# MA4264 Game Theory

make decisions simultanously　　all info about game is given

## Chapter 1. Static games of complete information

### The Prisoners' Dilemma.

- Two suspects are arrested and charged with a crime.
- The police lack sufficient evidence to convict the suspects, unless at least one confesses.

static　decision simultanously

The suspects are held in separate cells and told:

- If only one of you confesses and testifies against your partner, then

  1 confess
  1 holdout

  – the person who confesses will go free
  – the person who does not confess will be convicted and given a 20-year jail sentence.

- 2 confess If both of you confess, then both will be convicted and sent to prison for 5 years.

0 confess

- Finally, if neither of you confesses, then both of you will be convicted of a minor offense and sentenced to 1 year in jail.

**Question.** What should the suspects do?

This problem can be represented in a bi-matrix: table

(row) prisoner 1's payoff

(col) prisoner 2's payoff

choices

Case A

Case B

| | | Prisoner2 | |
|---|---|---|---|
| | | Holdout | Confess |
| Prisoner1 | Holdout | −1, −1 | −20, 0 |
| | Confess | 0, −20 | −5, −5 |

Case 1 Case 2

$u_1$ payoff to prisoner 1

| | C | H |
|---|---|---|
| C | −1 | −20 |
| H | 0 | −5 |

- Prisoner 1 is also called the row player.
- Prisoner 2 is also called the column player.
- Each entry of the bi-matrix has two numbers:
  - the first number is the payoff of the row player and
  - the second ~~is that of the~~ column player.

$u_1$ : (prisoner 1's choice, prisoner 2's choices) → year in jail for prisoner 1

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To get the values, recall:

- If only one of you confesses and testifies against your partner, then
  - the person who confesses will go free while

– the person who does not confess will be convicted and given a 20-year jail sentence.

- If both of you confess, then both will be convicted and sent to prison for 5 years.

- Finally, if neither of you confesses, then both of you will be convicted of a minor offense and sentenced to 1 year in jail.

always good in all cases

**Conclusion.** Both players have a dominant strategy, $\boxed{\text{Confess}}$. I.e., no matter what other player does, confess is best move.

**Definition 1.** *The* **normal-form** *(also called* **strategic-form***) representation of an n-player game specifies the players'*

2 (prisoners' dilemma)

$\{C, H\}$ $=$ $S_1$ $\quad S_2 = \{C, H\}$

choices

- $\boxed{\text{strategy}}$ **spaces** $S_1, \ldots, S_n$ *and*
- *their* **payoff functions** $u_1, \ldots, u_n$.

$u_1$ payoff to prisoner 1

$u_2$ payoff to prisoner 2

*We denote this game by*

$$G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}.$$

$G = \{S_1, S_2; u_1, u_2\}$

$(S_1, S_2)$

*Let* $(s_1, \ldots S_i, s_n)$ *be a combination of strategies, one for each player.*

P1 P2

$(C, H)$ $(C, C)$

$(H, C)$ $(H, H)$

(choice) strategy of player $i$

$u_2(C,H) = -20$



- *I.e., player $i$ plays strategy $s_i$.*

combination of strategies

*Then $u_i(s_1,\ldots,s_n)$ is the payoff to player $i$ if for each $j = 1,\ldots,n$, player $j$ chooses strategy $s_j$.*

$\boxed{\text{prisoner 1 } C}$ payoff to player 1 (prisoner 1) ?

The payoff of a player depends not only on his own action but also on the actions of others! This inter-dependence is the essence of games!

In The Prisoners' Dilemma:

(choices) strategy spaces

- $G = \{S_1, S_2, u_1, u_2\}$, ← payoff to prisoner 2

  payoff to prisoner 1

- $S_1 = \{$ Holdout, Confess $\} = S_2$

- $u_1(H, C) = -20$, $u_2(H, C) = 0, \ldots$

When

- there are only two players and

- each player has a finite number of strategies,

then the payoff functions can be represented by a bi-matrix, e.g.

strategies of player 2

strategies of player 1

| | L | R |
|---|---|---|
| T | $u_1(T,L), u_2(T,L)$ | $u_1(T,R), u_2(T,R)$ |
| B | $u_1(B,L), u_2(B,L)$ | $u_1(B,R), u_2(B,R)$ |

player 2

$$
\text{player 1} \quad = \begin{array}{c|c|c|}
 & \overbrace{L \quad\quad R} & \\
\hline
T & 2,1 & 0,2 \\
\hline
B & 1,2 & 3,0 \\
\hline
\end{array}
$$

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**Splitting the bill:**

Two friends are going to dinner and plan to split the bill no matter who orders what.

- There are two meals,

  pay

  – a cheap one priced at $10, which gives each of them $12 of pleasure, and

  benefit

  pay

  – an expensive dinner priced at $20, which gives them each $18 of pleasure.

  benefit

**Question.** What will they order?

normal form

Representation of the game:

strategy spaces

payoff to player 1

- $G = \{\underbrace{S_1, S_2}, u_1, u_2\}$, payoff to player 2

payoff functions

- $S_1 = \{$ Cheap, Expensive $\} = S_2$, The payoff functions $u_1$ and $u_2$ are given in the bimatrix:

pleasure \$12 → cheap \$10
total spent \$20
each pay \$10
pleasure \$12

net payoff

Total cost \$10+\$20 = \$30
each pay \$15

| | | (col) Peter | |
|---|---|---|---|
| | | Cheap | Expensive |
| (row) John | Cheap | 2 2 | −3 3 |
| | Expensive | 3 −3 | −2 −2 |

Total 20+20=40
each pay 20
pleasure 18

John pleasure \$18

peter pleasure \$12

John pleasure \$12
peter pleasure \$18

When John orders cheap and Peter orders expensive,

always a better choice
=dominating strategy

- both have to pay \$15, John gets \$12 of pleasure while Peter gets \$18.
- John's net pleasure is $12 - 15 = -3$.
- Peter's net pleasure is $18 - 15 = 3$.

Notations:

$(S_2, S_{-2}) = (S_1, S_2) = (C, H)$

combination of strategies

player 1, player i-1, player n

$$s = (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) = (s_i, s_{-i})$$

prisoner 1

player i

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

$(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots s_n)$

$$S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$$

strategies space of other players

**Definition 2.** *In a normal-form game* $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, *let* $s_i', s_i'' \in S_i$. *Strategy* $s_i'$

strategy spaces payoff functions

player i dominating strategy $s_i''$ $s_i'$ (always better)

strategy $s_i''$ is strictly dominating strategy $s_i'$
dominant

*is* **strictly dominated** *by strategy $s_i''$ (or strategy $s_i''$ strictly dominates strategy $s_i'$), if*

for all strategies of other player

using $s_i''$ always has a higher payoff to player i comparing to $s_i'$

$$u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i}), \quad \forall s_{-i} \in S_{-i}.$$

*i.e. for each feasible combination of the other players' strategies, player i's payoff from playing $s_i'$ is strictly less than the payoff from playing $s_i''$.*

$s_i'$

- Rational players do not play strictly dominated strategies since they are always not optimal no matter what strategies others would choose.

- In the Prisoner's Dilemma Game, Holdout is a strictly dominated strategy while Confess is a strictly dominant strategy.

- In the game of splitting the bill, Expensive is a strictly dominant strategy for both players.

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**Iterated elimination of strictly dominated strategies:** (IESDS)

Left v.s. Middle

Left v.s. Right

Middle v.s. Right

|  |  | Player2 |  |  |
|---|---|---|---|---|
|  |  | Left | Middle | Right |
| Player1 | Up | 1, 0 | 1, 2 | ~~0, 1~~ |
|  | Down | 0, 3 | 0, 1 | ~~2, 0~~ |

- For player 1, neither Up or Down is strictly dominated.
- For player 2, $\boxed{\text{Right}}$ is strictly dominated by $\boxed{\text{Middle.}}$

< Middle

Both players know that Right will not be played by player 2. Thus they can play the game as if it were the following game:

| | | Player2 Left | Player2 Middle |
|---|---|---|---|
| Player1 | Up | 1, 0 | 1, 2 |
| | Down | ~~0, 3~~ | ~~0, 1~~ |

Now, Down is strictly dominated by Up for player 1. Down can be eliminated from player 1's strategy space, resulting in

| | | Player2 Left | Player2 Middle |
|---|---|---|---|
| Player1 | Up | ~~1, 0~~ | 1, 2 |

< σ.s.

Left is strictly dominated by Middle for player 2. Eliminate Left from player 2's strategy space leaving (Up, Middle) as the $\boxed{\text{outcome}}$ of the game. □

action of player 2

action of player 1

Q. Iterated elimination of strictly deminated strategies may not always work.

- In the following game, no strategy is strictly dominated for any player.

IFSDS

"dominating" B

A > B for all cases

L v.s. C

C v.s. R

L v.s R

T v.s. M

T v.s. B

M v.s. B

| | | | Player2 | |
|---|---|---|---|---|
| | | L | C | R |
| | T | $0,4$ | $4,0$ | $5,3$ |
| Player1 | M | $4,0$ | $0,4$ | $5,3$ |
| | B | $3,5$ | $3,5$ | $6,6$ |

Ans best strategy in some cases some strategies of other players

**Definition 3.** *In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, the* **best response** *for player i to a combination of other players' strategies $s_{-i} \in S_{-i}$ is*

payoff function

strategy spaces

fix choice of other players

set of strategies of player i when others play $S_i$

$$R_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

$\subseteq S_i$

payoff to player i

*i.e. $R_i(s_{-i})$ is the set of* **best responses** *by player i to the other player's strategies $s_{-i}$.* when other play $S_{-i}$

$R_i(S_{-i}) = \{ S_i \in S_i \mid S_i$ is a maximizer of $U_i(S_i, S)\}$

Payoff to Player i

**Remarks:** $R_i(s_{-i}) \subset S_i$ can be an empty set, a singleton, or a finite or infinite set.

**Example:**

Prisoner's Dilemma

| | | Prisoner2 | |
|---|---|---|---|
| | | Holdout | Confess |
| Prisoner1 | Holdout | $-1, -1$ | $-20, 0$ |
| | Confess | $0, -20$ | $-5, -5$ |

best response of player 1

$R_1$(Holdout by 2) $=$ {Confess}, $R_1$(Confess by 2) $=$ Confess.

strategy of other players'

**Definition 4.** *In the n-player normal-form game* $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, *the strategies* $(s_1^*, \ldots, s_n^*)$ *are a* **Nash equilibrium** *if*

$s_1^* \in R_1(s_{-1}^*)$

each player best response

when other players play $s_{-i}^*$

$$s_i^* \in R_i(s_{-i}^*) \quad \forall\, i = 1, \ldots, n,$$

for all players

best response of player $i$

*equivalently,*

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \quad \forall\, i = 1, \ldots, n.$$

pay off of player $i$ when others $s_{-i}^*$

- In other words, no player has incentive to deviate from a Nash equilibrium.

## Find a Nash equilibrium

Let $G(R_i)$ denote the **graph** of $R_i$, defined by

player $i$ best response, other players can play with any strategy

$$G(R_i) = \{(s_i, s_{-i}) \mid s_i \in R_i(s_{-i}), s_{-i} \in S_{-i}\}$$

combinations of strategies

Then

$$(s_1^*, \ldots, s_n^*) \in \cap_{i=1}^n G(R_i)$$

player $1, 2, \cdots, n$

iff

best response of player $i$

player $i$

$$s_i^* \in R_i(s_{-i}^*),\ s_{-i}^* \in S_{-i}, \quad \underline{i = 1, \ldots, n},$$

others

iff $(s_1^*, \ldots, s_n^*)$ is a Nash equilibrium.

**Two ways for finding Nash equilibria.**

best respon of 1

trial and error

(i) For any guess $(s_1^*, s_2^*) \in S_1 \times S_2$, compute $s_1^* \in$ $R_1(s_2^*)$ and $s_2^* \in$ $R_2(s_1^*)$. Then, $(s_1^*, s_2^*)$ is a Nash equilibrium if

best response of 2

$$s_1^* \in R_1(s_2^*) \text{ and } s_2^* \in R_2(s_1^*).$$

combination of strategies

player 1 best response

(ii) Compute $\underline{R_1(s_2)}$ for all $s_2 \in S_2$ and $R_2(s_1)$ for all $s_1 \in S_1$. Then, any $(s_1^*, s_2^*) \in \boxed{\underline{G(R_1)}} \cap G(R_2)$ is a Nash equilibrium.

Combination of strategies player2 best response

**Finding Nash equilibria for a bimatrix game:**

(1) Underline the payoff to player $j$'s best response to each of player $i$'s strategies.

(2) If both $u_1(s_1^*, s_2^*)$ and $u_2(s_1^*, s_2^*)$ are underlined, then $(s_1^*, s_2^*)$ is an intersection of $\underline{G(R_1)}$ and $\underline{G(R_2)}$, thus is a Nash equilibrium.

Example:

| | | Player2 | | |
|---|---|---|---|---|
| | | L | C | R |
| Player1 | T | 0, 4 | 4, 0 | 5, 3 |
| | M | 4, 0 | 0, 4 | 5, 3 |
| | B | 3, 5 | 3, 5 | 6, 6 |

- When player 2 plays L, the best strategy by player 1 is M. Underline the 4.
- When player 2 plays C, the best strategy by player 1 is T. Underline the 4.
- When player 2 plays C, the best strategy by player 1 is B. Underline the 6.
- When player 1 plays T, the best strategy by player 2 is L. Underline the 4.
- When player 1 plays M, the best strategy by player 2 is C. Underline the 4.
- When player 1 plays B, the best strategy by player 2 is B. Underline the 6.

The unique Nash equilibrium is (B, R).

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## The Battle of the Sexes

- Mary and Peter are deciding on an evening's entermainment, attending either the opera or a prize fight.

- Both of them would rather spend the evening together than apart,
  - Peter would rather they be together at the prize fight while
  - Mary would rather they be together at the opera.

| | | (col) Peter | |
|---|---|---|---|
| | | Opera | Fight |
| (row) Mary | Opera | 2, 1 | 0, 0 |
| | Fight | 0, 0 | 1, 2 |

Both (Opera, Opera) and (Fight, Fight) are Nash equilibria. Thus a game can have multiple Nash equilibria.

## A game with 3 players

NE: fix other player(s) strategy and find best response for one player

player 2&3: $A_2A_3$, $A_2B_3$, $B_2A_3$, $B_2B_3$

$S_1 = \{A_1, B_1\}$
$S_2 = \{A_2, B_2\}$
$S_3 = \{A_3, B_3\}$

Player 3

Player 1 & Player 2 (combinations of strategies of player 1&2)

payoff to player 1, player 2, player 3

| | $A_3$ | $B_3$ |
|---|---|---|
| $A_1A_2$ | $-1, 0, 2$ | $3, 5, 1$ |
| $A_1B_2$ | $5, -2, 3$ | $-2, 1, -5$ |
| $B_1A_2$ | $0, 2, 1$ | $2, 2, 5$ |
| $B_1B_2$ | $2, -3, -3$ | $-1, 4, 0$ |

player 1&3: $A_1A_3$, $A_1B_3$, $B_1A_3$, $B_1B_3$

This game can also be represented as

| player1 | player2 | $A_3$ | $B_3$ |
|---|---|---|---|
| $A_1$ | $A_2$ | $-1, 0, 2$ | $3, 5, 1$ |
| | $B_2$ | $5, -2, 3$ | $-2, 1, -5$ |
| $B_1$ | $A_2$ | $0, 2, 1$ | $2, 2, 5$ |
| | $B_2$ | $2, -3, -3$ | $-1, 4, 0$ |

The Nash equilibrium is $(B_1, B_2, B_3)$.

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**The relation between Nash equilibrium and iterated elimination of strictly dominated strategies:**

**Example:**

IESDS

| | | Player2 | | |
|---|---|---|---|---|
| | | Left | Middle > Right | |
| Player1 | Up | 1, 0 | 1, 2 | 0, 1 |
| | Down | 2, 3 | 0, 1 | 2, 0 |

Right is strictly dominated by Middle, thus the game can be reduced to

NE is kept same after IESDS is applied.

| | | Player2 | |
|---|---|---|---|
| | | Left | Middle |
| Player1 | Up | 1, 0 | 1, 2 |
| | Down | 2, 3 | 0, 1 |

There is no strictly dominated strategy, hence IESDS has 4 outcomes (HW, check!)

up left Down Middle

$$\{(U, L), (U, M), (D, L), (D, M)\}.$$

However, there are only 2 Nash equilibria

$$\{(U, M), (D, L)\}.$$

NE $\subseteq$ outcome of IESDS $\in$ NE

**Proposition 1.** *If the strategies* $(s_1^*, \ldots, s_n^*)$ *are a Nash equilibrium in an n-player normal-form game* $G = \{S_1, \ldots, S_n; u_1 \ldots, u_n\}$, *then each* $s_i^*$ *cannot be eliminated in iterated elimination of strictly dominated strategies.*

prove by contradiction! suppose ~~some~~ first when $s_i^*$ eliminated why?

$s_{-i}^*$ not eliminated

*Proof.* Suppose $s_i^*$ is the first of the strategies $(s_1^*, \ldots, s_n^*)$ to be eliminated or being strictly dominated.

Then there exists $s_i''$ not yet eliminated from $S_i$, strictly dominates $s_i^*$, i.e.,

You have $s_i''$ gives a better payoff to player $i$

$\forall s_{-i} \in S_{-i}$ and not yet eliminated,

in particular consider $s_{-i}^*$

$$\begin{aligned} & u_i(s_1, \ldots, s_{i-1}, s_i^*, s_{i+1}, \ldots, s_n) \\ < \; & u_i(s_1, \ldots, s_{i-1}, s_i'', s_{i+1}, \ldots, s_n) \end{aligned}$$

$s_{-i}$

$\Rightarrow s^*$ not best response of $s_{-i}^*$

for all strategies $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ that have not been eliminated from the other player's strategy spaces. Since $s_i^*$ is the first equilibrium strategy to be eliminated, we have

$$\begin{aligned} & u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \\ < \; & u_i(s_1^*, \ldots, s_{i-1}^*, s_i'', s_{i+1}^*, \ldots, s_n^*) \end{aligned}$$

which contradicts to the fact that $s_i^*$ is a best response to $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$. $= s_{-i}^*$ □

The proposition tells us that

- any Nash equilibrium can survive the iterated elimination of strictly dominated strategies (IESDS)
- hence must be an outcome of IESDS, i.e.,

$$\{\text{Nash equilibria}\} \subseteq \{\text{Outcomes of IESDS}\}.$$

when NE $\overset{\subset}{=}$ outcome of IESDS ?

Ans: when unique IESDS outcome.

**Proposition 2.** *In the $n$-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$ where $S_1, \ldots, S_n$ are finite sets,*

unique IESDS

- *if iterated elimination of strictly dominated strategies eliminates all but the strategies $\boxed{(s_1^*, \ldots, s_n^*)}$ then these strategies are the unique Nash equilibrium of the game.*

NE

NE $\subseteq$ IESDS $\leftarrow$ unique

*Proof.* By $\boxed{\text{Proposition 1}}$, equilibrium strategies can never be eliminated in IESDS. Since $(s_1^*, \ldots, s_n^*)$ are the only strategies which are not eliminated, $s_i^*$ is thus the only possible equilibrium strategy for player $i$. Hence, we cannot find two different Nash equilibria.

To prove $(s_1^*, \ldots, s_n^*)$ are indeed a Nash equilibrium, we use proof by contradiction. Suppose $s_i^*$ is not a best response for player $i$ to $(S_1^*, S_i^*, S_n^*)$ not NE

some is not best respons $\notin$ IESDS outcome

$S_{-i}^* = (s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$. Let the relevant best response be $b_i \neq s_i^*$, i.e., for any $s_i \in S_i$,

best response

$$
\begin{aligned}
& u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \\
(1) \qquad & \leq u_i(s_1^*, \ldots, s_{i-1}^*, b_i, s_{i+1}^*, \ldots, s_n^*)
\end{aligned}
$$

best response $b_i$ better than $S_i^*$

Because the only strategy in $S_i$ which survives eliminations is $s_i^*$, thus $b_i$ must be strictly dominated by some

$b_i$ eliminated

$(S_1^*, S_2^*, \cdots, S_n^*) \in$ IESDS outcome

$S_i^*$

$t_i$ better than $b_i$

$b_i$ is not the best response

strategy $t_i$ at some stage of the process of iterated elimination. So we have

$\forall s_{-i} \in S_{-i}$ not eliminated yet,

$$\begin{aligned} & u_i(s_1, \ldots, s_{i-1}, b_i, s_{i+1} \ldots, s_n) \\ < \; & u_i(s_1, \ldots, s_{i-1}, t_i, s_{i+1} \ldots, s_n) \end{aligned}$$

for all strategies $(s_1, \ldots, s_{i-1}, s_{i+1} \ldots, s_n)$ that have not been eliminated from the other player's strategy spaces. Since $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$ are not eliminated,

$$\begin{aligned} & u_i(s_1^*, \ldots, s_{i-1}^*, b_i, s_{i+1}^* \ldots, s_n^*) \\ < \; & u_i(s_1^*, \ldots, s_{i-1}^*, t_i, s_{i+1}^* \ldots, s_n^*) \end{aligned}$$

which contradicts to (1), the best response property of $b_i$. $\square$

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**Infinite strategy spaces $S_i$.**

- We looked at strategy spaces $(S_1, ..., S_n)$ that are finite so that a table can be drawn.

- We now look at when $S_i$ need not be finite.

**Splitting a dollar**

Players 1 and 2 are bargaining over how to split one dollar. Both players simultaneously name shares they would like to have, $s_1$ and $s_2$, where $0 \leq s_1, s_2 \leq 1$.

each player name a number $\in [0,1]$
$\frac{1}{2}$ $\frac{1}{3}$

- If $s_1 + s_2 \le 1$, then the players receive the shares they named;
- if $s_1 + s_2 > 1$, then both players receive zero.

no prob

**Question.** What are the $\boxed{\text{pure-strategy}}$ Nash equilibria of this game?

$s_1$ v.s. mixed-strategy ← prob

Recall find NE

step 1 best response

step 2

$NE = \cap$ best respon

best response of player

- Given any $\boxed{s_2} \in [0, 1)$, the best response for player 1 is $R_1(s_2) = \{1 - s_2\}$. $0 \le s_2 < 1$ $R_1(s_2) = \{1 - s_2\}$
- To $s_2 = 1$, the player 1's best response is the set $[0, 1]$ because player 1's payoff is 0 no matter what she chooses. infinite

$s_1 \in [0,1]$ $s_1 = 0, \frac{1}{2}$ or $1$
payoff $= 0$

The diagram of

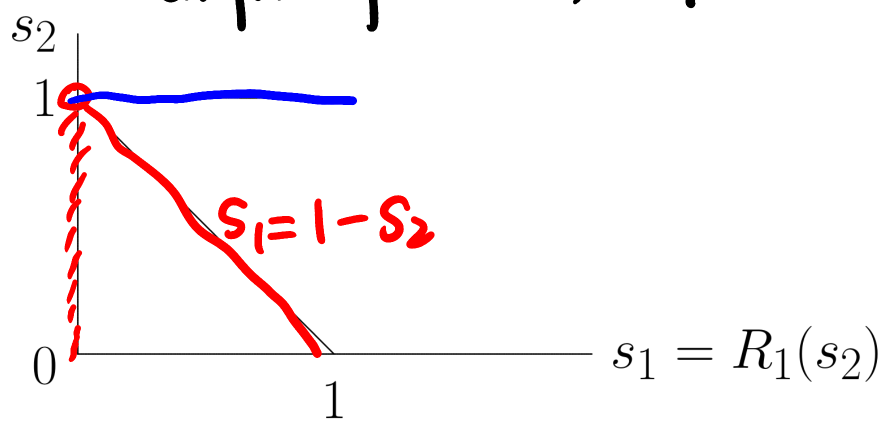
player 2 $s_2$

$$s_1 = R_1(s_2) = \begin{cases} 1 - s_2 & \text{if } 0 \le s_2 < 1 \\ [0, 1] & \text{if } s_2 = 1 \end{cases}$$

best response of 1

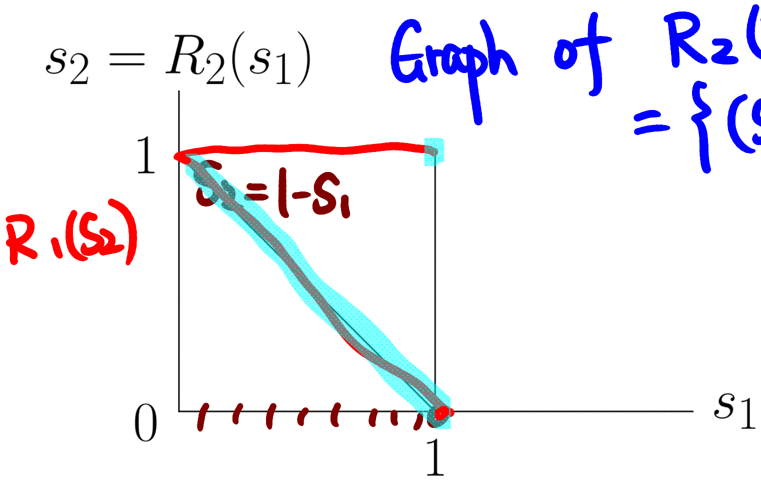
is

Graph of $R_1(s_2) = \{(s_1, s_2) :$ $s_1$ best response of $s_2\}$



Similarly, we have the best response for player 2:

$$s_2 = R_2(s_1) = \begin{cases} 1 - s_1 & \text{if } 0 \le s_1 < 1 \\ [0, 1] & \text{if } s_1 = 1 \end{cases}$$




Graph of $R_2(S_1)$ $= \{(S_1, S_2) : S_2$ best response of $S_1\}$

Intersections of $s_1 = R_1(s_2)$ and $s_2 = R_2(s_1)$ are

NE $= \{(s_1, s_2) : s_1 + s_2 = 1,\ s_1, s_2 \ge 0\}$ and $(1, 1)$.

They are the pure-strategy Nash equilibria.

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**Cournot Model of Duopoly** 2 firms (same product) by choosing quantity $q$ max net profit

If a firm produces $q$ units of a product

- at a cost of $c$ (known) per unit and
- can sell it at a price of $p$ per unit, (decided by total $Q$ quantity in market)

then the firm makes a net profit

$$\pi = pq - \overset{\text{cost}}{c}q - c_0, = (p-c)q - c_0$$

max $pq - cq - c_0$ $= \max \boxed{(p-c)q}$ net profit

p prime

where $c_0$ is the fixed cost.

The firm can decide $p$ and $q$ to maximize its net profit.

- ignore $c_0$, because $c_0$ does not affect the decision of $p$ and $q$.

Suppose firms 1 and 2 produce the same product.

$q_1$ $q_2$

- Let $q_i$ be the quantity of the product produced by firm $i$, $i = 1, 2$.
- Let $Q = q_1 + q_2$, the aggregate quantity of the product.
- Since firms produce the same product, the firm which sets the higher price has no market.

they sell product in the price $P_1 = P_2 = P$

- Thus, they sell the product at the same price, the market clearing price $P = a - Q$ if $Q < a$

$$P(Q) = \begin{cases} a - Q, & \text{if } Q < a \\ 0, & \text{if } Q \geq a. \end{cases}$$

$P \leq a$

- Let the cost of producing a unit of the product be $c$.

$c \geq a \Rightarrow c \geq a \geq P$

$\Rightarrow$ no one makes product

  – We assume $\boxed{c < a}$ and is same for both firms.

**Question.** How much shall each firm produce?

Firm $i$ ($i = 1, 2$) shall maximize its net profit

$$P\overbrace{(q_i + q_j)}^{p} \cdot q_i - c \cdot q_i$$

$q_i$

for $q_i \in [0, \infty)$ given $q_j$.

Formulate the problem into a normal-form game:

2 players = 2 firms

(1) The players of the game are the two firms.

possible values of $q_i$

(2) Each firm's strategy space is $S_i = [0, \infty)$, $i = 1, 2$. (Any value of $q_i$ is a strategy.)

$\pi_i = (P - c)\, q_i$

(3) The payoff to firm $i$ as a function of the strategies chosen by it and by the other firm, is simply its profit function:

payoff function.

2 players

one player $i$

the other player $j$

$$
\begin{aligned}
\pi_i(q_i, q_j) &= \boxed{P(q_i + q_j)} \cdot q_i - c \cdot q_i \\
&= \begin{cases} q_i[a - \underbrace{\overbrace{(q_i + q_j)}^{Q}}_{p} - c], & \text{if } \overbrace{q_i + q_j}^{Q} < a \\ -cq_i, & \text{if } q_i + q_j \geq a. \end{cases}
\end{aligned}
$$

$(q_i, q_j)$

$q_i, q_j < a$

Q: Can $q_i > a - q_j$ a best response?

A: No!

Recall:

Find NE

① best response

② $\cap$ best response

- For any $q_i > a - q_j$, the profit $\pi_i = -cq_i < 0$, such $q_i$ cannot be optimum.

$< a$

Q: Can $q_i \geq a$ be a best response?

Ans: No.

- Therefore, we need only consider $q_i \in [0, a - q_j]$.

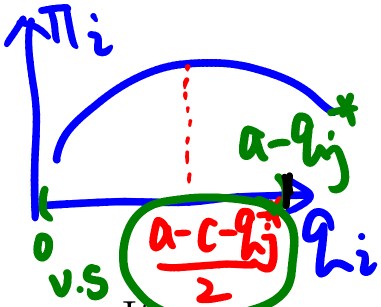
$q_i \geq a \to \pi_i = -cq_i < 0$

$q_i = 0 \to \pi_i = 0$ better response

$0 \le q_1, q_2 < a$ , $q_i \le a - q_j$ for best response

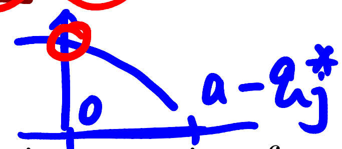
The quantity pair $(q_1^*, q_2^*)$ is a Nash equilibrium if, for each firm $i$, $q_i^*$ solves

$q_i$ max payoff to player $i$ ?




$$\max_{0 \le q_i \le a - q_j^*} q_i[\overset{P}{a - c - q_j^* - q_i}].$$

$a - c - q_j^* < 0$




If $a - c - q_j^* < 0$, then the profit is negative for any $q_i > 0$. Thus $\boxed{q_i = 0}$ is the optimum. $\frac{a-c-q_j^*}{2} < a-c-q_j^* < a-q_j^*$

If $a - c - q_j^* \ge 0$, the local maximum $\bar{q}_i$ can be determined by the optimal condition (the first derivative equals zero) $a - q_j^* - c - 2q_i = 0$, thus

$$\bar{q}_i = \frac{1}{2}(a - q_j^* - c). \in [0, a - q_j^*]$$

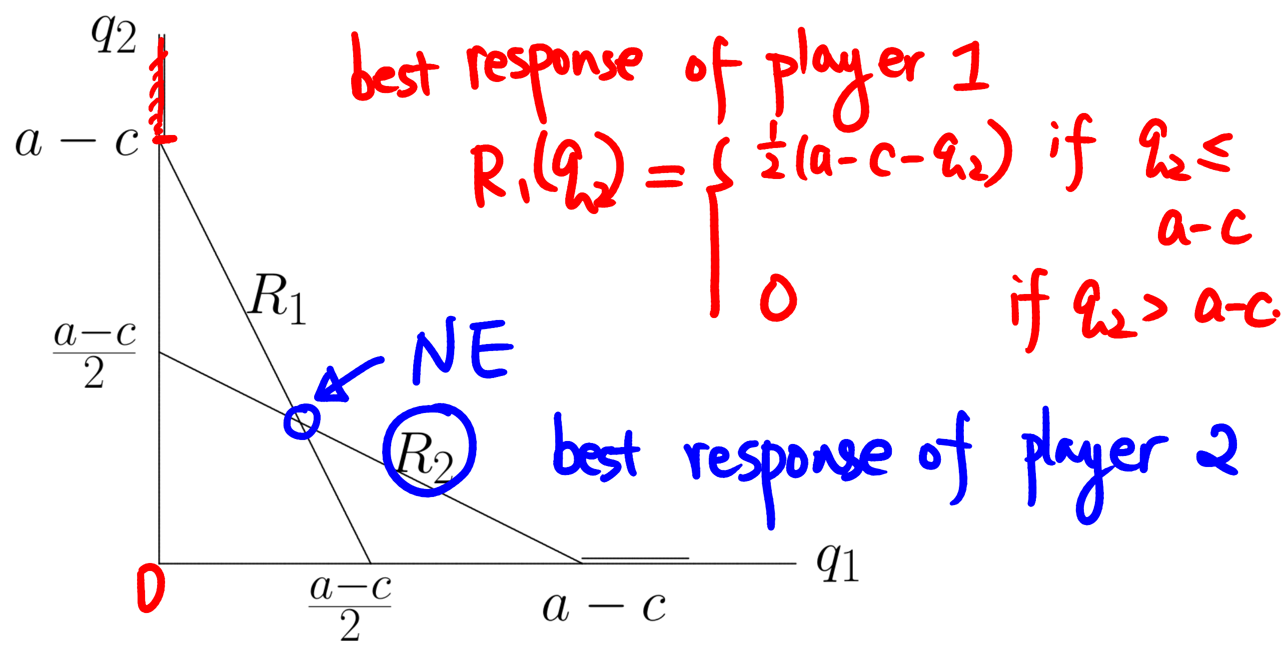
best response.

For a fixed $q_j$, the function $q_i[a - (q_i + q_j) - c]$ is concave for $q_i$, because its second derivative is $-2 < 0$. Therefore, $\bar{q}_i$ is the global maximum.

Thus, the best response of firm $i$ to the given quantity $q_j$ should be

the best response of player $i$

$$q_i = R_i(q_j) = \begin{cases} \frac{1}{2}(a - q_j - c), & \text{if } q_j \le a - c \quad (a - c - q_j \ge 0) \\ 0 & \text{if } q_j > a - c. \quad (a - c - q_j < 0) \end{cases}$$




The Nash equilibrium $(q_1^*, q_2^*)$ are the best responses, hence can be determined by the intersection of the two response curves, i.e.,

$$\begin{cases} q_1 = \frac{1}{2}(a - q_2 - c) & ① \\ q_2 = \frac{1}{2}(a - q_1 - c) & ② \end{cases}$$

$$q_1 = \frac{1}{2}\left(a - \frac{1}{2}(a - q_1 - c) - c\right)$$

$$\Rightarrow q_1 = \frac{1}{3}(a-c)$$

$$\Rightarrow q_2 \overset{②}{=} \frac{1}{2}(a - c - q_1) = \frac{1}{3}(a-c)$$

Solving the equations, we obtain

$$q_1^* = \frac{1}{3}(a-c), \qquad q_2^* = \frac{1}{3}(a-c).$$

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## Bertrand Model of Duopoly

Suppose now the two firms produce **<u>different</u>** products.