MA2213 Cheatsheet 19/20 Sem 1 Finals

Lecture 8: Lagrange Interpolation

Solving via linear system.

$$\begin{bmatrix}1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n\end{bmatrix}\begin{bmatrix}a_0 \\ a_1 \\ \vdots \\ a_n\end{bmatrix} = \begin{bmatrix}f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n)\end{bmatrix}$$

Solving via basis polynomials. Let $L_k(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$ for $k = 0, 1, \cdots, n$. Then $P_n(x) = y_0L_0(x) + y_1L_1(x) + \cdots + y_nL_n(x)$.

Error analysis.
$\forall x \in [a, b], \exists \xi \in (\min\{x, x_0, x_1, \cdots, x_n\}, \max\{x, x_0, x_1, \cdots, x_n\})$ such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Tutorial 3: Lagrange Interpolation

Lecture 9: Divided Differences

How to find Lagrange polynomial.
$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$ where $a_k = f[x_0, x_1, \cdots, x_k]$ and $f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}$.
$a_0 = f(x_0)$.

Lecture 10: Cubic Spline Interpolation

How to find μ_k and λ_k.
$\mu_k = \frac{x_k - x_{k-1}}{x_{k+1}-x_{k-1}}, \quad \lambda_k = \frac{x_{k+1}-x_k}{x_{k+1}-x_{k-1}}, \quad k = 1, 2, \cdots, n-1$

Natural Boundary Conditions. $M_0 = M_n = 0$.

$$\begin{bmatrix}2 & \lambda_1 & & & \\ \mu_2 & 2 & \lambda_2 & & \\ & \mu_3 & 2 & \ddots & \\ & & \ddots & \ddots & \lambda_{n-2} \\ & & & \mu_{n-1} & 2\end{bmatrix}\begin{bmatrix}M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-2} \\ M_{n-1}\end{bmatrix} = \begin{bmatrix}6f[x_0, x_1, x_2] \\ 6f[x_1, x_2, x_3] \\ 6f[x_2, x_3, x_4] \\ \vdots \\ 6f[x_{n-3}, x_{n-2}, x_{n-1}] \\ 6f[x_{n-2}, x_{n-1}, x_n]\end{bmatrix}$$

Clamped Boundary Conditions.
$2M_0 + M_1 = 6f[x_0, x_0, x_1], \quad M_{n-1} + 2M_n = 6f[x_{n-1}, x_n, x_n]$.

$$\begin{bmatrix}2 & \lambda_0 & & & \\ \mu_1 & 2 & \lambda_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-1} & 2 & \lambda_{n-1} \\ & & & \mu_n & 2\end{bmatrix}\begin{bmatrix}M_0 \\ M_1 \\ \vdots \\ M_{n-1} \\ M_n\end{bmatrix} = \begin{bmatrix}6f[x_0, x_0, x_1] \\ 6f[x_0, x_1, x_2] \\ \vdots \\ 6f[x_{n-2}, x_{n-1}, x_n] \\ 6f[x_{n-1}, x_n, x_n]\end{bmatrix}$$

How to find S_k.
$S_k(x) = M_{k-1}\frac{(x-x_k)^3}{6(x_{k-1}-x_k)} + M_k\frac{(x-x_{k-1})^3}{6(x_k-x_{k-1})} + A_kx + B_k$.
$A_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} - \frac{1}{6}(M_k - M_{k-1})(x_k - x_{k-1})$.
$B_k = \frac{x_kf(x_{k-1}) - x_{k-1}f(x_k)}{x_k - x_{k-1}} + \frac{1}{6}(M_kx_{k-1} - M_{k-1}x_k)(x_k - x_{k-1})$.

Ascertaining number of points for error $< \epsilon$. TODO

Tutorial 4: Divided diff and CSI

Lecture 11: Least Squares Approximation

Finding the coefficients. Let

$$X = \begin{pmatrix}1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^n\end{pmatrix}$$ and

$\mathbf{a} = (a_0, a_1, \cdots, a_n)^T, \quad \mathbf{y} = (y_0, y_1, \cdots, y_m)^T$. Solve $X^TXa = X^Ty$.

Proof for Exercise 1. TODO

Weighted LSA. $W = \operatorname{diag}\{w_0, w_1, \cdots, w_n\}$. Solve $X^TWX\mathbf{a} = X^TW\mathbf{y}$.

Lecture 12: Newton-Cotes Formulae

Linear Interpolation. $P(x) = f(a)\frac{x-b}{b-a} + f(b)\frac{x-a}{b-a}$.

Error analysis for linear interpolation. TODO

Ascertaining number of points for error $< \epsilon$. TODO

Trapezoidal Rule. $\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{2}[f(a) + f(b)]$.

Error for Trapezoidal Rule.
$\int_a^b f(x)\mathrm{d}x = \frac{b-a}{2}[f(a) + f(b)] - \frac{1}{12}(b-a)^3f''(\xi)$

Simpson's Rule. Assume that three data points $f(a), f\left(\frac{a+b}{2}\right)$ and $f(b)$ are given.Then $P(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) + \left[2f\left(\frac{a+b}{2}\right) - f(b) - f(a)\right]\frac{2(x-a)(x-b)}{(b-a)^2}$ whose integral is $\left[\frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b) + \frac{1}{6}f(a)\right](b-a)$.

Error for Simpson's Rule.
$\int_a^b f(x)\mathrm{d}x = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi)$

Lecture 13: Composite Numerical Integration

Tutorial 5: LSA and Integration

Miscellaneous

The Gamma function. $\Gamma(x) = \int_0^{+\infty} t^{x-1}\mathrm{e}^{-t}\mathrm{d}t$ and $\Gamma(n) = (n-1)!$ for any positive integer n.