Lecture 8: Lagrange Interpolation

Solving via linear system.

$$\begin{bmatrix}1 & x_0 & x_0^2 & \cdots & x_0^n\\ 1 & x_1 & x_1^2 & \cdots & x_1^n\\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n\end{bmatrix}\begin{bmatrix}a_0\\a_1\\ \vdots\\ a_n\end{bmatrix} = \begin{bmatrix} f(x_0)\\ f(x_1)\\ \vdots \\ f(x_n)\end{bmatrix}$$

Solving via basis polynomials. Let $L_k(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$ for $k=0,1,\cdots,n$. Then $P_n(x) = y_0L_0(x)+y_1L_1(x)+\cdots+y_nL_n(x)$.

Error analysis. $\forall x\in[a,b], \exists \xi \in (\min\{x,x_0,x_1,\cdots,x_n\}, \max\{x,x_0,x_1,\cdots,x_n\})$ such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Tutorial 3: Lagrange Interpolation

Example 1. Construct the LIP for $f(x)=\log_2 x$ using $x_0=1/2, x_1=1, x_2=2, x_3=4$. Find a bound of the absolute error for any $x\in[1/2,+\infty)$.
Solution. $f(0.5)=-1,\quad f(1)=0,\quad f(2)=1,\quad f(4)=2$. Solve using linear system to get $a_0=-\frac{52}{21}$, $a_1=\frac{7}{2}$, $a_2=-\frac{7}{6}$, $a_3=\frac{1}{7}$. So $P(x)=\frac17x^3-\frac76x^2+\frac72x-\frac{52}{21}$. To get an u.b for error, first find $f^{(4)}(x)=-\frac{6}{x^4\ln 2}$. Note monotonicity. Hence $|f^{(4)}(x)|\leqslant \frac{6}{(1/2)^4\ln 2}=\frac{96}{\ln 2}$. Thus an u.b for absolute error $|P(x)-f(x)|$ is $\frac{1}{4!}\times\frac{96}{\ln 2}|(x-1/2)(x-1)(x-2)(x-4)| = \frac{4}{\ln 2}|(x-1/2)(x-1)(x-2)(x-4)|$. $\square$

Result from Example 2. If nodes are equidistributed, the maximum value of $g(x)=|(x-x_0)(x-x_1)\cdots(x-x_N)|$ must be attained in (x_0,x_1) and (x_{N-1},x_N) (due to the symmetry). $|g(x^*)|\leqslant \frac14 N!h^{N+1}$.

Error estimation for equidistributed nodes. $|P_N(x)-f(x)|\leqslant \frac{h^{N+1}}{4(N+1)}\max_{\xi\in[a,b]}\left|f^{(N+1)}(\xi)\right|$, for all $x\in[a,b]$

Exercise 2. Let $P_n(x)$ be the LIP for $f(x)=\cos x$ with $x_k=kh,\quad k=0,1,\cdots,n$ where $h=\pi/(2n)$. **1.** Find a positive integer N such that $|P_N(x)-f(x)|<0.005,\quad$ for all $x\in[0,\pi/2]$.
Solution. For $f(x)=\cos(x)$, $\max_{\xi\in[0,\pi/2]}\left|f^{(N+1)}(\xi)\right|=1$. Hence it suffices to find $\frac{h^{N+1}}{4(N+1)}<0.005\implies N\geq 3$.

Lecture 9: Divided Differences

How to find the Lagrange polynomial. $P_n(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+\cdots+a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$ where $a_k=f[x_0,x_1,\cdots,x_k]$ and $f[x_0,x_1,\cdots,x_n]=\frac{f[x_1,x_2,\cdots,x_n]-f[x_0,x_1,\cdots,x_{n-1}]}{x_n-x_0}$. $a_0=f(x_0)$.

Lecture 10: Cubic Spline Interpolation (CSI)

How to find μ_k and λ_k. $\mu_k=\frac{x_k-x_{k-1}}{x_{k+1}-x_{k-1}},\quad \lambda_k=\frac{x_{k+1}-x_k}{x_{k+1}-x_{k-1}},\quad k=1,2,\cdots,n-1$

Natural Boundary Conditions. $M_0=M_n=0$.

$$\begin{bmatrix}2 & \lambda_1 & & & \\ \mu_2 & 2 & \lambda_2 & & \\ & \mu_3 & 2 & \ddots & \\ & & \ddots & \ddots & \lambda_{n-2} \\ & & & \mu_{n-1} & 2\end{bmatrix}\begin{bmatrix}M_1\\M_2\\M_3\\ \vdots\\ M_{n-2}\\ M_{n-1}\end{bmatrix}=\begin{bmatrix}6f[x_0,x_1,x_2]\\6f[x_1,x_2,x_3]\\6f[x_2,x_3,x_4]\\ \vdots\\ 6f[x_{n-3},x_{n-2},x_{n-1}]\\6f[x_{n-2},x_{n-1},x_n]\end{bmatrix}$$

Clamped Boundary Conditions. $2M_0+M_1=6f[x_0,x_0,x_1],\quad M_{n-1}+2M_n=6f[x_{n-1},x_n,x_n]$.

$$\begin{bmatrix}2 & \lambda_0 & & & \\ \mu_1 & 2 & \lambda_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-1} & 2 & \lambda_{n-1} \\ & & & \mu_n & 2\end{bmatrix}\begin{bmatrix}M_0\\M_1\\ \vdots\\ M_{n-1}\\ M_n\end{bmatrix}=\begin{bmatrix}6f[x_0,x_0,x_1]\\6f[x_0,x_1,x_2]\\ \vdots\\ 6f[x_{n-2},x_{n-1},x_n]\\6f[x_{n-1},x_n,x_n]\end{bmatrix}$$

How to find S_k.
$S_k(x)=M_{k-1}\frac{(x-x_k)^3}{6(x_{k-1}-x_k)}+M_k\frac{(x-x_{k-1})^3}{6(x_k-x_{k-1})}+A_kx+B_k$.
$A_k=\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}-\frac16(M_k-M_{k-1})(x_k-x_{k-1})$.
$B_k=\frac{x_kf(x_{k-1})-x_{k-1}f(x_k)}{x_k-x_{k-1}}+\frac16(M_kx_{k-1}-M_{k-1}x_k)(x_k-x_{k-1})$.

Piecewise Linear Interpolation (PLI). If $S(x)=S_k(x),\quad$ for $x\in[x_{k-1},x_k], k=1,2,\cdots,n$ then $S_k(x)=f(x_{k-1})\frac{x-x_k}{x_k-x_{k-1}}+f(x_k)\frac{x-x_{k-1}}{x_k-x_{k-1}}$

Error analysis for PLI on equidistributed nodes. If $x_k=x_0+kh$, then for $x\in[x_0,x_n]$, $|f(x)-S(x)|\leq\frac18h^2\max_{\xi\in[x_0,x_n]}|f''(\xi)|$

Tutorial 4: Divided Diff and CSI

Example 2: Quadratic spline interpolation. Given $n+1$ nodes $x_0<x_1<\cdots<x_{n-1}<x_n$ and a continuous function $f(x)$, find a function $S(x)$ such that **1.** $S(x)$ is first-order differentiable on (x_0,x_n) **2.** $S(x)$ is a quadratic polynomial on (x_{k-1},x_k) for any $k=2,3,\cdots,n$; **3.** $S(x)$ is a linear function on (x_0,x_1). **4.** $S(x_k)=f(x_k)$ for all $k=0,1,\cdots,n$.
Solution. $S_1(x)=f(x_0)\frac{x-x_1}{x_0-x_1}+f(x_1)\frac{x-x_0}{x_1-x_0}$.
$S_k(x)=\frac12M_k\frac{(x-x_{k-1})^2}{x_k-x_{k-1}}+\frac12M_{k-1}\frac{(x-x_k)^2}{x_{k-1}-x_k}+C_k,\quad k=2,3,\cdots,n$.
$C_k=f(x_k)-\frac12M_k(x_k-x_{k-1})$. $M_k=2f[x_{k-1},x_k]-M_{k-1}$. Since $M_1=S_1'(x_1)=f[x_0,x_1]$, M_k for $k=2,3,\cdots,n$ can be obtained iteratively.

Lecture 11: Least Squares Approximation

Proof of optimality. Suppose a satisfies $X^TX\mathbf{a}=X^T\mathbf{y}$. Then for any vector b with the same length as a, we have $(X\mathbf{b}-\mathbf{y})^T(X\mathbf{b}-\mathbf{y})\geqslant(X\mathbf{a}-\mathbf{y})^T(X\mathbf{a}-\mathbf{y})$. **Proof.**

$$(X\mathbf b-\mathbf y)^T(X\mathbf b-\mathbf y)-(X\mathbf a-\mathbf y)^T(X\mathbf a-\mathbf y)$$
$$=\mathbf b^TX^TX\mathbf b-2\mathbf b^TX^T\mathbf y-\mathbf a^TX^TX\mathbf a+2\mathbf a^TX^T\mathbf y$$
$$=\mathbf b^TX^TX\mathbf b-2\mathbf b^TX^TX\mathbf a-\mathbf a^TX^TX\mathbf a+2\mathbf a^TX^TX\mathbf a$$
$$=\mathbf b^TX^TX\mathbf b-2\mathbf b^TX^TX\mathbf a+\mathbf a^TX^TX\mathbf a=(\mathbf b-\mathbf a)^TX^TX(\mathbf b-\mathbf a)\geqslant 0$$

Finding the coefficients. Let $X=\begin{pmatrix}1&x_0&x_0^2&\cdots&x_0^n\\1&x_1&x_1^2&\cdots&x_1^n\\ \vdots&\vdots&\vdots&&\vdots\\1&x_m&x_m^2&\cdots&x_m^n\end{pmatrix}$ and $\mathbf a=(a_0,a_1,\cdots,a_n)^T,\quad \mathbf y=(y_0,y_1,\cdots,y_m)^T$. Solve $X^TXa=X^Ty$.

Lecture Exercise 1. Show that the least squares approximation is unique if and only if the matrix X has full column rank, *i.e.*, the rank of X equals its number of columns. **Proof.** Note rank$(X^TX)=$ rank(X). ($\implies$)If the LSA is unique, the columns are linearly independent since we can write Xa as a unique linear combination of the columns of X. Hence the rank of the matrix is equal to the number of columns.($\impliedby$) If $Xa=b$ and $Xa'=b$ then $X(a-a')=0$. Since X has full rank, $Xc=0$ iff $c=0\implies a=a'$ (only the trivial solution to the homogeneous equation of linear combination of its columns exists).

Weighted LSA. $W=\operatorname{diag}\{w_0,w_1,\cdots,w_n\}$. Solve $X^TWX\mathbf a=X^TW\mathbf y$.

Lecture 12: Newton-Cotes Formulae (NCF)

Trapezoidal Rule. $\int_a^bf(x)\mathrm dx\approx\frac{b-a}{2}[f(a)+f(b)]$.

Error for Trapezoidal Rule. $\int_a^bf(x)\mathrm dx=\frac{b-a}{2}[f(a)+f(b)]-\frac1{12}(b-a)^3f''(\xi)$

Simpson's Rule. Given $f(a), f\left(\frac{a+b}{2}\right)$ and $f(b)$, $P(x)=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)+\left[2f\left(\frac{a+b}2\right)-f(b)-f(a)\right]\frac{2(x-a)(x-b)}{(b-a)^2}$ whose integral is $\left[\frac23f\left(\frac{a+b}2\right)+\frac16f(b)+\frac16f(a)\right](b-a)$.

Error for Simpson's Rule. $\int_a^bf(x)\mathrm dx=\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}2\right)+f(b)\right]-\frac1{90}\left(\frac{b-a}2\right)^5f^{(4)}(\xi)$

Theorem 3 on Page 4. For closed Newton-Cotes formula with $n+1$ nodes, when n is odd and $f(x)$ is $(n+1)$-th order differentiable, there exists $\xi\in(a,b)$ such that

Gaussian Elimination to find weights. TODO

How to find degree of accuracy. If $\int_a^bx^jdx=\sum_{k=0}^nw_kx_k^j$ for $j=0,\cdots,n$ but $\int_a^bx^{n+1}dx\neq\sum_{k=0}^nw_kx_k^{n+1}$ then n is the degree of accuracy.

General form of NCF. $\int_a^b f(x)\mathrm{d}x \approx \int_a^b P(x)\mathrm{d}x = \sum_{k=0}^n f(x_k) \int_a^b L_k(x)\mathrm{d}x = \sum_{k=0}^n w_k f(x_k)$ where $w_k = \int_a^b L_k(x)\mathrm{d}x$

Result from Exercise 2.
$w_k = \frac{b-a}{n} \int_0^n \prod_{j=0, j\neq k}^n \frac{x-j}{k-j} dx, \quad \forall\ k = 0, \cdots, n.$

Lecture 13: Composite Numerical Integration

Composite Trapezoidal Rule (CTR). Confusing use of h. Read carefully first

Error analysis for CTR. TODO

Composite Simpson's Rule (CSR). TODO

Error analysis for CSR. TODO

Tutorial 5: LSA and Integration

Figuring out the data points used in LSA. If the $p(x)$ is given but the data points are incomplete, form the $X^T X a = X^T y$ system and select rows that look very similar to eliminate as many variables at once as possible.

Miscellaneous

The Gamma function. $\Gamma(x) = \int_0^{+\infty} t^{x-1}\mathrm{e}^{-t}\mathrm{d}t$ and $\Gamma(n) = (n-1)!$ for any positive integer n.

Linear Algebra and Calculus. TODO

General formulae for higher derivatives. Higher order derivatives

Error proofs from lectures/tutorials. LSA proof of optimality, the proof of errors, tutorial identities

Finding the set of x and y values for the lowest error LIP. First find the lowest-error polynomial (LEP), which is $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$. From Homework Q3,

$$M = \begin{pmatrix} b-a & \frac{b^2-a^2}{2} & \cdots & \frac{b^{m+1}-a^{m+1}}{m+1} \\ \frac{b^2-a^2}{2} & \frac{b^3-a^3}{3} & \cdots & \frac{b^{m+2}-a^{m+2}}{m+2} \\ \vdots & \vdots & & \vdots \\ \frac{b^{m+1}-a^{m+1}}{m+1} & \frac{b^{m+2}-a^{m+2}}{m+2} & \cdots & \frac{b^{2m+1}-a^{2m+1}}{2m+1} \end{pmatrix}$$ and

b $= \left(\int_a^b x^0 f(x)dx, \cdots, \int_a^b x^m f(x)dx\right)^T$. Solve $Ma = b$ to get the coefficients of the LEP. Then reverse-engineer the Gaussian Elimination process for finding a when constructing the LIP to get x and y (possibly non-unique).