PHY 420 Spring 2021

Week 9—Thursday, May 27—Discussion Worksheet

Transformation of Electromagnetic Fields (continued)

For the specific Lorentz transformation corresponding to a ***boost along the x^1 axis with speed $c\beta$*** from the unprimed frame to the primed frame, the explicit **equations of transformation** are

$$\begin{aligned} E'_1 &= E_1 & B'_1 &= B_1 \\ E'_2 &= \gamma\Big(E_2 - \beta B_3\Big) & B'_2 &= \gamma\Big(B_2 + \beta E_3\Big) \\ E'_3 &= \gamma\Big(E_3 + \beta B_2\Big) & B'_3 &= \gamma\Big(B_3 - \beta E_2\Big) \end{aligned} \tag{11.148}$$

where $\gamma = (1-\beta^2)^{-1/2}$.

1. In the previous class, you verified the transformation relations for E'_1 and E'_2. Let's begin today by working out the rest of the transformations.

(a) Start by writing $F' = AF\tilde{A}$ from the worksheet of the previous class for reference:

$$\begin{pmatrix} 0 & -E'_1 & -E'_2 & -E'_3 \\ E'_1 & 0 & -B'_3 & B'_2 \\ E'_2 & B'_3 & 0 & -B'_1 \\ E'_3 & -B'_2 & B'_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\gamma^2 E_1 + \gamma^2\beta^2 E_1 & -\gamma E_2 + \gamma\beta B_3 & -\gamma E_3 - \gamma\beta B_3 \\ -\gamma^2\beta^2 E_1 + \gamma^2 E_1 & 0 & \gamma\beta E_2 - \gamma B_3 & \gamma\beta E_3 + \gamma B_2 \\ \gamma E_2 - \gamma\beta B_3 & -\gamma\beta E_2 + \gamma B_3 & 0 & -B_1 \\ \gamma E_3 + \gamma\beta B_2 & -\gamma\beta E_3 - \gamma B_2 & B_1 & 0 \end{pmatrix}$$

(b) Verify by comparing the elements of the matrices written in part (a) above that

$$E'_3 = \gamma\Big(E_3 + \beta B_2\Big)$$

$$E'_3 = \gamma E_3 + \gamma\beta B_2$$

so,

$$E'_3 = \gamma(E_3 + \beta B_2)$$

2. We are engaged in verifying the transformation equations in equation (11.148).

(a) Verify that

$$B_1' = B_1$$

(i) By a direct comparison of $(F')^{32}$, the element in the bottom row (remember, $F^{\alpha\beta}$ indexes from $\alpha = 0$ to 3), the third column, with F^{32}, we see

$$B_1' = B_1$$

(ii) Compare $(F')^{23}$ with F^{23}, to get

$$-B_1' = -B_1$$

which gives the same result

(b) Verify that

$$B_2' = \gamma\left(B_2 + \beta E_3\right)$$

Comparing $(F')^{13}$ and F^{13}, we get

$$B_2' = \gamma\beta E_3 + \gamma B_2$$
$$= \gamma B_2 + \gamma\beta E_3$$

Therefore, $B_2' = \gamma(B_2 + \beta E_3)$

Comparing $(F')^{31}$ and F^{31}

$$-B_2' = -\gamma\beta E_3 - \gamma B_2$$

thus,

$$B_2' = \gamma B_2 + \gamma\beta E_3 = \gamma(B_2 + \beta E_3)$$

3. After verifying the equation for B'_3 in equation (11.148), you will write the inverse transformation equations.

(a) Verify

$$B'_3 = \gamma\left(B_3 - \beta E_2\right)$$

Comparing $(F')^{21}$ and F^{21}, we get

$$B'_3 = -\gamma\beta E_2 + \gamma B_3$$
$$= \gamma B_3 - \gamma\beta E_2$$

Therefore,

$$B'_3 = \gamma(B_3 - \beta E_2)$$

Again, we get the same result by comparing $(F')^{12}$ and F^{12}

$$-B'_3 = \gamma\beta E_2 - \gamma B_3$$
$$= \gamma(B_3 - \beta E_2)$$

(b) Now, let's write the inverse transformation equations. For reference, here is equation (11.148) again:

$$\begin{aligned}
E'_1 &= E_1 & B'_1 &= B_1 \\
E'_2 &= \gamma\left(E_2 - \beta B_3\right) & B'_2 &= \gamma\left(B_2 + \beta E_3\right) \\
E'_3 &= \gamma\left(E_3 + \beta B_2\right) & B'_3 &= \gamma\left(B_3 - \beta E_2\right)
\end{aligned} \tag{11.148}$$

The inverse of equation (11.148) is found, as usual, by **interchanging primed and unprimed quantities and putting $\boldsymbol{\beta \to -\beta}$.** Write down the inverse transformation equations in the spaces below.

$E_1 =$ E'_1 $\qquad$ $B_1 =$ B'_1

$E_2 =$ $\gamma(E'_2 + \beta B'_3)$ $\qquad$ $B_2 =$ $\gamma(B'_2 - \beta E'_3)$

$E_3 =$ $\gamma(E'_3 - \beta B'_2)$ $\qquad$ $B_3 =$ $\gamma(B'_3 + \beta E'_2)$

For a general Lorentz transformation from K to a frame K' moving with velocity $\vec{v}$ relative to K, the transformation of the fields is

$$\begin{aligned} \vec{E}' &= \gamma\left(\vec{E} + \vec{\beta} \times \vec{B}\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right) \\ \vec{B}' &= \gamma\left(\vec{B} - \vec{\beta} \times \vec{E}\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right) \end{aligned} \tag{11.149}$$

where $\vec{\beta} = \vec{v}/c$, and $\gamma = (1-\beta^2)^{-1/2}$.

4. Consider the transformation equations written above in equation (11.149).

(a) The transformations in equation (11.149) demonstrate clearly that $\vec{E}$ and $\vec{B}$ ***have no independent existence***. Justify this qualitatively, first by taking $\vec{E} \neq 0, \vec{B} = 0$ in frame K and showing that K' will have components for both the electric and magnetic fields.

$$\vec{E}' = \gamma\left(\vec{E} + \vec{\beta} \times 0\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right) = \gamma\vec{E} - \frac{\gamma}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)$$

$$B' = \gamma\left(0 - \vec{\beta} \times \vec{E}\right) \cancel{- \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot 0\right)} = -\gamma\vec{\beta} \times \vec{E}$$

(b) **Write down** *the inverse of the relations* in equation (11.149), which we'll need for proofs later. Recall that the inverse transformation can be written by swapping the primes and setting $\beta \to -\beta$. Be careful with the second term.

$$\vec{E} = \gamma\left(\vec{E}' - \vec{\beta} \times \vec{B}'\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{E}'\right)$$

$$\vec{B} = \gamma\left(\vec{B}' - \vec{\beta} \times \vec{E}'\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{B}'\right)$$

You have already seen that the transformations in equation (11.149) demonstrate clearly that $\vec{E}$ and $\vec{B}$ have no independent existence. It is worth keeping in mind that:

- A purely electric or magnetic field in one coordinate system will appear as a mixture of electric and magnetic fields in another coordinate frame.
- Certain restrictions, apply, of course — as you will show on the homework; a purely electrostatic field in one coordinate system cannot be transformed into a purely magnetostatic field in another.
- In summary, the fields are interrelated, and one should speak of the electromagnetic field $F^{\alpha\beta}$, rather than $\vec{E}$ and $\vec{B}$ separately.

5. Suppose in a certain frame K' there exists an electric field $\vec{E}'$, but ***no magnetic field***; e.g., you could think of one or more point charges at rest in K' for a concrete example.

Then, show that the relation between the magnetic field $\vec{B}$ and electric field $\vec{E}$ in frame K is given by

$$\vec{B} = \vec{\beta} \times \vec{E}$$

Note that the field $\vec{E}$ is ***not*** the electrostatic field in K', but instead the field transformed from K' to K.

$$\vec{E} = \gamma(\vec{E}' - \vec{\beta} \times \vec{B}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{E}')$$

$$\vec{B} = \gamma(\vec{B}' + \vec{\beta} \times \vec{E}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{B}')$$

$$\vec{B}' = 0$$

$$\vec{B} = \gamma(\vec{\beta} \times \vec{E}')$$

$$\vec{E} = \gamma(\vec{E}' - 0) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{E}')$$

Cross product of $\vec{\beta}$ on Both sides

$$\vec{\beta} \times \vec{E} = \gamma(\vec{\beta} \times \vec{E}') - \frac{\gamma^2}{\gamma+1}\underbrace{\vec{\beta} \times \vec{\beta}}_{0}(\vec{\beta} \cdot \vec{E}')$$

So that $\vec{\beta} \times \vec{E} = \gamma(\beta \times \vec{E}') = \vec{B}$

Thus,

$$\vec{B} = \vec{\beta} \times \vec{E}$$