National University of Singapore
Department of Mathematics

2020/2021 Semester 2 | MA3110 Mathematical Analysis II | Tutorial 2

Homework 1

1. Please submit your solutions of **Questions H1-H4** before 8pm on **1 Feb 2021** (Monday).
   (a) Your submission must be one single PDF file.
   (b) Name the PDF file by **Your Name(HW1).pdf**.
   (c) Upload your PDF file into the LumiNUS folder **Homework1_Submission**.
2. You may discuss the problems with other students, but you must write up your solutions by yourself. *Copying is a breach of academic honesty.*

H1. Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(x) = 2 + 3x^2 + 4\ln x \quad \text{for } x > 0.$$

(i) Prove that f is strictly increasing on $(0, \infty)$.

(ii) Let $g : \mathbb{R} \to \mathbb{R}$ be the inverse function of f. Find $g'(5)$.

H2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^x + x^2 \cos\left(\dfrac{1}{2x}\right) & x \neq 0 \\ 1 & x = 0. \end{cases}$$

(i) Find $f'(x)$ for each $x \in \mathbb{R}$.

(ii) Is $f \in C^1(\mathbb{R})$? Justify your answer.

H3. Use the Mean Value Theorem to prove the Bernoulli's inequality:

$$(1 + x)^n > 1 + nx, \quad \text{for all } x \in (-1, 0) \cup (0, \infty) \text{ and } n = 2, 3, 4, \ldots$$

H4. Suppose that the function f is continuous on $[a, b]$ and differentiable on (a, b). Prove that if

$$(f(b))^2 - (f(a))^2 = b^2 - a^2,$$

then there exists $c \in (a, b)$ such that

$$f'(c)f(c) = c.$$

Tutorial 2 Questions

1. Prove that the converse of Part (i) of Theorem 6.3.3 is true. That is, prove that if f is differentiable on (a, b) and is increasing on (a, b), then

$$f'(x) \geq 0 \qquad \forall x \in (a, b).$$

2. *If a function f has a positive derivative at a point, does it follow that f is increasing in a neighborhood of c?*

   Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x + 3x^2 \cos \dfrac{1}{x} & x \neq 0 \\ \\ 0 & x = 0. \end{cases}$$

   (i) Show that $f'(0) = 2$.

   (ii) Does there exist $\delta > 0$ such that f is increasing on $(-\delta, \delta)$? Justify your answer.

3. **(Intermediate Value Theorem for Derivatives)** Let the function f be differentiable on $[a, b]$ and $f'(a) < f'(b)$. Prove that if k is a real number such that $f'(a) < k < f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = k$.

4. Let I be an interval. Prove that if f is differentiable on I and if the derivative f' is bounded on I, then f satisfies the ***Lipschitz condition*** on I, that is, there is a $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|, \quad \forall x, y \in I.$$

   (Consequently, f is uniformly continuous on I.)

   Deduce that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

5. Suppose that the function f has the following properties:

   (i) f is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

   (ii) f' is strictly increasing on $(0, 1)$, that is, for $0 < x < y < 1$, $f'(x) < f'(y)$.

   (iii) $f(0) = 0$.

   Prove that the function $f(x)/x$ is strictly increasing on $(0, 1]$.

6. Suppose that the function $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and $f'(a) = f'(b) = 0$. By using the function

$$h(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} & a < x \leq b \\ \\ f'(a) = 0 & x = a, \end{cases}$$

prove that there exists a point c in (a, b) such that

$$\frac{f(c) - f(a)}{c - a} = f'(c).$$