MA3110 Homework 1

Question 1

Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(x) = 2 + 3x^2 + 4 \ln x \quad \text{for } x > 0.$$

(i) Prove that f is strictly increasing on $(0, \infty)$.

> From the lecture exercise on Chapter 6, page 14, we concluded that if $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b). Since $f'(x) = 6x + \frac{4}{x} > 0 \;\text{ for } x > 0$, f is strictly increasing on $(0, \infty)$.

(ii) Let $g : \mathbb{R} \to \mathbb{R}$ be the inverse function of f. Find $g'(5)$.

> Note that $f(1) = 5$. Since f is strictly monotone and continuous on $(0, \infty)$, by the Inverse Function Theorem,
>
> $$\begin{aligned} g'(5) &= g'(f(1)) \\ &= \frac{1}{f'(1)} \\ &= \frac{1}{6(1) + \frac{4}{1}} \\ &= 0.1. \end{aligned}$$

Question 2

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^x + x^2 \cos\left(\frac{1}{2x}\right) & \text{if } x \neq 0 \\ 1 & \text{if x} = 0. \end{cases}$$

(1) Find $f'(x)$ for each $x \in \mathbb{R}$.

> For $x \neq 0$,
>
> $$\begin{aligned} f'(x) &= \frac{d}{dx}\left(e^x + x^2 \cos\left(\frac{1}{2x}\right)\right) \\ &= e^x + \frac{1}{2}\sin\frac{1}{2x} + 2x\cos\frac{1}{2x}. \end{aligned}$$
>
> For $x = 0$,

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{e^x + x^2\cos\left(\frac{1}{2x}\right) - 1}{x - 0} \\ &= \lim_{x\to 0}\frac{e^x}{x} + \lim_{x\to 0} x\cos\left(\frac{1}{2x}\right) - \lim_{x\to 0}\frac{1}{x} \\ &= 1 + 0 - 0 \text{ (by L'Hospital's rule and the Squeeze Theorem)} \\ &= 1. \end{aligned}$$

(ii) is $f \in C^1(\mathbb{R})$? Justify your answer.

> We have to check if $f'(x)$ is continuous on $\mathbb{R}$. For $x \neq 0$, $f'(x)$ is clearly continuous.
>
> However, f' is not continuous at $x = 0$. If f' was continuous at $x = 0$, then for any sequence $x_n \to 0$, we must have $f'(x_n) \to f'(0) = 1$. We show that there is a sequence (x_n) which violates sequential criterion, and so f' is not continuous at $x = 0$.
>
> The sequence (x_n) is defined by $x_n = \frac{1}{\left(4n+\frac{1}{2}\right)\pi}$, $n \in \mathbb{N}$. Then $x_n \to 0$, but
>
> $$\begin{aligned} f'(x_n) &= e^{x_n} + \frac{1}{2}\sin\frac{1}{2x_n} + 2x_n\cos\frac{1}{2x_n} \\ &= e^{\frac{1}{\left(4n+\frac{1}{2}\right)\pi}} + \frac{1}{2}\sin\left(2n\pi + \frac{\pi}{4}\right) + \frac{1}{\left(4n+\frac{1}{2}\right)\pi}\cos\left(2n\pi + \frac{\pi}{4}\right) \\ &\to 1 + \frac{\sqrt{2}}{4} + 0 \\ &\neq 1 = f'(0). \end{aligned}$$

Question 3

Use the Mean Value Theorem to prove the Bernoulli's inequality:

$$(1+x)^n > 1 + nx, \ \forall x \in (-1,0) \cup (0,\infty) \text{ and } n = 2,3,4\cdots$$

> Let $f(x) = (1+x)^n$. $f(x)$ is differentiable in the range given above.
>
> First we consider the range $(0,\infty)$. For any x, there exists a point $c \in (0,x)$ such that $f'(c) = \frac{f(x)-f(0)}{x-0}$. This means
>
> $$\begin{aligned} n(1+c)^{n-1} &= \frac{(1+x)^n - 1}{x} \\ \implies nx(1+c)^{n-1} &= (1+x)^n - 1 \\ \implies (1+x)^n &= 1 + nx(1+c)^{n-1} \end{aligned}$$
>
> Since $c > 0$ and $n - 1 > 0$, $(1+c)^{n-1} > 1$ so $(1+x)^n > 1 + nx$.
>
> Next we consider the range $(-1,0)$. For any x, there exists a point $c \in (x,0)$ such that $f'(c) = \frac{f(0)-f(x)}{0-x}$. This means
>
> $$\begin{aligned} n(1+c)^{n-1} &= \frac{1-(1+x)^n}{-x} \\ \implies -nx(1+c)^{n-1} &= 1 - (1+x)^n \\ \implies (1+x)^n &= 1 + nx(1+c)^{n-1} \end{aligned}$$
>
> Since $x < 0$, $0 < 1 + c < 1$ and $n - 1 > 0$, $n(1+c)^{n-1}x > nx$, proving the inequality.

Question 4

Suppose that the function f is continuous on $[a, b]$ and differentiable on (a, b). Prove that if

$$(f(b))^2 - (f(a))^2 = b^2 - a^2,$$

then there exists $c \in (a, b)$, such that

$$f'(c)f(c) = c.$$

> Define $g(x) = (f(x))^2 - x^2$. Then $g'(x) = 2f(x)f'(x) - 2x$, $g(b) = (f(b))^2 - b^2$ and $g(a) = (f(a))^2 - a^2$.
>
> Since $(f(b))^2 - (f(a))^2 = b^2 - a^2 \implies (f(b))^2 - b^2 = (f(a))^2 - a^2$, we see that $g(a) = g(b)$. Hence by Rolle's Theorem there exists $c \in (a, b)$ such that $g'(c) = 0$.
>
> $$\begin{aligned} & g'(c) = 0 \\ \implies & 2f(c)f'(c) - 2c = 0 \\ \implies & c = f'(c)f(c). \end{aligned}$$