Where Should a Pilot Start Descent?

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Overview

Introduction

- Application of calculus to real-world aviation problems
- Determine a cubic polynomial that models aircraft descent during landing
- Calculate how far from the airport a pilot should start descent
- Given: acceleration limit (k), altitude (h), and speed (v)

Problem Statement

An aircraft begins descent from cruising altitude h at horizontal distance ℓ from touchdown.

Three Conditions:

1. Cruising altitude is h when descent starts at distance ℓ from touchdown
2. Pilot must maintain constant horizontal speed throughout descent
3. Absolute value of vertical acceleration should not exceed constant k

Goal: Find cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$

Setting Up the Coordinate System

- Origin $(0, 0)$ at touchdown point on runway
- Positive x-axis points right (aircraft approach direction)
- $x =$ horizontal distance from touchdown
- $P(x) =$ altitude (vertical height) above runway
- $\ell =$ horizontal distance where descent begins
- $h =$ cruising altitude at start of descent
- Aircraft flies right to left, approaching origin

Boundary Conditions

At Start of Descent ($x = \ell$):

- Position: $P(\ell) = h$ (at cruising altitude)
- Velocity: $P'(\ell) = 0$ (horizontal flight path)

At Touchdown ($x = 0$):

- Position: $P(0) = 0$ (on the ground)
- Velocity: $P'(0) = 0$ (smooth landing)

These four conditions ensure smooth transition from cruising to landing.

Step 1: Finding the Derivative

Given cubic polynomial:

$$P(x) = ax^3 + bx^2 + cx + d$$

Using the power rule:

$$P'(x) = 3ax^2 + 2bx + c$$

Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$

Step 2-3: Applying Conditions at Touchdown

Condition 1: $P(0) = 0$

$$P(0) = a(0)^3 + b(0)^2 + c(0) + d = d$$

$$\boxed{d = 0}$$

Condition 2: $P'(0) = 0$

$$P'(0) = 3a(0)^2 + 2b(0) + c = c$$

$$\boxed{c = 0}$$

Our polynomial simplifies to: $P(x) = ax^3 + bx^2$

Step 4-5: Applying Conditions at Start

Condition 3: $P(\ell) = h$

$$a\ell^3 + b\ell^2 = h \quad \text{(Equation 1)}$$

Condition 4: $P'(\ell) = 0$

$$3a\ell^2 + 2b\ell = 0 \quad \text{(Equation 2)}$$

Step 6: Solving the System

From Equation 2:

$$b = -\frac{3a\ell}{2}$$

Substitute into Equation 1:

$$a\ell^3 - \frac{3a\ell^3}{2} = h$$

$$a\ell^3 \left(-\frac{1}{2}\right) = h$$

$$\boxed{a = -\frac{2h}{\ell^3}} \quad \text{and} \quad \boxed{b = \frac{3h}{\ell^2}}$$

Final Answer - Part 1

$$\boxed{P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2}$$

Or in factored form:

$$P(x) = \frac{h}{\ell^3}\left(3\ell x^2 - 2x^3\right)$$

This polynomial describes a smooth descent from altitude h at distance ℓ down to the ground at $x = 0$.

Part 2: Deriving the Acceleration Constraint

Goal: Show that $\frac{6hv^2}{\ell^2} \leq k$

Given:

- Constant horizontal speed: $\frac{dx}{dt} = -v$
- Acceleration limit: $\left|\frac{d^2y}{dt^2}\right| \leq k$
- Vertical position: $y = P(x)$

Relating Acceleration to Polynomial

Using the chain rule:

$$\frac{dy}{dt} = P'(x) \cdot \frac{dx}{dt} = -vP'(x)$$

Taking derivative again:

$$\begin{aligned} \frac{d^2y}{dt^2} &= -v \cdot P''(x) \cdot (-v) \\ &= v^2 P''(x) \end{aligned}$$

$$\boxed{\frac{d^2y}{dt^2} = v^2 P''(x)}$$

Finding the Second Derivative

From $P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2$

First derivative:

$$P'(x) = -\frac{6h}{\ell^3}x^2 + \frac{6h}{\ell^2}x$$

Second derivative:

$$P''(x) = -\frac{12h}{\ell^3}x + \frac{6h}{\ell^2}$$

Factor:

$$P''(x) = \frac{6h}{\ell^2}\left(1 - \frac{2x}{\ell}\right)$$

Finding Maximum Acceleration

Evaluate $P''(x)$ at key points:

At $x = 0$**:** $P''(0) = \frac{6h}{\ell^2}$
At $x = \ell$**:** $P''(\ell) = -\frac{6h}{\ell^2}$

$P''(x)$ is linear, decreasing from $\frac{6h}{\ell^2}$ to $-\frac{6h}{\ell^2}$

Maximum absolute value: $|P''(x)|_{\text{max}} = \frac{6h}{\ell^2}$

Deriving the Constraint

Maximum vertical acceleration:

$$\left|\frac{d^2y}{dt^2}\right|_{\max} = v^2 \cdot \frac{6h}{\ell^2} = \frac{6hv^2}{\ell^2}$$

Must not exceed k:

$$\boxed{\frac{6hv^2}{\ell^2} \leq k}$$

Part 3: Given Information

Airline Specifications:

- Maximum vertical acceleration: $k = 860$ mi/h^2
- Cruising altitude: $h = 35,000$ ft
- Horizontal speed: $v = 300$ mi/h

Question: How far from the airport should the pilot start descent?

Unit Conversion

Convert altitude from feet to miles:

$$h = 35,000 \text{ ft} \times \frac{1 \text{ mi}}{5,280 \text{ ft}}$$

$$h = \frac{35,000}{5,280} \text{ mi}$$

$$h = 6.6288 \text{ mi}$$

Now all units are consistent (miles and hours).

Solving for Distance

From Part 2: $\frac{6hv^2}{\ell^2} = k$ (equality for minimum distance)

Solve for ℓ:

$$\ell = \sqrt{\frac{6hv^2}{k}}$$

Substitute values:

$$\begin{aligned}
\ell &= \sqrt{\frac{6(6.6288)(300)^2}{860}} \\
&= \sqrt{\frac{3,579,552}{860}} \\
&= \sqrt{4,162.27} \\
&= 64.52 \text{ mi}
\end{aligned}$$

Answer - Part 3

64.52 miles

The pilot should start descent approximately **64.52 miles** away from the airport.

This ensures vertical acceleration never exceeds the safety limit of 860 mi/h^2.

Part 4: The Polynomial Function

Using our calculated values:

- $h = 6.6288$ mi
- $\ell = 64.52$ mi

The approach path polynomial becomes:

$$P(x) = -0.0000494x^3 + 0.00478x^2$$

where x is in miles and $P(x)$ is altitude in miles.

Key Points on the Path

At start of descent ($x = 64.52$ mi):

$$P(64.52) = 6.6288 \text{ mi} = 35,000 \text{ ft}$$

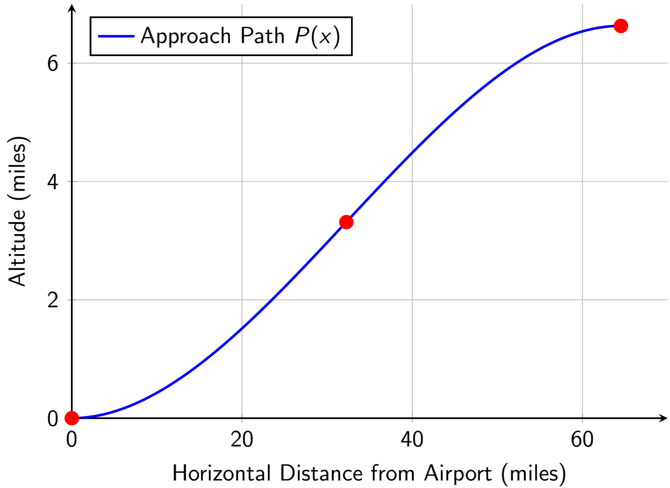
At halfway ($x = 32.26$ mi):

$$P(32.26) = 3.313 \text{ mi} \approx 17,493 \text{ ft}$$

At touchdown ($x = 0$ mi):

$$P(0) = 0 \text{ mi} = 0 \text{ ft}$$

Aircraft Approach Path Graph



Graph Analysis

The graph shows:

- Aircraft starts at (64.52, 6.63 mi) $=$ 35,000 ft altitude
- Path curves smoothly downward
- Touches down at origin (0, 0)
- Horizontal tangent at both endpoints (smooth transitions)
- Gradual descent with controlled acceleration

The cubic curve creates a smooth S-shaped profile that minimizes passenger discomfort while maintaining safety.