Physics 460—Homework Report 7
Due Tuesday, May 19, 1 pm **Name:** Timothy Holmes

Complete all the problems on the accompanying assignment.

List all the problems you worked on in the space below. Circle the ones you fully completed:

1a c 2abc

Please place the problems into the following categories:

- These problems helped me understand the concepts better: ____________
- I found these problems fairly easy: X
- I found these problems very challenging: ____________

In the space below, show your work (even if not complete) for any problems you still have questions about. Indicate where in your work the question(s) arose, and ask specific questions that I can answer.

Use the back of this sheet or attach additional paper, if necessary.

If you have no remaining questions about this homework assignment, use this space for one of the following:

- Write one or two of your solutions here so that I can give you feedback on its clarity.
- Explain how you checked that your work is correct.

Wolfram
Course Notes

(1) In thinking about Bell's inequality, I started to wonder what assumptions were really necessary for the result to hold. This problem investigates that.

Recall that for a spin-1/2 system the z-basis states can be written in terms of the basis state along a second direction $\hat{n}$ as This transformation can be reversed, and the z-state basis vectors can be written as

$$|+_z\rangle = \cos\frac{\theta}{2}\,|+_{\hat{n}}\rangle + \sin\frac{\theta}{2}\,|-_{\hat{n}}\rangle, \quad \text{and} \quad |-_z\rangle = e^{-i\phi}\sin\frac{\theta}{2}\,|+_{\hat{n}}\rangle - e^{-i\phi}\cos\frac{\theta}{2}\,|-_{\hat{n}}\rangle.$$

Here θ is the angle of $\hat{n}$ relative to the z axis (the co-latitude) and ϕ is the azimuthal angle. We showed in an in-class activity that the correlation for the singlet state was

$$\epsilon(\hat{k}, \hat{n}) = -\cos\theta.$$

(a) The state $|\Psi\rangle = |+;-\rangle$ is like the singlet state in that the spins of the two particles are anti-correlated, but unlike the singlet state, this state is not entangled. Calculate the correlation coefficient for this state, assuming the first particle is measured along the z axis and the second is measured along the $\hat{n}$ axis.

(b) Since the spins of the particles in this state are anti-correlated, we can use the standard Bell inequality, Eq. (4.40),

$$|\epsilon(\hat{n}_1, \hat{n}_2) - \epsilon(\hat{n}_1, \hat{n}_3)| \le 1 + \epsilon(\hat{n}_2, \hat{n}_3),$$

with this state. Using the angles from the course notes, with $\hat{n}_1$ and $\hat{n}_2$ perpendicular, and $\hat{n}_3$ at a 45° angle to both $\hat{n}_1$ and $\hat{n}_2$, does quantum mechanics predict that measurements of the correlations for this state violate Bell's inequality?

(c) Suppose instead that the state of the system is $|\Psi\rangle = |+;+\rangle$. This state is not entangled, and the spins of the particles are correlated, not anti-correlated. Calculate the correlation coefficient for this state, assuming the first particle is measured along the z axis and the second is measured along the $\hat{n}$ axis.

(d) Since the spins of the particles in this state are correlated, we have to modify the derivation of Bell's inequality. Explain how to modify it, and show that the proper inequality for this state is

$$|\epsilon(\hat{n}_1, \hat{n}_2) - \epsilon(\hat{n}_1, \hat{n}_3)| \le 1 - \epsilon(\hat{n}_2, \hat{n}_3).$$

Again using the angles from the course notes, with $\hat{n}_1$ and $\hat{n}_2$ perpendicular, and $\hat{n}_3$ at a 45° angle to both $\hat{n}_1$ and $\hat{n}_2$, does quantum mechanics predict that measurements of the correlations for this state violate Bell's inequality?

(e) Given your answers above, are either entanglement or anti-correlation of the spins of the two particles important for predictions of violations of Bell's inequality? Explain.

(2) To better understand Bell's argument, it might be useful to invent a hidden variable theory. Here's a particularly simple-minded one.

Let the hidden variable λ have the range $-1 \le \lambda \le 1$, and let the "measurement" function $f(\hat{n}, \lambda)$ be

$$f(\hat{n}, \lambda) = \begin{cases} +1, & \lambda \ge \cos\theta, \\ -1 & \lambda < \cos\theta, \end{cases}$$

where θ is the angle between $\hat{n}$ and the z axis (so $0 \le \theta \le \pi$). This says that the result of the measurement depends on a comparison of the value of λ to the orientation of the measurement axis. We will apply Bell's result, so the measurement functions for the two particles obey the property $f_1(\hat{n}, \lambda) = -f_2(\hat{n}, \lambda)$.

(a) Let the probability density be uniform, $\rho(\lambda) = A$, where A is a constant. Find A.

(b) Using the three angles from Figure 4.2 of the course notes, with $\hat{n}_1$ along the z axis, $\hat{n}_2$ along the x axis, and $\hat{n}_3$ at a 45° angle in the x–z plane, calculate $\epsilon(\hat{n}_1, \hat{n}_2)$, $\epsilon(\hat{n}_1, \hat{n}_3)$, and $\epsilon(\hat{n}_2, \hat{n}_3)$ for this hidden variable theory.

(c) Show that (for these three angles at least) this hidden variable theory obeys Bell's inequality,

$$|\epsilon(\hat{n}_1, \hat{n}_2) - \epsilon(\hat{n}_1, \hat{n}_3)| \le 1 + \epsilon(\hat{n}_2, \hat{n}_3).$$

Homework 7

(1)

$$|+z\rangle = \cos\frac{\theta}{2}|+\hat{n}\rangle + \sin\frac{\theta}{2}|-\hat{n}\rangle$$

$$|-z\rangle = e^{i\varphi}\sin\frac{\theta}{2}|+\hat{n}\rangle - e^{i\varphi}\cos\frac{\theta}{2}|-\hat{n}\rangle$$

Correlation for the singlet state

$$E(\hat{k},\hat{n}) = -\cos\theta$$

(a) $|\psi\rangle = |+;-\rangle$

$P_{++} = |\langle ++|\psi\rangle|^2 = \cos^2\theta/2$

$P_{+-} = |\langle +-|\psi\rangle|^2 = \sin^2\theta/2$

$P_{-+} = |\langle -+|\psi\rangle|^2 = (e^{-i\varphi})^2\sin^2\theta/2 \rightarrow e^{-2i\varphi}\sin^2\theta/2$

$P_{--} = |\langle --|\psi\rangle|^2 = (e^{-i\varphi})^2\cos^2\theta/2 \rightarrow e^{-2i\varphi}\cos^2\theta/2$

$C = P_{++} + P_{+-} - P_{-+} + P_{--}$

$C = \cos^2\theta/2 + \sin^2\theta/2 - (e^{-2i\varphi}\sin^2\theta/2) + (e^{-2i\varphi}\cos^2\theta/2)$

(B) $|E(\hat{n}_1,\hat{n}_2) - E(\hat{n}_1,\hat{n}_3)| \leq 1 + E(n_2,n_3)$

(c) $|\psi\rangle = |+;+\rangle$

$P_{++} = |\langle ++|\psi\rangle|^2 = \cos^2\theta/2$

$P_{+-} = |\langle +-|\psi\rangle|^2 = \sin^2\theta/2$

$P_{-+} = |\langle -+|\psi\rangle|^2 = 0$

$P_{--} = |\langle --|\psi\rangle|^2 = 0$

$C = P_{++} + P_{+-} - P_{-+} + P_{--}$

$C = \cos^2\theta/2 + \sin^2\theta/2$

(D)

(e)

(2) $$f(\hat{n}, \lambda) = \begin{cases} +1 , & \lambda \geq \cos\theta \\ -1 , & \lambda < \cos\theta \end{cases}$$

(a) $\rho(\lambda) = A$ $$1 = \int_{-1}^{1} A \, d\lambda \Rightarrow 1 = A \left[\lambda \right]_{-1}^{1}$$

$$\rightarrow 1 = A \left[1 - (-1) \right] \rightarrow 1 = 2A \quad A = 1/2 \quad \rho(\lambda) = \frac{1}{2}$$

(B)

$$E(\hat{n}_1, \hat{n}_2) = -\int \frac{1}{2} (1)(1) = -1$$

$$E(\hat{n}_1, \hat{n}_3) = -\int \frac{1}{2} (1)(1) = -1$$

$$E(\hat{n}_2, \hat{n}_3) = -\int \frac{1}{2} (+1)(1) = -1$$

(c) $$\left| E(\hat{n}_1, \hat{n}_2) - E(\hat{n}_1, \hat{n}_3) \right| \leq 1 + E(n_2, n_3$$

$$= \left| -1 - (-1) \right| \leq 1 + (-1)$$

$$\boxed{0 \leq 0}$$