MA2108 Cheatsheet 19/20 Sem 1 Finals

Result 1. $2^{n-1} \leq n!$ and $n < 2^n \ \forall n \in \mathbb{N}$

Result 2. $n^2 \leq 2^n \ \forall n \in \mathbb{N} \backslash \{2,3,4\}$

Well-ordering principle. Every nonempty subset A of $\mathbb{N}$ has a least (first) element, i.e. there exists $p \in A$ such that $p \leq a$ for all $a \in A$

Lemma 1.5.1.

1. If $c > 1$, then $c^n > c$ for every natural number $n \geq 2$.
2. If $0 < c < 1$, then $c^n < c$ for every natural number $n \geq 2$.

Theorem 1.5.3. If $a \in \mathbb{R}$ is such that $0 \leq a < \varepsilon$ for every positive number ε, then $a = 0$

Bernoulli's Inequality. If $x > -1$, then $(1+x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}$

Harmonic mean. The *harmonic mean* of $a_1, a_2, \cdots, a_n$ is defined as $H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$.

The AM-GM-HM Inequality. $H \leq G \leq A$. Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Triangle Inequality. For $a, b \in \mathbb{R}, |a+b| \leq |a| + |b|$

Corollary 1.8.2.

$$||a| - |b|| \leq |a - b|$$
$$|a - b| \leq |a| + |b|$$

Lemma 1.9.1. Let u be an upper bound of $S \subseteq \mathbb{R}$. Then $u = \sup S$ if and only if $\forall \varepsilon > 0$, $\exists x_\varepsilon \in S$ such that $u - \varepsilon < x_\varepsilon$. (The infimum version of this statement was proven in Homework 2)

Supremum property of $\mathbb{R}$. Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum.

Archimedean Property. If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ such that $x < n_x$

Corollary 1.9.2. For any $\varepsilon > 0, \exists n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$

Corollary 1.9.3. If $x > 0$, then $\exists n \in \mathbb{N}$ such that $n - 1 \leq x < n$

Theorem 1.11.1. Let $a > 0$ and $n \in \mathbb{N}$. There exists a unique positive real number u with $u^n = a$. We call the number u the positive nth root of a and write $u = \sqrt[n]{a}$ or $a^{1/n}$.

Result. If $a > 0$ and $n, m \in \mathbb{N}$, then

$$\left(a^{1/n}\right)^m = (a^m)^{1/n}$$

Density Theorem of $\mathbb{Q}$. If $a, b \in \mathbb{R}$ is such that $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Corollary 1.12.1. If $a, b \in \mathbb{R}$ is such that $a < b$, then there exists an irrational number x such that $a < x < b$.

Corollary 1.12.2. Every interval $I \subseteq \mathbb{R}$ contains infinitely many rational numbers and infinitely many irrational numbers.

Theorem 2.1.1. If (x_n) converges, then it has exactly one limit.

Theorem 2.2.1. Every convergent sequence is bounded.

Theorem 2.2.2. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

1. $\lim_{n\to\infty}(x_n + y_n) = x + y$
2. $\lim_{n\to\infty}(x_n - y_n) = x - y$
3. $\lim_{n\to\infty}(x_n y_n) = xy$
4. $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$, provided $y_n \neq 0, \forall n \in \mathbb{N}$, and $y \neq 0$

Corollary 2.2.3. If (x_n) converges and $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k$$

Classic limit. $\lim_{n\to\infty} \frac{\sin n}{n} = 0$

Theorem 2.2.4. If $|x_n| \to 0$, then $x_n \to 0$.

Theorem 2.2.5. If $0 < b < 1$, then $\lim_{n\to\infty} b^n = 0$

Remark on Theorem 2.2.4 and 2.2.5. Theorems 2.2.4 and 2.2.5 together imply that $b^n \to 0$ for all b with $|b| < 1$

Theorem 2.2.6. If $c > 0$, then $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$

Theorem 2.2.7.

1. If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} |x_n| = |x|$
2. If all $x_n \geq 0$ and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$

Theorem 2.2.8. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

Theorem 2.2.9.

1. If $x_n \geq 0$ for all $n \in \mathbb{N}$ and (x_n) converges, then $\lim_{n\to\infty} x_n \geq 0$
2. If (x_n) and (y_n) are convergent and $x_n \geq y_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n$
3. If $a, b \in \mathbb{R}$ and $a \leq x_n \leq b$ for all n and (x_n) is convergent, then $a \leq \lim_{n\to\infty} x_n \leq b$

Monotone Convergence Theorem. If (x_n) is monotone and bounded, then it converges.

$$\lim_{n\to\infty} x_n = \begin{cases} \sup\{x_n : n \in \mathbb{N}\} & \text{if } x_n \uparrow \\ \inf\{x_n : n \in \mathbb{N}\} & \text{if } x_n \downarrow \end{cases}$$

Nested Interval Theorem. Let $I_n = [a_n, b_n], n \in \mathbb{N}$ be a nested sequence of closed bounded intervals, that is, $I_n \supseteq I_{n+1}$ for $n \in \mathbb{N}$. Then the intersection $\bigcap_{n=1}^{\infty} I_n = \{x : x \in I_n \forall n \in \mathbb{N}\}$ is nonempty. In addition, if length of $I_n = b_n - a_n \to 0$ then $\bigcap_{n=1}^{\infty} I_n$ contains exactly one point.

Theorem 2.4.1. If (x_n) converges to x, then any subsequence (x_{n_k}) also converges to x.

Corollary 2.4.2. If (x_n) has a subsequence which is divergent, then (x_n) diverges.

Corollary 2.4.3. If (x_n) has two convergent subsequences whose limits are not equal, then (x_n) diverges.

Monotone Subsequence Theorem. Every sequence has a monotone subsequence.

Bolzano-Weierstrass Theorem. Every bounded sequence has a convergent subsequence.

Lemma 2.5.1. Let $x \in \mathbb{R}$. Then there exists an increasing rational sequence (r_n) which con- verges to x.

Theorem 2.5.3. If $a \geq 1$ and (r_n) is a decreasing rational sequence with limit x, then $\lim_{n\to\infty} a^{r_n} = a^x$.

Theorem 2.5.4 (Properties of exponents).

1. $a^{x+y} = a^x a^y$
2. $(a^x)^y = a^{xy}$
3. If $a > 1$ and $x < y$, then $a^x < a^y$

Theorem 2.6.1. Let (x_n) be a bounded sequence and let $M = \limsup x_n$.

1. For each $\varepsilon > 0$, there are at most finitely many n' such that $x_n \geq M + \varepsilon$. Equivalently, there exists $K \in \mathbb{N}$ such that $n \geq K \Longrightarrow x_n < M + \varepsilon$.
2. For each $\varepsilon > 0$, there are infinitely many $n's$ such that $x_n > M - \varepsilon$.

Theorem 2.6.2. Let (x_n) be a bounded sequence and let $m = \liminf x_n$.

1. For each $\varepsilon > 0$, there are at most only finitely many n 's such that $x_n \leq m - \varepsilon$. Equivalently, there exists $K \in \mathbb{N}$ such that $n \geq K \Longrightarrow x_n > m - \varepsilon$.
2. For each $\varepsilon > 0$, there are infinitely many n' s such that $x_n < m + \varepsilon$.

Theorem 2.6.3. Let (x_n) be a bounded sequence. Then (x_n) converges if and only if $\limsup x_n = \liminf x_n$.

Theorem 2.6.4. Let (x_n) and (y_n) be bounded sequence such that $x_n \leq y_n$ for every $n \in \mathbb{N}$. Then $\limsup x_n \leq \limsup y_n$ and $\liminf x_n \leq \liminf y_n$.

Definition of a Cauchy sequence. A sequence (x_n) is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon, \quad \forall n, m \geq K$.

Theorem 2.7.1. Every convergent sequence is Cauchy.

Cauchy criterion. Every Cauchy sequence is convergent (and thus bounded).

Contractive sequences. A sequence (x_n) is said to be contractive if $\exists C$ with $0 < C < 1$ such that $|x_{n+2} - x_{n+1}| \leq C\,|x_{n+1} - x_n|, \quad \forall n \in \mathbb{N}$.

Theorem 2.7.3. Every contractive sequence is Cauchy (and so is convergent).

Partial fraction. $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

Theorem 3.1.1. If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then the series $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$.

Theorem 3.1.2. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The n-th term divergence test. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Cauchy criterion for series. The series $\sum_{n=1}^{\infty} a_n$ converges if and only iffor every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $|a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon, \quad \forall m > n \geq K$.

Theorem 3.2.1. If $a_n \geq 0$ for all n, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence (s_n) of partial sums is bounded.

Theorem 3.2.2. If $p > 1$, then the p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Theorem 3.2.3. If $0 < p \leq 1$, then the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Comparison Test. Suppose that $0 \leq a_n \leq b_n, \quad \forall n \geq K$ for some $K \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges. $\sum_{n=1}^{\infty} a_n$ diverges $\Longrightarrow \sum_{n=1}^{\infty} b_n$ diverges.

Limit Comparison Test. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with positive terms. Suppose $\rho = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists. If $\rho > 0$, then either the two series both converge or both diverge. If $\rho = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Alternating Series Test. If (a_n) is a decreasing sequence such that $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Theorem 3.4.1. If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

Ratio Test. Suppose that all the terms of the series $\sum_{n=1}^{\infty} a_n$ are nonzero and the limit $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists. If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. No conclusion if $\rho = 1$.

Ross pg. 79. $\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$.

Proof. Let $\alpha = \limsup |s_n|^{1/n}$ and $L = \limsup \left|\frac{s_{n+1}}{s_n}\right|$. It suffices to show $\alpha \leq L_1$ for any $L_1 > L$. $\exists K$ such that $\sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n \geq K\right\} < L_1$. Thus $\left|\frac{s_{n+1}}{s_n}\right| < L_1$ for $n \geq K$, and we write $|s_n| = \left|\frac{s_n}{s_{n-1}}\right| \cdot \left|\frac{s_{n-1}}{s_{n-2}}\right| \cdots \left|\frac{s_{K+1}}{s_K}\right| \cdot |s_K|$, such that $|s_n| < L_1^{n-K} |s_K|$. Let $a = L_1^{-N} |s_N| > 0$. Then $|s_n|^{1/n} < L_1 a^{1/n} \to L_1$ for $n > K$. So $\alpha = \limsup |s_n|^{1/n} \leq L_1$.

Result from Ch 3 Pg 16. If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Result from Ch 3 Pg 17. Let $\rho = \limsup |a_n|^{1/n}$. If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. No conclusion if $\rho = 1$.

Proof of Ch 3 Pg 16. $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \limsup |a_n|^{1/n} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$. Prove using Pg 17.

Root Test. Suppose $\rho = \lim_{n\to\infty} |a_n|^{1/n}$ exists. If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. No conclusion if $\rho = 1$.

Theorem 3.6.1. If the series $\sum_{n=1}^{\infty} a_n$ converges, then any series obtained by grouping the terms of $\sum_{n=1}^{\infty} a_n$ also converges and has the same value as $\sum_{n=1}^{\infty} a_n$.

Theorem 3.7.1. If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any rearrangement of $\sum_{n=1}^{\infty} a_n$ also converges and has the same sum as $\sum_{n=1}^{\infty} a_n$.

Theorem 3.8.1. $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ and for each $n \in \mathbb{N}, e - \sum_{j=0}^{n} \frac{1}{j!} < \frac{1}{n(n!)}$

Theorem 3.8.2. e is irrational.

Sequential criterion. $\lim_{x\to a} f(x) = L \Longleftrightarrow$ If (x_n) is any sequence in the domain of f such that $x_n \neq a$ for all n and $x_n \to a$, then $f(x_n) \to L$. Note that L and a can be infinity.

Corollary 4.2.2. $\lim_{x\to a} f(x) \neq L \Longleftrightarrow$ there is a sequence (x_n) in the domain of f such that $x_n \neq a$ for all n and $x_n \to a$, but $f(x_n) \not\to L$

Divergent Criterion. Method 1. Find a sequence (x_n) in the domain of f such that $x_n \neq a$ for all n and $x_n \to a$, but $(f(x_n))$ diverges. **Method 2**. Find two sequences (x_n) and (y_n) in the domain of f such that $x_n \neq a$ and $y_n \neq a$ for all n and $x_n \to a, y_n \to a$, but $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$.

Dirichlet Function. $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ is continuous nowhere.

Lemma 4.2.3. There exists a sequence (x_n) such that x_n is rational for all $n, x_n \neq c$ for all n and $x_n \to c \in \mathbb{R}$, and a sequence (y_n) such that y_n is irrational for all $n, y_n \neq c$ for all n and $y_n \to c$

Theorem 4.3.1. Suppose f is defined in a deleted neighborhood of $x = a$. If $\lim_{x\to a} f(x)$ exists, then f is bounded in a deleted neighborhood of $x = a$,that is, $\exists M > 0$ and $\delta > 0$ such that $0 < |x - a| < \delta \Longrightarrow |f(x)| \leq M$.

Theorem 4.3.2. Limit laws apply to functions.

Theorem 4.3.3. If $f(x) \leq g(x)$ for all x in a deleted neighborhood of $x = a$ and both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

Theorem 4.3.4. If f is defined in a deleted neighborhood of $x = a$ and $\lim_{x\to a} f(x) = L$ exists and $L > 0$, then $\exists \delta > 0$ such that $f(x) > 0 \quad \forall x$ such that $0 < |x - a| < \delta$

Functions proven to meet seq crit. Polynomial, abs, sqrt, x^r, a^x, sin and cos, rational functions (f(x)/q(x)),

Thomae's function. Let $f : (0, 1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, p, q \in \mathbb{N} \text{ and } \gcd(p, q) = 1 \end{cases}$$

which is discontinuous at all rational points and continuous at all irrational points.

Theorem 5.2.3. Theorem 5.2.3. Suppose that $f : A \to \mathbb{R}, g : B \to \mathbb{R}$ and $f(A) \subseteq B$, so that $g \circ f$ is defined. If f is continuous on A, and g is continuous on B, then $g \circ f$ is continuous on A.

Remarks on continuous functions.. $\sqrt{\sin x}$ is continuous on $(0, \pi)$.

Theorem 5.3.1. If f is continuous on $[a, b]$, then f is bounded on $[a, b]$

Extreme-value Theorem.. If f is continuous on $[a, b]$, then there exists $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a, b]$

Location of Roots Theorem. If f is continuous on $[a, b], f(a) < 0 < f(b)$, then there exists a point c, in (a, b) such that $f(c) = 0$.

Intermediate Value Theorem. If f is continuous on $[a, b]$, and k is between $f(a)$ and $f(b)$, then there exists a point c in (a, b) such that $f(c) = k$.

Theorem 5.3.2. If f is continuous on $[a, b]$, then $f([a, b]) = [m, M]$, where $m = \inf f([a, b])$ and $M = \sup f([a, b])$

Theorem 5.4.1. Let I $\subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be an increasing function. If $c \in I$ is not an end point of I, then $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and they are given by $\lim_{x\to c^-} f(x) = \sup\{f(x) : x \in I, x < c\}$ and $\lim_{x\to c^+} f(x) = \inf\{f(x) : x \in I, x > c\}$

Continuous Inverse Theorem. Let I $\subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a strictly monotone function. If f is continuous on I and $J = f(I)$, then its inverse function $f^{-1} : J \to \mathbb{R}$ is strictly monotone and continuous on J.

Uniform Continuity. $x, y \in I, |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$.

Sequential Criterion for Uniform Continuity. The function $f : I \to \mathbb{R}$ is uniformly continuous on I if and only if for any two sequences (x_n) and (y_n) in I such that $x_n - y_n \to 0$, we have $f(x_n) - f(y_n) \to 0$. Corollary: f not continuous on I if you find two sequences (x_n) and (y_n) in I such that $x_n - y_n \to 0$ but $f(x_n) - f(y_n) \to 0$.

Theorem 5.5.3. If f is continuous on a closed bounded interval $[a, b]$, then it is uniformly continuous on $[a, b]$.

Theorem 5.5.4. If I is an interval and $f : I \to \mathbb{R}$ satisfies the Lipschitz condition on I, that is, there is a $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|, \quad \forall x, y \in I$$

then f is uniformly continuous on I.

Theorem 5.5.5. If $f : I \to$ R is uniformly continuous on I and (x_n) is a Cauchy sequence in I, then $(f(x_n))$ is a Cauchy sequence.

Theorem 5.5.6. If the function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b), then $f(a)$ and $f(b)$ can be defined so that the extended function is continuous on $[a, b]$. Take a sequence (x_n) in (a, b) such that $x_n \to a$. Define $f(a) = \lim_{n\to\infty} f(x_n)$. Same for b.