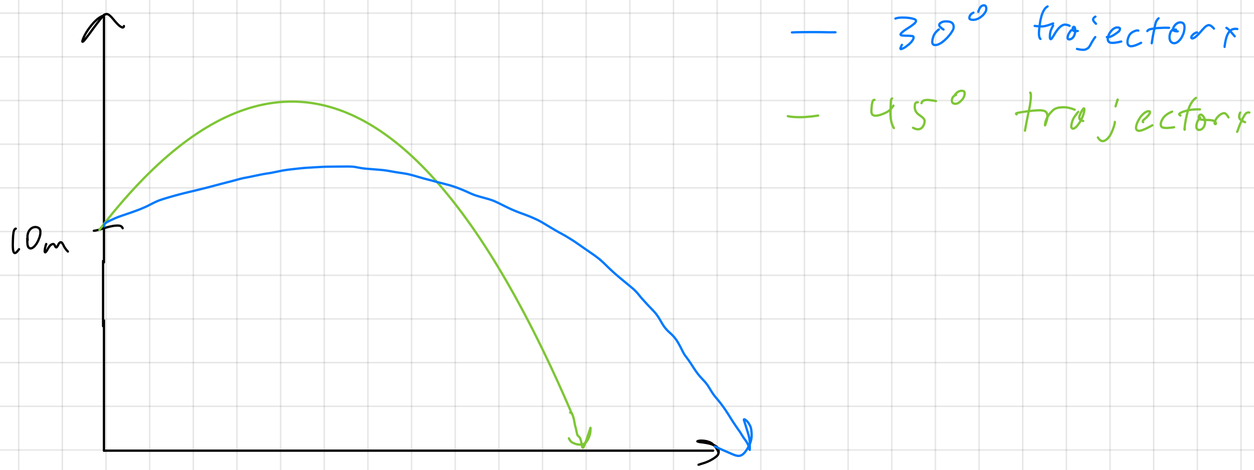
Question #14

A. we know that without initial height, 45° would be the furthest, but since there is an initial height this angle may not have the optimal distance. From the graph below it seems like 30° is the optimal angle.



B. we will first gather information we know about the acceleration and initial velocity vector.

From APSC 111 we know that the acceleration vector of gravity is

$$g = \langle 0, -9.81 \rangle$$

we can break the initial velocity into x and y components as follows:

$$v_i = \langle 12\cos(\alpha), 12\sin(\alpha) \rangle$$

From APSC 111 we have the formula

$$x - x_0 = v_i t + \tfrac{1}{2} a t^2 \quad (\text{for constant } a)$$

Although we have only used this formula with scalars, this formula would still work with vectors since acceleration is constant in both the x and y component. Hence we have,

$$\vec{r}(t) - \langle 0, 10 \rangle = \langle 12\cos(\alpha), 12\sin(\alpha) \rangle \cdot t + \tfrac{1}{2} t^2 \cdot \langle 0, -9.81 \rangle$$

$$\Rightarrow \vec{r}(t) = \langle 0, 10 \rangle + \langle 12\cos(\alpha), 12\sin(\alpha) \rangle \cdot t + \tfrac{1}{2} t^2 \cdot \langle 0, -9.81 \rangle$$

$$\Rightarrow \vec{r}(t) = \langle 12\cos(\alpha)\cdot t, 10 + 12\sin(\alpha)\cdot t - \tfrac{1}{2}(9.81)t^2 \rangle$$

C. At 45 degrees ($\frac{\pi}{4}$ rads)

$$\vec{r}(t) = \langle x, 0 \rangle = \langle 12\cos(\tfrac{\pi}{4})\cdot t, 10 + 12\sin(\tfrac{\pi}{4})\cdot t - \tfrac{1}{2}(9.81)t^2 \rangle$$

$$0 = 10 + 12\sin(\tfrac{\pi}{4})\cdot t - \frac{9.81}{2} t^2$$

By the quadratic formula,

$$t = 2.53 \text{ or } -0.80$$

But negative values don't make sense so

$$t = 2.53$$

with this value we can then solve for x

$$x = 12\cos(\tfrac{\pi}{4})\cdot t$$

$$= 21.47 \text{ meters}$$

At 30° ($\pi/6$ rads):

$$\vec{r}(t) = \langle x, 0 \rangle = \langle 12\cos(\tfrac{\pi}{6})\cdot t, 10 + 12\sin(\tfrac{\pi}{6})\cdot t - \tfrac{1}{2}(9.81)t^2 \rangle$$

$$\Rightarrow 0 = 10 + 12\sin(\tfrac{\pi}{6})\cdot t - \tfrac{1}{2}(9.81)t^2$$

$$\Rightarrow t = 2.17 \cancel{\text{or } -0.942}$$

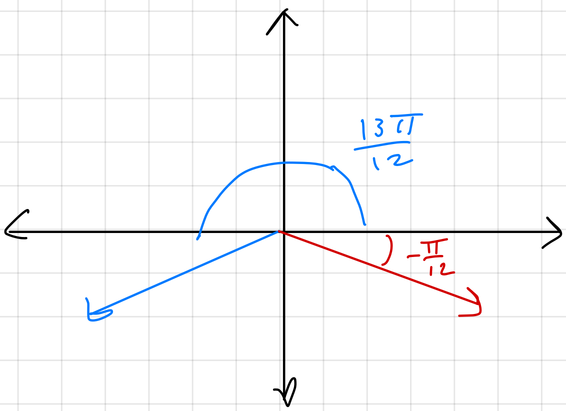
$$x = 12\cos(\tfrac{\pi}{6})(2.17)$$

$$x = 22.55 \text{ meters}$$

Question 15:

A.

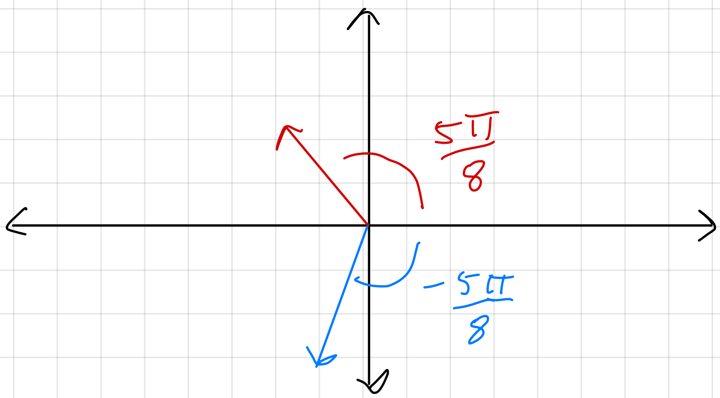
Note that $\arcsin(\theta)$ must be in $[-\frac{\pi}{2}, \frac{\pi}{2}]$



Note that the 2 vectors have the same sin value but $-\pi/12$ is the only value in the range

Hence, $\arcsin\left(\sin\left(\frac{13\pi}{12}\right)\right) = \frac{-\pi}{12}$

B.



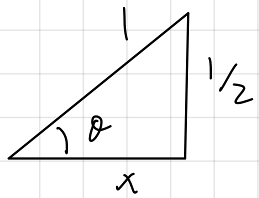
Since arccos has a range $[0, \pi]$

$\arccos\left(\cos\left(\frac{-5\pi}{8}\right)\right) = \frac{5\pi}{8}$

C.

From $\arcsin\left(\frac{1}{2}\right)$ we can construct the following triangle

$$\theta = \arcsin(1/2)$$



$\cos(\theta) = \frac{x}{1}$ and we can solve for x using the pythagorean theorem

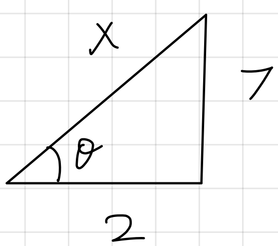
$$1 = \sqrt{(1/2)^2 + x^2}$$

$$\Rightarrow \quad x = \frac{\sqrt{3}}{2}$$

Hence $\cos\left(\arcsin\left(\frac{1}{2}\right)\right) = \frac{\sqrt{3}}{2}$

D.

$$\theta = \arctan(7/2)$$

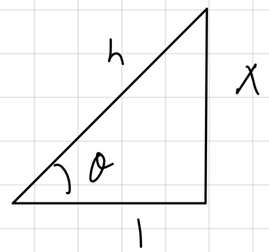


$$\sin(\theta) = \frac{7}{x}$$

$$x = \sqrt{7^2 + 2^2} = \sqrt{53}$$

$$\Rightarrow \quad \sin(\theta) = \frac{7}{\sqrt{53}}$$

E. $\theta = \arctan(x)$



$$\sin(\theta) = \frac{x}{h}$$

$$h = \sqrt{x^2+1}$$

$$\Rightarrow \sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}}$$

Question 16:

note:

$$\frac{d}{dx} \arctan(x) = \frac{1}{x^2+1}$$

$$\frac{d}{dx} b^x = b^x \ln b$$

$$\frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

A.

$$f'(s) = \frac{4^s \frac{1}{1+s^2} - \arctan(s)\, 4^s \ln(s)}{(4^s)^2}$$

B.

$$\vec{r}'(t) = \left\langle \frac{-e^{2t} \frac{1}{\sqrt{1-x^2}} - \arccos(t)(2e^{2t})}{(e^{2t})^2}, \right.$$

$$\left. \frac{t^{-2/3}}{3}\, 10^{(t^{1/3})} \ln(10) \right\rangle$$

Question 17:

A.

$a = \langle -9, -9.8 \rangle$

$v_0 = v(0) = \langle 10, 6 \rangle$

$x_0 = \langle 0, 0 \rangle$

$x - x_0 = v_i t + \frac{1}{2} a t^2$

$r(t) - \langle 0, 0 \rangle = \langle 10, 6 \rangle t + \frac{1}{2} \langle -9, -9.8 \rangle t^2$

$r(t) = \langle 10t - \frac{9}{2} t^2, 6t - \frac{9.8}{2} t^2 \rangle$

B.

$r(t) = \langle x, 0 \rangle = \langle 10t - \frac{9}{2} t^2, 6t - \frac{9.8}{2} t^2 \rangle$

$\Rightarrow 0 = 6t - \frac{9.8}{2} t^2$

$\Rightarrow t = 1.22, 0$

$t = 0$ is just where we started though

C.

$\vec{a} = \langle -9, -9.8 \rangle$

D.

$|\vec{a}| = \sqrt{(-9)^2 + (-9.8)^2} = 13.3$

Hence, neptune