Chapter 6: Differentiable Functions

6.1 Limits of functions

We review some standard facts on limits of functions. See Chapter 4 of MA2108 for more details.

Roughly speaking, we say that a function f has a limit L at the point $x = a$ if

$$x \approx a \Longrightarrow f(x) \approx L.$$

$\boxed{\textbf{Definition}}$ Let the function f be defined in a deleted neighborhood of a. We say that the real number L is the limit of f at a if for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

In this case, we write

$$\lim_{x \to a} f(x) = L.$$

Sequential Criterion for limits

$\lim\limits_{x \to a} f(x) = L \Longleftrightarrow$ If (x_n) is any sequence in the domain of f such that $x_n \neq a$ for all n and $x_n \to a$, then $f(x_n) \to L$.

Consequences:

1. If there is a sequence (x_n) in the domain of f such that $x_n \neq a$ for all n and $x_n \to a$, but $f(x_n) \not\to L$, then $\lim\limits_{x \to a} f(x) \neq L$.

2. *Divergent Criterion.* To prove that $\lim\limits_{x \to a} f(x)$ does not exist:

   (a) *Method 1.* Find a sequence (x_n) in the domain of f such that $x_n \neq a$ for all n and $x_n \to a$, but $(f(x_n))$ diverges.

   (b) *Method 2.* Find two sequences (x_n) and (y_n) in the domain of f such that $x_n \neq a$ and $y_n \neq a$ for all n and $x_n \to a$, $y_n \to a$, but $\lim\limits_{n \to \infty} f(x_n) \neq \lim\limits_{n \to \infty} f(y_n)$.

$\boxed{\textbf{Exercise}}$ (Revision) Suppose that the limit $\lim\limits_{x \to a} f(x) = L$ exists.

(i) Prove that if $L > 0$, then there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow f(x) > 0.$$

(ii) Prove that if $L \neq 0$, then there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow f(x) \neq 0.$$

$\boxed{\textbf{Definition}}$ A function f is said to be *continuous* at a if $\lim_{x \to a} f(x) = f(a)$.

Other types of limits:

Two sided limit at a	**One-sided limits at a**	**Limit at infinity**
$\lim_{x \to a} f(x) = L$	*(left-hand limit)* $\lim_{x \to a^-} f(x) = L$	$\lim_{x \to \infty} f(x) = L$
	(right-hand limit) $\lim_{x \to a^+} f(x) = L$	$\lim_{x \to -\infty} f(x) = L$

(Finite limit) $L \in \mathbb{R}$
(Infinite limit) $L = \infty, \ -\infty$

6.2 The derivative

$\boxed{\textbf{Definition}}$ A function f is said to be *differentiable* at the point a if f is defined in some open interval containing a and the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. In this case, $f'(a)$ is called the *derivative* of f at a.

$\boxed{\textbf{Remark}}$

(i) By letting $h = x - a$, we can write $f'(a)$ as the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

(ii) Geometrically, $f'(a)$ is the slope of the tangent line to the curve $y = f(x)$ at $x = a$.

$\boxed{\textbf{Definition}}$

(a) **(Differentiable functions on open intervals)**
If f is differentiable at every point in (a, b), then we say that f is differentiable on (a, b).

(b) **(Differentiable functions on closed intervals)**
If the function $f : [a, b] \to \mathbb{R}$ is such that

  (i) f is differentiable on (a, b); and

  (ii) the one-sided limits

$$L_1 = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}, \quad L_2 = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$$

  exist,

then we say that f is differentiable on $[a, b]$.

In this case, we define

$$f'(a) := L_1, \quad f'(b) = L_2.$$

Question: How should we define differentiable functions on other types of intervals $[a, b)$, $[a, \infty)$, $(a, b]$, $(-\infty, b]$ and $(-\infty, \infty)$?

Definition Let f be differentiable on the interval I. Then the derivative of f is the function $f' : I \to \mathbb{R}$ given by

$$x \to f'(x), \quad x \in I.$$

We also write $f'(x) = \frac{d}{dx} f(x)$ as a function of x, and write the derivative of f at the point a as

$$f'(a) = \frac{d}{dx} f(x)\bigg|_{x=a}.$$

Definition We say that f is *continuously differentiable* on I if f is differentiable on I and f' is continuous on I.

Notation The collection of all functions which are continuously differentiable on I is denoted by $C^1(I)$.

Example For any constant c, $\frac{d}{dx}(c) = 0$, i.e. the derivative of a constant function is the zero function.

Proof. Let $f(x) = c$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = 0. \quad \square$$

Example $\frac{d}{dx}(x) = 1$, i.e. the derivative of x is the constant function 1.

Proof. Let $f(x) = x$. Then

$$f'(x) = \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1. \quad \square$$

Exercise Prove that for $n \in \mathbb{N}$, $\frac{d}{dx}(x^n) = nx^{n-1}$. So $x^n \in C^1(\mathbb{R})$.

Theorem 6.2.1. *If f is differentiable at a, then it is continuous at a.*

Proof. We have

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}\{f(x)-f(a)\} + \lim_{x\to a} f(a) \\
&= \lim_{x\to a}\left\{\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right\} + f(a) \\
&= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a) + f(a) \\
&= f'(a)\cdot 0 + f(a) \\
&= f(a).
\end{aligned}$$

□

Remark

(a) By the above theorem, if f is not continuous at a, then it is not differentiable at a.

(b) f continuous at $a \not\Longrightarrow f$ differentiable at a.

Example Let $f(x) = |x|$. Then f is continuous at 0.

Note that

$$\begin{aligned}
\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+}\frac{|x|-0}{x-0} = \lim_{x\to 0^+}\frac{x}{x} = 1, \\
\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^-}\frac{|x|-0}{x-0} = \lim_{x\to 0^-}\frac{-x}{x} = -1.
\end{aligned}$$

So

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$$

does not exist, and f is not differentiable at 0.

Remark

(i) By extending the corners of the graph of $|x|$, we obtain continuous functions which are not differentiable at any finite number of points or at countably infinite number of points.

(ii) There exists a continuous function on $\mathbb{R}$ which is not differentiable anywhere. We will discuss the construction of such a function in Chapter 8.

Theorem 6.2.2. *Let f and g be differentiable at a. Then*

(a) $\left.\dfrac{d}{dx}[f(x) \pm g(x)]\right|_{x=a} = f'(a) \pm g'(a).$

(b) (Product Rule)

$$\left.\frac{d}{dx} f(x)g(x)\right|_{x=a} = f'(a)g(a) + f(a)g'(a).$$

(c) (Quotient Rule) If $g(a) \neq 0$, then

$$\left.\frac{d}{dx}\frac{f(x)}{g(x)}\right|_{x=a} = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Proof. (a) Exercise.

(b) We have

$$\begin{aligned}
\left.\frac{d}{dx} f(x)g(x)\right|_{x=a} &= \lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \left\{ f(x) \cdot \frac{g(x) - g(a)}{x-a} \right\} + \lim_{x\to a} \left\{ \frac{f(x) - f(a)}{x-a} \cdot g(a) \right\} \\
&= f(a)g'(a) + f'(a)g(a).
\end{aligned}$$

(c) We have

$$\begin{aligned}
\left.\frac{d}{dx}\frac{f(x)}{g(x)}\right|_{x=a} &= \lim_{x\to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(a) - g(x)f(a)}{g(x)g(a)(x-a)} \\
&= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - g(x)f(a)}{g(x)g(a)(x-a)} \\
&= \lim_{x\to a} \frac{1}{g(x)g(a)} \left\{ \frac{f(x) - f(a)}{x-a} \cdot g(a) - f(a)\frac{g(x) - g(a)}{x-a} \right\} \\
&= \frac{1}{(g(a))^2} \left\{ f'(a)g(a) - f(a)g'(a) \right\}.
\end{aligned}$$

□

Remark

You can also prove the formula $\frac{d}{dx}(x^n) = nx^{n-1}$ $(n \in \mathbb{N})$ using the Product Rule and induction, and the formula

$$\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}} \quad (n \in \mathbb{N})$$

using the Quotient Rule. The second formula can be written as $\frac{d}{dx}(x^{-n}) = (-n)x^{-n-1}$. Hence we can combine the two formulas into one:

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \forall n \in \mathbb{Z} \setminus \{0\}.$$

Remark We will define the functions e^x, $\ln x$, $\sin x$, $\cos x$ precisely in Chapter 9. For now, we will assume the following results:

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}\ln x = \frac{1}{x}.$$

The derivatives of other trigonometric functions can be obtained by using the above facts and Theorem 6.2.2.

Theorem 6.2.3. (Carathéodory's Theorem) *Let I be an interval, $f : I \to \mathbb{R}$ and $c \in I$. Then $f'(c)$ exists if and only if there exists a function φ on I such that φ is continuous at c and*

$$f(x) - f(c) = \varphi(x)(x - c) \qquad \forall x \in I.$$

In this case, $f'(c) = \varphi(c)$.

Proof. $(\Longrightarrow)$ Define

$$\varphi(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c} & x \in I, x \neq c, \\ f'(c) & x = c. \end{cases}$$

Then clearly $f(x) - f(c) = \varphi(x)(x - c)$ for all $x \in I$. Moreover,

$$\lim_{x \to c} \varphi(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) = \varphi(c).$$

So φ is continuous at c.

$(\Longleftarrow)$ Since φ is continuous at c,

$$\varphi(c) = \lim_{x \to c} \varphi(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

So $f'(c) = \varphi(c)$ exists.

□

Chain Rule. *Let I and J be intervals, and let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be such that $f(J) \subseteq I$. If $a \in J$, f is differentiable at a and g is differentiable at $f(a)$, then $h = g \circ f$ is differentiable at a, and*

$$h'(a) = g'(f(a))f'(a).$$

Proof. Since $f'(a)$ exists, by Theorem 6.2.3, there exists a function φ on J such that φ is continuous at a and $f(x) - f(a) = \varphi(x)(x - a)$ for all $x \in J$, and $\varphi(a) = f'(a)$.

Next, $g'(f(a))$ exists. By Theorem 6.2.3 again, there exists a function ψ on I such that ψ is continuous at $b = f(a)$ and $g(y) - g(b) = \psi(y)(y - b)$ for all $y \in I$, and $\psi(b) = g'(b)$.

We now define the function $\alpha : J \to \mathbb{R}$ by

$$\alpha(x) = \psi(f(x))\varphi(x) \qquad \forall x \in J.$$

Then α is continuous at a. Moreover, for each $x \in J$, by putting $y = f(x)$, we obtain

$$g(f(x)) - g(f(a)) = \psi(f(x))(f(x) - f(a)) = \psi(f(x))\varphi(x)(x - a) = \alpha(x)(x - a).$$

By Theorem 6.2.3, $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = \alpha(a) = \psi(f(a)\varphi(a) = g'(f(a))f'(a).$$

□

Remark

(1) Another form of the Chain Rule:
Let $y = g(f(x)) = g(u)$, where $u = f(x)$. Then

$$\frac{dy}{dx} = (g \circ f)'(x) = g'(f(x))f'(x) = g'(u)u' = \frac{dy}{du}\frac{du}{dx}.$$

$$\textit{Formally}: \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

(2) We usually rewrite a complicated function as a composite function (by making a substitution) and use the Chain Rule to find its derivative.

Example Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Determine the points at which f is differentiable.

Solutions: Let $h(x) = 1/x$ and $g(x) = \sin x$. If $c \neq 0$, then h is differentiable at c and g is differentiable at $h(c)$. By the Chain Rule, the composite function

$$s(x) = (g \circ h)(x) = \sin(1/x)$$

is differentiable at c, and

$$s'(c) = g'(h(c))h'(c) = \cos(1/c)(-1/c^2) = -\frac{\cos(1/c)}{c^2}.$$

By the Product Rule, $f(x) = x^2 s(x)$ is differentiable at c, and

$$f'(c) = 2cs(c) + c^2 s'(c) = 2c\sin(1/c) - \cos(1/c).$$

For $c = 0$,

$$f'(0) = \lim_{x\to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x\to 0} x\sin(1/x) = 0$$

by the Squeeze Theorem. Hence f is differentiable on $\mathbb{R}$.

Question: Is f continuously differentiable on $\mathbb{R}$? That is, is $f \in C^1(\mathbb{R})$?

Since $f'(x) = 2x\sin(1/x) - \cos(1/x)$ for $x \neq 0$, f' is clearly continuous on $\mathbb{R}\backslash\{0\}$, i.e. $f \in C(\mathbb{R}\backslash\{0\})$.

However, f' is not continuous at $x = 0$. So $f \notin C(\mathbb{R})$.
We prove this using sequential criterion: If f' is continuous at $x = 0$, then for any sequence $x_n \to 0$, we must have $f'(x_n) \to f'(0) = 0$. We show that there is a sequence (x_n) which violates sequential criterion, and so f' is not continuous at $x = 0$.

The sequence (x_n) is defined by $x_n = \dfrac{1}{2n\pi}$, $n \in \mathbb{N}$. Then $x_n \to 0$, but

$$f'(x_n) = 2x_n \sin\frac{1}{x_n} - \cos\frac{1}{x_n} = \frac{1}{n\pi}\sin(2n\pi) - \cos(2n\pi) = -1 \not\to 0 = f'(0).$$

Inverse Functions. Recall that if $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is strictly monotone and continuous on I, then $J = f(I)$ is also an interval and the inverse function $g = f^{-1} : J \to \mathbb{R}$ satisfies the relation

$$g(f(x)) = x \quad \forall x \in I.$$

By the Continuous Inverse Theorem (c.f. Section 5.4), g is strictly monotone and continuous on J.

Theorem 6.2.4. (Inverse Function Theorem) *Let I be an interval, and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J := f(I)$ and let $g : J \to \mathbb{R}$ be the inverse function of f. If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $d := f(c)$ and*

$$g'(d) = \frac{1}{f'(c)}.$$

Proof. By Theorem 6.2.3, there exists a function φ defined on I such that φ is continuous at c, $\varphi(c) = f'(c) \neq 0$ and

$$f(x) - f(c) = \varphi(x)(x - c) \quad \forall x \in I.$$

Since φ is continuous at c and $\varphi(c) \neq 0$, there exists $\delta > 0$ such that

$$\varphi(x) \neq 0 \quad \forall x \in V := (c - \delta, c + \delta) \subseteq I.$$

Let $U := f(V) \subseteq J$. Then U is an interval containing $d := f(c)$ and $g(U) = V$.
For all $y \in U$, we have $g(y) \in V$ and $y = f(g(y))$, so

$$\varphi(g(y)) \neq 0 \quad \text{ and } \quad y - d = f(g(y)) - f(c) = \varphi(g(y))(g(y) - g(d)).$$

Hence

$$g(y) - g(d) = \frac{1}{\varphi(g(y))}(y - d), \quad y \in U.$$

Now the function $\dfrac{1}{\varphi \circ g}$ is defined on U and it is continuous at d. By Theorem 6.2.3, g is differentible at d and

$$g'(d) = \frac{1}{\varphi(g(d))} = \frac{1}{\varphi(c)} = \frac{1}{f'(c)}.$$

□

Example Let $n \in \mathbb{N}$, and let $f(x) = x^n$ for $x \in [0, \infty)$. Then f is strictly increasing and continuous on $[0, \infty)$. The inverse function of f is given by

$$g(y) = y^{1/n}, \qquad y \in [0, \infty).$$

Now $f'(x) = nx^{n-1}$ for $x \in [0, \infty)$. So by Theorem 6.2.4,

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{n(g(y))^{n-1}} = \frac{1}{ny^{(n-1)/n}} = \frac{1}{n}y^{\frac{1}{n}-1}, \qquad y \in (0, \infty).$$

Rational exponents: Recall that if $x > 0$ and $r = \dfrac{n}{m}$ where $n, m \in \mathbb{N}$, then

$$x^r = x^{\frac{n}{m}} := \left(x^{\frac{1}{m}}\right)^n.$$

Example Let r be a positive rational number, and let $f(x) = x^r$ for $x > 0$. Prove that

$$f'(x) = rx^{r-1} \quad \text{for all } x > 0.$$

Proof. Tutorial 1. □

6.3 Mean Value Theorem and applications

The circle of related theorems: Rolle's Theorem, Mean Value Theorem, Cauchy's Mean Value Theorem and Taylor's Theorem are fundamental results in differential calculus. In this section, we will discuss these results and a few standard applications.

Definition Let I be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$.

(a) (*Absolute Maximum*) We say that $f(x_0)$ is the *absolute maximum* of f on I if $f(x_0) \geq f(x)$ for all $x \in I$.

(b) (*Absolute Minimum*) We say that $f(x_0)$ is the *absolute minimum* of f on I if $f(x_0) \leq f(x)$ for all $x \in I$.

(c) (*Relative Maximum*) We say $f(x_0)$ is a *relative maximum* of f if there exists $\delta > 0$ such that

$$f(x) \leq f(x_0), \qquad \forall x \in (x_0 - \delta, x_0 + \delta) \subseteq I.$$

(d) (*Relative Minimum*) We say $f(x_0)$ is a *relative minimum* of f if there exists $\delta > 0$ such that

$$f(x) \geq f(x_0), \qquad \forall x \in (x_0 - \delta, x_0 + \delta) \subseteq I.$$

(e) (*Relative Extremum*) We say $f(x_0)$ is a *relative extremum* of f if $f(x_0)$ is either a relative maximum or a relative minimum of f.

Remark (i) A relative extremum can only occur at an interior point, but an absolute extremum may occur at one of the end points of the interval. So if a function has an absolute maximum at a point x_0, it may not have a relative maximum at x_0.

(ii) If f has a absolute maximum at an interior point x_0 of I, then $f(x_0)$ is also a relative maximum of f.

Lemma 6.3.1. *Let $f : (a, b) \to \mathbb{R}$ and $f'(c)$ exists for some $c \in (a, b)$.*

(i) If $f'(c) > 0$, then there exists $\delta > 0$ such that

$$f(x) < f(c) \quad \text{for every } x \in (c - \delta, c), \text{ and}$$

$$f(x) > f(c) \quad \text{for every } x \in (c, c + \delta).$$

(ii) If $f'(c) < 0$, then there exists $\delta > 0$ such that

$$f(x) > f(c) \quad \text{for every } x \in (c - \delta, c), \text{ and}$$

$$f(x) < f(c) \quad \text{for every } x \in (c, c + \delta).$$

Proof. (i) Since $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \Longrightarrow \frac{f(x) - f(c)}{x - c} > 0.$$

If $x \in (c - \delta, c)$, then $x - c < 0$, so that

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c) < 0 \quad \text{and} \quad f(x) < f(c).$$

If $x \in (c, c + \delta)$, then $x - c > 0$, so that

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c) > 0 \quad \text{and} \quad f(x) > f(c).$$

The proof for (ii) is similar.

□

Interior Extremum Theorem. *Suppose that c is an interior point of an interval I and $f : I \to \mathbb{R}$ is differentiable at c. If f has a relative extremum at c, then $f'(c) = 0$.*

Proof. Suppose f has a relative maximum at c.

(1) If $f'(c) > 0$, by Lemma 6.3.1, there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq I$ (since c is an interior point of I), $f(x) < f(c)$ for every $x \in (c - \delta, c)$, and $f(x) > f(c)$ for every $x \in (c, c + \delta)$. This contradicts the assumption that f has a relative maximum at c.

(2) If $f'(c) < 0$, then likewise we get a contradiction using Lemma 6.3.1.

Hence $f'(c) = 0$.

The case of a relative minimum at c is similar. □

Remark

(1) A function f may have a relative extremum at x_0 but $f'(x_0)$ does not exist.

Example : Let $f(x) = |x|, \ x \in \mathbb{R}$.
Then f has a relative (indeed absolute) minimum at 0, but $f'(0)$ does not exist.

(2) The converse of the Interior Extremum Theorem is ***false***: $f'(c) = 0$ does not imply that f has a relative extremum at c.

Example : Let $f(x) = x^3, \ x \in \mathbb{R}$.
Then $f'(0) = 0$ but 0 is *not* a relative extremum point of f.

Rolle's Theorem. *If f is continuous on $[a,b]$ and differentiable on (a,b) and $f(a) = f(b)$, then there exists $c \in (a,b)$ such that*

$$f'(c) = 0.$$

Proof. **Case 1:** f is a constant function.
In this case, $f'(x) = 0$ for all $x \in (a,b)$.

Case 2: f is not a constant function.
By the Extrema-value Theorem, there exists $x_1, x_2 \in [a,b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2), \qquad \forall x \in [a,b].$$

Since f is not constant, $f(x_1) \neq f(x_2)$. Since $f(a) = f(b)$, at least one of x_1 and x_2 is in (a,b), and we call this point c. So f has a absolute extremum at $c \in (a,b)$. In particular, f has a relative extremum at c. By the Interior Extremum Theorem, $f'(c) = 0$. □

Mean Value Theorem. *If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define $g : [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then g is continuous on $[a,b]$, differentiable on (a,b),

$$g(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0,$$

and

$$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = 0.$$

By Rolle's Theorem, $\exists c \in (a,b)$ such that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0.$$

So

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

Example Use the Mean Value Theorem to show that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$.

Solution: Let $f(x) = e^x$, $x \in \mathbb{R}$. Then $f'(x) = e^x$ for all $x \in \mathbb{R}$.

If $x > 0$, then by the Mean Value Theorem, there exists $c_1 \in (0, x)$ such that

$$e^x - 1 = f(x) - f(0) = f'(c_1)(x - 0) = e^{c_1}x.$$

Since $e^{c_1}x > e^0 x = x$, we obtain $e^x > 1 + x$.

Similarly, if $x < 0$, then there exists $c_2 \in (x, 0)$ such that $e^x - 1 = e^{c_2}x$. Since $x < 0$ and $e^{c_2} < 1$, $e^{c_2}x > x$. Again we obtain $e^x > 1 + x$.

Finally, if $x = 0$, then $e^x = 1 = 1 + x$.

Theorem 6.3.2. *If f is continuous on $[a, b]$ and differentiable on (a, b), and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.*

Proof. If $a < x \leq b$, then by the Mean Value Theorem, there is a point $c \in (a, x)$ such that

$$f'(c) = \frac{f(x) - f(a)}{x - a}.$$

But $f'(c) = 0$, so $f(x) = f(a)$. □

Definition Let $f : I \to \mathbb{R}$.

- f is said to be *increasing* on I if

$$x_1, x_2 \in I \text{ and } x_1 < x_2 \implies f(x_1) \leq f(x_2).$$

- f is said to be *decreasing* on I if

$$x_1, x_2 \in I \text{ and } x_1 < x_2 \implies f(x_1) \geq f(x_2).$$

Theorem 6.3.3. *Let f be differentiable on (a, b).*

(i) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing on (a, b).

(ii) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).

Proof. We only prove (i). Let $a < x_1 < x_2 < b$. By the Mean Value Theorem, there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

So

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since $f'(c) \geq 0$ and $x_2 - x_1 > 0$, $f(x_2) - f(x_1) \geq 0$. So $f(x_2) \geq f(x_1)$. □

Remark The converse of (i) and (ii) in Theorem 6.3.3 are true. See Tutorial 2.

Exercise

(i) Prove that if $f'(x) > 0$ for all $x \in (a, b)$, then f is ***strictly increasing*** on (a, b), i.e.

$$a < x_1 < x_2 < b \Longrightarrow f(x_1) < f(x_2).$$

Give an example to show that the converse is false.

(ii) Prove that if $f'(x) < 0$ for all $x \in (a, b)$, then f is ***strictly decreasing*** on (a, b).

First Derivative Test. *Let f be continuous on $[a, b]$ and $c \in (a, b)$. Suppose that f is differentiable on (a, b) except possibly at c.*

(i) If there is a neighborhood $(c - \delta, c + \delta) \subseteq I$ of c such that $f'(x) \geq 0$ for $x \in (c - \delta, c)$ and $f'(x) \leq 0$ for $x \in (c, c + \delta)$, then

$$f(c) \geq f(x), \qquad \forall x \in (c - \delta, c + \delta).$$

Hence f has a relative maximum at c.

(ii) If there is a neighborhood $(c - \delta, c + \delta) \subseteq I$ of c such that $f'(x) \leq 0$ for $x \in (c - \delta, c)$ and $f'(x) \geq 0$ for $x \in (c, c + \delta)$, then

$$f(c) \leq f(x), \qquad \forall x \in (c - \delta, c + \delta).$$

Hence f has a relative minimum at c.

Proof. We only prove (i). Let $x \in (c - \delta, c)$. By applying the Mean Value Theorem to $[x, c]$, we obtain a point $x_0 \in (x, c)$ such that

$$f'(x_0) = \frac{f(c) - f(x)}{c - x},$$

i.e.

$$f(c) - f(x) = f'(x_0)(c - x).$$

Since $c - \delta < x_0 < c$, $f'(x_0) \geq 0$. This together with $c - x > 0$ give $f(c) - f(x) \geq 0$. So $f(c) \geq f(x)$.

Next we let $x \in (c, c + \delta)$. Then there exists $x_1 \in (c, x)$ such that

$$f(x) - f(c) = f'(x_1)(x - c).$$

Since $f'(x_1) \leq 0$ and $x - c > 0$, $f(x) - f(c) \leq 0$. So $f(c) \geq f(x)$. □

Higher derivatives

If f is differentiable on an interval I, then its derivative f' is a function on I. So we can consider the differentiability of f'. If $c \in I$ and f' is differentiable at c, then we call the derivative of f' at c the ***second derivative*** of f at c and denote it by $f''(c)$ or $f^{(2)}(c)$. That is,

$$f''(c) := (f')'(c).$$

Similarly, we define the third derivative $f'''(c) = f^{(3)}(c) := (f'')'(c)$. In general, if $n \in \mathbb{N}$, then the nth derivative $f^{(n)}(c)$ of f at c is defined as

$$f^{(n)}(c) := (f^{(n-1)})'(c).$$

So $f^{(n)}(c)$ exists if $f^{(n-1)}$ exists in a neighborhood of c, and $f^{(n-1)}$ is differentiable at c.

Notation Let I be an interval.

(i) For $n \in \mathbb{N}$, let
$$C^n(I) = \{f : f^{(n)} \text{ exists and is continuous on } I\}.$$

(ii) Let
$$C^\infty(I) = \{f : f^{(n)} \text{ exists and is continuous on } I \text{ for all } n \in \mathbb{N}\} = \bigcap_{n=1}^{\infty} C^n(I).$$

If $f \in C^\infty(I)$, then we say that f is *infinitely differentiable* on I.

Remark

(i) We denote the collection of all continuous functions on I by $C^0(I)$ or $C(I)$.

(ii) For integers $m > n \geq 1$, we have

$$C^\infty(I) \subset C^m(I) \subset C^n(I) \subset C(I).$$

Exercise Suppose that the function $f : [a,b] \to \mathbb{R}$ has the following properties:

(1) f is continuous on $[a,b]$.

(2) f'' exists on (a,b).

(3) The graph of f and the line segment joining the points $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0))$, where $a < x_0 < b$.

Prove that there exists $c \in (a,b)$ such that $f''(c) = 0$.

Second Derivative Test. *Let f be defined on an interval I and let its derivative f' exist on I. Suppose that c is an interior point of I such that $f'(c) = 0$ and $f''(c)$ exists.*

(i) If $f''(c) > 0$, then f has a relative minimum at c.

(ii) If $f''(c) < 0$, then f has a relative maximum at c.

Proof. (i) Suppose that $(f')'(c) = f''(c) > 0$.
By Lemma 6.3.1 (applied to f'), there exists $\delta > 0$ such that

$$f'(x) \begin{cases} < f'(c) = 0 & \text{if } x \in (c-\delta, c) \\ > f'(c) = 0 & \text{if } x \in (c, c+\delta). \end{cases}$$

By the First Derivative Test, f has a relative minimum at c.

The proof for (ii) is similar. $\square$

Remark

(a) If $f''(c) = 0$, then the Second Derivative Test is inconclusive.

Example: Let $f(x) = x^3$. Then $f'(0) = f''(0) = 0$, but 0 is not a relative extremum point for f.

(b) It is easier to apply the Second Derivative Test, but it is less powerful than the First Derivative Test.

Example: Let $g(x) = x^4$. Then $g'(0) = g''(0) = 0$. So the Second Derivative Test cannot be used here.
On the other hand,

$$g'(x) = 4x^3 \begin{cases} < 0 & x < 0 \\ > 0 & x > 0. \end{cases}$$

By the First Derivative Test, 0 is a relative minimum point for g.

Cauchy Mean Value Theorem. *Let f and g be continuous on $[a,b]$ and differentiable on (a,b), and assume that $g'(x) \neq 0$ for all $x \in (a,b)$. Then there exists $c \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. First we claim that $g(a) \neq g(b)$.
If this is not the case, then $g(a) = g(b)$, and by Rolle's Theorem, there exists $x_0 \in (a,b)$ such that $g'(x_0) = 0$. But this contradicts the assumption that $g'(x) \neq 0$ for all $x \in (a,b)$.

We now define $h : [a,b] \to \mathbb{R}$ by

$$h(x) := \frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a)) - (f(x)-f(a)), \qquad x \in [a,b]. \quad (*)$$

Then h is continuous on $[a,b]$, differentiable on (a,b), $h(a) = h(b) = 0$. So by Rolle's Theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$. On the other hand, by $(*)$,

$$h'(c) = \frac{f(b)-f(a)}{g(b)-g(a)}g'(c) - f'(c).$$

□

L'Hospital's Rule for right-hand limit (0/0 case)

Let f and g be differentiable on (a,b) and assume that $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose that $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$.

(i) If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, where $L \in \mathbb{R}$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$.

(ii) If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \pm\infty$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \pm\infty$.

Proof. (i) Let $\varepsilon > 0$. Then there exists $c \in (a,b)$ such that

$$t \in (a,c) \Longrightarrow L - \frac{\varepsilon}{2} < \frac{f'(t)}{g'(t)} < L + \frac{\varepsilon}{2}.$$

Now let $a < x < y < c$.
By Rolle's Theorem, $g(x) \neq g(y)$.
Further, by the Cauchy Mean Value Theorem, there exists $u \in (x,y)$ such that

$$\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(u)}{g'(u)}.$$

Since $u \in (x,y) \subseteq (a,c)$,

$$L - \frac{\varepsilon}{2} < \frac{f(y)-f(x)}{g(y)-g(x)} < L + \frac{\varepsilon}{2}. \quad (*)$$

Now, since $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$, we have for each $y \in (a,c)$,

$$\lim_{x\to a^+} \frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f(y)}{g(y)}.$$

It follows that as $x \to a^+$ in $(*)$,

$$L - \varepsilon < L - \frac{\varepsilon}{2} \le \frac{f(y)}{g(y)} \le L + \frac{\varepsilon}{2} < L + \varepsilon, \quad \text{for all } y \in (a,c).$$

(ii) Exercise. □

Example Find $\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}}$.

Solutions: By L'Hospital's Rule,

$$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+} \frac{\cos x}{\frac{1}{2}\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+} 2\sqrt{x}\cos x = 0.$$

Remark L'Hospital's Rule (with obvious modifications) remains valid if the right-hand limits as "$x \to a^+$" are replaced throughout by the left-hand limits as "$x \to b^-$", or replaced throughout by the limits as "$x \to c$", or as "$x \to \pm\infty$". For other forms of L'Hospital's Rule, see Section 6.3 of the textbook.

L'Hospital's Rule for right-hand limit (∞/∞ case)

Let f and g be differentiable on (a,b) and assume that $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose that $\lim_{x\to a^+} g(x) = \infty$.

(i) If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, where $L \in \mathbb{R}$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$.

(ii) If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \pm\infty$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \pm\infty$.

Example Find $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

Solutions:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

$\boxed{\textbf{Example}}$ Find $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.
Solutions:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

$\boxed{\textbf{Example}}$ Find $\lim_{x\to\infty} \frac{x^n}{e^x}$ where $n \in \mathbb{N}$.
Solutions:

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

$\boxed{\textbf{Example}}$ Find $\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n}$ where $n \in \mathbb{N}$.
Solutions: Using the fact that $\lim_{x\to 0^+} f(x) = \lim_{y\to\infty} f(1/y)$, we have

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = \lim_{y\to\infty} \frac{y^n}{e^{y^2}}.$$

For $y > 1$,

$$0 \le \frac{y^n}{e^{y^2}} \le \frac{y^{2n}}{e^{y^2}} = \frac{(y^2)^n}{e^{y^2}} \to 0 \quad \text{as } y \to \infty,$$

using the result of the previous example. By the squeeze theorem, $\frac{y^n}{e^{y^2}} \to 0$ as $y \to \infty$.
So $\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = 0$.

Taylor's Theorem
Let f be a function such that $f \in C^n([a,b])$ and $f^{(n+1)}$ exists on (a,b). If $x_0 \in [a,b]$, then for any $x \in [a,b]$, there exists a point c between x and x_0 such that

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\quad + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \end{aligned}$$

Proof. Fix $x \in [a,b]$. We may assume that $x_0 \neq x$. Let M be the unique number such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + M(x - x_0)^{n+1}.$$

Define $F : [a,b] \to \mathbb{R}$ by

$$F(t) := f(t) + f'(t)(x - t) + \frac{f''(t)}{2!}(x - t)^2 + \cdots + \frac{f^{(n)}(t)}{n!}(x - t)^n + M(x - t)^{n+1}. \quad (*)$$

Then F is continuous on $[a, b]$ and is differentiable on (a, b).

Now $F(x) = f(x)$. By the choice of M, we also have $F(x_0) = f(x)$. So by Rolle's Theorem, there exists a point c between x and x_0 such that

$$F'(c) = 0.$$

On the other hand, by differentiating $(*)$, we obtain

$$F'(c) = \frac{f^{(n+1)}(c)}{n!}(x - c)^n - M(n + 1)(x - c)^n.$$

So

$$M = \frac{f^{(n+1)}(c)}{(n + 1)!}$$

□

Remark

(a) If $n = 0$, then by Taylor's Theorem,

$$f(x) = f(x_0) + f'(c)(x - x_0),$$

that is,

$$f'(c) = \frac{f(x) - f(x_0)}{x - x_0}.$$

So Taylor's Theorem can be regarded as an extension of the Mean Value Theorem

(b) The polynomial

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the ***nth Taylor polynomial*** for f at x_0. It has the property that

$$P_n^{(j)}(x_0) = f^{(j)}(x_0) \quad \text{for } j = 0, 1, 2..., n.$$

So we can use it to estimate the value of f at points near x_0. In this context, we see that Taylor's Theorem give us the error (or remainder) of this estimation:

$$R_n(x) := f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - x_0)^{n+1}$$

where c is a point between x and x_0. The formula for R_n is called the ***Lagrange form*** (or the ***derivative form***) of the remainder.

Example Show that $\cos x \geq 1 - \frac{1}{2}x^2$ for all $x \in \mathbb{R}$.

Solutions: Let $f(x) = \cos x$, $x \in \mathbb{R}$ and set $x_0 = 0$. Then by Taylor's Theorem,

$$\begin{aligned} f(x) &= f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + R_2(x) \\ &= 1 - \frac{1}{2}x^2 + R_2(x) \end{aligned}$$

where

$$R_2(x) = \frac{f'''(c)}{3!}x^3 = \frac{\sin c}{6}x^3$$

for some c between 0 and x.

We consider 3 cases: $0 \le x \le \pi$, $-\pi \le x \le 0$ and $|x| \ge \pi$.
If $0 \le x \le \pi$, then $0 < c < \pi$ and $\sin c > 0$, so that $R_2(x) \ge 0$. Hence $\cos x = 1 - \frac{1}{2}x^2 + R_2(x) \ge 1 - \frac{1}{2}x^2$.
The other cases are left as an exercise.

$\boxed{\textbf{Example}}$ If $f(x) = e^x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$ for each x_0 and each x.

Solution: Since $f'(x) = e^x = f(x)$ for all $x \in \mathbb{R}$, $f^{(j)}(x) = f(x)$ for all $j \in \mathbb{N}$ and for all $x \in \mathbb{R}$. For fixed x_0 and x,

$$R_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x - x_0)^{n+1} = \frac{f(c_n)}{(n+1)!}(x - x_0)^{n+1}$$

and c_n is a point between x and x_0. Let I be the closed interval with end points x_0 and x. Since f is continuous on I, there exists $M > 0$ such that $|f(u)| \le M$ for all $u \in I$. It follows that

$$|R_n(x)| = \frac{|f(c_n)|}{(n+1)!}\left|(x - x_0)^{n+1}\right| \le \frac{M}{(n+1)!}|x - x_0|^{n+1}.$$

We now recall the following fact from MA2108: If (x_n) is a positive sequence and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} < 1$, then $x_n \to 0$.

Let $x_n = \frac{M}{(n+1)!}|x - x_0|^{n+1}$, $n \in \mathbb{N}$. Then

$$\frac{x_{n+1}}{x_n} = \frac{|x - x_0|}{n+2} \to 0.$$

Hence $x_n \to 0$. By the Squeeze Theorem, $R_n(x) \to 0$.

$\boxed{\textbf{Remark}}$ Let $f(x) = e^x$. For each $n \in \mathbb{N}$, by Taylor's Theorem,

$$f(x) = P_n(x) + R_n(x)$$

where

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j = e^{x_0} \sum_{j=0}^{n} \frac{(x - x_0)^j}{j!}$$

is the nth Taylor polynomial for f at x_0. By letting $n \to \infty$, we obtain

$$f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} P_n(x) + \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} P_n(x) + 0 = e^{x_0} \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!}.$$

In particular, if $x_0 = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Remark In general, if $R_n(x) \to 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

i.e., $f(x)$ is equal to its *Taylor series*.

Question: Does $R_n(x) \to 0$ all the time?

Example Let

$$h(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

For $x \neq 0$,

$$h'(x) = e^{-1/x^2} \frac{d}{dx}\left(-\frac{1}{x^2}\right) = \frac{2}{x^3} e^{-1/x^2}.$$

At $x = 0$,

$$h'(0) = \lim_{x \to 0} \frac{e^{-1/x^2} - 0}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = 0.$$

Hence, h is differentiable on $\mathbb{R}$.

Moreover, it can be proved that (Tutorial 3) for any $j \in \mathbb{N}$,

$$h^{(j)}(0) = 0.$$

Now by Taylor's Theorem,

$$h(x) = P_n(x) + R_n(x)$$

where

$$P_n(x) = h(0) + h'(0)x + \frac{h''(0)}{2!}x^2 + \cdots + \frac{h^{(n)}(0)}{n!}x^n = 0.$$

It follows that

$$h(x) = R_n(x) \not\to 0 \quad \text{for } x \neq 0.$$

In particular,

$$h(x) \neq \sum_{j=0}^{\infty} \frac{h^{(j)}}{j!} x^j \quad \text{for } x \neq 0.$$

That is, $h(x)$ **is not equal to its Taylor series except at** $x = 0$**.**