Physics 460—Particle in a Sphere: Normalization Spring 2020

The energy eigenstates for the particle in a sphere are given in Eq. (2.89) of the course notes. These states are not normalized, unfortunately.

Note: Because the spherical harmonics are already normalized, you don't have to worry about them when answering these questions. In other words, we already know that

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \, \left| Y_l^m(\theta,\phi) \right|^2 = 1.$$

So you really only have to worry about the normalization of the radial part.

(1) Explain in detail how to normalize these states, assuming that the radius of the sphere is a. In other words, assuming that the system is in a state with a given n and l, how would you calculate the normalization constant for Eq. (2.89)?

(2) Find explicitly the normalization constants for the states with $(n,l) = (1,0)$, $(n,l) = (2,0)$, $(n,l) = (1,1)$, and $(n,l) = (2,1)$. It's fine to use WolframAlpha to do the integrals and have the normalization constants be numbers, rather than algebraic expressions, though you should include the dependence on a.

(3) Suppose the system is in a state with $(n,l) = (2,1)$. If you measure the position of the particle, what is the probability that you will find it inside the first zero of j_1?

(1) Normalization in the radial direction is

$$A^2 \int_0^a r^2 \left| j_l(k_{nl} r) \right|^2 dr = 1 \quad \text{where } k_{nl} = \frac{z_{nl}}{a}$$

Let $x = \frac{r}{a} \Rightarrow dr = a\,dx \quad r^2 = a^2 x^2 \quad k_{nl} r = z_{nl} x$

$$A^2 a^3 \int_0^1 x^2 \left| j_l(z_{nl} x) \right|^2 dx = 1$$

(2) $\underline{n=1,\ l=0}$, $z_{10} = \pi$, $j_0(\pi x) = \frac{\sin \pi x}{\pi x}$ (2.87a)

$$A^2 a^3 \int_0^1 x^2 \left| j_0(\pi x) \right|^2 dx = 1 = \frac{1}{2\pi^2} A^2 a^3$$

$$\boxed{A = \sqrt{\frac{2\pi^2}{a^3}}}$$

$\underline{n=2,\ l=0}$, $z_{20} = 2\pi$, $j_0(2\pi x) = \dfrac{\sin(2\pi x)}{2\pi x}$

$$A^2 a^3 \int_0^1 x^2 \left|j_0(2\pi x)\right|^2 dx = 1 = \frac{1}{8\pi^2} A^2 a^3$$

$$\boxed{A = \sqrt{\frac{8\pi^2}{a^3}}}$$

$\underline{n=1,\ l=1}$, $z_{11} = 1.43\pi$, $j_1(z_{11}x) = \dfrac{\sin(z_{11}x)}{(z_{11}x)^2} - \dfrac{\cos(z_{11}x)}{z_{11}x}$

$$A^2 a^3 \int_0^1 x^2 \left|j_1(z_{11}x)\right|^2 dx = 1 = 0.0603\, A^2 a^3$$

$$\boxed{A = \frac{1}{\sqrt{0.0603\, a^3}}}$$

$\underline{n=2,\ l=1}$, $z_{12} = 2.459\pi$ $j_1(z_{12}x) = \dfrac{\sin(z_{12}x)}{(z_{12}x)^2} - \dfrac{\cos(z_{12}x)}{z_{12}x}$

$$A^2 a^3 \int_0^1 x^2 \left|j_1(z_{12}x)\right|^2 dx = 1 = 0.00824\, a^3 A^2$$

$$\boxed{A = \frac{1}{\sqrt{0.00824\, a^3}} = 11\, a^{-3/2}}$$

(3) In the state with $n=2, \ell=1$, the wavenumber is $k_{21} = \frac{z_{21}}{a}$. The first zero occurs at radius r_{11}, where $k_{21} r_{11} = z_{11}$, or $r_{11} = \frac{z_{11}}{k_{21}} = \frac{z_{11}}{z_{21}} a$.

The probability of finding the particle inside this position is

$$P(r \le r_{11}) = A^2 \int_0^{r_{11}} r^2 \left| j_1(k_{21} r) \right|^2 dr$$

As before, let

$$r = xa \Rightarrow r_{11} = x_{11} a = \frac{z_{11}}{z_{21}} a \Rightarrow x_{11} = \frac{z_{11}}{z_{21}}$$

$$P(r \le a z_{11}) = A^2 a^3 \int_0^{z_{11}/z_{21}} x^2 \left| j_1(z_{21} x) \right|^2 dx$$

using the value for A from above, and using WolframAlpha for the integral,

$$P(r \le a z_{11}) = (11)^2 (0.00464) = 0.56$$

seems reasonable...