PHY 411 – Electrodynamics I

Winter 2021

In Preparation for the Final Exam

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Final Examination

These slides were presented in class on Thursday (Mar 11), and are now the *Study Guide for the Final Examination*.

- Final Exam is on Tuesday (March 16) at class time (begins 4:20 PM).
- You must be present at the Zoom meeting at the beginning of class in order to take the Final Exam.
- Final Exam will have Part 1 (due within about the first hour of class) and Part II (due on a slightly longer time schedule); details will be announced the day before the exam.
- During the exam, you will have access to all class materials posted on D2L; looking at any other text, online, or talking to anyone (other than Dr. Sarma) will be grounds for academic integrity violations.

Warning!

This is *not a substitute* for class summaries and homework solutions. It is only a list of ideas and/or equations I want you to remember. If this is the only thing you look at, expect to do very badly on the exam!

Caveat: In the following slides, I've mentioned several relations that will be supplied on formula sheets. To avoid any confusion, I'd like to add here that *an exception would be if I asked you to derive any of these relations on the test, in which case, they would obviously not be on the Formula Sheet*.

Maxwell's Equations

- $\vec{\nabla} \cdot \vec{D} = \rho$
- $\vec{\nabla} \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}$
- $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$
- $\vec{\nabla} \cdot \vec{B} = 0$

I will supply on Formula Sheet for Final Exam, but hope you have already committed to memory.

Throughout this document, formulas you need to remember and additional commentary will be provided in blue font; this is extra from what was presented in class.

Expect you to remember the constitutive relations:

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

Note that remembering these is not just for the exam, but for your own good. Not knowing them gives off the wrong vibe!

See Class Summaries for more details

Maxwell's Equations in source-free space

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} - i\omega \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \vec{\nabla} \times \vec{B} + i\omega\mu\epsilon \vec{E} = 0$$

assumes harmonic time dependence

You should know how to derive these from the supplied Maxwell equations.

See Class Summaries and Worksheets for detailed discussions, especially on deriving these.

Helmholtz Wave Equation

$$\left(\nabla^2 + \mu\epsilon\omega^2\right) \vec{E} = 0$$
$$\left(\nabla^2 + \mu\epsilon\omega^2\right) \vec{B} = 0 \qquad (7.3)$$

I will supply as

$$\left(\nabla^2 + k^2\right)\vec{u} = 0$$

Expect you to know

$$k = \omega\sqrt{\mu\epsilon}$$

Jackson, equation (7.5)

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}, \qquad \text{where } n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

This is a key set of relationships. Make sure you learn it, and remember it well!

How do you remember? It's enough to remember that $k = \omega\sqrt{\mu\epsilon}$

Plane Electromagnetic Waves

Plane Wave Solutions:

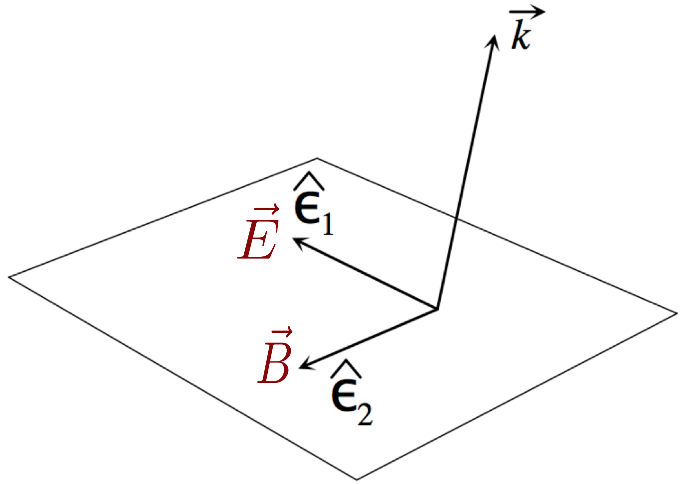
$$\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\vec{B} = \sqrt{\mu\epsilon}\, \frac{\vec{k} \times \vec{E}}{k}$$

$\vec{B}$ will be on the Formula Sheet, but you must be able to write $\vec{E}$ ***by yourself***, especially in 1-D: see HW 4

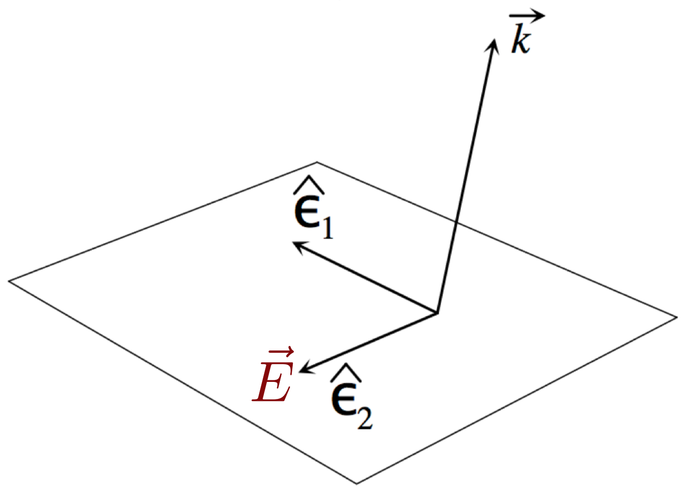
Polarization of Waves

$$\vec{E}_1 = \hat{\epsilon}_1 \, E_1 \, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$\vec{E}_2 = \hat{\epsilon}_2 \, E_2 \, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.18)$$



most general homogenous plane wave

$$\vec{E}(\vec{x},t) = \left(\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.19)$$



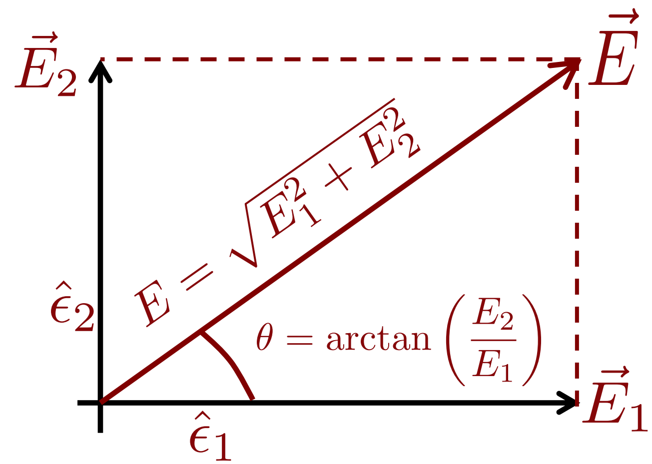
You must know how to write these – there is nothing new to memorize here since you'd be able to write these if you knew how to write a plane wave. However, you must understand why you can represent waves in this manner.

Polarization of Waves

$$\vec{E}_1 = \hat{\epsilon}_1\, E_1\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$\vec{E}_2 = \hat{\epsilon}_2\, E_2\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad (7.18)$$

$$\vec{E}(\vec{x},t) = \left(\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad (7.19)$$

Linear Polarization:



Again, there is nothing new to memorize here. However, you must understand what is a linearly polarized wave and how it is represented in this formalism.

Likewise for elliptically and circularly polarized waves on the next slide.

For details, see Class Summaries.

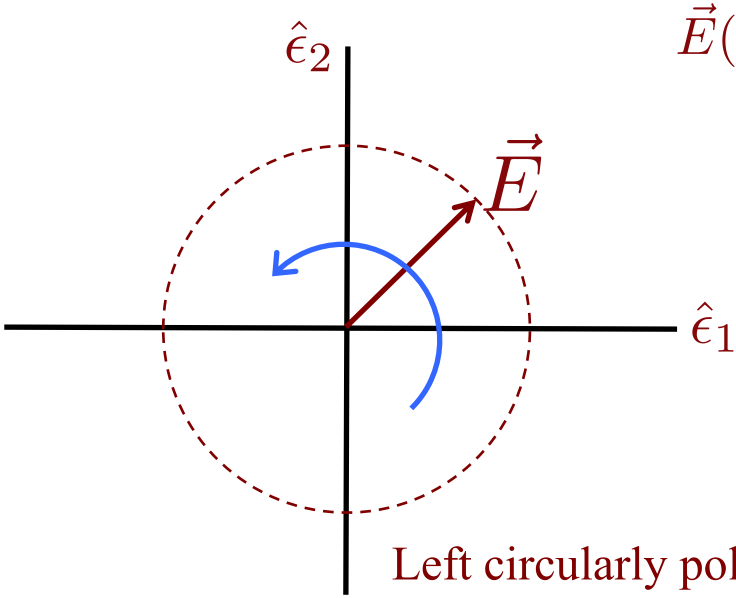
Polarization of Waves

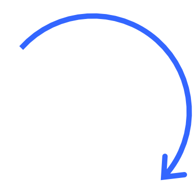
Must know everything on this page!

Elliptical Polarization: E_1 and E_2 have different phases

Circular Polarization: E_1 and E_2 are equal in magnitude but differ in phase by $\pi/2$

$$\vec{E}(\vec{x},t) = E_0 \left(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.20)$$



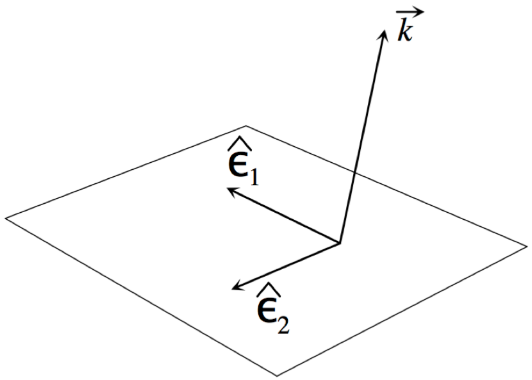


Right circularly polarized $\left(\hat{\epsilon}_1 - i\hat{\epsilon}_2\right)$

Left circularly polarized $\left(\hat{\epsilon}_1 + i\hat{\epsilon}_2\right)$

Polarization: Another set of basis vectors

$$\hat{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}\left(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2\right) \qquad (7.22)$$



$$\vec{E}(\vec{x},t) = \left(E_+\hat{\epsilon}_+ + E_-\hat{\epsilon}_-\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.24)$$

$$\vec{E}(\vec{x},t) = \left(\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.19)$$

I don't expect you to memorize how the new set of basis vectors is constructed – I'll provide that on the test, if there is a problem/question on it. But I do expect you to be able to appreciate that they are a completely equivalent set of basis vectors to describe polarization.

Stokes Parameters

Linear polarization basis:

$$\vec{E}(\vec{x},t) = \left(\hat{\epsilon}_1 \underbrace{E_1}_{a_1 e^{i\delta_1}} + \hat{\epsilon}_2 \underbrace{E_2}_{a_2 e^{i\delta_2}}\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.19)$$

Can then write Stokes parameters s_0, s_1, s_2, s_3, in linear polarization basis

- See eq. (7.27) on page 301

Circular polarization basis:

$$\vec{E}(\vec{x},t) = \left(\underbrace{E_+}_{a_+ e^{i\delta_+}} \hat{\epsilon}_+ + \underbrace{E_-}_{a_- e^{i\delta_-}} \hat{\epsilon}_-\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.24)$$

Can then write Stokes parameters s_0, s_1, s_2, s_3, in circular polarization basis

- See eq. (7.28) on page 301

Reflection and Refraction: Kinematic Properties

Incident wave:

$$\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$\vec{B} = \sqrt{\mu\epsilon}\,\frac{\vec{k}\times\vec{E}}{k}$$

Refracted wave:

$$\vec{E}' = \vec{E}_0'\, e^{i(\vec{k}'\cdot\vec{x}-\omega t)}$$

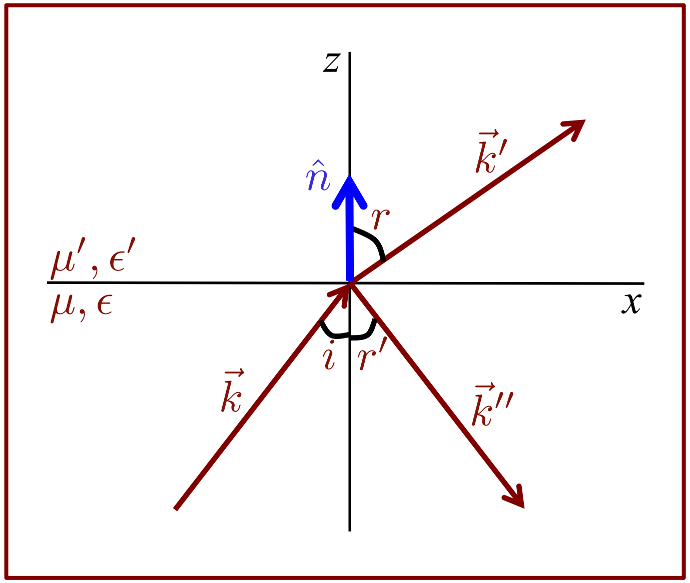
$$\vec{B}' = \sqrt{\mu'\epsilon'}\,\frac{\vec{k}'\times\vec{E}'}{k'}$$

Reflected wave:

$$\vec{E}'' = \vec{E}_0''\, e^{i(\vec{k}''\cdot\vec{x}-\omega t)}$$

$$\vec{B}'' = \sqrt{\mu\epsilon}\,\frac{\vec{k}''\times\vec{E}''}{k}$$

You must remember and be able to write *E* and *B* for incident, refracted and reflected waves (especially in 1-D): see HW 2.



Boundary conditions must be satisfied at all points on the interface $z = 0$ at all times

$$(\vec{k}\cdot\vec{x})_{z=0} = (\vec{k}'\cdot\vec{x})_{z=0} = (\vec{k}''\cdot\vec{x})_{z=0} \qquad (7.34)$$

Must know everything else on this page.

$$k \sin\, i = k' \sin\, r = k'' \sin\, r' \qquad (7.35)$$

- All three wave vectors lie in the same plane
- Law of reflection: angle i = angle r'
- Snell's Law: $n \sin\, i = n' \sin\, r$

Reflection and Refraction: Dynamic Properties

Boundary conditions:

$$\left[\epsilon\left(\vec{E}_0+\vec{E}_0''\right)-\epsilon'\,\vec{E}_0'\right]\cdot\hat{n}=0 \longrightarrow (7.37.a)$$

$$\left[\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0''-\vec{k}'\times\vec{E}_0'\right]\cdot\hat{n}=0 \longrightarrow (7.37.b)$$

$$\left[\vec{E}_0+\vec{E}_0''-\vec{E}_0'\right]\times\hat{n}=0 \longrightarrow (7.37.c)$$

$$\left[\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0''\right)-\frac{1}{\mu'}\left(\vec{k}'\times\vec{E}_0'\right)\right]\times\hat{n}=0 \longrightarrow (7.37.d)$$

Must know:

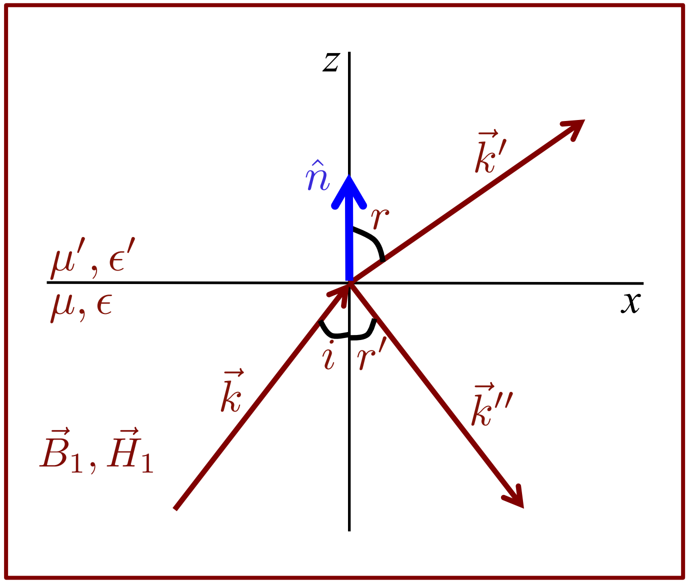
Normal components of $\vec{D}$ and $\vec{B}$ are continuous

Tangential components of $\vec{E}$ and $\vec{H}$ are continuous

I don't expect you to memorize the equations above (it would be useless), but I do expect that you will be able to *construct/derive* them as needed by remembering whose normal components are continuous, and whose tangential components are continuous. See Class Summaries and HW 3.

The way ahead: Break into 2 problems

- $\vec{E}$ perpendicular to plane of incidence
- $\vec{E}$ parallel to plane of incidence



Jackson Figure 7.6 (a) on page 305

Reflection and Transmission Coefficients

$$\boxed{\vec{S}\cdot\hat{n} = = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left|E_0\right|^2 \cos\, i}$$

$$\vec{S}'\cdot\hat{n} = \frac{1}{2}\sqrt{\frac{\epsilon'}{\mu'}}\left|E_0'\right|^2 \cos\, r$$

$$\vec{S}''\cdot\hat{n} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left|E_0''\right|^2 \cos\, r'$$

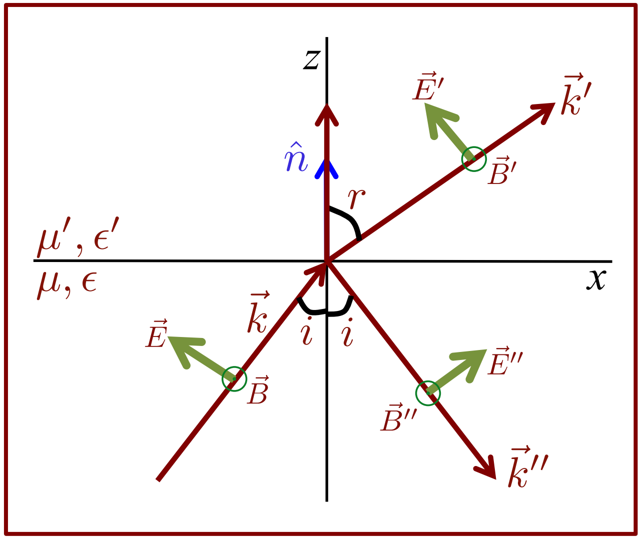
$$T = \frac{\vec{S}'\cdot\hat{n}}{\vec{S}\cdot\hat{n}}$$

$$R = \frac{\vec{S}''\cdot\hat{n}}{\vec{S}\cdot\hat{n}}$$

You must remember and be able to write these. See Class Summaries and HW 3 for how to use them.

This generic version will be supplied on Formula Sheet.
You must be able to build the other two by analogy.

Do not memorize derived transmission and reflection coefficients for perpendicular and parallel cases. You must obtain them, if there is such a problem, by starting from scratch (i.e., by writing E and B fields).



Brewster's Angle

For $\vec{E}$ parallel to the plane of incidence

$$\frac{E_0''}{E_0} = \frac{\boxed{\frac{\mu}{\mu'} n'^{\,2} \cos i - n\sqrt{n'^{\,2} - n^2 \sin^2 i}}}{\frac{\mu}{\mu'} n'^{\,2} \cos i + n\sqrt{n'^{\,2} - n^2 \sin^2 i}} \qquad = 0$$

I don't expect you to memorize the equation above (it would be useless), but I do expect that you will know about the Brewster angle and how it is calculated (see below), and understand how the reflection coefficient curves will look (see HW 3).

The amplitude of the reflected wave will be zero

The angle of incidence for which the amplitude of the reflected wave will be zero is given by:

$$i_B = \arctan\left(\frac{n'}{n}\right)$$

Total Internal Reflection

If $n > n'$, then Snell's law tells us there is an i_0 for which $r = 90°$.

$$\sin\, i_0 = \frac{n'}{n} \quad \Rightarrow \quad i_0 = \arcsin\left(\frac{n'}{n}\right) \qquad (7.43)$$

What happens if $i > i_0$?

- $\sin\, r = \left(\frac{n}{n'}\right) \sin\, i = \frac{\sin\, i}{\sin\, i_0} \quad \Rightarrow \quad \sin\, r > 1$

- $\cos\, r = i\sqrt{\left(\frac{\sin\, i}{\sin\, i_0}\right)^2 - 1}$

  Caution! i = imaginary (if in blue-colored font)

- $e^{i\vec{k}'\cdot\vec{x}} = e^{ik'(x\sin\, r + z\cos\, r)} = \underbrace{e^{-k'\left(\sqrt{(\sin\, i/\sin\, i_0)^2 - 1}\right)z}}_{\text{The refracted wave is attenuated exponentially beyond the interface } z = 0} e^{ik'(\sin\, i/\sin\, i_0)\,x}$

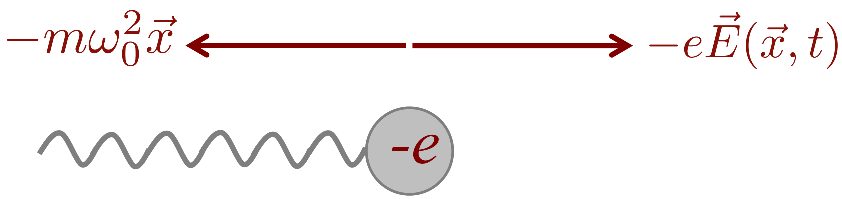
I expect you to remember and be able to apply Snell's law to find the critical angle of incidence for total internal reflection, and derive what happens to the refracted wave, as done on this page

For $i > i_0$, get Total Internal Reflection

Dispersion
Model for time-varying fields

$$m\left[\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}\right] = -e\vec{E}(\vec{x},t) \qquad (7.49)$$

I expect you to understand how this, and the static model before it, are set up. See Class Summaries for details.



$$\sum F \equiv -e\vec{E} - m\omega_0^2\vec{x} = \quad m\ddot{\vec{x}}$$

Electron interacts with other electrons, losing energy; so account for this by introducing a damping term

$$-m\gamma\dot{\vec{x}}$$

You must also remember:

$$\epsilon = \epsilon_0\left(1+\chi_e\right)$$

$$\vec{P} = \epsilon_0\,\chi_e\,\vec{E}$$

$$\vec{p}_{\rm mol} = e\vec{x}$$

$$\vec{P} = N\left\langle\vec{p}_{\rm mol}\right\rangle$$

Dispersion

Complex Dielectric Constant:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\boxed{N}e^2}{\epsilon_0 m} \sum_j \frac{\boxed{f_j}}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \tag{7.51}$$

Number of molecules per unit volume

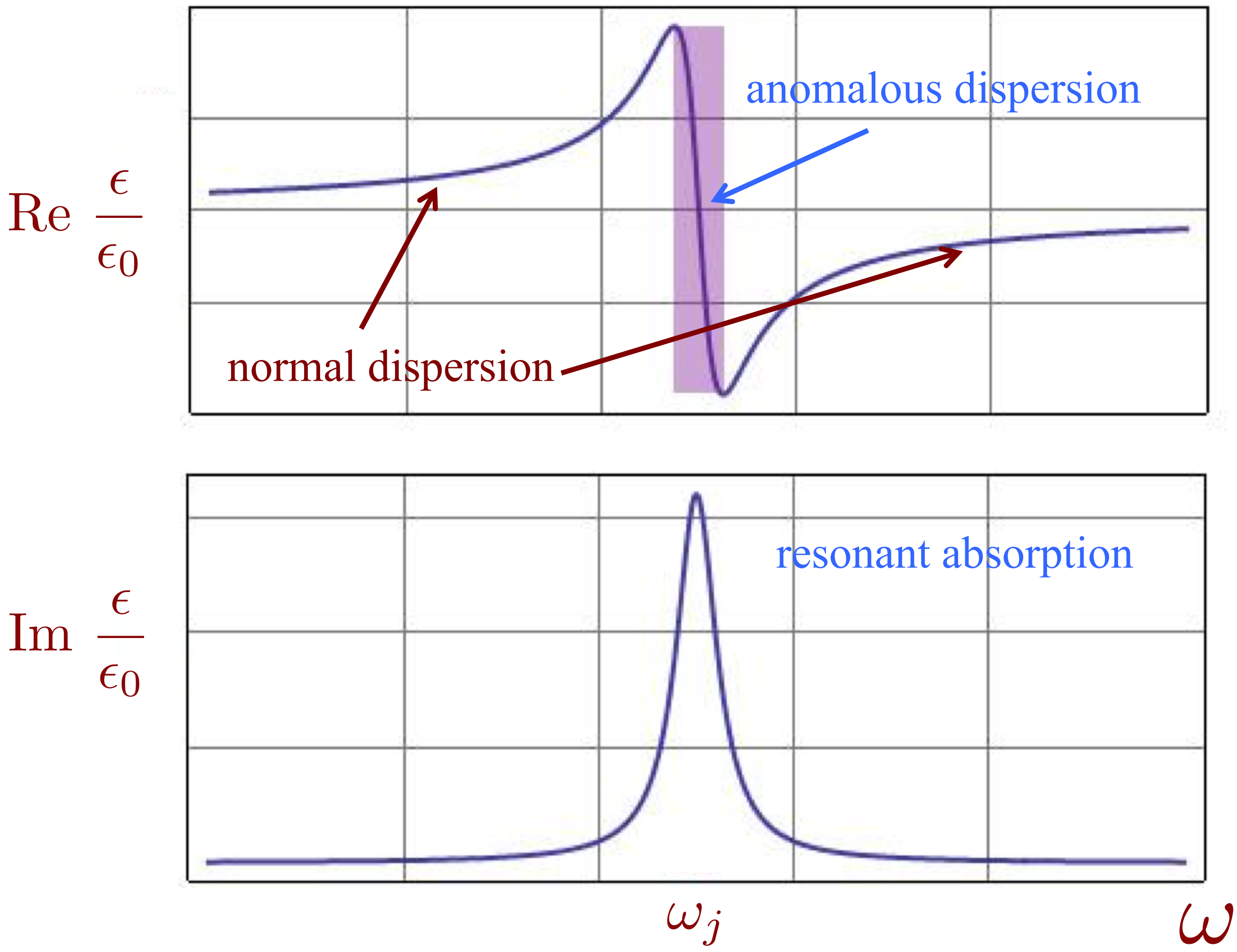
Number of electrons per molecule with binding frequency ω_j

I don’t expect you to memorize this equation, but you must understand it enough to be able to explain what all the terms mean.

You must also understand the consequences of a complex dielectric constant, one example being the graph on the next page, another being the attenuation of a plane wave as it propagates.

Read the Class Summaries and refer to the Worksheets for details.

Normal and Anomalous Dispersion



Source: *Wolfram*

Attenuation of a plane wave

What is the most appropriate quantity to pick in order to describe the attenuation of a plane wave?

Complex $\epsilon(\omega)$ ⇒ Complex $n(\omega)$ ⇒ Complex k

Be able to express k in this form, and then be aware of what you get, as shown in the expression below, and the boxes in this slide.

$$k = \beta + i\frac{\alpha}{2} \qquad (7.53)$$

$$\vec{E}(\vec{x}) \sim e^{ikz} = e^{-\frac{\alpha}{2}z} e^{i\beta z}$$

intensity of the wave ($|\vec{E}|^2$) falls off as $e^{-\alpha z}$

α is called the *attenuation constant* or *absorption coefficient*

Affects traveling part of the wave; its dependence on frequency means that waves with different frequencies travel at different speeds

Low Frequency Behavior

In the limit $\omega \to 0$, there is a qualitative difference in the response of the medium depending on whether or not there is a resonance at zero.

- If a resonance does not exist at $\omega_i = 0$ (i.e., the lowest resonant frequency is different from zero), we have a dielectric insulator whose details match what we wrote while discussing the static case.
- If there is a resonance $\omega_0 = 0$, then $\epsilon(\omega)$ has a complex component that attenuates the propagation of electromagnetic energy. We know now that this describes conduction.

Suppose some fraction f_0 of the electrons per molecule have their lowest resonance frequency at $\omega_0 = 0$.

$\Downarrow$

"free" electrons

Low Frequency Behavior

Suppose some fraction f_0 of the electrons per molecule have their lowest resonance frequency at $\omega_0 = 0$.

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \left[\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \right] \tag{7.51}$$

$$\epsilon(\omega) = \epsilon_b(\omega) + \underbrace{i \left[\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \right]}_{\text{“free” electrons}} \tag{7.56}$$

“bound” dipoles

I don’t expect you to memorize (7.56) but you must understand where it came from, and what it means. See the Class Summaries and Worksheets for details of this and the previous slide.

Low Frequency Behavior

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = -i\omega \left(\epsilon_b + i\,\frac{\sigma}{\omega} \right) \vec{E}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial (\epsilon \vec{E})}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = -i\omega \left[\epsilon_b + i \left\{ \frac{N e^2 f_0}{m\omega(\gamma_0 - i\omega)} \right\} \right] \vec{E}$$

Drude model for electrical conductivity

$$\sigma = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)} \qquad (7.58)$$

Tells us that the “conductivity” is closely related to the complex dielectric constant when the lowest resonant frequency is zero.

Nothing to memorize here, but be aware of what you need to do to get to the Drude model (as you did on the Discussion Worksheet).

The High Frequency Limit

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \left[\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \right] \qquad (7.51)$$

- At frequencies >> highest resonant frequency

$$\frac{\epsilon(\omega)}{\epsilon_0} \simeq 1 - \frac{\omega_P^2}{\omega^2} \qquad (7.59)$$

where

$$\omega_P^2 = \frac{NZe^2}{\epsilon_0 m} \qquad (7.60)$$

Plasma frequency

Number of molecules per unit volume

Number of electrons per molecule

NZ = number of electrons per unit volume

I don't expect you to memorize (7.59), but you must know where it came from and how to obtain it, and hence to find (7.60). You must also understand the consequences of this limit, as described on this slide and in the class summary.

The High Frequency Limit

- At frequencies >> highest resonant frequency

$$\frac{\epsilon(\omega)}{\epsilon_0} \simeq 1 - \frac{\omega_P^2}{\omega^2} \qquad (7.59)$$

where

$$\omega_P^2 = \frac{NZe^2}{\epsilon_0 m} \qquad (7.60)$$

In particular, you must understand that starting from (7.59) and (7.60), you can obtain a dispersion relation, as written below.

- Wave number k in this limit:

$$ck = \sqrt{\omega^2 - \omega_p^2} \qquad (7.61)$$

sometimes written as

$$\omega^2 = \omega_p^2 + c^2 k^2 \longrightarrow$$ dispersion relation for $\omega = \omega(k)$

The High Frequency Limit

- At frequencies >> highest resonant frequency

$$\frac{\epsilon(\omega)}{\epsilon_0} \simeq 1 - \frac{\omega_P^2}{\omega^2} \qquad (7.59)$$

- In dielectric media, (7.59) holds only for $\omega^2 \gg \omega_p^2$

- In certain situations (ionosphere, tenuous lab plasmas) where all electrons are essentially free so that damping is negligible, (7.59) can hold for a wide range of frequencies, including $\omega < \omega_p$

But if $\omega < \omega_p$, then wave number k is purely imaginary.

$$ck = \sqrt{\omega^2 - \omega_p^2} \qquad (7.61)$$

- Such waves incident on a plasma are reflected and the fields inside fall off exponentially with distance from the surface.

Attenuation constant at $\omega = 0$

$$\alpha_{\text{plasma}} \simeq \frac{2\omega_p}{c} \qquad (7.62)$$

The High Frequency Limit: Metals

- The reflectivity of metals at optical and higher frequencies is caused by essentially the same behavior as for the tenuous plasma

$$\epsilon(\omega) = \epsilon_b(\omega) + i\left[\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}\right] \qquad (7.56)$$

You must know about these results.

- At frequencies for which $\omega \gg \gamma_0$

$$\epsilon(\omega) \simeq \epsilon_b(\omega) - \frac{\omega_p^2}{\omega^2}\epsilon_0 \qquad (7.59.a)$$

$$\Downarrow$$

$k = \omega\sqrt{\mu\epsilon}$ is imaginary

- Same behavior as plasma for $\omega \ll \omega_p$ $\longrightarrow$ Light penetrates only a short distance into the metal and is almost completely reflected.

The High Frequency Limit: Metals

$$\vec{E} \sim e^{i(kz-\omega t)}$$

$$k = \beta + i\alpha/2$$

$$\vec{E} \sim e^{-\boxed{\frac{\alpha}{2}}z}\, e^{\cdots}$$

dimensions of 1/length

- Therefore, the penetration into the metal is characterized by a parameter known as the *skin depth* δ, given by

$$\delta = \frac{2}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Don't memorize, but you must be able to derive such an expression for the skin depth.

- However, when the frequency is increased into the domain where the dielectric constant becomes positive, which is typically in the ultraviolet, metals are suddenly able to transmit light, and become transparent.

Chandra X-ray Observatory, NASA

Propagation through Dispersive Media

There are no truly monochromatic waves!

But we can superpose monochromatic plane-wave solutions

Need to consider:

- If medium is dispersive (i.e., dielectric constant is a function of frequency), then the phase velocity is no longer the same for each frequency component of the wave. As a result, different components of the wave travel with different speeds and tend to change phase with respect to one another.

- If the medium is dispersive, the velocity of energy flow may differ greatly from the phase velocity: $$v_p = \frac{\omega}{k}$$

- If the medium is dissipative, a pulse of radiation will be attenuated as it travels with (or without) distortion, depending on whether the dissipative effects are (or are not) functions of frequency.

Propagation through Dispersive Media

- Build solution via Fourier series

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx - i\omega(k)t}\, dk \tag{7.80}$$

Don't memorize (7.80) and (7.81), but understand what we are doing here.

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx \tag{7.81}$$

- Group velocity

$$v_g = \left.\frac{d\omega}{dk}\right|_{k_0} \tag{7.86}$$

Like (7.5) for the phase velocity, you must not only understand the idea of group velocity (see Class Summaries for details), but also remember (7.86).

Vector and Scalar Potentials

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad (6.7)$$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \qquad (6.9)$$

It is absolutely essential to remember how to arrive at these two relations, in addition to remembering the relations themselves.

Vector and Scalar Potentials

- Maxwell's equations

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{6.6}$$

There is no need to memorize the equations below, but you must know how to get to them (as you worked out on the worksheet).

$$\nabla^2 \Phi + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -\frac{\rho}{\epsilon_0} \tag{6.10}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla} \left[\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right] = -\mu_0 \vec{J} \tag{6.11}$$

4 equations reduced to 2

But still coupled ...

Vector and Scalar Potentials

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0} \tag{6.10}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right] = -\mu_0\vec{J} \tag{6.11}$$

$\vec{E}, \vec{B}$ are left unchanged by the transformation

You must know the implications of this transformation, even though you might not remember the exact relations.

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \tag{6.12}$$

$$\Phi \to \Phi - \frac{\partial\Lambda}{\partial t} \tag{6.13}$$

Choose $(\vec{A}, \Phi)$ to satisfy the Lorenz condition

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0 \tag{6.14}$$

You must be able to pick this out as the Lorenz condition if it is presented to you.

Vector and Scalar Potentials

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \qquad (6.15)$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \qquad (6.16)$$

$\Phi, \vec{A}$
uncoupled

What we just did is called a *gauge transformation*

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \qquad (6.12)$$

$$\Phi \to \Phi - \frac{\partial \Lambda}{\partial t} \qquad (6.13)$$

and the invariance of the fields under such transformations is called *gauge invariance*

I don't expect you to memorize (6.15) or (6.16) but you must understand where they came from, and how to get to them. See the Class Summary for details.

Gauge Transformations

Gauge transformation:

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \tag{6.12}$$

$$\Phi \to \Phi - \frac{\partial \Lambda}{\partial t} \tag{6.13}$$

Restricted gauge transformation:

I don't expect you to memorize (6.19) or (6.20) but you must understand their difference from the gauge transformation in (6.12) & (6.13). See the Class Summary for details.

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \qquad \Phi \to \Phi - \frac{\partial \Lambda}{\partial t} \tag{6.19}$$

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0 \tag{6.20}$$

All potentials in this restricted class are said to belong to the *Lorenz gauge*

Preparing for Chapter 9
Green Functions

- Green functions were preparatory to Chapter 9, so they will not be included on the Final Exam.

- We will see them in PHY 420, just before we begin Chapter 9.

Chapters 1-5

Poisson's equation and Laplace's equation

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \qquad (1.28)$$

$$\nabla^2\Phi = 0 \qquad (1.29)$$

Laplace's Equation

- We will now study several techniques that involve working with Laplace's equation.
- First, though, why Laplace's equation?
- Many problems in electrostatics involve boundary surfaces on which *Dirichlet* or *Neumann* conditions are specified. The formal solution of such problems involves Green functions, discussed by Jackson in Chapter 1.
- In practical situations, or even idealized approximations to practical situations, however, it might be difficult sometimes to build the Green function.
- That's why a number of approaches to the boundary value problems have been developed in electrostatics.

Approaches to Boundary Value Problems

- Method of images
- Expansion in orthogonal functions → Laplace equation
- Numerical methods

Relevant when we have a charge-free region of space bounded by surfaces on which the potential is known

Laplace's Equation in rectangular coordinates

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{2.48}$$

- Apply separation of variables

$$\Phi(\vec{x}) = X(x)Y(y)Z(z) \tag{2.49}$$

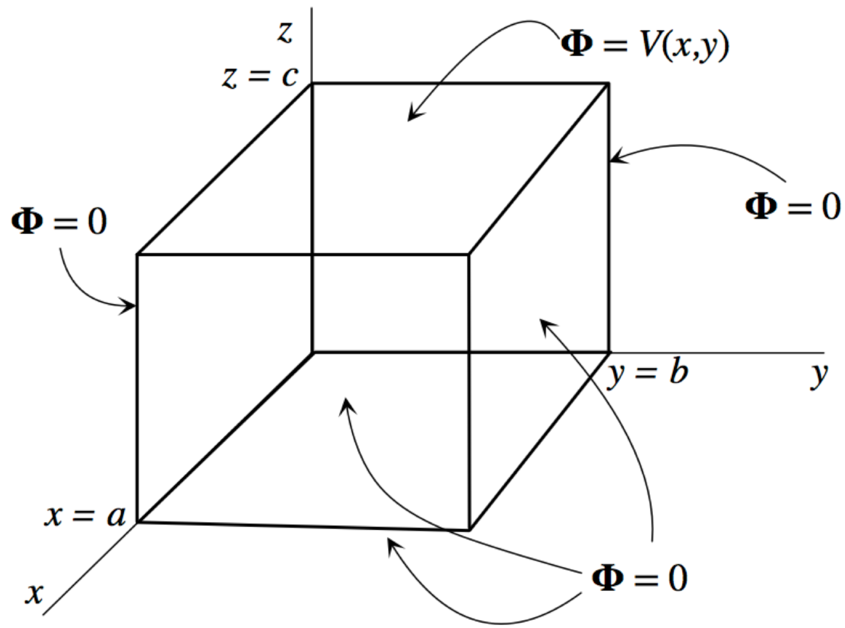
Substitute this in (2.48)

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}} = 0 \tag{2.50}$$

Why can we equate each of these to a constant?

Laplace's Equation in rectangular coordinates

- Worksheet Problem: Use the separation of variables method to solve Laplace's equation when the potential is zero on all faces of a cubical enclosure (see figure below), except on the face $z = c$, where it is $V(x,y)$.



Orthogonal Functions and Expansions

- Solutions to, e.g., Laplace's equation involve handling orthogonal expressions
- Go over Section 1.8 (pages 67-69) in Jackson to review
- In particular, you should be familiar and comfortable with the following:
  - Orthonormality of a set of real or complex functions $U_n(\xi)$ — equations (2.28) and (2.29)
  - Expanding an arbitrary function $f(\xi)$ in a series of orthonormal functions $U_n(\xi)$ — equations (2.30) and (2.33)
  - Evaluating the coefficients in the expansion of $f(\xi)$ — equation (2.32)
  - Being aware of the most famous orthogonal functions, the sines and cosines — equations (2.36) and (2.37)
  - Generalization to higher dimensions — equations (2.38) and (2.39)
  - Expansion of a continuum of functions through the Fourier integral — equations (2.40) up to (2.46)

Laplace's Equation in spherical coordinates

$$\frac{1}{r}\,\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\,\sin\,\theta}\,\frac{\partial}{\partial\theta}\left(\sin\,\theta\,\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\,\sin^2\,\theta}\,\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{3.1}$$

- Apply separation of variables by assuming

$$\Phi = \frac{U(r)}{r}\,P(\theta)Q(\phi) \tag{3.2}$$

Extra factor of *r* included to force the same dimensionality into each factor

- Substitute in (3.1) and show that we get:

$$r^2\,\sin^2\theta\left[\frac{1}{U}\,\frac{d^2U}{dr^2} + \frac{1}{P\,r^2\sin\,\theta}\,\frac{d}{d\theta}\left(\sin\,\theta\,\frac{dP}{d\theta}\right)\right] + \underbrace{\frac{1}{Q}\,\frac{d^2Q}{d\phi^2}}_{\text{function of } \phi \text{ only}} = 0 \tag{3.3}$$

Laplace's Equation in spherical coordinates

$$r^2 \sin^2\theta \left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{P\, r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right)\right] + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0 \qquad (3.3)$$

- Set

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2 \qquad (3.4)$$

- The solution is

$$Q = e^{\pm i m \phi} \qquad (3.5)$$

where we take *m* to be an integer, so we can allow the full azimuthal range of values

- So far, the solution is

$$\Phi = \frac{U(r)}{r}\, P(\theta)\, e^{\pm i m \phi}$$

Laplace's Equation in Spherical Coordinates

- Other part separates into:

$$\frac{1}{\sin\,\theta}\,\frac{d}{d\theta}\left(\sin\,\theta\,\frac{dP}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\,\theta}\right]P=0 \tag{3.6}$$

$$\frac{d^2U}{dr^2}-\frac{l(l+1)}{r^2}\,U=0 \tag{3.7}$$

- Solution of radial equation: $$U = A\,r^{l+1} + B\,r^{-l}$$

  where A and B are constants to be determined from boundary conditions, and l is still undetermined

- So far, the solution is

$$\Phi = \frac{U(r)}{r}\,P(\theta)Q(\phi) = \frac{A\,r^{l+1}+B\,r^{-l}}{r}\,P(\theta)\,e^{\pm im\phi}$$

or

$$\Phi_{lm} = \left[A\,r^l + \frac{B}{r^{l+1}}\right]\,P(\theta)\,e^{\pm im\phi}$$

Laplace's Equation in Spherical Coordinates

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P=0 \qquad (3.6)$$

- Customary to write the θ-equation in terms of $x = cos\,\theta$ instead of θ itself

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right]+\left[l\left(l+1\right)-\frac{m^2}{1-x^2}\right]P=0 \qquad (3.9)$$

$$\Downarrow$$

Generalized Legendre equation

Solutions are the associated Legendre functions

- Let's do $m = 0$ first:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right]+l\left(l+1\right)P=0 \qquad (3.10)$$

$$\Downarrow$$

Legendre equation

Solutions are Legendre polynomials

Laplace's Equation in Spherical Coordinates

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right]+l\left(l+1\right)P=0 \tag{3.10}$$

- The solution to P(x), where $x = \cos\theta$, turns out to be a standard math function known as Legendre polynomials.

Legendre polynomials

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \\
P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right) \\
P_4(x) &= \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)
\end{aligned} \tag{3.15}
$$

Do ***not*** memorize; will be supplied
(unless I ask you to work them out)

Legendre Polynomials

- The Legendre polynomials form a complete orthogonal set of functions in the interval $-1 \le x \le 1$.
- The orthogonality condition is: (will be supplied)

$$\int_{-1}^{1} P_{l'}(x)\, P_l(x)\, dx = \frac{2}{2l+1}\, \delta_{l'l} \qquad (3.21)$$

- Any function $f(x)$ in $-1 \le x \le 1$ can be expanded in terms of these polynomials:

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \qquad (3.23)$$

where

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} f(x)\, P_l(x)\, dx \qquad (3.24)$$

(3.23) and (3.24) are standard stuff you should already know.
Do not memorize (3.24); you must be able to work it out.

Boundary value problems with azimuthal symmetry

- The general solution to the Laplace equation for a problem with azimuthal symmetry ($m = 0$) is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\,\theta) \qquad (3.33)$$

where the coefficients A_l and B_l are determined from boundary conditions.

- **Problem:** Suppose the potential is specified to be $V(\theta)$ on the surface of a sphere of radius a. Find the potential *inside* the sphere. Find the potential *outside* the sphere.

Associated Legendre Functions

- Previously dealt with potential problems having azimuthal symmetry, so $m = 0$.

- In general, though, $m \neq 0$. Then we must work with the generalized Legendre equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[l\,(l+1)-\frac{m^2}{1-x^2}\right]P=0 \qquad (3.9)$$

Do not memorize this equation; it will be provided, if needed.

- In order to have finite solutions on the interval $-1 \leq x \leq 1$, the parameter l must be zero or a positive integer and the integer m can only take the values

$-l, -(l-1), \ldots, 0, \ldots, (l-1), l$ You must remember this.

- The solution having these properties is called an *associated Legendre function* $P_l^m(x)$

Spherical Harmonics

- In terms of the spherical harmonics, the solution to the generalized Legendre equation (3.9) is

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm}\, r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_{lm}(\theta,\phi) \tag{3.61}$$

where

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi} \tag{3.53}$$

Do not memorize;
will be supplied if needed.

Expansion in Spherical Harmonics

- Since $Y_{lm}(\theta,\phi)$ form a complete set of functions, an arbitrary function $f(\theta,\phi)$ can be expanded in spherical harmonics

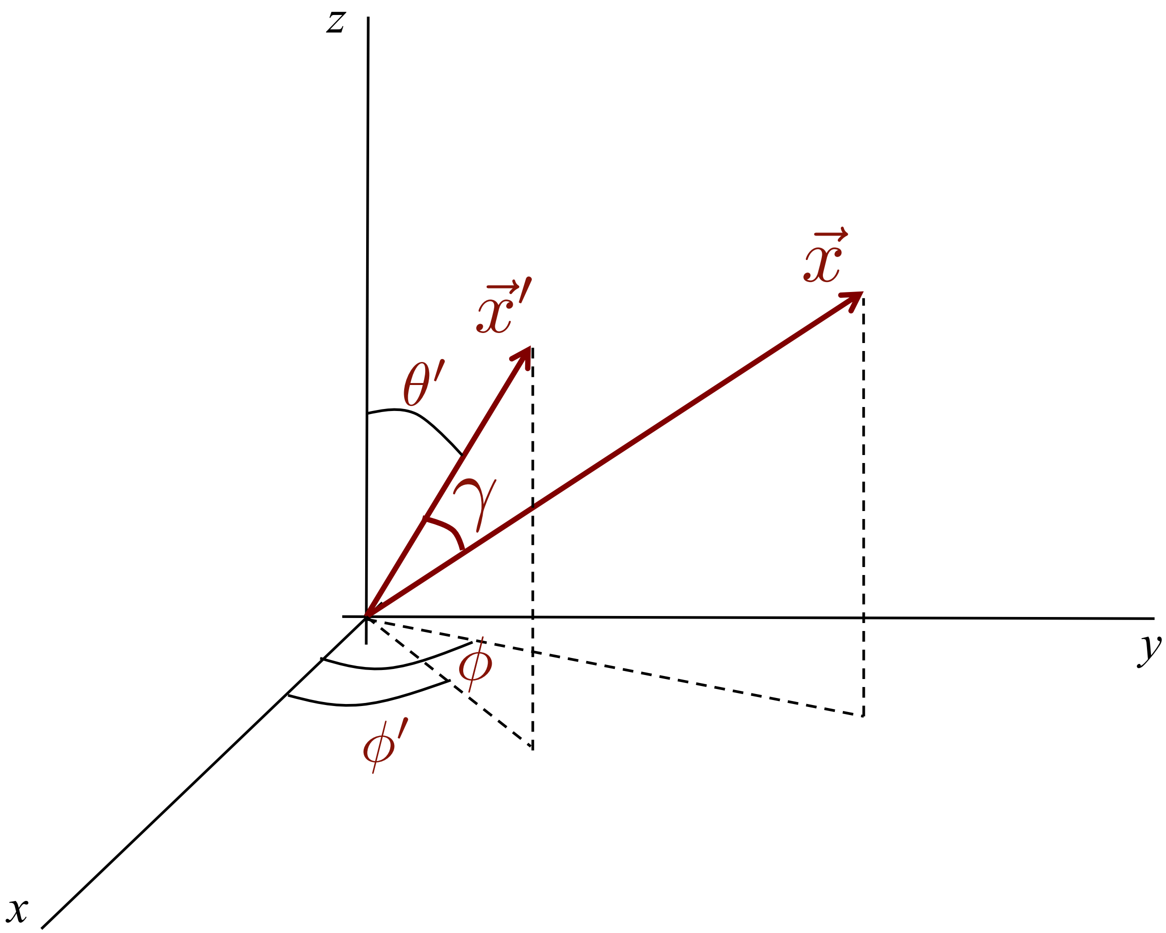
$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm}\, Y_{lm}(\theta,\phi)$$

$$C_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, f(\theta,\phi)\, Y^*_{lm}(\theta,\phi)$$

Addition theorem for spherical harmonics

- Consider the two vectors separated by angle γ as in the figure below:

$$P_l(\cos\,\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi')\, Y_{lm}(\theta, \phi)$$



A useful expansion

- In general, for points not on the z-axis:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\cos\,\gamma) \qquad (3.38)$$

- Using the addition theorem for spherical harmonics, we get:

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}}\right) Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi) \qquad (3.70)$$

Multipole Expansion

- We will now study the potential due to a localized charge distribution and its expansion in multipoles.

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm}\, r^l + \frac{B_{lm}}{r^{l+1}} \right] Y_{lm}(\theta,\phi)$$

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}$$

Set $A_l = 0$
because we want the
distribution far away

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{B_{lm}}{r^{l+1}}\, Y_{lm}(\theta,\phi)$$

Multipole Expansion

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{B_{lm}}{r^{l+1}}Y_{lm}(\theta,\phi)$$

To match Jackson's expression, write

$$B_{lm} = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{2l+1}q_{lm}$$

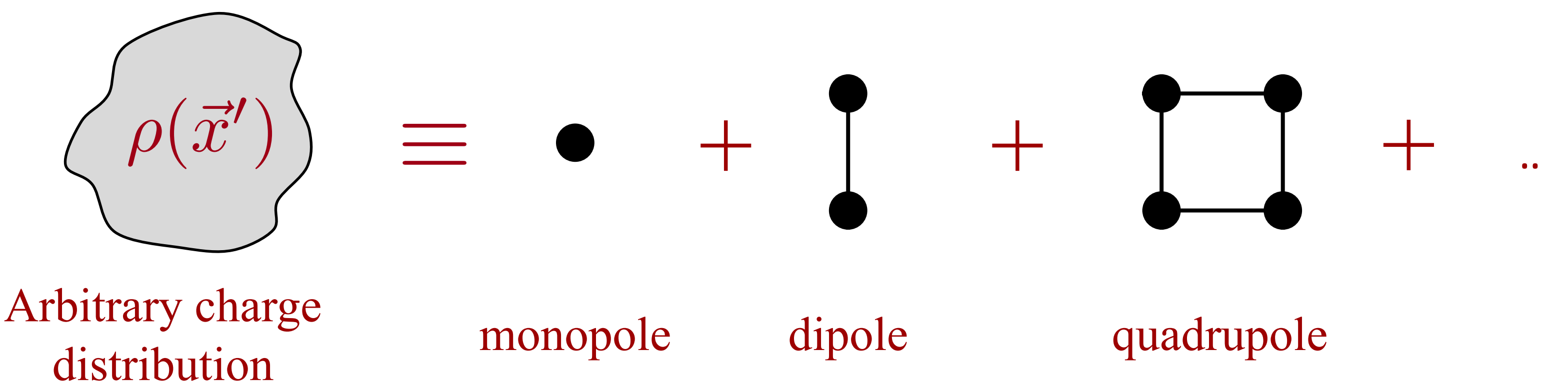
$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \tag{4.1}$$

- Eq. (4.1) is called the *multipole expansion*. See Week 8-Tue class summary for the form of the multipole moments q_{lm}, and how to express select q_{lm}'s in terms of the dipole moment and quadrupole moment (also see Homework 7).

Multipole Expansion

What is the use of the multipole expansion?

- Instead of evaluating the whole integral, we can find only the multipole moments, and stop when we have enough terms.
- We now have a convenient description of an arbitrary charge distribution in terms of multipole moments.



Magnetostatics

- Basic entity is the magnetic dipole (because no “magnetic charge”).

- Dipole aligns itself in direction of $\vec{B}$

$$\text{torque on dipole: } \vec{N} = \underbrace{\vec{\mu}}_{\text{magnetic moment}} \times \vec{B} \tag{5.1}$$

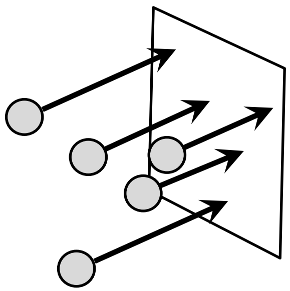
- Thus, more complicated situation than electrostatics

- Quantitative explanation had to wait until after connection between currents and magnetic fields was established.

Magnetostatics

- Recall that a current corresponds to charges in motion and is described by a current density $\vec{J}$

- The current density $\vec{J}$ is measured in units of positive charge crossing unit area per unit time (amperes per m^2), the direction of motion of the charges being the direction of $\vec{J}$



- So $\vec{J}$ is a vector field that describes the flow of charge at any point in space, the integral of its component perpendicular to dA over the surface is the current I.

- Continuity equation (for magnetostatics):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

- In magnetostatics:

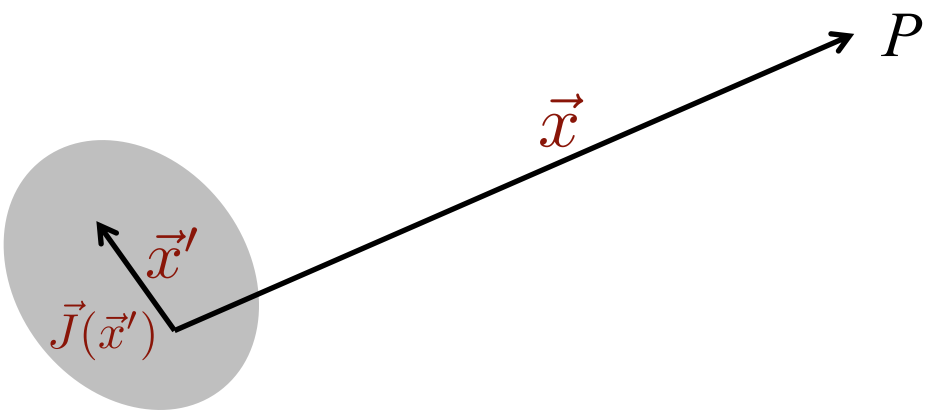
$$\vec{\nabla} \cdot \vec{J} = 0$$

Magnetostatics

Biot-Savart Law: $$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\,\vec{\nabla} \times \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x'$$

Ampere's Law: $$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$$

Vector Potential

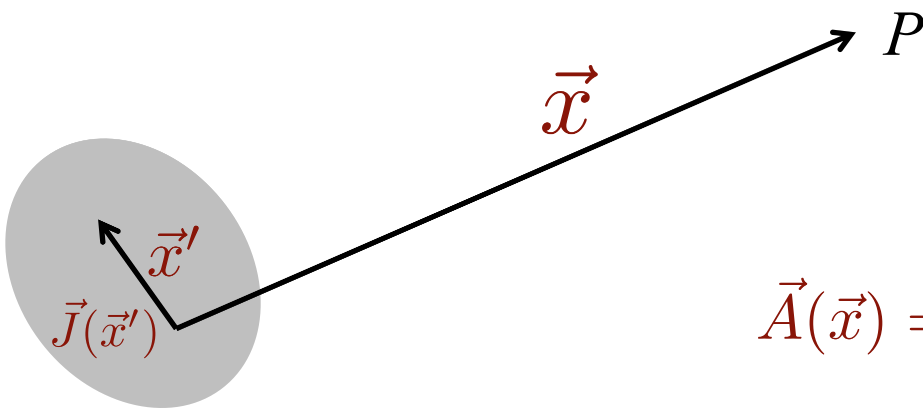


- Vector potential:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x' \qquad (5.32)$$

Do not memorize;
will be supplied if needed.

Magnetic Field of a localized current distribution



$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x' \tag{5.32}$$

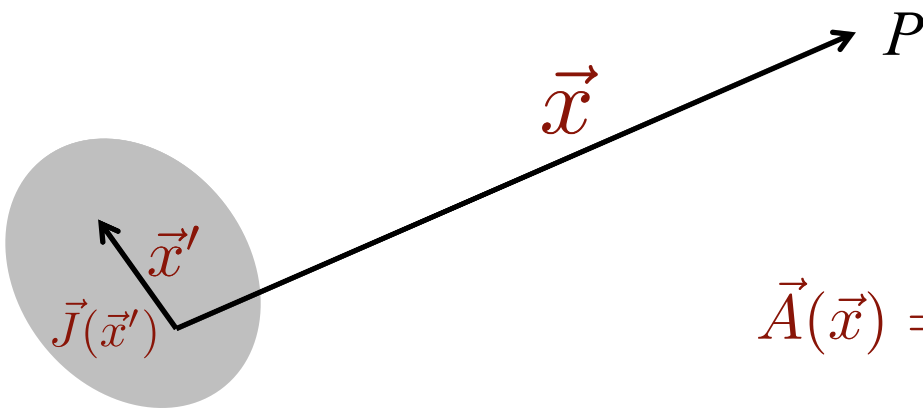
- Expand the denominator of (5.32) in powers of $\vec{x}'$, assuming $|\vec{x}| \gg |\vec{x}'|$.

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{(|\vec{x}|^2 - 2\vec{x} \cdot \vec{x}\,' + |\vec{x}\,'|^2)^{1/2}}$$

$$= \frac{1}{|\vec{x}|} \left[1 - \frac{2\vec{x} \cdot \vec{x}\,'}{|\vec{x}|^2} + \frac{|\vec{x}\,'|^2}{|\vec{x}|^2} + \dots\right]^{-1/2}$$

$$\approx \frac{1}{|\vec{x}|} \left[1 + \frac{\vec{x} \cdot \vec{x}\,'}{|\vec{x}|^2} + \dots\right]$$

Magnetic Field of a localized current distribution



$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x' \qquad (5.32)$$

- Expanding the denominator of (5.32) in powers of $\vec{x}'$, assuming $|\vec{x}| \gg |\vec{x}'|$. get that the *i*th component of the vector potential will have the expansion

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{1}{|\vec{x}|}\underbrace{\int J_i(\vec{x}')\, d^3x'}_{\substack{\text{monopole term}\\ = 0}} + \frac{\vec{x}}{|\vec{x}|^3}\cdot \underbrace{\int J_i(\vec{x}')\,\vec{x}'\, d^3x'}_{\text{dipole term}} + \dots\right] \qquad (5.51)$$

Magnetic Field of a localized current distribution

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\left[\boxed{= 0} + \underbrace{\frac{\vec{x}}{|\vec{x}|^3} \cdot \int J_i(\vec{x}')\, \vec{x}'\, d^3x' }_{\text{can show that}} + \ldots \right]$$

$$\vec{x} \cdot \int \vec{x}'\, J_i(\vec{x}')\, d^3x' = -\frac{1}{2}\left[\vec{x} \times \int \left(\vec{x}' \times \vec{J}\right) d^3x'\right]_i$$

- Keeping only the 2nd term, dropping all higher terms, then putting back $i = x, y, z$ to write the full vector, we get

$$\vec{A}(\vec{x}) = -\frac{1}{2}\,\frac{\mu_0}{4\pi}\,\frac{1}{|\vec{x}|^3}\underbrace{\left\{\vec{x} \times \int \left(\vec{x}' \times \vec{J}\right) d^3x'\right\}}_{\text{Flip order, to get rid of minus sign outside}}$$

Will be supplied if needed.

$$\vec{A}(\vec{x}) = \frac{1}{2}\,\frac{\mu_0}{4\pi}\,\frac{1}{|\vec{x}|^3}\left[\int \left(\vec{x}' \times \vec{J}\right) d^3x'\right] \times \vec{x}$$

Magnetic Field of a localized current distribution

$$\vec{A}(\vec{x}) = \frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\left[\int \left(\vec{x}\,' \times \vec{J}\right) d^3x'\right] \times \vec{x}$$

Define the magnetic dipole moment:

$$\vec{m} = \frac{1}{2}\int \vec{x}\,' \times \vec{J}(\vec{x}\,')\, d^3x'$$

Can interpret this as current times area of loop in which current is flowing

So that the vector potential is now

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}$$

Recall eq. (4.10) in Chapter 4, where we wrote the dipole term in the multipole expansion for the electrostatic field.

$$\left[\Phi(\vec{x})\right]_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{|\vec{x}|^3}$$

Macroscopic Equations in Magnetostatics

- Just as in electrostatics, going from free space to matter requires adjustments.

- We have assumed current density is a function of position, but this is often not true in macroscopic problems.

- The electrons in matter often cause effective atomic currents, and their current density is a rapidly fluctuating quantity; only its average over a macroscopic volume is known or pertinent.

- Moreover, electrons can contribute intrinsic magnetic moments that can give rise to dipole fields that vary appreciably on atomic scales.

- Fortunately, $\vec{\nabla} \cdot \vec{B} = 0$ also holds for the macroscopic magnetostatic field, so we can still use the concept of a vector potential $\vec{A}(\vec{x})$.

Macroscopic Equations in Magnetostatics

- First, $\vec{\nabla} \cdot \vec{B} = 0$ also holds for the macroscopic magnetostatic field, so we can still use the concept of a vector potential $\vec{A}(\vec{x})$.

- The large number of atoms or molecules per unit volume, each with its molecular magnetic moment $\vec{m}_i$ gives rise to an average macroscopic magnetization or magnetic moment density

$$\vec{M}(\vec{x}) = \sum_i N_i \langle \vec{m}_i \rangle \tag{5.76}$$

where N_i is the average number per unit volume of molecules of type i and $\langle \vec{m}_i \rangle$ is the average molecular moment in a small volume at the point $\vec{x}$.

- We find that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \left[\underbrace{\frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}}_{\substack{\text{Macroscopic current density} \\ \text{from the flow of free} \\ \text{charges in the medium}}} + \underbrace{\frac{\vec{M}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}}_{\substack{\text{Due to magnetization} \\ \text{written in (5.76) above}}} \right] d^3x' \tag{5.77}$$

Macroscopic Equations in Magnetostatics

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \left[\frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \frac{\vec{M}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \right] d^3x' \qquad (5.77)$$

- Can show that from (5.77), we get

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}') + \vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' \qquad (5.78)$$

- Therefore, the magnetization is contributing an effective current density

Must know: $\vec{J}_M = \vec{\nabla} \times \vec{M}$ (5.79)

- So macroscopic equivalent of $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Must know: $\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \vec{\nabla} \times \vec{M} \right]$ (5.80)

Often combined with $\vec{B}$ to
define new magnetic field $\vec{H}$

- Therefore, $(\vec{J} + \vec{J}_M)$ plays the role of current density in macroscopic equations.

Macroscopic Equations in Magnetostatics

- We define the magnetic field

Must know: $$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \tag{5.81}$$

- Then, macroscopic equations:

$$\begin{aligned} \vec{\nabla} \times \vec{H} &= \vec{J} \\ \vec{\nabla} \cdot \vec{B} &= 0 \end{aligned} \tag{5.82}$$

- Constitutive relation in linear media:

Must know: $$\vec{B} = \mu \vec{H} \tag{5.84}$$

- Boundary conditions:

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0$$

$$\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K}$$

where $\hat{n}$ is a unit normal pointing from region 1 to region 2, and $\vec{K}$ is the surface current density.

Method of Images

- In principle, all boundary value problems in electrostatics can be solved using Green functions.
- In practice, however, it may not always be possible to find a Green function by simple means.
- This has led to the development of a number of special techniques to the solution of boundary value problems.
- One of these, the expansion in orthogonal functions, we have already learned.
- Another special technique is the method of images.

Method of Images

- The method of images can be applied when we have one or more point charges in the presence of boundary surfaces, e.g., conductors that are grounded or held at fixed potentials.

- If the geometry is favorable, we can place charges of appropriate magnitudes in suitable locations external to the region of interest to simulate the required boundary conditions. These charges are known as image charges.

- Substitution of the actual problem by a larger region with image charges is known as the method of images.

Method of Images

- Point charge and infinite conducting plane: Homework 8
- Point charge outside grounded conducting sphere

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  - Point charge *inside* grounded conducting sphere
  - Point charge in presence of charged conducting sphere
  - Point charge near conducting sphere at fixed potential

Corollaries to grounded sphere; pages 60-62

- Conducting sphere in uniform electric field
- Image charges in dielectric materials