Where Should a Pilot Start Descent?

Honors Proj Prompt 2

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Math 161H Honors Calculus I

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Fall 2025

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1 Introduction

The project I chose explores the application of calculus to real-world aviation problems. Specifically, I will determine a cubic polynomial that models the descent path of an aircraft during landing and solve how far away from the airport a pilot should start descent, given a k limit, altitude, and speed.

2 Problem Statement

An aircraft begins its descent from a cruising altitude of h at a horizontal distance ℓ from the touchdown point. The pilot must design a flight path $P(x)$ that satisfies the following conditions:

1. The cruising altitude is h when descent starts at a horizontal distance from touchdown at the origin
2. The pilot must maintain a constant horizontal speed throughout descent
3. The absolute value of the vertical acceleration should not exceed a constant k (which is much less than the acceleration due to gravity).

My task is to find a cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ that describes this approach path.

3 Setting Up the Coordinate System

I establish a coordinate system with the following conventions:

- The origin $(0, 0)$ is located at the touchdown point on the runway
- The positive x-axis points to the right (in the direction from which the aircraft approaches)

- The variable x represents the horizontal distance from the touchdown point
- The function $P(x)$ represents the altitude (vertical height) of the aircraft above the runway
- Distance ℓ is the horizontal distance from touchdown where descent begins (so the aircraft starts at position $x = \ell$ to the right of the origin)
- Height h is the cruising altitude at the start of descent
- The aircraft flies from right to left, approaching the origin

4 Identifying the Boundary Conditions

To determine the four unknown coefficients (a, b, c, and d) in the cubic polynomial, I need four conditions. These conditions come from the physical requirements of a safe landing.

4.1 Conditions at the Start of Descent ($x = \ell$)

At position $x = \ell$ (to the right of the touchdown point), the aircraft begins its descent:

Position Condition: The aircraft is at cruising altitude

$$P(\ell) = h$$

Velocity Condition: The aircraft transitions smoothly from level flight, meaning the vertical velocity is zero (the flight path is horizontal)

$$P'(\ell) = 0$$

4.2 Conditions at Touchdown ($x = 0$)

At position $x = 0$ (the origin), the aircraft touches down:

Position Condition: The aircraft is on the ground

$$P(0) = 0$$

Velocity Condition: The aircraft lands smoothly with no vertical velocity (the flight path is horizontal at touchdown)

$$P'(0) = 0$$

These four conditions ensure a smooth transition from cruising flight to landing.

5 Mathematical Solution

5.1 Step 1: Finding the Derivative

Given the cubic polynomial:

$$P(x) = ax^3 + bx^2 + cx + d$$

I can use the power rule to find the derivative (which represents the slope of the flight path):

$$P'(x) = 3ax^2 + 2bx + c$$

5.2 Step 2: Applying Condition 1 - $P(0) = 0$

Substituting $x = 0$ into $P(x)$:

$$P(0) = a(0)^3 + b(0)^2 + c(0) + d$$
$$0 = d$$

$$\boxed{d = 0}$$

5.3 Step 3: Applying Condition 2 - $P'(0) = 0$

Substituting $x = 0$ into $P'(x)$:

$$\begin{aligned} P'(0) &= 3a(0)^2 + 2b(0) + c \\ 0 &= c \end{aligned}$$

Therefore: $\boxed{c = 0}$

Now the polynomial simplifies to:

$$P(x) = ax^3 + bx^2$$

$$P'(x) = 3ax^2 + 2bx$$

5.4 Step 4: Applying Condition 3 - $P(\ell) = h$

Substituting $x = \ell$ into the simplified $P(x)$:

$$\begin{aligned} P(\ell) &= a\ell^3 + b\ell^2 \\ h &= a\ell^3 + b\ell^2 \end{aligned}$$

This gives me the first equation:

$$\boxed{a\ell^3 + b\ell^2 = h} \quad \text{(Equation 1)}$$

5.5 Step 5: Applying Condition 4 - $P'(\ell) = 0$

Substituting $x = \ell$ into the simplified $P'(x)$:

$$\begin{aligned} P'(\ell) &= 3a\ell^2 + 2b\ell \\ 0 &= 3a\ell^2 + 2b\ell \end{aligned}$$

This gives me the second equation:

$$\boxed{3a\ell^2 + 2b\ell = 0} \quad \text{(Equation 2)}$$

5.6 Step 6: Solving the System of Equations

From Equation 2, I can solve for b in terms of a:

$$\begin{aligned} 3a\ell^2 + 2b\ell = 0 \\ 2b\ell &= -3a\ell^2 \\ b &= -\frac{3a\ell}{2} \end{aligned}$$

Now I substitute this expression for b into Equation 1:

$$\begin{aligned} a\ell^3 + b\ell^2 &= h \\ a\ell^3 + \left(-\frac{3a\ell}{2}\right)\ell^2 &= h \\ a\ell^3 - \frac{3a\ell^3}{2} &= h \end{aligned}$$

Factor out $a\ell^3$:

$$\begin{aligned}
a\ell^3\left(1-\frac{3}{2}\right) &= h \\
a\ell^3\left(\frac{2}{2}-\frac{3}{2}\right) &= h \\
a\ell^3\left(-\frac{1}{2}\right) &= h \\
-\frac{a\ell^3}{2} &= h \\
a\ell^3 &= -2h \\
a &= -\frac{2h}{\ell^3}
\end{aligned}$$

Therefore: $\boxed{a=-\frac{2h}{\ell^3}}$

Now I can find b:

$$\begin{aligned}
b &= -\frac{3a\ell}{2} \\
b &= -\frac{3}{2}\cdot\left(-\frac{2h}{\ell^3}\right)\cdot\ell \\
b &= \frac{3\cdot 2h\ell}{2\ell^3} \\
b &= \frac{3h}{\ell^2}
\end{aligned}$$

Therefore: $\boxed{b=\frac{3h}{\ell^2}}$

6 Final Answer for Part 1

Substituting all coefficients back into the polynomial, I obtain:

$$\boxed{P(x)=-\frac{2h}{\ell^3}x^3+\frac{3h}{\ell^2}x^2}$$

This can also be written in factored form as:

$$\boxed{P(x) = \frac{h}{\ell^3}\left(3\ell x^2 - 2x^3\right)}$$

Or factoring out x^2:

$$\boxed{P(x) = \frac{hx^2}{\ell^3}\left(3\ell - 2x\right)}$$

7 Interpretation

This polynomial describes the descent path where:

- At $x = \ell$ (far from touchdown, to the right): The aircraft is at altitude h with horizontal flight path
- As x decreases from ℓ to 0 (moving left toward touchdown): The aircraft descends smoothly
- At $x = 0$ (touchdown): The aircraft reaches the ground with a horizontal flight path

Note that the coefficient $a = -\frac{2h}{\ell^3}$ is negative, which makes sense because the altitude must decrease as the aircraft approaches touchdown from the right.

8 Verification

1. $P(0) = -\frac{2h}{\ell^3}(0)^3 + \frac{3h}{\ell^2}(0)^2 = 0$
2. $P'(x) = -\frac{6h}{\ell^3}x^2 + \frac{6h}{\ell^2}x$, so $P'(0) = 0$
3. $P(\ell) = -\frac{2h}{\ell^3}\ell^3 + \frac{3h}{\ell^2}\ell^2 = -2h + 3h = h$
4. $P'(\ell) = -\frac{6h}{\ell^3}\ell^2 + \frac{6h}{\ell^2}\ell = -\frac{6h}{\ell} + \frac{6h}{\ell} = 0$

9 Part 2: Deriving the Acceleration Constraint

Now I need to use conditions (ii) and (iii) to show that:

$$\frac{6hv^2}{\ell^2} \leq k$$

9.1 Understanding Vertical Acceleration

Condition (ii) states that the pilot maintains a constant horizontal speed v throughout descent. This means:

$$\frac{dx}{dt} = -v$$

(The negative sign indicates the aircraft is moving from right to left, toward the origin.)

Condition (iii) states that the absolute value of the vertical acceleration should not exceed k:

$$\left|\frac{d^2y}{dt^2}\right| \leq k$$

where $y = P(x)$ is the vertical position (altitude).

9.2 Relating Vertical Acceleration to the Polynomial

I need to find the vertical acceleration $\frac{d^2y}{dt^2}$ in terms of the polynomial $P(x)$.

Using the chain rule:

$$\frac{dy}{dt} = \frac{dP}{dx} \cdot \frac{dx}{dt} = P'(x) \cdot (-v) = -vP'(x)$$

Taking the derivative again:

$$\begin{aligned}
\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(-vP'(x)\right) \\
&= -v \cdot \frac{d}{dt}\left(P'(x)\right) \\
&= -v \cdot \frac{dP'(x)}{dx} \cdot \frac{dx}{dt} \\
&= -v \cdot P''(x) \cdot (-v) \\
&= v^2 P''(x)
\end{aligned}$$

Vertical acceleration:

$$\boxed{\frac{d^2y}{dt^2} = v^2 P''(x)}$$

9.3 Finding the Second Derivative of $P(x)$

From the previous work, I found:

$$P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2$$

The first derivative is:

$$P'(x) = -\frac{6h}{\ell^3}x^2 + \frac{6h}{\ell^2}x$$

The second derivative is:

$$P''(x) = -\frac{12h}{\ell^3}x + \frac{6h}{\ell^2}$$

I can factor this as:

$$P''(x) = \frac{6h}{\ell^2}\left(1 - \frac{2x}{\ell}\right)$$

9.4 Finding the Maximum Vertical Acceleration

The vertical acceleration is:

$$\frac{d^2y}{dt^2} = v^2 P''(x) = v^2 \cdot \frac{6h}{\ell^2}\left(1 - \frac{2x}{\ell}\right)$$

To find where this is maximum (in absolute value), I need to examine $P''(x)$ over the interval $0 \leq x \leq \ell$.

Let's evaluate $P''(x)$ at key points:

At $x = 0$:

$$P''(0) = \frac{6h}{\ell^2}(1 - 0) = \frac{6h}{\ell^2}$$

At $x = \ell$:

$$P''(\ell) = \frac{6h}{\ell^2}\left(1 - \frac{2\ell}{\ell}\right) = \frac{6h}{\ell^2}(1 - 2) = -\frac{6h}{\ell^2}$$

I can see that $P''(x)$ is a linear function that decreases from $\frac{6h}{\ell^2}$ at $x = 0$ to $-\frac{6h}{\ell^2}$ at $x = \ell$.

The maximum absolute value occurs at the endpoints:

$$\max_{0 \leq x \leq \ell} |P''(x)| = \frac{6h}{\ell^2}$$

9.5 Applying the Acceleration Constraint

The absolute value of the vertical acceleration is:

$$\left|\frac{d^2y}{dt^2}\right| = |v^2 P''(x)| = v^2 |P''(x)|$$

Since v^2 is always positive, the maximum vertical acceleration is:

$$\max_{0 \leq x \leq \ell} \left|\frac{d^2y}{dt^2}\right| = v^2 \cdot \max_{0 \leq x \leq \ell} |P''(x)| = v^2 \cdot \frac{6h}{\ell^2} = \frac{6hv^2}{\ell^2}$$

Condition (iii) requires that this maximum not exceed k:

$$\boxed{\frac{6hv^2}{\ell^2} \leq k}$$

9.6 Physical Interpretation

This constraint tells me that:

- Higher cruising altitude h requires larger k (more vertical acceleration capability)
- Higher horizontal speed v requires larger k (more vertical acceleration capability)
- Longer approach distance ℓ reduces the required k (gentler descent)

In practice, this means that for a given maximum acceptable vertical acceleration k, the pilot must either:

- Begin descent farther from the airport (increase ℓ), or
- Reduce horizontal speed (decrease v), or
- Start from a lower altitude (decrease h)

10 Part 3: Calculating the Descent Distance

10.1 Given Information

A flight is in the folowing state:

- Maximum vertical acceleration: $k = 860$ mi/h^2
- Cruising altitude: $h = 35,000$ ft
- Horizontal speed: $v = 300$ mi/h

I need to find the minimum distance ℓ from the airport where the pilot should start descent.

10.2 Unit Conversion

First, I need to convert the altitude from feet to miles so all the units are consistent.

$$h = 35,000 \text{ ft} \times \frac{1 \text{ mi}}{5,280 \text{ ft}} = \frac{35,000}{5,280} \text{ mi} = 6.6288 \text{ mi}$$

10.3 Solving for ℓ

From Part 2, I derived that:

$$\frac{6hv^2}{\ell^2} \leq k$$

To find the minimum descent distance, I can use the equality (the most restrictive case):

$$\frac{6hv^2}{\ell^2} = k$$

Solving for ℓ:

$$\begin{aligned}
\frac{6hv^2}{\ell^2} &= k \\
6hv^2 &= k\ell^2 \\
\ell^2 &= \frac{6hv^2}{k} \\
\ell &= \sqrt{\frac{6hv^2}{k}}
\end{aligned}$$

10.4 Substituting Values

Now I substitute the values:

$$\begin{aligned}
\ell &= \sqrt{\frac{6hv^2}{k}} \\
&= \sqrt{\frac{6(6.6288)(300)^2}{860}} \\
&= \sqrt{\frac{6(6.6288)(90,000)}{860}} \\
&= \sqrt{\frac{3,579,552}{860}} \\
&= \sqrt{4,162.27} \\
&= 64.52 \text{ mi}
\end{aligned}$$

10.5 Answer

The pilot should start descent approximately $\boxed{\text{64.52 miles}}$ away from the airport.

This means the aircraft will begin its descent when it is about 64.52 miles horizontally from the touchdown point, ensuring that the vertical acceleration never exceeds the airline's safety limit of 860 mi/h^2.

11 Part 4: Graphing the Approach Path

Now I will graph the approach path using the conditions from Part 3.

11.1 The Polynomial Function

Using the values:

- $h = 6.6288$ mi
- $\ell = 64.52$ mi

The approach path polynomial is:

$$P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2$$

Substituting the values:

$$P(x) = -\frac{2(6.6288)}{(64.52)^3}x^3 + \frac{3(6.6288)}{(64.52)^2}x^2$$

$$P(x) = -\frac{13.2576}{268,560.96}x^3 + \frac{19.8864}{4,162.83}x^2$$

$$P(x) = -0.0000494x^3 + 0.00478x^2$$

where x is in miles and $P(x)$ is in miles (altitude).

11.2 Key Points on the Graph

Let's calculate some key points:

At start of descent ($x = 64.52$ mi):

$$P(64.52) = 6.6288 \text{ mi} = 35,000 \text{ ft}$$

At halfway ($x = 32.26$ mi):

$$\begin{aligned} P(32.26) &= -0.0000494(32.26)^3 + 0.00478(32.26)^2 \\ &= -1.656 + 4.969 \\ &= 3.313 \text{ mi} \approx 17,493 \text{ ft} \end{aligned}$$

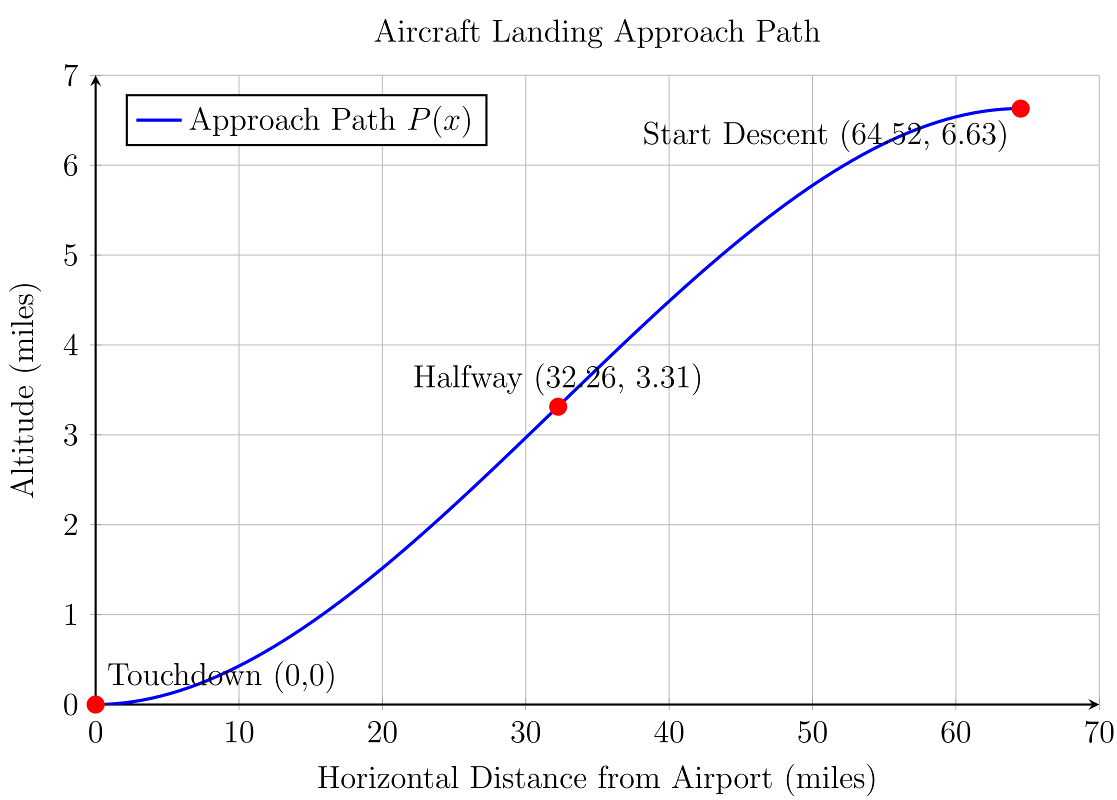
At touchdown ($x = 0$ mi):

$$P(0) = 0 \text{ mi} = 0 \text{ ft}$$

11.3 Graph

The graph of the approach path shows:

- The aircraft starts at position $(64.52, 6.6288)$ - that is, 64.52 miles horizontally from the airport at an altitude of 6.6288 miles (35,000 ft)
- The path curves smoothly downward as the aircraft approaches the airport
- The aircraft touches down at the origin $(0, 0)$
- At both the start and end of the path, the curve is horizontal (zero slope), indicating smooth transitions



12 Conclusion

1. Derived a cubic polynomial $P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2$ that models a safe and smooth aircraft landing path

2. Used the constant horizontal speed condition and the vertical acceleration constraint to derive that $\frac{6hv^2}{\ell^2} \leq k$

3. Calculated that for an airline with $k = 860$ mi/h^2, cruising altitude of 35,000 ft, and speed of 300 mi/h, the pilot should begin descent approximately 64.52 miles from the airport

4. Described the graph of the approach path, which shows a smooth cubic curve from the starting altitude down to the runway

This mathematical model demonstrates how calculus can be applied to solve practical engineering problems in aviation. By understanding the relationship between speed, altitude, distance, and acceleration, pilots and engineers can design safe landing approaches. The specific calculation shows that even for a relatively modest cruising altitude, aircraft need to begin their descent many miles before reaching the airport to maintain speed restrictions and excess speed on the airframe in the approach configuration.