PHY 411 **Winter 2021**

Week 6—Tuesday, Feb 9—Discussion Worksheet

Laplace's Equation in Rectangular Coordinates

1. The Laplace equation in rectangular coordinates is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{2.48}$$

(a) Show that by using $\Phi(\vec{x}) = X(x)Y(y)Z(z)$ to separate variables, one obtains

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0 \tag{2.50}$$

$$\frac{\partial^2}{\partial x^2} X(x)Y(y)Z(z) + \frac{\partial}{\partial y^2} X(x)Y(y)Z(z) + \frac{\partial}{\partial z^2} X(x)Y(y)Z(z) = 0$$

Divide out each term

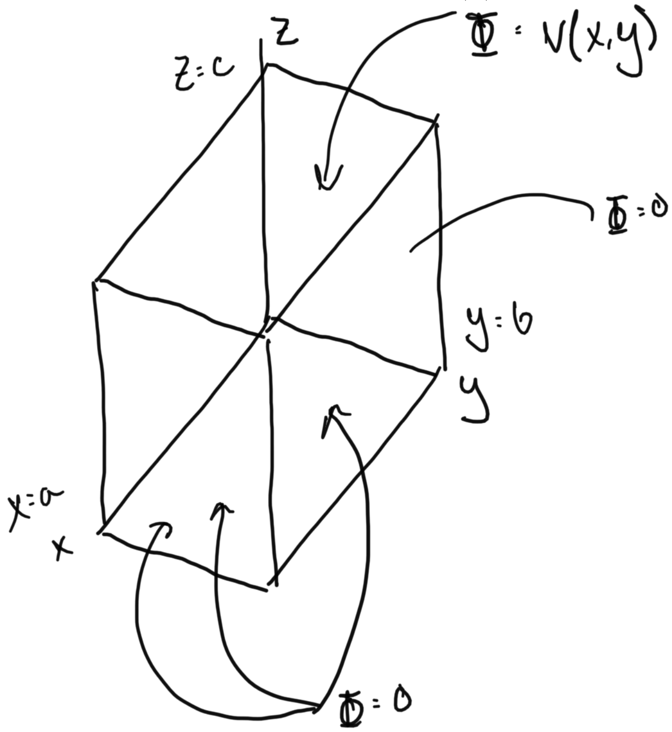
$$\frac{1}{X}\frac{\partial^2}{\partial x^2} X(x)\cancel{Y(y)}\cancel{Z(z)} + \frac{1}{Y}\frac{\partial}{\partial y^2}\cancel{X(x)}Y(y)\cancel{Z(z)} + \frac{1}{Z}\frac{\partial}{\partial z^2}\cancel{X(x)}\cancel{Y(y)}Z(z) = 0$$

$$\rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{\partial y^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2}$$

(b) Why can you set each of these terms equal to a constant? Are there any relations between one or more of these constants?

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\alpha^2 \qquad \frac{1}{Y}\frac{d^2 Y}{dy^2} = -\beta^2 \qquad \frac{1}{Z}\frac{d^2 Z}{dz^2} = \gamma^2$$

2. Consider a rectangular box with dimensions (a, b, c) along the (x, y, z) directions respectively; the figure is shown on the PowerPoint slide (and in the posted class summary). All surfaces of the box are kept at zero potential except the surface $z = c$, which is kept at a potential $V(x, y)$. Find the potential $\Phi(\vec{x})$ everywhere inside the box.



Laplace's Equation in Spherical Coordinates

3. In spherical coordinates (r, θ, ϕ), the Laplace equation can be written as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{3.1}$$

(a) By assuming a product form for the potential: $\Phi = \dfrac{U(r)}{r}P(\theta)Q(\phi)$, show that equation (3.1) separates to

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{P\,r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right] + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0 \tag{3.3}$$

(b) Set the ϕ-dependent term equal to $-m^2$ and solve the equation for $Q(\phi)$.

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2$$

(c) If you set the ϕ-dependent term equal to $-m^2$, the sum of the r and θ terms in equation (3.3) must be set equal to what? Why?

4. Separate the r and θ equations in equation (3.3) and show that you obtain

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P=0 \tag{3.6}$$

$$\frac{d^2U}{dr^2}-\frac{l(l+1)}{r^2}U=0 \tag{3.7}$$

From the form in equation (3.6) and (3.7), you must be able to figure out what to set each equation to after you've separated the r and θ equations. If you're wondering why we set it equal to such a weird combination, it is because we're anticipating a standard equation in the θ part, and that is what drives our choice.

5. Solve the r-equation in equation (3.7) by trying a power solution $U = r^{\alpha}$ and show that the solution is

$$U = A\, r^{l+1} + B\, r^{-l}$$