- In-class exam 2 on Tuesday
  - Hamill chapters 3 and 4
  - Homework assignments 4, 5, 6
  - In-class activities 10-17
  - (No canonical transformations)

- Sorry, I didn't manage to finish grading homework 6. I will post the sample solutions tomorrow morning and will put the graded exams into your mailboxes tomorrow afternoon. If that doesn't work for you let me know and we'll find an alternative way for you to get the homework before the exam.
- There will be a short homework assignment due on Thursday (on D2L posted tomorrow)
- The 1-pager for your term project is also due on Thursday.
- Let's postpone the next reading assignment to Tuesday November 12. I *think* we won't get to the material from sections 5.3-5.5 before then anyway. We'll have one more reading assignment after that (chapter 6), which will be due on the last day of class (November 19).

Let's do activity 17 together: We want to plot phase space trajectories and find the equilibrium solutions.

$$V(r) = \frac{k}{n+1} r^{n+1}$$

$n = -2$:

$$V(r) = -kr^{-1}$$

Gravitational or Coulomb potential.

$n = 1$:

$$V(r) = \frac{1}{2} kr^2$$

Two-dimensional harmonic oscillator

Before we jump in, what do you expect the phase space trajectories to look like in the case of $n = -2$? (For example, what happens to the p_r for very large and very small r? How does the value of the Hamiltonian affect the trajectories? What happens for very large and for very small H? What does $H < 0$ and $H > 0$ represent?)

Let's derive the Hamiltonian. Same as always!

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{n+1}r^{n+1}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = const$$

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L$$

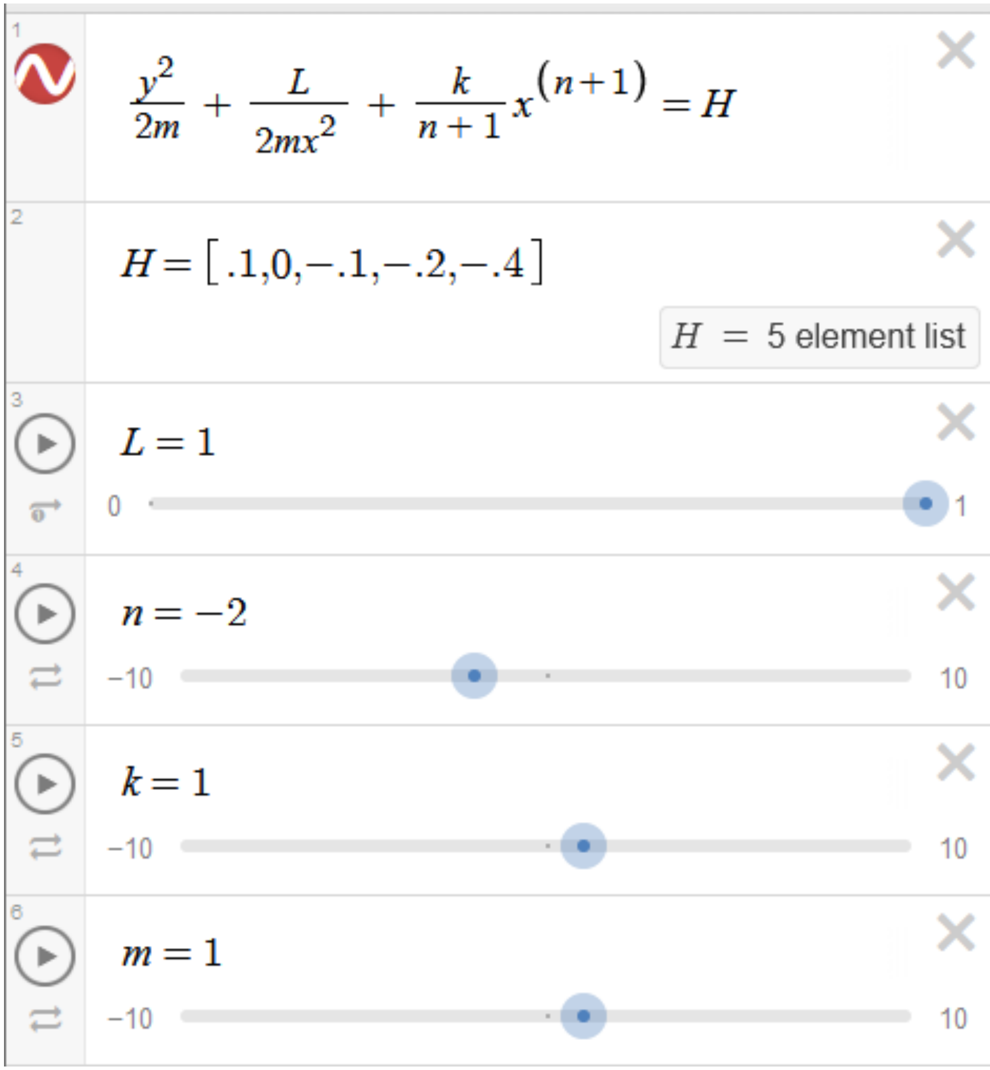
$$= p_r\frac{p_r}{m} + p_\theta\frac{p_\theta}{mr^2} - \frac{m}{2}\left(\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2r^4}\right) + \frac{k}{n+1}r^{n+1}$$

$$= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2mr^2} + \frac{k}{n+1}r^{n+1}$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{k}{n+1}r^{n+1}$$

$$H = \frac{p_r^2}{2m} + \frac{\mathcal{L}^2}{2mr^2} + \frac{k}{n+1}r^{n+1}\ (p_\theta = const = \mathcal{L})$$

Let's plot the phase space trajectories



Let’s find the equilibrium solutions:

$$H = \frac{p_r^2}{2m} + \frac{\mathcal{L}^2}{2mr^2} + \frac{k}{n+1} r^{n+1}$$

Equations of motion:

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{\mathcal{L}^2}{mr^3} - kr^n$$

Equilibrium when

$$\dot{r} = \dot{p}_r = 0 \Rightarrow p_r = 0, r^{n+3} = \frac{\mathcal{L}^2}{mk} \Rightarrow r = const$$

Not surprisingly, these are circular trajectories

New Topic: Canonical Transformations

(will not be on Tuesday's exam, but will be on the final)

Generating Functions

$$F = F_1(q_i, Q_i, t)$$

$$P_i = -\frac{\partial F_1}{\partial Q_i}$$

$$p_i = \frac{\partial F_1}{\partial q_i}$$

$$K = H + \frac{\partial F_1}{\partial t}$$

$$F = F_2(q_i, P_i, t)$$

$$p_i = \frac{\partial F_2}{\partial q_i},$$

$$Q_i = \frac{\partial F_2}{\partial P_i},$$

$$K = H + \frac{\partial F_2}{\partial t}$$

$$F = F_3(p_i, Q_i, t)$$

$$q_i = -\frac{\partial F_3}{\partial p_i},$$

$$P_i = -\frac{\partial F_3}{\partial Q_i},$$

$$K = H + \frac{\partial F_3}{\partial t}$$

$$F = F_4(p_i, P_i, t)$$

$$q_i = -\frac{\partial F_4}{\partial p_i},$$

$$Q_i = \frac{\partial F_4}{\partial P_i},$$

$$K = H + \frac{\partial F_4}{\partial t}.$$

Activity 18: You are given the canonical transformation equations and are asked to derive the generating function and the new Hamiltonian:

$$H\left(x, y, z, p_x, p_y, p_z\right) \rightarrow K(r, \phi, Z, P_r, P_\phi, P_Z)$$

Cartesian Coordinates

Cylindrical Coordinates

Canonical transformation

We already know that we want he following transformation equations:

$$x = r\cos\phi\,,\, y = r\sin\phi, z = Z$$

We also already know that we want to use $Q_i = r, \phi, Z$

But we still need to find expressions for the new momenta $P_i = P_r, P_\phi, P_Z$

Which of the four generating functions should we use?

$$F = F_3(p_i, Q_i, t)$$

$$q_i = -\frac{\partial F_3}{\partial p_i},$$

$$P_i = -\frac{\partial F_3}{\partial Q_i},$$

$$K = H + \frac{\partial F_3}{\partial t}$$

The q_i equation allows us to determine the generating function because we know the expressions we need to get are:

$$x = r\cos\phi\,, y = r\sin\phi, z = Z$$

Once we have derived the generating function we can then use to find expressions for the new momenta and the new Hamiltonian from the second and third equations.

Can you figure out what F_3 needs to be, so that

$x = -\frac{\partial F_3}{\partial p_x} = r\cos\phi$ $y = -\frac{\partial F_3}{\partial p_y} = r\sin\phi$ and $z = -\frac{\partial F_3}{\partial px} = Z$

We need to satisfy $x = -\frac{\partial F_3}{\partial p_x} = r\cos\phi$

The following term would give us this partial derivative: $-p_x(r\cos\phi)$

We also need to satisfy $y = -\frac{\partial F_3}{\partial p_y} = r\sin\phi$

This term would do that: $-p_y(r\sin\phi)$

Finally, we also need $z = -\frac{\partial F_3}{\partial p_z} = Z$

This term would do that: $-p_z Z$

Combining these three terms we get: $F_3(p_i, Q_i, t) = -p_x(r\cos\phi) - p_y(r\sin\phi) - p_z Z$

Now that we have the generating function, we can use it to derive the new momenta P_i and the new Hamiltonian K.

$$F_3(p_i, Q_i, t) = -p_x(r\cos\phi) - p_y(r\sin\phi) - p_z Z$$

$$P_r = -\frac{\partial F_3}{\partial r} = p_x\cos\phi + p_y\sin\phi$$

$$P_\phi = -\frac{\partial F_3}{\partial \phi} = -rp_x\sin\phi + rp_y\cos\phi$$

$$P_Z = -\frac{\partial F_3}{\partial Z} = p_z$$

$$K = H + \frac{\partial F_3}{\partial t} = H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

The last step is to express the new Hamiltonian using the new momenta and coordinates $K = K(r, \phi, Z, P_r, P_\phi, P_Z)$

Can you find and expression for $p_x^2 + p_y^2 + p_z^2$ in terms of $r, \phi, Z, P_r, P_\phi, P_Z$?

$$P_r^2 = p_x^2 \cos^2\phi + p_y^2 \sin^2\phi + 2p_x p_y \cos\phi \sin\phi$$

$$\frac{P_\phi^2}{r^2} = p_x^2 \sin^2\phi + p_y^2 \cos^2\phi - 2p_x p_y \cos\phi \sin\phi$$

$$\Rightarrow P_r^2 + \frac{P_\phi^2}{r^2} = p_x^2 + p_y^2$$

So canonical transformation gives us the new Hamiltonian of a free particle in cylindrical coordinates:

$$K = \frac{1}{2m}\left(P_r^2 + \frac{P_\phi^2}{r^2} + P_z^2\right)$$

Now try this yourself for spherical coordinates:

$$H\left(x, y, z, p_x, p_y, p_z\right) \to K(r, \theta, \phi, P_r, P_\theta, P_\phi)$$

Cartesian Coordinates

Spherical Coordinates

Step 1: Find the generating function so that

$$x = -\frac{\partial F_3}{\partial p_x} = r \sin\theta \cos\phi \qquad y = -\frac{\partial F_3}{\partial p_y} = r \sin\theta \sin\phi \qquad z = -\frac{\partial F_3}{\partial p_z} = r \cos\theta$$

Step 2: Derive expressions for P_r, P_θ, P_ϕ and for K

Step 3: Express K in terms of $r, \theta, \phi, P_r, P_\theta, P_\phi$

Activity 19

In this problem we are given the old and the new Hamiltonian are, and are asked to derive the canonical transformation from $(y, p) \longrightarrow (Y, P)$

$$H(y, p) = \frac{p^2}{2m} + mgy \quad K(Y, P) = P$$

So assuming the generating function does not have an explicate time dependence,

$$K(Y, P) = H(y, p) = P, \text{ so } P = \frac{p^2}{2m} + mgy$$

If we use F_4 as our generating function, this becomes

$$P = \frac{p^2}{2m} + mg\,\frac{\partial F_4}{\partial p}$$

Which you can integrate to obtain F_4.

Once you have F_4, you can compute the canonical transformations.

You can also derive the equations of motion for the new coordinates $\dot{Y}$ and $\dot{P}$ because the new Hamiltonian obeys Hamilton's equations. These will be simple expressions because the new Hamiltonian is so simple $(\mathrm{K} = \mathrm{P})$, which you can easily integrate.

You can determine what the values of the two integration constant are by initial conditions $y(0) = h$ and $p(0) = mv_0$