and $f'(c) = 0$, then g is not differentiable at $d := f(c)$.

5. Let $f(x) = x^5 + 4x + 3$, $x \in \mathbb{R}$, and let $g = f^{-1} : \mathbb{R} \to \mathbb{R}$ be the inverse function of f. Calculate $g'(8)$.

6. Use the Mean Value Theorem to prove that
$$\sqrt{1+x} < 1 + \frac{1}{2}x \quad \text{for all } x > 0.$$

7. Let I be an open interval and let $c \in I$. Let $f : I \to \mathbb{R}$ be continuous and define $g : I \to \mathbb{R}$ by $g(x) = |f(x)|$ for $x \in I$.
   (a) Give an example where f is differentiable at c but g is not differentiable at c.
   (b) Prove that if g is differentiable at c, then f is also differentiable at c.
   (c) Compute $f'(c)$ in terms of $g'(c)$.

8. Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$. Establish the **Straddle Lemma:** Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in I$ satisfy $c - \delta < u \le c \le v < c + \delta$, then we have
$$|f(v) - f(u) - (v-u)f'(c)| \le \varepsilon(v-u).$$