MA2108 Cheatsheet 19/20 Sem 1 Midterm

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The Real Numbers

Result 1. $2^{n-1} \leq n!$ and $n < 2^n \ \forall n \in \mathbb{N}$

Result 2. $n^2 \leq 2^n \ \forall n \in \mathbb{N}\backslash\{2,3,4\}$

Well-ordering principle. Every nonempty subset A of $\mathbb{N}$ has *a* least (first) element, i.e. there exists $p \in A$ such that $p \leq a$ for all $a \in A$

Lemma 1.5.1.

1. If $c > 1$, then $c^n > c$ for every natural number $n \geq 2$.
2. If $0 < c < 1$, then $c^n < c$ for every natural number $n \geq 2$.

Theorem 1.5.2. For any non-zero number a, $a^2 > 0$.

Theorem 1.5.3. If $a \in \mathbb{R}$ is such that $0 \leq a < \varepsilon$ for every positive number ε, then $a = 0$

Bernoulli's Inequality. If $x > -1$, then

$$(1+x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}$$

Some useful properties of absolute value.

1. $|a| \leq c \to -c \leq a \leq c$
2. $-|a| \leq a \leq |a|$

Definition of means.

1. The *arithmetic mean* of $a_1, a_2, \cdots, a_n$ is defined as $A = \frac{a_1 + a_2 + \cdots + a_n}{n}$.
2. The *geometric mean* of $a_1, a_2, \cdots, a_n$ is defined as $G = (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$.
3. The *harmonic mean* of $a_1, a_2, \cdots, a_n$ is defined as $H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$.

The AM-GM-HM Inequality. Let A, G, H be the arithmetic mean, the geometric mean and the harmonic mean of the positive numbers $a_1, a_2, \ldots, a_n$ respectively. Then

$$H \leq G \leq A.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Example for finding a set given an inequality. Solve $|x| + |x+1| < 2$

Solution: Note that x and $x+1$ change signs at 0 and -1.

Case $1: x \leq -1$

In this case, $|x| + |x+1| = -x + (-x-1) = -2x - 1 < 2$, so that $2x > -3$ and $x > -3/2$. Thus the points in $(-3/2, \infty) \cap (-\infty, -1] = (-3/2, -1]$ satisfy the inequality.

Case $2: -1 < x < 0$

In this case, $|x| + |x+1| = -x + (x+1) = 1 < 2$ which is always true. So all the points in $(-1, 0)$ satisfy the inequality.

Case $3: x \geq 0$

In this case, $|x| + |x+1| = x + (x+1) = 2x + 1 < 2$, so that $2x < 1$ and $x < 1/2$. Thus the points in $(-\infty, 1/2) \cap [0, \infty) = [0, 1/2)$ satisfy the inequality.

So the solution set is $(-3/2, -1] \cup (-1, 0) \cup [0, 1/2) = (-3/2, 1/2)$.

Triangle Inequality. For $a, b \in \mathbb{R}, |a+b| \leq |a| + |b|$

Corollary 1.8.2.

$$||a| - |b|| \leq |a - b|$$
$$|a - b| \leq |a| + |b|$$

Proving supremums. Let L be the supposed supremum. First prove that L is an upper bound. Then prove that L is the smallest upper bound.

Example for proving supremums. Let S be a nonempty subset of $\mathbb{R}$ and $a \in \mathbb{R}$. Let

$$a + S = \{a + x : x \in S\}.$$

Prove that if S is bounded above, then $\sup(a+S) = a + \sup S$.

Solution:

$$a + x \leq a + \sup S \quad \forall x \in S \tag{1}$$

This says that $a + \sup S$ is an upper bound for $a + S$. Next suppose v is any upper bound of $a + S$. Then

$$a + x \leq v, \quad \forall x \in S \tag{2}$$

$$x \leq v - a \quad \forall x \in S \tag{3}$$

So v-a is an upper bound for S. Thus

$$\sup S \leq v - a \tag{4}$$

$$a + \sup S \leq v \tag{5}$$

We have shown that $a + \sup S$ is an upper bound for $a + S$ and is less than or equal to any other upper bound for $a + S$. Thus $\sup(a+S) = a + \sup S$. $\square$

Lemma 1.9.1. Let u be an upper bound of $S \subseteq \mathbb{R}$. Then $u = \sup S$ if and only if $\forall \varepsilon > 0$, $\exists x_\varepsilon \in S$ such that $u - \varepsilon < x_\varepsilon$.

Supremum property of $\mathbb{R}$. Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum.

Archimedean Property. If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ such that $x < n_x$

Corollary 1.9.2. For any $\varepsilon > 0, \exists n \in \mathbb{N}$ such that

$$\frac{1}{n} < \varepsilon$$

Corollary 1.9.3. If $x > 0$, then $\exists n \in \mathbb{N}$ such that

$$n - 1 \leq x < n$$

Theorem 1.11.1. Let $a > 0$ and $n \in \mathbb{N}$. There exists a unique positive real number u with

$$u^n = a \ .$$

We call the number u the positive nth root of a and write $u = \sqrt[n]{a}$ or $a^{1/n}$.

Result. If $a > 0$ and $n, m \in \mathbb{N}$, then

$$\left(a^{1/n}\right)^m = (a^m)^{1/n}$$

Theorem 1.11.2 (Properties of rational exponents).

1. If $a > 0$ and $r, s \in \mathbb{Q}$, then $a^{r+s} = a^r a^s$ and $(a^r)^s = a^{rs}$
2. If $0 < a < b$ and $r \in \mathbb{Q}$ with $r > 0$, then $a^r < b^r$
3. If $a > 1, r, s \in \mathbb{Q}$ with $r < s$, then $a^r < a^s$

Density Theorem of $\mathbb{Q}$. If $a, b \in \mathbb{R}$ is such that $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Corollary 1.12.1. If $a, b \in \mathbb{R}$ is such that $a < b$, then there exists an irrational number x such that $a < x < b$.

Corollary 1.12.2. Every interval $I \subseteq \mathbb{R}$ contains infinitely many rational numbers and infinitely many irrational numbers.

Sequences

Theorem 2.1.1. If (x_n) converges, then it has exactly one limit.

Theorem 2.2.1. Every convergent sequence is bounded.

Theorem 2.2.2. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

1. $\lim_{n\to\infty} (x_n + y_n) = x + y$
2. $\lim_{n\to\infty} (x_n - y_n) = x - y$
3. $\lim_{n\to\infty} (x_n y_n) = xy$
4. $\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \frac{x}{y}$, provided $y_n \neq 0, \forall n \in \mathbb{N}$, and $y \neq 0$

Corollary 2.2.3. If (x_n) converges and $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k$$

Classic limit. $\lim_{n\to\infty} \frac{\sin n}{n} = 0$

Theorem 2.2.4. If $|x_n| \to 0$, then $x_n \to 0$.

Theorem 2.2.5. If $0 < b < 1$, then $\lim_{n\to\infty} b^n = 0$

Remark on Theorem 2.2.4 and 2.2.5. Theorems 2.2.4 and 2.2.5 together imply that $b^n \to 0$ for all b with $|b| < 1$

Theorem 2.2.6. If $c > 0$, then $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$

Theorem 2.2.7.

1. If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} |x_n| = |x|$
2. If all $x_n \geq 0$ and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$

Theorem 2.2.8.

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Theorem 2.2.9.

1. If $x_n \geq 0$ for all $n \in \mathbb{N}$ and (x_n) converges, then $\lim_{n\to\infty} x_n \geq 0$
2. If (x_n) and (y_n) are convergent and $x_n \geq y_n$ for all $n \in \mathbb{N}$, then
$$\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n$$
3. If $a, b \in \mathbb{R}$ and $a \leq x_n \leq b$ for all n and (x_n) is convergent, then
$$a \leq \lim_{n\to\infty} x_n \leq b$$

Past Midterm Questions

18/19 Question 5. If (x_n) converges to 0 and (y_n) is bounded, then $(x_n y_n)$ converges to 0.

Solution: Since (y_n) is bounded, there exists $M > 0$ such that $|y_n| \leq M$ for all $n \in \mathbb{N}$. Now let $\varepsilon > 0$. since $x_n \to 0$, there exists $K \in \mathbb{N}$ such that

$$|x_n - 0| < \frac{\varepsilon}{M} \quad \forall n \geq K$$

Then

$$n \geq K \implies |x_n y_n - 0| = |x_n||y_n| \leq \frac{\varepsilon}{M} \cdot M = \varepsilon$$

18/19 Question 6. Suppose (a_n) is convergent and $a = \lim_{n\to\infty} a_n$. For each $n \in \mathbb{N}$, let

$$b_n = \frac{1}{n^2} \sum_{j=1}^{n} (n-j+1)a_j = \frac{na_1 + (n-1)a_2 + \cdots + 2a_{n-1} + a_n}{n^2}$$

.

1. Prove that for each $n \in \mathbb{N}$,

$$b_n = \frac{1}{n^2} \sum_{j=1}^{n} (n-j+1)(a_j - a) + \frac{n+1}{2n} a$$

.
Proof skipped.

2. Prove that (b_n) converges.
Solution. Let $c_n = \frac{1}{n^2}\sum_{j=1}^{n}(n-j+1)(a_j - a)$ such that $b_n = c_n + \frac{n+1}{2n}a$. We want to prove that $\lim_{n\to\infty} c_n = 0$.
Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, $\exists K_1 \in \mathbb{N}$ s.t.

$$n \geq K_1 \to |a_n - a| < \frac{\epsilon}{2}$$

.
Let $C = \sum_{j=1}^{K_1} |a_j - a|$. By A.P $\exists K_2 \in \mathbb{N}$ s.t. $K_2 > \frac{2C}{\epsilon}$. Let $K = \max(K_1, K_2)$. Then for $n \geq K$, we have

$$\begin{aligned}
|c_n - 0| &= \left| \frac{1}{n^2} \sum_{j=1}^{K_1} (n-j+1)(a_j - a) \right| \\
&\leq \frac{1}{n^2} \sum_{j=1}^{K_1} (n-j+1)|a_j - a| \\
&\quad + \frac{1}{n^2} \sum_{j=K_1+1}^{n} (n-j+1)|a_j - a| \\
&\leq \frac{1}{n^2} \sum_{j=1}^{K_1} n|a_j - a| + \frac{1}{n^2} \sum_{j=K_1+1}^{n} n \cdot \frac{\epsilon}{2} \\
&\leq \frac{C}{n} + \frac{1}{n^2}(n - K_1)n\frac{\epsilon}{2} \\
&\leq \frac{C}{K} + \frac{\epsilon}{2} \\
&< \frac{C}{2C/\epsilon} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

13/14 Question 6. Let (a_n) and (b_n) be defined by setting

$$a_1 = 3, b_1 = 2, a_{n+1} = a_n + 2b_n \text{ and } b_{n+1} = a_n + b_n \text{ for } n \in \mathbb{N}$$

. Moreover, let $c_n = \frac{a_n}{b_n}$.

1. Express c_{n+1} in terms of c_n.
Solution.

$$\begin{aligned}
c_{n+1} &= \frac{a_{n+1}}{b_{n+1}} \\
&= \frac{a_n + 2b_n}{a_n + b_n} \\
&= \frac{\frac{a_n}{b_n} + 2}{\frac{a_n}{b_n} + 1} \\
&= \frac{c_n + 2}{c_n + 1}
\end{aligned}$$

2. Prove that $\left|c_{n+1} - \sqrt{2}\right| < r\left|c_n - \sqrt{2}\right|$, $r = \sqrt{2} - 1$.
Solution.

$$\begin{aligned}
\left|c_{n+1} - \sqrt{2}\right| &= \left|\frac{c_n + 2}{c_n + 1} - \sqrt{2}\right| \\
&= \frac{1}{c_n + 1}\left|\left(1 - \sqrt{2}\right)c_n + \left(2 - \sqrt{2}\right)\right| \\
&= \frac{1}{c_n + 1}\left|\left(1 - \sqrt{2}\right)\left(c_n - \sqrt{2}\right)\right| \\
&= \frac{\sqrt{2} - 1}{c_n + 1}\left|c_n - \sqrt{2}\right| \\
&= \frac{r}{c_n + 1}\left|c_n - \sqrt{2}\right| \\
&< r\left|c_n - \sqrt{2}\right|
\end{aligned}$$

3. Prove that c_n converges and find its limit.
Solution. For $n \geq 2$ we have by (ii),

$$\left|c_n - \sqrt{2}\right| < r\left|c_{n-1} - \sqrt{2}\right| < \cdots < r^{n-1}\left|c_1 - \sqrt{2}\right|$$

Since $0 < r < 1$, $e^{n-1} \to 0$. So

$$r^{n-1}\left|c_1 - \sqrt{2}\right| \to 0.$$

By the Squeeze Theorem, $\left|c_n - \sqrt{2}\right| \to 0$. It follows that $\lim_{n\to\infty} c_n = \sqrt{2}$.

Tutorial Questions

Ratio theorem from Tutorial 4. If

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$$

where $L < 1$, then

$$\lim_{n\to\infty} x_n = 0$$

.

Expansion of $x^n - y^n$.

$$x^n - y^n = (x - y)(x^{p-1} + x^{p-2}y + x^{p-3}y^2 + \cdots + xy^{p-2} + y^{p-1})$$

Tutorial 3 Question 2d. Prove

$$\lim_{n\to\infty} \frac{4^n}{n!} = 0$$

.

Solution. Notice that

$$\frac{4^n}{n!} = \frac{4}{1} \cdot \frac{4}{2} \cdot \frac{4}{3} \cdot \underbrace{\frac{4}{4} \cdot \frac{4}{5} \cdot \cdots \cdot \frac{4}{n-1}}_{\leq 1} \cdot \frac{4}{n}$$

when $n \geq 5$. So

$$\frac{4^n}{n!} \leq \frac{256}{6n} < \frac{43}{n}$$

. Let $\epsilon > 0$. By A.P, $\exists K \in \mathbb{N}$ s.t. $K > \max(5, \frac{43}{\epsilon})$...

Result from Tutorial 1 Question 7.

$$\max(a, b) = \frac{1}{2}(a + b + |a - b|)$$

$$\min(a, b) = \frac{1}{2}(a + b - |a - b|)$$

Tutorial 1 Question 3. For any $n \in \mathbb{N}, n > 1$,

$$\left(1 + \frac{1}{n-1}\right)^{n-1} < \left(1 + \frac{1}{n}\right)^n$$

Tutorial 4 Question 1g.

$$\begin{aligned}
\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) &= \frac{2^2 - 1}{2^2} \cdot \frac{3^2 - 1}{3^2} \cdot \frac{4^2 - 1}{4^2} \cdots \frac{(n-1)^2 - 1}{(n-1)^2} \cdot \frac{n^2 - 1}{n^2} \\
&= \frac{3}{2^2} \cdot \frac{2 \cdot 4}{3^2} \cdot \frac{3 \cdot 5}{4^2} \cdots \frac{(n-2)n}{(n-1)^2} \cdot \frac{(n-1)(n+1)}{n^2} \\
&= \frac{1}{2} \cdot \frac{n+1}{n} \\
&= \frac{1 + \frac{1}{n}}{2} \to \frac{1}{2}.
\end{aligned}$$