1. Basis in $\mathbb{R}^n$
2. Introduction to *piecewise continuous functions*
3. Introduction to basis for *piecewise continuous functions* and *Fourier series*

We have seen already that *Taylor expansions* of functions are useful in a variety of ways.

However, Taylor expansions are not adequate in *at least* three ways

1. If a function is *periodic*, a Taylor expansions *misses* this fact.
2. If a function is *discontinuous*, Taylor expansions cannot be used.
3. Information is *encoded* in different ways by different expansions. Some information is more conveniently extracted using other expansions
4. We will now develop an entirely new kind of expansion that overcomes these difficulties. The result proves to be extremely useful in an enormous number of physical situations.

The expansion we introduce now is called a *Fourier Series.*

We introduce it in a bit of round about way by talking about basis of $\mathbb{R}^n$ using *vectors in 2-D.*

Some definitions:

- A vector space is a collection of objects, called *vectors*, that obey a set of *rules*
- A *basis* of a vector space, *V,* is a subset of vectors in *V* that are both *linearly independent and span the space*.

To see what all this means, do questions (1) – (5) on the worksheet.

(1) $\vec{\boldsymbol{A}} = A_x \hat{\boldsymbol{x}} + A_y \hat{\boldsymbol{y}}$

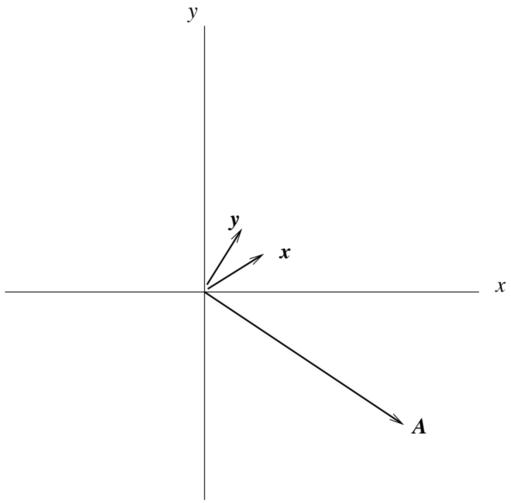
(2) A general vector *cannot* be represented by just *one* of these vectors.

(3-4) *Any vector* in the plane can be represented by just *two vectors*.

(5) Completeness: The *minimum set* of vectors that can be used to represent *any vector*

The figure shows the vector ***A*** but now with two other unit vectors, **x** and **y**.

$$\vec{\boldsymbol{A}} = A_x \hat{\boldsymbol{x}} + A_y \hat{\boldsymbol{y}}$$



Both sets of unit vectors are *complete.* That is, any general vector can be written as a linear combination of either set. The vectors are said to be *basis* for the 2-D vector space. Yet the basis vectors have an important difference

The set of basis vectors used on the worksheet is *orthogonal,* this second is not. Mathematically we can express an orthogonal basis for vectors in 2D as

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} = 0$$

while for the non-orthogonal basis we have that

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} \neq 0$$

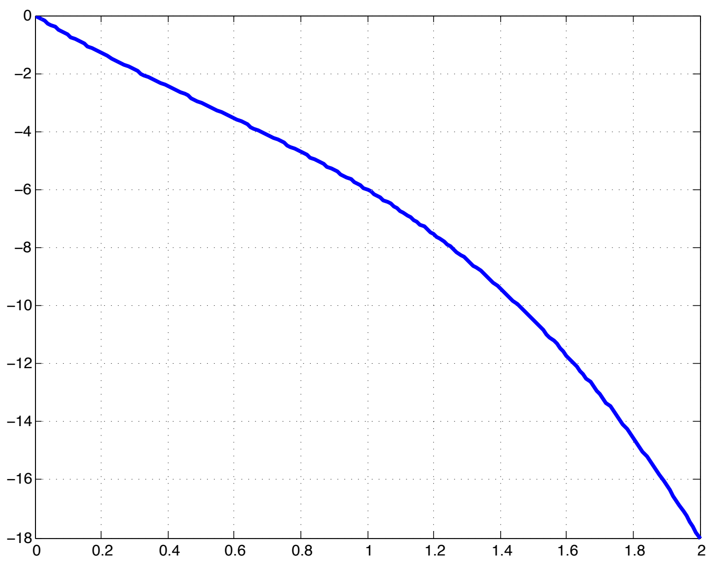
So what?

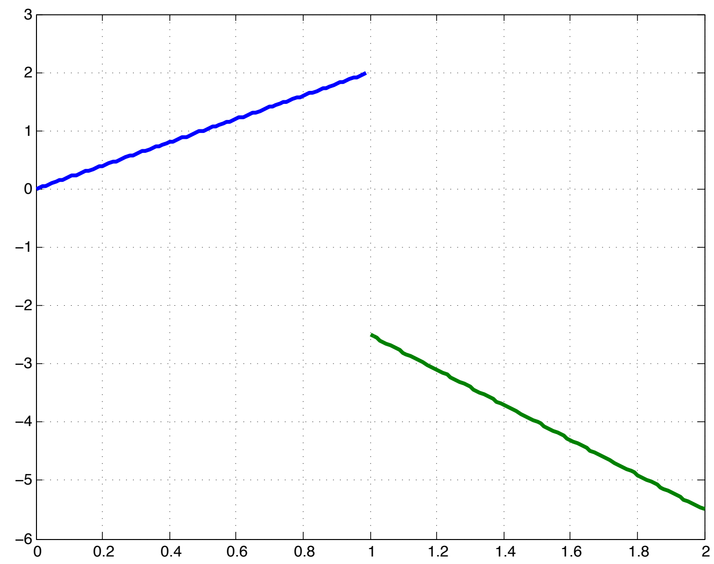
known

$$\begin{aligned} \vec{A} &= A_x \hat{x} + A_y \hat{y} \quad \text{multiply both sides by } \hat{x} \\ \vec{A} \cdot \hat{x} &= A_x \hat{x} \cdot \hat{x} + A_y \hat{y} \cdot \hat{x} \\ \vec{A} \cdot \hat{x} &= A_x \end{aligned}$$

In the 19th and early 20th centuries, the idea of basis was extended to more kinds of objects.

In particular, it was found that these ideas could be extended to *piecewise continuous functions* on an interval *[a,b]*. These functions are called *piecewise continuous functions*





It turns out that one can show that *piecewise continuous functions* form a vector space on the interval [-*L*, *L*]. ←

If the space of piecewise smooth functions is a vector space, how do we define an inner product?

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

Orthogonality is then

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx = 0$$

Show that *$f(x) = -x$, $g(x) = 2$* are orthogonal on the interval $[-4, 4]$.

Determine if these two functions are orthogonal on the interval $[0, 8]$

As you go through the next few questions on the worksheet, keep in mind what you just did in parts (1) – (5).

What about a basis for this space? Do question (6) on the worksheet and S T O P.

(6) $$f(x) = a_o(1) + \sum_n a_n \cos(nx) + \sum_m b_m \sin(mx)$$

Do question 7 and S T O P

(7) $$\int_{-\pi}^{\pi} \cos(nt)\cos(mt) = \begin{cases} \pi\delta_{m,n}, & m \neq n \\ 2\pi, & m = n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nt)\sin(mt) = 0$$

$$\int_{-\pi}^{\pi} \sin(nt)\sin(mt) = \begin{cases} 1 & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

Do question 8 on worksheet and S T O P

(8) $$f(x) = a_o(1) + \sum_n a_n \cos(nx) + \sum_m b_m \sin(mx)$$

I'll find just the b_m the *a's* work similarly. First multiply both sides by sin (nx) and integrate to give

$$\int_{-\pi}^{\pi} f(x)\sin(nx) = \sum_m \int_{-\pi}^{\pi} b_m \sin(mx)\sin(nx)$$

The *a's* drop out due to orthogonally. The right hand side is also subject to orthogonal conditions and the sum is equal to 0 except when *m = n.* Thus, we have

$$b_n = \int_{-\pi}^{\pi} f(x)\sin(nx)$$

Finish the worksheet