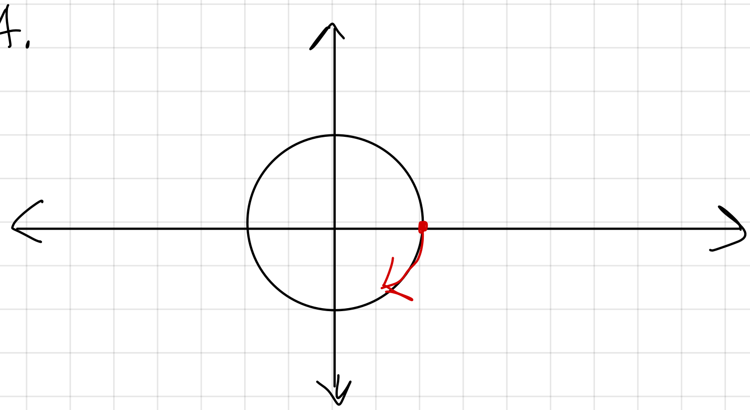
Question 10:

A.



we start with our unit circle going CCW and apply transformations to get to our desired trajectory:

$$\langle x, y \rangle = \langle \cos t, \sin t \rangle$$

Transform the radius so $r = 2$

$$\langle x, y \rangle = \langle 2\cos t, 2\sin t \rangle$$

Change the direction from CCW to CW

$$\langle x, y \rangle = \langle 2\cos(-t), 2\sin(-t) \rangle$$
$$= \langle 2\cos(t), -2\sin(t) \rangle$$

Now we want to change the speed without changing the trajectory of the curve. we can achieve this by multiplying t by a constant α.

$$r(t) = \langle 2\cos(\alpha t), -2\sin(\alpha t) \rangle$$

From the problem we want the speed to be 3 m/s, so we have the following equation:

$$\left| \frac{d}{dt} r(t) \right| = \left| \left\langle \frac{d}{dt} 2\cos(\alpha t), \frac{d}{dt}(-2\sin(\alpha t)) \right\rangle \right|$$

$$= \left| \langle -2\alpha \sin(\alpha t), -2\alpha \cos(\alpha t) \rangle \right|$$

$$= \sqrt{(-2\alpha \sin(\alpha t))^2 + (-2\alpha \cos(\alpha t))^2}$$

$$= \sqrt{4\alpha^2 (\sin^2(\alpha t) + \cos^2(\alpha t))}$$

$$= \sqrt{4\alpha^2}$$

$$= \pm 2\alpha$$

$$\Rightarrow \alpha = 3/2, \cancel{-3/2}$$

While it may seem that $\alpha = -3/2$ is a valid solution, this would reverse the direction and make our trajectory CCW and hence is not valid in the context of this problem making $\alpha = 3/2$ our only valid value

Putting everything together we get,

$$\vec{r}(t) = \langle 2\cos(\tfrac{3}{2}t), -2\sin(\tfrac{3}{2}t) \rangle$$

B. (we already did this when solving A)

$$\frac{d}{dt}\vec{r}(t) = \left\langle \frac{d}{dt}(2\cos(\tfrac{3}{2}t)), \frac{d}{dt}(-2\sin(\tfrac{3}{2}t)) \right\rangle$$

$$= \langle -3\sin(\tfrac{3}{2}t), -3\cos(\tfrac{3}{2}t) \rangle$$

C. From the problem we need to find when $r'(t)$ is a constant multiple of $\langle -2, 2 \rangle$

$$r'(t) = \left\langle -3\sin\left(\tfrac{3}{2}t\right), -3\cos\left(\tfrac{3}{2}t\right) \right\rangle = \alpha \langle -2, 2 \rangle$$

$$\Rightarrow \begin{array}{l} -3\sin\left(\tfrac{3}{2}t\right) = -2\alpha \\ -3\cos\left(\tfrac{3}{2}t\right) = 2\alpha \end{array}$$

$$\Rightarrow -3\sin\left(\tfrac{3}{2}t\right) - 3\cos\left(\tfrac{3}{2}t\right) = 0$$

$$\Rightarrow -\sin\left(\tfrac{3}{2}t\right) = \cos\left(\tfrac{3}{2}t\right)$$

$$\Rightarrow \frac{\sin\left(\frac{3}{2}t\right)}{\cos\left(\frac{3}{2}t\right)} = -1$$

$$\Rightarrow \tan\left(\tfrac{3}{2}t\right) = -1$$

$$\Rightarrow \frac{3}{2}t = \tan^{-1}(-1)$$

$$\Rightarrow t = -\frac{1}{6}\pi$$

Question 11:

A. There are multiple ways of doing this: we could eliminate the parameter, t, and just graph x in terms of y (just need to be careful of domain restrictions) or we could construct a table of values.

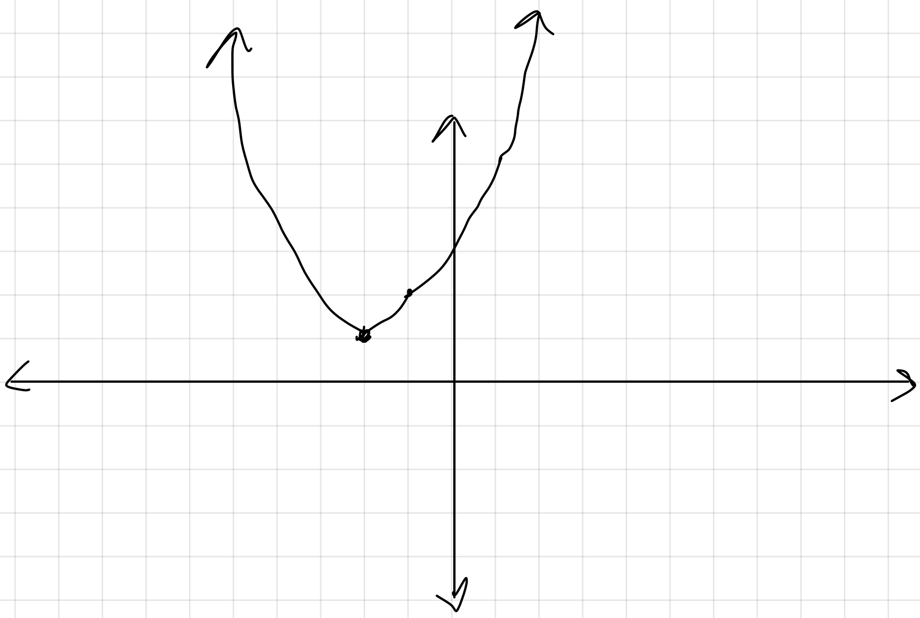
There aren't any domain restrictions here so we will eliminate t

$$\langle x, y\rangle = \langle t-2, t^2+1\rangle$$

$$\Rightarrow \quad x = t-2, \quad y = t^2+1$$

$$\Rightarrow \quad t = x+2, \quad y = t^2+1$$

$$\Rightarrow \quad y = (x+2)^2+1$$



B.

$$r(t) = \langle t-2,\ t^2+1 \rangle$$

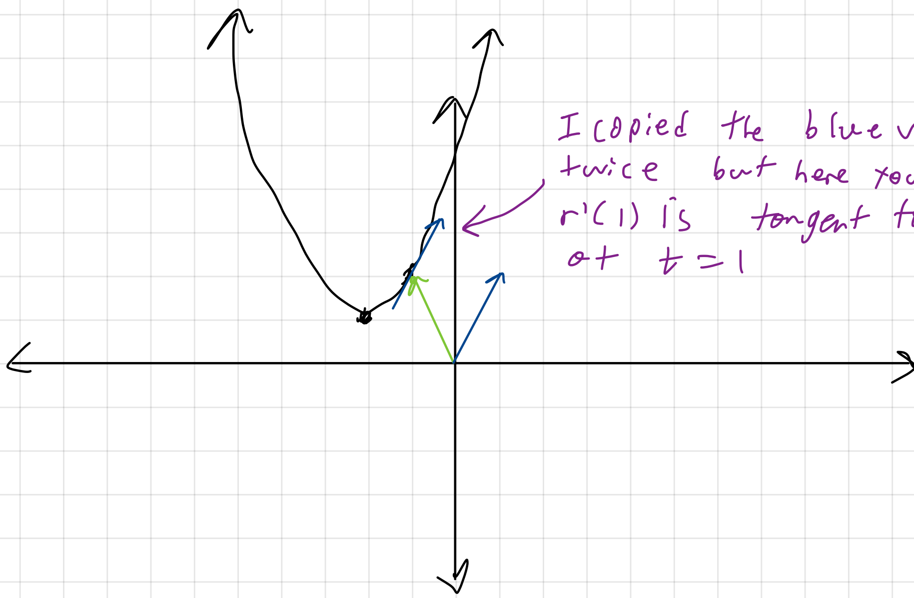
$$r'(t) = \left\langle \frac{d}{dt}(t-2),\ \frac{d}{dt}(t^2+1) \right\rangle = \langle 1,\ 2t \rangle$$

$$r(1) = \langle -1,\ 2 \rangle$$

$$r'(1) = \langle 1,\ 2 \rangle$$

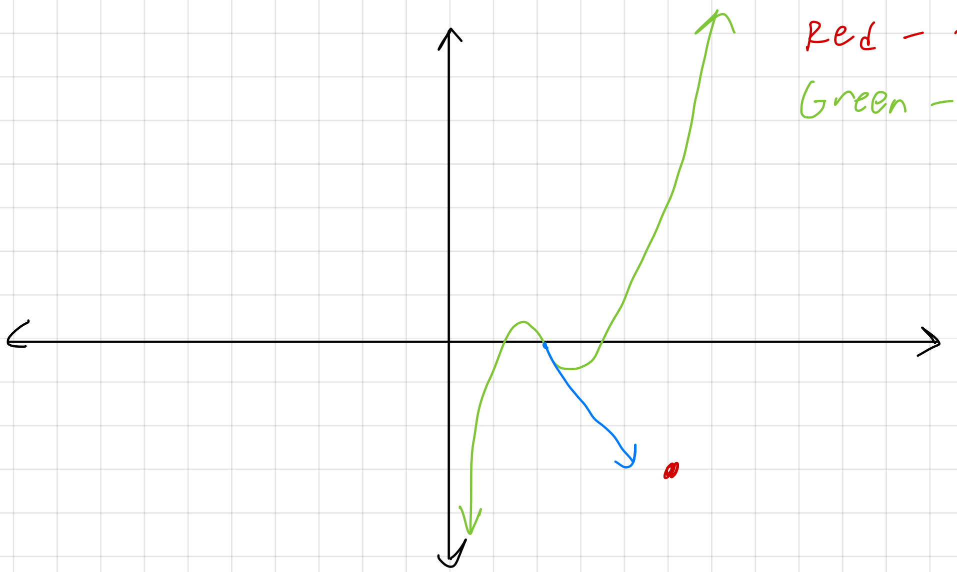
Green vector is $r(1)$

Blue vector is $r'(1)$



I copied the blue vector twice but here you can see $r'(1)$ is tangent to the curve at $t=1$

Question 12:



Let $\vec{L}(t)$ be the position vector at time t and let α be the time Indy jumps off the cart

$$\vec{L}(t) = \vec{r}(\alpha) + \frac{d}{dt}\vec{r}(\alpha) \cdot (t-\alpha)$$

It may seem confusing that we have 2 times here. Just remember that α is a constant and t is the independent variable. Soon this will reduce to a system of equations

$$\vec{L}(t) = \langle 2+\alpha, \alpha^3-\alpha \rangle + \langle 1, 3\alpha^2-1 \rangle (t-\alpha)$$

for some t $\vec{L}(t) = (5,-3)$. Hence,

$$\langle 5,-3 \rangle = \langle 2+\alpha, \alpha^3-\alpha \rangle + \langle 1, 3\alpha^2-1 \rangle (t-\alpha)$$

$$5 = 2 + \alpha + 1(t - \alpha) = 2 + t$$

$$\Rightarrow -3 = \alpha^3 - \alpha + (t - \alpha)(3\alpha^2 - 1)$$

$\Rightarrow t = 3$ (we substitute this value into this equation)

$$-3 = \alpha^3 - \alpha + (3 - \alpha)(3\alpha^2 - 1)$$

$$\Rightarrow -3 = \alpha^3 - \alpha + 9\alpha^2 - 3\alpha^3 - 3 + \alpha$$

$$\Rightarrow 0 = -2\alpha^3 + 4\alpha^2$$

$$= \alpha^2(-2\alpha + 4)$$

$$\Rightarrow \alpha = 0 \text{ or } 4/2$$

As all solutions, lets test each point to see if they are logical in the context of the problem

$\alpha = 0$, $t = 3$ makes sense since Indy lets go at time $= 0$ and gets the villian 3 seconds later

$\alpha = 9/2$, $t = 3$ doesn't make sense since Indy lets go at time $= 4.5$ and reaches the villian at time $= 3$. As far as we know now, we cannot go back in time so this answer doesn't make sense in terms of the question's context. Hence Indy should jump at time $= 0$.