A1 Differential Equations

Michaelmas Term 2019

Sheet 1

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Contents

1 Solutions

1.1 Revison Problem 1

The sequence $\{y_n\}_{n\geq 0}$ is said to converge uniformly on $[a,b]$ to a limit function y if for any $\varepsilon > 0$ $\exists\mathbb{N}$ k that depends only on ε such that:

$$\forall x \in [a,b], \mid y_n(x) - y(x) \mid < \varepsilon \ \forall n \geq k$$

We need to show that if each y_n is continuous on $[a,b]$, then the uniform limit is continuous on $[a,b]$. To do this, we will pick an arbitrary value, c, such that $c \in [a,b]$ and chose $\varepsilon > 0$. Since $\{y_n\}_{n\geq 0}$ is uniformly convergent on $[a,b]$ to the function y, then $\exists k \in \mathbb{N}$ such that $\forall x \in [a,b], \mid y_n(x) - y(x) \mid < \frac{\varepsilon}{3}, \forall n \geq k$. Therefore,
$y_k(x) - y(x) < \frac{\varepsilon}{3}\ \forall x \in [a,b]$ and $\mid y_k(c) - y(c) \mid < \frac{\varepsilon}{3}$ since y_k is continuous at $x = c$, $\exists\delta > 0$ such that $\mid y_k(x) - y_k(c) \mid < \frac{\varepsilon}{3}\ \forall x \in N(c,\delta) \cap [a,b]$. Now, we can continue, $\mid y(x) - y(c) \mid$ $\leq \mid y(x) - y_k(x) \mid + \mid y_k(x) - y_k(c) \mid + \mid y_k(c) - y(c) \mid \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\ \forall x \in N(c,\delta) \cap [a,b]$. This is true by the triangle inequality, i.e. $\mid y(x) - y(c) \mid < \varepsilon\ \forall x \in N(c,\delta) \cap [a,b]$. This proves that y is continuous at $c \in [a,b]$. Since we have picked c arbitrarily, we have proven that y is continuous on $[a,b]$.
Given that $\lim\limits_{n\to\infty} \int_a^b \mid y_n(x) - y(x) \mid dx \to 0$, we have:
$\Big| \lim\limits_{n\to\infty} \int_a^b y_n(x)dx - \int_a^b y(x)dx \Big| = \Big| \lim\limits_{n\to\infty} \int_a^b \big(y_n(x) - y(x)\big)dx \Big| \leq$
$\leq \lim\limits_{n\to\infty} \int_a^b \mid y_n(x) - y(x) \mid dx = 0 \quad$ (because $\int_a^b \mid f \mid \geq \Big| \int_a^b f \Big|$).
Thus, $\lim\limits_{n\to\infty} \int_a^b y_n(x)dx = \int_a^b y(x)$ and $\int_a^b y_n(x) \to \int_a^b y(x)$.
Given $[a,b] = [0,1]$ and $y_n(x) = nxe^{-nx^2}$, if $x = 0$, then $y_n(x) \to 0$. We know that if u_n is a sequence of positive real valued functions such that $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$, then $\lim\limits_{n\to\infty} u_n = 0$. Now, $u_n = nxe^{-nx^2} > 0\ \forall x \in (0,1]$. $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{(n+1)xe^{-(n+1)x^2}}{nxe^{-nx^2}} = \lim\limits_{n\to\infty} (1+\frac{1}{n})e^{-x^2}$ $= e^{-x^2} < 1$. Therefore, $\lim\limits_{n\to\infty} u_n = 0\ \forall x \in (0,1]$. Thus $y_n(x) \to 0$, $\forall x \in [0,1]$. Thus, $\{y_n\}_{n\geq 0}$ converges pointwise on $[0,1$ to the function $y(x) = 0$, $\forall x \in [0,1]$. $\int_0^1 y_n(x)dx$ $= \int_0^1 nxe^{-nx^2}dx = \int_0^1 \frac{n}{2}e^{-nz}dz$, letting $z = x^2$ and $dz = 2xdx$. Therefore, $\int_0^1 \frac{n}{2}e^{-nz}dz$ $= \frac{n}{2}\left[\frac{e^{-nz}}{-n}\right]_0^1 = \frac{1}{2}[1 - e^{-n}]$. Therefore, $\int_0^1 y_n(x)dx \to \frac{1}{2}$ and also $\int_0^1 y(x)dx = 0$. Thus,

$\int_0^1 y_n(x)dx$ does not converge to $\int_0^1 y(x)dx$, so the convergence is non-uniform. To show $\max_{0\leq x\leq 1} y_n(x) = \sqrt{\frac{n}{2e}}$, we will apply the second derivative test on $y_n(x) = nxe^{-nx^2}$.

$y_n'(x) = ne^{-nx^2} + nxe^{-nx^2}(-2nx)$, so $y_n'(x) = 0 \Rightarrow x = \pm\sqrt{\frac{1}{2n}}$.

$$y_n''(x) = -2n^2xe^{-nx^2} - 2n^2 \cdot 2x \cdot e^{-nx^2} - 4n^2x^2e^{-nx^2}(-2nx) =$$

$$= -2n^2xe^{-nx^2} - 4n^2xe^{-nx^2} - 6n^3x^2e^{-nx^2}$$

$$= -6n^2xe^{-nx^2}(1 - nx)$$

Therefore, $y_n''\left(\frac{1}{\sqrt{2n}}\right) < 0$ and $y_n''\left(-\frac{1}{\sqrt{2n}}\right) > 0$, therefore y_n'' is maximum at $x = \frac{1}{\sqrt{2n}}$.

Thus, $y_n\left(\frac{1}{\sqrt{2n}}\right) = n \cdot \frac{1}{\sqrt{2n}} \cdot e^{-n\cdot\frac{1}{2n}} = \sqrt{\frac{n}{2}} \cdot e^{-\frac{1}{2}} = \sqrt{\frac{n}{2e}}$ and $\max_{0\leq x\leq 1} y_n(x) = \sqrt{\frac{n}{2e}}$. The graph of $y_n(x)$ versus x is shown in Figure 1.

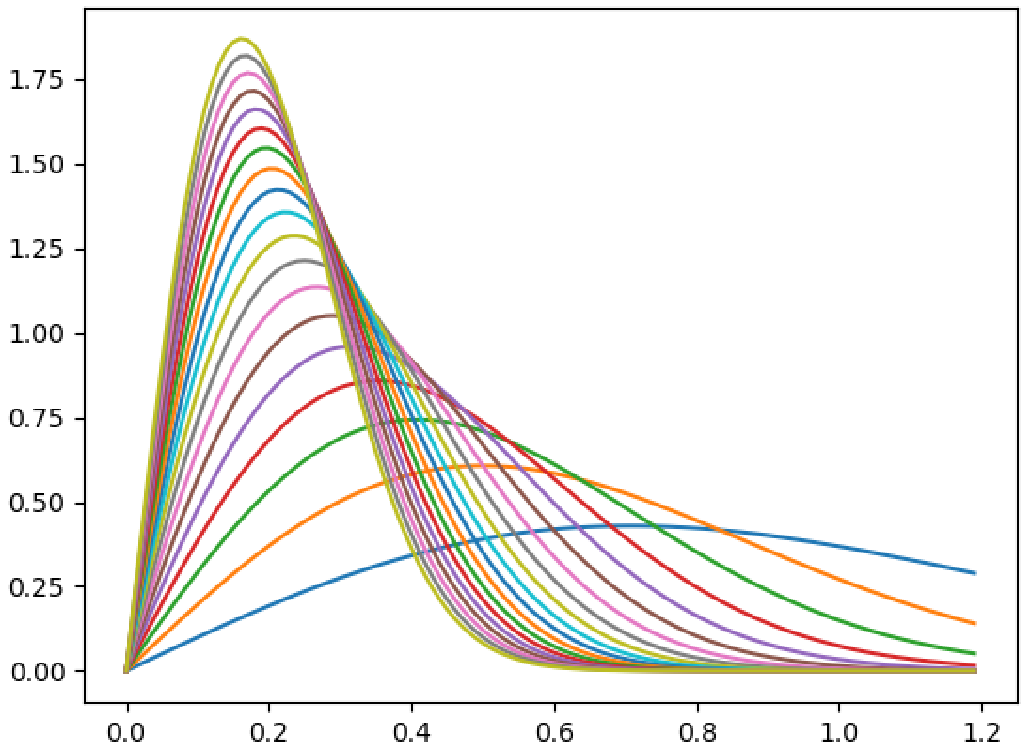

Figure 1: The graph of $y_n(x)$ versus x is shown below, for $1 \leq n \leq 20$.

I will include a second plot in Figure 2 because it demonstrates nicely that there is no uniform conversion. While the plot in Figure 1 included the first *non-trivial* 20 members of the sequence (y_1 through y_{20}), the second graph shows y_{200} to y_{600}, skipping every 50. This lets us visualize what is happening *a bit further*. We can clearly see that the functions y_n are getting *closer to y=0* for smaller values of x as n grows, however, the max peak is getting higher and higher, albeit closer and closer to 0.

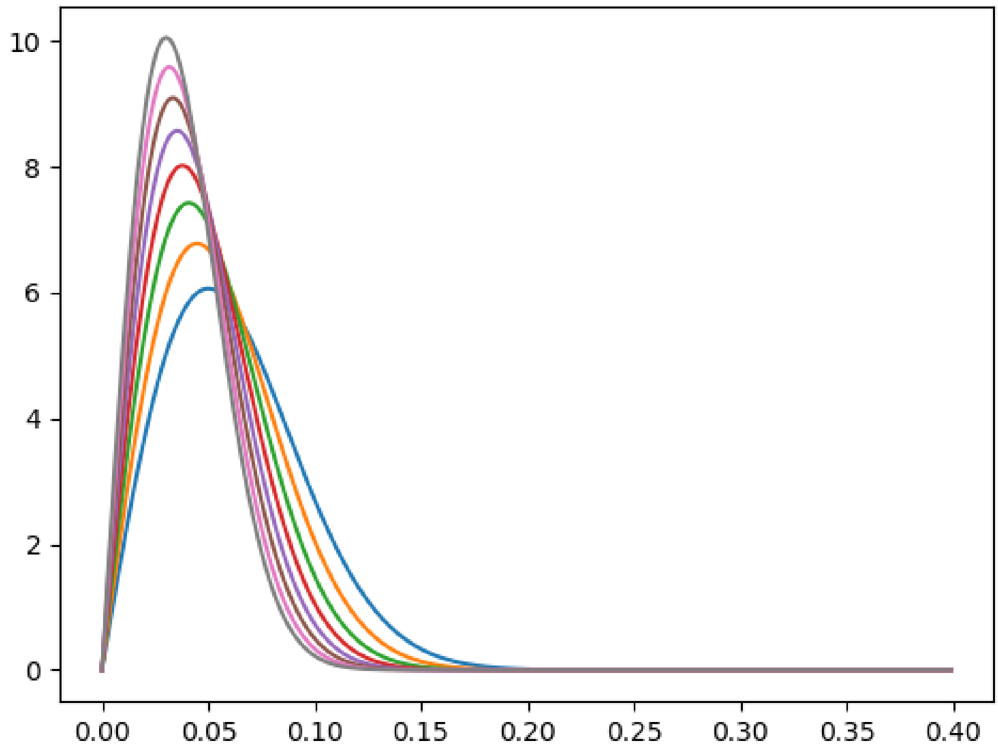

Figure 2: The graph of $y_n(x)$ versus x is shown below, for $1 \leq n \leq 20$.

1.2 Revison Problem 2

Weirstrauss M-test. Let $f_1, f_2, f_3, ...x \to R$ be a sequence of functions from a set X to the real numbers. Assume that there are constants, M_k, such that the following two conditions hold.

1. $\mid f_k(x) \mid \leq M_k\ \forall x \in X$ and $k \geq 1$
2. $\sum_{k=1}^{\infty} M_k < \infty$

Then the series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely and uniformly on X. We will use this test to show that the series $\sum_{n=0}^{\infty}(-1)^n \cdot \frac{\boldsymbol{cos}(nx)}{1+n^2}$ converges uniformly on $[-\pi, \pi]$.

$$\left| (-1)^n \cdot \frac{\boldsymbol{cos}(nx)}{1+n^2} \right| = \left| \frac{\boldsymbol{cos}(nx)}{1+n^2} \right| = \frac{\mid \boldsymbol{cos}(nx) \mid}{1+n^2} \leq \frac{\mid \boldsymbol{cos}(nx) \mid}{n^2} \leq \frac{1}{n^2} \Rightarrow M_k = \frac{1}{n^2}$$

Obviously, $\sum_{n=0}^{\infty} \frac{1}{n^2}$ (by the p-series test). Therefore, the series $\sum_{k=1}^{\infty}(-1)^n \cdot \frac{\boldsymbol{cos}(nx)}{1+n^2}$ converges uniformly.

1.3 ODEs and Picard's Theorem Question 3

Consider the IVP: $y' = x^2 + y^2, y(0) = 0$. The sequence of Picard approximations:

$$y_0 = 0$$

$$y_1 = \int_0^x t^2 dt = \frac{x^3}{3}$$

$$y_2 = \int_0^x \left[t^2 + \left(\frac{t^3}{3}\right)^2 \right] dt = \frac{x^3}{3} + \frac{x^7}{7 \cdot 9}$$

$$y_3 = \int_0^x \left[t^2 + \left(\frac{t^3}{3} + \frac{x^7}{7 \cdot 9}\right)^2 \right] dt = \frac{x^3}{3} + \frac{x^7}{7 \cdot 9} + \frac{2x^{11}}{3 \cdot 7 \cdot 9 \cdot 11} + \frac{x^{15}}{(7 \cdot 9)^2 \cdot 15}$$

Consider $R = \{(x, y) :\mid x \mid < \frac{1}{\sqrt{2}}, \mid y \mid < \frac{1}{\sqrt{2}}\}$. We will show that Picard's Theorem is satisfied on R.

i $x^2 + y^2$ is continuous on R and $max \mid x^2 + y^2 \mid = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$, so $M = 1$ and $hM = \frac{1}{\sqrt{2}}M = \frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{2}} = k$.

ii $x^2 + y^2$ is bounded on R and differentiable with respect to y. Therefore, by *1.10* from the notes, $x^2 + y^2$ is Lipschitz continuous on R. We could have also checked this condition explicitly:

$$| f(x,u) - f(x,v) | = | x^2 + u^2 - x^2 - v^2 | = | u^2 - v^2 | = | u+v || u-v | \leq \sqrt{2} | u-v |$$

for u, v such that $(x,u) \in R$ and $(x,v) \in R$.

Therefore, both conditions (i) and (ii) are satisfied so y_n converges as $n \to \infty$.

Consider the IVP $y' = (1-2x)y, y(0) = 1$. The sequence of Picard approximations:

$$y_0 = 1$$

$$y_1 = 1 + \int_0^x (1-2t)dt = 1 + \int_0^x d(t-t^2) = 1 + x - x^2$$

$$y_2 = 1 + \int_0^x (1-2t)(1+t-t^2)dt = 1 + \int_0^x (1+(t-t^2))d(t-t^2) = 1 + (x-x^2) + \frac{(x-x^2)^2}{2}$$

$$y_3 = 1 + \int_0^x (1-2t)(1+t-t^2) + \frac{(t-t^2)^2}{2} dt = 1 + \int_0^x (1+(t-t^2) + \frac{(t-t^2)^2}{2} d(t-t^2)$$

$$= 1 + (x-x^2) + \frac{(x-x^2)^2}{2} = 1 + (x-x^2) + \frac{(x-x^2)^2}{2} + \frac{(x-x^2)^3}{2 \cdot 3}$$

Clearly, $y_{n+1} = 1 + \frac{(x-x^2)}{1!} + \frac{(x-x^2)^2}{2!} + ... + \frac{(x-x^2)^n}{n!}$. We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, x \in \mathbb{R}$, and, therefore, we can conclude that our IVP has a unique solution $\forall x \in \mathbb{R}$, which is $y = e^{x-x^2}$.

We could have easily obtained such a solution directly by separating variables:

$$\frac{dy}{dx} = (1-2x)y$$

$$\int \frac{dy}{y} = \int (1-2x)dx$$

$$ln(y) = x - x^2 + C$$

$$y = e^{x-x^2+C} = e^C e^{x-x^2} = Ce^{x-x^2} \text{ (considering } c = e^c\text{)}$$

And, because $y(0) = 1, c = 1$, so $y = e^{x-x^2}$. Now, we will show how to apply Picard's theorem to show that this problem has a unique solution $\forall x$. Note that $F(x,y) = (1-2x)y$ is continuous $\forall x \in [a-h, a+k]$. Actually, we can consider any interval $[c,d]$ as the interval does not have to be balanced. Because F is continuous on the interval, it is also bounded. Thus, we have condition (i), and now we have to show that $F(x,y)$ satisfies global Lipschitz condition (iii) $\forall \mathbb{R} y$ and $x \in [c,d]$.

$| F(x,u) - F(x,v) | = | (1-2x)u - (1-2x)v | = | 1-2x || u-v | \leq L | u-v |$. Therefore, $| 1-2x | \leq L$ can be resolved on any $[c,d]$. Therefore, there exists a unique global solution for this IVP.

1.4 ODEs and Picard's Theorem Question 4

(a) (i) $F(x,y) = xy^{\frac{1}{3}}$ does not satisfy the Lipschitz condition on any rectangle of the form $R = \{(x,y) :\mid x \mid \leq h, \mid y \mid \leq k\}$, where $h > 0$ and $k > 0$.

Proof by contradiction. Suppose it does, then there exists a constant L such that $\forall x :\mid x \mid \leq h$ and $\forall y :\mid y \mid \leq k$:

$$\mid F(x,y) - F(x,0) \mid = \mid x \mid\mid y^{\frac{1}{3}} \mid \leq L \mid y \mid$$

Therefore, $\mid h \mid\mid y^{-\frac{2}{3}} \mid$ is unbounded as $y \to 0$. Therefore, $F(x,y) = xy^{\frac{1}{3}}$ is not Lipschitz continuous on R. So the Picard theorem will fail if $b = 0$. The derivative of $F(x,y)$ with respect to y is unbounded as $y \to 0$ and does not exist at $y = 0$.

(ii) Consider $R = \{(x,y) :\mid x \mid \leq h, \mid y - b \mid \leq k\}$ where $h > 0$ and $0 < k < b$ so that y cannot be zero in R. Consider a derivative of $F(x,y)$ with respect to y: $F_y(x,y) = \dfrac{xy^{-\frac{2}{3}}}{3}$, which is bounded, therefore $F(x,y)$ is Lipschitz continuous and condition (ii) is satisifed. $F(x,y)$ is also continuous on R and $max_R \mid xy^{\frac{1}{3}} \mid \leq h(b+k)^{\frac{1}{3}} = M$. From this, we can find h to satisfy condition (i): $M_h \leq k$ and $h^2(b+k)^{\frac{1}{3}} \leq k$ or $h^2 \leq \dfrac{k}{(b+k)^{\frac{1}{3}}}$. Therefore, for $b > 0$, Picard's theorem is satisfied on $R = \{(x,y) :\mid x \mid \leq h, \mid y - b \mid \leq k\}$, where $0 < k < b$ and $0 < h < \sqrt{\dfrac{k}{(b+k)^3}}$. Therefore, Picard's theorem guarantees the existance of a unique solution of a given IVP on the rectangle $R = \{(x,y) :\mid x \mid \leq h, \mid y - b \mid \leq k\}$ where $0 < k < b$ and $0 < h < \sqrt{\dfrac{k}{(b+k)^3}}$.

(iii) We have the IVP $y' = xy^{\frac{1}{3}}, y(0) = 0$. Separating the variables, we have:

$$\int \frac{dy}{y^{\frac{1}{3}}} = \int x dx$$

$$\frac{3}{2}y^{\frac{2}{3}} = \frac{x^2}{2} + C$$

$$y = \frac{(x^2 + 2C)^{\frac{3}{2}}}{3\sqrt{3}}$$

So, if $y(0) = 0$, there is a solution $y = \dfrac{x^3}{3\sqrt{3}}$. $y = 0$ is also a solution on $[-c, c]$, so we can define the solution to be:

$$y = \left\{ \begin{array}{ll} 0, & \text{if } -c \leq x \leq c \\ \dfrac{(x^2-c^2)^{\frac{3}{2}}}{3\sqrt{3}}, & \text{if } \mid x \mid > c \end{array} \right\}$$

Because $\mid x \mid > c, x^2 - c^2$ will always be positive.

(b) To prove that there exists a unique solution on $\mid x \mid \leq h, y \geq b : R = \{(x,y) : \mid x \mid \leq h, y \geq b\}$, we will check if the global Lipschitz condition on R is satisfied. $\mid f(x,u) - f(x,v) \mid = \mid xu^{\frac{1}{3}} - xv^{\frac{1}{3}} \mid = \mid x \mid\mid u^{\frac{1}{3}} - v^{\frac{1}{3}} \mid = \dfrac{\mid x \mid\mid u - v \mid}{u^{\frac{2}{3}} + u^{\frac{1}{3}}v^{\frac{1}{3}} + v^{\frac{2}{3}}} \leq$
$\leq \dfrac{h}{3b^{\frac{2}{3}}} \mid u - v \mid$. Therefore, we have $f(x,y) = xy^{\frac{1}{3}}$ is continuous $\forall x \in [-h,h]$ and we have shown that $f(x,y)$ satisfies the Lipschitz condition $\forall y > b$ ($L = \frac{h}{3b^{\frac{2}{3}}}$). So, now we can state that there exists a unique solution of the given IVP on R.

1.5 ODEs and Picard's Theorem Question 5

First, we will show that $y_n(x)$ converges uniformly to a continuous function. Consider differences between successive approximations:

$$\begin{cases} \mathrm{e}_0(x) = f(x) \\ \mathrm{e}_{n+1}(x) = y_{n+1}(x) - y_n(x) \end{cases}$$

Note that $y_n(x) = \sum_{k=0}^{n} e_k(x)$. In order for $y_n(x)$ to converge, we need the series $\sum_{k=0}^{n} e_k(x)$ to converge. Notice that $e_{n+1}(x) = y_{n+1}(x) - y_n(x) = \int_a^x k(x,t)\big(y_n(t) - y_{n-1}(t)\big)dt$. Because the absolute value of an integral is less than or equivalent to the integral of the absolute value, we can conclude that $\mid e_{n+1}(x) \leq \left| \int_a^\infty \mid k(x,t) \mid\mid y_n(t) - y_{n-1}(t) \mid dt \right|$. Because $k(x,t)$ is continuous on $[a,b]$x$[a,b]$, there exists $M > 0$ such that $\mid k(x,t) \leq M$ $\forall (x,t) \in [a,b]$x$[a,b]$. Therefore, $\mid e_{n+1}(x) \leq M \left| \int_a^x \mid y_n(t) - y_{n-1}(t) \mid dt \right|$. Notice also that $f(x)$ is also continuous on $[a,b]$ and, therefore, is also bounded. Let's assume $f(x)$ is bounded on $[a,b]$ by a constant L: $\mid f(x) \mid \leq L \ \forall x \in [a,b]$. We will now show that $\forall x \in [a,b]$:

$$\mid e_n(x) \mid \leq \frac{LM^n}{n!} \mid x - a \mid^n \leq \frac{LM^n}{n!}(b-a)^n \tag{1}$$

We have already established that $| e_{n+1}(x) |\leq M \left| \int_a^x | y_n(t) - y_{n-1}(t) dt \right|$, and, therefore, $| e_{n+1}(x) |\leq M \left| \int_a^x | e_n(t) dt \right|$. We will prove (1) by induction.

$$e_1(x) = y_1(x) - y_0(x) = f(x) + \int_a^x k(x,t)f(t)dt - f(x) = \int_a^x k(x,t)f(t)dt \leq$$
$$\leq ML \mid x-a \mid$$

Therefore (1) is true for n=1.Assume $| e_n(x) | \leq \frac{LM^n}{n!} \mid x-a \mid^n$.

$$e_{n+1}(x) = \int_a^x k(x,t)\big(y_n(t) - y_{n-1}(t)\big)dt = \int_a^x k(x,t)e_n(t)dt \leq M \int_a^x \mid e_n(t) \mid dt$$
$$\leq \frac{LM^{n+1}}{n!} \int_a^x \mid t-a \mid dt \leq \frac{LM^{n+1}}{(n+1)!} \mid x-a \mid^{n+1}$$

Therefore, $| e_n(x) |\leq \dfrac{LM^n}{n!} \mid x-a \mid^n \leq \dfrac{LM^n}{n!}(b-a)^n$ is true by induction. The series $\sum_{k=0}^{n} e_k(x)$ and, therefore y_n converges uniformly to a continuous function. The convergence is immediate from the Weierstrauss M-test, because $\dfrac{LM^n(b-a)}{n!}$ converges and is independent on x. Therefore, as $y_n(x) \to y(x)$, $n \to \infty$. The continuity of $y(x)$ follows from the continuity of $y_n(x)$. So, we have proven that the sequence $y_n(x)$ has a limit y(x) which is a continuous function. We will now prove that $y(x)$ is a solution. Taking a limit in $y_{n+1}(x) = f(x) + \int_a^x k(x,t)y_n(t)dt$, we get $y(x) = f(x) + \int_a^x k(x,t)y(t)dt$. We will now prove the uniqueness of the solution. Assume that there exists two solutions of the problem: $y(x)$ and $z(x)$.

$$e(x) = y(x) - z(x) = \int_a^k k(x,t)\big(y(t) - z(t)\big)dt$$

$$\mid e(x) \mid \leq \left| \textstyle\int_a^x \mid k(x,t) \mid\mid y(t) - z(t) \mid dt \right| \leq M \left| \int_a^x \mid y(t) - z(t) \mid dt \right| \leq$$
$$\leq M \left| \int_a^x \mid e(t) \mid dt \right|,$$ where we used the fact that $k(x,t)$ is continuous on $[a,b]$x$[a,b]$. Because $y(x)$ and $z(x)$ are continuous, $e(x)$ is also continuous, and, therefore, bounded: $\mid e(x) \mid \leq P$. So, $\mid e(x) \mid \leq M \left| \int_a^x \mid e(t) \mid dt \right| \leq MP \mid x-a \mid$. By induction, we can show that $\mid e(x) \mid \leq \dfrac{m^n p(b-a)}{n!}$. But as $\dfrac{m^n p(b-a)}{n!} \to 0$, $n \to \infty$. This shows that $e(x) = 0$ and, therefore, $y(x) = z(x)$. Finally, we have to show that the solution continuously depends on $f(x)$, in other words, *small* changes in $f(x)$ can cause only *small* changes in the solution. More precisely, if $y_1(x)$ is a solution of $y(x) = f_1(x) + \int_a^b k(x,t)y(t)dt$ and $y_2(x)$ is a solution of $y(x) = f_2(x) + \int_a^b k(x,t)y(t)dt$, then we can make $\mid y_1(x) - y_2(x) \mid$ as small as we like by making $\mid f_1(x) - f_2(x)$ small

enough: $\forall\varepsilon > 0\ \exists\sigma > 0 :\mid f_1(x) - f_2(x) \mid< \sigma \Rightarrow\mid y_1(x) - y_2(x) \mid< \varepsilon\ \forall x \in [a, b]$. Note that $y_1(x) = f_1(x) + \displaystyle\int_a^b k(x,t)y_1(t)dt \leq L_1 + M \left| \int_a^x y_1(t)dt \right|$, where L_1 and M are upper boundaries for $f_1(x)$ and $k(x,t)$ respectively (remember, they are continuous on $[a, b]$ and $[a, b]$x$[a, b]$, respectively). Therefore, by Gronwall's inequality, $y_1(x) \leq L_1 e^{M|x-a|}$. Similarly, $y_2(x) \leq L_2 e^{M|x-a|}$, where $L_2 = \max f_2(x), x \in [a, b]$. So, we have $\mid y_1(x) - y_2(x) \mid \leq \mid L_1 - L_2 \mid e^{M(b-a)}$, where $L_1 = \max f_1(x), x \in [a, b]$ and $L_2 = \max f_2(x), x \in [a, b]$. We can see that $\mid y_1(x) - y_2(x) \mid< \varepsilon$, whenever $\mid L_1 - L_2 \mid< e^{-M(b-a)}\varepsilon$, which will hold true when $\mid f_1(x) - f_2(x) \mid< \varepsilon e^{-M(b-a)}\ \forall x \in [a, b]$ (this is even a bit stronger than needed, but if that's the way to show that f_1 and f_2 are *close* enough, then we can go with it).