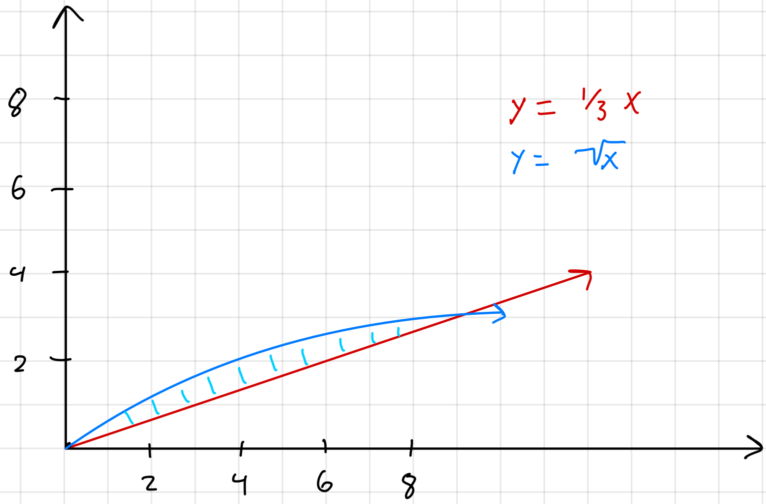
Question 32:

A.



B.

Finding the interval:

$y = \frac{1}{3}x$
$y = \sqrt{x}$ $\Rightarrow$ $\sqrt{x} = \frac{1}{3}x$

$\Rightarrow$ $x = \frac{x^2}{9}$

$\Rightarrow$ $x^2 - 9x = 0$

$\Rightarrow$ $x(x-9) = 0$

$\Rightarrow$ $x = 0$ or $x = 9$

Hence the interval we are integrating on is $[0, 9]$

When integrating by vertical slices the area is

$$A = \int (\text{Top}(x) - \text{Bottom}(x))\, dx$$

$$A = \int_0^9 \left(\sqrt{x} - \frac{1}{3}x\right) dx = \frac{2}{3}x^{3/2} - \frac{1}{6}x^2 \Big|_0^9$$

$$= \left[\frac{2}{3}(9)^{3/2} - \frac{1}{6}(9)^2\right] - \left[\frac{2}{3}(0)^{3/2} - \frac{1}{6}(0)^2\right]$$

$$= \frac{9}{2}$$

Problem does not ask you to integrate this but in case you want an extra example

C. Finding the y interval

$y = \frac{1}{3}x$
$y = \sqrt{x}$ $\Rightarrow$ $x = 3y$
$x = y^2$

$\Rightarrow 3y = y^2$

$\Rightarrow y^2 - 3y = 0$

$\Rightarrow y(y-3) = 0$

$\Rightarrow y = 0$ or 3

Hence the interval is $[0, 3]$

when doing horizontal slices we have the following formula

lower x value

$$A = \int_0^3 (\text{Right}(x) - \text{left}(x))\, dy$$

remember to integrate by y

Higher X value

when we did vertical integration, we did Top - Bottom or Higher Y values - Lower Y values. Similarly we are taking the higher X values and subtracting with the lower X values

$$A = \int_0^3 (3y - y^2)\, dx$$

$$= \frac{3}{2}y^2 - \frac{1}{3}y^3 \Big]_0^3$$

$$= 9/2$$

Question 33.

$$\int f(x)\, g'(x)\, dx = -\int f'(x)\, g(x)\, dx + f(x)\, g(x)$$

we make $f(x) = x$ $\quad g'(x) = e^{-4x}$

we know that $f'(x) = 1$ $\quad g(x) = -\frac{1}{4} e^{-4x}$

$$\int x e^{-4x} dx = -\int 1 \cdot \left(-\frac{1}{4} e^{-4x}\right) dx + x\left(-\frac{1}{4} e^{-4x}\right)$$

$$\int x e^{-4x} dx = -\frac{1}{16} e^{-4x} - \frac{x}{4} e^{-4x} + C$$

Differentiate to confirm!

$$\frac{d}{dx}\left(-\frac{1}{16} e^{-4x} - \frac{x}{4} e^{-4x}\right) = \frac{1}{4} e^{-4x} - \frac{1}{4} e^{-4x} + x e^{-4x}$$

$$= x e^{-4x}$$

Choose u in this order

Logs
Inverse
Algebraic
Trig
Exponentials

Question 34:

$$\int \frac{2+x}{1+x^2}\,dx = \int \frac{2}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx$$

$$= 2 \cdot \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx$$

Lets solve each integral separately

$$\boxed{\int \frac{1}{1+x^2}\,dx = \arctan(x) + C}$$

Let $u = (1+x^2)$ $\quad \frac{du}{dx} = 2x \implies dx = \frac{1}{2x}\,du$

$$\int \frac{x}{1+x^2}\,dx = \int \frac{x}{u}\left(\frac{1}{2x}\right)du$$

$$= \int \frac{1}{2}\left(\frac{1}{u}\right)du$$

$$= \frac{1}{2}\ln|u| + C$$

$$\implies \boxed{\int \frac{x}{1+x^2} = \frac{1}{2}\ln|1+x^2| + C}$$

Substituting these values back into the original equation:

$$\int \frac{2+x}{1+x^2}\,dx = \int \frac{2}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx$$

$$= 2 \cdot \arctan(x) + \frac{1}{2}\ln|1+x^2| + C$$

Question 35:

A. u-Substitution:

Let $\boxed{u = -3x^2}$ $\qquad \frac{du}{dx} = -6x \Rightarrow \boxed{dx = \frac{1}{-6x}\,du}$

$$\int x e^{-3x^2}\,dx = \int x e^{u}\left(-\tfrac{1}{6x}\right) du$$

$$= \int -\frac{1}{6} e^{u}\,du$$

$$= -1/6\, e^{u} + C$$

$$= -1/6\, e^{-3x^2} + C$$

B. Integration by parts:

$$\int f(x)\,g'(x)\,dx = -\int f'(x)\,g(x)\,dx + f(x)\,g(x)$$

Let $g'(x) = 1$ and $f(x) = \arccos(x)$

$g(x) = x$ $\qquad f'(x) = -\frac{1}{\sqrt{1-x^2}}$

u-Substitution:

$$\int \arccos(x)\,dx = -\int -\frac{x}{\sqrt{1-x^2}}\,dx + x\arccos(x)$$

Let $u = 1-x^2$ and $du = -2x\,dx$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{x}{\sqrt{u}} \cdot \frac{1}{-2x}\,du$$

$$= -\frac{1}{2}\int u^{-1/2}\,du$$

$$= -u^{1/2} + C$$

$$= -\sqrt{1-x^2} + C$$

$$\int \arccos(x)\, dx = -\sqrt{1-x^2} + x \arccos(x) + C$$

$x^2 y^2 + xy = 2$ Find where slope $= -1$

$$\frac{d}{dx}(x^2y^2 + xy) = \frac{d}{dx}(2)$$

$$2xy^2 + 2x^2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-y - 2xy^2}{x^2 2y + x} = -1$$

$$\Rightarrow \frac{-y(1 + 2xy)}{x(1 + 2xy)} = -1$$

$$\Rightarrow \frac{-y}{x} = -1$$

$$\Rightarrow y = x$$

$$x^2y^2 + xy = 2$$

$$y^4 + y^2 - 2 = 0$$

$$(y^2+2)(y^2-1) = 0$$

$$(y^2+2)(y-1)(y+1) = 0$$

$$\Rightarrow y = 1, -1$$

Hence points are $(1, 1)$ and $(-1, -1)$