Physics 440, Autumn 2019 Activity 19: Canonical transformation (free fall)

The Hamiltonian for a particle falling in one dimension is

$$H = \frac{p^2}{2m} + mgy$$

a. Find a canonical transformation from (y, p) to (Y, P) such that the new Hamiltonian is $K(Y, P) = P$. Use a generating function of type $F_4(p, P, t)$.

b. Write down Hamilton's equations for the new variables (Y, P), and solve them for $Y(t)$ and $P(t)$. You should have two unknown constants in your solutions.

c. Using your canonical transformation, find $y(t)$ and $p(t)$ from your expressions for $Y(t)$ and $P(t)$. Evaluate your unknown constants for initial conditions $y(0) = h$ and $p(0) = mv_0$. Do your equations for $y(t)$ and $p(t)$ look familiar?

d. In part a. we made the new Hamiltonian pretty simple, $K = P$. But we can do better! Investigate the possibility of finding a canonical transformation $(p, y) \rightarrow (R, Z)$ so that the new Hamiltonian is $K = 0$. (Hint: try the generating function $F_4(p, R, t) = \frac{p^3}{6m^2 g} - p\frac{R}{mg} - Rt$.) Solve the resulting equations of motion for $Y(t)$ and $P(t)$ for the same initial conditions as before. What is the physical interpretation of this transformation?