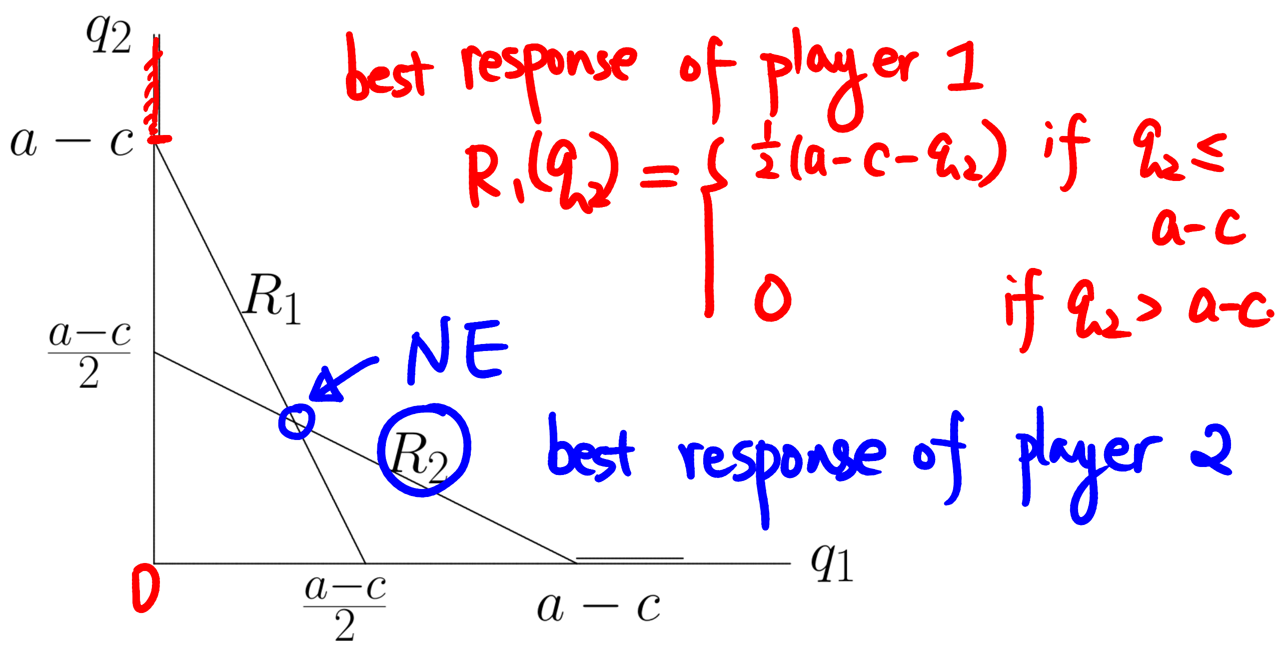


The Nash equilibrium (q_1^*, q_2^*) are the best responses, hence can be determined by the intersection of the two response curves, i.e.,

$$\begin{cases} q_1 = \frac{1}{2}(a - q_2 - c) & ① \\ q_2 = \frac{1}{2}(a - q_1 - c) & ② \end{cases}$$

$$q_1 = \frac{1}{2}\left(a - \frac{1}{2}(a - q_1 - c) - c\right)$$

$$\Rightarrow q_1 = \frac{1}{3}(a-c)$$

$$\Rightarrow q_2 \overset{②}{=} \frac{1}{2}(a-c-q_1) = \frac{1}{3}(a-c)$$

Solving the equations, we obtain

$$q_1^* = \frac{1}{3}(a-c), \qquad q_2^* = \frac{1}{3}(a-c).$$

Bertrand Model of Duopoly

Suppose now the two firms produce **different** products.

Recall

Normal-form representation of

Chap 1 a static Game with player $1, 2, \cdots, n$

Strategy spaces $S_1, S_2, \cdots, S_n$

possible choice/strategy of player 1

Payoff $u_1, u_2, \cdots, u_n$

payoff function of player i u_i : combination of strategies of n players $\longmapsto$ payoff to player i

best response of player i to strategies of other players $(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n)$ fix

R_i is the set of maximizers of $u_i(s_1, \cdots, s_{i-1}, s_i, s_{i+1}, \cdots, s_n)$ variable

payoff function of player i

R_i $\begin{cases} \phi \\ \text{finite} \\ \text{infinite} \end{cases}$

Nash Equilibrium $(s_1^*, \cdots, s_n^*)$

if every player plays with the best response

i.e. $\forall i = 1, \cdots, n$ s_i^* is a best response to $(s_1^* \cdots s_{i-1}^*, s_{i+1}^*, \cdots s_n^*)$

Cournot's model What if making diff product?

2 firms making same product $Q = q_1 + q_2$

payoff to each firm

$$\boxed{\pi = pq - cq} = (p-c)\,q$$

cost per unit

price per unit quantity

p is a function of Q ← total quantity

Aim chose q of each firm to max payoff of that firm

choices/ strategy

strategy space $S_1 = S_2 = [0, +\infty)$

Fix q_2 find q_1 ← best response of firm 1

Fix q_1 find q_2 ← best response of firm 2

→ NE

Bertrand Model use p as strategy/choice

q is taken as a function of p

- In this case, we cannot use the aggregate quantity to determine market prices as in Cournot's model.

- Thus, instead of using quantities as variables, here we use prices as variables.

strategies

strategies

$p_1 \ p_2 \in S_1 = S_2 = [0, +\infty)$

- Assume that the cost and the reservation price for the two firms are the same and they are c and a respectively. Assume $c < a$.

Cournot's model

$P = \begin{cases} a - Q & \text{if } Q \le a \\ 0 & \text{if } Q > a \end{cases} \Rightarrow P \geq 0$

cost per unit

If firms 1 and 2 choose prices p_1 and p_2, respectively, the quantity that consumers demand from firm i is

quantity   variable / strategy $\{i, j\} = \{1, 2\}$

payoff

$\pi_i = p_i q_i - c q_i = (p_i - c) q_i$

$$q_i(p_i, p_j) = a - p_i + bp_j$$

p_j

where $b > 0$ reflects the extent to which firm i's product is a "substitute" for firm j's product.

Question. How to find the Nash equilibrium?

① Fix strategy of player j, find best response of player i

② $\cap$ best response $= NE$

The strategy space of firm i consists of all possible prices, thus $S_i = [0, \infty)$, $i = 1, 2$.

The profit of firm i is

payoff   parameters

$$\pi_i(p_i, p_j) = q_i(p_i, p_j) \cdot p_i - c \cdot q_i(p_i, p_j) = q_i(p_i - c)$$
$$\overset{(*)}{=} \underbrace{[a - p_i + bp_j]}_{= q_i}(p_i - c).$$

NE is (p_1^*, p_2^*) then p_i^* best response of p_j^* fix strategy of firm j

Thus, the price pair (p_1^*, p_2^*) solves

maximizers of $\pi_i(p_i, p_j^*)$ variable

$$\max_{0\le p_i<\infty} \boxed{\pi_i(p_i,p_j^*)} = \max_{0\le p_i<\infty} (a - p_i + bp_j^*)(p_i - c), \quad (*)$$

Quad in p_i

whose solution is

maximizer of $\pi_i(p_i, p_j^*)$ is the sol'n of $\frac{\partial \pi_i}{\partial p_i} = 0 \Rightarrow \frac{\partial}{\partial p_i}(a - p_i + bp_j^*)(p_i - c)$

$$p_i^* = \frac{1}{2}(a + bp_j^* + c). \quad (☆)$$

$= -(p_i - c) + (a - p_i + bp_j^*) = 0 \Rightarrow (☆)$

Hence

$$\begin{array}{l} i=1, j=2 \Rightarrow \\ i=2, j=1 \Rightarrow \end{array} \begin{cases} p_1^* = \frac{1}{2}(a + bp_2^* + c) \\ p_2^* = \frac{1}{2}(a + bp_1^* + c). \end{cases}$$

$p_1^* = \frac{1}{2}\left[a + b \cdot \frac{1}{2}(a + bp_1^* + c) + c\right]$

Solving the equations, we obtain

$p_1, p_2 \in [0, +\infty)$

$q_i^* = a - p_i^* + bp_j^*$

$q_1^* = q_2^* = \frac{a - c + bc}{2-b}$

$$p_1^* = \frac{\underset{>0}{a} + \underset{>0}{c}}{2-b}, \qquad p_2^* = \frac{a+c}{2-b}.$$

The problem makes sense only if $b < 2$! □

Q: What if $b > 2$? A: No Nash Equilibrium.

Bertrand Model with Homogeneous Products

e.g. rice & noodle (replace)

In the previous example, we analyzed the Bertrand duopoly model with **differentiated** products.

e.g rice & juice

- The case of **homogeneous products** yields a stark conclusion.

Bertrand Model strategy: price $P_1, P_2 \in [0, +\infty)$
quantity is a function of price.

Homogeneous (replace)

- Suppose that the quantity that consumers demand from firm i is

demand goes to i only

demand $\Rightarrow$
$$q_i(p_i, p_j) = \begin{cases} \boxed{a - p_i} & \text{if } p_i < p_j \\ 0 & \text{if } p_i > p_j \\ (a - p_i)/2 & \text{if } p_i = p_j \end{cases}$$

product i cheaper

product i expensive

"buy low"

  – i.e., all customers buy the product from the firm who offers a lower price.

$\pi = pq - cq - C_0$

- Suppose also that there are no fixed costs and that marginal costs are constant at c, where $c < a$.

$\uparrow$ cost per unit

$\pi_i = q_i(P_i - c)$

Fix other firm's j strategy P_j

Find the Nash equilibrium (p_1^*, p_2^*).

Given firm j's price p_j, firm i's payoff function is

find the best response (P_i) of firm i to

max π_i

$$\pi_i(p_i, p_j) = \begin{cases} \cancel{(a - p_i)(p_i - c)} & \cancel{\text{if } p_i < p_j} \\ \frac{1}{2}(a - p_i)(p_i - c) = 0 & \text{if } p_i = p_j = c \\ 0 & \text{if } p_i > p_j \end{cases}$$

$\uparrow$ variable

lower price

higher price

π_i, P_i

$P_j = c \because P_i \in [c, a] \therefore P_i \geq P_j$

$\therefore \pi_i = 0$ always

Since the payoff will be negative if $p_i < c$ or $> a$, we can assume the strategy space $S_i = [c, a]$.

$[0, +\infty)$ get same NE

We find three cases from the observation of the payoff curves

Claim If P_1 or $P_2 \in [0,+\infty) \setminus [c,a]$, then (P_1, P_2) is not a NE.

$\Rightarrow$ (If (P_1, P_2) is NE, then $P_1, P_2 \in [c,a]$)

proof $P_1 < c$ $\quad P_1 > a \quad P_1 \in [c,a]$

$P_2 < c \quad P_2 > a \quad P_2 \in [c,a]$

① $P_1, P_2 < c$ ② $P_1 < c$, $P_2 > a$
③ $P_1 < c$ $\quad P_2 \in [c,a]$ ④ $P_1 > a$, $P_2 < c$
⑤ $P_1, P_2 > a$ ⑥ $P_1 > a \quad P_2 \in [c,a]$
⑦ $P_1 \in [c,a] \quad P_2 < c$ ⑧ $P_1 \in [c,a]$, $P_2 > a$

① $P_1, P_2 < c$

Then

not best response to P_2

not NE

$P_1 < P_2 < c < a$

For player 1 (firm 1), $q_1 = a - P_1$

$\pi_1 = \underbrace{(a-P_1)}_{+}\underbrace{(P_1 - c)}_{-} < 0$

Consider $P_1' = c \Rightarrow$ if take (P_1', P_2) then $\pi_1' = 0$ giving player 1 (firm 1) better payoff

not best response to P_2

$P_1 = P_2 < c$

For player 1, $\pi_1 = \frac{1}{2}(a-P_1)(P_1-c) < 0$

not best response to P_1

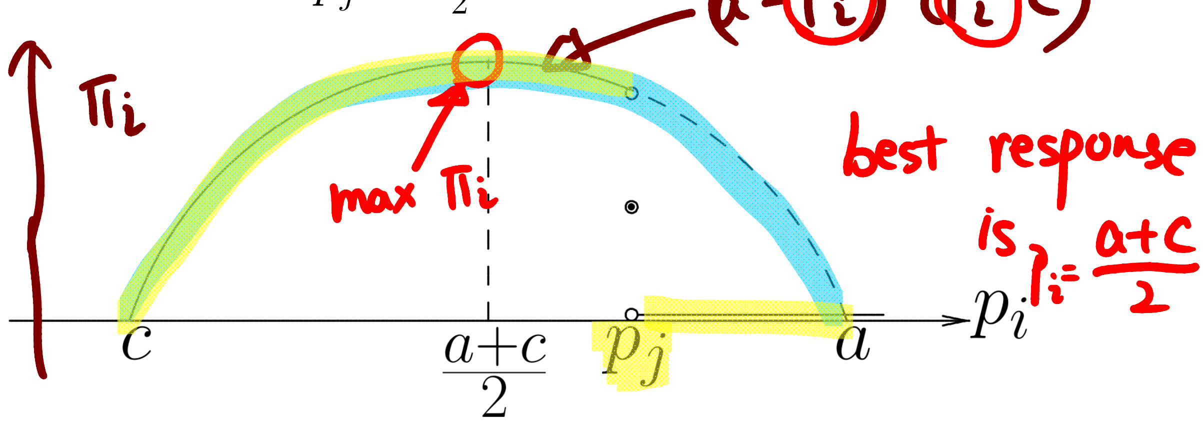
$P_2 < P_1 < c < a$

For player 2 (firm 2), $q_2 = a - P_2$

$\pi_2 = \underbrace{(a-P_2)}_{+}\underbrace{(P_2 - c)}_{-} < 0$ consider $P_2' = c$

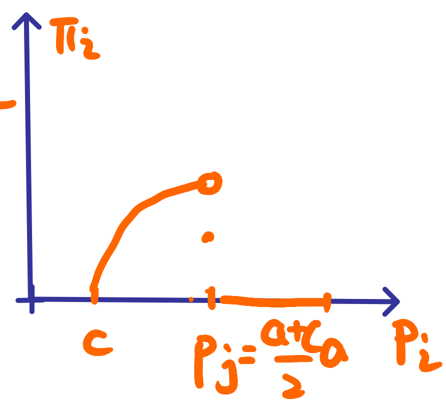
using (P_1, P_2') $\pi_2' = 0$ firm 2 better payoff

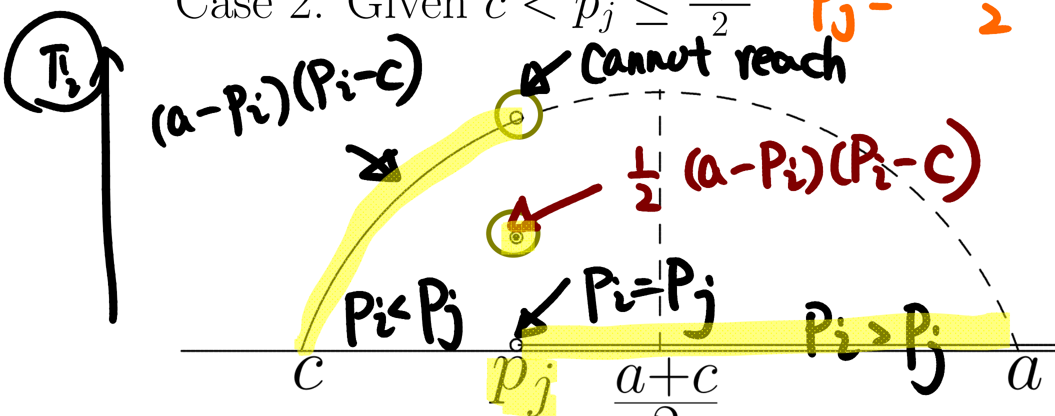
Case 1: Given $p_j > \frac{a+c}{2}$



The maximum payoff is reached at $p_i = \frac{a+c}{2}$. Thus, the best response $R_i(p_j) = \frac{a+c}{2}$.

Case 2: Given $c < p_j \leq \frac{a+c}{2}$





Ri in Case 2 is ∅

In this case, $\max_{p_i} \pi_i(p_i, p_j)$ **has no solution**.

proof: It is easy to see from the second figure that

$$\sup_{p_i} \pi_i(p_i, p_j) = (a - p_j)(p_j - c).$$

Cannot reach

However, no $p_i \in [c, a]$ can make $\pi_i(p_i, p_j) = (a - p_j)(p_j - c)$. (For $p_i \in (c, p_j)$, the function $\pi_i(p_i, p_j) = (a - p_i)(p_i - c)$ is strictly increasing. For $p_i > p_j$, best response $\begin{cases} \phi \checkmark \\ \text{finite} \checkmark \\ \text{infinite} \end{cases}$

$\pi_i(p_i, p_j) = 0$. For $p_i = p_j$, $\pi_i(p_i, p_j) = \frac{1}{2}(a - p_i)(p_i - c)$.) Thus, there is no maximizer. $\square$

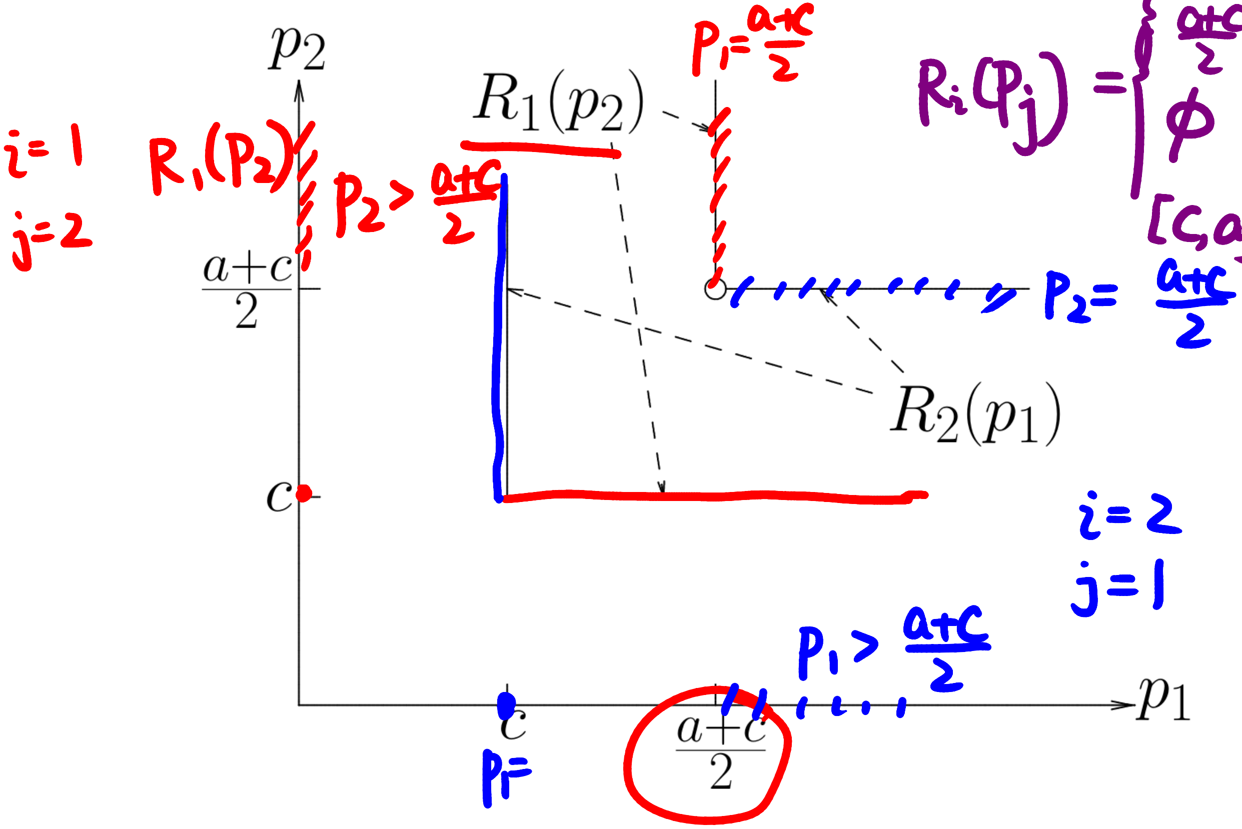
This means that $R_i(p_j) \underline{= \emptyset.}$

Case 3: Given $p_j = c$

$\pi_i(p_i, c) = 0$ for any $p_i \in [c, a]$. Thus any p_i is a maximizer, and $R_i(c) = [c, a]$. ← infinite

The best-response correspondences are sketched below

$$R_i(p_j) = \begin{cases} \{\frac{a+c}{2}\} & \text{if } p_j > \frac{a+c}{2} \\ \phi & \text{if } c < p_j \le \frac{a+c}{2} \\ [c,a] & \text{if } p_j = c \end{cases}$$



- The only intersection of the two correspondences is $\underline{(c, c)}$. NE $\quad P_1 = c = P_2$

This shows that if the firms choose prices simultaneously, then

$$\pi_1 = \pi_2 = \frac{1}{2}(a - P_1)\underbrace{(P_1 - c)}_{=0}$$

- the unique Nash equilibrium is that both firms charge the price $p_i = c$ and
- the net profit $\pi_i = 0$.

↑ 2 players

↓
The problem of the commons

$n \geq 2$ players

If citizens respond only to private incentives, public goods will be underprovided and public resources overutilized.

- This is also our first game with more than 2 players.

Example:

n farmers in a village, $(n \geq 2)$.

- For $i = 1, \ldots, n$, g_i is the number of private goats owned by farmer i.

Green — common

- Let $G = g_1 + \cdots + g_n$.

c = cost per unit

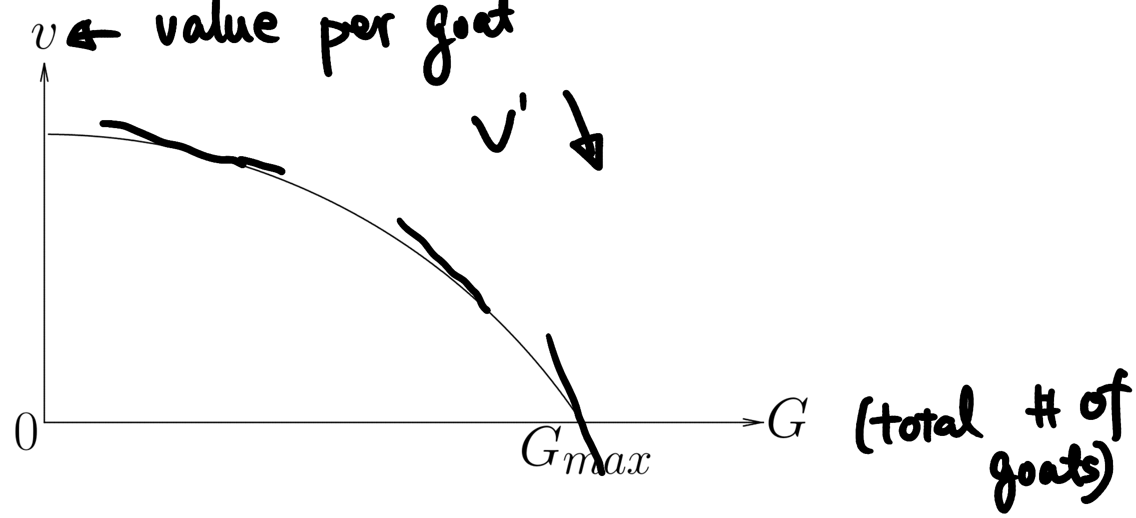
- The cost of buying and caring for a goat is c.
- The value to a farmer of grazing a goat on the green when a total number of G goats are grazing

is $v(G)$ per goat.

$$v(G) = \begin{cases} 0, & \text{if } G \geq G_{max} \\ > 0, & \text{if } G < G_{max}. \end{cases}$$

v ↓

When $G < G_{max}$, $\boxed{v'(G) < 0}$, $\cancel{v''(G) < 0}$.



- The farmers simultaneously choose how many goats to own.

Claim: If $S_i \in [0,+\infty)$ then same NE

- Assume goats are continuously divisible; let $S_i = [0, G_{max})$. If $g_i > G_{max}$ then $\pi_i = (\underset{0}{\underset{\uparrow}{v}} - c)\, g_i$ $= -c g_i < 0$

- The payoff to farmer i is

$$g_i \cdot v(g_1 + \cdots + g_n) - c \cdot g_i.$$

- Thus, if $(g_1^*, \ldots, g_n^*)$ is to be a Nash equilibrium, then, for each i, g_i^* must maximize → best response to other player's strategies $(g_1^*, \dots, g_{i-1}^*, g_{i+1}^*, \dots, g_n^*)$

farmer i strategy

Fix other players' strategy

variable $\underbrace{g_i \cdot v(g_i + \boxed{g_{-i}^*}) - c\, g_i}_{\text{payoff to farmer } i}$

max/min is at turning point

where $\boxed{g^*_{-i} = g^*_1 + \cdots + g^*_{i-1} + g^*_{i+1} + \cdots + g^*_n}$.

diff w.r.t. g_i

①

order important!

② sub $g_i = g_i^*$ ($\because g_i^*$ turning point)

- Differentiate the function above with respect to g_i resulting in

$$(2) \qquad v(g_i^* + g^*_{-i}) + g_i^* \cdot v'(g_i^* + g^*_{-i}) - c = 0$$

n eqns

n unknowns

for $i = 1, 2, \ldots, n$.

- Solve this system of equations, we obtain Nash equilibrium $(g_1^*, \ldots, g_n^*)$.

For example, take

$$v(x) = a - x^2, \qquad a > c.$$

$v'(x) = -2x < 0$

$v''(x) = -2 < 0$

Claim. Nash equilibrium is

$$g_i^* = \sqrt{\frac{a-c}{n^2+2n}}, \quad i = 1, \ldots, n.$$

Proof. $v'(x) = -2x$. The equations (2) are

$(g_1, \cdots, g_n)$ for $(g_1^*, \cdots, g_n^*)$

$$(2) \Rightarrow \underbrace{a - (g_1 + \cdots + g_n)^2}_{v(g_1, \cdots, g_n)} - 2g_i\underbrace{(g_1 + \cdots + g_n)}_{v'} - c = 0 \Rightarrow$$

$$\boxed{g_i = \frac{a - c - (g_1 + \cdots + g_n)^2}{2(g_1 + \cdots + g_n)}}$$

for $i = 1, \ldots, n$. Obviously $g_1 = \cdots = g_n$. Thus

$g_1 =$

$g_2 =$

$\vdots$

$g_n =$

$$g_1 + \cdots + g_n = ng_1 \Rightarrow a - n^2 g_1^2 - 2n g_1^2 - c = 0.$$

$$\Rightarrow g_1^2 = \frac{a-c}{n^2+2n} = g_2^2 = \cdots = g_n^2$$

We obtain Nash equilibrium

n- players (each player max own payoff)
selfish

$$g_i^* = \sqrt{\frac{a-c}{n^2+2n}}, \quad i = 1, \ldots, n.$$

A comparison between the game solution and the social optimum solution

take village as one group making one decision of total # of goats

If the whole village decides the total number of goats to graze, then the social optimum G^{**} should solve

total payoff to village

$$\max_{0 \le G < G_{max}} \; Gv(G) - Gc,$$

total # of goats

diff w.r.t. G

which implies

optimum sol'n at turning point

sub $G = G^{**}$ ($\because G^{**}$ turning point)

$$(3) \qquad \boxed{v(G^{**}) + G^{**}v'(G^{**})} - c = 0.$$

On the other hand, by adding together n equations in (2), we obtain

$$\left.\begin{aligned} &v(G^*) + g_1^* \cdot v'(G^*) - c = 0 \\ &v(G^*) + g_2^* \cdot v'(G^*) - c = 0 \\ &\vdots \\ &v(G^*) + g_n^* \cdot v'(G^*) - c = 0 \end{aligned}\right\} \text{n-eqn}$$

$$nv(G^*) + G^*v'(G^*) - nc = 0,$$

which implies

$$(4) \qquad \boxed{v(G^*) + \frac{1}{n}G^*v'(G^*)} - c = 0,$$

where $G^* = g_1^* + \cdots + g_n^*$.

Claim: $G^* > G^{**}$.

NE social optimum

(3)&(4) $\Rightarrow v(G^{**}) + G^{**} v'(G^{**}) = v(G^*) + \frac{1}{n} G^* v'(G^*)$

$G^* > G^{**}$

proof by contradiction

Proof of Claim. Suppose $G^* \le G^{**}$. Then $\overbrace{v(G^*)}^{3} \ge \overbrace{v(G^{**})}^{4}$ $(v\downarrow)$ since $v' < 0$. Similarly, $0 > v'(G^*) \ge v'(G^{**})$ $(v' \downarrow)$ since $v'' < 0$. $\because n \ge 2$

Next $G^*/n < G^* \le G^{**}$, since $n \ge 2$.

Thus

$$\underbrace{\frac{G^*}{n} v'(G^*)}_{1} \ge \frac{G^*}{n} v'(G^{**}) > \underbrace{G^{**} v'(G^{**})}_{2}. \qquad (\because v' < 0)$$

This implies

$$\overbrace{v(G^*)}^{3} + \overbrace{\frac{G^*}{n} v'(G^*)}^{1} > \overbrace{v(G^{**})}^{4} + \overbrace{G^{**} v'(G^{**})}^{2},$$

which contradicts to (4) and (3). Therefore

$$G^* > G^{**}.$$

This means that too many goats are grazed in the Nash equilibrium, compared to the social optimum.

For example, take

$$v(x) = a - x^2, \qquad a > c.$$

$g_i^* = \sqrt{\frac{a-c}{n^2+2n}}$

$\Downarrow$

$G^* = n\sqrt{\frac{a-c}{n^2+2n}} = \sqrt{\frac{a-c}{1+2/n}}$

Then $v'(x) = -2x$.

$$a - (G^*)^2 + \frac{G^*}{n}(-2G^*) - c = 0.$$

(3)
$$\underbrace{a - (G^{**})^2}_{v(G^{**})} + G^{**}\underbrace{(-2G^{**})}_{v'(G^{**})} - c = 0.$$

These imply

$$G^* = \frac{\sqrt{a-c}}{\sqrt{1+2/n}} > G^{**} = \frac{\sqrt{a-c}}{\sqrt{3}}.$$

$1+2/n < 3 \quad (n \geq 2)$

In fact,

$= \sqrt{\frac{3}{1+2/n}}$

$$\lim_{n\to\infty} \frac{G^*}{G^{**}} = \sqrt{3} \approx 1.73.$$

$n \to \infty$

This means that, when there is a large number of farmers, who choose the number of goats simultanuously by their own, the village grazes 73% more goats than the social optimum.

G^*

Final-Offer Arbitration

2 player

Suppose the wage dispute of a firm and a union is settled by the final-offer arbitration.

- The firm wants to minimize the expected wage settlement and the union wants to maximize it.

- First the firm and the union simultaneously make offers, denoted by w_f and w_u respectively ($w_f < w_u$).

  not player

- Second, the arbitrator chooses one of the two offers as the settlement.

near to w_f ← x → near to w_u

w_f $\frac{w_f+w_u}{2}$ w_u

Assume prob $(x = \frac{w_f+w_u}{2}) = 0$

- Assume that the arbitrator has an ideal settlement x and simply chooses as the settlement the offer that is closer the x:

  players: firm, union, arbitrator (random) → expected payoff

  – the arbitrator chooses w_f if $x < (w_f+w_u)/2$, and chooses w_u if $x > (w_f + w_u)/2$.

  $F(a) = \text{prob}(\text{arbitrator settle } x < a)$

- Both players believe that x is randomly distributed according to a cumulative probability distribution denoted by $F(x)$, with associated probability density function denoted by $f(x)$ $(= F'(x))$.

  $f(x) = F'(x)$

  – Let x_m be the median of x, i.e., $\underbrace{F(x_m)}_{\text{prob (arb settle } < x_m)} = 1/2$.

Question: Why not choose $(w_f + w_u)/2$ as the final settlement?

What is best response of firm? $w_f = 0$

What is best response of union? $w_u = \infty$

Note that the expected wage settlement is

payoff to firm $= -\phi$

payoff to union $= \phi$

$$\phi(w_f, w_u) = w_f \underbrace{F(\frac{w_f + w_u}{2})}_{\text{prob}(w_f \text{ near to } x)} + w_u \underbrace{[1 - F(\frac{w_f + w_u}{2})]}_{\text{prob}(w_u \text{ near to } x)}.$$

$\text{prob}(w_f \text{ near to } x) \Leftrightarrow x < \frac{w_f+w_u}{2}$

Because

$$\begin{aligned} Prob(w_f \text{ is chosen}) &= Prob(x < \frac{w_f + w_u}{2}) \\ &= F(\frac{w_f + w_u}{2}), \end{aligned}$$

$$Prob(w_u \text{ is chosen}) = 1 - F(\frac{w_f + w_u}{2}),$$

$$\phi = w_f F\left(\frac{w_f+w_u}{2}\right) + w_u\left[1 - F\left(\frac{w_f+w_u}{2}\right)\right]$$

$$\frac{\partial \phi}{\partial w_f} = F\left(\frac{w_f+w_u}{2}\right) + w_f \cdot \frac{\partial}{\partial w_f} F\left(\frac{w_f+w_u}{2}\right)$$
$$+ w_u\left[-\frac{\partial}{\partial w_f} F\left(\frac{w_f+w_u}{2}\right)\right]$$

$$\frac{\partial}{\partial w_f} F\left(\frac{w_f+w_u}{2}\right) = f\left(\frac{w_f+w_u}{2}\right) \cdot \frac{\partial}{\partial w_f}\left(\frac{w_f+w_u}{2}\right)$$
$$= \frac{1}{2} f\left(\frac{w_f+w_u}{2}\right)$$

$$\therefore \frac{\partial \phi}{\partial w_f} = F\left(\frac{w_f+w_u}{2}\right) + \frac{1}{2} w_f f\left(\frac{w_f+w_u}{2}\right)$$
$$- \frac{1}{2} w_u f\left(\frac{w_f+w_u}{2}\right) = 0 \Rightarrow (\star 1)$$
$$= F\left(\frac{w_f+w_u}{2}\right) - \frac{1}{2}(w_u - w_f) f\left(\frac{w_f+w_u}{2}\right)$$

similarly,

$$\frac{\partial \phi}{\partial w_u} = w_f \cdot f\left(\frac{w_f+w_u}{2}\right) \cdot \frac{1}{2} + 1 - F\left(\frac{w_f+w_u}{2}\right)$$
$$+ w_u\left(-f\left(\frac{w_f+w_u}{2}\right) \cdot \frac{1}{2}\right)$$
$$= (w_f - w_u) \cdot \frac{1}{2} \cdot f\left(\frac{w_f+w_u}{2}\right) - F\left(\frac{w_f+w_u}{2}\right)$$
$$+ 1 = 0 \Rightarrow (\star 2)$$

Fix w_u^* firm $\min_{w_f} \phi(w_f, w_u^*)$

Fix w_f^* union $\max_{w_u} \phi(w_f^*, w_u)$

(w_f^*, w_u^*) are a Nash equilibrium if and only if they solve

$$\min_{w_f} \phi(w_f, w_u^*) \quad \text{and} \quad \max_{w_u} \phi(w_f^*, w_u),$$

$$0 = \left.\frac{\partial \phi(w_f^*, w_u)}{\partial w_u}\right|_{w_u = w_u^*}$$

variable $\left.\frac{\partial \phi(w_f, w_u^*)}{\partial w_f}\right|_{w_f = w_f^*} = 0$

respectively.

Claim. The Nash equilibrium is

$$w_f^* = x_m - \frac{1}{2f(x_m)}, \qquad w_u^* = x_m + \frac{1}{2f(x_m)}.$$

Proof of Claim. From the optimality conditions $\frac{\partial \phi}{\partial w_f} = 0$ and $\frac{\partial \phi}{\partial w_u} = 0$, it follows that

$$\frac{\partial \phi}{\partial w_f} = 0 \ (\star 1) \quad \frac{1}{2}(w_u - w_f) f\left(\frac{w_f + w_u}{2}\right) = F\left(\frac{w_f + w_u}{2}\right)$$

$$\frac{\partial \phi}{\partial w_u} = 0 \ (\star 2) \quad \frac{1}{2}(w_u - w_f) f\left(\frac{w_f + w_u}{2}\right) = 1 - F\left(\frac{w_f + w_u}{2}\right)$$

$$\Rightarrow F\left(\frac{w_f + w_u}{2}\right) = 1 - F\left(\frac{w_f + w_u}{2}\right)$$

So we have $F\left(\frac{w_f + w_u}{2}\right) = \frac{1}{2}$, which implies

$F(x_m) = \frac{1}{2}$

prob (arb settle $< x_m$) $= \frac{1}{2}$

$$x_m \quad \therefore \quad \frac{w_f + w_u}{2} = x_m.$$

(★3) $w_f + w_u = 2x_m$

Furthermore we have

(★1) $\cancel{\frac{1}{2}}(w_u - w_f) f(x_m) = \cancel{\frac{1}{2}}$

$$w_u - w_f = \frac{1}{f(x_m)} \quad (\star 4)$$

(★3)+(★4) $2w_u = 2x_m + \frac{1}{f(x_m)}$

Solving the above two equations, we obtain the Nash equilibrium

$$w_f^* = x_m - \frac{1}{2f(x_m)}, \qquad w_u^* = x_m + \frac{1}{2f(x_m)}. \qquad \square$$

Q: Does NE always exist in a game?

A: Sometimes NE does not exist.

Matching Pennies

Each player has a penny and must choose whether to display it with heads or tails facing up.

- If the two pennies match (i.e., both are heads up or both are tails up) then player 2 wins player 1's penny; payoff (−1, 1) player 1 & 2
- if the pennies do not match then 1 wins 2's penny. (1, −1)

		Player2	
		Heads	Tails
Player1	Heads	$-1, \underline{1}$	$\underline{1}, -1$
	Tails	$\underline{1}, -1$	$-1, \underline{1}$

Player 1 & 2 (pure) strategy

strategy space

$S_1 = S_2 = \{H, T\}$

prob(T)

mixed strategy of player 1 $(p, 1-p)$

Prob(H)

There is no pure-strategy Nash equilibrium in this game. The reason is that the solution to such a game necessarily involves uncertainty about what the other players will do. □

In order to find equilibria to such problems, we introduce the notion of a mixed strategy. ← prob distribution

Mixed Strategies pure strategy

Definition 5. *In the normal-form game*

$$G = \{\underbrace{S_1, \ldots, S_n}_{\text{strategy spaces}}; \underbrace{u_1, \ldots, u_n}_{\text{payoff function}}\},$$

strategy space of player i

suppose $\overset{\downarrow}{S_i} = \{\underbrace{s_{i1}, \ldots, s_{iK}}_{\text{pure strategy}}\}$. *Then*

- *each strategy* s_{ik} *in* S_i *is called a* **pure strategy** *for player* i.
- *A* **mixed strategy** *for player* i *is a probability distribution* $p_i = (p_{i1}, \ldots, p_{iK})$, *where* $p_{i1} + \cdots + p_{iK} = 1$ *and* $p_{ik} \geq 0$.

  P_{i1} = prob (S_{i1} is used) ... P_{ik} = prob (S_{ik} is used)

In Matching Pennies, $S_i = \{\overbrace{Heads, Tails}^{\text{pure strategies}}\}$.

- So Heads and Tails are the pure strategies;
- a mixed strategy for player i is the probability distribution $(\overset{\text{prob(H)}}{q}, \overset{\text{prob(T)}}{1 - q})$, where q is the probability of playing Heads, $1 - q$ is the probability of playing Tails, and $0 \leq q \leq 1$.
- The mixed strategy $(\overset{\text{prob(H)}}{1}, \overset{\text{prob(T)}}{0})$ is simply the pure strategy Heads; and the mixed strategy $(\underset{\text{prob(H)}}{0}, \overset{}{\underset{\text{prob(T)}}{1}})$ is simply the pure strategy Tails.

Recall that the definition of Nash equilibrium guarantees that each player's pure strategy is a best response to the other players' pure strategies.

NE (pure-strategy NE)

To extend the definition to include mixed strategies,

mixed-strategy NE

best response of a player (mixed strategy) to other players' mixed strategy (mixed strategy)

if every player plays with best response, then we get NE mixed strategy best response of a player i to other players' strategy mixed-strategy (mixed strategy)

↳ fix others' mixed strategy, find mixed strategy maximizers of expected payoff function of player i

prob distribution

$u_i(s_1, \cdots, s_{i-1}, s_i, s_{i+1}, \cdots, s_n)$

∵ each combination of pure-strategies is associated with some prob

- we simply require that each player's <u>mixed</u> strategy be a best response to the other players' <u>mixed</u> strategies.

We illustrate the notion of Nash equilibria by considering the case of two players.

Let

$$S_1 = \{\underbrace{s_{11}, s_{12}, \ldots, s_{1J}}_{\text{player 1's pure strategies}}\} \qquad S_{1j}$$

and

$$S_2 = \{\underbrace{s_{21}, s_{22}, \ldots, s_{2K}}_{\text{player 2's pure strategies}}\} \qquad S_{2k}$$

be the respective sets of pure strategies for players 1 and 2. We use s_{1j} and s_{2k} to denote arbitrary pure strategies from S_1 and S_2 respectively.

Fix player 2's mixed strategy

If player 1 believes that player 2 will play the strategies $(s_{21}, s_{22}, \ldots, s_{2K})$ with the probabilities $p_2 = (p_{21}, p_{22}, \ldots, p_{2K})$, then player 1's **expected payoff** from playing the $\boxed{\text{pure strategy}}$ $\boxed{s_{1j}}$ is

expected payoff to player 1 when $\boxed{\text{pure strategy}}$ is used

$$\underbrace{v_1(\boxed{s_{1j}}, p_2)}_{\text{expected payoff to player 1 when } (S_{1j},\, P_2) \text{ is used}} = \sum_{k=1}^{K} \overbrace{p_{2k} u_1(s_{1j},}^{\text{prob}} s_{2k}),$$

and player 1's **expected payoff** from playing the mixed strategy $p_1 = (p_{11}, p_{12}, \ldots, p_{1J})$ is

prob

$$\begin{aligned} v_1(p_1, p_2) &= \sum_{j=1}^{J} p_{1j} v_1(s_{1j}, p_2) \\ &= \sum_{j=1}^{J}\sum_{k=1}^{K} p_{1j} p_{2k} u_1(s_{1j}, s_{2k}). \end{aligned}$$

expected payoff to 1 when (P_1, P_2) used

last page expected payoff to 1 when (S_{1j}, P_2) use

Fix mixed strategy of player 1

If player 2 believes that player 1 will play the strategies $(s_{11}, s_{12}, \ldots, s_{1J})$ with the probabilities $p_1 = (p_{11}, p_{12}, \ldots, p_{1J})$, then player 2's **expected payoff** from playing the pure strategies $(s_{21}, \ldots, s_{2K})$ with the probability $p_2 = (p_{21}, \ldots, p_{2K})$ is

expected payoff to player 2 via mixed strategy $P_2 = (P_{21}, \cdots, P_{2K})$

$V_2(P_1, S_{2k}) = \sum_{j=1}^{J} P_{1j} U_2(S_{1j}, S_{2k})$

$\boxed{\text{via pure strategy } S_{2k}}$

$$\begin{aligned} v_2(p_1, p_2) &= \sum_{k=1}^{K} p_{2k} v_2(p_1, s_{2k}) \\ &= \sum_{k=1}^{K}\sum_{j=1}^{J} p_{2k} p_{1j} u_2(s_{1j}, s_{2k}) \end{aligned}$$

prob

expected payoff to player 2 when S_{2k} is used

Definition 6. *In the two-player normal-form game $G = \{S_1, S_2; u_1, u_2\}$, the mixed strategies (p_1^*, p_2^*) are a* **Nash equilibrium** *if each player's mixed strategy is a best response to the other player's mixed strategy:*

expected payoff to player 1 when P_1 used

$$\forall P_1 \qquad v_1(p_1^*, p_2^*) \geq v_1(p_1, p_2^*)$$

Fix mixed strategy P_2^* of player 2

expected payoff to player 1 when P_1^* used

Fix mixed strategy p_1^* of player 1

expected payoff to player 2 when p_2^* used

and

$$\forall p_2 \quad v_2(p_1^*, p_2^*) \geq v_2(p_1^*, p_2)$$

expected payoff to player 2 when p_2 used

for all probability distributions p_1 and p_2 on S_1 and S_2 respctively.

Find mixed-strategy Nash equilibria

We only consider two-player games in which each player has 2 pure strategies, so that we can use the graphical approach.

prob (s_{11} used)

s_{11} s_{12} pure strategies of 1

Let $p_1 = (r, 1-r)$ be a mixed strategy for player 1 and $p_2 = (q, 1-q)$ a mixed strategy for player 2. Then

prob(s_{21} used) s_{21} s_{22} pure strategies of 2

prob

when (s_{12}, p_2)

$$v_1(p_1, p_2) = r v_1(s_{11}, p_2) + (1-r) v_1(s_{12}, p_2).$$

expected payoff to player 1 when (p_1, p_2) used

expected payoff to 1 when (s_{11}, p_2)

expected payoff to 1

For each given p_2, i.e., for each given q, we compute the value of r, denoted $r^*(q)$, such that $p_1 = (r, 1-r)$ is a best response for player 1 to $p_2 = (q, 1-q)$. That is, $r^*(q)$ is the set of solutions to

Fix mixed strategy of player 2 $p_2 = (q, 1-q)$

best response of player 1

Find the best response of player 1 (mixed strategy) choose the pure strategy which gives a higher expected payoff

$$\underbrace{\max_{0 \leq r \leq 1} v_1(p_1, p_2)}_{\text{expected payoff to player 1 (r variable)}}.$$

$p_1 = (r, 1-r)$

$r^*(\cdot)$ is called the best-response correspondence. It may contain more than one value.

The best response of player 1

in pure strategy $\begin{cases} S_{11} \to \text{expected payoff to 1 } V_1(S_{11}, P_2) \\ S_{12} \to \text{expected payoff to 1 } V_1(S_{12}, P_2) \end{cases}$

choose $\underset{r=1}{S_{11}}$ as best response if $V_1(S_{11}, P_2) > V_1(S_{12}, P_2)$

.... $\underset{r=0}{S_{12}}$ .... $<$

choose any .... $=$

$r \in [0,1]$

$$U_1(P_1, P_2) = r\, V_1(S_{11}, P_2) + (1-r)\, V_1(S_{12}, P_2)$$

$$= V_1(S_{12}, P_2)$$

Fix strategy of player 1 $P_1 = (r, 1-r)$

find best response of player 2

2's pure strategies $\begin{cases} S_{21} \to \text{expected payoff to 2 is } V_2(P_1, S_{21}) \\ S_{22} \to \text{.......... is } V_2(P_1, S_{22}) \end{cases}$

best response of player 2 is $(\underset{prob(S_{21})}{q}, \underset{prob(S_{22})}{1-q})$

- $S_{21} \Leftrightarrow q = 1$ if $V_2(P_1, S_{21}) > V_2(P_1, S_{22})$
- $S_{22} \Leftrightarrow q = 0$ .... $<$ ....
- any $\Leftrightarrow q \in [0,1]$ $\underbrace{=}_{V_2(P_1, P_2) = U_2(P_1, S_{22})}$

(r, 1-r) (q, 1-q) fixed

Observing $v_1(p_1, p_2)$, we obtain

best response r of player 1

$$r^*(q) = \begin{cases} 1, & \text{if } v_1(s_{11}, p_2) > v_1(s_{12}, p_2) \\ 0, & \text{if } v_1(s_{11}, p_2) < v_1(s_{12}, p_2) \\ [0,1], & \text{if } v_1(s_{11}, p_2) = v_1(s_{12}, p_2) \end{cases}$$

Similarly, maximizing

expected payoff to player 2 when (p_1, p_2) used

$$v_2(p_1, p_2) = \overset{\text{prob}}{q}\, v_2(p_1, s_{21}) + \overset{\text{prob}}{(1-q)}\, v_2(p_1, s_{22})$$

expected payoff to player 2 when (p_1, s_{21}) used

expected payoff to 2 when (p_1, s_{22})

yields

best response of player 2

$$q^*(r) = \begin{cases} 1, & \text{if } v_2(p_1, s_{21}) > v_2(p_1, s_{22}) \\ 0, & \text{if } v_2(p_1, s_{21}) < v_2(p_1, s_{22}) \\ [0,1], & \text{if } v_2(p_1, s_{21}) = v_2(p_1, s_{22}) \end{cases}$$

A mixed-strategy Nash equilibrium is an intersection of the two best-response conrrespondences $r^*(q)$ and $q^*(r)$.

Find a Nash equilibrium for the Game of Matching Pennies

		Player2	
		Heads (q)	Tails (1-q)
Player1	Heads (s_{11}) r	−1, 1	1, −1
	Tails (s_{12}) 1-r	1, −1	−1, 1

→ $q \cdot (-1) + (1-q) \cdot 1$

→ $1 \cdot q + (-1) \cdot (1-q)$

H T

Let $p_1 = (r, 1-r)$ be a mixed strategy in which player 1 plays Heads with probability r. Let $p_2 = (q, 1-q)$ be a mixed strategy for player 2. Then

Fix strategy of player 2 $P_2 = (q, 1-q)$

Find best response of player 1

expected payoff to 1 when S_{11}

$$
\begin{aligned}
v_1(\overset{H}{s_{11}}, p_2) &= q \cdot (-1) + (1-q) \cdot 1 = 1 - 2q \quad \text{v.s.} \\
v_1(\overset{T}{s_{12}}, p_2) &= q \cdot 1 + (1-q) \cdot (-1) = -1 + 2q.
\end{aligned}
$$

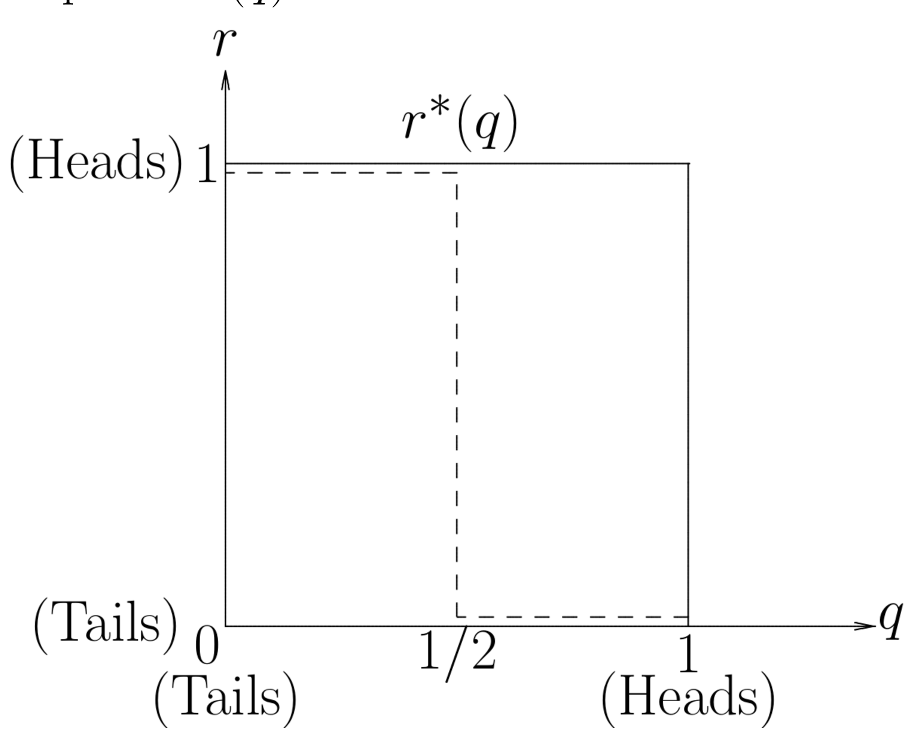
expected payoff to 1 when S_{12}

Player 1 chooses Heads (Tails), i.e., $r^*(q) = 1$ (0) $\iff v_1(s_{11}, p_2) >$ (<) $v_1(s_{12}, p_2) \iff q < 1/2$ (>). Hence

$1 - 2q > -1 + 2q$

$$
r^*(q) = \begin{cases} 1, & \text{if } 0 \le q < 1/2 \\ 0, & \text{if } 1/2 < q \le 1 \\ [0,1], & \text{if } q = 1/2 \end{cases}
$$

Graph of $r^*(q)$:



Fix strategy of player 1 $p_1 = (r, 1-r)$

find the best response of player 2

	H (q)	T ($1-q$)
H ← r	-1, 1	1, -1
T ← $1-r$	1, -1	-1, 1

$r \cdot (-1) + (1-r) \cdot 1$

$r \cdot 1 + (1-r) \cdot (-1)$

To determine $q^*(r)$, we use

expected payoff to 2 when $s_{21} = H$

$$v_2(p_1, s_{21}) = \underline{r \cdot 1 + (1-r) \cdot (-1)} = 2r - 1$$

expected payoff to 2 when T

$$v_2(p_1, s_{22}) = r \cdot (-1) + (1-r) \cdot 1 = -2r + 1.$$

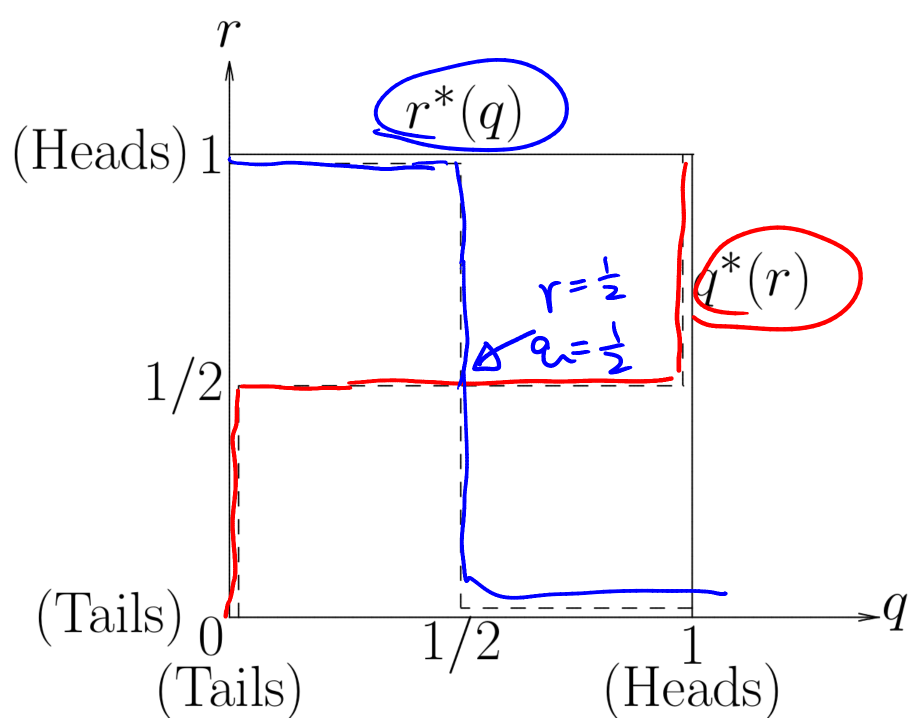
Player 2 chooses Heads (T), i.e., $q^*(r) = 1 \iff v_2(p_1, s_{21}) > v_2(p_1, s_{22}) \iff r > 1/2$, hence

$2r-1$ $<$ $-2r+1$ $<$

best response of player 2

$$q^*(r) = \begin{cases} 1, & \text{if } 1/2 < r \leq 1 \\ 0, & \text{if } 0 \leq r < 1/2 \\ [0, 1], & \text{if } r = 1/2 \end{cases}$$

We draw the graphs of $r^*(q)$ and $q^*(r)$ together.



The graphs of the best response correspondences $r^*(q)$ and $q^*(r)$ intersect at only one point $q = \frac{1}{2}$ and $r = \frac{1}{2}$.

- Thus $p_1^* = (\overbrace{\tfrac{1}{2}}^{r}, \overbrace{\tfrac{1}{2}}^{1-r})$, $p_2^* = (\overbrace{\tfrac{1}{2}}^{q}, \overbrace{\tfrac{1}{2}}^{1-q})$ are the only Nash equilibrium in mixed strategies for the game.
  - This means that randomly choosing Tails and Heads with equal probability is the best strategy.

- If there are more than 2 strategies for a player, we can first eliminate strictly dominated strategies.

Claim: If a pure strategy $s_{kj} \in S_{kj}$ is eliminated by IESDS, then the strategy will be played with zero probability $p_{kj} = 0$ in any mixed strategy Nash equilibrium.

If there are only 2 strategies left for each player, then we can use the approach discussed before.

Example.

		Player2		
		L	C	R
	T	4, 4	4, 0	5, 1
Player1	M	2, 0	0, 4	5, 3
	B	3, 5	3, 5	6, 6

Eliminate the dominated strategies M and C, resulting in