# Chapter 1: The Real Numbers

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## 1.1 Set Operations

Given two sets $A$ and $B$.

- If every element of $A$ also belongs to $B$, then we say that $A$ is a *subset* of $B$ and write $A \subseteq B$.
- The *union* of $A$ and $B$ is defined by
$$A \cup B = \{x : \ x \in A \text{ or } x \in B\}.$$
- The *intersection* of $A$ and $B$ is defined by
$$A \cap B = \{x : \ x \in A \text{ and } x \in B\}.$$
- The *complement of $B$ relative to $A$* is defined by
$$A \backslash B = \{x : \ x \in A \text{ and } x \notin B\}.$$

The set with no element is called the *empty set*, and is denoted by $\emptyset$.

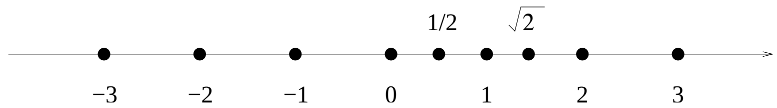
## 1.2 Number systems

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, ...\} = \text{the set of all natural numbers} \\
\mathbb{Z} &= \{0, \pm 1, \pm 2, ...\} = \text{the set of all integers} \\
\mathbb{Q} &= \left\{\frac{n}{m} : \ n, m \in \mathbb{Z}, \ m \neq 0\right\} = \text{the set of all rational numbers} \\
\mathbb{R} &= \text{the set of all real numbers.}
\end{aligned}$$

We have
$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

**Remark** There is a formal construction of the system of real numbers. See the books by Rudin and by Parzynski and Zipse.

**The real line:** It is convenient to identify real numbers with points on a line.




There are real numbers which are not rational. These numbers are called *irrational numbers*. So $\mathbb{R}\backslash\mathbb{Q}$ is the set of all irrational numbers.

**Theorem 1.2.1.** *$\sqrt{2}$ is irrational.*

**Definition** An integer $n$ is said to be

- *even* if $n = 2k$ for some integer $k$;
- *odd* if $n = 2k - 1$ some integer $k$.

What can you say about the parity of $n^2$ if $n$ is even/odd?

*Proof of Theorem 1.2.1:* Suppose $\sqrt{2}$ is rational. Then

$$\sqrt{2} = \frac{n}{m}$$

where $n$ and $m$ are integers with no common factor other than 1. Then

$$2 = \frac{n^2}{m^2}$$

and

$$2m^2 = n^2.$$

This says that $n^2$ is even. So $n$ is also even, and $n = 2k$ for some integer $k$. Substituting this into the last equation, we get

$$2m^2 = 4k^2.$$

So

$$m^2 = 2k^2.$$

But this says $m^2$ is even. So $m$ is also even. It follows that 2 is a common factor for $n$ and $m$. This contradicts our assumption on $n$ and $m$. So $\sqrt{2}$ is not rational. □

## 1.3 The natural numbers

We shall assume that $\mathbb{N}$ has the following fundamental property:

**Principle of mathematical induction.** *Let $S \subseteq \mathbb{N}$. If*

*(i) $1 \in S$; and*
*(ii) $k \in S \implies k + 1 \in S$;*

*then $S = \mathbb{N}$.*

**Principle of mathematical induction (application version):** *For each $n \in \mathbb{N}$, let $P(n)$ be a statement about $n$. If*

*(i) $P(1)$ is true; and*
*(ii) for $k \in \mathbb{N}$, $P(k)$ is true $\implies P(k + 1)$ is true;*

*then $P(n)$ is true for all $n \in \mathbb{N}$.*

*Proof:* Apply the principle of mathematical induction to the set $S = \{n \in \mathbb{N} : P(n) \text{ is true}\}$. □

**Example** Prove that $2^{n-1} \leq n!$ for all $n \in \mathbb{N}$.

*Solution:* Let $P(n)$ be the statement $2^{n-1} \leq n!$.

When $n = 1$, we have $2^{1-1} = 2^0 = 1 \leq 1!$. So $P(1)$ is true.

Suppose $P(k)$ is true, i.e. $2^{k-1} \leq k!$. (This is called the induction hypothesis.) Then since $2 \leq k + 1$,

$$2^{(k+1)-1} = 2 \cdot 2^{k-1} \leq (k + 1)k! = (k + 1)!.$$

So $P(k + 1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

**Exercise** Prove that $n < 2^n$ for all $n \in \mathbb{N}$.

**Principle of mathematical induction (second version):** *Let $n_0 \in \mathbb{N}$. If*

*(i) $P(n_0)$ is true; and*
*(ii) for each natural number $k \geq n_0$, $P(k)$ is true $\Longrightarrow P(k+1)$ is true;*

*then $P(n)$ is true for all natural numbers $n \geq n_0$.*

*Proof.* Exercise.

**Well-ordering principle for $\mathbb{N}$:** *Every nonempty subset $A$ of $\mathbb{N}$ has a least (first) element,* i.e. there exists $p \in A$ such that $p \leq a$ for all $a \in A$

*Proof.* Let $A$ be a non-empty subset of $\mathbb{N}$ and assume that $A$ has no least element. Define $S \subseteq \mathbb{N}$ by

$$S = \{n \in \mathbb{N} : \ n < a \text{ for each } \ a \in A\}.$$

Then $S \cap A = \emptyset$. We shall use the principle of mathematical induction to show that $S = \mathbb{N}$.

First we have $1 \notin A$, for otherwise 1 would be the least element of $A$. Hence, $1 < a$ for each $a \in A$ and so $1 \in S$.

Next, we assume that $p \in S$. Then $p < a$ for each $a \in A$. If $p + 1 \in A$, then it would be the least element of $A$. Hence, $p + 1 \notin A$ and $p + 1 < a$ for each $a \in A$. It follows that $p + 1 \in S$. By the principle of mathematical induction, $S = \mathbb{N}$. Since $S \cap A = \emptyset$, this implies that $A = \emptyset$, which is a contradiction. So $A$ must have a least element.

□

## 1.4 The algebraic properties of $\mathbb{R}$

$\mathbb{R}$ is a *complete ordered field.*

$\mathbb{R}$ is a field because it has the following algebraic properties:

**1.** $a + b = b + a, \quad \forall a, b \in \mathbb{R}$.

**2.** $(a + b) + c = a + (b + c), \quad \forall a, b, c \in \mathbb{R}$.

**3.** $\exists 0 \in \mathbb{R}$ such that $0 + a = a + 0 = a, \quad \forall a \in \mathbb{R}$.

**4.** For each $a \in \mathbb{R}$, $\exists\ -a \in \mathbb{R}$ such that

$$a + (-a) = (-a) + a = 0.$$

**5.** $ab = ba, \quad \forall a, b \in \mathbb{R}$.

**6.** $(ab)c = a(bc), \quad \forall a, b, c \in \mathbb{R}$.

**7.** $\exists 1 \in \mathbb{R}$ such that $1 \neq 0$ and $1a = a1 = a \ \ \forall a \in \mathbb{R}$.

**8.** If $a \in \mathbb{R}$ and $a \neq 0$, then $\exists\ \dfrac{1}{a} \in \mathbb{R}$ such that

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1.$$

**9.** $a(b + c) = ab + ac, \quad \forall a, b, c \in \mathbb{R}$.

**Notation**

- $\forall$ means "for every";
- $\exists$ means "there exists".

**Remark**

Any nonempty set $F$ together with two binary operations called addition and multiplication satisfying conditions 1-9 is called a *field*. So $\mathbb{Q}$ and $\mathbb{C}$ are fields, but $\mathbb{Z}$ and $\mathbb{N}$ are not.

## 1.5 The ordered properties of $\mathbb{R}$

There is a relation "<" on $\mathbb{R} \times \mathbb{R}$ which has the following properties:

(a) (The trichotomy property) If $a, b \in \mathbb{R}$, then exactly one of the following holds:

$$a < b, \quad b < a \ \text{ or } \ a = b.$$

(b) $a < b$ and $b < c \implies a < c$.
(c) $a < b \implies a + c < b + c, \ \forall c \in \mathbb{R}$.
(d) $a < b$ and $c > 0 \implies ac < bc$, and
$a < b$ and $c < 0 \implies ac > bc$.

**Notation**

We write $a \leq b$ if $a < b$ or $a = b$.

**Definition** We call a real number $a$

(i) *positive* if $a > 0$,
(ii) *nonnegative* if $a \geq 0$,
(iii) *negative* if $a < 0$,
(iv) *nonpositive* if $a \leq 0$.

**Lemma 1.5.1.** *(i) If $c > 1$, then $c^n > c$ for every natural number $n \geq 2$.*

*(ii) If $0 < c < 1$, then $c^n < c$ for every natural number $n \geq 2$.*

*Proof:*

(i) For each $n \in \mathbb{N}$, let $P(n)$ be the statement $c^n > c$.
By multiplying the inequality $c > 1$ by $c$, we obtain $c^2 > c$. So $P(2)$ holds.
Assume that for some $k \geq 2$, $P(k)$ is true, i.e. $c^k > c$. Multiplying this inequality by $c$, we obtain

$$c^{k+1} = c \cdot c^k > c \cdot c = c^2.$$

But $c^2 > c$. So by transitivity,

$$c^{k+1} > c.$$

Thus $P(k+1)$ also holds.

By the principle of mathematical induction, $c^n > c$ holds for all natural number $n \geq 2$.

(ii) We can prove this statement by induction as in (i). Alternatively, observe that if $0 < c < 1$, then $1/c > 1$. By (i),

$$\frac{1}{c^n} = \left(\frac{1}{c}\right)^n > \frac{1}{c}.$$

Multiplying this inequality by $c^{n+1}$, we obtain $c > c^n$. □.

**Theorem 1.5.2.** *For any nonzero real number $a$, $a^2 > 0$.*

*Proof:* Since $a \neq 0$, either $a > 0$ or $a < 0$ by the trichotomy property.

If $a > 0$, then $a \cdot a > a \cdot 0$. So $a^2 > 0$.
If $a < 0$, then we need to switch the sign when multiplying $a$ to both side of $a < 0$, i.e. $a \cdot a > a \cdot 0$. Again we obtain $a^2 > 0$. $\square$

**Exercise** Prove that every natural number is positive.

**Theorem 1.5.3.** *If $a \in \mathbb{R}$ is such that $0 \leq a < \varepsilon$ for **every** positive number $\varepsilon$, then $a = 0$.*

*Proof:* Since $a \geq 0$, either $a > 0$ or $a = 0$. Suppose to the contrary that $a > 0$.

Take $\varepsilon_0 = a/2$. Then $\varepsilon_0$ is positive and $\varepsilon_0 < a$ (Why?). But this contradicts the assumption on $a$. So we must have $a = 0$. $\square$.

**Exercise** Let $a, b \in \mathbb{R}$. Prove that if $a - \varepsilon < b$ for **every** $\varepsilon > 0$, then $a \leq b$.

## 1.6 Intervals

An interval is a subset $I$ of $\mathbb{R}$ with the following property: if $x, y \in I$ and $x < y$, then

$$x < t < y \Longrightarrow t \in I.$$

**Types of intervals:**

$(a, b) = \{x \in \mathbb{R} : \ a < x < b\}$ (open interval).
$[a, b] = \{x \in \mathbb{R} : \ a \leq x \leq b\}$ (closed interval).
$[a, b) = \{x \in \mathbb{R} : \ a \leq x < b\}$.
$(a, b] = \{x \in \mathbb{R} : \ a < x \leq b\}$.
$[a, \infty) = \{x \in \mathbb{R} : \ x \geq a\}$.
$(a, \infty) = \{x \in \mathbb{R} : \ x > a\}$.
$(-\infty, b] = \{x \in \mathbb{R} : \ x \leq b\}$.
$(-\infty, b) = \{x \in \mathbb{R} : \ x < b\}$.
$(-\infty, \infty) = \mathbb{R}$.

## 1.7 Solving inequalities

Two important rules used in solving inequalities:

**Rule 1:** *If $ab > 0$, then either*
*(i) $a > 0$ and $b > 0$, or*
*(ii) $a < 0$ and $b < 0$.*

**Rule 2:** *If $ab < 0$, then either*
*(i) $a < 0$ and $b > 0$, or*
*(ii) $a > 0$ and $b < 0$.*

*Proof:* See page 28 of the textbook. □

**Example** Solve $2x^2 + 3x > 2$.

*Solution:* We have

$$2x^2 + 3x > 2 \iff 2x^2 + 3x - 2 > 0 \iff (2x - 1)(x + 2) > 0.$$

So by Rule 1, either (i) $2x - 1 > 0$ and $x + 2 > 0$, or (ii) $2x - 1 < 0$ and $x + 2 < 0$.

For (i) $x > \frac{1}{2}$ and $x > -2 \iff x > \frac{1}{2}$.
For (ii) $x < \frac{1}{2}$ and $x < -2 \iff x < -2$.

So the solution set is $\{x \in \mathbb{R} : x > 1/2\} \cup \{x \in \mathbb{R} : x < -2\}$, that is, $(-\infty, -2) \cup (1/2, \infty)$.

**Example** Solve $\dfrac{3x + 1}{2x + 3} < \dfrac{1}{2}$.

*Solution:* We have

$$\begin{aligned}
\frac{3x+1}{2x+3} < \frac{1}{2} \quad &\iff \quad \frac{3x+1}{2x+3} - \frac{1}{2} < 0 \\
&\iff \quad \frac{4x-1}{2(2x+3)} < 0 \\
&\iff \quad 2(2x+3)^2 \cdot \frac{4x-1}{2(2x+3)} < 2(2x+3)^2 \cdot 0 \\
&\iff \quad (2x+3)(4x-1) < 0.
\end{aligned}$$

By Rule 2, we either have (i) $2x + 3 > 0$ and $4x - 1 < 0$, or (i) $2x + 3 < 0$ and $4x - 1 > 0$.

For (i), $x > -\frac{3}{2}$ and $x < \frac{1}{4} \iff -\frac{3}{2} < x < \frac{1}{4}$.

For (ii), $x < -\dfrac{3}{2}$ and $x > \dfrac{1}{4}$. But this is impossible.

So the solution set is $\{x : \ -3/2 < x < 1/4\} = (-3/2, 1/4)$.

**Bernoulli's inequality.** *If* $x > -1$*, then*

$$(1+x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}.$$

*Proof:* Use induction (Tutorial 1). □

**Definition** Let $n \geq 2$ and let $a_1, a_2, ..., a_n$ be positive numbers.

- The *arithmetic mean* of $a_1, a_2, ..., a_n$ is defined as $A = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$.
- The *geometric mean* of $a_1, a_2, ..., a_n$ is defined as $G = (a_1 a_2 \cdots a_n)^{1/n}$.
- The *harmonic mean* of $a_1, a_2, ..., a_n$ is defined as $H = \dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$.

**The AM-GM-HM inequality:** *Let* $A, G, H$ *be the arithmetic mean, the geometric mean and the harmonic mean of the positive numbers* $a_1, a_2, ..., a_n$ *respectively. Then*

$$H \leq G \leq A.$$

*Equality holds if and only if* $a_1 = a_2 = \cdots = a_n$.

*Proof.* Tutorial 2.

## 1.8 Absolute value

**Definition** Let $a \in \mathbb{R}$. The *absolute value* of $a$ is defined by

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \\ 0 & \text{if } a = 0. \end{cases}$$

**Example** $|3| = 3$, $|-2| = 2$, $|0| = 0$.

**Theorem 1.8.1.** *(Properties of absolute value)*

*(i)* $|a| \geq 0$, $a \leq |a|$ *and* $-a \leq |a|$, $\forall a \in \mathbb{R}$.

*(ii)* $|a| = 0 \Longleftrightarrow a = 0$.

*(iii)* $|-a| = |a|$, $\forall a \in \mathbb{R}$.

*(iv)* $|ab| = |a||b|$, $\forall a, b \in \mathbb{R}$.

*(v)* $|a|^2 = a^2$, $\forall a \in \mathbb{R}$.

*(vi) If* $c \geq 0$, *then* $|a| \leq c \Longleftrightarrow -c \leq a \leq c$.

*(vii)* $-|a| \leq a \leq |a|$, $\forall a \in \mathbb{R}$.

*Proof:* (vi) ($\Longrightarrow$) Assume that $|a| \leq c$. Since $a \leq |a|$ and $-a \leq |a|$, by transitivity,

$$a \leq c \quad \text{and} \quad -a \leq c.$$

So we have $a \leq c$ and $a \geq -c$. Combining these inequalities, $-c \leq a \leq c$.

($\Longleftarrow$) Assume that $-c \leq a \leq c$. Then

$$a \leq c \quad \text{and} \quad a \geq -c.$$

The second inequality is equivalent to $-a \leq c$. Since $a \leq c$ and $-a \leq c$, $|a| \leq c$.

The proofs for the remaining parts are left as exercise. □

**Example** Solve $|x| + |x+1| < 2$.

*Solution:* **Case 1:** $x \leq -1$
In this case, $|x| + |x+1| = -x + (-x-1) = -2x - 1 < 2$, so that $2x > -3$ and $x > -3/2$. Thus the points in $(-3/2, \infty) \cap (-\infty, -1] = (-3/2, -1]$ satisfy the inequality.

**Case 2:** $-1 < x < 0$
In this case, $|x| + |x+1| = -x + (x+1) = 1 < 2$ which is always true. So all the points in $(-1, 0)$ satisfy the inequality.

**Case 3:** $x \geq 0$
In this case, $|x| + |x + 1| = x + (x + 1) = 2x + 1 < 2$, so that $2x < 1$ and $x < 1/2$. Thus the points in $(-\infty, 1/2) \cap [0, \infty) = [0, 1/2)$ satisfy the inequality.

So the solution set is $(-3/2, -1] \cup (-1, 0) \cup [0, 1/2) = (-3/2, 1/2)$.

**Triangle inequality:** *For $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.*

*Proof:* We have

$$-|a| \leq a \leq |a|$$

$$-|b| \leq b \leq |b|.$$

Adding the inequalities gives

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

By part (vi) of Theorem 1.8.1, we obtain

$$|a + b| \leq |a| + |b|. \quad \square$$

**Corollary 1.8.2.** *For $a, b \in \mathbb{R}$, we have*

*(a)* $||a| - |b|| \leq |a - b|$,

*(b)* $|a - b| \leq |a| + |b|$.

*Proof:* (a) By the triangle inequality,

$$|a| = |(a - b) + b| \leq |a - b| + |b|,$$

so

$$|a| - |b| \leq |a - b| \qquad (1)$$

Interchanging the roles of $a$ and $b$, we obtain

$$|b| - |a| \leq |b - a|,$$

which can be written as

$$-(|a| - |b|) \leq |a - b| \qquad (2)$$

(1) and (2) gives $||a| - |b|| \leq |a - b|$.

(b) By the triangle inequality,

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|. \quad \square$$

**Corollary 1.8.3.** *For $a_1, a_2, ..., a_n \in \mathbb{R}$,*

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

*Proof:* This follows by induction. $\square$

## 1.9 The completeness property of $\mathbb{R}$

**Definition** Let $S \subseteq \mathbb{R}$ be nonempty. A number $u$ is called

(i) an *upper bound* of $S$ if $x \leq u$ for all $x \in S$.

(ii) a *lower bound* of $S$ if $x \geq u$ for all $x \in S$.

**Example** Let $S = (0, 1]$.

1, 1.5 and 10 are upper bounds.
0, $-0.7$ and $-2$ are lower bounds.

**Exercise** Do $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ have upper bounds and lower bounds?

**Definition** We say that a nonempty set $S \subseteq \mathbb{R}$ is

(i) *bounded above* if $S$ has an upper bound.

(ii) *bounded below* if $S$ has a lower bound.

(iii) *bounded* if $S$ has an upper bound and a lower bound.

(iv) *unbounded* if $S$ is not bounded, that is, either it does not have any upper bound or it does not have any lower bound.

**Example** Let $S_1 = (0, 1]$, $S_2 = (-\infty, 0)$ and $S_3 = [72, \infty)$. Then

- $S_1$ and $S_2$ are bounded above.
- $S_1$ and $S_3$ are bounded below.
- $S_1$ is bounded.
- $S_2$ and $S_3$ are unbounded.

**Definition** Let $S$ be a nonempty subset of $\mathbb{R}$.

(a) A real number $M$ is called the *supremum* (or least upper bound) of $S$ if

(i) $M$ is an upper bound of $S$;

(ii) $M \leq u$ for every upper bound $u$ of $S$.

In this case, we write $M = \sup S$.

(b) A real number $L$ is called the *infimum* (or greatest lower bound) of $S$ if

(i) $L$ is a lower bound of $S$;

(ii) $L \geq v$ for every lower bound $v$ of $S$.

In this case, we write $L = \inf S$.

The supremum and infimum of of a set may or may not be elements of the set.

**Example**

(a) If $S_1 = \{1, 2, 3, 4\}$, then $\sup S_1 = 4$ and $\inf S_1 = 1$. Both 4 and 1 are elements of $S_1$.

(b) If $S_2 = (0, 1)$, then $\sup S_2 = 1$ and $\inf S_2 = 0$. Both 0 and 1 are not elements of $S_2$.

(c) If $S_3 = (0,2) \cup [3,5]$. Then $\sup\ S_3 = 5$ and $\inf S_3 = 0$. Note that 5 is an element of $S_3$ but 0 is not.

(d) If $S_4 = [72, \infty)$, then $\inf S_4 = 72$ but $\sup\ S_4$ does not exist.

(e) $\mathbb{R} = (-\infty, \infty)$ has no supremum and no infimum.

**Definition** Let $S$ be a nonempty subset of $\mathbb{R}$.

(i) If $u = \sup\ S$ and $u \in S$, then $u$ is also called the *maximum* of $S$. In this case, we write $u = \max S$.

(ii) If $v = \inf S$ and $v \in S$, then $v$ is also called the *minimum* of $S$. In this case, we write $v = \min S$.

In the example above, $\max S_1 = 4$ and $\min S_1 = 1$, but the set $S_2$ has no maximum and no minimum.

**Question:** Which kind of sets always have a maximum and a minimum?

**Exercise** For each of the following subsets $S$ of $\mathbb{R}$, determine by inspection $\sup S$, $\inf S$, $\max S$ and $\min S$ when they exist.

(a) $\left\{x \in \mathbb{R} :\ x \neq 2 \text{ and } 2 + x \geq \dfrac{2}{2-x}\right\}$.

(b) $\{x \in \mathbb{R} :\ |2x+1| < |x-1| + 1\}$.

(c) $\left\{x \in \mathbb{R} : \left|\dfrac{x}{x-1}\right| < \dfrac{1}{2}\right\}$.

(d) $\left\{1 - \dfrac{(-1)^n}{n}\ :\ n \in \mathbb{N}\right\}$.

**Answers:**
(a) $S = [-\sqrt{2},\ \sqrt{2}] \cup (2, \infty)$, $\inf S = \min S = -\sqrt{2}$, but $\sup S$ and $\max S$ do not exist.
(b) $S = (-3, 1/3)$, $\sup S = 1/3$, $\inf S = -3$, but both $\max S$ and $\min S$ do not exist.
(c) $S = (-1, 1/3)$, $\sup S = 1/3$ and $\inf S = -1$, but both $\max S$ and $\min S$ do not exist.
(d) $\sup S = \max S = 2$ and $\inf S = \min S = 1/2$.

**Lemma 1.9.1.** *Let $u$ be an upper bound of $S \subseteq \mathbb{R}$. Then $u = \sup S$ if and only if $\forall \varepsilon > 0$, $\exists\, x_\varepsilon \in S$ such that $u - \varepsilon < x_\varepsilon$.*

*Proof:* ($\Longrightarrow$) Suppose $u = \sup S$. Let $\varepsilon > 0$. Then $u - \varepsilon < u$, so $u - \varepsilon$ cannot be an upper bound for $S$. Hence $\exists\, x_\varepsilon \in S$ such that $x_\varepsilon > u - \varepsilon$.

($\Longleftarrow$) Suppose $\forall \varepsilon > 0$, $\exists\, x_\varepsilon \in S$ such that $u - \varepsilon < x_\varepsilon$. Assume that $u \neq \sup S$. Then there is an upper bound $v$ of $S$ such that $v < u$.

We now take $\varepsilon = u - v > 0$. Then $\exists\, x_\varepsilon \in S$ such that $u - \varepsilon < x_\varepsilon$. But $u - \varepsilon = u - (u - v) = v$. So $v < x_\varepsilon$. This contradicts the fact that $v$ is an upper bound for $S$. □

**Exercise** Let $u$ be a lower bound of $S \subseteq \mathbb{R}$. Prove that $u = \inf S$ if and only if for every $\varepsilon > 0$, there exists $x_\varepsilon \in S$ such that $x_\varepsilon < u + \varepsilon$.

Our final assumption on $\mathbb{R}$ is the following:

**The supremum property of $\mathbb{R}$** (or the completeness property/axiom)

*Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum.*

This means that

$$\boxed{S \text{ has an upper bound} \Longrightarrow \sup S \text{ exists.}}$$

The supremum property implies the following:

**The infimum property of $\mathbb{R}$:** *Every nonempty subset of $\mathbb{R}$ which is bounded below has a infimum.*

*Proof:* Let $S$ be a nonempty subset of $\mathbb{R}$ and it has a lower bound $b$. Let $A = \{-x : x \in S\}$. We have

$$x \geq b \quad \forall x \in S.$$

So

$$-x \le -b \quad \forall x \in S,$$

and this says $-b$ is an upper bound for $A$. Since $A$ is bounded above, by the supremum property of $\mathbb{R}$, $A$ has a supremum $u$.

Claim: $\inf S = -\sup A = -u$.

$u$ is an upper bound for $A$, so

$$-x \le u, \quad \forall -x \in A, \text{ or equivalently } \forall x \in S.$$

This gives

$$x \ge -u \quad \forall x \in S.$$

Hence $-u$ is a lower bound for $S$.

Let $v$ be another lower bound for $S$. Then $-v$ is an upper bound for $A$. Since $u = \sup A$, $u \le -v$. So $-u \ge v$. Hence $\inf S = -u$. □

**Example** Let $S$ be a nonempty subset of $\mathbb{R}$ and $a \in \mathbb{R}$. Let

$$a + S = \{a + x : x \in S\}.$$

Prove that if $S$ is bounded above, then $\sup(a + S) = a + \sup S$.

*Solution:* We have $x \le \sup S$, $\forall x \in S$. So

$$a + x \le a + \sup S \quad \forall x \in S.$$

This says that $a + \sup S$ is an upper bound for $a + S$.

Next suppose $v$ is any upper bound of $a + S$. Then

$$a + x \le v, \quad \forall x \in S.$$

So

$$x \le v - a \quad \forall x \in S,$$

and $v - a$ is an upper bound for $S$. Thus

$$\sup S \le v - a,$$

$$a + \sup S \le v.$$

We have shown that $a + \sup S$ is an upper bound for $a + S$ and is less than or equal to any other upper bound for $a + S$. Thus $\sup(a + S) = a + \sup S$. □

**Example** Let $A$ and $B$ be nonempty bounded subsets of $\mathbb{R}$, and let

$$C = \{a + b : \ a \in A, b \in B\}.$$

Prove that

$$\sup\ C = \sup\ A + \sup\ B.$$

*Solution:* Let $c \in C$. Then $c = a + b$ for some $a \in A$ and $b \in \sup B$. Now $a \leq \sup A$ and $b \leq \sup B$, so that

$$c = a + b \leq \sup A + \sup B.$$

Hence $\sup A + \sup B$ is an upper bound of $C$.

Next let $u$ be an upper bound of $C$. Then for all $a \in A$ and all $b \in B$,

$$a + b \leq u$$

or

$$a \leq u - b.$$

Thus for each $b \in B$, $u - b$ is an upper bound of $A$. Consequently,

$$\sup A \leq u - b.$$

This gives

$$b \leq u - \sup A \qquad \forall b \in B,$$

indicating that $u - \sup A$ is an upper bound of $B$. So

$$\sup B \leq u - \sup A$$

and

$$\sup A + \sup B \leq u.$$

This shows that $\sup A + \sup B$ is the smallest upper bound of $C$, i.e. $\sup C = \sup A + \sup B$.

**Archimedean property:** *If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ such that $x < n_x$.*

*Proof:* Suppose the statement is not true. Then there is a real number $x$ such that $x \geq n$ for all $n \in \mathbb{N}$. So $x$ is an upper bound for $\mathbb{N}$. By the supremum property, $u = \sup \mathbb{N}$ exists.

By taking $\varepsilon = 1$ and applying Lemma 1.9.1, $\exists\ m \in \mathbb{N}$ such that

$$u - 1 < m.$$

So

$$u < m + 1.$$

Since $m + 1 \in \mathbb{N}$, this says that $u$ is not an upper bound for $\mathbb{N}$. But $u = \sup S$, so we have obtained a contradiction. □

**Remark** The Archimedean Property implies that $\mathbb{N}$ is not bounded above.

**Corollary 1.9.2.** *For any $\varepsilon > 0$, $\exists\, n \in \mathbb{N}$ such that*

$$\frac{1}{n} < \varepsilon.$$

*Proof:* Let $x = 1/\varepsilon$. Then by the Archimedean property, $\exists\, n \in \mathbb{N}$ such that

$$x = \frac{1}{\varepsilon} < n.$$

Mulitplying $\varepsilon/n$ to the inequality gives

$$\frac{1}{n} = \left(\frac{\varepsilon}{n}\right) \cdot x < \left(\frac{\varepsilon}{n}\right) \cdot n = \varepsilon. \ \square$$

**Example** Let

$$S = \left\{\frac{1}{n} : \ n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}.$$

Prove that $\inf S = 0$.

*Solution:* Since 0 is a lower bound for $S$, $\inf S \geq 0$.

If $\inf S > 0$, then by Corollary 1.9.2, $\exists\, n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \inf S.$$

But this contradicts the definition of $\inf S$. So $\inf > 0$ is false and we must have $\inf S = 0$. □

**Corollary 1.9.3.** *If $x > 0$, then $\exists\, n \in \mathbb{N}$ such that*

$$n - 1 \leq x < n.$$

*Proof:* Let $S = \{m \in \mathbb{N} : \ x < m\}$. By the Archimedean property, $S \neq \emptyset$. By the well-ordering principle, $S$ has a least element $n$, that is,

$$n \in S, \ \text{ and } \ n \leq m \ \ \forall m \in S.$$

It follows that $n - 1 \notin S$, that is, $n - 1 \leq x$. So $n - 1 \leq x < n$. □

**Notation:** For any real number $x$, $[x]$ denotes the greatest integer less than or equal to $x$. In the above corollary, $[x] = n - 1$.

## 1.10 The existence of square root

**Theorem 1.10.1.** *There exists a unique positive real number $b$ with $b^2 = 2$.*

*Proof:* : Let $S = \{x \in \mathbb{R} : x > 0, \ x^2 < 2\}$. Then $S \neq \emptyset$ because $1 \in S$. On the other hand, if $y > 2$, then $y^2 > 4$ so that $y \notin S$. Thus if $x \in S$, then $x \leq 2$. So 2 is an upper bound of $S$. Since $S$ is bounded above, $b = \sup S$ exists.

We claim that $b^2 = 2$. We shall prove this by showing that it is impossible to have $b^2 < 2$ or $b^2 > 2$.

Suppose that $b^2 < 2$. Then

$$\frac{2b+1}{2-b^2} > 0.$$

By the Archimedean Property, $\exists n \in \mathbb{N}$ such that

$$n > \frac{2b+1}{2-b^2}.$$

Then

$$\left(b + \frac{1}{n}\right)^2 = b^2 + \frac{2b}{n} + \frac{1}{n^2} < b^2 + \frac{2b+1}{n} < b^2 + (2 - b^2) = 2.$$

Hence $b + 1/n \in S$. But $b + 1/n > b$. This contradicts the fact that $b = \sup \ S$.

Next assume that $b^2 > 2$. By the Archimedean Property, $\exists m \in \mathbb{N}$ such that

$$m > \frac{2b}{b^2 - 2}.$$

Then

$$\left(b - \frac{1}{m}\right)^2 = b^2 - \frac{2b}{m} + \frac{1}{m^2} > b^2 - \frac{2b}{m} > b^2 - (b^2 - 2) = 2.$$

If $x \in S$, then $x^2 < 2 < (b - 1/m)^2$, so that $x < b - 1/m$. Hence $b - 1/m$ is an upper bound of $S$. But $b - 1/m < b$, which again contradicts the fact $b = \sup S$.

Since the statements $b^2 < 2$ and $b^2 > 2$ are both false, we must have $b^2 = 2$.

Uniqueness: Is it possible to have a positive number $a$ such that $a \neq b$ and $a^2 = 2$?□

Using similar reasoning, we can prove that for any positive real number $c$, there exists a unique positive real number $b$ such that $b^2 = c$. We call $b$ the *positive square root* of $c$ and write

$$b = \sqrt{c}.$$

**Remark** The reasoning used in the proof for Theorem 1.10.1 can also be used to show that the supremum property does not hold for $\mathbb{Q}$. In fact, the set $A = \{r \in \mathbb{Q} : \ r \geq 0,\ r^2 < 2\}$ does not have a supremum in $\mathbb{Q}$.

## 1.11 The existence of nth root and rational exponents

**Theorem 1.11.1.** *Let $a > 0$ and $n \in \mathbb{N}$. There exists a unique positive real number $u$ with*

$$u^n = a.$$

*We call the number $u$ the positive nth root of $a$ and write $u = \sqrt[n]{a}$ or $a^{1/n}$.*

*Sketch of proof:* The proof is similar to the square root case. Let

$$S = \{t \in \mathbb{R} : \ t > 0,\ t^n < a\}.$$

Then one can show that $\dfrac{a}{1+a} \in S$ and $1 + a$ is an upper bound for $S$. Hence $S$ is nonempty and is bounded above. By the supremum property, $u = \sup S$ exists. We claim that $u^n = a$. We prove this by showing that it is impossible to have $u^n < a$ or $u^n > a$. Details are left as an exercise. □

**Exercise** Prove that if $a > 0$ and $n, m \in \mathbb{N}$, then

$$(a^{1/n})^m = (a^m)^{1/n}.$$

We can now define $a^r$ where $a > 0$ and $r$ is a rational number.

**Definition** For $a > 0$ and $n, m \in \mathbb{N}$, we define

$$a^{m/n} := (a^{1/n})^m$$

and

$$a^{-m/n} := \frac{1}{a^{m/n}}.$$

(We also define $a^0 = 1$.)

We need to check that the above definition of $a^r$ is well defined. That is, if $m, n, p, q$ are natural numbers such that $m/n = p/q$ , then is it true that

$$(a^{1/n})^m = (a^{1/q})^p?$$

To see this, note that $mq = np$ and

$$\left\{(a^{1/n})^m\right\}^q = (a^{1/n})^{mq} = (a^{1/n})^{np} = a^p.$$

Thus $(a^{1/n})^m$ is the $q$th root of $a^p$, that is,

$$(a^{1/n})^m = (a^p)^{1/q}.$$

**Theorem 1.11.2. (Properties of rational exponents)**

*(i) If $a > 0$ and $r, s \in \mathbb{Q}$, then $a^{r+s} = a^r a^s$ and $(a^r)^s = a^{rs}$.*

*(ii) If $0 < a < b$ and $r \in \mathbb{Q}$ with $r > 0$, then $a^r < b^r$.*

*(iii) If $a > 1$, $r, s \in \mathbb{Q}$ with $r < s$, then $a^r < a^s$.*

*Proof.* Exercise. □

## 1.12 Density of $\mathbb{Q}$

**The Density Theorem.** *If $a, b \in \mathbb{R}$ is such that $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

*Proof:* There are three cases to consider.

**Case 1:** $0 < a < b$.
In this case, $b - a > 0$. By Corollary 1.9.2, $\exists\, k \in \mathbb{N}$ such that

$$\frac{1}{k} < b - a.$$

Let $A = \{n \in \mathbb{N} : \frac{n}{k} > a\}$. By the Archimedean property, $\exists n_1 \in \mathbb{N}$ such that $n_1 > ak$. So $\frac{n_1}{k} > a$ and $n_1 \in A$. Thus $A \neq \emptyset$.

By the well-ordering principle, $A$ has a least element $n_0$. So

$$\frac{n_0}{k} > a \;\text{ and }\; \frac{n_0 - 1}{k} \leq a.$$

Then

$$a < \frac{n_0}{k} = \frac{n_0 - 1}{k} + \frac{1}{k} \leq a + \frac{1}{k} < a + (b - a) = b.$$

So $r = n_0/k$ is a rational number satisfying $a < r < b$.

**Case 2:** $a \leq 0 < b$. By Corollary 1.9.2, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < b$. So take $r = \frac{1}{n}$.

**Case 3:** $a < b \leq 0$. Then $0 \leq -b < -a$. By case 1 and 2, there is a rational number $r'$ satisfying $-b < r' < -a$. Take $r = -r'$. □

**Corollary 1.12.1.** *If $a, b \in \mathbb{R}$ is such that $a < b$, then there exists an irrational number $x$ such that $a < x < b$.*

*Proof:* By the density theorem, $\exists r \in \mathbb{Q}$ such that $r \neq 0$ and $\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$. So $a < r\sqrt{2} < b$ and $r\sqrt{2}$ is irrational. □

**Corollary 1.12.2.** *Every interval $I \subseteq \mathbb{R}$ contains infinitely many rational numbers and infinitely many irrational numbers.*

**Definition** A subset $D$ of $\mathbb{R}$ is said to be *dense* if for any $a, b \in \mathbb{R}$ with $a < b$, $D \cap (a, b) \neq \emptyset$.

We have proved that both $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ are dense. Clearly any set containing a dense set is also dense. Can you find a smaller dense set than $\mathbb{Q}$?

# Chapter 2: Sequences

## 2.1 Definition and examples

Informally, a sequence is an infinite list of numbers

$$(x_1, x_2, x_3, ..., x_n, x_{n+1}, ...)$$

defined according to some rule.

**Example** For the sequence $(2, 4, 6, 8, ...)$,

$$x_1 = 2, \quad x_2 = 4 = 2 \cdot 2, \quad x_3 = 6 = 2 \cdot 3, \; ... \,, \; x_n = 2n, \; ...$$

We can denote the sequence by $(2n)$.
$(2n)$ can be regarded as the function $X : \mathbb{N} \to \mathbb{R}$, $X(n) = 2n$, $n \in \mathbb{N}$.

**Definition** A *sequence* in $\mathbb{R}$ is a real-valued function $X$ with domain $\mathbb{N}$, that is,

$$X : \mathbb{N} \to \mathbb{R}.$$

The numbers $X(n)$ for $n = 1, 2, 3, ...$ are called the *terms* of the sequence.

**Notation** We usually write $x_n$ for $X(n)$ and denote the sequence $X$ either by

$$(x_n), \quad (x_n)_{n=1}^{\infty}, \quad \{x_n\} \text{ or } \{x_n\}_{n=1}^{\infty}.$$

**More examples**

$$\begin{aligned}
(\frac{1}{n}) &= (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ......) \\
(n+1) &= (2, 3, 4, 5, 6, ......) \\
((-1)^{n+1}) &= (1, -1, 1, -1, 1, -1, ......) \\
(\frac{n}{n+1}) &= (\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, ......) \\
(1 + (-\frac{1}{2})^n) &= (\frac{1}{2}, \frac{5}{4}, \frac{7}{8}, \frac{17}{16}, \frac{31}{32}, ......) \\
(1) &= (1, 1, 1, 1, 1, 1, ......)
\end{aligned}$$

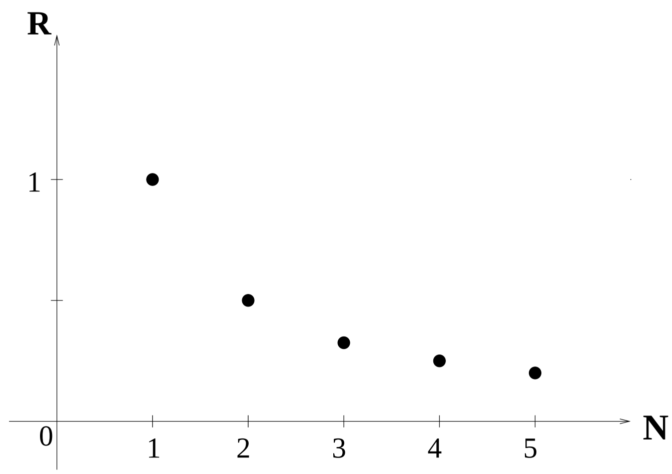
**Definition** A sequence of the form

$$(c) = (c, c, c, c, ......)$$

is called a *constant sequence*.

Given a sequence $(x_n)$. We are most interested in its limiting behavior, i.e. the pattern of $x_n$ when $n$ gets large.

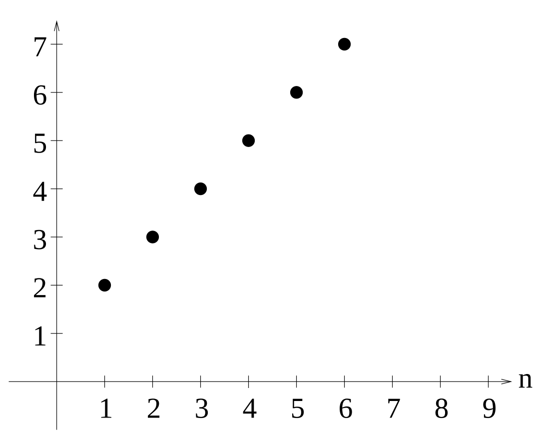
**Example** We examine the graph of the sequence $(1/n)$.



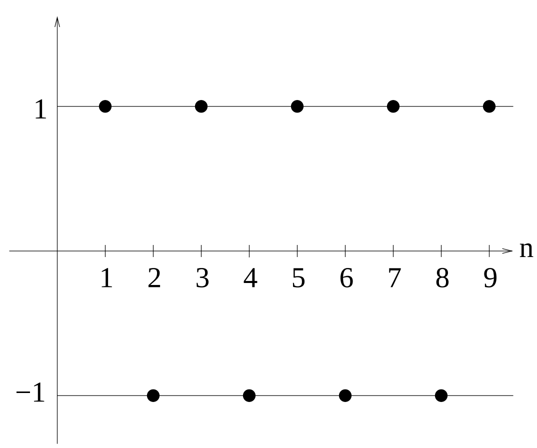

We note that as $n$ gets larger and larger, $1/n$ gets closer and closer to 0, that is, it tends to a "limiting value" of 0. We say $(1/n)$ *converges* to 0 and write

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

**Example** The two sequences below are "divergent".

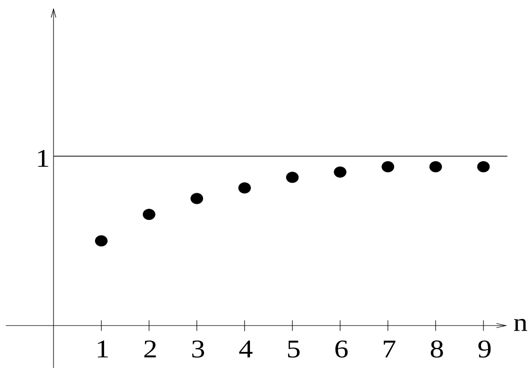



(n+1)
(increases without bound)



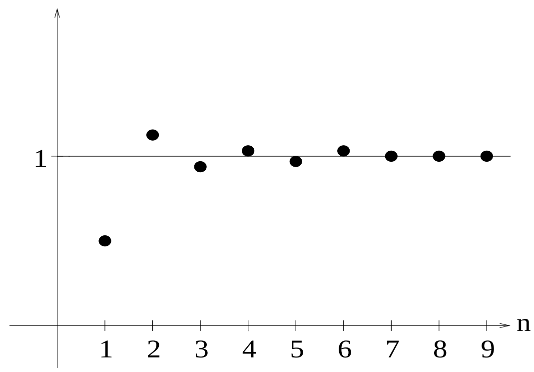

$((-1)^{n+1})$
(oscillates between −1 and 1)

The following two sequences are “convergent”.




$(\frac{n}{n+1})$

(increases towards a"limiting value" of 1)




$(1+(-1/2)^n)$

(tends towards a "limiting value" of 1 in an oscillating fashion)

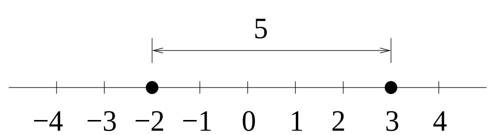
We say that the sequences $(\frac{n}{n+1})$ and $(1 + (-\frac{1}{2})^n)$ *converge* to 1. We also say 1 is the limit of $(\frac{n}{n+1})$ and $(1 + (-\frac{1}{2})^n)$. We will later give a precise definition of limit.

For $a, b \in \mathbb{R}$,

$$|a - b| = \text{ distance between } a \text{ and } b.$$

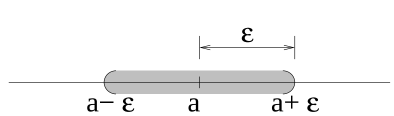
**Example** If $a = -2$ and $b = 3$, then

$$\text{distance between } -2 \text{ and } 3 \ = |(-2) - (3)| = |-5| = 5.$$




**Definition** Let $a \in \mathbb{R}$ and $\varepsilon > 0$. The $\varepsilon$-neighborhood of $a$ is the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} : \ |x - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon).$$

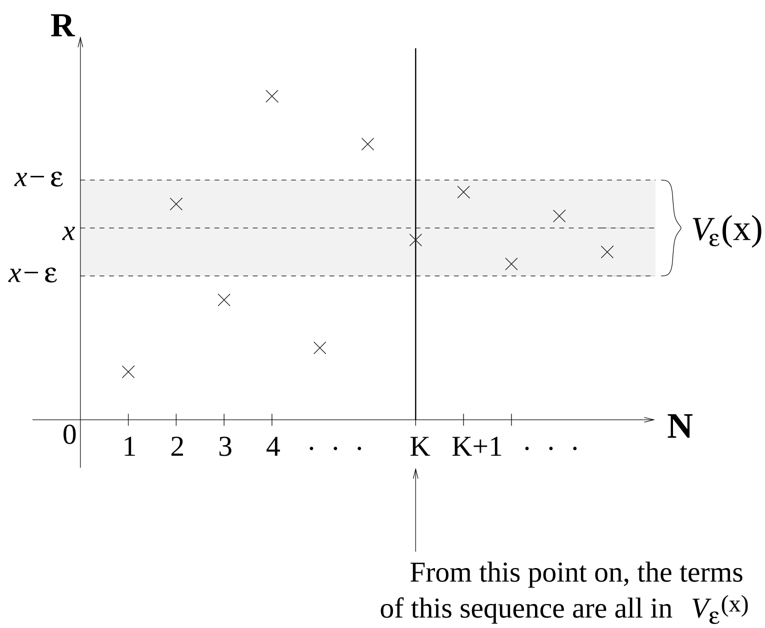


Note that $V_\varepsilon(a)$ contains points whose distance from $a$ is less than $\varepsilon$. Thus if $\varepsilon$ is very small and $x \in V_\varepsilon(a)$, then $x$ is very close to $a$.

**Definition** We say that $x$ is the *limit* of $(x_n)$ if for every $\varepsilon > 0$, there exists $K = K(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \quad \forall n \geq K.$$

(or equivalently, $x_n \in V_\varepsilon(x)$, $\forall n \geq K$.)




Roughly speaking, if $x$ is the limit of $(x_n)$, then we can make $x_n$ as close to $x$ as we wish by choosing $n$ big enough.

**Definition** If $(x_n)$ has a limit, then we say it is *convergent*. Otherwise, we say it is *divergent*.

**Theorem 2.1.1.** *If $(x_n)$ converges, then it has exactly one limit.*

*Proof:* Suppose $x$ and $x'$ are limits of $(x_n)$. Let $\varepsilon > 0$ be arbitrary, and let $\varepsilon' = \varepsilon/2$. Since $x_n \to x$, $\exists K_1 = K_1(\varepsilon') \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon' = \frac{\varepsilon}{2} \quad \forall n \geq K_1.$$

Similarly, since $x_n \to x'$, $\exists K_2 = K_2(\varepsilon') \in \mathbb{N}$ such that

$$|x_n - x'| < \varepsilon' = \frac{\varepsilon}{2} \quad \forall n \geq K_2.$$

Let $K = \max(K_1, K_2)$. Then

$$\begin{aligned} |x - x'| &= |(x - x_n) + (x_n - x')| \\ &\leq |x_n - x| + |x_n - x'| \quad \text{(by the triangle inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \qquad \forall n \geq K. \end{aligned}$$

Since $\varepsilon$ is arbitrary, $|x - x'| = 0$, and so $x = x'$. □

**Definition** If $x$ is the limit of $(x_n)$, then we say $(x_n)$ *converges* to $x$, and we write

$$\lim_{n\to\infty} x_n = x, \quad \text{or} \quad \lim x_n = x, \quad \text{or}$$

$$x_n \to x \text{ as } n \to \infty,$$

or simply

$$x_n \to x.$$

**Example** Prove that if $(x_n) = (c)$ is a constant sequence, then $\lim\limits_{n\to\infty} x_n = c$.

*Proof:* Let $\varepsilon > 0$ be given. Take $K = 1$. Then

$$|x_n - c| = |c - c| = 0 < \varepsilon, \qquad \forall n \geq K = 1.$$

**Example** Prove that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

*Proof:* Let $\varepsilon > 0$. By the Archimedean property, $\exists K = K(\varepsilon) \in \mathbb{N}$ such that $K > 1/\varepsilon$. Thus if $n \geq K$, then $n > 1/\varepsilon$, and $1/n < \varepsilon$. Thus

$$\left|\frac{1}{n} - 0\right| < \varepsilon, \quad \forall n \geq K.$$

**Exercise** Prove that $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$.

$\boxed{\textbf{Example}}$ Prove that $\lim_{n\to\infty} \frac{2n^2+1}{n^2+3n} = 2.$

*Proof:* We have

$$\begin{aligned} \left|\frac{2n^2+1}{n^2+3n} - 2\right| &= \left|\frac{1-6n}{n^2+3n}\right| \\ &\le \frac{1+6n}{n^2+3n} \\ &< \frac{n+6n}{n^2} \\ &= \frac{7n}{n^2} = \frac{7}{n}. \end{aligned}$$

Let $\varepsilon > 0$ be given. Choose $K \in \mathbb{N}$ such that $K > 7/\varepsilon$. Then

$$n \ge K \Longrightarrow \left|\frac{2n^2+1}{n^2+3n} - 2\right| < \frac{7}{n} \le \frac{7}{K} < \varepsilon.$$

$\boxed{\textbf{Remark}}$ To prove that a given sequence $(x_n)$ converges to $x$:

**Step 1:** Express $|x_n - x|$ in terms of $n$, and find a simple upper bound $L = L(n)$ for it, i.e., $|x_n - x| \le L$.

**Step 2:** Let $\varepsilon > 0$ be arbitrary. Find $K \in \mathbb{N}$ such that for all $n \ge K$, $L = L(n) < \varepsilon$. Then

$$n \ge K \Longrightarrow |x_n - x| \le L < \varepsilon.$$

In the previous example, $L(n) = \frac{7}{n}$.

$\boxed{\textbf{Exercise}}$ Prove that $\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}.$

**Reading** **The $K(\varepsilon)$ Game** (page 58-59 of the textbook)

**Example** Let $x_n = (-1)^n,\ n \in \mathbb{N}$. Prove that $(x_n)$ diverges.

*Proof:* Suppose $\lim_{n\to\infty} x_n = x$. Take $\varepsilon = 1/2$. Then $\exists K \in \mathbb{N}$ such that

$$|x_n - x| < \frac{1}{2}, \quad \forall n \geq K.$$

Since $x_n = 1$ or $-1$,

$$|1 - x| < \frac{1}{2} \quad \text{and} \quad |-1 - x| < \frac{1}{2}.$$

What is wrong here?

## 2.2 Limit theorems

**Definition** A sequence $(x_n)$ is said to be *bounded* if $\exists M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

**Example** The sequences $(1/n)$ and $(\frac{n}{2n+1})$ are bounded because

$$\left|\frac{1}{n}\right| \leq 1 \quad \text{and} \quad \left|\frac{n}{2n+1}\right| = \frac{n}{2n+1} \leq \frac{n}{2n} = \frac{1}{2}, \quad \forall n \in \mathbb{N}.$$

The sequences $(2n)$ and $(n+1)$ are unbounded.

**Theorem 2.2.1.** *Every convergent sequence is bounded.*

*Proof:* Let $(x_n)$ be a convergent sequence and $\lim_{n\to\infty} x_n = x$. Take $\varepsilon = 1$. Then $\exists K \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon = 1, \qquad \forall n \geq K.$$

Thus

$$n \geq K \Longrightarrow |x_n| = |(x_n - x) + x| \leq |x_n - x| + |x| \leq 1 + |x|.$$

Let $M = \max(|x_1|, ...|x_{K-1}|, |x| + 1)$. Then

$$|x_n| \leq M, \qquad \forall n \in \mathbb{N}.$$

So $(x_n)$ is bounded. □

**Remark** By the above theorem, any unbounded sequence is divergent. Thus $(2n)$ and $(n+1)$ are divergent.

**Question** Is it true that

$$(x_n) \text{ is bounded} \Longrightarrow (x_n) \text{ converges?}$$

---

Given two sequences $(x_n)$ and $(y_n)$. We can form the sequences $(x_n + y_n)$, $(x_n - y_n)$, $(x_n y_n)$ and $(x_n/y_n)$.

**Example** Let $(x_n) = (n^2)$ and $(y_n) = (2/n)$. Then

$$\begin{aligned}
(x_n + y_n) &= \left(n^2 + \frac{2}{n}\right) = \left(\frac{n^3+2}{n}\right) = \left(3, 5, \frac{29}{3}, ......\right) \\
(x_n - y_n) &= \left(n^2 - \frac{2}{n}\right) = \left(\frac{n^3-2}{n}\right) = \left(-1, 3, \frac{25}{3}, ......\right) \\
(x_n y_n) &= (2n) = (2, 4, 6, ......) \\
\left(\frac{x_n}{y_n}\right) &= \left(\frac{n^3}{2}\right) = \left(\frac{1}{2}, 4, \frac{27}{2}, ......\right).
\end{aligned}$$

**Theorem 2.2.2.** *If* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$*, then*

*(i)* $\lim_{n\to\infty} (x_n + y_n) = x + y$*;*

*(ii)* $\lim_{n\to\infty} (x_n - y_n) = x - y$*;*

*(iii)* $\lim_{n\to\infty} (x_n y_n) = xy$*;*

*(iv)* $\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \frac{x}{y}$*, provided* $y_n \neq 0$, $\forall n \in \mathbb{N}$*, and* $y \neq 0$.

**Remark**

(a) An important special case is when one of the sequence is a constant sequence. For example, if $x_n \to x$ and $c$ is a constant, then

- $\lim_{n\to\infty} (c + x_n) = c + \lim_{n\to\infty} x_n = c + x$,
- $\lim_{n\to\infty} cx_n = c \lim_{n\to\infty} x_n = cx$.

(b) Using induction, the theorem extends naturally to $k$ sequences.

**Corollary 2.2.3.** *If* $(x_n)$ *converges and* $k \in \mathbb{N}$*, then*

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k.$$

Using these rules, we can now compute the limits of a large number of sequences very efficiently.

**Example** Show that for any $c \in \mathbb{R}$ and $k \in \mathbb{N}$, $\lim_{n\to\infty} \frac{c}{n^k} = 0$.

*Solution.*

$$\lim_{n\to\infty} \frac{c}{n^k} = c \lim_{n\to\infty} \left(\frac{1}{n}\right)^k = c\left(\lim_{n\to\infty} \frac{1}{n}\right)^k = c \cdot 0^k = 0.$$

**Example** Compute $\lim_{n\to\infty} \dfrac{2n^3+n^2}{n^3+5}$.

*Solution.*

$$\lim_{n\to\infty} \frac{2n^3+n^2}{n^3+5} = \lim_{n\to\infty} \frac{\frac{2n^3+n^2}{n^3}}{\frac{n^3+5}{n^3}} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{1+\frac{5}{n^3}} \overset{\text{(iv)}}{=} \frac{\lim_{n\to\infty}\left(2+\frac{1}{n}\right)}{\lim_{n\to\infty}\left(1+\frac{5}{n^3}\right)} \overset{\text{(i)}}{=} \frac{2+\lim_{n\to\infty}\frac{1}{n}}{1+\lim_{n\to\infty}\frac{5}{n^3}} = 2.$$

**Exercise** Compute $\lim_{n\to\infty} \left(\dfrac{n}{1+2n}\right)^5$.

---

***Proof of Theorem 2.2.2.***

(i) Let $\varepsilon > 0$. Since $x_n \to x$ and $y_n \to y$, there exist $K_1, K_2 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{2} \qquad \forall n \geq K_1,$$

$$|y_n - y| < \frac{\varepsilon}{2} \qquad \forall n \geq K_2.$$

Let $K = \max(K_1, K_2)$. Then

$$\begin{aligned} |(x_n+y_n)-(x+y)| &= |(x_n-x)+(y_n-y)| \\ &\leq |x_n-x|+|y_n-y| \quad \text{(by triangle inequality)} \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &= \varepsilon, \qquad \forall n \geq K. \end{aligned}$$

The Proof for (ii) is similar.

(iii) Since $(x_n)$ converges, it is bounded. Thus there exists $M_1 > 0$ such that

$$|x_n| \leq M_1, \qquad \forall n \in \mathbb{N}.$$

Now

$$\begin{aligned} |x_ny_n - xy| &= |(x_ny_n - x_ny) + (x_ny - xy)| \\ &\leq |x_n(y_n-y)| + |(x_n-x)y| \\ &= |x_n||y_n-y| + |x_n-x||y| \\ &\leq M_1|y_n-y| + |y||x_n-x| \\ &\leq M\left(|y_n-y|+|x_n-x|\right), \end{aligned}$$

where $M = \max(M_1, |y|)$.

Now let $\varepsilon > 0$ be given. Since $x_n \to x$ and $y_n \to y$, there exist $K_1, K_2 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{2M}, \qquad \forall n \geq K_1,$$

$$|y_n - y| < \frac{\varepsilon}{2M}, \qquad \forall n \geq K_2.$$

Let $K = \max(K_1, K_2)$. Then

$$|x_n y_n - xy| < M\left(\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M}\right) = \varepsilon, \quad \forall n \geq K.$$

This shows $\lim x_n y_n = xy$.

(iv) We first show that $\lim\limits_{n\to\infty} \left(\frac{1}{y_n}\right) = \frac{1}{y}$.

Let $\varepsilon_1 = \frac{|y|}{2} > 0$. Since $y_n \to y$, there exists $K_1 \in \mathbb{N}$ such that

$$|y_n - y| < \varepsilon_1 = \frac{|y|}{2}, \qquad \forall n \geq K_1.$$

Now we have

$$|y_n - y| \geq ||y_n| - |y|| \geq |y| - |y_n|.$$

Thus for $n \geq K_1$,

$$|y| - |y_n| < \frac{|y|}{2}$$

which gives

$$|y_n| > \frac{|y|}{2}.$$

Now let $\varepsilon > 0$ be given. Then there exists $K_2 \in \mathbb{N}$ such that

$$|y_n - y| < \frac{|y|^2}{2} \cdot \varepsilon, \qquad \forall n \geq K_2.$$

Let $K = \max(K_1, K_2)$. Then

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{|y_n - y|}{|y_n y|} < \frac{\frac{|y|^2 \varepsilon}{2}}{\frac{|y|}{2}|y|} = \varepsilon, \quad \forall n \geq K.$$

This shows $\lim\limits_{n\to\infty} \left(\frac{1}{y_n}\right) = \frac{1}{y}$.

Now it follows that

$$\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \lim_{n\to\infty} \left(x_n \cdot \frac{1}{y_n}\right) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} \frac{1}{y_n}\right) = x \cdot \frac{1}{y} = \frac{x}{y}.$$

**Squeeze Theorem.** *If $x_n \leq y_n \leq z_n$, $\forall n$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$, then*

$$\lim_{n\to\infty} y_n = a.$$

*Proof:* Let $\varepsilon > 0$. Since $x_n \to a$ and $z_n \to a$, $\exists K \in \mathbb{N}$ such that for $n \geq K$,

$$|x_n - a| < \varepsilon \quad \text{and} \quad |z_n - a| < \varepsilon,$$

$$\text{i.e.,} \quad -\varepsilon < x_n - a < \varepsilon \quad \text{and} \quad -\varepsilon < z_n - a < \varepsilon.$$

So

$$-\varepsilon < x_n - a \leq y_n - a \leq z_n - a < \varepsilon \quad \forall n \geq K,$$

and

$$|y_n - a| < \varepsilon \quad \forall n \geq K. \ \square$$

**Classic Example** Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

*Solution.* We have $-1 \leq \sin n \leq 1$. So

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

Now

$$\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \left(-\frac{1}{n}\right) = 0.$$

By the squeeze theorem, $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

**Theorem 2.2.4.** *If $|x_n| \to 0$, then $x_n \to 0$.*

*Proof:* Let $\varepsilon > 0$. Then there exists $K \in \mathbb{N}$ such that

$$n \geq K \Longrightarrow ||x_n| - 0| < \varepsilon.$$

But $||x_n| - 0| = |x_n - 0|$. So $|x_n - 0| < \varepsilon$ for all $n \geq K$. $\square$

**Theorem 2.2.5.** *If* $0 < b < 1$*, then* $\lim\limits_{n\to\infty} b^n = 0$.

*Proof:* Let $a = \frac{1}{b} - 1$. Note that $a > 0$, and

$$b = \frac{1}{1+a}.$$

Now for all $n \in \mathbb{N}$, by Bernoulli's inequality

$$(1+a)^n \geq 1 + na,$$

so that

$$0 < b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} \leq \frac{1}{na}.$$

Now

$$\lim_{n\to\infty} \frac{1}{na} = \frac{1}{a} \lim_{n\to\infty} \frac{1}{n} = 0.$$

So by the Squeeze theorem, $\lim\limits_{n\to\infty} b^n = 0$. □

**Example** By the above theorem, we now know that $\lim\limits_{n\to\infty} \frac{1}{2^n} = 0$ and $\lim\limits_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$.

**Remark** Theorems 2.2.4 and 2.2.5 together imply that $b^n \to 0$ for all $b$ with $|b| < 1$.

**Question:** If $b > 1$, what can say about the sequence $(b^n)$?

**Theorem 2.2.6.** *If* $c > 0$*, then* $\lim\limits_{n\to\infty} c^{\frac{1}{n}} = 1$.

*Proof:* We shall consider two cases.

**Case 1:** $c \geq 1$.
In this case, $c^{\frac{1}{n}} \geq 1$. Let $d_n = c^{\frac{1}{n}} - 1$. Then $d_n \geq 0$, $c^{\frac{1}{n}} = 1 + d_n$ and so $c = (1 + d_n)^n$. By Bernoulli's inequality,

$$c = (1 + d_n)^n \geq 1 + nd_n$$

so that

$$0 \leq d_n \leq \frac{c-1}{n}.$$

Now

$$\lim_{n\to\infty} \frac{c-1}{n} = (c-1)\lim_{n\to\infty} \frac{1}{n} = 0.$$

Thus by the Squeeze theorem, $\lim_{n\to\infty} d_n = 0$. Consequently,

$$\lim_{n\to\infty} c^{\frac{1}{n}} = \lim_{n\to\infty} (1 + d_n) = 1 + \lim_{n\to\infty} d_n = 1.$$

**Case 2:** $0 < c < 1$.
In this case, $\frac{1}{c} > 1$. By Case 1, $\lim_{n\to\infty} (1/c)^{\frac{1}{n}} = 1$. Consequently,

$$\lim_{n\to\infty} c^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{(1/c)^{\frac{1}{n}}} = \frac{1}{\lim_{n\to\infty} (1/c)^{\frac{1}{n}}} = 1. \square$$

$\boxed{\textbf{Example}}$ By the theorem, we now know that $\lim_{n\to\infty} 2^{\frac{1}{n}} = 1$.

**Theorem 2.2.7.** *(a) If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} |x_n| = |x|$.*

*(b) If all $x_n \geq 0$ and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.*

*Proof:* (a) Let $\varepsilon > 0$. Since $x_n \to x$, $\exists K \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon, \qquad \forall n \geq K.$$

Since $|x_n - x| \geq ||x_n| - |x||$ for all $n$,

$$n \geq K \Longrightarrow ||x_n| - |x|| \leq |x_n - x| < \varepsilon.$$

So $|x_n| \to |x|$.

(b) We will only prove the case $x > 0$. Let $\varepsilon > 0$. There exists $K \in \mathbb{N}$ be such that

$$n \geq K \Longrightarrow |x_n - x| < \sqrt{x}\varepsilon.$$

Then

$$n \geq K \Longrightarrow \left|\sqrt{x_n} - \sqrt{x}\right| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} < \frac{1}{\sqrt{x}}|x_n - x| < \frac{\sqrt{x}\varepsilon}{\sqrt{x}} = \varepsilon. \square$$

**Theorem 2.2.8.**
$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

*Proof:* First note that $n^{\frac{1}{n}} > 1$ for $n \geq 2$. Let $k_n = n^{\frac{1}{n}} - 1$. Then $n^{\frac{1}{n}} = 1 + k_n$.

By the Binomial theorem, for $n \geq 2$,

$$n = (1 + k_n)^n = 1 + k_n + \frac{n(n-1)}{2}k_n^2 + \cdots + k_n^n \geq \frac{n(n-1)}{2}k_n^2,$$

i.e.

$$n \geq \frac{n(n-1)}{2}k_n^2.$$

So for $n \geq 2$,

$$0 \leq k_n^2 \leq \frac{2}{n-1}.$$

By the squeeze theorem, $\lim_{n\to\infty} k_n^2 = 0$. So $\lim_{n\to\infty} k_n = 0$ and

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} (1 + k_n) = 1. \ \square$$

**Exercise** Evaluate the following limits:

(i) $\displaystyle\lim_{n\to\infty} \frac{2^n + 3^{n+1} + 5^{n+2}}{2^{n+2} + 3^n + 5^{n+1}}$.

(ii) $\displaystyle\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right)$.

**Theorem 2.2.9.** *(a) If $x_n \geq 0$ for all $n \in \mathbb{N}$ and $(x_n)$ converges, then $\lim_{n\to\infty} x_n \geq 0$.*

*(b) If $(x_n)$ and $(y_n)$ are convergent and $x_n \geq y_n$ for all $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n.$$

*(c) If $a, b \in \mathbb{R}$ and $a \leq x_n \leq b$ for all $n$ and $(x_n)$ is convergent, then*

$$a \leq \lim_{n\to\infty} x_n \leq b.$$

*Proof:* (a) Let $x = \lim_{n\to\infty} x_n$. Assume to the contrary that $x < 0$. Take $\varepsilon = -x > 0$. Then since $x_n \to x$, there exists $K \in \mathbb{N}$ such that

$$n \geq K \Longrightarrow |x_n - x| < \varepsilon = -x.$$

So for $n \geq K$,

$$x_n < x + \varepsilon = x - x = 0.$$

But this contradicts the assumption that $x_n \geq 0$ for all $n \in \mathbb{N}$. Hence $x \geq 0$.

(b) Let $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Note that $x_n - y_n \geq 0$ for all $n$ and $x_n - y_n \to x - y$. By (a), $x - y \geq 0$. So $x \geq y$.

(c) Exercise. □

## 2.3 Monotone sequences

**Definition** We say the sequence $(x_n)$ is

- *increasing* if
$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$$
- *decreasing* if
$$x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots.$$
- *monotone* if it is either increasing or decreasing.

**Example** Which of the following sequences are increasing, decreasing or monotone?

$$\begin{aligned}
(2n+1) &= (3, 5, 7, 9, 11, \ldots\ldots) \\
(1 + (-1)^n) &= (0, 2, 0, 2, 0, 2, \ldots\ldots) \\
\left(\frac{1}{n}\right) &= \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots\right) \\
\left(\frac{n}{n+1}\right) &= \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\ldots\right) \\
(c) &= (c, c, c, c, c, \ldots\ldots) \\
(a^n) &= (a, a^2, a^3, a^4, a^5, \ldots\ldots).
\end{aligned}$$

Recall that if a sequence $(x_n)$ is convergent, then it is bounded. The converse in general is false. However if $(x_n)$ is monotone and bounded, then it is convergent.

**Monotone Convergence Theorem**

*If $(x_n)$ is monotone and bounded, then it converges. In this case,*

$$\lim_{n\to\infty} x_n = \begin{cases} \sup\{x_n : n \in \mathbb{N}\} & \text{if } x_n \uparrow \\ \\ \inf\{x_n : n \in \mathbb{N}\} & \text{if } x_n \downarrow \end{cases}$$

*Proof:* **Case 1:** $(x_n)$ is increasing and bounded.

Let $S = \{x_n : n \in \mathbb{N}\}$. Since $(x_n)$ is bounded, there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Thus $M$ is an upper bound of the set $S$. By the supremum property of $\mathbb{R}$, $x = \sup S$ exists. We shall prove that $\lim\limits_{n\to\infty} x_n = x$.

Let $\varepsilon > 0$. Since $x = \sup S$, $x - \varepsilon$ is not an upper bound of $S$. So there exists $x_K \in S$ such that $x_K > x - \varepsilon$. Thus $0 \leq x - x_K < \varepsilon$.

Since $(x_n)$ is increasing, $x_K \leq x_n$ for all $n \geq K$. It follows that for all $n \geq K$, we have

$$0 \leq x - x_n \leq x - x_K < \varepsilon.$$

So $|x - x_n| < \varepsilon$ for all $n \geq K$, and this says $\lim\limits_{n\to\infty} x_n = x$.

**Case 2:** $(x_n)$ is decreasing and bounded.

Use similar reasoning or consider the sequence $(-x_n)$. □

**Example** Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{x_n + 2}$ for all $n \in \mathbb{N}$, i.e.

$$(x_n) = (\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, ......).$$

Prove that $(x_n)$ converges and find its limit.

*Solution:* **Step 1:** Prove that $x_n \leq 2$ for all $n \in \mathbb{N}$ by induction.

Let P($n$) be the statement $x_n \leq 2$. Clearly P(1) holds.

Assume that P($k$) holds, i.e. $x_k \leq 2$. Then

$$x_{k+1} = \sqrt{x_k + 2} \leq \sqrt{2+2} = \sqrt{4} = 2.$$

So P($k+1$) holds. By the principle of mathematical induction, $x_n \leq 2$ for all $n \in \mathbb{N}$.

**Step 2:** Prove that $(x_n)$ is increasing by induction.

Let P($n$) be the statement $x_n \leq x_{n+1}$. Then since $x_1 = \sqrt{2} \leq x_2 = \sqrt{2+\sqrt{2}}$, P(1) holds.

Assume that P($k$) holds, i.e. $x_k \leq x_{k+1}$. Then

$$x_{k+1} = \sqrt{x_k + 2} \leq \sqrt{x_{k+1} + 2} = x_{k+2}.$$

So P($k+1$) holds. By the principle of mathematical induction, $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, that is, $(x_n)$ is increasing.

**Step 3:** Apply the monotone convergence theorem.

Since $(x_n)$ is increasing and bounded, it converges. Let $x$ be its limit.
For all $n \in \mathbb{N}$, $x_{n+1} = \sqrt{x_n + 2}$, so that

$$x_{n+1}^2 = x_n + 2.$$

Taking limits on both sides,

$$\lim_{n\to\infty} x_{n+1}^2 = \lim_{n\to\infty} (x_n + 2) = (\lim_{n\to\infty} x_n) + 2,$$

which gives

$$x^2 = x + 2, \quad \text{or} \quad x^2 - x - 2 = (x-2)(x+1) = 0.$$

So either $x = -1$ or $x = 2$. But $x_n \geq x_1 = \sqrt{2}$ for all $n$, so $\lim_{n\to\infty} x_n = x \geq \sqrt{2}$. It is impossible to have $x = -1$. So $x = 2$. □

**Example** Let $0 < b < 1$ and $y_n = b^n$ for $n \in \mathbb{N}$. Then

$$y_{n+1} = b^{n+1} = b \cdot b^n = by_n < y_n \qquad \forall n \in \mathbb{N}.$$

So $(y_n)$ is decreasing. It is also bounded below by 0. So $(y_n)$ converges by the Monotone Convergence Theorem.

If $y = \lim_{n\to\infty} y_n$, then $y = by$ and $y(1-b) = 0$. Since $1 - b \neq 0$, $y = 0$. This gives an alternative proof of Theorem 2.2.5.

**Nested Interval Theorem**

*Let $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ be a nested sequence of closed bounded intervals, that is, $I_n \supseteq I_{n+1}$ for $n \in \mathbb{N}$. Then the intersection*

$$\bigcap_{n=1}^{\infty} I_n = \{x : \ x \in I_n \ \forall n \in \mathbb{N}\}$$

*is nonempty. In addition, if*

$$\text{length of } I_n = b_n - a_n \to 0,$$

*then $\bigcap_{n=1}^{\infty} I_n$ contains exactly one point.*

*Proof.* First we have $a_n \leq b_n$ for all $n \in \mathbb{N}$. Since $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, the sequence $(a_n)$ is increasing and $(b_n)$ is decreasing. So

$$a_1 \leq a_n \leq b_n \leq b_1 \qquad \forall n \in \mathbb{N}.$$

Thus $(a_n)$ is bounded above by $b_1$ and $(b_n)$ is bounded below by $a_1$. By the Monotone Convergence Theorem, both $(a_n)$ and $(b_n)$ converge. Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then $a \geq a_n$ and $b \leq b_n$for all $n \in \mathbb{N}$. Moreover, $a \leq b$. So $a, b \in \bigcap_{n=1}^{\infty} I_n$.

Next assume that $a_n - b_n \to 0$. Then $a = b$. Suppose $c \in \bigcap_{n=1}^{\infty} I_n$. Then

$$a_n \leq c \leq b_n \qquad \forall n \in \mathbb{N}.$$

By letting $n \to \infty$, we obtain

$$a \leq c \leq a.$$

So $c = a$ and $\bigcap_{n=1}^{\infty} I_n = \{a\}$. □

**The harmonic series**

For each $n \in \mathbb{N}$, let

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Then $(x_n)$ is clearly increasing.

*Question: Is $(x_n)$ bounded?*

Sample computations show $x_{50,000} \approx 11.4$ and $x_{100,000} \approx 12.1$, suggesting that $(x_n)$ is likely to be bounded.

$$\begin{aligned}
x_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}} + \cdots + \frac{1}{2^n}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right) \\
&= 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \cdots + 2^{n-1} \cdot \frac{1}{2^n} \\
&= 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \frac{1}{2}}_{n} = 1 + \frac{n}{2}.
\end{aligned}$$

For any $M > 0$, by the Archimedean property, $\exists n \in \mathbb{N}$ such that

$$1 + \frac{n}{2} > M \iff n > 2(M-1).$$

So $M$ is not an upper bound for $\{x_n : n \in \mathbb{N}\}$. Consequently $(x_n)$ is not bounded, and it diverges.

**Remark** We say that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

---

**The Euler number**

Consider the sequence

$$e_n = \left(1 + \frac{1}{n}\right)^n, \quad \forall n \in \mathbb{N}.$$

We claim that $(e_n)$ is increasing and bounded.

**Why is $(e_n)$ increasing?**

Recall in Tutorial 1, we have proved:

$$\left(1 + \frac{1}{n-1}\right)^{n-1} < \left(1 + \frac{1}{n}\right)^n.$$

This says that $e_{n-1} < e_n$.

**Why is $(e_n)$ bounded?**

We have proved in page 4 of Chapter 1 that $2^{k-1} \leq k!$ for all $k \in \mathbb{N}$. So

$$\frac{1}{k!} \leq \frac{1}{2^{k-1}}.$$

Using this and the Binomial expansion of $e_n$, for $n \geq 2$,

$$\begin{aligned}
2 = e_1 < e_n &= \left(1 + \frac{1}{n}\right)^n \\
&= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \cdots + \frac{n(n-1)\cdots 2 \cdot 1}{n!} \cdot \frac{1}{n^n} \\
&< 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 2 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\
&= 2 + \left(1 - \frac{1}{2^{n-1}}\right) < 3.
\end{aligned}$$

Since $(e_n)$ is increasing and bounded, by the monotone convergence theorem, $(e_n)$ converges.

**Definition** The limit of $(e_n)$ is denoted by $e$ and is called the *Euler number*.

It is known that $e \approx 2.718$.

**Example** Compute $\lim_{n\to\infty} \left(1 + \frac{1}{n+3}\right)^{2n}$.

*Solution:*

$$\begin{aligned}
\lim_{n\to\infty} \left(1 + \frac{1}{n+3}\right)^{2n} &= \lim_{n\to\infty} \left(1 + \frac{1}{n+3}\right)^{2(n+3)-6} \\
= \lim_{n\to\infty} \frac{\left[\left(1 + \frac{1}{n+3}\right)^{n+3}\right]^2}{\left[1 + \frac{1}{n+3}\right]^6} &= \frac{\lim_{n\to\infty} (e_{n+3})^2}{\lim_{n\to\infty} \left(1 + \frac{1}{n+3}\right)^6} = \frac{e^2}{1} = e^2.
\end{aligned}$$

## 2.4 Subsequences and the Bolzano-Weierstrass theorem

The sequence $(2n)$ can be obtained by deleting the odd indexed terms from $(n)$:

$$(2, 4, 6, 8, 10, ......) = (\cancel{1}, 2, \cancel{3}, 4, \cancel{5}, 6, ...).$$

We say that the $(2n)$ is a subsequence of $(n)$.

In general, a subsequence of a sequence is obtained by deleting certain terms from the sequence (without messing up the original ordering!)

**Example** The following are subsequences of $(1/n)$:

$$\begin{aligned}
\left(\frac{1}{n+3}\right) &= \left(\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, ......\right) = \left(\cancel{1}, \cancel{\frac{1}{2}}, \cancel{\frac{1}{3}}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, ......\right) \\
\left(\frac{1}{3n-2}\right) &= \left(1, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, ......\right) = \left(1, \cancel{\frac{1}{2}}, \cancel{\frac{1}{3}}, \frac{1}{4}, \cancel{\frac{1}{5}}, \cancel{\frac{1}{6}}, \frac{1}{7}, \cancel{\frac{1}{8}}, \cancel{\frac{1}{9}}, \frac{1}{10}, ......\right) \\
\left(\frac{1}{n^2}\right) &= \left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, ......\right) = \left(1, \cancel{\frac{1}{2}}, \cancel{\frac{1}{3}}, \frac{1}{4}, \cancel{\frac{1}{5}}, \cancel{\frac{1}{6}}, \cancel{\frac{1}{7}}, \cancel{\frac{1}{8}}, \frac{1}{9}, ......\right)
\end{aligned}$$

The following are **not** subsequences of $(1/n)$:

$$\begin{gathered}
\left(\frac{1}{3}, 1, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, ......\right) \\
\left(1, \frac{1}{3}, \frac{1}{5}, \frac{2}{3}, \frac{1}{7}, ......\right).
\end{gathered}$$

**Definition** Let $(x_n)$ be a sequence and let

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots.$$

be an increasing sequence of natural numbers. The sequence

$$(x_{n_k}) = (x_{n_k})_{k=1}^{\infty} = (x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}, ......, x_{n_k}, ......)$$

is called a *subsequence* of $(x_n)$.

**Remark** Recall that formally a sequence is a real-valued function on $\mathbb{N}$. If $X, Y : \mathbb{N} \to \mathbb{R}$ are sequences, then $Y$ is a subsequence of $X$ if there is a strictly increasing function $Z : \mathbb{N} \to \mathbb{N}$ such that $Y = X \circ Z$, that is, $Y$ is the composition of $X$ with $Z$. In our notation, $X(n) = x_n$, $Z(k) = n_k$, so $Y(k) = X(Z(k)) = X(n_k) = x_{n_k}$.

**Example** In the previous example, $(x_n) = (1/n)$, and

- $\left(\dfrac{1}{n+3}\right) = \left(\dfrac{1}{4}, \dfrac{1}{5}, \dfrac{1}{6}, \dfrac{1}{7}, ......\right) = (x_4, x_5, x_6, x_7, ......) = (x_{n_k})$ with $n_k = k + 3$.

- $\left(\dfrac{1}{3n-2}\right) = \left(1, \dfrac{1}{4}, \dfrac{1}{7}, \dfrac{1}{10}, \dfrac{1}{13}, ......\right) = (x_1, x_4, x_7, x_{10}, x_{13}, ......) = (x_{n_k})$ with $n_k = 3k - 2$.

- $\left(\dfrac{1}{n^2}\right) = \left(1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \dfrac{1}{25}, ......\right) = (x_1, x_4, x_9, x_{16}, x_{25}, ......) = (x_{n_k})$ with $n_k = k^2$.

**Example**

- If $n_k = 2k$, then $(x_{n_k}) = (x_{2k}) = (x_2, x_4, x_6, x_8, x_{10}, ......)$ is the subsequence of "even terms".

- If $n_k = 2k - 1$, then $(x_{n_k}) = (x_{2k-1}) = (x_1, x_3, x_5, x_7, x_9, ......)$ is the subsequence of "odd terms".

- If $m \in \mathbb{N}$ and $n_k = m + k$, then the subsequence $(x_{n_k}) = (x_{m+k}) = (x_{m+1}, x_{m+2}, x_{m+3}, ......)$ is called the $m$-tail of $(x_n)$.

**Note:** If $(x_{n_k})$ is a subsequence of $(x_n)$, then $n_k \geq k$.

**Theorem 2.4.1.** *If $(x_n)$ converges to $x$, then any subsequence $(x_{n_k})$ also converges to $x$.*

*Proof:* Let $\varepsilon > 0$. Then $\exists K \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq K$.

Now $n_k \geq k$ for all $k \in \mathbb{N}$. So if $k \geq K$, $n_k \geq K$. It follows that

$$|x_{n_k} - x| < \varepsilon, \quad \forall k \geq K. \ \square$$

**Example** What is the limit of $\left(\left(1+\frac{1}{2n^2}\right)^{2n^2}\right)$?

*Solution:* We observe that $\left(\left(1+\frac{1}{2n^2}\right)^{2n^2}\right)$ is a subsequence of $\left(\left(1+\frac{1}{n}\right)^{n}\right)$. By Theorem 2.4.1,

$$\lim_{n\to\infty}\left(1+\frac{1}{2n^2}\right)^{2n^2}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e.$$

**Corollary 2.4.2.** *If $(x_n)$ has a subsequence which is divergent, then $(x_n)$ diverges.*

**Example** Let

$$x_n=\begin{cases}\dfrac{1}{n} & \text{when } n \text{ is odd}\\[2ex] n & \text{when } n \text{ is even,}\end{cases}$$

that is,

$$(x_n)=\left(1,2,\frac{1}{3},4,\frac{1}{5},6,\frac{1}{7},......\right).$$

Then the even terms forms the subsequence

$$(x_{2n})=(2,4,6,8,......)$$

which is divergent. So $(x_n)$ is divergent.

**Corollary 2.4.3.** *If $(x_n)$ has two convergent subsequences whose limits are not equal, then $(x_n)$ diverges.*

**Special case of Corollary 2.4.3:** *If the odd terms and the even terms of $(x_n)$ do not converge to the same limit, then $(x_n)$ diverges.*

**Example** Show that the sequence $((-1)^n)$ diverges.

- Odd terms $\to\ -1$.
- Even terms $\to\ 1$.

So $((-1)^n)$ diverges.

**Example** Is the sequence $\left(\dfrac{(-1)^n n}{n+1}\right)$ convergent?

- Odd terms $x_{2k-1} = -\dfrac{2k-1}{2k} = -1 + \dfrac{1}{2k} \to\ -1$.
- Even terms $x_{2k} = \dfrac{2k}{2k+1} \to\ 1$.

So $\left(\dfrac{(-1)^n n}{n+1}\right)$ diverges.

**Exercise** For each $n \in \mathbb{N}$, let $x_n = \dfrac{n\sin(n\pi/3)}{n+1}$. Is the sequence $(x_n)$ convergent?

**Example** For each $n \in \mathbb{N}$, let $x_n = \sin n$. Then $(x_n)$ diverges because it has two subsequences $(x_{n_k})$ and $(x_{m_k})$ with the property that $x_{n_k} > 1/2$ and $x_{m_k} < -1/2$ for all $k \in \mathbb{N}$.

**Monotone Subsequence Theorem**

*Every sequence has a monotone subsequence.*

*Proof:* Let $(x_n)$ be a sequence. We call a natural number $m$ a *peak point* of $(x_n)$ if

$$x_m \geq x_n, \qquad \forall n \geq m.$$

**Case 1:** $(x_n)$ has infinitely many peak points.

If $m_1 < m_2 < m_3 < \cdots$ are the peak points, then

$$x_{m_1} \geq x_{m_2} \geq x_{m_3} \geq x_{m_4} \geq x_{m_5} \geq \cdots$$

So $(x_{m_k})$ is a decreasing subsequence of $(x_n)$.

**Case 2:** $(x_n)$ has only finitely many peak points.

Let $m_1 < m_2 < \cdots < m_j$ be all the peak points.

Let $n_1 = m_j + 1$. Then

$$n_1 \text{ is not a peak point } \implies \exists\, n_2 > n_1 \text{ such that } x_{n_2} > x_{n_1}.$$
$$n_2 \text{ is not a peak point } \implies \exists\, n_3 > n_2 \text{ such that } x_{n_3} > x_{n_2}.$$

Continuing this way, we obtain an increasing subsequence $(x_{n_k})$ of $(x_n)$. □

**Bolzano-Weierstrass Theorem**

*Every bounded sequence has a convergent subsequence.*

*Proof:* Let $(x_n)$ be a bounded sequence.

By the monotone subsequence theorem, $(x_n)$ has a monotone subsequence $(x_{n_k})$.

Since $(x_n)$ is bounded, so is $(x_{n_k})$.

By the monotone convergence theorem, $(x_{n_k})$ converges. □

**Example** The sequence $((-1)^n)$ diverges, but

$$((-1)^{2n-1}) = (-1, -1, -1, -1, ......) \qquad \text{and} \qquad ((-1)^{2n}) = (1, 1, 1, 1, ......)$$

are convergent subsequences.

## 2.5 Real exponents

Let $a > 0$ and $r \in \mathbb{Q}$. We have defined $a^r$ in Section 1.11 of Chapter 1: If $r > 0$ and $r = m/n$ where $n, m \in \mathbb{N}$, then

$$a^r = a^{m/n} := (a^{1/n})^m$$

and

$$a^{-r}; = \frac{1}{a^r}.$$

**How to define $a^x$ when the exponent $x$ is irrational?** For example, what is $3^{\sqrt{2}}$?

Note that $(1, 1.4, 1.41, 1.414, ...)$ is a rational sequence with limit $\sqrt{2}$. So $3^{\sqrt{2}}$ should be the limit of the sequence

$$3^1, 3^{1.4}, 3^{1.41}, 3^{1.414}, ...$$

**Definition** Let $a > 0$ and let $x$ be a real number.

(i) If $a \geq 1$, then we define

$$a^x := \lim_{n\to\infty} a^{r_n}$$

where $(r_n)$ is an increasing rational sequence which converges to $x$.

(ii) If $0 < a < 1$, we define

$$a^x = \left(\frac{1}{a}\right)^{-x}.$$

**Lemma 2.5.1.** *Let $x \in \mathbb{R}$. Then there exists an increasing rational sequence $(r_n)$ which converges to $x$.*

*Proof.* By the Density Theorem, there exists $r_1 \in \mathbb{Q}$ such that $x - 1 < r_1 < x$. We proceed with induction: assume that $r_{n-1}$ has been chosen for some natural number $n > 1$. By the Density theorem again, there exists $r_n \in \mathbb{Q}$ such that

$$\max(r_{n-1}, x - \frac{1}{n}) < r_n < x.$$

In this way, we obtained an increasing rational sequence $(r_n)$ such that

$$x - \frac{1}{n} < r_n < x.$$

Note that $x - 1/n \to x$. So by the Squeeze Theorem, $r_n \to x$. □

**Theorem 2.5.2.** *The above definition of $a^x$ is well defined.*

*Proof.* Assume that $a \geq 1$, and let $(r_n)$ be an increasing rational sequence with limit $x$. By Part (iii) of Theorem 1.11.2 in Chapter 1, the sequence $(a^{r_n})$ is increasing. Take a rational number $r$ such that $r > x$. Then $a^{r_n} < a^r$ for all $n$, so $(a^{r_n})$ is also bounded. By the Monotone Convergence Theorem, $(a^{r_n})$ converges. Let $L = \lim_{n\to\infty} a^{r_n}$.

We need to show that the definition of $a^x$ does not depend on the choice of the sequence $(r_n)$. So let $(s_n)$ be another increasing rational sequences with limit $x$. We claim that $(a^{s_n})$ also converges to $L$.

To see this, let

$$R_n = r_n - \frac{1}{n}, \quad S_n = s_n - \frac{1}{n}, \quad n \in \mathbb{N}.$$

Then $(R_n)$ and $(S_n)$ are increasing rational sequence such that

$$R_n < x, \quad R_n \to x, \quad S_n < x \text{ and } S_n \to x.$$

We now construct two subsequences $(R_{n_k})$ and $(S_{m_k})$ as follows: Let $n_1 = 1$. Since $R_{n_1} < x$, there exists $m_1$ such that

$$R_{n_1} < S_{m_1} < x.$$

Similarly there exists $n_2 > n_1$ such that $S_{m_1} < R_{n_2} < x$. Continuing this process, we obtain

$$R_{n_1} < S_{m_1} < R_{n_2} < S_{m_2} < \cdots$$

We now let

$$t_n = \begin{cases} R_{n_k} & n = 2k-1 \\ S_{m_k} & n = 2k. \end{cases}$$

Then $(t_n)$ is an increasing rational sequence with limit $x$, and as before the sequence $(a^{t_n})$ is convergent.

Now note that

$$a^{R_n} = a^{r_n - 1/n} = a^{r_n} a^{-1/n} = \frac{a^{r_n}}{a^{1/n}} \to \frac{L}{1} = L$$

and similarly

$$\lim_{n\to\infty} a^{S_n} = \lim_{n\to\infty} a^{s_n}.$$

It follows that

$$L = \lim_{n\to\infty} a^{R_n} = \lim_{k\to\infty} a^{R_{n_k}} = \lim_{n\to\infty} a^{t_n} = \lim_{k\to\infty} a^{t_{2k}} = \lim_{k\to\infty} a^{S_{m_k}} = \lim_{n\to\infty} a^{S_n} = \lim_{n\to\infty} a^{s_n}. \ \square$$

**Theorem 2.5.3.** *If $a \geq 1$ and $(r_n)$ is a decreasing rational sequence with limit $x$, then*

$$\lim_{n\to\infty} a^{r_n} = a^x.$$

*Proof.* Exercise.

**Theorem 2.5.4. (Properties of exponents)**

*(i)* $a^{x+y} = a^x a^y$. *(ii)* $(a^x)^y = a^{xy}$.

*(iii) If $a > 1$ and $x < y$, then $a^x < a^y$.*

*Proof.* (i) Only need to prove the case when $a \geq 1$. Let $(x_n)$ and $(y_n)$ are increasing rational sequences such that $x_n \to x$ and $y_n \to y$. Then $a^{x_n} \to a^x$ and $a^{y_n} \to a^y$. Now $(x_n + y_n)$ is an increasing rational sequence and $x_n + y_n \to x + y$, so $a^{x_n+y_n} \to a^{x+y}$. On the other hand,

$$a^{x_n+y_n} = a^{x_n}a^{y_n} \to a^x a^y.$$

By the uniqueness of limit,

$$a^{x+y} = \lim_{n\to\infty} a^{x_n+y_n} = a^x a^y.$$

The proof of (ii) and (iii) are left as exercise. □

## 2.6 Limit superior and limit inferior

**Definition** Let $(x_n)$ be a sequence. A point $x$ is called a *cluster point* of $(x_n)$ if $(x_n)$ has a subsequence $(x_{n_k})$ which converges to $x$, that is,

$$x_{n_k} \to x.$$

**Example** Let $x_n = (-1)^n + \frac{1}{n}$, $n \in \mathbb{N}$. Then

$$x_{2k} = 1 + \frac{1}{2k} \to 1 \qquad \text{and} \qquad x_{2k-1} = -1 + \frac{1}{2k-1} \to -1.$$

So 1 and $-1$ are cluster points of $(x_n)$.

**Notation** Let $C(x_n)$ be the set of all cluster points of $(x_n)$.

**Definition** Let $(x_n)$ be a bounded sequence. By the Bolzalno-Weiestrass Theorem, $(x_n)$ has a convergent subsequence. So in this case $C(x_n)$ is nonempty. Moreover, $C(x_n)$ is bounded.

(i) We define the *limit superior* of $(x_n)$ to be

$$\limsup x_n = \sup C(x_n).$$

(ii) We define the *limit inferior* of $(x_n)$ to be

$$\liminf x_n = \inf C(x_n).$$

**Remarks:** Some books used the notation $\overline{\lim}_{n\to\infty} x_n$ for $\limsup x_n$ and $\underline{\lim}_{n\to\infty} x_n$ for $\liminf x_n$.

**Example** If $x_n = (-1)^n + \dfrac{1}{n}$ $(n \in \mathbb{N})$, then $C(x_n) = \{-1, 1\}$. So

$$\limsup x_n = 1 \qquad \text{and} \qquad \liminf x_n = -1.$$

**Exercise** Let

$$x_n = \frac{(2n^2 + 3)\sin(n\pi/4)}{\sqrt{4n^4 + 5n^3 - 1}}, \qquad n \in \mathbb{N}.$$

Find $\limsup x_n$ and $\liminf x_n$.

**Theorem 2.6.1.** *Let $(x_n)$ be a bounded sequence and let $M = \limsup x_n$.*

*(i) For each $\varepsilon > 0$, there are at most finitely many $n$'s such that $x_n \geq M + \varepsilon$. Equivalently, there exists $K \in \mathbb{N}$ such that*

$$n \geq K \Longrightarrow x_n < M + \varepsilon.$$

*(ii) For each $\varepsilon > 0$, there are infinitely many $n$'s such that $x_n > M - \varepsilon$.*

*Proof.* Suppose (i) is false. Then there exists $\varepsilon > 0$ such that there are infinitely many $n$'s such that $x_n \geq M + \varepsilon$. We now choose a subsequence $(x_{n_k})$ from these terms. Then

$$x_{n_k} \geq M + \varepsilon, \qquad \forall k \in \mathbb{N}.$$

Since $(x_{n_k})$ is bounded, it has a convergent subsequence $x_{n_{k_\ell}} \to x$ and $x \geq M + \varepsilon$. So $x \in C(x_n)$ and $x > M$. But this contradicts the fact $M = \sup C(x_n)$. This proves (i).

Next suppose (ii) is false. Then there exists $\varepsilon > 0$ such that there are only finitely many $n$'s such that $x_n > M - \varepsilon$. It follows that no subsequence of $(x_n)$ can have a limit greater than $M - \varepsilon$. So $M - \varepsilon$ is an upper bound for $C(x_n)$. But this again contradicts the fact that $M = \sup C(x_n)$. This proves (ii). $\square$

**Exercise** Prove that the converse of Theorem 2.6.1 is also true, that is, if $M$ is a real number satisfying conditions (i) and (ii), then $M = \limsup x_n$.

**Theorem 2.6.2.** *Let $(x_n)$ be a bounded sequence and let $m = \liminf x_n$.*

*(i) For each $\varepsilon > 0$, there are at most only finitely many $n$'s such that $x_n \le m - \varepsilon$. Equivalently, there exists $K \in \mathbb{N}$ such that*

$$n \ge K \Longrightarrow x_n > m - \varepsilon.$$

*(ii) For each $\varepsilon > 0$, there are infinitely many $n$'s such that $x_n < m + \varepsilon$.*

*Proof.* Exercise. □

**Theorem 2.6.3.** *Let $(x_n)$ be a bounded sequence. Then $(x_n)$ converges if and only if*

$$\limsup x_n = \liminf x_n.$$

*Proof.* ($\Longrightarrow$): If $x_n \to x$, then by Theorem 2.4.1, every subsequence of $(x_n)$ also converges to $x$. Consequently $C(x_n) = \{x\}$ and $\limsup x_n = \liminf x_n = x$.

($\Longleftarrow$): Let $M = \limsup x_n = \liminf x_n$ and $\varepsilon > 0$. Then by Theorems 2.6.1 and 2.6.2, there exists $K \in \mathbb{N}$ such that

$$n \ge K \Longrightarrow \begin{cases} x_n < M + \varepsilon \\ \\ x_n > M - \varepsilon \end{cases} \Longrightarrow -\varepsilon < x_n - M < \varepsilon \Longrightarrow |x_n - M| < \varepsilon.$$

Hence $\lim x_n = M$. □

**Theorem 2.6.4.** *Let $(x_n)$ and $(y_n)$ be bounded sequence such that $x_n \le y_n$ for every $n \in \mathbb{N}$. Then*

$$\limsup x_n \le \limsup y_n$$

*and*

$$\liminf x_n \le \liminf y_n.$$

*Proof.* Let $x$ be a cluster point of $(x_n)$ and $x_{n_k} \to x$. Consider the subsequence $(y_{n_k})$ of $(y_n)$. Since it is bounded, it has a convergent subsequence $(y_{n_{k_\ell}})$. Then since $x_{n_{k_\ell}} \le y_{n_{k_\ell}}$ for all $\ell$,

$$x = \lim_{k\to\infty} x_{n_k} = \lim_{\ell\to\infty} x_{n_{k_\ell}} \le \lim_{\ell\to\infty} y_{n_{k_\ell}} \le \limsup y_n.$$

This shows that $\limsup y_n$ is an upper bound for $C(x_n)$. It follows that $\limsup x_n \leq \limsup y_n$.

The proof of the second inequality is left as an exercise. □

**Exercise** Let $(a_n)$ be a bounded sequence. Prove that there is a subsequence of $(a_n)$ which converges to $\limsup\ a_n$.

**Exercise** **(Alternative definition of limit superior)**
Let $(x_n)$ be a bounded sequence. For each $n \in \mathbb{N}$, let

$$y_n = \sup\{x_j : \ j \geq n\}.$$

(i) Prove that the sequence $(y_n)$ is convergent.

(ii) Let $y = \lim_{n\to\infty} y_n$. Prove that there is a subsequence $(x_{n_k})$ of $(x_n)$ such that $x_{n_k} \to y$.

(iii) Prove that $\limsup x_n = y$.

## 2.7 The Cauchy criterion

**Definition** A sequence $(x_n)$ is called a *Cauchy sequence* if for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon, \qquad \forall n, m \geq K.$$

(This means that for large $n$, the $x_n$'s are very close to each other.)

**Theorem 2.7.1.** *Every convergent sequence is Cauchy.*

*Proof:* Suppose $x_n \to x$.
Let $\varepsilon > 0$. Then $\exists\ K \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{2}, \qquad \forall n \geq K.$$

It follows that

$$\begin{aligned} |x_n - x_m| &= |(x_n - x) - (x_m - x)| \\ &\leq |x_n - x| + |x_m - x| \text{ (triangle inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n, m \geq K. \end{aligned}$$

Thus $(x_n)$ is Cauchy. □

The Converse of the above theorem is also true!

**Theorem 2.7.2.** *Every Cauchy sequence is bounded .*

*Proof:* Take $\varepsilon = 1$. Then $\exists\, K \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon = 1, \qquad \forall n, m \geq K.$$

In particular, putting $m = K$, we obtain

$$|x_n - x_K| < \varepsilon = 1, \qquad \forall n \geq K.$$

It follows that for $n \geq K$, we have

$$\begin{aligned} |x_n| &= |(x_n - x_K) + x_K| \\ &\leq |x_n - x_K| + |x_K| \\ &< 1 + |x_K|. \end{aligned}$$

Let $M = \max(|x_1|, |x_2|, ..., |x_{K-1}|, 1 + |x_K|)$. Then

$$|x_n| \leq M, \qquad \forall n \in \mathbb{N}.$$

So $(x_n)$ is bounded. □

**Cauchy criterion.** *Every Cauchy sequence is convergent.*

*Proof:* Let $(x_n)$ be a Cauchy sequence.

By Theorem 2.7.2, it is bounded.

By the Bolzano-Weierstrass theorem, it has a convergent subsequence $(x_{n_k})$.

Let $x = \lim_{k\to\infty} x_{n_k}$.

**Claim:** $x_n \to x$.

Let $\varepsilon > 0$. Since $(x_n)$ is Cauchy, $\exists\, K_1 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2}, \qquad \forall n, m \geq K_1.$$

Now $x_{n_k} \to x$. So $\exists K_2 \in \mathbb{N}$ such that $K_2 \geq K_1$ and

$$|x_{n_k} - x| < \frac{\varepsilon}{2}, \qquad \forall k \geq K_2.$$

In particular,

$$|x_{n_{K_2}} - x| < \frac{\varepsilon}{2}.$$

Since $K_2 \geq K_1$, $n_{K_2} \geq K_1$, so that

$$|x_n - x_{n_{K_2}}| < \frac{\varepsilon}{2}, \qquad \forall n \geq K_1.$$

It follows that for all $n \geq K_1$,

$$\begin{aligned} |x_n - x| &= |(x_n - x_{n_{K_2}}) + (x_{n_{K_2}} - x)| \\ &\leq |x_n - x_{n_{K_2}}| + |x_{n_{K_2}} - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \ \square \end{aligned}$$

$\boxed{\textbf{Definition}}$ A sequence $(x_n)$ is said to be *contractive* if $\exists\, C$ with $0 < C < 1$ such that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|, \qquad \forall n \in \mathbb{N}.$$

**Theorem 2.7.3.** *Every contractive sequence is Cauchy (and so is convergent).*

*Proof:* Suppose that $(x_n)$ is a contractive sequence and

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|, \qquad \forall n \in \mathbb{N},$$

for some $0 < C < 1$.

By applying the above inequality repeatedly, we obtain for all $n \geq 2$,

$$\begin{aligned} |x_{n+1} - x_n| &\leq C|x_n - x_{n-1}| \\ &\leq C^2|x_{n-1} - x_{n-2}| \\ &\leq \cdots \\ &\leq C^{n-1}|x_2 - x_1|. \end{aligned}$$

Now if $m > n$, then

$$\begin{aligned} |x_m - x_n| &\leq |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)| \\ &\leq |(x_m - x_{m-1})| + |(x_{m-1} - x_{m-2})| + \cdots + |(x_{n+1} - x_n)| \\ &\leq \left(C^{m-2} + C^{m-3} + \cdots + C^{n-1}\right)|x_2 - x_1| \\ &= C^{n-1}\left(1 + C + \cdots + C^{m-n-1}\right)|x_2 - x_1| \\ &= C^{n-1} \cdot \frac{1 - C^{m-n}}{1 - C}|x_2 - x_1| \\ &\leq \frac{C^{n-1}}{1 - C}|x_2 - x_1| \\ &= \frac{C^n}{C(1 - C)}|x_2 - x_1|. \end{aligned}$$

Now let $\varepsilon > 0$. Since $0 < C < 1$, $C^n \to 0$. So $\exists\, K \in \mathbb{N}$ such that

$$C^n = |C^n - 0| < \frac{C(1 - C)}{|x_2 - x_1|} \cdot \varepsilon, \qquad \forall n \geq K.$$

It follows that for $m > n \geq K$,

$$|x_m - x_n| \leq \frac{|x_2 - x_1|}{C(1 - C)} C^n < \frac{|x_2 - x_1|}{C(1 - C)} \cdot \frac{C(1 - C)}{|x_2 - x_1|} \cdot \varepsilon = \varepsilon.$$

Thus $(x_n)$ is Cauchy. □

$\boxed{\textbf{Example}}$ Prove that the sequence $(x_n)$ defined by

$$x_1 = 2, \qquad x_{n+1} = \frac{1}{2 + x_n}, \quad \forall n \in \mathbb{N},$$

is convergent and find its limit.

*Solution:* Clearly $x_n > 0$ for all $n \in \mathbb{N}$. We have

$$
\begin{aligned}
|x_{n+2} - x_{n+1}| &= \left|\frac{1}{2+x_{n+1}} - \frac{1}{2+x_n}\right| = \left|\frac{2+x_n-2-x_{n+1}}{(2+x_{n+1})(2+x_n)}\right| \\
&= \frac{|x_{n+1}-x_n|}{(2+x_{n+1})(2+x_n)} \\
&\leq \frac{|x_{n+1}-x_n|}{2\cdot 2} \\
&= \frac{1}{4}|x_{n+1}-x_n|.
\end{aligned}
$$

Thus $(x_n)$ is contractive. By the previous theorem, it is Cauchy, and so is convergent.

Let $x = \lim_{n\to\infty} x_n$. Then since $x_{n+1} = \dfrac{1}{2+x_n}$,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2+x_n} = \frac{1}{2+\lim_{n\to\infty} x_n}$$

and we obtain

$$x = \frac{1}{2+x}$$

or

$$x^2 + 2x - 1 = 0.$$

The solutions of this equations are $\pm\sqrt{2} - 1$. Since $x_n > 0$ for all $n \in \mathbb{N}$, $x \geq 0$. So $x = \sqrt{2} - 1$.

## 2.8 Properly divergent sequences

$\boxed{\textbf{Definition}}$ We say that a sequence $(x_n)$ *tends to* $\infty$ if for every $M > 0$, there exists $K \in \mathbb{N}$ such that

$$x_n > M, \qquad \forall n \geq K.$$

In this case, we write

$$\lim_{n\to\infty} x_n = \infty$$

or

$$x_n \to \infty \quad \text{as} \quad n \to \infty.$$

**Remark**

- $\infty$ is not a real number!
- Sequences which tend to $\infty$ are clearly divergent (although we write $\lim_{n\to\infty} x_n = \infty$ for such a sequence $(x_n)$.)

**Example** Prove that if $(x_n)$ is increasing and unbounded, then $x_n \to \infty$.

*Proof.* Let $M > 0$. Since $(x_n)$ is unbounded, $\exists K \in \mathbb{N}$ such that

$$x_K > M.$$

Since $(x_n)$ is increasing, $x_n \geq x_K$ for all $n \geq K$. Thus

$$x_n > M, \qquad \forall n \geq K. \quad \square$$

**Example** The following are special cases of the above examples:

- $\lim_{n\to\infty} n = \infty$.
- $\lim_{n\to\infty} n^k = \infty$ where $k \in \mathbb{N}$.
- $\lim_{n\to\infty} b^n = \infty$ where $b > 1$.
- $\lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) = \infty$.

**Definition** We say that a sequence $(x_n)$ *tends to* $-\infty$ if for every $M < 0$, there exists $K \in \mathbb{N}$ such that

$$x_n < M, \qquad \forall n \geq K.$$

In this case, we write

$$\lim_{n\to\infty} x_n = -\infty,$$

or

$$x_n \to -\infty \quad \text{as} \quad n \to \infty.$$

**Definition** We call a sequence $(x_n)$ *properly divergent* if either $x_n \to \infty$ or $x_n \to -\infty$.

# Chapter 3: Infinite Series

## 3.1 Definition and examples

**Summation notation:**

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

**Example**

$$\begin{aligned}\sum_{k=1}^{n} \frac{3}{10^k} &= \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n} \\ &= \frac{\frac{3}{10}\left[1 - \left(\frac{1}{10}\right)^n\right]}{1 - \frac{1}{10}} \\ &= \frac{1}{3}\left(1 - \frac{1}{10^n}\right).\end{aligned}$$

Given a sequence $(a_n)$, we can form an *infinite series* which is the "sum"

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots$$

**Example** Consider the infinite series

$$\sum_{k=1}^{\infty} \frac{3}{10^k} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$$

Let

$$\begin{aligned} s_1 &= \frac{3}{10} \\ s_2 &= \frac{3}{10} + \frac{3}{10^2} \\ s_3 &= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} \\ &\vdots \\ s_n &= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n}.\end{aligned}$$

Intuitively,

$$\sum_{k=1}^{\infty} \frac{3}{10^k} \approx s_n$$

for very large $n$, and for *larger* $n$, $s_n$ gives *better* approximation for the series. So we define

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{3}{10^k} &= \lim_{n\to\infty} s_n \\
&= \lim_{n\to\infty} \left( \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} \right) \\
&= \lim_{n\to\infty} \frac{1}{3} \left( 1 - \frac{1}{10^n} \right) \\
&= \frac{1}{3}.
\end{aligned}$$

This agrees with our intuition:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{3}{10^k} &= 0.3 + 0.03 + 0.003 + 0.0003 + \cdots \\
&= 0.3333\cdots \\
&= \frac{1}{3}.
\end{aligned}$$

**Definition** Let $(a_n)$ be a sequence. The infinite series generated by $(a_n)$ is the sequence $(s_n)$ defined by

$$s_n = a_1 + a_2 + \cdots + a_n, \qquad n = 1, 2, 3, ...$$

It is denoted by

$$\sum_{n=1}^{\infty} a_n.$$

(i) We say that $a_n$ is a *term* of the series and $s_n$ is a *partial sum* of the series.

(ii) If $(s_n)$ converges to a limit $s$, then we say the series $\sum_{n=1}^{\infty} a_n$ converges to $s$, and we write

$$s = \sum_{n=1}^{\infty} a_n.$$

The limit $s$ is called the *sum* of the series.

(iii) If $(s_n)$ diverges, then we say the series $\sum_{n=1}^{\infty} a_n$ diverges (and it has no sum).

$\boxed{\textbf{Example}}$ Does the series $\sum_{k=1}^{\infty}(-1)^{k+1}$ converge?

*Solution:* Note that $s_1 = 1$, $s_2 = 0$, $s_3 = 1$, $s_4 = 0, \ldots$. So the sequence of partial sums $(s_n)$ is the oscillating sequence

$$(1, 0, 1, 0, ...)$$

which is divergent. So the series $\sum_{k=1}^{\infty}(-1)^{k+1}$ diverges (and it has no sum).

$\boxed{\textbf{Example}}$ In Chapter 2, we proved that the sequence

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is unbounded, so is divergent.

Note that $x_n$ is the partial sum of the the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

So the harmonic series diverges.

**Geometric series**

$$\sum_{k=1}^{\infty} r^{k-1} = 1 + r + r^2 + r^3 + \cdots$$

$r$ is called the *common ratio* of the series.

**Two uninteresting cases:**

What can you say about the geometric series when $r = 1$ and when $r = -1$?

Now assume $r \neq \pm 1$. Then

$$\begin{aligned} s_n &= 1 + r + r^2 + \cdots + r^{n-1} \\ &= \frac{1 - r^n}{1 - r} \qquad \text{(sum of a G.P.).} \end{aligned}$$

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, so that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - r^n}{1 - r} = \frac{1 - 0}{1 - r} = \frac{1}{1 - r},$$

that is,

$$\sum_{k=1}^{\infty} r^{k-1} = \frac{1}{1 - r}.$$

If $|r| > 1$, then $(r^n)$ diverges, so that $(s_n)$ diverges. In this case, we say that the geometric series $\sum_{k=1}^{\infty} r^{k-1}$ diverges.

**Example** The series $\sum_{k=1}^{\infty} 2^{k-1}$ diverges because its common ratio is $r = 2$ and $|r| > 1$.

**Example** The series $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$ converges since its common ratio is $r = 1/2$ and $|r| < 1$:

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

**Example** $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots = ?$

*Solution:* Use “partial fractions” techniques to write

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Then

$$
\begin{aligned}
s_n &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\
&= 1 - \frac{1}{n+1},
\end{aligned}
$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

**Theorem 3.1.1.** *(a) If the series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent, then the series* $\sum_{n=1}^{\infty}(a_n + b_n)$ *is also convergent and*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

*(b) If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent and* $c \in \mathbb{R}$*, then the series* $\sum_{n=1}^{\infty} ca_n$ *is also convergent and*

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$

*Proof.* (a) For each $n \in \mathbb{N}$, let

$$s_n = a_1 + \cdots + a_n,\ t_n = b_1 + \cdots + b_n \text{ and } r_n = (a_1 + b_1) + \cdots + (a_n + b_n).$$

The series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent. This means that $(s_n)$ and $(t_n)$ are convergent. Now

$$r_n = s_n + t_n \ \ \forall n \in \mathbb{N}.$$

So $(r_n)$ is also convergent, and $\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$.

The proof for (b) is similar. □

**Theorem 3.1.2.** *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

*Proof:* Let $s_n = a_1 + \cdots + a_n$. Since $\sum_{n=1}^{\infty} a_n$ converges, $(s_n)$ converges to a limit $s$, i.e. $\lim_{n\to\infty} s_n = s$. Now for each $n$,

$$s_{n+1} = (a_1 + \cdots + a_n) + a_{n+1} = s_n + a_{n+1},$$

and

$$a_{n+1} = s_{n+1} - s_n.$$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} s_{n+1} - \lim_{n\to\infty} s_n = s - s = 0. \ \square$$

**The n-th term divergence test:** *If* $\lim_{n\to\infty} a_n \neq 0$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

*Proof:* This is the contrapositive of Theorem 3.1.2. $\square$

**Example** Does the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ converge?

*Solution:* Since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, the series diverges by the $n$-th term divergence test.

**Warning:** $\lim_{n\to\infty} a_n = 0$ does not imply that $\sum_{n=1}^{\infty} a_n$ converges.

**Example** The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges although $\lim_{n\to\infty} \frac{1}{n} = 0$.

**Cauchy criterion for series:**

*The series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if for every* $\varepsilon > 0$*, there exists* $K \in \mathbb{N}$ *such that*

$$|a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon, \qquad \forall m > n \geq K.$$

*Proof:* Let $(s_n)$ be the sequence of partial sums. Then

$$|s_n - s_m| = |(a_1 + \cdots + a_n) - (a_1 + \cdots + a_n + a_{n+1} + \cdots + a_m)| = |a_{n+1} + a_{n+2} + \cdots + a_m|.$$

Now apply the Cauchy criterion for sequences to $(s_n)$. □

## 3.2 Series with nonnegative terms

**Theorem 3.2.1.** *If* $a_n \geq 0$ *for all n, then the series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if the sequence* $(s_n)$ *of partial sums is bounded.*

*Proof:* For each $n \in \mathbb{N}$,

$$s_{n+1} - s_n = a_n \geq 0,$$

so that

$$s_{n+1} \geq s_n.$$

Thus $(s_n)$ is increasing. By the Monotone Convergence Theorem, $(s_n)$ converges if and only if it is bounded. □

**Remark** If $a_n \geq 0$ for all $n$ and the series $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} a_n = \infty$.

For example, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

**Theorem 3.2.2.** *If $p > 1$, then the $p$-series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges.*

*Proof.* Let $s_n = \sum_{m=1}^{n} \frac{1}{m^p}$. Since $1/m^p > 0$ for all $m$, $(s_n)$ is an increasing sequence. We consider the subsequence

$$(s_{n_k}) = (s_1, s_3, s_7, s_{15}, ...)$$

where $n_k = 2^k - 1$ for $k \in \mathbb{N}$.

**Claim:** $(s_{n_k})$ is bounded.

Let $r = \frac{1}{2^{p-1}}$. Then

$$s_{n_2} = s_3 = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) < 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) = 1 + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}} = 1 + r,$$

$$\begin{aligned} s_{n_3} = s_7 &= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) \\ &< 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) \\ &= 1 + \frac{2}{2^p} + \frac{4}{4^p} \\ &= 1 + r + r^2. \end{aligned}$$

By induction, we have

$$s_{n_k} < 1 + r + r^2 + \cdots + r^{k-1} < \frac{1}{1-r}$$

for all $k \in \mathbb{N}$. This proves the claim.

It follows that $(s_n)$ is also bounded (why?). So by the monotone convergence theorem, the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. □

**Example** By this theorem, the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^3}$, $\sum_{n=1}^{\infty} \frac{1}{n^4}$, etc, are all convergent.

**Question:** What about the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $0 < p \leq 1$?

**Comparison Test.** *Suppose that*

$$0 \leq a_n \leq b_n, \qquad \forall n \geq K$$

*for some $K \in \mathbb{N}$. Then*

*(i)* $\sum_{n=1}^{\infty} b_n$ *converges* $\Longrightarrow \sum_{n=1}^{\infty} a_n$ *converges.*

*(ii)* $\sum_{n=1}^{\infty} a_n$ *diverges* $\Longrightarrow \sum_{n=1}^{\infty} b_n$ *diverges.*

*Proof:* (i) We use the Cauchy criterion. Let $\varepsilon > 0$. Then there exists $K_1 \in \mathbb{N}$ such that

$$m > n \geq K_1 \Longrightarrow b_{n+1} + \cdots + b_m < \varepsilon.$$

Let $K_2 = \max(K, K_1)$. Then

$$m > n \geq K_2 \Longrightarrow a_{n+1} + \cdots + a_m \leq b_{n+1} + \cdots + b_m < \varepsilon.$$

By the Cauchy criterion again, $\sum_{n=1}^{\infty} a_n$ converges.

(ii) This is the contrapositive of (i). □

**Theorem 3.2.3.** *If* $0 < p \leq 1$, *then the* $p$*-series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *diverges.*

*Proof.* Since $0 < p \leq 1$, $n^p \leq n^1 = n$, so that

$$\frac{1}{n} \leq \frac{1}{n^p} \qquad \forall n \in \mathbb{N}.$$

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by the comparison test. □

$\boxed{\textbf{Example}}$ The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a $p$-series with $p = 1/2$. So it diverges.

$\boxed{\textbf{Example}}$ Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + 2}$ convergent?

*Solution:* Note that

$$0 \leq \frac{1}{3^n + 2} \leq \frac{1}{3^n}, \qquad \forall n \in \mathbb{N},$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges because it is a geometric series with $r = 1/3$. So by the comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + 2}$ converges. □

$\boxed{\textbf{Exercise}}$ Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n - 1}$ convergent?

**Limit Comparison Test.** *Let* $\displaystyle\sum_{n=1}^{\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *be series with* <u>*positive*</u> *terms, that is,*

$$a_n > 0, \;\; b_n > 0 \qquad \forall n \in \mathbb{N},$$

*and suppose that the limit*

$$\rho = \lim_{n \to \infty} \frac{a_n}{b_n}$$

*exists.*

*(i) If* $\rho > 0$*, then either the two series both converge or both diverge.*

*(ii) If* $\rho = 0$ *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *converges, then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges.*

$\boxed{\textbf{Example}}$ Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - n + 1}$ convergent?

*Solution:* We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a $p$-series with $p = 2 > 1$, so it converges. It seems reasonable to compare the given series with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, but unfortunately

$$\frac{1}{n^2 - n + 1} \geq \frac{1}{n^2}.$$

So comparison test fails. We use the limit comparison test instead:

$$\rho = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2 - n + 1}} = \lim_{n\to\infty} \frac{n^2 - n + 1}{n^2} = \lim_{n\to\infty} \left(1 - \frac{1}{n} + \frac{1}{n^2}\right) = 1 > 0.$$

So either the two series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - n + 1}$ both converge or both diverge.

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - n + 1}$.

$\boxed{\textbf{Example}}$ Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 2}$ convergent?

*Solution:* We know that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but

$$\frac{1}{n + 2} \leq \frac{1}{n}, \qquad \forall n \in \mathbb{N}.$$

So comparison test fails. We use the limit comparison test instead:

$$\rho = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n + 2}} = \lim_{n\to\infty} \frac{n + 2}{n} = \lim_{n\to\infty} \left(1 + \frac{2}{n}\right) = 1 > 0.$$

So either the two series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 2}$ both converge or both diverge.

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 2}$.

*Proof of the limit comparison test:* (i) There exists $K \in \mathbb{N}$ such that

$$\left|\frac{a_n}{b_n} - \rho\right| < \frac{\rho}{2}, \qquad \forall n \geq K.$$

Thus

$$\begin{aligned} n \geq K \quad &\Longrightarrow \quad -\frac{\rho}{2} < \frac{a_n}{b_n} - \rho < \frac{\rho}{2} \\ &\Longrightarrow \quad \frac{\rho}{2} < \frac{a_n}{b_n} < \frac{3\rho}{2} \\ &\Longrightarrow \quad \text{(I)} \;\; a_n < \left(\frac{3\rho}{2}\right) b_n \;\; \text{and} \;\; \text{(II)} \;\; b_n < \left(\frac{2}{\rho}\right) a_n. \end{aligned}$$

By (I) and the comparison test,

$$\sum_{n=1}^{\infty} b_n \text{ is convegent} \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ is convegent},$$

$$\sum_{n=1}^{\infty} a_n \text{ is divegent} \Longrightarrow \sum_{n=1}^{\infty} b_n \text{ is divegent}.$$

By (II) and the comparison test,

$$\sum_{n=1}^{\infty} a_n \text{ is convegent} \Longrightarrow \sum_{n=1}^{\infty} b_n \text{ is convegent},$$

$$\sum_{n=1}^{\infty} b_n \text{ is divegent} \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ is divegent}.$$

The proof for (ii) is left as an exercise. □

## 3.3 Alternating series

**Alternating Series Test.** *If $(a_n)$ is a decreasing sequence such that $a_n > 0$ for all $n$ and $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.*

*Proof:* Let $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$, $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$,

$$s_{2(n+1)} = s_{2n} + (a_{2n+1} - a_{2n+2}) \geq s_{2n}$$

and

$$s_{2n+1} = s_{2n-1} - (a_{2n} - a_{2n+1}) \leq s_{2n-1}.$$

Moreover,

$$0 \leq s_2 \leq s_{2n} \leq s_{2n} + a_{2n+1} = s_{2n+1} \leq s_1 = a_1.$$

By the monotone convergent theorem, both $(s_{2n})$ and $(s_{2n-1})$ are convergent. Now

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} s_{2n}.$$

Since $(s_{2n})$ and $(s_{2n-1})$ have the same limit, $(s_n)$ converges.

**Example** By the alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

## 3.4 Absolute convergence

Given a series $\sum_{n=1}^{\infty} a_n$. If we take the absolute values of its terms, we obtain the series $\sum_{n=1}^{\infty} |a_n|$.

**Example** If the given series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, then taking the absolute values of its terms gives the harmonic series:

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}.$$

**Natural question:**

(a) Is it true that $\sum_{n=1}^{\infty} a_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} |a_n|$ converges ?

(b) Is it true that $\sum_{n=1}^{\infty} |a_n|$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges ?

**Counter-example for (a):** The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test but the series $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So

$$\sum_{n=1}^{\infty} a_n \text{ converges } \not\Longrightarrow \sum_{n=1}^{\infty} |a_n| \text{ converges.}$$

**Definition**

(i) We say the series $\sum_{n=1}^{\infty} a_n$ converges *absolutely* if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

(ii) We say the series $\sum_{n=1}^{\infty} a_n$ converges *conditionally* if it converges but the series $\sum_{n=1}^{\infty} |a_n|$ diverges.

So the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

Assertion (b) is always true:

**Theorem 3.4.1.** *If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.*

*Proof:* We use the Cauchy criterion. Let $\varepsilon > 0$. Since the series $\sum_{n=1}^{\infty} |a_n|$ converges, there exists $K \in \mathbb{N}$ such that

$$m > n \geq K \Longrightarrow |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \varepsilon. \qquad (*)$$

By the triangle inequality,

$$|a_{n+1} + a_{n+2} + \cdots + a_m| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m|.$$

This together with $(*)$ give

$$m > n \geq K \Longrightarrow |a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon.$$

Since the series $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy criterion, it converges. □

**Remarks:** The above theorem suggests that one way to test a series for convergence is to first test it for absolute convergence.

## 3.5 Additional tests for convergence

**Ratio Test.** *Suppose that all the terms of the series* $\sum_{n=1}^{\infty} a_n$ *are nonzero and the limit*

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

*exists.*

*(i) If* $\rho < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

*(ii) If* $\rho > 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

*(iii) No conclusion if* $\rho = 1$*.*

*Proof:* (i) Let $\varepsilon = (1-\rho)/2 > 0$ and $r = (1+\rho)/2 < 1$. Then there exists $K \in \mathbb{N}$ such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \varepsilon = \frac{1-\rho}{2}, \qquad \forall n \geq K.$$

Thus

$$n \geq K \Longrightarrow \frac{|a_{n+1}|}{|a_n|} < \rho + \frac{1-\rho}{2} = r \Longrightarrow |a_{n+1}| < |a_n| r.$$

It follows that for any $m \in \mathbb{N}$,

$$|a_{K+m}| < |a_{K+m-1}| r < |a_{K+m-2}| r^2 < \cdots < |a_K| r^m.$$

Equivalently,

$$|a_n| < Cr^n, \qquad \forall n \geq K$$

where

$$C = \frac{|a_K|}{r^K}.$$

Since $0 < r < 1$, the series $\sum_{n=1}^{\infty} Cr^n = C\sum_{n=1}^{\infty} r^n$ converges. Thus by the comparison test, the series $\sum_{m=1}^{\infty} |a_n|$ converges.

(ii) Take $\varepsilon = \rho - 1$. Then there exists $K \in \mathbb{N}$ such that

$$\left|\frac{|a_{n+1}|}{|a_n|} - \rho\right| < \varepsilon = \rho - 1, \qquad \forall n \geq K.$$

Thus

$$n \geq K \Longrightarrow \frac{|a_{n+1}|}{|a_n|} > \rho - (\rho - 1) = 1 \Longrightarrow |a_{n+1}| > |a_n|.$$

By induction,

$$|a_n| > |a_K| > 0 \qquad \forall n \geq K.$$

Thus $\lim_{n\to\infty} a_n \neq 0$. By the $n$-th term test, the series $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but

$$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1. \ \square$$

The following exercise shows that we can replace the limit in Part (i) of the Ratio Test by the limit superior.

**Exercise**

(i) Prove that if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| > 1$, then does the series $\sum_{n=1}^{\infty} a_n$ necessarily diverge?

**Root Test.** *Suppose that the limit*

$$\rho = \lim_{n\to\infty} |a_n|^{1/n}$$

*exists.*

*(i) If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

*(ii) If $\rho > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

*(iii) No conclusion if $\rho = 1$.*

*Proof:* (i) Let $r$ be such that $\rho < r < 1$. Since $|a_n|^{1/n} \to \rho$, there exists $K \in \mathbb{N}$ such that $|a_n|^{1/n} < r$ for all $n \geq K$. Then

$$n \geq K \Longrightarrow |a_n| < r^n.$$

Since $0 < r < 1$, $\sum_{n=1}^{\infty} r^n$ converges . So the series $\sum_{n=1}^{\infty} |a_n|$ also converges by the comparison test.

(ii) There exists $K \in \mathbb{N}$ such that $|a_n|^{1/n} > 1$ for all $n \geq K$. So for all $n \geq K$, $|a_n| > 1$. It follows that $a_n \nrightarrow 0$. By the $n$-term test, the series $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{(n^{1/n})^2} = 1. \square$$

Again, we can replace the limit in the Root Test by the limit superior.

**Exercise** Let $\rho = \limsup |a_n|^{1/n}$.

(i) If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $\rho > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) No conclusion if $\rho = 1$.

$\boxed{\textbf{Example}}$ Are the following series convergent?

(i) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!}$.

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{[2(n+1)]^n}{n^{n+1}}$.

## 3.6 Grouping of series

Consider the series

$$\sum_{n=1}^{\infty}(-1)^{n+1} = 1 - 1 + 1 - 1 + \cdots$$

We have seen that it is divergent. What happens if we insert parentheses? We consider the following two ways:

$$(1-1)+(1-1)+(1-1)+\cdots = 0 \qquad (a)$$

$$1+(-1+1)+(-1+1)+(-1+1)+\cdots = 1 \qquad (b)$$

How are their partial sums related? Let $s_n = \displaystyle\sum_{k=1}^{n}(-1)^{k+1}$ be the partial sums of the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}$. Then the sequence of partial sums for series (a) is

$$(s_{2n}) = (s_2, s_4, s_6, ...) = (0, 0, 0, ...)$$

and the sequence of partial sums for series (b) is

$$(s_{2n-1}) = (s_1, s_3, s_5, ...) = (1, 1, 1, ...).$$

More generally, let

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

with partial sums $s_n = a_1 + \cdots + a_n$, $n \in \mathbb{N}$. If keep the ordering of the terms and group the terms in some way , we obtain a new series

$$(a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + \cdots$$

If we denote the partial sums of the new series by $(t_n)$, then

$$t_1 = a_1 + \cdots + a_{n_1} = s_{n_1}$$

$$t_2 = (a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) = s_{n_2}$$

$$t_3 = (a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) = s_{n_3}$$

$$\cdots\cdots$$

Thus $(t_k) = (s_{n_k})$ is a subsequence of $(s_n)$. It follows that if $(s_n)$ converges, then so is $(t_k)$. Moreover they have the same limit. Thus we have proved the following theorem:

**Theorem 3.6.1.** *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then any series obtained by grouping the terms of* $\sum_{n=1}^{\infty} a_n$ *is also convergent and has the same value as* $\sum_{n=1}^{\infty} a_n$.

## 3.7 Rearrangements of series

Consider the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Let us rearrange its terms as follows:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \cdots$$

This series also converges (exercise). Moreover, observe that

$$\begin{aligned} & \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} + \cdots \\ = \; & \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} + \cdots \\ = \; & \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots\right). \end{aligned}$$

This is half of the original sum! Thus rearrangements of a series may change its sum.

**Remark** In fact, for any real number $c$, it is possible to rearrange the terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ so that the new sum is exactly $c$. See page 255 of the textbook.

**Definition** A series $\sum_{n=1}^{\infty} b_n$ is a *rearrangement* of the series $\sum_{n=1}^{\infty} a_n$ if there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{f(n)}$ for all $n \in \mathbb{N}$.

**Exercise** The series

$$\frac{1}{2^2} + \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^3} + \cdots$$

is a rearrangement of the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Does it converge? If $a_n$ denotes its $n$th term, then what is $\limsup \frac{a_{n+1}}{a_n}$?

**Theorem 3.7.1.** *If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ also converges and has the same sum as $\sum_{n=1}^{\infty} a_n$.*

*Proof:* Let $\varepsilon > 0$. Since the series $\sum_{n=1}^{\infty} |a_n|$ converges, there exists $K \in \mathbb{N}$ such that

$$m > n \geq K \Longrightarrow |a_{n+1}| + |a_{n+2}| + \cdots |a_m| < \varepsilon.$$

In particular, by taking $n = K$, we have

$$m > K \Longrightarrow |a_{K+1}| + |a_{K+2}| + \cdots |a_m| < \varepsilon.$$

We now let

$$s_n = a_1 + \cdots + a_n, \qquad s'_n = b_1 + \cdots + b_n$$

be the partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Suppose that the integers $i_1, ..., i_K$ are such that

$$b_{i_1} = a_1,\ b_{i_2} = a_2,\ \cdots,\ b_{i_K} = a_K.$$

Let $M = \max(i_1, ..., i_K)$. Then for any $n \geq M$, the terms $a_1, ..., a_K$ will all appear in both the partial sums $s_n$ and $s'_n$. Consequently, in the difference $s_n - s'_n$, all the terms $a_1, ..., a_K$ will disappear. It follows that if $n \geq M$, then there exists $m > K$ such that

$$|s_n - s'_n| \leq |a_{K+1}| + |a_{K+2}| + \cdots + |a_m| < \varepsilon.$$

Hence $\lim\limits_{n\to\infty} (s_n - s'_n) = 0$. If $s = \lim\limits_{n\to\infty} s_n$, then

$$\lim_{n\to\infty} s'_n = \lim_{n\to\infty} (s'_n - s_n) + \lim_{n\to\infty} s_n = 0 + s = s. \square$$

**Question:** Does the series $\sum\limits_{n=1}^{\infty} b_n$ converge absolutely?

## 3.8 Why is $e$ irrational?

Recall that the Euler's number $e$ is defined as the limit

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

**Theorem 3.8.1.** *(a)* $e = \sum\limits_{n=0}^{\infty} \frac{1}{n!}$.

*(b) For each* $n \in \mathbb{N}$, $e - \sum\limits_{j=0}^{n} \frac{1}{j!} < \frac{1}{n(n!)}$.

*Proof.* (a) By the Binomial formula,

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} \\
&\quad + \cdots + \frac{n(n-1)\cdots 2 \cdot 1}{n!} \cdot \frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \\
&\quad + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\
&\leq 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = \sum_{k=1}^{n} \frac{1}{k!}.
\end{aligned}$$

By the ratio test, the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. So by letting $n \to \infty$, we obtain

$$e \leq \sum_{n=0}^{\infty} \frac{1}{n!}. \quad (*)$$

Next we fix $k \in \mathbb{N}$. For $n > k$, we have

$$\begin{aligned}\left(1 + \frac{1}{n}\right)^n \quad > \quad & 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \\ & + \cdots + \frac{1}{k!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$e \geq 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!}.$$

Now letting $k \to \infty$ gives

$$e \geq \sum_{k=0}^{\infty} \frac{1}{k!}.$$

This together with $(*)$ proves (a).

(b) Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$. Then for $m > n$, we have

$$\begin{aligned} s_m - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!} \\ &= \frac{1}{(n+1)!}\left\{1 + \frac{1}{(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(n+2)(n+3)\cdots(m)}\right\} \\ &< \frac{1}{(n+1)!}\left\{1 + \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+1)^{m-n-1}}\right\} \\ &< \frac{1}{(n+1)!}\left\{\frac{1}{1 - \frac{1}{n+1}}\right\} \\ &= \frac{1}{n(n!)}. \end{aligned}$$

It follows that

$$e - \sum_{j=0}^{n} \frac{1}{j!} = e - s_n = \lim_{m \to \infty} (s_m - s_n) < \frac{1}{n(n!)}.$$

**Theorem 3.8.2.** *The Euler number e is irrational.*

*Proof.* Assume that $e$ is rational. Then $e = \dfrac{p}{q}$ for some natural numbers $p$ and $q$. Then by Part(b) of Theorem 3.8.1,

$$0 < q!(e - s_q) < \frac{q!}{q(q!)} = \frac{1}{q} \leq 1.$$

Observe that both

$$q!e = p(q-1)!$$

and

$$q!s_q = q!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!}\right)$$

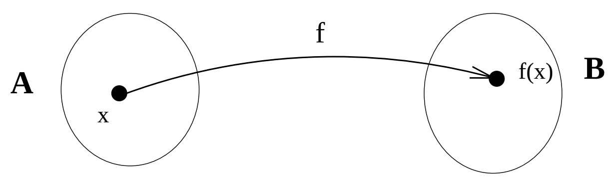
are integer, so $q!(e - s_q)$ is also an integers. We have shown that this integer is between 0 and 1, which is a contradiction. □

# Chapter 4: Limits of Functions

## 4.1 Real-valued functions

Let $A$ and $B$ be sets. A function $f$ from $A$ into $B$ is a rule which assigns to each element $x$ in $A$ a **unique** element $f(x)$ in $B$. In this case, we write

$$f : A \to B.$$




- $A$ is the *domain* of $f$.
- $B$ is the *codomain* of $f$.
- The set $f(A) = \{f(x) : x \in A\}$ is the *range* of $f$.

If $A \subseteq \mathbb{R}$, then

$$f : A \to \mathbb{R}$$

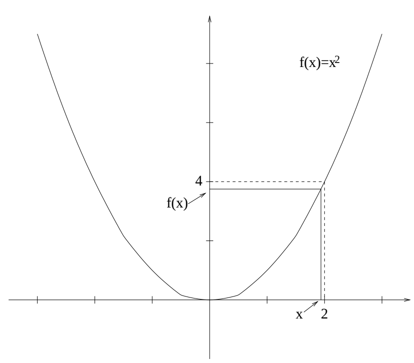
is called a *real-valued function of a real variable*. We shall only consider real-valued functions whose domain is either an interval or a union of intervals.

## 4.2 Definition of limits and examples

Roughly speaking, we say that a function $f$ has a limit $L$ at a point $x = a$ if

$$x \text{ is sufficiently close to } a \implies f(x) \text{ is close to } L \text{ (as close as we like).}$$

$\boxed{\textbf{Example}}$ Examine the behavior of $f(x) = x^2$ near the point $x = 2$.



• If $x = 1.99$, then $f(x) = 3.9601$.
• If $x = 1.999$, then $f(x) = 3.996001$.
• If $x = 2.000003$, then $f(x) = 4.000012$.

We see that

$$x \approx 2 \Longrightarrow f(x) \approx 4,$$

that is, $f(x)$ approaches 4 as $x$ approaches 2.

So we say that the limit of $f$ at $x = 2$ is 4, and write

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 = 4.$$

$\boxed{\textbf{Example}}$ Examine the behavior of $f(x) = (\sin x)/x$ near the point $x = 0$. Note that $f(0)$ is not defined.

| $x$ | $f(x)$ |
|---|---|
| ±1.0 | 0.84147 |
| ±0.9 | 0.87036 |
| ±0.8 | 0.89670 |
| ±0.7 | 0.92031 |
| ±0.6 | 0.94107 |
| ±0.5 | 0.95885 |
| ±0.4 | 0.97355 |
| ±0.3 | 0.98507 |
| ±0.2 | 0.99355 |
| ±0.1 | 0.99833 |
| ±0.01 | 0.99998 |

We note that as $x$ approaches 0, $f(x)$ approaches 1. So we write

$$\lim_{x\to 2} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

---

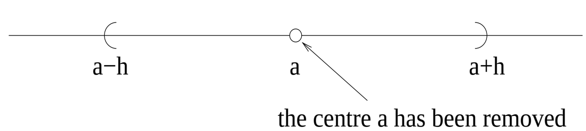
If $h > 0$, then the $h$-neighborhood of the point $a$ is the set

$$V_h(a) = \{x : \ |x - a| < h\} = (a - h, a + h).$$

Define

$$V_h^*(a) = V_h(a)\backslash\{a\} = \{x : \ 0 < |x - a| < h\}.$$

$V_h^*(a)$ is called a *deleted neighborhood* of $a$.




Note that

$$V_h^*(a) = (a - h, a) \cup (a, a + h).$$

$\boxed{\textbf{Definition}}$ Let $f$ be defined in a deleted neighborhood of $a$. We say that the number $L$ is the *limit* of $f$ at $x = a$ if for any given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$x \in V_\delta^*(a) \Longrightarrow f(x) \in V_\varepsilon(L).$$

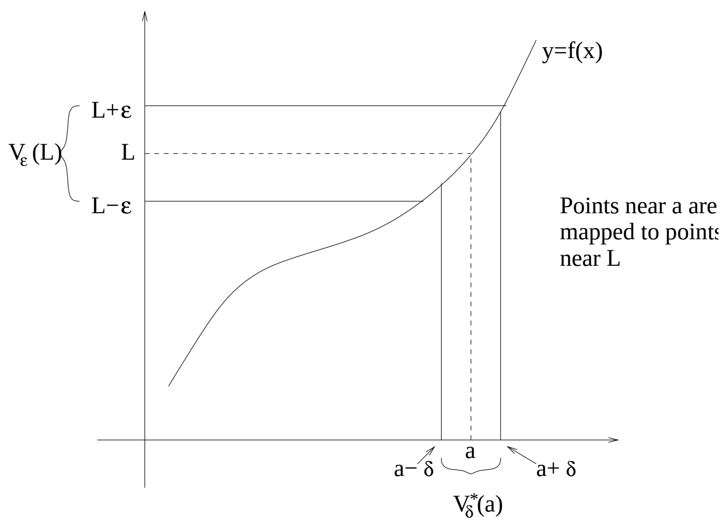
In this case, we write

$$\lim_{x \to a} f(x) = L$$

or

$$f(x) \to L \qquad \text{as } x \to a.$$

We also say "$f$ converges to $L$ at $a$" or "$f$ approaches $L$ as $x$ approaches $a$".




**Note:** To discuss the limit of $f$ at a point $x = a$, we do not require $f$ be defined at $x = a$. So even if $f(a)$ is defined, its value has no bearing on $\lim_{x \to a} f(x)$.

$\boxed{\textbf{Definition}}$ If $f$ has no limit at $x = a$, then we say $f$ diverges at $a$.

We note that

$$x \in V_\delta^*(a) \Longleftrightarrow 0 < |x - a| < \delta$$

and

$$f(x) \in V_\varepsilon(L) \Longleftrightarrow |f(x) - L| < \varepsilon.$$

$\varepsilon - \delta$ **definition of limit:** We say $L$ is the limit of $f$ at $x = a$ if for any given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

**Remark** Recall that proving a sequence converges is a $K(\varepsilon)$ game. In a very similar way, proving a function converges at a point is a $\varepsilon - \delta$ game.

**Remark** When discussing limit, the textbook assumes that the given function $f$ is defined on a set $A$ and $a$ is a *cluster point* of $A$. This means that there is a sequence $(x_n)$ in $A$ such that $x_n \neq a$ for all $n$ and $x_n \to a$. This condition is more general than ours: we only assume that $f$ is defined in a deleted neighborhood $V_h^*(a)$ of $a$. In our case, we may take $A = V_h^*(a)$. Then $a$ is a cluster point of $A$.

**Example** **Limit of a constant function**

Let $f(x) = c$, $\forall x \in \mathbb{R}$. Prove that for any $a \in \mathbb{R}$,

$$\lim_{x\to a} f(x) = c.$$

*Proof:* Let $\varepsilon > 0$ be given. Then

$$|f(x) - c| = |c - c| = 0 < \varepsilon, \qquad \forall x \in \mathbb{R}.$$

So $\delta$ can be **any** positive number, and

$$|f(x) - c| = |c - c| = 0 < \varepsilon, \qquad \forall 0 < |x - a| < \delta. \ \square$$

**Example** **Limit of the identity function**

Let $f(x) = x$, $\forall x \in \mathbb{R}$. Prove that for any $a \in \mathbb{R}$,

$$\lim_{x\to a} f(x) = a.$$

*Proof:* Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then

$$0 < |x - a| < \delta \Longrightarrow |f(x) - a| = |x - a| < \delta = \varepsilon. \ \square$$

**Exercise** Prove that for any $a \in \mathbb{R}$,

$$\lim_{x\to a} |x| = |a|.$$

**Example** Let $f(x) = x^2$. Use $\varepsilon - \delta$ definition to prove

$$\lim_{x \to 2} f(x) = 4.$$

*Proof:* Let $\varepsilon > 0$ be given. We want to find a $\delta > 0$ such that

$$0 < |x - 2| < \delta \Longrightarrow |x^2 - 4| < \varepsilon.$$

Now $|x^2 - 4| = |x + 2||x - 2|$. We will make an initial restriction on $x$ which will bound the factor $|x + 2|$. Restrict $x$ to $|x - 2| < 1$. Then

$$|x + 2| = |(x - 2) + 4| \le |x - 2| + |4| < 1 + 4 = 5.$$

Thus if $|x - 2| < 1$, then

$$|f(x) - 4| = |x + 2||x - 2| < 5|x - 2|.$$

So we can choose

$$\delta = \min\left(1, \frac{\varepsilon}{5}\right).$$

Then

$$0 < |x - 2| < \delta \Longrightarrow |f(x) - 4| < 5|x - 2| < 4 \cdot \frac{\varepsilon}{5} = \varepsilon. \ \square$$

**Remark** The above arguments can be easily modified to prove $\lim_{x \to a} x^2 = a^2$.

**Example** Use $\varepsilon - \delta$ definition to prove

$$\lim_{x \to 3} \frac{x}{x + 2} = \frac{3}{5}.$$

*Proof:* We have

$$\left|\frac{x}{x + 2} - \frac{3}{5}\right| = \left|\frac{2(x - 3)}{5(x + 2)}\right| = \frac{2}{5} \cdot \frac{1}{|x + 2|} \cdot |x - 3|.$$

First restrict $x$ to $|x - 3| < 1$. Then $2 < x < 4$, so $4 < x + 2 < 6$. In particular, $|x + 2| > 4$, so that

$$\frac{1}{|x + 2|} < \frac{1}{4}.$$

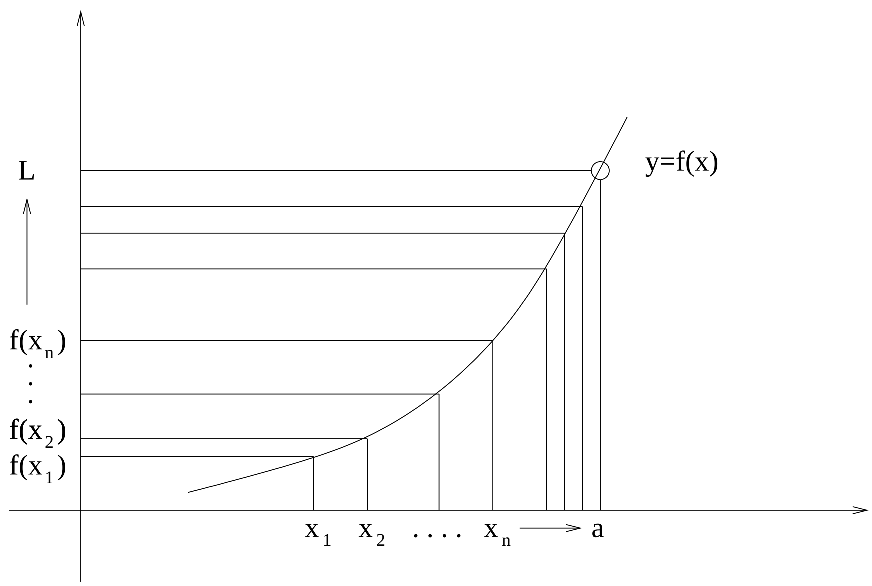
It follows that

$$0 < |x - 3| < 1 \Longrightarrow \left|\frac{x}{x + 2} - \frac{3}{5}\right| < \frac{2}{5} \cdot \frac{1}{4} \cdot |x - 3| = \frac{|x - 3|}{10}.$$

Now let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon)$. Then

$$0 < |x-3| < \delta \Longrightarrow \left|\frac{x}{x+2} - \frac{3}{5}\right| < \frac{|x-3|}{10} < \frac{\varepsilon}{10} < \varepsilon. \square$$

**Theorem 4.2.1. (Sequential Criterion for limits)**
$\lim\limits_{x\to a} f(x) = L \Longleftrightarrow$ *If $(x_n)$ is any sequence in the domain of $f$ such that $x_n \neq a$ for all $n$ and $x_n \to a$, then $f(x_n) \to L$.*




*Proof:* ($\Longrightarrow$) Assume that $\lim\limits_{x\to a} f(x) = L$.

Let $(x_n)$ be a sequence in the domain of $f$ such that $x_n \neq a$ for all $n$ and $x_n \to a$.

Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to a} f(x) = L$, there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \Longrightarrow |f(x) - L| < \varepsilon \qquad (*)$$

Now $x_n \neq a$ for all $n$, so $|x_n - a| > 0$ for all $n$.

On the other hand, since $x_n \to a$, $\exists K \in \mathbb{N}$ such that

$$0 < |x_n - a| < \delta, \qquad \forall n \geq K.$$

It follows from $(*)$ that

$$n \geq K \Longrightarrow |f(x_n) - L| < \varepsilon.$$

This proves $f(x_n) \to L$.

($\Longleftarrow$) We prove its contrapositive statement.

Suppose $\lim_{x\to a} f(x) \neq L$.

Then there exists $\varepsilon_0 > 0$ such that for each $\delta > 0$, $\exists x = x(\delta)$ such that

$$0 < |x - a| < \delta, \qquad \text{but} \qquad |f(x) - L| \geq \varepsilon_0.$$

For each $n \in \mathbb{N}$, take $\delta = 1/n$. Then $\exists x_n$ such that

$$0 < |x_n - a| < \frac{1}{n}, \qquad \text{but} \qquad |f(x_n) - L| \geq \varepsilon_0.$$

So we have obtained a sequence $(x_n)$ with the property that $x_n \to a$ but $f(x_n) \not\to L$. $\square$

Sequential criterion can now be used to deduce a large number of limits.

$\boxed{\textbf{Example}}$ **(Limit of polynomials)**

Let $k \in \mathbb{N}$, and let $f(x) = x^k$, $x \in \mathbb{R}$. If $(x_n)$ is a sequence such that $x_n \neq a$ and $x_n \to a$, then

$$f(x_n) = x_n^k \to a^k.$$

By the sequential criterion,

$$\lim_{x\to a} f(x) = \lim_{x\to a} x^k = a^k = f(a).$$

More generally, let $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k$ be a polynomial. Then for any sequence $(x_n)$ such that $x_n \neq a$ and $x_n \to a$, we have

$$p(x_n) = c_0 + c_1 x_n + c_2 x_n^2 + \cdots + c_k x_n^k \to p(a) = c_0 + c_1 a + c_2 a^2 + \cdots + c_k a^k.$$

So by the sequential criterion,

$$\lim_{x\to a} p(x) = p(a).$$

That is, the limit of a polynomial $p(x)$ at a point $a$ is given by its value $p(a)$ at $a$.

$\boxed{\textbf{Example}}$ Evaluate $\lim_{x\to 1}(2 - 5x^3 - 4x^7 + 13x^9)$.

*Solution:* $\lim_{x\to 1}(2 - 5x^3 - 4x^7 + 13x^9) = 2 - 5 - 4 + 13 = 6$.

**Example** **(Limit of the square root function)**

Prove that for any $a > 0$, $\lim_{x\to a} \sqrt{x} = \sqrt{a}$.

*Proof:* Let $f(x) = \sqrt{x}$, and let $(x_n)$ be a sequence such that $x_n > 0$ and $x_n \to a$. Then by Theorem 2.2.7 of Chapter 2,

$$f(x_n) = \sqrt{x_n} \to \sqrt{a}.$$

By the sequential criterion, $\lim_{x\to a} f(x) = \sqrt{a}$. □

**Exercise** Prove $\lim_{x\to a} \sqrt{x} = \sqrt{a}$ using the $\varepsilon - \delta$ definition of limit.

**Example** More generally, if $a > 0$, then for any $k \in \mathbb{N}$, $\lim_{x\to a} x^{1/k} = a^{1/k}$.
It follows from this that for any $r \in \mathbb{Q}$,

$$\lim_{x\to a} x^r = a^r.$$

**Example** For any $a > 0$ and any $b \in \mathbb{R}$, $\lim_{x\to b} a^x = a^b$.

*Proof.* For any sequence $(x_n)$ such that $x_n \neq b$ for all $n$ and $x_n \to b$, we have

$$a^{x_n} \to a^b$$

by Question 7 of Tutorial 5. □

**Corollary 4.2.2.** $\lim_{x\to a} f(x) \neq L \iff$ *there is a sequence* $(x_n)$ *in the domain of* $f$ *such that* $x_n \neq a$ *for all* $n$ *and* $x_n \to a$, *but* $f(x_n) \not\to L$.

**Divergent Criterion.** To prove that $\lim_{x\to a} f(x)$ does not exist:

**Method 1.** Find a sequence $(x_n)$ in the domain of $f$ such that $x_n \neq a$ for all $n$ and $x_n \to a$, but $(f(x_n))$ diverges.

**Method 2.** Find two sequences $(x_n)$ and $(y_n)$ in the domain of $f$ such that $x_n \neq a$ and $y_n \neq a$ for all $n$ and $x_n \to a$, $y_n \to a$, but $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$.

**Example** Prove that $\lim_{x\to 0} 1/x^2$ does not exist.

*Proof:* Use Method 1. Let $f(x) = 1/x^2$ and $x_n = 1/n$, $n \in \mathbb{N}$.

Then $x_n \neq 0$ for all $n$, $x_n \to 0$, but $(f(x_n)) = (n^2)$ is divergent.

So $\lim_{x\to 0} 1/x^2$ does not exist.

**Example** Prove that $\lim_{x\to 0} \sin(1/x)$ does not exist.

*Proof:* Use Method 1. Let $f(x) = \sin(1/x)$ and

$$x_n = \frac{2}{(2n+1)\pi}, \qquad \forall n \in \mathbb{N}.$$

Then $x_n \neq 0$ for all $n$, $x_n \to 0$ and

$$f(x_n) = \sin(n + \frac{1}{2})\pi = (-1)^n, \qquad \forall n \in \mathbb{N},$$

that is,

$$(f(x_n)) = (-1, 1, -1, 1, -1, 1, -1, ......).$$

So $(f(x_n))$ diverges. Hence $\lim_{x\to 0} \sin(1/x)$ does not exist.

**Example** Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that $\lim_{x\to a} f(x)$ does not exist for any $a \in \mathbb{R}$.

*Proof:* Use Method 2. Take a rational sequence $(x_n)$ and an irrational sequence $(y_n)$ such that $x_n \neq a$, $y_n \neq a$ for all $n$, $x_n \to a$ and $y_n \to a$. Then $f(x_n) = 1$ and $f(y_n) = 0$ for all $n$. So

$$\lim_{n\to\infty} f(x_n) = 1 \qquad \text{and} \qquad \lim_{n\to\infty} f(y_n) = 0.$$

So $\lim_{x\to a} f(x)$ does not exist. □

**Lemma 4.2.3.** *Let $c \in \mathbb{R}$.*

*(i) There exists a sequence $(x_n)$ such that $x_n$ is rational for all $n$, $x_n \neq c$ for all $n$ and $x_n \to c$.*

*(ii) There exists a sequence $(y_n)$ such that $y_n$ is irrational for all $n$, $y_n \neq c$ for all $n$ and $y_n \to c$.*

*Proof:* Similar to Lemma 2.5.1 of Chapter 2. □

### 4.3 Limit theorems

**Theorem 4.3.1.** *Suppose $f$ is defined in a deleted neighborhood of $x = a$. If $\lim_{x\to a} f(x)$ exists, then $f$ is bounded in a deleted neighborhood of $x = a$, that is, $\exists M > 0$ and $\delta > 0$ such that*

$$0 < |x - a| < \delta \Longrightarrow |f(x)| \leq M.$$

*Proof.* Suppose $f$ is defined in $V_h^*(a)$.

Take $\varepsilon = 1$. Then $\exists\delta > 0$ such that $\delta < h$ and

$$\begin{aligned} 0 < |x - a| < \delta \quad &\Longrightarrow \quad |f(x) - L| < \varepsilon = 1 \\ &\Longrightarrow \quad |f(x)| = |(f(x) - L) + L| \leq |f(x) - L| + |L| < 1 + |L|. \end{aligned}$$

Thus we can take $M = 1 + |L|$. □

**Theorem 4.3.2.** *Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.*

*(i)* $\lim_{x\to a}(f \pm g)(x) = \lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M.$

*(ii)* $\lim_{x\to a}(f \cdot g)(x) = \lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)] = LM.$

*(iii) If $g(x) \neq 0$ in a deleted neighborhood of $a$ and $\lim_{x\to a} g(x) = M \neq 0$, then*

$$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}.$$

*Proof:* The $\varepsilon - \delta$ proof is similar to the proofs for the limit theorems for sequences.

Alternatively, we can use the sequential criterion.

Let $(x_n)$ be a sequence in the domain of $f$ and $g$ such that $x_n \neq a$ for all $n$ and $x_n \to a$.

Since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, $f(x_n) \to L$ and $g(x_n) \to M$. It follows from the limit theorems for sequences that

- $(f \pm g)(x_n) = f(x_n) \pm g(x_n) \to L \pm M.$
- $(f \cdot g)(x_n) = f(x_n)g(x_n) \to LM.$
- $(f/g)(x_n) = f(x_n)/g(x_n) \to L/M$, provided the conditions given in (iii) are satisfied.

**Remark** The following are special cases of Theorem 4.3.2:

- If $c \in \mathbb{R}$, then
$$\lim_{x\to a} c f(x) = c \cdot \lim_{x\to a} f(x).$$

- For any $k \in \mathbb{N}$,
$$\lim_{x\to a} [f(x)]^k = [\lim_{x\to a} f(x)]^k.$$

**Example** Evaluate $\lim\limits_{x\to 2} \dfrac{x^2+3}{4-x}$.

*Solution:* $\lim\limits_{x\to 2} \dfrac{x^2+3}{4-x} = \dfrac{\lim_{x\to 2}(x^2+3)}{\lim_{x\to 2}(4-x)} = \dfrac{2^2+3}{4-2} = \dfrac{7}{2}.$

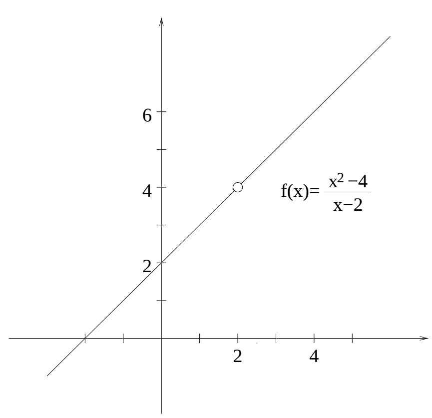
**Example** More generally, if $f(x)$ and $g(x)$ are polynomials and $g(a) \neq 0$, then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}.$$

The quotient $\dfrac{f(x)}{g(x)}$ of two polynomials is called a *rational function*.

**Example** Does the limit $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$ exist?

*Solution:* Can't apply part (iii) of Theorem 4.3.2 because $\lim\limits_{x\to 2}(x-2) = 0$.

Note that for $x \neq 2$,
$$\frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2} = x+2.$$



Since the limit of a function at $x = 2$ does not involve the value of the function at that point (see the basic principle stated below),

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2}(x + 2) = 4. \square$$

**Basic Principle:** If $f(x) = g(x)$ for all $x$ in a deleted neighborhood of $x = a$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

provided one of these limits exist.

*Proof.* Suppose that $f(x) = g(x)$ for all $x \in V_h^*(a)$ and $L = \lim_{x\to a} f(x)$ exists. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\delta < h$ and

$$0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

For each $x$ which satisfies $0 < |x - a| < \delta$, $x \in V_h^*(a)$ so that $f(x) = g(x)$. Hence

$$0 < |x - a| < \delta \Longrightarrow |g(x) - L| = |f(x) - L| < \varepsilon. \square$$

**Theorem 4.3.3.** *If $f(x) \leq g(x)$ for all $x$ in a deleted neighborhood of $x = a$ and both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then*

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

*Proof:* Use sequential criterion. $\square$

**Squeeze Theorem.** Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x$ in a deleted neighborhood of $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then

$$\lim_{x\to a} g(x) = L.$$

*Proof:* Use sequential criterion and the squeeze theorem for sequences. $\square$

**Example** Prove that $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

*Solution:* For $x \neq 0$,

$$|\sin\frac{1}{x}| \leq 1,$$

so that

$$0 \leq |x\sin\frac{1}{x}| \leq |x|.$$

Now $\lim_{x\to 0}|x| = 0$. By the Squeeze theorem, $|x\sin(1/x)| \to 0$, and so $x\sin(1/x) \to 0$ as $x \to 0$.

**Theorem 4.3.4.** *If $f$ is defined in a deleted neighborhood of $x = a$ and $\lim_{x\to a} f(x) = L$ exists and $L > 0$, then $\exists\delta > 0$ such that*

$$f(x) > 0 \qquad \forall x \text{ such that } 0 < |x-a| < \delta.$$

*Proof:* Take $\varepsilon = L/2$. □

## 4.4 One-sided limits

**Definition** (i) We say $L$ is the *right-hand limit* of $f$ at $a$ if for any given $\varepsilon > 0$, $\exists\delta > 0$ such that

$$0 < x-a < \delta \ \text{ (i.e. } x \in (a, a+\delta)) \Longrightarrow |f(x) - L| < \varepsilon.$$

In this case, we write

$$\lim_{x\to a^+} f(x) = L.$$

(ii) We say $L$ is the *left-hand limit* of $f$ at $a$ if for any given $\varepsilon > 0$, $\exists\delta > 0$ such that

$$-\delta < x-a < 0 \ \text{ (i.e. } x \in (a-\delta, a)) \Longrightarrow |f(x) - L| < \varepsilon.$$

In this case, we write

$$\lim_{x\to a^-} f(x) = L.$$

The limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^+} f(x)$ are called *one-sided limits* at the point $x = a$.

**Theorem 4.4.1.** $\lim_{x\to a} f(x) = L$ *exists if and only if both* $\lim_{x\to a^+} f(x)$ *and* $\lim_{x\to a^-} f(x)$ *exist and*

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

*Proof:* ($\Longrightarrow$) Let $\varepsilon > 0$ be given. Then $\exists \delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

In particular, if $a < x < a + \delta$, then $0 < x - a < \delta$ and $|x - a| = x - a < \delta$, so that $|f(x) - L| < \varepsilon$. This shows $\lim_{x \to a^+} f(x) = L$. Similarly, $\lim_{x \to a^-} f(x) = L$.

($\Longleftarrow$) Let $\varepsilon > 0$ be given. Then $\exists \delta_1, \delta_2 > 0$ such that

$$0 < x - a < \delta_1 \implies |f(x) - L| < \varepsilon,$$

$$-\delta_2 < x - a < 0 \implies |f(x) - L| < \varepsilon.$$

Take $\delta = \min(\delta_1, \delta_2)$. Then

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon. \square$$

**Corollary 4.4.2.** *If either one of the one-sided limits of $f$ at $x = a$ does not exist or $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.*

The one-sided limits share many properties with the usual two-sided limits.

**Basic Principle:** If $f(x) = g(x)$ for all $x$ in $(a, a + h)$, then

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x)$$

provided one of these limits exist.

There is a parallel statement for left-hand limit.

**Example** Let

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Show that $\lim_{x \to 0} f(x) = 0$.

*Solution:* We use the basic principle above. Since $f(x) = x^2$ for $x \in (0, \infty)$,

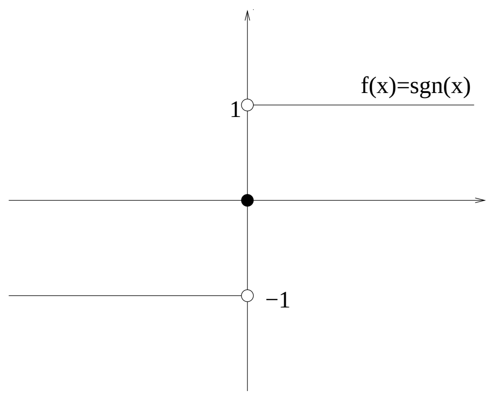
$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x^2 = \lim_{x \to 0} x^2 = 0.$$

On the other hand, $f(x) = -x$ for $x \in (-\infty, 0)$. So

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (-x) = \lim_{x\to 0}(-x) = 0.$$

Since $\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = 0$, $\lim_{x\to 0} f(x) = 0$.

$\boxed{\textbf{Example}}$ Let $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$




Then $\lim_{x\to 0^+} \operatorname{sgn}(x) = 1$, $\lim_{x\to 0^-} \operatorname{sgn}(x) = -1$ but $\lim_{x\to 0} \operatorname{sgn}(x)$ does not exist. What about $\lim_{x\to a} \operatorname{sgn}(x)$ for $a \neq 0$?

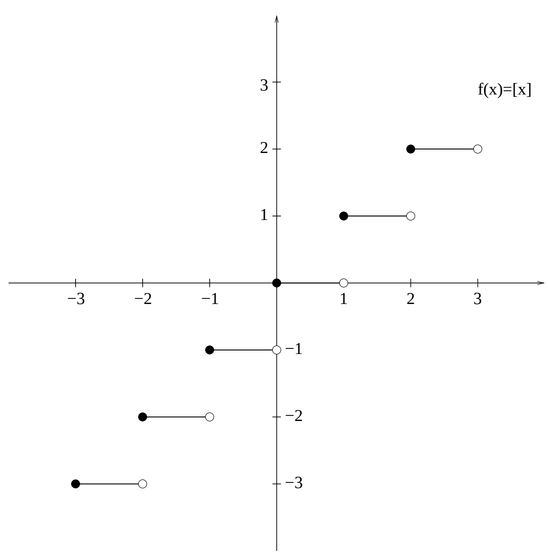
$\boxed{\textbf{Definition}}$ **The greatest integer function**

For $x \in \mathbb{R}$,

$$[x] = \text{greatest integer less than or equal to } x.$$

So for each $n \in \mathbb{Z}$,

$$[x] = n \qquad \text{if } \ x \in [n, n+1).$$



For example, $[1.2] = 1$, $[2] = 2$ and $[-2.3] = -3$.

$\boxed{\textbf{Example}}$ Find $\lim_{x\to 2^+}[x]$ and $\lim_{x\to 2^-}[x]$. Does $\lim_{x\to 2}[x]$ exist?

*Solution:* Since $[x] = 2$ for all $x \in (2, 3)$,

$$\lim_{x\to 2^+}[x] = \lim_{x\to 2^+} 2 = 2,$$

and since $[x] = 1$ for all $x \in (1, 2)$,

$$\lim_{x\to 2^-}[x] = \lim_{x\to 2^-} 1 = 1.$$

Since $\lim_{x\to 2^+}[x] \neq \lim_{x\to 2^-}[x]$, $\lim_{x\to 2}[x]$ does not exist.

$\boxed{\textbf{Remark}}$ Similarly, for each integer $n$, $\lim_{x\to n^+}[x] = n$ and $\lim_{x\to n^-}[x] = n - 1$.

$\boxed{\textbf{Example}}$ Evaluate the following limits or show that they do not exist.

(i) $\displaystyle\lim_{x\to 3^+}\frac{[2x]+x}{[x^2]+1}$.

(ii) $\displaystyle\lim_{x\to 3}\frac{[2x]+x}{[x^2]+1}$.

*Solution:* (i) For $x \in (3, 3.1)$, $6 < 2x < 6.2$ and $9 < x^2 < 9.61$, so that

$$\lim_{x\to 3^+}\frac{[2x]+x}{[x^2]+1} = \lim_{x\to 3^+}\frac{6+x}{9+1} = \frac{9}{10}.$$

(ii) For $x \in (2.9, 3)$, $5.8 < 2x < 6$ and $8.41 < x^2 < 9$, so that

$$\lim_{x\to 3^-}\frac{[2x]+x}{[x^2]+1} = \lim_{x\to 3^-}\frac{5+x}{8+1} = \frac{8}{9}.$$

Since

$$\lim_{x\to 3^+}\frac{[2x]+x}{[x^2]+1} \neq \lim_{x\to 3^-}\frac{[2x]+x}{[x^2]+1},$$

the limit $\displaystyle\lim_{x\to 3}\frac{[2x]+x}{[x^2]+1}$ does not exist.

**Theorem 4.4.3. (Sequential Criterion for right-hand limits)**
$\lim_{x\to a^+} f(x) = L \iff$ *If $(x_n)$ is any sequence in the domain of $f$ such that $x_n > a$ for all $n$ and $x_n \to a$, then $f(x_n) \to L$.*

*Proof.* Exercise. □

There is a similar sequential criterion for the left-hand limit.

Using these sequential criteria, one can prove similar limit theorems for one-sided limits.

**Theorem 4.4.4.** *If* $\lim_{x\to a^\pm} f(x) = L$ *and* $\lim_{x\to a^\pm} g(x) = M$*, then* $\lim_{x\to a^\pm} (f \pm g)(x) = L \pm M$ *and* $\lim_{x\to a^\pm} (f \cdot g)(x) = LM$*. If in addition,* $g(x) \neq 0$ *near* $a$ *and* $M \neq 0$*, then we also have* $\lim_{x\to a^\pm} f(x)/g(x) = L/M$*.*

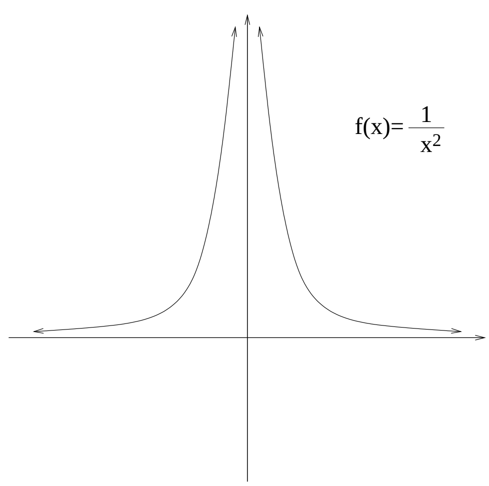
**Squeeze Theorem for right-hand limit.** Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$. If $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} h(x) = L$, then

$$\lim_{x\to a^+} g(x) = L.$$

There is also a squeeze theorem for left-hand limit. Both theorems can be proved using the sequential criterion.

## 4.5 Infinite limits and limits at infinity

Consider the function $f(x) = 1/x^2$.




We note that $f(x)$ becomes arbitrarily large when $x$ gets near 0. We say that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

$\boxed{\textbf{Definition}}$ (**Infinite limits**) Let $f$ be defined in a deleted neighborhood of $x = a$.

(i) We say that the function $f$ *tends to* $\infty$ as $x \to a$ if for any $M > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow f(x) > M.$$

In this case, we write

$$\lim_{x \to a} f(x) = \infty.$$

(ii) We say that the function $f$ *tends to* $-\infty$ as $x \to a$ if for any $M < 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow f(x) < M.$$

In this case, we write

$$\lim_{x \to a} f(x) = -\infty.$$

$\boxed{\textbf{Example}}$ Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

*Proof:* Let $M > 0$ be given. Choose $\delta = 1/\sqrt{M}$. Then

$$0 < |x - 0| < \delta \Longrightarrow |x| < \frac{1}{\sqrt{M}} \Longrightarrow x^2 < \frac{1}{M} \Longrightarrow \frac{1}{x^2} > M. \ \square$$

**Theorem 4.5.1. (Sequential Criterion for infinite limits)**
$\lim_{x \to a} f(x) = \infty \Longleftrightarrow$ *If* $(x_n)$ *is any sequence in the domain of* $f$ *such that* $x_n \neq a$ *for all* $n$ *and* $x_n \to a$, *then* $f(x_n) \to \infty$.

*Proof.* Exercise. $\square$

There is also a sequential criterion for $\lim_{x \to a} f(x) = -\infty$.

**Question:** Is there a squeeze theorem for $\lim_{x \to a} f(x) = \pm\infty$?

$\boxed{\textbf{Exercise}}$

(a) Formulate the definitions for the following statements:

- $\lim_{x \to a^+} f(x) = \infty$,
- $\lim_{x \to a^-} f(x) = \infty$,
- $\lim_{x \to a^+} f(x) = -\infty$,
- $\lim_{x \to a^-} f(x) = -\infty$.

(b) For each of these types of limit, state and prove a sequential criterion.

$\boxed{\textbf{Example}}$ We have

$$\lim_{x \to 0^+} \frac{1}{x} = \infty, \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty,$$

but

$$\lim_{x \to 0} \frac{1}{x} \neq \infty, \qquad \text{and} \qquad \lim_{x \to 0} \frac{1}{x} \neq -\infty.$$

$\boxed{\textbf{Exercise}}$ Prove that

$$\lim_{x \to 2^+} \frac{x}{x-2} = \infty, \qquad \text{and} \qquad \lim_{x \to 2^-} \frac{x}{x-2} = -\infty.$$

$\boxed{\textbf{Definition}}$ (**Limit at infinity**) Let $f$ be defined in $(a, \infty)$ for some $a \in \mathbb{R}$. We say that $L$ is the limit of $f$ as $x \to \infty$ if for any $\varepsilon > 0$, there exists $M > a$ such that

$$x > M \implies |f(x) - L| < \varepsilon.$$

In this case we write

$$\lim_{x \to \infty} f(x) = L.$$

$\boxed{\textbf{Exercise}}$ Formulate the definition of the statement

$$\lim_{x \to -\infty} f(x) = L.$$

**Example** Prove that for $k \in \mathbb{N}$, $\lim_{x\to\infty} \dfrac{1}{x^k} = 0$.

*Proof:* Let $\varepsilon > 0$ be given. Choose $M = \dfrac{1}{\varepsilon^{1/k}}$. Then

$$x > M \Longrightarrow x^k > M^k = \frac{1}{\varepsilon} \Longrightarrow \left|\frac{1}{x^k} - 0\right| = \frac{1}{x^k} < \varepsilon. \ \square$$

**Theorem 4.5.2. (Sequential criterion for limit at infinity)**
$\lim_{x\to\infty} f(x) = L$ *if and only if for any sequence* $(x_n)$ *in the domain of* $f$ *such that* $x_n \to \infty$, $f(x_n) \longrightarrow L$.

*Proof.* Exercise.

**Exercise** State and prove a sequential criterion for $\lim_{x\to-\infty} f(x) = L$.

**Theorem 4.5.3.** *If* $\lim_{x\to\infty} f(x) = L$ *and* $\lim_{x\to\infty} g(x) = M$, *then*

$$\lim_{x\to\infty}(f \pm g)(x) = L \pm M \quad \text{and} \quad \lim_{x\to\infty}(f \cdot g)(x) = LM.$$

*If, in addition, there exists* $K > 0$ *such that* $g(x) \neq 0$ *for* $x > K$ *and* $M \neq 0$, *then we also have*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

*Proof.* Use the sequential criterion. $\square$

**Example** Evaluate $\lim_{x\to\infty} \dfrac{2x^2+3}{3x^2+x}$.

*Solution:* We have

$$\lim_{x\to\infty} \frac{2x^2+3}{3x^2+x} = \lim_{x\to\infty} \frac{2+\frac{3}{x^2}}{3+\frac{1}{x}} = \frac{\lim_{x\to\infty}(2+\frac{3}{x^2})}{\lim_{x\to\infty}(3+\frac{1}{x})} = \frac{2}{3}.$$

**Squeeze Theorem for limit at infinity.** If $f(x) \le g(x) \le h(x)$ for all $x > M$ for some $M > 0$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = L$, then
$$\lim_{x\to\infty} g(x) = L.$$

*Proof.* Use the sequential criterion and the squeeze theorem for sequences. □

**Definition** We say the function $f$ tends to $\infty$ as $x \to \infty$ if any $M > 0$, there exists $K > 0$ such that
$$x > K \implies f(x) > M.$$
In this case, we write
$$\lim_{x\to\infty} f(x) = \infty.$$

**Exercise** Formulate and prove a sequential criterion for the limit $\lim_{x\to\infty} f(x) = \infty$.

**Exercise**

(a) Formulate the definitions for the following statements:

- $\lim_{x\to\infty} f(x) = -\infty$.
- $\lim_{x\to-\infty} f(x) = \infty$.
- $\lim_{x\to-\infty} f(x) = -\infty$.

(b) For each of these types of limit, state and prove a sequential criterion.

**Example** Prove that for any $n \in \mathbb{N}$, $\lim_{x\to\infty} x^n = \infty$.

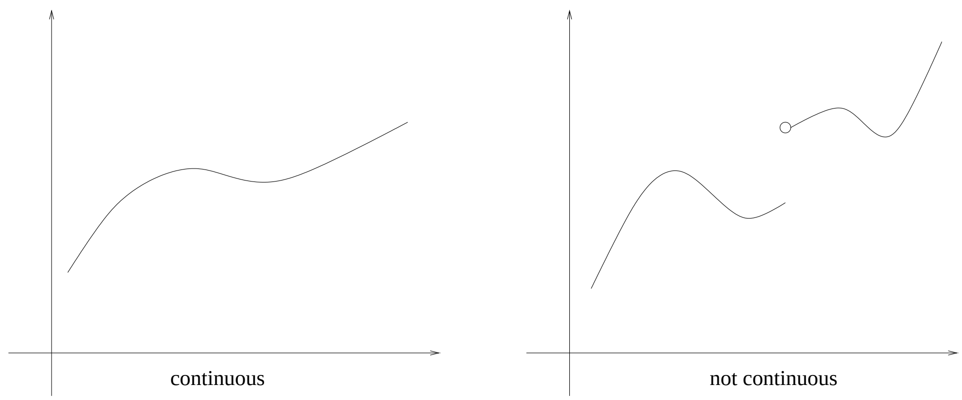
*Proof:* Let $M > 0$. Choose $K = \sqrt[n]{M}$. Then
$$x > K \implies x^n > K^n = M. \square$$

**Exercise** Let $n \in \mathbb{N}$. Prove that
$$\lim_{x\to-\infty} x^n = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd.} \end{cases}$$

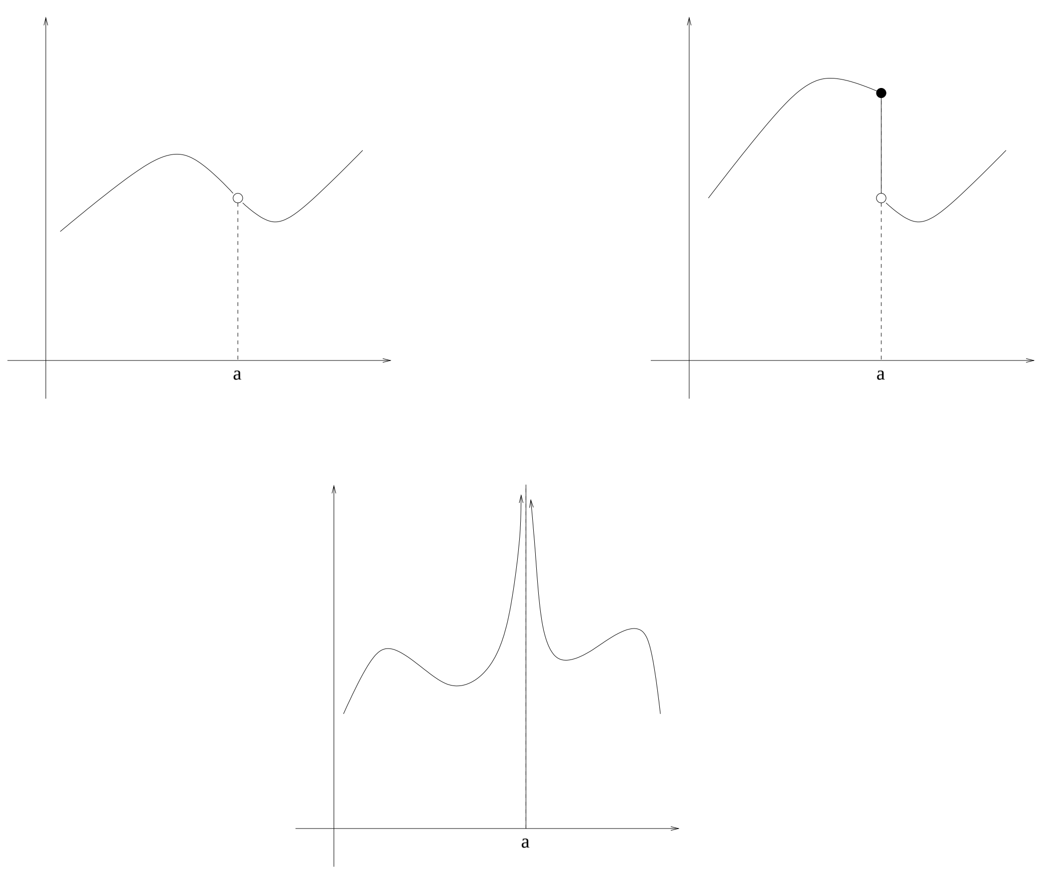
# Chapter 5: Continuous Functions

## 5.1 Definitions and examples

Intuitively, a continuous function is one such that its graph is an unbroken curve.




Some reasons why a curve is broken:



**Definition** $f$ is said to be *continuous* at $a$ if the following conditions are satisfied:

(i) $f$ is defined in a neighborhood $V_h(a)$ of $a$.

(ii) $\lim_{x \to a} f(x)$ exists.

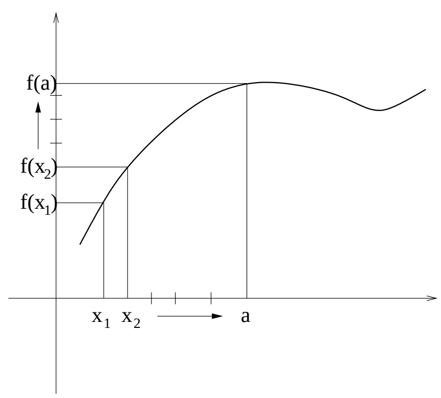
(iii) $\lim_{x \to a} f(x) = f(a)$.

If $f$ is not continuous at $a$, then we say $f$ is *discontinuous* at $a$.

$\varepsilon - \delta$ **definition of continuity:** $f$ is continuous at $a$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \qquad \text{whenever} \quad |x - a| < \delta.$$

**Theorem 5.1.1. (Sequential Criterion for Continuity)** *$f$ is continuous at $x = a$ if and only if for every sequence $(x_n)$ in the domain of $f$ such that $x_n \to a$, we have $f(x_n) \to f(a)$.*




*Proof:* Follows from the sequential criterion for limit of functions and the definition of continuity. □

**Definition** If $f$ is continuous at every point in a set $S$, then we say $f$ is continuous on $S$.

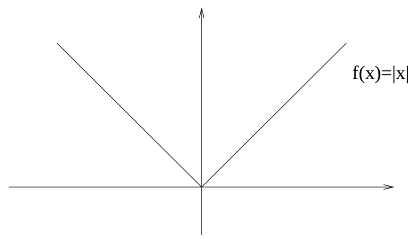
**Example** **Polynomial.** Recall that if $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ is a polynomial, then for any $a \in \mathbb{R}$,

$$\lim_{x \to a} p(x) = p(a).$$

Thus $p$ is continuous at every point $a$ in $\mathbb{R}$, that is, $p$ is continuous on $\mathbb{R}$.

Some specific examples: $3x^2 + 4x + 5$, $4x^7 + \frac{1}{2}x - 3$ and $2x + 3$ are continuous on $\mathbb{R}$.

$\boxed{\textbf{Example}}$ **Absolute-value function.** Let $f(x) = |x|$.



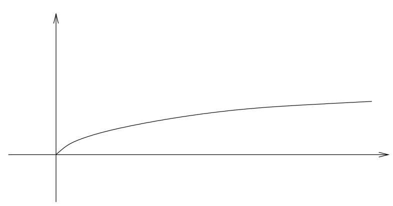

For all $a \in \mathbb{R}$,

$$\lim_{x\to a} f(x) = \lim_{x\to a} |x| = |a| = f(a).$$

So $f$ is continuous everywhere.

$\boxed{\textbf{Example}}$ **Square-root function.**

Let $f(x) = \sqrt{x}$, $x \geq 0$.



Then for all $a > 0$,

$$\lim_{x\to a} f(x) = \lim_{x\to a} \sqrt{x} = \sqrt{a} = f(a).$$

So $f$ is continuous on $(0, \infty)$.

$\boxed{\textbf{Example}}$ More generally, for $r \in \mathbb{Q}$, the function $g(x) = x^r$ is continuous on $(0, \infty)$. This follows from

$$\lim_{x\to a} x^r = a^r \quad \text{for any } a > 0.$$

(See page 8 of Chapter 4)

$\boxed{\textbf{Exercise}}$ In fact, the rational exponent in the previous example may be replaced by any real exponent. Let $\alpha \in \mathbb{R}$ and let $h(x) = x^\alpha$, $x > 0$. Prove that $h$ is continuous on $(0, \infty)$.

$\boxed{\textbf{Example}}$ Let $a > 0$, and let $f(x) = a^x$ for $x \in \mathbb{R}$. Then for any $b \in \mathbb{R}$,

$$\lim_{x\to b} f(x) = \lim_{x\to b} a^x = a^b = f(b)$$

(see page 8 of Chapter 4). So $f$ is continuous on $\mathbb{R}$.

**The exponential function:** An important special case is when $a = e$, the Euler number. The function

$$E : \mathbb{R} \to \mathbb{R}$$

$$E(x) = e^x, \qquad x \in \mathbb{R}$$

is called the *exponential function.* It is continuous on $\mathbb{R}$.

**Example** The functions $\sin x$ and $\cos x$ are continuous on $\mathbb{R}$.

(The proofs will be discussed in MA3110.)

**Example** **Rational functions.**

Let $p$ and $q$ be polynomials, and let $f(x) = p(x)/q(x)$. Then $f$ is defined everywhere except at the zeros of $q$.

If $q(a) \neq 0$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{p(x)}{q(x)} = \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} = \frac{p(a)}{q(a)} = f(a).$$

So $f$ is continuous everywhere except at the zeros of $q$.

**Example** Let $f(x) = [x]$.

For any $n \in \mathbb{Z}$, $\lim\limits_{x\to n}[x]$ does not exist. So $f$ is discontinuous at all the integral points. It is continuous everywhere else, i.e., on $\mathbb{R} \setminus \mathbb{Z}$.

Sometimes we can "save" a function which is discontinuous at a point:

- If $\lim\limits_{x\to a} f(x) = L$ exists but $f(a)$ is not defined, then we can simply define $f(a) = L$. The resulting function will be continuous at $a$.

**Example** Let $f(x) = x \sin \frac{1}{x}$, $x \neq 0$.

Since $f(0)$ is not defined, $f$ is discontinuous at $x = 0$.

However, by the Squeeze theorem, we obtain

$$\lim_{x\to 0} x \sin \frac{1}{x} = 0.$$

So we define $f(0) = 0$, i.e. the new $f$ is

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then the new $f$ is continuous at $x = 0$, so it is continuous on $\mathbb{R}$.

Some functions are hopeless!

- If $\lim_{x\to a} f(x)$ does not exist, then there is no way to make $f$ continuous at $a$.

$\boxed{\textbf{Example}}$ Let $f(x) = x/(x-1)$, $x \neq 1$.

Since $\lim_{x\to 1} f(x)$ does not exist, we cannot define $f(1)$ in such a way that $f$ is continuous at $x = 1$.

$\boxed{\textbf{Example}}$ A very bad function: Let

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

We have proved before that $\lim_{x\to a} f(x)$ does not exist for any $a \in \mathbb{R}$. So $f$ is not continuous anywhere.

$\boxed{\textbf{Example}}$ Let

$$f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ x+3 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Find the points at which $f$ is continuous.

*Solution:* Let $a \in \mathbb{R}$. Take a rational sequence $(x_n)$ and an irrational sequence $(y_n)$ such that $x_n \to a$, and $y_n \to a$. Then

$$f(x_n) = 2x_n \to 2a, \quad f(y_n) = y_n + 3 \to a + 3.$$

If $f$ is continuous at $x = a$, then

$$2a = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = a + 3$$

so that $a = 3$. It follows that if $a \neq 3$, then $f$ is not continuous at $x = a$.

Next we prove that $f$ is continuous at $x = 3$, i.e. $\lim_{x\to 3} f(x) = f(3) = 6$.
Let $\varepsilon > 0$. We choose $\delta = \varepsilon/2$. Then if $|x-3| < \delta$, we have

$$|f(x) - 6| = \begin{cases} |2x-6| = 2|x-3| < 2\cdot\dfrac{\varepsilon}{2} < \varepsilon & \text{if } x \text{ is rational} \\ |x+3-6| = |x-3| < \dfrac{\varepsilon}{2} < \varepsilon & \text{if } x \text{ is irrational.} \end{cases}$$

In other words,

$$|x-3| < \delta \Longrightarrow |f(x) - f(3)| < \varepsilon.$$

So $f$ is continuous at $x = 3$.

Hence, $f$ is continuous at only one point $x = 3$.

**Example** **Thomae's function.**

Let $f : (0, 1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \\ \dfrac{1}{q} & \text{if } x = \dfrac{p}{q},\ p, q \in \mathbb{N} \text{ and } \gcd(p, q) = 1. \end{cases}$$

So $f(1/\sqrt{2}) = 0$, $f(2/3) = 1/3$, $f(0.6) = f(6/10) = f(3/5) = 1/5$.

Where is $f$ continuous?

*Solution.* **Claim:** $f$ is not continuous at all rational points.
In fact, if $a \in \mathbb{Q}$ and $(x_n)$ is a irrational sequence in $(0, 1)$ such that $x_n \to a$, then $f(x_n) = 0 \to 0 \neq f(a)$.

Is $f$ continuous at the irrational points?

Equivalently, let $a$ be an irrational point in $(0, 1)$ and we ask: Is $\lim_{x \to a} f(x) = f(a) = 0$?

Let $\varepsilon > 0$. We need to find $\delta > 0$ such that $f(x) < \varepsilon$ for all $x \in (a - \delta, a + \delta)$.

**Observation 1:** The irrational $x$'s in $(0, 1)$ do not cause any trouble.

**Observation 2:** $f(p/q)$ is small if $q$ is large.

**Observation 3:** There are only finitely many rational numbers in $(0, 1)$ with small denominators, i.e. $p/q$ with $q \leq 1/\varepsilon$.

**Observation 4:** We can choose $\delta > 0$ so small that the interval $(a - \delta, a + \delta)$ will miss all the rational numbers with small denominators.

**Observation 5:** Now convince yourself that all $x \in (a - \delta, a + \delta)$ are such that $f(x) < \varepsilon$.

## 5.2 Combinations of continuous functions

**Theorem 5.2.1.** *Suppose that $f$ and $g$ are continuous at $x = a$.*

*(a) $f \pm g$, $f \cdot g$ and $cf$ are also continuous at $x = a$, where $c$ is a constant.*

*(b) If $g(a) \neq 0$, then $f/g$ is also continuous at $x = a$.*

*Proof:* Follows from the limit theorem for functions. □

**Example** Let $f(x) = \tan x$. Where is $f$ continuous?

*Solution:* We have

- $f(x) = \dfrac{\sin x}{\cos x}$.
- $\sin x$ and $\cos x$ are continuous everywhere.
- $\cos x = 0$ if and only if $x = (n + \frac{1}{2})\pi$ for some $n \in \mathbb{Z}$.

Thus $f$ is continuous on $\mathbb{R} \backslash \{(n + \frac{1}{2})\pi : \; n \in \mathbb{Z}\}$.

**Exercise** Where is $\cot x$ continuous?

**Definition** **Composite Functions.**

Suppose that $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subseteq B$.

We define the *composite function* $g \circ f : A \to \mathbb{R}$ by

$$(g \circ f)(x) = g[f(x)], \qquad \forall x \in A.$$

**Theorem 5.2.2.** *Suppose the functions $f$ and $g$ are such that $g \circ f$ is defined. If $f$ is continuous at $a$, and $g$ is continuous at $f(a)$, then $g \circ f$ is continuous at $a$.*

*Proof.* Let $\varepsilon > 0$. Since $g$ is continuous at $f(a)$, there exists $\delta_1 > 0$ such that

$$|y - f(a)| < \delta_1 \Longrightarrow |g(y) - g[f(a)]| < \varepsilon. \qquad (*)$$

Now $f$ is continuous at $a$, so there exists $\delta > 0$ such that

$$|x - a| < \delta \Longrightarrow |f(x) - f(a)| < \delta_1.$$

Consequently, by putting $y = f(x)$ in $(*)$, we have

$$|x - a| < \delta \Longrightarrow |f(x) - f(a)| < \delta_1 \overset{(*)}{\Longrightarrow} |g[f(x)] - g[f(a)]| < \varepsilon. \; \square$$

Alternatively, we can prove the above theorem using the sequential criterion.

**Theorem 5.2.3.** *Suppose that $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subseteq B$, so that $g \circ f$ is defined. If $f$ is continuous on $A$, and $g$ is continuous on $B$, then $g \circ f$ is continuous on $A$.*

**Example** The function $h(x) = \sin(e^x)$ is continuous everywhere because $h = g \circ f$ with $g(y) = \sin y$ and $f(x) = e^x$.

**Example** Let $g(x) = |x|$. If $f : A \to \mathbb{R}$, then

$$(g \circ f)(x) = g[f(x)] = |f(x)| = |f|(x), \qquad \forall x \in A.$$

- If $f$ is continuous at $a$, then by Theorem 5.2.2, $|f|$ is continuous at $a$.
- If $f$ is continuous on $A$, then by Theorem 5.2.3, $|f|$ is continuous on $A$.

Some specific examples:

(a) $f(x) = \left|\dfrac{x^3 - 2x + 5}{x - 1}\right|$ is continuous on $\mathbb{R}\backslash\{1\}$.

(b) $g(x) = |\sin x|$ is continuous everywhere.

**Exercise** Is it true that

$$|f| \text{ continuous} \Longrightarrow f \text{ continuous?}$$

**Example** Let $g(x) = \sqrt{x}$. If $f : A \to \mathbb{R}^+$, i.e $f(x) > 0$ for each $x \in A$, then

$$(g \circ f)(x) = g[f(x)] = \sqrt{f(x)} = \sqrt{f}(x), \qquad \forall x \in A.$$

- If $f$ is continuous at $a$, then by Theorem 5.2.2, $\sqrt{f}$ is continuous at $a$.
- If $f$ is continuous on $A$, then by Theorem 5.2.3, $\sqrt{f}$ is continuous on $A$.

Some special cases:

(a) $f(x) = \sqrt{x^2 + x + 1}$ is continuous everywhere.

(b) $g(x) = \sqrt{\sin x}$ is continuous on $(0, \pi)$.

### 5.3 Continuous functions on intervals

**Definition** If $f$ is defined on the closed interval $[a,b]$, then we say that $f$ is continuous on $[a,b]$ if

(i) $f$ is continuous on $(a,b)$ in the usual sense, ie. $\lim_{x\to c} f(x) = f(c)$ for all $c \in (a,b)$;

(ii) $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

**Definition** A function $f : A \to \mathbb{R}$ is said to be *bounded* on $A$ if there exists $M > 0$ such that

$$|f(x)| \leq M, \qquad \forall x \in A.$$

So in this case, the set $f(A)$ is bounded.

**Example** Is the function $1/x$ bounded on $[2,\infty)$?

*Solution:* If $x \in [2,\infty)$, then $x \geq 2$, so that

$$\left|\frac{1}{x}\right| = \frac{1}{x} \leq \frac{1}{2}.$$

So $1/x$ is bounded on $[2,\infty)$ (take $M = 1/2$).

**Theorem 5.3.1.** *If $f$ is continuous on $[a,b]$, then $f$ is bounded on $[a,b]$.*

*Proof:* Suppose $f$ is not bounded. Then for each $n \in \mathbb{N}$, $\exists x_n \in [a,b]$ such that

$$|f(x_n)| > n.$$

Since $(x_n)$ is bounded, by the Bolzano-Weierstrass theorem, $(x_n)$ has a convergent subsequence $(x_{n_k})$.

Let $c = \lim_{k\to\infty} x_{n_k}$.
Since $a \leq x_{n_k} \leq b$ for all $k \in \mathbb{N}$, $a \leq c \leq b$, i.e. $c \in [a,b]$.

Since $f$ is continuous at $c$, $f(x_{n_k}) \to f(c)$.

On the other hand, $|f(x_{n_k})| > n_k \geq k \uparrow \infty$, i.e $(f(x_{n_k}))$ is unbounded. So $(f(x_{n_k}))$ is divergent.
But this contradicts the fact $f(x_{n_k}) \to f(c)$.

So $f$ is bounded on $[a,b]$. □

**Extreme values:** Suppose that a function $f$ is bounded on $A$, and

$$M = \sup f(A), \quad m = \inf f(A).$$

**Question:** Do there exist $x_1, x_2 \in A$ such that

$$f(x_1) = m, \quad f(x_2) = M?$$

**Example** Consider the function $f(x) = 1/x$, $x \in A$.

(i) If $A = (1, 2)$, then

$$\sup f(A) = 1, \qquad \inf f(A) = \frac{1}{2}.$$

But there are no points $x_1, x_2 \in A$ such that $f(x_1) = 1$ and $f(x_2) = \frac{1}{2}$.

(ii) If $A = [1, 2]$, then

$$f(1) = \sup f(A) = 1, \quad f(2) = \inf f(A) = \frac{1}{2}.$$

**Extreme-value Theorem.** *If $f$ is continuous on $[a, b]$, then there exists $x_1, x_2 \in [a, b]$ such that*

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a, b].$$

*Proof:* By Theorem 5.3.1, $f$ is bounded on $[a, b]$. Let

$$M = \sup f([a, b]) = \sup\{f(x) : \ x \in [a, b]\}.$$

We need to find $x_2 \in [a, b]$ such that $f(x_2) = M$.

Since $M = \sup f([a, b])$, for each $n \in \mathbb{N}$, there exists $a_n \in [a, b]$ such that

$$M - \frac{1}{n} < f(a_n) \leq M.$$

By the Squeeze Theorem, $f(a_n) \to M$.

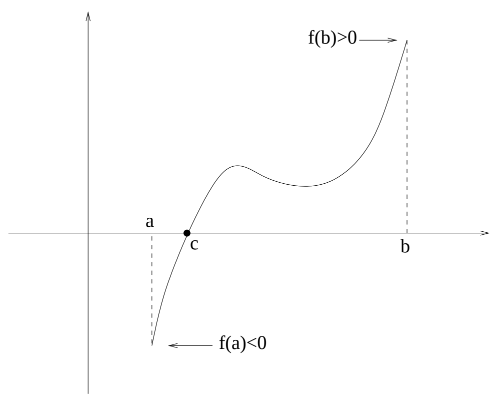
On the other hand, $(a_n)$ is bounded. By the Bolzano-Weiestrass Theorem, it has a convergent subsequence $(a_{n_k})$. Let $x_2 = \lim\limits_{k\to\infty} a_{n_k}$. Then since $a \leq a_{n_k} \leq b$ for all $k$, $a \leq x_2 \leq b$, i.e. $x_2 \in [a, b]$.
Since $f$ is continuous at $x_2$ and $a_{n_k} \to x_2$,

$$f(a_{n_k}) \to f(x_2).$$

But $(f(a_{n_k}))$ is a subsequence of $(f(a_n))$ and $f(a_n) \to M$, we also have $f(a_{n_k}) \to M$. By the uniqueness of limit, $f(x_2) = M$.

Similar arguments show that there exists $x_1 \in [a, b]$ such that $f(x_1) = \inf f([a, b])$. $\square$

**Location of Roots Theorem.** *If $f$ is continuous on $[a, b]$, $f(a) < 0 < f(b)$, then there exists a point $c$ in $(a, b)$ such that $f(c) = 0$.*




*Proof:* Let $A = \{x \in [a, b] : \ f(x) \leq 0 \}$.

Since $a \in A$, $A \neq \emptyset$. Moreover, $A$ is bounded. So $c = \sup A$ exists. Note that $c \in (a, b)$. (Why?)

**Claim:** $f(c) = 0$.
Suppose $f(c) \neq 0$. Then either $f(c) < 0$ and $f(c) > 0$.

**Case 1:** $f(c) < 0$
$\exists \delta > 0$ such that

$$f(x) < 0, \qquad \forall x \in (c - \delta, c + \delta).$$

In particular,

$$f\left(c + \frac{\delta}{2}\right) < 0$$

so that $c + \dfrac{\delta}{2} \in A$. But $c + \dfrac{\delta}{2} > c$ contradicts the fact $c = \sup A$.

**Case 2:** $f(c) > 0$
Again there exists $\delta > 0$ such that

$$f(x) > 0, \qquad \forall x \in (c - \delta, c + \delta).$$

In particular, for $c - \dfrac{\delta}{2} \leq x \leq c$, $x \notin A$. It follows that $c - \dfrac{\delta}{2}$ is an upper bound of $A$.
But $c - \dfrac{\delta}{2} < c$, and this contradicts $c = \sup A$.

Since both case 1 and case 2 cannot occur, we must have $f(c) = 0$ $\square$

$\boxed{\textbf{Example}}$ Show that the equation

$$x^3 - x + 2 = 0$$

has a solution between $-2$ and 1.

**Intermediate Value Theorem.** *If $f$ is continuous on $[a, b]$, and $k$ is between $f(a)$ and $f(b)$, then there exists a point $c$ in $(a, b)$ such that $f(c) = k$.*

*Proof:* Assume that $f(a) < k < f(b)$. Let $g(x) = f(x) - k$. Then $g$ is continuous on $[a, b]$, $g(a) = f(a) - k < 0$ and $g(b) = f(b) - k > 0$. By the Location of Roots Theorem, $\exists c \in (a, b)$ such that

$$g(c) = f(c) - k = 0.$$

So $f(c) = k$. $\square$

$\boxed{\textbf{Exercise}}$ Let $f$ be a continuous function on $[a, b]$ and suppose that $f(a) \neq f(b)$. Prove that there is a number $c$ in $(a, b)$ such that

$$f(c) = \frac{1}{5}f(a) + \frac{4}{5}f(b).$$

Hint: $\dfrac{1}{5} + \dfrac{4}{5} = 1$.

**Theorem 5.3.2.** *If $f$ is continuous on $[a, b]$, then*

$$f([a, b]) = [m, M],$$

*where $m = \inf f([a, b])$ and $M = \sup f([a, b])$.*

*Proof:* By the Extreme-value Theorem, there exist $x_1$ and $x_2$ in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$.

Suppose that $k \in [m, M]$. Then

$$m = f(x_1) \leq k \leq M = f(x_2).$$

By the Intermediate Value Theorem, there exists a point $c$ between $x_1$ and $x_2$ such that $f(c) = k$. This shows that $k \in f([a, b])$. Since $k \in [m, M]$ is arbitrary, $f([a, b]) = [m, M]$. $\square$

Theorem 5.3.2 states that the image of a closed bounded interval under a continuous function is also a closed bounded interval.

In general, if $I$ is an interval and $f$ is continuous on $I$, then $f(I)$ is also an interval. However, $I$ and $f(I)$ may not be of the same type.

**Example** Let $f(x) = 1/(x^2+1)$, $I_1 = (-1, 1)$ and $I_2 = [0, \infty)$. Then $f(I_1) = (1/2, 1]$ and $f(I_2) = (0, 1]$.

## 5.4 Monotone and inverse functions

**Definition** Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$.

(a) $f$ is *increasing* on $A$ if

$$x_1, x_2 \in A \text{ and } x_1 \leq x_2 \Longrightarrow f(x_1) \leq f(x_2).$$

(b) $f$ is *strictly increasing* on $A$ if

$$x_1, x_2 \in A \text{ and } x_1 < x_2 \Longrightarrow f(x_1) < f(x_2).$$

(c) $f$ is *decreasing* on $A$ if

$$x_1, x_2 \in A \text{ and } x_1 \leq x_2 \Longrightarrow f(x_1) \geq f(x_2).$$

(d) $f$ is *strictly decreasing* on $A$ if

$$x_1, x_2 \in A \text{ and } x_1 < x_2 \Longrightarrow f(x_1) > f(x_2).$$

(e) $f$ is *monotone* if it is either increasing or decreasing.

(f) $f$ is *strictly monotone* if it either strictly increasing or strictly decreasing.

It turns out that a monotone function defined on an interval always has one-sided limits.

**Theorem 5.4.1.** *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be an increasing function. If $c \in I$ is not an end point of $I$, then $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and they are given by*

$$\lim_{x\to c^-} f(x) = \sup\{f(x) : \ x \in I, \ x < c\} \quad \textit{and} \quad \lim_{x\to c^+} f(x) = \inf\{f(x) : \ x \in I, \ x > c\}.$$

*Proof:* Let $S = \{f(x) : \ x \in I, \ x < c\}$. If $x \in I$ and $x < c$, then $f(x) \leq f(c)$. Thus $f(c)$ is an upper bound of $S$. By the Supremum property of $\mathbb{R}$, $L = \sup S$ exists. We shall prove that $\lim_{x\to c^-} f(x) = L$.

Let $\varepsilon > 0$. Then since $L - \varepsilon$ is not an upper bound of $S$, there exists $x_\varepsilon \in I$ such that $x_\varepsilon < c$ and

$$L - \varepsilon < f(x_\varepsilon) \leq L. \qquad (*)$$

Let $\delta = c - x_\varepsilon > 0$. If $c - \delta < x < c$, then $x > x_\varepsilon$, so that $f(x) \geq f(x_\varepsilon)$. This together with $(*)$ gives

$$L - \varepsilon < f(x_\varepsilon) \leq f(x) \leq L.$$

Hence

$$c - \delta < x < c \Longrightarrow |f(x) - L| < \varepsilon.$$

This proves $\lim\limits_{x \to c^-} f(x) = L$. The proof for the other formula is similar. □

**Remark** By the above theorem, if $f$ is increasing and is discontinuous at $c$, then

$$\lim_{x \to c^-} f(x) < \lim_{x \to c^+} f(x).$$

The difference

$$j_f(c) = \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x)$$

is called the *jump* of $f$ at $c$.

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a strictly monotone function. Then $f$ is injective. Let $J = f(I)$. Then we can define a function $g : J \to \mathbb{R}$ as follows: For each $y \in f(I)$, there is a unique $x \in I$ such that $f(x) = y$. We set

$$g(y) = x.$$

In other words, $g$ is the *inverse function* of $f$. It is denoted by $f^{-1}$.

**Continuous Inverse Theorem.** *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a strictly monotone function. If $f$ is continuous on $I$ and $J = f(I)$, then its inverse function $f^{-1} : J \to \mathbb{R}$ is strictly monotone and continuous on $J$.*

*Proof:* We shall assume that $f$ is strictly increasing. The other case is similar.

Since $f$ is continuous on $I$ and $I$ is an interval, $J = f(I)$ is also an interval. Let $f^{-1} : J \to \mathbb{R}$ be the inverse function of $f$.

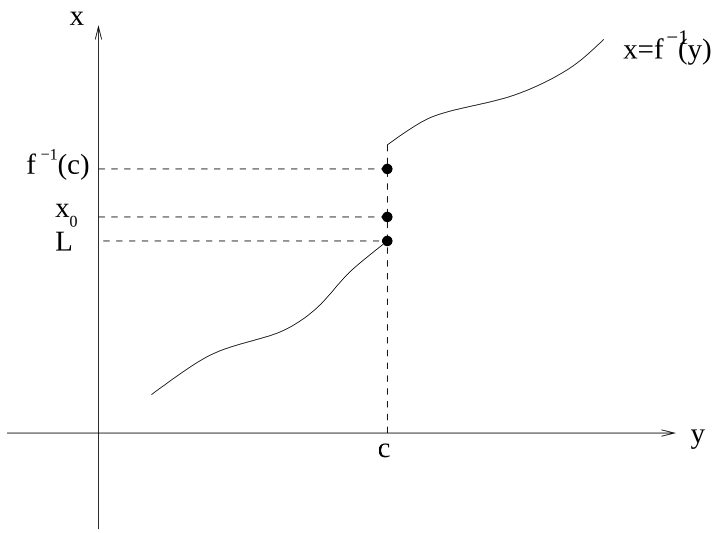
**Claim:** $f^{-1}$ is strictly increasing on $J$.

In fact, let $y_1 < y_2$ be two points in $J$. Then there exist $x_1, x_2 \in I$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. By the definition of $f^{-1}$, $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$. If $x_1 \geq x_2$, then since $f$ is strictly increasing, $y_1 = f(x_1) \geq f(x_2) = y_2$, which contradicts the fact that $y_1 < y_2$. So we must have $x_1 < x_2$, that is, $f^{-1}(y_1) < f^{-1}(y_2)$. This proves the claim.

Next we shall prove that $f^{-1}$ is continuous on $J$. By contradiction: assume that $f^{-1}$ is discontinuous at $c \in J$. Then either

$$\lim_{y \to c^-} f^{-1}(y) < f^{-1}(c) \qquad \text{or} \qquad f^{-1}(c) < \lim_{y \to c^+} f^{-1}(y).$$

Assume that the first case occurs (the second case can be treated in a similar way). Write $L = \lim_{y \to c^-} f^{-1}(y)$ and choose any point $x_0$ such that $L < x_0 < f^{-1}(c)$. Then $x_0 \in I$.




Next we shall show that $f(x_0) \notin J$: Let $y_1 \in J$. If $y_1 < c$, then

$$f^{-1}(y_1) < \lim_{y \to c^-} f^{-1}(y) = L < x_0.$$

On the other hand, if $y_1 \geq c$, then

$$f^{-1}(y_1) \geq f^{-1}(c) > x_0.$$

Hence $x_0 \neq f^{-1}(y_1)$ for all $y_1 \in J$, and consequently $f(x_0) \notin J$. Since $x_0 \in I$, this contradicts $f(I) = J$. □

**Example** Let $a > 1$, and let $f(x) = a^x$, $x \in \mathbb{R}$. Then $f$ is continuous and is strictly increasing on $\mathbb{R}$.
In addition, the range of $f$ is $(0, \infty)$.
By the Continuous Inverse Theorem, its inverse function $f^{-1}$ is also strictly increasing and continuous on $(0, \infty)$.

**The natural logarithm function:** The inverse function of the exponential function $E(x) = e^x$, $x \in \mathbb{R}$ is given by

$$\ln : (0, \infty) \to \mathbb{R}$$

$$\ln(y) = x \qquad \text{if } y = e^x,$$

and is called *the natural logarithm function*. It is strictly increasing and continuous on $(0, \infty)$.

## 5.5 Uniform continuity

Consider the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$ and $g : (0, \infty) \to \mathbb{R}$, $g(x) = 1/x$. Both function are continuous on their domains.

Let $\varepsilon > 0$.

(i) If $a \in \mathbb{R}$ and $\delta = \varepsilon/2$, then

$$|x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon.$$

Note that $\delta$ depends only on $\varepsilon$, and is independent of the point $a$.

(ii) If $a \in (0, \infty)$, then

$$|g(x) - g(a)| = \left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x - a|}{a|x|}.$$

If $|x - a| < a/2$, then $x > a/2$ and

$$|g(x) - g(a)| = \frac{|x - a|}{a|x|} < \frac{2}{a^2}|x - a|.$$

If we take $\delta = \min\left(\frac{a}{2}, \frac{1}{2}a^2\varepsilon\right)$, then

$$|x - a| < \delta \Longrightarrow |g(x) - g(a)| < \varepsilon.$$

Note that $\delta$ depends on both the point $a$ and $\varepsilon$.

We say that $f$ is uniformly continuous on $\mathbb{R}$, but $g$ is not uniformly continuous on $(0, \infty)$.

**Definition** Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. We say that $f$ is *uniformly continuous* on $I$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in I, \ |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon.$$

(This says that for a given $\varepsilon > 0$, a $\delta$ can be chosen such that it works for all the points in $I$.)

Clearly if $f$ is uniformly continuous on $I$, then it is continuous on $I$, The converse is false.

**Example** Let the function $g : [0, \infty) \to \mathbb{R}$ be uniformly continuous on $[0, \infty)$ and $g(0) = 0$. Prove that there exists $C > 0$ such that

$$|g(x)| < 1 + Cx \qquad \text{for all } x > 0.$$

*Solution:* Let $\delta_1 > 0$ be such that

$$x, y \geq 0, \ |x - y| < \delta_1 \Longrightarrow |g(x) - g(y)| < 1.$$

Let $x > 0$, and set $n = [x/\delta]$ where $\delta = \delta_1/2$. Then

$$\begin{aligned}
|g(x)| &= |g(x) - g(0)| \\
&= |\{g(x) - g(n\delta)\} + \{g(n\delta) - g((n-1)\delta)\} + \cdots + \{g(\delta) - g(0)\}| \\
&\leq |g(x) - g(n\delta)| + |g(n\delta) - g((n-1)\delta)| + \cdots + |g(\delta) - g(0)| \\
&< \overbrace{1 + 1 + \cdots + 1}^{n+1} \\
&= 1 + n \\
&\leq 1 + \frac{x}{\delta} \\
&= 1 + Cx
\end{aligned}$$

where $C = 1/\delta$.

**Theorem 5.5.1. (Sequential criterion for uniform continuity)**

*The function $f : I \to \mathbb{R}$ is uniformly continuous on $I$ if and only if for any two sequences $(x_n)$ and $(y_n)$ in $I$ such that $x_n - y_n \to 0$, we have $f(x_n) - f(y_n) \to 0$.*

*Proof.* ($\Longrightarrow$) Assume that $f : I \to \mathbb{R}$ is uniformly continuous on $I$, and let $(x_n)$ and $(y_n)$ be sequences in $I$ such that $x_n - y_n \to 0$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$x, a \in I,\ |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon. \qquad (*)$$

Since $x_n - y_n \to 0$, there exists $K \in \mathbb{N}$ such that

$$n \geq K \Longrightarrow |x_n - y_n| < \delta.$$

By this and $(*)$, we have

$$n \geq K \Longrightarrow |f(x_n) - f(y_n)| < \varepsilon.$$

($\Longleftarrow$) Assume that $f : I \to \mathbb{R}$ is not uniformly continuous on $I$. Then there exists $\varepsilon_0 > 0$ such that for each $\delta > 0$, there exist $x_\delta, y_\delta \in I$ such that

$$|x_\delta - y_\delta| < \delta, \qquad \text{but} \qquad |f(x_\delta) - f(y_\delta)| \geq \varepsilon_0.$$

In particular, for each $n \in \mathbb{N}$, we may take $\delta = 1/n$. Then there are $x_n$ and $y_n$ in $I$ such that

$$|x_n - y_n| < \frac{1}{n}, \qquad \text{but} \qquad |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

So $x_n - y_n \to 0$ but $f(x_n) - f(y_n) \nrightarrow 0$. $\square$

**Corollary 5.5.2.** *The function $f : I \to \mathbb{R}$ is not uniformly continuous on $I$ if and only if there exist two sequences $(x_n)$ and $(y_n)$ in $I$ such that $x_n - y_n \to 0$ but $f(x_n) - f(y_n) \nrightarrow 0$.*

**Example** Consider the function $g(x) = 1/x$ on $(0, 1]$. Let $x_n = 1/n$ and $y_n = 1/(n+1)$, $n \in \mathbb{N}$. Then the sequences $(x_n)$ and $(y_n)$ are in $(0, 1]$,

$$x_n - y_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \to 0,$$

but

$$|g(x_n) - g(y_n)| = |n - (n+1)| = 1 \nrightarrow 0.$$

Hence $g$ is not uniformly continous on $(0, 1]$.

**Exercise** Is the function $f(x) = x^2$ uniformly continuous on $(0, \infty)$?

**Theorem 5.5.3.** *If $f$ is continuous on a closed bounded interval $[a, b]$, then it is uniformly continuous on $[a, b]$.*

*Proof.* Suppose $f$ is not uniformly continuous on $[a, b]$. Then from the proof of Theorem 5.5.1, there exist $\varepsilon_0 > 0$ and two sequences $(x_n)$ and $(y_n)$ in $[a, b]$ such that $x_n - y_n \to 0$ but

$$|f(x_n) - f(y_n)| \geq \varepsilon_0 \qquad \forall n \in \mathbb{N}. \qquad (*)$$

Now $a \leq x_n \leq b$ for all $n$, so $(x_n)$ is a bounded sequence. By the Bolzano-Weiestrass Theorem, $(x_n)$ has a convergent sequence $(x_{n_k})$. Let $c = \lim x_{n_k}$. Then since $a \leq x_n \leq b$, we have $a \leq c \leq b$. Moreover,

$$y_{n_k} = x_{n_k} - (x_{n_k} - y_{n_k}) \to c - 0 = c.$$

Since $f$ is continuouse at $c$,

$$f(x_{n_k}) \to f(c) \qquad \text{and} \qquad f(y_{n_k}) \to f(c).$$

Consequently

$$f(x_{n_k}) - f(y_{n_k}) \to f(c) - f(c) = 0.$$

On the other hand, by $(*)$,

$$|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0 \qquad \forall k \in \mathbb{N},$$

so that $f(x_{n_k}) - f(y_{n_k}) \nrightarrow 0$. This is a contradiction. □

**Theorem 5.5.4.** *If $I$ is an interval and $f : I \to \mathbb{R}$ satisfies the **Lipschitz condition** on $I$, that is, there is a $K > 0$ such that*

$$|f(x) - f(y)| \le K|x - y|, \qquad \forall x, y \in I,$$

*then $f$ is uniformly continuous on $I$.*

*Proof.* Let $\varepsilon > 0$. Take $\delta = \varepsilon/K$. Then

$$x, y \in I, |x - y| < \delta \Longrightarrow |f(x) - f(y)| \le K|x - y| < K\delta = \varepsilon. \square$$

**Example** Let $f(x) = ax + b$ be a linear function, where $a$ and $b$ are real constants. Then

$$|f(x) - f(y)| = |a||x - y| \qquad \forall x, y \in \mathbb{R}.$$

So $f(x) = ax + b$ satisfies the Lipschitz condition on $\mathbb{R}$. Consequently it is uniformly continuous on $\mathbb{R}$.

**Example** It can be proved that

$$|\sin x - \sin a| \le |x - a| \qquad \forall x, a \in \mathbb{R}.$$

So $f(x) = \sin x$ satisfies the Lipschitz condition on $\mathbb{R}$. Consequently it is uniformly continuous on $\mathbb{R}$.

**Example** (Uniform Continuity does not imply Lipschitz)

The function $g(x) = \sqrt{x}$ is continuous on $[0, 1]$, so it is uniformly continuous on $[0, 1]$. But there does not exists $K > 0$ for which

$$|g(x) - g(0)| \le K|x - 0| \qquad \forall x \in (0, 1].$$

Why?

**Continuous functions may not preserve Cauchy sequences!**

This means that if $(x_n)$ is a Cauchy sequence and $f$ is a continuous function, then $(f(x_n))$ may not be a Cauchy sequence.

Here is an example: Consider $f(x) = 1/x$, $x > 0$. The sequence $(1/n)$ is a Cauchy sequence in $(0, \infty)$, but

$$f\left(\frac{1}{n}\right) = n, \qquad n \in \mathbb{N}.$$

Since $(f(1/n))$ diverges, it is not a Cauchy sequence.

**Theorem 5.5.5.** *If $f : I \to \mathbb{R}$ is uniformly continuous on $I$ and $(x_n)$ is a Cauchy sequence in $I$, then $(f(x_n))$ is a Cauchy sequence.*

*Proof.* Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$x, y \in I,\ |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon. \qquad (*)$$

Now $(x_n)$ is Cauchy, so there exists $K \in \mathbb{N}$ such that

$$n, m \geq K \Longrightarrow |x_n - x_m| < \delta.$$

This together with condition $(*)$ gives

$$n, m \geq K \Longrightarrow |f(x_n) - f(x_m)| < \varepsilon.$$

This shows that $(f(x_n))$ is a Cauchy sequence. $\square$.

**Theorem 5.5.6.** *If the function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on $(a, b)$, then $f(a)$ and $f(b)$ can be defined so that the extended function is continuous on $[a, b]$.*

*Proof.* Take a sequence $(x_n)$ in $(a, b)$ such that $x_n \to a$. Then $(x_n)$ is a Cauchy sequence. By Theorem 5.5.5, $(f(x_n))$ is also a Cauchy sequence, so it converges.

Define $f(a) = \lim_{n\to\infty} f(x_n)$.

**Claim:** $f(a)$ is well defined (i.e. it does not depends on the choice of the sequence $(x_n)$).
Let $(y_n)$ be another sequence in $(a, b)$ converging to $a$. Then

$$y_n - x_n \to a - a = 0.$$

Since $f$ is uniformly continuous on $(a, b)$, this implies

$$f(y_n) - f(x_n) \to 0.$$

Consequently,

$$f(y_n) = [f(y_n) - f(x_n)] + f(x_n) \to 0 + f(a) = f(a).$$

This shows that $f(a)$ is well defined.

The proof of the claim also shows that $\lim_{x\to a^+} f(x) = f(a)$, so that $f$ is continuous at $a$.

Next we take a sequence $(u_n)$ in $(a, b)$ which converges to $b$ and define $f(b) = \lim_{n\to\infty} f(u_n)$. Using similar arguments, we can show $\lim_{x\to b^-} f(x) = f(b)$. $\square$

**Example** Is the function $f(x) = \cos(1/x)$ uniformly continuous on $(0, 1)$?