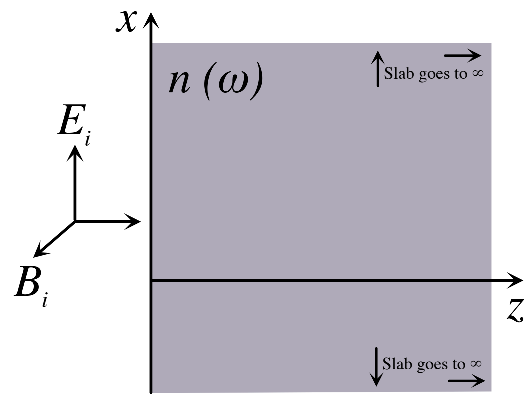
PHY 411 **Winter 2021**

Homework 4 solutions

1. A plane wave of frequency ω is incident normally from vacuum (see figure below) on a semi-infinite slab of material with a complex index of refraction $n(\omega)$, where $n^2(\omega) = \epsilon(\omega)/\epsilon_0$.



Show that the reflection coefficient is given by

$$R = \left| \frac{1 - n(\omega)}{1 + n(\omega)} \right|^2$$

whereas the transmission coefficient is given by

$$T = \frac{4\,\mathrm{Re}\,n(\omega)}{|1 + n(\omega)|^2}$$

Solution: As always, begin by setting up the incident, reflected and refracted (transmitted) waves. The incident wave travels in vacuum (μ_0, ϵ_0) along the $\hat{z}$ direction and has its $\vec{E}_i$ along the $\hat{x}$ direction; likewise for the refracted (transmitted) wave, except it is in the medium with $n(\omega)$ so we use $\mu, \epsilon(\omega)$. The reflected wave is along in vacuum (μ_0, ϵ_0), and we must take care to have it traveling along $-\hat{z}$ and specify $\vec{B}_r$ along $-\hat{y}$ to get the directions right.

The incident wave is then

$$\vec{E}_i = \hat{x} E_i \, e^{ikz - i\omega t} \tag{H4.1}$$

$$\vec{B}_i = \hat{y} \sqrt{\mu_0 \epsilon_0} \, E_i \, e^{ikz - i\omega t} \tag{H4.2}$$

The reflected wave is

$$\vec{E}_r = \hat{x} E_r \, e^{-ikz - i\omega t} \tag{H4.3}$$

$$\vec{B}_r = -\hat{y} \sqrt{\mu_0 \epsilon_0} \, E_r \, e^{-ikz - i\omega t} \tag{H4.4}$$

The transmitted (refracted) wave is

$$\vec{E}_t = \hat{x} E_t \, e^{ik_1 z - i\omega t} \tag{H4.5}$$

$$\vec{B}_t = \hat{y} \sqrt{\mu \epsilon(\omega)} \, E_t e^{ik_1 z - i\omega t} \tag{H4.6}$$

where, again, I've explicitly labeled $\epsilon(\omega)$ to indicate its dependence on the frequency ω.

Let the interface between the vacuum and the semi-infinite slab be at $z = 0$.

Apply boundary conditions at $z = 0$. Of the four boundary conditions we could apply (e.g., see equation (7.37) in Jackson), the normal components of $\vec{D}$ and $\vec{B}$ are of no use here because we've written expressions for the fields polarized parallel to the interface. Therefore, we'll need to apply boundary conditions on the tangential components of $\vec{E}$ and $\vec{H}$. I'll write the boundary condition with $\vec{B}$ instead of $\vec{H}$ by canceling $\mu = \mu_0$ without writing it explicitly.

So, setting the tangential components of $\vec{E}$ equal at $z = 0$, we get

$$E_i + E_r = E_t \tag{H4.7}$$

and setting the tangential components of $\vec{H} = \vec{B}/\mu$ equal at $z = 0$, we get

$$\frac{\sqrt{\mu_0\epsilon_0}}{\mu_0}\Big(E_i - E_r\Big) = \frac{\sqrt{\mu\epsilon(\omega)}}{\mu}\,E_t$$

so that

$$E_i - E_r = \sqrt{\frac{\mu_0\epsilon(\omega)}{\mu\epsilon_0}}\,E_t$$

As usual, we'll assume nonpermeable media, so $\mu = \mu_0$, and the above equation becomes

$$E_i - E_r = \sqrt{\frac{\epsilon(\omega)}{\epsilon_0}}\,E_t$$

and since we're given that $n^2(\omega) = \epsilon(\omega)/\epsilon_0$, we get finally that

$$E_i - E_r = n(\omega)E_t \tag{H4.8}$$

As always, it'll be easier to operate if we divide by E_i on both sides. So, equation (H4.7) becomes

$$1 + \frac{E_r}{E_i} = \frac{E_t}{E_i} \tag{H4.9}$$

whereas we get from equation (H4.8) that

$$1 - \frac{E_r}{E_i} = n(\omega)\,\frac{E_t}{E_i} \tag{H4.10}$$

To keep our procedures consistent, let us set the ratios to

$$\frac{E_r}{E_i} = r, \qquad \frac{E_t}{E_i} = t \tag{H4.11}$$

Then, equation (H4.9) and equation (H4.10) can be rewritten using equation (H4.11) as

$$1 + r = t \tag{H4.12}$$

$$1 - r = n(\omega)\,t \tag{H4.13}$$

At normal incidence ($i = 0$), the reflection coefficient R is given by

$$R = \frac{\vec{S}_r\cdot\hat{n}}{\vec{S}_i\cdot\hat{n}} = \frac{\frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\Big|E_r\Big|^2\cos\left(r' = 0\right)}{\frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\Big|E_i\Big|^2\cos\left(i = 0\right)} = \left|\frac{E_r}{E_i}\right|^2 = \Big|r\Big|^2 \tag{H4.14}$$

Again, to save time, I won't work out the steps to derive the expressions for $\vec{S}_i\cdot\hat{n}$ and $\vec{S}_r\cdot\hat{n}$ here, because I derived $\vec{S}'\cdot\hat{n}$ in the Class Summary for Week 4—Day 1.

Equation (H4.14) tells us that to find R, we need r; to find it, subtract equation (H4.13) from equation (H4.12):

$$1 + r - (1 - r) = t - n(\omega)\,t$$

Again, substitute equation (H4.12) for t:

$$2r = \Big[1 - n(\omega)\Big]\Big(1 + r\Big)$$

and gather terms with r on the left hand side:

$$2r - r + n(\omega)r = 1 - n(\omega)$$

so that

$$r = \frac{1 - n(\omega)}{1 + n(\omega)}$$

From equation (H4.14), the reflection coefficient is $R = |r|^2$, so that finally

$$R = \left|\frac{1 - n(\omega)}{1 + n(\omega)}\right|^2$$

which is what we're asked to derive.

Meanwhile, in writing the transmission coefficient in previous problems, we've been a little careless with our symbols since we were only dealing with real n. But now that we have a complex n, we will have to explicitly state that the flow of energy is the real part of the complex Poynting vector, so that

$$T = \frac{\mathrm{Re}\,\vec{S}_t \cdot \hat{n}}{\mathrm{Re}\,\vec{S}_i \cdot \hat{n}} = \frac{\frac{1}{2}\mathrm{Re}\sqrt{\frac{\epsilon(\omega)}{\mu}}\Big|E_t\Big|^2 \cos(r = 0)}{\frac{1}{2}\mathrm{Re}\sqrt{\frac{\epsilon_0}{\mu_0}}\Big|E_i\Big|^2 \cos(i = 0)} = \mathrm{Re}\sqrt{\frac{\epsilon(\omega)}{\epsilon_0}}\left|\frac{E_t}{E_i}\right|^2 = \mathrm{Re}\,n(\omega)\Big|t\Big|^2 \qquad \text{(H4.15)}$$

To get t, add together equation (H4.12) and equation (H4.13)

$$1 + r + (1 - r) = t + n(\omega)\,t$$

so that

$$t = \frac{2}{1 + n(\omega)} \qquad \text{(H4.16)}$$

Putting equation (H4.16) in equation (H4.15), we get the transmission coefficient:

$$T = \mathrm{Re}\,n(\omega)\Big|t\Big|^2 = \mathrm{Re}\,n(\omega)\left|\frac{2}{1 + n(\omega)}\right|^2$$

which demonstrates that the transmission coefficient is given by

$$T = \frac{4\,\mathrm{Re}\,n(\omega)}{|1 + n(\omega)|^2}$$

2. In class we discussed a static model for a substance in the presence of an electric field. The polarization of neighboring molecules gives rise to an internal field $\vec{E}_i$ in addition to the average macroscopic field $\vec{E}$, so that the dipole moment is modified to

$$\left\langle \vec{p}_{\text{mol}} \right\rangle = \epsilon_0 \gamma_{\text{mol}} \left(\vec{E} + \vec{E}_i \right)$$

where γ_{mol} is the molecular polarizability. Jackson finds that $E_i = \vec{P}/3\epsilon_0$.

Starting from the definition that $\vec{P} = N\langle \vec{p}_{\text{mol}} \rangle$, where N is the number of molecules per unit volume, derive the Clausius-Mossotti equation

$$\gamma_{\text{mol}} = \frac{3}{N} \frac{(\epsilon/\epsilon_0 - 1)}{(\epsilon/\epsilon_0 + 2)}$$

Solution: To figure out how to proceed, consider that the problem says to start from the definition that $\vec{P} = N\langle \vec{p}_{\text{mol}} \rangle$, and we know from Jackson that $E_i = \vec{P}/3\epsilon_0$. Finally, $\vec{E}$ is also connected to $\vec{P}$ via $\vec{P} = \epsilon_0 \chi_e \vec{E}$. Thus, since $\vec{P}$ appears to be a common thread, let us begin by putting everything in terms of $\vec{P}$.

Substituting the expression for $\langle \vec{p}_{\text{mol}} \rangle$ given above into $\vec{P} = N\langle \vec{p}_{\text{mol}} \rangle$, we get

$$\begin{aligned} \vec{P} = N\left\langle \vec{p}_{\text{mol}} \right\rangle &= N\epsilon_0 \gamma_{\text{mol}} \left(\vec{E} + \vec{E}_i \right) \\ &= N\epsilon_0 \gamma_{\text{mol}} \left[\frac{\vec{P}}{\epsilon_0 \chi_e} + \frac{\vec{P}}{3\epsilon_0} \right] \\ \text{so that} \quad \vec{P} &= N\gamma_{\text{mol}} \left[\frac{1}{\chi_e} + \frac{1}{3} \right] \vec{P} \end{aligned}$$

where I've taken $\vec{P}$ outside the parentheses on the right hand side, and canceled ϵ_0 in the numerator and denominator. Effectively, this relation implies that the scalar quantities multiplying the vector on both sides of the equation must be equal to each other, and so we obtain that

$$1 = N\gamma_{\text{mol}} \left[\frac{1}{\chi_e} + \frac{1}{3} \right]$$

which can be simplified to

$$1 = N\gamma_{\text{mol}} \left[\frac{3 + \chi_e}{3\chi_e} \right]$$

so that

$$\gamma_{\text{mol}} = \frac{1}{N} \left[\frac{3\chi_e}{3 + \chi_e} \right]$$

Now, since $\dfrac{\epsilon}{\epsilon_0} = 1 + \chi_e$, we can substitute $(\epsilon/\epsilon_0 - 1)$ in the expression above for χ_e to get

$$\gamma_{\text{mol}} = \frac{1}{N} \left[\frac{3(\epsilon/\epsilon_0 - 1)}{3 + (\epsilon/\epsilon_0 - 1)} \right]$$

so that, finally

$$\gamma_{\text{mol}} = \frac{3}{N} \frac{(\epsilon/\epsilon_0 - 1)}{(\epsilon/\epsilon_0 + 2)}$$

which is the Clausius-Mossotti equation.

3. Consider the following experimental data for nitrogen.

Temperature (K)	296.9	296.9	296.9	296.9
Pressure (Pa)	1.0200×10^5	57.50×10^5	221.6×10^5	1011.6×10^5
Density (kg m^{-3})	1.180	66.04	236.1	578.0
Dielectric constant	1.00052	1.03109	1.11413	1.29633

(a) Calculate γ_{mol} for each of the four sets of data given above.

Solution: To calculate γ_{mol}, we'll need the number of molecules per unit volume N. If we use the ideal gas law, $PV = \mathcal{N}kT$, then the number of molecules per unit volume $N = \mathcal{N}/V$ is given by P/kT, where $k = 1.38 \times 10^{-23}$ J K^{-1}, so that

$$\gamma_{\text{mol}} = \frac{3kT}{P} \frac{(\epsilon/\epsilon_0 - 1)}{(\epsilon/\epsilon_0 + 2)}$$

I've copied the table below, with the first four rows repeated from above. I've also added a row at the bottom of the table with the values of γ_{mol} (in dimensions of m^3).

Temperature (K)	296.9	296.9	296.9	296.9
Pressure (Pa)	1.0200×10^5	57.50×10^5	221.6×10^5	1011.6×10^5
Density (kg m^{-3})	1.180	66.04	236.1	578.0
Dielectric constant	1.00052	1.03109	1.11413	1.29633
Answer (b): γ_{mol}	2.088×10^{-29}	2.193×10^{-29}	2.033×10^{-29}	1.092×10^{-29}

(b) In principle, γ_{mol} is a function of the electric field, but for a wide range of field strengths, it is a constant that characterizes the response of the molecules to an applied field. Did you find that γ_{mol} is constant in all the four instances that you calculated above? Comment.

Solution: γ_{mol} is largely constant for the first 3 cases, but not for the fourth. This might be because our assumption of the ideal gas law breaks down at such a high pressure.

Space left blank for student notes; homework solutions continue on the next page.

4. Answer the following. *You must support your answer with appropriate calculations and explanations if you want full credit. In both parts (a) and (b) below, a yes/no answer that is not supported by calculations and explanations will be awarded zero credit.*

The electron density in the ionosphere reaches a maximum value of 1.5×10^{12} m^{-3}.

(a) Can a 2 MHz wave be used to communicate with a satellite?

Solution: In the high frequency limit, when the frequency of the electromagnetic waves is well above the highest resonant frequency ($\omega \gg \omega_j$), we know that the wave number is given by

$$ck = \sqrt{\omega^2 - \omega_p^2}$$

where ω_p is the plasma frequency defined below. In a plasma for which all the electrons are essentially "free" this equation can hold for a wide range of frequencies, including $\omega < \omega_p$. So, we have two possibilities. If $\omega > \omega_p$ in a plasma, then ck will be real and electromagnetic waves will propagate in the plasma. On the other hand, if $\omega < \omega_p$ in a plasma, then ck will be imaginary and electromagnetic waves will not propagate in the plasma, but will instead be reflected.

The plasma frequency is given by

$$\omega_p = \sqrt{\frac{NZe^2}{\epsilon_0 m}}$$

where N is the number of molecules per unit volume and Z is the number of electrons per molecule so that NZ is the electron density, e is the electron charge, ϵ_0 is the permittivity of free space, and m is the mass of a charge (i.e., the electron mass). Thus, we get

$$\omega_p = \sqrt{\frac{(1.5 \times 10^{12}\ \text{m}^{-3})\,(1.6 \times 10^{-19}\ \text{C})^2}{(8.854 \times 10^{-12}\ \text{farad/m})\,(9.11 \times 10^{-31}\ \text{kg})}} = 6.8998 \times 10^7\ \text{rad/s}$$

so that

$$f_p = \frac{\omega_p}{2\pi} = \frac{6.8998 \times 10^7\ \text{rad/s}}{2\pi} = 1.098 \times 10^7\ \text{Hz} = 11\ \text{MHz}$$

Since 2 MHz < 11 MHz, we find that $\omega < \omega_p$, so that the wave number ck will be imaginary. Therefore, a 2 MHz wave **will not propagate** through the ionosphere and hence $\boxed{\textbf{cannot be used to communicate}}$ with a satellite.

(b) Can a 2 GHz wave be used to communicate with a satellite?

Solution: We don't need to do any new calculations for this part. Since 2 GHz > 11 MHz, and thus $\omega > \omega_p$, we get that the wave number k is real, which means that a 2 GHz wave **will propagate** through the ionosphere and hence $\boxed{\textbf{can be used to communicate}}$ with a satellite.