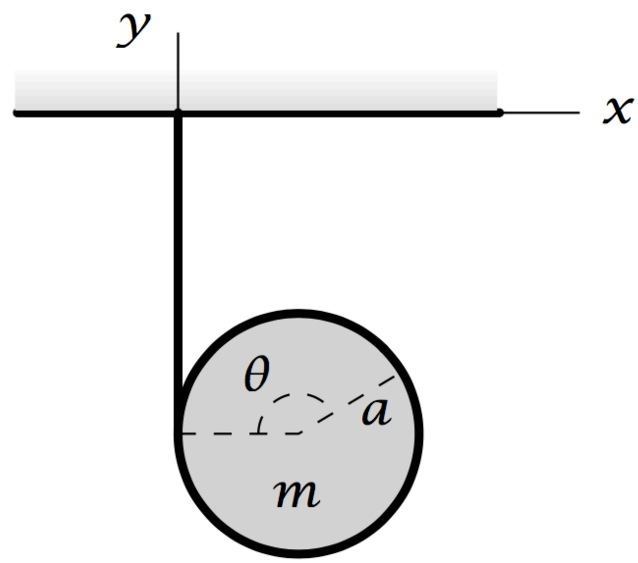
A string that is attached to a fixed support point is wrapped around a disk of radius a and mass m as shown below. The disk is allowed to fall from rest, and falls straight down. The moment of inertia of the disk is $I_{cm} = \frac{1}{2}ma^2$.



a. Using the coordinates shown, find the Lagrangian and the equation of constraint.
b. Using the Lagrange multiplier method, find the equations of motion and the forces of constraint.
c. Compare the forces of constraint to what you know from elementary physics.
d. Solve the equations of motion to find the motion of the disk.

As in activity 12 (the bead on the hemisphere), we will find the generalized forces using the equation on p. 80, and find the equation for λ by setting up equations for the constraints (Hamill equation 3.10) and differential equations for the generalized coordinates (Hamill equation 3.16).

We'll use y and θ as our generalized coordinates. The equation of constraint for rolling without slipping is $f = y + a\theta = 0$. (I am assuming that θ increases as the disk rotates clockwise.)

a.

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 - mgy$$

$$f = y + a\theta = 0$$

b.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \lambda\frac{\partial f}{\partial y} \Leftrightarrow m\ddot{y} + mg = \lambda \qquad (i)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial f}{\partial \theta} \Leftrightarrow I\ddot{\theta} - 0 = a\lambda \Leftrightarrow ma\ddot{\theta} = 2\lambda \quad (ii)$$

$f = y + a\theta = 0 \; \frac{\partial f}{\partial y} = 1, \frac{\partial f}{\partial \theta} = a; \ddot{y} = -a\ddot{\theta}$, plugging this into (i)

$$-ma\ddot{\theta} + mg = \lambda \Leftrightarrow -2\lambda + mg = \lambda \Leftrightarrow \lambda = \frac{1}{3}mg$$

We can now plug λ into the equations for the generalized forces:

$Q_y = \lambda\frac{\partial f}{\partial y} = \frac{1}{3}mg \quad Q_\theta = \frac{1}{3}mga$ (torque)

In the elementary Newtonian approach we identify two forces acting on the disk in vertical direction: tension and gravity. Plugging the net force $F_{net} = mg - T$ into Newton's second law we get

$$mg - T = m\ddot{y} \quad (iii)$$

We also identify the torque on the disk about its center of mass, which we plug into the rotational version of Newton's second law

$$\tau_{net} = Ta = I\ddot{\theta} = \frac{1}{2}ma^2\ddot{\theta} = \frac{1}{2}ma^2\frac{\ddot{y}}{a} \Leftrightarrow T = \frac{1}{2}m\ddot{y}$$

plugging this expression for T into (iii) we get

$$mg - \frac{1}{2}m\ddot{y} = m\ddot{y} \Leftrightarrow \ddot{y} = \frac{2}{3}g \Rightarrow T = \frac{1}{2}m\ddot{y} = \frac{1}{3}mg, \tau = \frac{1}{3}mga$$

So $Q_y = T$ and $Q_\theta = \tau$. The Newtonian approach gives the same results for tension and torque as the Lagrangian multiplier approach gives for the constraint forces.

c.

Solving the equations of motion for y and θ we get

$$\ddot{y} = \frac{\lambda}{m} - g = \frac{\frac{1}{3}mg}{m} - g = -\frac{2}{3}g \Rightarrow y(t) = y_0 + v_0 t - \frac{1}{3}gt^2$$

$$\ddot{\theta} = \frac{2\lambda}{ma} = \frac{\frac{2}{3}mg}{ma} = \frac{2}{3}\frac{g}{a} \Rightarrow \theta(t) = \theta_0 + \omega_0 t + \frac{1}{3}\frac{g}{a}t^2$$