MA2213 AY19/20 Semester 1 Finals

Lecture 8: Lagrange Interpolation

Solving via linear system.

$$\begin{bmatrix}1 & x_0 & x_0^2 & \cdots & x_0^n\\ 1 & x_1 & x_1^2 & \cdots & x_1^n\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_n & x_n^2 & \cdots & x_n^n\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ \vdots\\ a_n\end{bmatrix}=\begin{bmatrix}f(x_0)\\ f(x_1)\\ \vdots\\ f(x_n)\end{bmatrix}$$

Solving via basis polynomials. Let $L_k(x)=\frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$ for $k=0,1,\cdots,n$. Then $P_n(x)=y_0L_0(x)+y_1L_1(x)+\cdots+y_nL_n(x)$.

Error analysis. $\forall x\in[a,b],\exists\xi\in(\min\{x,x_0,x_1,\cdots,x_n\},\max\{x,x_0,x_1,\cdots,x_n\})$ such that

$$f(x)=P_n(x)+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Tutorial 3: Lagrange Interpolation

Example 1. Construct the LIP for $f(x)=\log_2 x$ using $x_0=1/2,x_1=1,x_2=2,x_3=4$. Find a bound of the absolute error for any $x\in[1/2,+\infty)$.
Solution. $f(0.5)=-1,\quad f(1)=0,\quad f(2)=1,\quad f(4)=2$. Solve using linear system to get $a_0=-\frac{52}{21}$, $a_1=\frac{7}{2}$, $a_2=-\frac{7}{6}$, $a_3=\frac{1}{7}$. So $P(x)=\frac{1}{7}x^3-\frac{7}{6}x^2+\frac{7}{2}x-\frac{52}{21}$. To get an u.b for error, first find $f^{(4)}(x)=-\frac{6}{x^4\ln 2}$. Note monotonicity. Hence $\left|f^{(4)}(x)\right|\leqslant\frac{6}{(1/2)^4\ln 2}=\frac{96}{\ln 2}$. Thus an u.b for absolute error $|P(x)-f(x)|$ is $\frac{1}{4!}\times\frac{96}{\ln 2}|(x-1/2)(x-1)(x-2)(x-4)|=\frac{4}{\ln 2}|(x-1/2)(x-1)(x-2)(x-4)|$. $\square$.

Result from Example 2. If nodes are equidistributed, the maximum value of $g(x)=|(x-x_0)(x-x_1)\cdots(x-x_N)|$ must be attained in (x_0,x_1) and (x_{N-1},x_N) (due to the symmetry). $|g(x^*)|\leqslant\frac{1}{4}N!h^{N+1}$.

Error estimation for equidistributed nodes. $|P_N(x)-f(x)|\leqslant\frac{h^{N+1}}{4(N+1)}\max_{\xi\in[a,b]}\left|f^{(N+1)}(\xi)\right|$, for all $x\in[a,b]$

Exercise 2. Let $P_n(x)$ be the LIP for $f(x)=\cos x$ with $x_k=kh,\quad k=0,1,\cdots,n$ where $h=\pi/(2n)$. **1.** Find a positive integer N such that $|P_N(x)-f(x)|<0.005,\quad$ for all $x\in[0,\pi/2]$.
Solution. For $f(x)=\cos(x)$, $\max_{\xi\in[0,\pi/2]}\left|f^{(N+1)}(\xi)\right|=1$. Hence it suffices to find $\frac{h^{N+1}}{4(N+1)}<0.005\implies N\geq 3$.

Lecture 9: Divided Differences

How to find the Lagrange polynomial. $P_n(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+\cdots+a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$ where $a_k=f[x_0,x_1,\cdots,x_k]$ and $f[x_0,x_1,\cdots,x_n]=\frac{f[x_1,x_2,\cdots,x_n]-f[x_0,x_1,\cdots,x_{n-1}]}{x_n-x_0}$. $a_0=f(x_0)$.

Lecture 10: Cubic Spline Interpolation (CSI)

How to find μ_k and λ_k. $\mu_k=\frac{x_k-x_{k-1}}{x_{k+1}-x_{k-1}},\quad\lambda_k=\frac{x_{k+1}-x_k}{x_{k+1}-x_{k-1}},\quad k=1,2,\cdots,n-1$

Natural Boundary Conditions. $M_0=M_n=0$.

$$\begin{bmatrix}2 & \lambda_1 & & & \\ \mu_2 & 2 & \lambda_2 & & \\ & \mu_3 & 2 & \ddots & \\ & & \ddots & \ddots & \lambda_{n-2}\\ & & & \mu_{n-1} & 2\end{bmatrix}\begin{bmatrix}M_1\\ M_2\\ M_3\\ \vdots\\ M_{n-2}\\ M_{n-1}\end{bmatrix}=\begin{bmatrix}6f[x_0,x_1,x_2]\\ 6f[x_1,x_2,x_3]\\ 6f[x_2,x_3,x_4]\\ \vdots\\ 6f[x_{n-3},x_{n-2},x_{n-1}]\\ 6f[x_{n-2},x_{n-1},x_n]\end{bmatrix}$$

Clamped Boundary Conditions. $2M_0+M_1=6f[x_0,x_0,x_1],\quad M_{n-1}+2M_n=6f[x_{n-1},x_n,x_n]$.

$$\begin{bmatrix}2 & \lambda_0 & & & \\ \mu_1 & 2 & \lambda_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-1} & 2 & \lambda_{n-1}\\ & & & \mu_n & 2\end{bmatrix}\begin{bmatrix}M_0\\ M_1\\ \vdots\\ M_{n-1}\\ M_n\end{bmatrix}=\begin{bmatrix}6f[x_0,x_0,x_1]\\ 6f[x_0,x_1,x_2]\\ \vdots\\ 6f[x_{n-2},x_{n-1},x_n]\\ 6f[x_{n-1},x_n,x_n]\end{bmatrix}$$

How to find S_k.
$S_k(x)=M_{k-1}\frac{(x-x_k)^3}{6(x_{k-1}-x_k)}+M_k\frac{(x-x_{k-1})^3}{6(x_k-x_{k-1})}+A_kx+B_k$.
$A_k=\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}-\frac{1}{6}(M_k-M_{k-1})(x_k-x_{k-1})$.
$B_k=\frac{x_kf(x_{k-1})-x_{k-1}f(x_k)}{x_k-x_{k-1}}+\frac{1}{6}(M_kx_{k-1}-M_{k-1}x_k)(x_k-x_{k-1})$.

Piecewise Linear Interpolation (PLI). If $S(x)=S_k(x),\quad$ for $x\in[x_{k-1},x_k],k=1,2,\cdots,n$ then $S_k(x)=f(x_{k-1})\frac{x-x_k}{x_{k-1}-x_k}+f(x_k)\frac{x-x_{k-1}}{x_k-x_{k-1}}$

Error analysis for PLI on equidistributed nodes. If $x_k=x_0+kh$, then for $x\in[x_0,x_n]$, $|f(x)-S(x)|\leq\frac{1}{8}h^2\max_{\xi\in[x_0,x_n]}|f''(\xi)|$

Tutorial 4: Divided Diff and CSI

Example 2: Quadratic spline interpolation. Given $n+1$ nodes $x_0<x_1<\cdots<x_{n-1}<x_n$ and a continuous function $f(x)$, find a function $S(x)$ such that **1.** $S(x)$ is first-order differentiable on (x_0,x_n) **2.** $S(x)$ is a quadratic polynomial on (x_{k-1},x_k) for any $k=2,3,\cdots,n$; **3.** $S(x)$ is a linear function on (x_0,x_1). **4.** $S(x_k)=f(x_k)$ for all $k=0,1,\cdots,n$.
Solution. $S_1(x)=f(x_0)\frac{x-x_1}{x_0-x_1}+f(x_1)\frac{x-x_0}{x_1-x_0}$.
$S_k(x)=\frac{1}{2}M_k\frac{(x-x_{k-1})^2}{x_k-x_{k-1}}+\frac{1}{2}M_{k-1}\frac{(x-x_k)^2}{x_{k-1}-x_k}+C_k,\quad k=2,3,\cdots,n$.
$C_k=f(x_k)-\frac{1}{2}M_k(x_k-x_{k-1})$. $M_k=2f[x_{k-1},x_k]-M_{k-1}$. Since $M_1=S_1'(x_1)=f[x_0,x_1]$, M_k for $k=2,3,\cdots,n$ can be obtained iteratively.

Lecture 11: Least Squares Approximation

Proof of optimality. Suppose a satisfies $X^TX\mathbf{a}=X^T\mathbf{y}$. Then for any vector b with the same length as a, we have $(X\mathbf{b}-\mathbf{y})^T(X\mathbf{b}-\mathbf{y})\geqslant(X\mathbf{a}-\mathbf{y})^T(X\mathbf{a}-\mathbf{y})$. **Proof.**

$$\begin{aligned}&(X\mathbf{b}-\mathbf{y})^T(X\mathbf{b}-\mathbf{y})-(X\mathbf{a}-\mathbf{y})^T(X\mathbf{a}-\mathbf{y})\\ =&\mathbf{b}^TX^TX\mathbf{b}-2\mathbf{b}^TX^T\mathbf{y}-\mathbf{a}^TX^TX\mathbf{a}+2\mathbf{a}^TX^T\mathbf{y}\\ =&\mathbf{b}^TX^TX\mathbf{b}-2\mathbf{b}^TX^TX\mathbf{a}-\mathbf{a}^TX^TX\mathbf{a}+2\mathbf{a}^TX^TX\mathbf{a}\\ =&\mathbf{b}^TX^TX\mathbf{b}-2\mathbf{b}^TX^TX\mathbf{a}+\mathbf{a}^TX^TX\mathbf{a}=(\mathbf{b}-\mathbf{a})^TX^TX(\mathbf{b}-\mathbf{a})\geqslant 0\end{aligned}$$

Finding the coefficients. Let $X=\begin{pmatrix}1 & x_0 & x_0^2 & \cdots & x_0^n\\ 1 & x_1 & x_1^2 & \cdots & x_1^n\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & x_m & x_m^2 & \cdots & x_m^n\end{pmatrix}$ and $\mathbf{a}=(a_0,a_1,\cdots,a_n)^T,\quad\mathbf{y}=(y_0,y_1,\cdots,y_m)^T$. Solve $X^TXa=X^Ty$.

Lecture Exercise 1. Show that the least squares approximation is unique if and only if the matrix X has full column rank, *i.e.*, the rank of X equals its number of columns. **Proof.** Note $\operatorname{rank}(X^TX)=\operatorname{rank}(X)$. ($\implies$)If the LSA is unique, the columns are linearly independent since we can write Xa as a unique linear combination of the columns of X. Hence the rank of the matrix is equal to the number of columns.($\impliedby$) If $Xa=b$ and $Xa'=b$ then $X(a-a')=0$. Since X has full rank, $Xc=0$ iff $c=0\implies a=a'$ (only the trivial solution to the homogeneous equation of linear combination of its columns exists).

Weighted LSA. $W=\operatorname{diag}\{w_0,w_1,\cdots,w_n\}$. Solve $X^TWX\mathbf{a}=X^TW\mathbf{y}$.

Lecture 12: Newton-Cotes Formulae (NCF)

Trapezoidal Rule. $\int_a^b f(x)\mathrm{d}x\approx\frac{b-a}{2}[f(a)+f(b)]$.

Error for Trapezoidal Rule. $\int_a^b f(x)\mathrm{d}x=\frac{b-a}{2}[f(a)+f(b)]-\frac{1}{12}(b-a)^3f''(\xi)$

Simpson's Rule. Given $f(a),f\left(\frac{a+b}{2}\right)$ and $f(b)$, $P(x)=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)+\left[2f\left(\frac{a+b}{2}\right)-f(b)-f(a)\right]\frac{2(x-a)(x-b)}{(b-a)^2}$ whose integral is $\left[\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)+\frac{1}{6}f(a)\right](b-a)$.

Error for Simpson's Rule. $\int_a^b f(x)\mathrm{d}x=\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{90}\left(\frac{b-a}{2}\right)^5f^{(4)}(\xi)$

Theorem 3 on Page 4. For closed Newton-Cotes formula with $n+1$ nodes, when n is odd and $f(x)$ is $(n+1)$ -th order differentiable, there exists $\xi\in(a,b)$ such that $\int_a^b f(x)\mathrm{d}x=\sum_{k=0}^n w_kf(x_k)+\frac{h^{n+2}f^{(n+1)}(\xi)}{(n+1)!}\int_0^n s(s-1)\cdots(s-n)\mathrm{d}s$; when n is even and $f(x)$ is $(n+2)$-th order differentiable, there exists $\xi\in(a,b)$ such that $\int_a^b f(x)\mathrm{d}x=\sum_{k=0}^n w_kf(x_k)+\frac{h^{n+3}f^{(n+2)}(\xi)}{(n+2)!}\int_0^n s^2(s-1)\cdots(s-n)\mathrm{d}s$.

Gaussian Elimination to find weights.
$\int_a^b x^j dx = \sum_{k=0}^n w_k x_k^j, \quad j = 0, 1, \cdots, n$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1} \\ x_0^n & x_1^n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \\ w_n \end{bmatrix} = \begin{bmatrix} \int_a^b dx \\ \int_a^b x dx \\ \vdots \\ \int_a^b x^{n-1} dx \\ c^b x^n dx \end{bmatrix}$$

How to find degree of accuracy. If $\int_a^b x^j dx = \sum_{k=0}^n w_k x_k^j$ for $j = 0, \cdots, n$ but $\int_a^b x^{n+1} dx \neq \sum_{k=0}^n w_k x_k^{n+1}$ then n is the degree of accuracy.

General form of NCF. $\int_a^b f(x) \mathrm{d}x \approx \int_a^b P(x) \mathrm{d}x = \sum_{k=0}^n f(x_k) \int_a^b L_k(x) \mathrm{d}x = \sum_{k=0}^n w_k f(x_k)$ where $w_k = \int_a^b L_k(x) \mathrm{d}x$

Result from Exercise 2.
$w_k = \frac{b-a}{n} \int_0^n \prod_{j=0, j\neq k}^n \frac{x-j}{k-j} dx, \quad \forall\, k = 0, \cdots, n.$

Lecture 13: Composite Numerical Integration

Composite Trapezoidal Rule (CTR). Assume $x_k - x_{k-1} = h$ for all $k = 1, 2, \cdots, n$. Then
$\int_a^b S(x) \mathrm{d}x = \sum_{k=1}^n \frac{h}{2} [f(x_{k-1}) + f(x_k)] = h\left[\frac{1}{2} f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2} f(x_n)\right]$

Error analysis for CTR. Suppose $f(x)$ is second-order continuously differentiable on $[a, b]$, and $h = \frac{b-a}{\infty}, \quad x_k = kh, \quad k = 0, 1, \cdots, n$. There exists $\xi \in (a, b)$ such that
$\int_a^b f(x) \mathrm{d}x = h\left[\frac{1}{2} f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2} f(x_n)\right] - \frac{b-a}{12} h^2 f''(\xi).$

Composite Simpson's Rule (CSR).

$$\begin{aligned} \int_a^b f(x) \mathrm{d}x &\approx \sum_{k=1}^{n/2} \frac{h}{3} [f(x_{2k-2}) + 4 f(x_{2k-1}) + f(x_{2k})] \\ &= \frac{h}{3} \left(f(a) + f(b) + 2 \sum_{k=1}^{n/2-1} f(x_{2k}) + 4 \sum_{k=1}^{n/2} f(x_{2k-1}) \right) \end{aligned}$$

Error analysis for CSR. Suppose $f(x)$ is fourth-order continuously differentiable on $[a, b]$, and

$$h = \frac{b-a}{n}, \quad x_k = kh, \quad k = 0, 1, \cdots, n$$

When n is an even integer, there exists $\xi \in (a, b)$ such that
$\int_a^b f(x) \mathrm{d}x = \frac{h}{3}\left(f(a) + f(b) + 2\sum_{k=1}^{n/2-1} f(x_{2k}) + 4\sum_{k=1}^{n/2} f(x_{2k-1})\right) - \frac{b-a}{180} h^4 f^{(4)}(\xi)$

Achieve abs err $< \epsilon$. Approximate $\int_0^2 \sqrt{1+x^2} dx$ such that the absolute error is less than 10^{-3}. For composite trapezoidal rule, $b = 2, a = 0$, abs.err $\leq \frac{1}{6}(\frac{2}{n})^2 \max_{\xi \in [0,2]} |f''(\xi)| = \frac{1}{6}(\frac{2}{n})^2$. Hence $n \geq 26 \implies n = 26$. For composite Simpson's rule, abs.err $\leq \frac{1}{90}(\frac{2}{n})^4 \max_{\xi \in [0,2]} |f^{(4)}(\xi)| = \frac{8}{15n^4}$. Hence $n \geq 4.8 \implies n = 6$ because **we need n to be even.**

Tutorial 5: LSA and Integration

Q4. Consider the following numerical integration: $\int_{-1}^1 f(x) \mathrm{d}x \approx w_1 f(x_1) + w_2 f(x_2)$. How to choose x_1, x_2 and w_1, w_2 to achieve maximum degree of accuracy? **Solution.** For 4 unknowns, set 4 equations: set $f(x)$ to be $1, x, x^2, x^3$ and assume that the numerical integration is exact. Then we get equations 1. $w_1 + w_2 = \int_{-1}^1 1 \mathrm{d}x = 2$, 2. $w_1 x_1 + w_2 x_2 = \int_{-1}^1 x \mathrm{d}x = 0$, 3. $w_1 x_1^2 + w_2 x_2^2 = \int_{-1}^1 x^2 \mathrm{d}x = \frac{2}{3}$, 4. $w_1 x_1^3 + w_2 x_2^3 = \int_{-1}^1 x^3 \mathrm{d}x = 0$. Solve to get $w_1 = w_2 = 1, x_1 = -x_2 = \frac{1}{\sqrt{3}}$. Degree of accuracy is found to be 3.

Miscellaneous

Figuring out the data points used in LSA. If the $p(x)$ is given but the data points are incomplete, form the $X^T X a = X^T y$ system and select rows that look very similar to eliminate as many variables at once as possible.

The Gamma function. $\Gamma(x) = \int_0^{+\infty} t^{x-1} \mathrm{e}^{-t} \mathrm{d}t$ and $\Gamma(n) = (n-1)!$ for any positive integer n.

Linear Algebra and Calculus.
$\frac{d}{dt} \int_{a(t)}^{b(t)} q(x) dx = q(b(t)) b'(t) - q(a(t)) a'(t)$. To add LSA lemmas...

Result from T3 Eg. 2. For $f(x) = 1/\sqrt{x}$, $f^{(N+1)}(x) = \left(-\frac{1}{2}\right)^{N+1} (2N+1)!! x^{-N-3/2}$.

Error proofs from lectures/tutorials. Check the lecture error proofs

Tut 4 Eg. 1. Suppose $f(x)$ is nth order differentiable and $x_0, x_1, \cdots, x_n$ are $n+1$ distinct real numbers. Show that there exists ξ between $\min\{x_0, x_1, \cdots, x_n\}$ and $\max\{x_0, x_1, \cdots, x_n\}$ such that $f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}$.. **Solution.** Since the divided difference is independent of the order of its parameters, we can assume that $x_0 < x_1 < \cdots < x_n$. Let $P_n(x)$ be the Lagrange interpolating polynomial of $f(x)$ with nodes $x_0, x_1, \cdots, x_n$. Then

$$\begin{aligned} P_n(x) =& f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ \cdots + f[x_0, x_1, x_2, \cdots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}) \end{aligned}.$$

The n th derivative of $P_n(x)$ is $P_n^{(n)}(x) = n! f[x_0, x_1, x_2, \cdots, x_n]$. Now we define $g(x) = f(x) - P_n(x)$. By the definition of interpolation, we know that $g(x_0) = g(x_1) = \cdots = g(x_n) = 0$. By Rolle's theorem, there exists $\xi \in (x_0, x_n)$ such that $g^{(n)}(\xi) = 0$. since $g^{(n)}(\xi) = f^{(n)}(\xi) - P_n^{(n)}(\xi) = f^{(n)}(\xi) - n! f[x_0, x_1, \cdots, x_n]$ we prove the claim by equating the above equation to zero.

Tut 5 Eg. 1. Show that when f is second-order differentiable on $[a, b]$, there exists $\xi \in (a, b)$ such that $\int_a^b f(x) \mathrm{d}x - (b-a) f\left(\frac{a+b}{2}\right) = \frac{(b-a)^3}{24} f''(\xi)$. **Solution.** Let $\zeta = (a+b)/2$ and define $g(t) = \int_{\zeta-t}^{\zeta+t} f(x) \mathrm{d}x - 2t f(\zeta), \quad h(t) = g(t) - \left(\frac{2t}{b-a}\right)^3 g\left(\frac{b-a}{2}\right)$. It is obvious that $h(0) = h\left(\frac{b-a}{2}\right) = 0$. By Rolle's theorem, we can find $t_0 > 0$ such that $h'(t_0) = 0$, i.e., $g'(t_0) = 3t_0^2 \left(\frac{2}{b-a}\right)^3 g\left(\frac{b-a}{2}\right)$. Note that $g'(t_0) = f(\zeta + t_0) + f(\zeta - t_0) - 2f(\zeta) = 2t_0^2 f[\zeta - t_0, \zeta, \zeta + t_0]$. Therefore we can find $\xi \in (\zeta - t_0, \zeta + t_0) \subset (a, b)$ such that $f[\zeta - t_0, \zeta, \zeta + t_0] = \frac{1}{2} f''(\xi)$. Do some substitutions.

Finding the set of x and y values for the lowest error LIP. First find the lowest-error polynomial (LEP), which is $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$. From Homework Q3,

$$M = \begin{pmatrix} b-a & \frac{b^2-a^2}{2} & \cdots & \frac{b^{m+1}-a^{m+1}}{m+1} \\ \frac{b^2-a^2}{2} & \frac{b^3-a^3}{3} & \cdots & \frac{b^{m+2}-a^{m+2}}{m+2} \\ \vdots & \vdots & & \vdots \\ \frac{b^{m+1}-a^{m+1}}{m+1} & \frac{b^{m+2}-a^{m+2}}{m+2} & \cdots & \frac{b^{2m+1}-a^{2m+1}}{2m+1} \end{pmatrix} \text{ and}$$

b $= \left(\int_a^b x^0 f(x) dx, \cdots, \int_a^b x^m f(x) dx\right)^T$. Solve $Ma = b$ to get the coefficients of the LEP. Then reverse-engineer the Gaussian Elimination process for finding a when constructing the LIP to get x and y (possibly non-unique).