Physics 460—Angular Momentum Operators Spring 2020

A number of very useful results about the angular momentum operators and their eigenstates are given in the course notes without proof. It's excellent practice to derive these from the fundamental relationships between the operators. Here's your chance.

The fundamental properties of angular momentum are determined by the commutation relations between the operators:

$$\left[J_x, J_y\right] = i\hbar J_z, \quad \left[J_y, J_z\right] = i\hbar J_x, \quad \left[J_z, J_x\right] = i\hbar J_y.$$

The total angular momentum is $J^2 = {J_x}^2 + {J_y}^2 + {J_z}^2$, and the raising and lowering operators are defined by $J_\pm = J_x \pm iJ_y$.

(1) Using the properties above, derive the "useful commutation relations" given in Eq. (1.28) on page 8 of the course notes.

(2) Using the definition of the angular momentum operators, Eq. (1.21), along with what you know about the position and momentum operators, evaluate the following commutators:

$$\left[X, L_z\right], \quad \left[Y, L_z\right], \quad \left[Z, L_z\right], \quad \left[P_x, L_z\right], \quad \left[P_y, L_z\right], \quad \left[P_z, L_z\right].$$

① We want to show:

$$[J_z, J_\pm] = \pm\hbar J_\pm, \quad [J_+, J_-] = 2\hbar J_z, \quad [J^2, \vec{J}] = 0$$

$$[J_z, J_\pm] = [J_z, J_x \pm i J_y] = [J_z, J_x] \pm i[J_z, J_y]$$
$$= i\hbar J_y \pm i(-i\hbar J_x) = \hbar(\pm J_x + iJ_y)$$
$$= \pm\hbar(J_x \pm iJ_y) = \pm\hbar J_\pm \ \checkmark$$

$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y]$$
$$= [J_x, J_x] + i[J_y, J_x] - i[J_x, J_y] + [J_y, J_y]$$
$$= 0 + i(-i\hbar J_z) - i(i\hbar J_z) + 0$$
$$= 2\hbar J_z \ \checkmark$$

For the last part we will use the commutator identity:

$$[AB, C] = A[B,C] + [A,C]B$$

Then

$$[J^2, J_x] = [J_x^2 + J_y^2 + J_z^2, J_x]$$
$$= [J_x^2, J_x] + [J_y^2, J_x] + [J_z^2, J_x]$$
$$= 0 + J_y(-i\hbar J_z) + (-i\hbar J_z)J_y + J_z(i\hbar J_y) + (i\hbar J_y)J_z$$
$$= 0$$

$$[J^2, J_y] = [J_x^2, J_y] + [J_y^2, J_y] + [J_z^2, J_y]$$
$$= J_x(i\hbar J_z) + (i\hbar J_z)J_x + 0 + J_z(-i\hbar J_x) + (-i\hbar J_x)J_z$$
$$= 0$$

$$[J^2, J_z] = [J_x^2, J_z] + [J_y^2, J_z] + [J_z^2, J_z]$$
$$= J_x(-i\hbar J_y) + (-i\hbar J_y)J_x + J_y(i\hbar J_x) + (i\hbar J_x)J_y + 0$$
$$= 0$$

$$\therefore \; [J^2, \vec{J}] = 0$$

2) $L_z = XP_y - YP_x$

$$[X, L_z] = [X, XP_y] - [X, YP_x] = 0 - Y[X, P_x] = -i\hbar Y$$

$$[Y, L_z] = [Y, XP_y] - [Y, YP_x] = X[Y, P_y] - 0 = i\hbar X$$

$$[Z, L_z] = [Z, XP_y] - [Z, YP_x] = 0 - 0 = 0$$

$$[P_x, L_z] = [P_x, XP_y] - [P_x, YP_x] = [P_x, X]P_y - 0 = -i\hbar P_y$$

$$[P_y, L_z] = [P_y, XP_y] - [P_y, YP_x] = 0 - [P_y, Y]P_x = i\hbar P_x$$

$$[P_z, L_z] = [P_z, XP_y] - [P_z, YP_x] = 0 - 0 = 0$$