Learning Goals:

1. Fast Fourier Transform

We discussed somewhat the importance of the Fourier Transform in physics. We also discussed how in real applications in which resolution is always limited, it is the discrete Fourier transform that is used. That is

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt \;\rightarrow\; g(n\Delta\omega) = \sum_{m=0}^{N-1} f(m(\Delta t) e^{-i2\pi nmn/N}$$

- For each ω, there are N calculations
- There are a total of N ω*'s* since $\Delta\omega \sim \frac{1}{T}$
- So N^2 calculations.

Now let's *tinker* $$g(n\Delta\omega) = \sum_{m=0}^{N-1} f(m\Delta t) e^{-i2\pi mn/N}$$

$$g(n\Delta\omega) = \underbrace{\sum_{m=0,2,\ldots}^{N-1} f(m\Delta t) e^{-i2\pi mn/N}}_{\text{even terms}} + \underbrace{\sum_{m=1,3,\ldots}^{N-1} f(m\Delta t) e^{-i2\pi mn/N}}_{\text{odd terms}}$$

$$g(n\Delta\omega) = \underbrace{\sum_{m=0,2,\ldots}^{N-1} f(m\Delta t)e^{-i2\pi mn/N}}_{\text{even terms}} + \underbrace{\sum_{m=1,3,\ldots}^{N-1} f(m\Delta t)e^{-i2\pi mn/N}}_{\text{odd terms}}$$

$$g(n\Delta\omega) = \sum_{j=0}^{N/2-1} f(2j\Delta t)e^{-i2\pi 2jn/N} + \sum_{j=0}^{N/2-1} f((2j+1)\Delta t)e^{-i2\pi(2j+1)n/N}$$

even odd

$$g(n\Delta\omega) = \sum_{j=0}^{N/2-1} f(2j\Delta t)e^{-i2\pi jn/(N/2)} + e^{-i2\pi n/N} \sum_{j=0}^{N/2-1} f((2j+1)\Delta t)e^{-i2\pi jn/(N/2)}$$

$$g(n\Delta\omega) = g_{even}(n\Delta\omega) + e^{-i2\pi n/N} g_{odd}(n\Delta\omega)$$

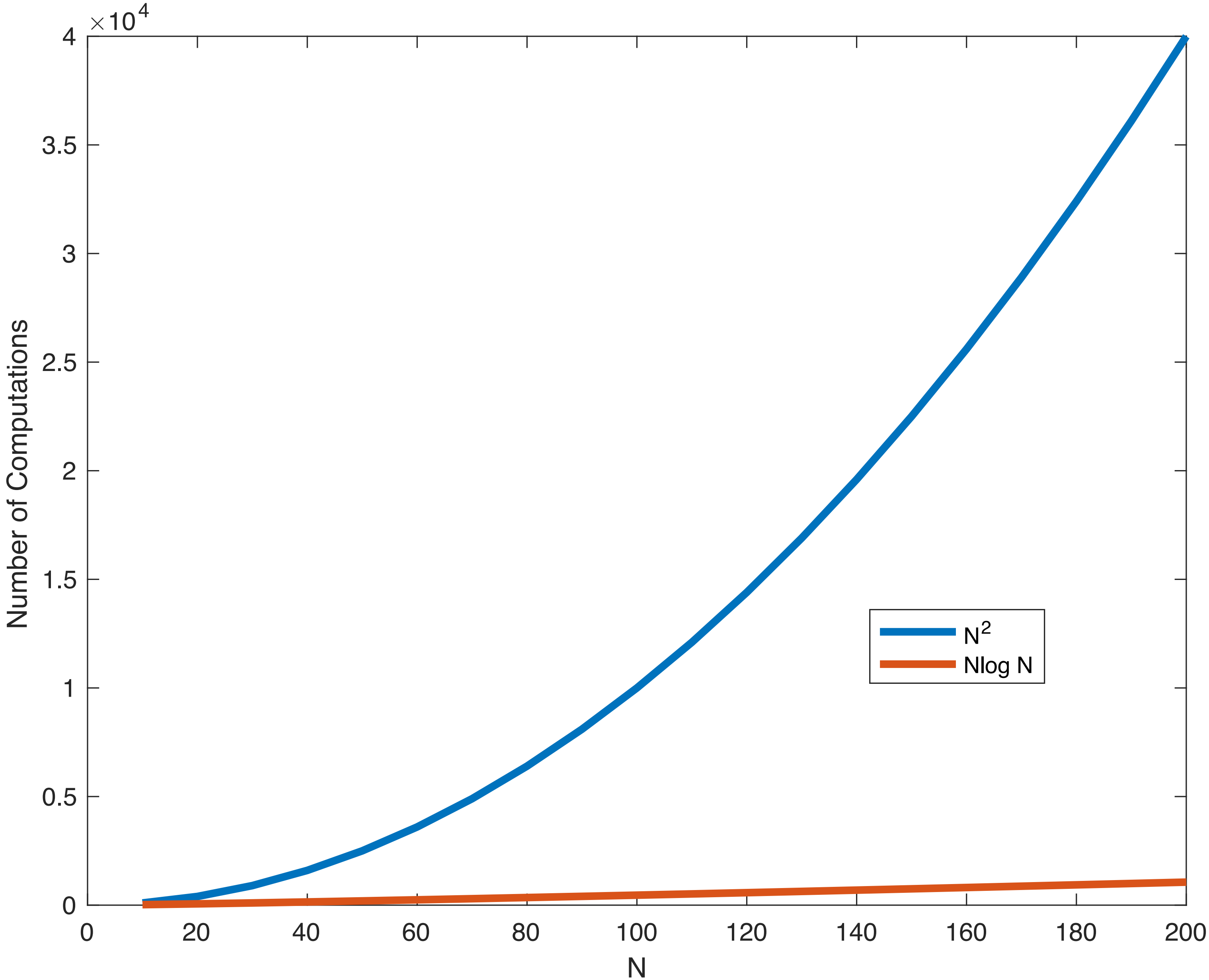
twiddle factor

Each of the sums now contains half as many terms

So what? To find out, do question 1 on the worksheet and S T O P

(1) 2 x $(N/2)^2 < N^2$ operations required.

Do question (2) and (3) on the worksheet.



Before we fully develop the fast Fourier transform, we need to introduce the idea of *recursive* programming. To see how this works, do question (4) on the worksheet and S T O P.

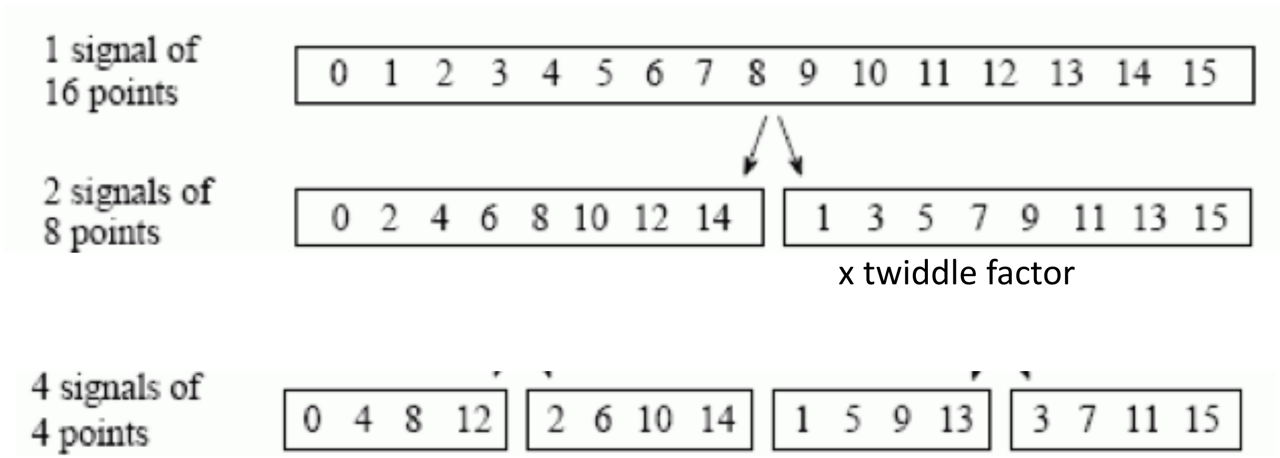
(4)

```
function f= MyFactorial (n)
%MyFactorial is a demonstration of recursive programming

if(n==1)
      f  =  1;
      return;
end
f = n*MyFactorial(n-1);

end
```

Okay, now back to the FFT. Consider the following data set. Each box has the value of the signal as shown. Assume the function is evenly sampled, say, 1 per second



x twiddle factor

Order
$2 \times (N/2)^2$

x twiddle factor

x twiddle factor

Order
$4 \times (N/4)^2$

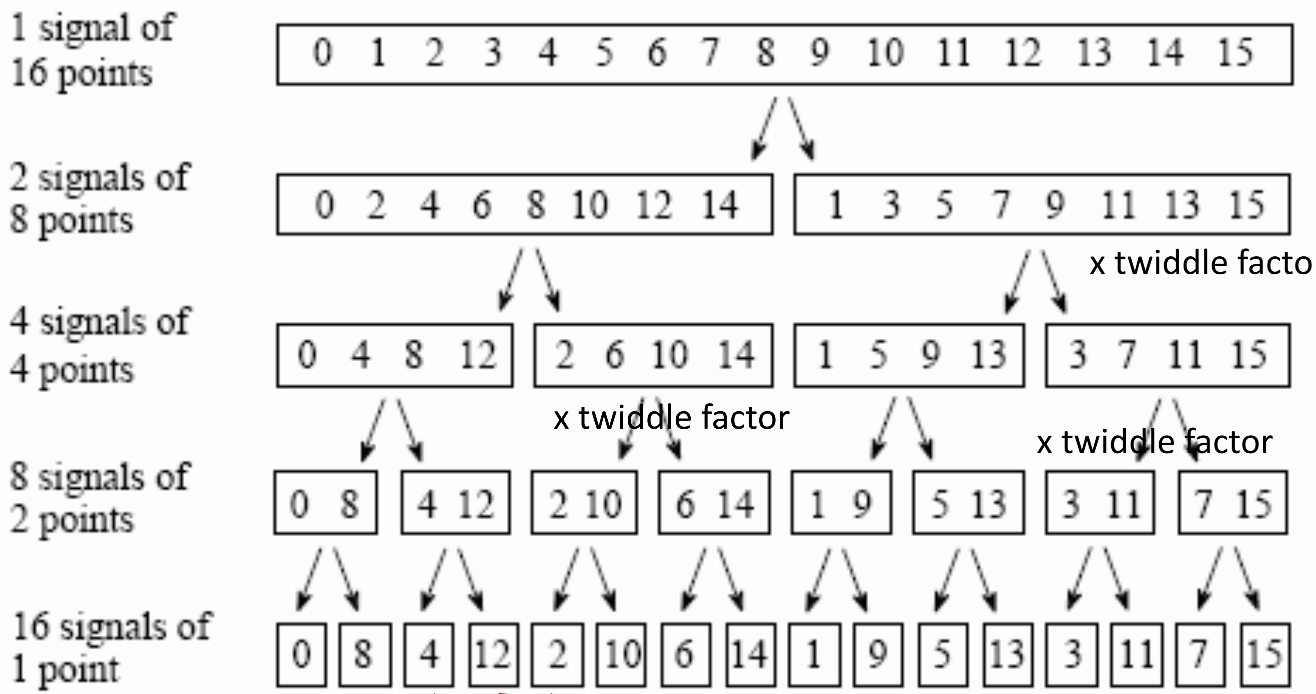


FIGURE 12-2
The FFT decomposition. An N point signal is decomposed into N signals each containing a single point. Each stage uses an *interlace decomposition*, separating the even and odd numbered samples.

$$g(n\Delta\omega) = \cancel{\sum_{j=0}^{N/2-1}} f(2j\Delta t)e^{-i2\pi jn/(N/2)} + e^{-i2\pi n/N} \cancel{\sum_{j=0}^{N/2-1}} f((2j+1)\Delta t)e^{-i2\pi jn/(N/2)}$$

Order is now $N \ln N$

Detailed example, N = 4

$$g(n\Delta\omega) = \underbrace{\sum_{j=0}^{N/2-1} f(2j\Delta t)e^{-i2\pi jn/(N/2)}}_{E} + \underbrace{W_N^n \sum_{j=0}^{N/2-1} f((2j+1)\Delta t)e^{-i2\pi jn/(N/2)}}_{O}$$

$$E = EE + E0$$

$$O = OE + OO.$$

$$\begin{aligned} EE &= \sum_{j=0}^{N/(2\cdot 2)-1} f(2(2j\Delta t))e^{-i2\pi(2j)n/(N/2)} \\ &= \sum_{j=0}^{N/4-1} f(4j\Delta t)e^{-i2\pi jn/(N/4)} \\ EO &= \sum_{j=0}^{N/(2\cdot 2)-1} f(2(2j+1)\Delta t)e^{\frac{-i2\pi(2j+1)n}{(N/2)}} \\ &= \sum_{j=0}^{N/4-1} f((4j+2)\Delta t)e^{\frac{-i2\pi jn}{(N/4)}} e^{\frac{-i2\pi n}{(N/2)}} \\ &= W_{\frac{N}{2}}^n \sum_{j=0}^{N/4-1} f((4j+2)\Delta t)e^{\frac{-i2\pi jn}{(N/4)}} \end{aligned}$$

$$\begin{aligned} OE &= W_N^n \sum_{j=0}^{N/(2\cdot 2)-1} f((2(2j)+1)\Delta t)e^{\frac{-i2\pi(2j)n}{(N/2)}} \\ &= W_N^n \sum_{j=0}^{N/4-1} f((4j+1)\Delta t)e^{\frac{-i2\pi jn}{(N/4)}} \\ OO &= W_N^n \sum_{j=0}^{N/(2\cdot 2)-1} f((2(2j+2)+1)\Delta t)e^{\frac{-i2\pi(2j+1)n}{(N/2)}} \\ &= W_N^n \sum_{j=0}^{N/4-1} f((4j+3)\Delta t)e^{\frac{-i2\pi jn}{(N/4)}} e^{\frac{-i2\pi n}{(N/2)}} \\ &= W_N^n W_{\frac{N}{2}}^n \sum_{j=0}^{N/4-1} f((4j+3)\Delta t)e^{\frac{-i2\pi jn}{(N/4)}} \end{aligned}$$

$$g(n\Delta\omega) = f(0\Delta t) + W_2^n f(2\Delta t) + W_4^n f(1\Delta t) + W_4^n W_2^n f(3\Delta t).$$

$$W_N^n = e^{-i2\pi n/N}$$

Do question (5) on the worksheet and S T O P

0^{th} frequency: 6. What do you notice about this value?

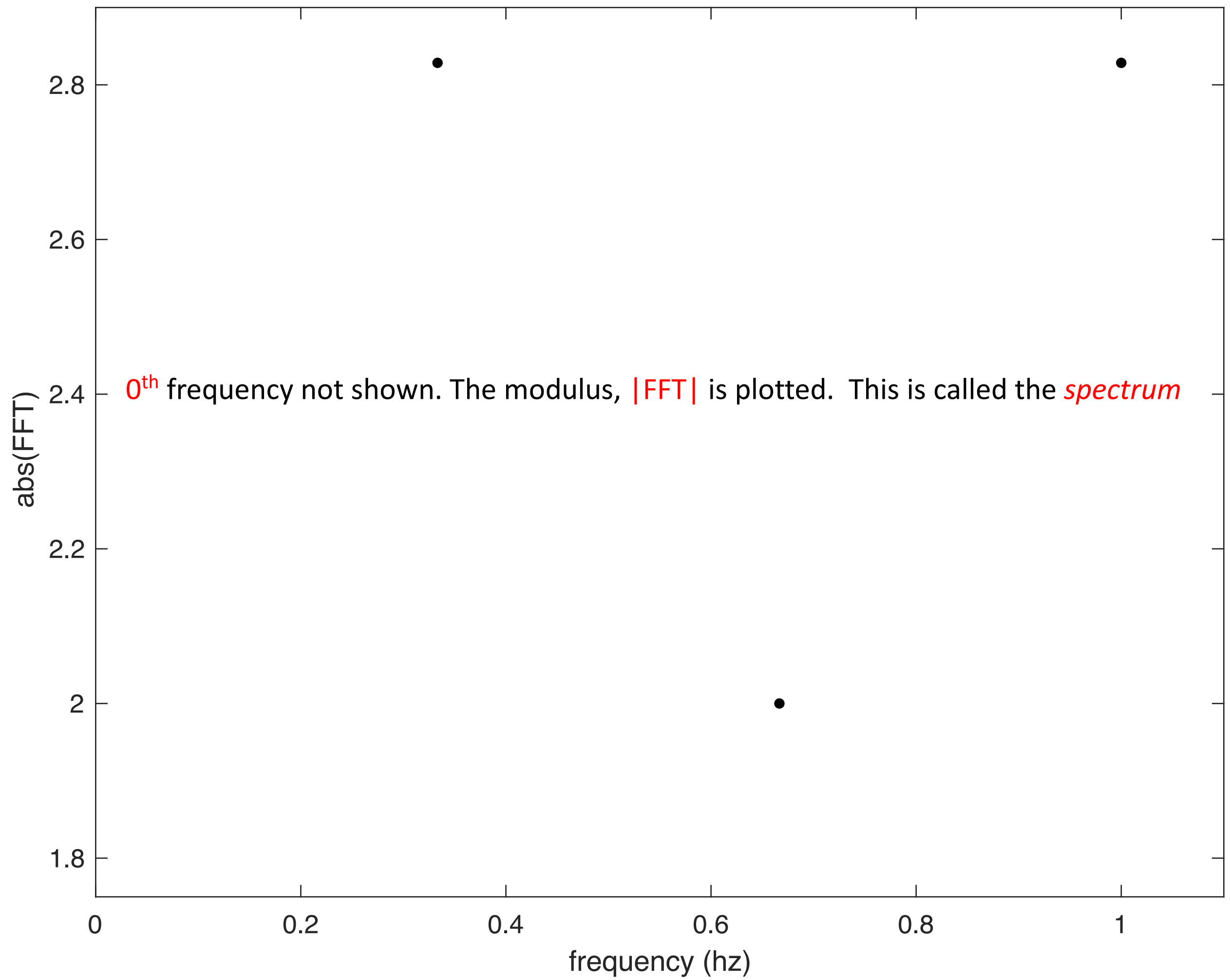
1^{st} frequency: -2 + 2*i* Notice that Fourier transform is complex even though the data is real. There is nothing wrong with this result, but one does need to adjust how one visualizes these results.

Full break down

$$\Delta\omega \sim \frac{1}{T}$$

Frequency	FFT
0	6
1/3 (also could 2π)	-2 + 2*i*
2/3 (also could 2π)	-2
3/3 (also could 2π)	-2 – 2*i*

Fourier-03



0th frequency not shown. The modulus, |FFT| is plotted. This is called the *spectrum*
abs(FFT)
2.8
2.6
2.4
2.2
2
1.8
0
0.2
0.4
0.6
0.8
1
frequency (hz)

We now have our basic *FFT* algorithm in place.

The *FFT* has many important properties that you need to understand. The first we'll explore has to do with what is called the *Nyquist frequency*. Do question (6) on the worksheet and S T O P

(6)
$$
\begin{aligned}
g(n\Delta\omega) &= \sum_{m=0}^{N-1} f(m\Delta t)e^{-i2\pi mn/N} \\
&= f(0) + f(1)e^{-i2\pi(1)n/8} + f(2)e^{-i2\pi(2)n/8} \\
&+ f(3)e^{-i2\pi(3)n/8} + f(4)e^{-i2\pi(4)n/8} \\
&+ f(5)e^{-i2\pi(5)n/8} + f(6)e^{-i2\pi(6)n/8} + f(7)e^{-i2\pi(7)n/8} \\
&= \\
&\quad f(0) + f(1)e^{-i\pi n/4} + f(2)e^{-i\pi n/2} \\
&+ f(3)e^{-i3\pi n/4} + f(4)e^{-i\pi n} \\
&+ f(5)e^{-i5\pi n/4} + f(6)e^{-i3\pi n/2} + f(7)e^{-i7\pi n/4}
\end{aligned}
$$

Same frequency in magnitude but negatives of each other

After frequency *N/2*, the frequencies are just negatives of each other

More things to keep in mind about *FFT's.* Sampling rate is very important and in general causes two kinds of problems.

The first is explored in question (7), do this question and S T O P

The last issue we'll explore having to do with sampling is the case when the sampling rate is not commensurate with the period of the function. To see how this happens, do question (8) on the worksheet.