PHY 411 **Winter 2021**

Class Summary—Week 4, Day 2—Thursday, Jan 28

Normal and Anomalous Dispersion

In the previous class, we derived an expression for the dielectric constant:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m}\sum_j \left[\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}\right] \tag{7.51}$$

We will now discuss what we can learn from this equation.

Let's consider the real and imaginary parts separately — we will be referring to Re $\epsilon(\omega)$ and Im $\epsilon(\omega)$ respectively below.

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m}\sum_j f_j \left\{\frac{(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2} + i\left[\frac{\omega\gamma_j}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2}\right]\right\} \tag{7.51.a}$$

To make the discussion easy to follow, let all the oscillators have the same frequency ω_1. Then, equation (7.51.a) becomes

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2 f_1}{\epsilon_0 m}\left\{\frac{(\omega_1^2 - \omega^2)}{(\omega_1^2 - \omega^2)^2 + \omega^2\gamma_1^2} + i\left[\frac{\omega\gamma_1}{(\omega_1^2 - \omega^2)^2 + \omega^2\gamma_1^2}\right]\right\} \tag{7.51.b}$$

Also in the discussion below, I will be referring to the sign of $\chi(\omega) = \epsilon/\epsilon_0 - 1$.

Now imagine scanning across in frequency ω, starting from $\omega = 0$.

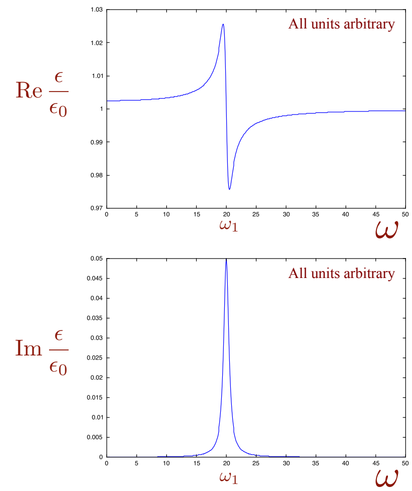
- For $\omega < \omega_1$, the sign of Re $\chi(\omega)$ is positive, and so $\dfrac{\epsilon(\omega)}{\epsilon_0} > 1$.
- For $\omega > \omega_1$, the sign of Re $\chi(\omega)$ is negative, and so $\dfrac{\epsilon(\omega)}{\epsilon_0} < 1$.
- For $\omega = \omega_1$, Re $\chi(\omega)$ is zero, and so $\dfrac{\epsilon(\omega)}{\epsilon_0} = 1$.

Qualitatively, we expect the real part of the dielectric constant to be positive for values of ω less than the oscillator frequency ω_1, and negative for ω greater than ω_1. At $\omega = \omega_1$, the real part of the dielectric constant is zero, and we can see from equation (7.51.b) that the imaginary part is non-zero; in fact, if you remember this equation from your study of damped harmonic oscillators, you'll remember that we have a resonance for $\omega = \omega_1$.

In order to see the behavior described above, I plotted the real and imaginary parts of the dielectric constant *vs.* the frequency ω. I used $\omega_j = 20$, $\gamma_j = 1$, and set all other coefficients equal to 1. The factor of 20 is actually a gross undercount; typically $\gamma_j \sim 10^9$ s^{-1}, whereas $\omega_j \sim 10^{15}$ s^{-1} at optical frequencies. Still, the graph is reasonably representative of what we might expect, and is shown on the next page.

The graph below shows the real and imaginary parts of the dielectric constant *vs.* the frequency ω, plotted using the parameters described on the previous page.

The behavior we predicted on the previous page is seen clearly in the graph: the real part of the dielectric constant is positive for $\omega < \omega_1$, zero at $\omega = \omega_1$, and negative for $\omega > \omega_1$. Moreover, for $\omega < \omega_1$, the real part of the dielectric constant increases gradually with ω, and then sharply increases as ω approaches ω_1 from below. The graph then dips, going to zero and then to a negative maximal value, until finally beginning to increase sharply again, followed by a leveling-off but the values of the real part of ϵ/ϵ_0 remain negative as ω continues to becomes larger and larger compared to ω_1. Meanwhile the imaginary part is always small, so that $\epsilon(\omega)$ is real for the most part, except when ω is near ω_1, at which point the value of the imaginary part of ϵ/ϵ_0 shoots up (i.e., we get a resonance).



In the general case with different values of oscillator frequencies ω_j, the behavior displayed in our restricted example above is repeated as we approach each ω_j. Keep in mind that the factor $(\omega_j^2 - \omega^2)$ is positive for $\omega < \omega_j$ and negative for $\omega > \omega_j$.

- Below the lowest ω_j, all the real terms in the sum in equation (7.51.a) are positive and the real part of $\epsilon(\omega)/\epsilon_0 > 1$.
- As we increase ω and successive ω_j values are passed, more and more terms in the sum become negative (in their real part), until finally, beyond the highest ω_j, the whole sum is negative and the real part of $\epsilon(\omega)/\epsilon_0 < 1$.

Let us look again at what happens in the vicinity of a resonance ω_i.

- At any $\omega = \omega_i$, the real part of $\epsilon(\omega)$ is zero, and $\epsilon(\omega)$ is purely imaginary at that point.
- As ω approaches ω_i from below, the real part increases rapidly, then becomes zero at $\omega = \omega_i$, then becomes large and negative, then increases back again.
- Meanwhile, the imaginary part, which is usually small, grows, reaching a peak at $\omega = \omega_i$, when the real part is zero — near $\omega = \omega_i$, the contribution of this term can dominate the rest of the sum.

Based on the discussion above, we define the following terms:

- **Normal dispersion** is seen to occur everywhere except in the neighborhood of a resonant frequency ω_i, and is associated with an increase in the real part of $\epsilon(\omega)$ with increasing ω.
- **Anomalous dispersion** is associated with a decrease in the real part of $\epsilon(\omega)$ with increasing ω. This is true only near a sufficiently strong resonance that dominates the sum.
- **Resonant absorption** — only where there is anomalous dispersion is the imaginary part of $\epsilon(\omega)$ appreciable. Such a positive imaginary part to $\epsilon(\omega)$ represents dissipation of energy from the electromagnetic wave into the medium, and so the regions where the imaginary part of $\epsilon(\omega)$ is large are called regions of resonant absorption.

Attenuation of a plane wave

One of the consequences of having a complex dielectric constant is that a traveling wave gets attenuated. We will now study this in greater detail. First, let us ask: what is the most appropriate quantity to pick in order to describe the attenuation of a plane wave?

In equation (7.51.a), we expressed the dielectric constant $\epsilon(\omega)/\epsilon_0$ as a complex number with real and imaginary parts.

But consider equation (7.5) that we wrote in a previous class: $v = \dfrac{\omega}{k} = \dfrac{c}{n}, \qquad n = \sqrt{\dfrac{\mu\epsilon}{\mu_0\epsilon_0}}$

Clearly, a complex $\epsilon(\omega)$ implies a complex index of refraction $n(\omega)$, which in turn implies a complex wave number k. Now, the wave number k features in the phase factor of a plane wave (as the magnitude of the wave vector $\vec{k}$), so it is definitely useful in describing the attenuation of a plane wave, as we shall see below.

Let us, therefore, express the attenuation of a plane wave by writing the wave number k in terms of its real and imaginary parts:

$$k = \beta + i\frac{\alpha}{2} \tag{7.53}$$

Then, assuming a one-dimensional wave propagating along the z-direction, the spatial part of the phase factor can be written as

$$\vec{E}(\vec{x}) \sim e^{ikz} = e^{-\frac{\alpha}{2}\,z}\,e^{i\beta z}$$

which means that the intensity of the wave ($|\vec{E}|^2$) falls off as $e^{-\alpha z}$, so that α is called the *attenuation constant* or *absorption coefficient*. If you were wondering why the imaginary part of the wave number k was defined with a factor of (1/2), now you know why!

It is the imaginary part α of the wave number k that is responsible for exponentially attenuating the wave. As the wave propagates through the material, the resonances of the material absorb energy away from the wave and the fields are attenuated. Of course, there also exists the possibility of a negative α amplifying the wave, but that is the subject of lasers, which lends itself to a quantum mechanical approach.

Starting from $k = \omega\sqrt{\mu\epsilon}$, ***you showed on the worksheet that***

$$\begin{aligned} \beta^2 - \frac{\alpha^2}{4} &= \frac{\omega^2}{c^2}\,\mathrm{Re}\left(\frac{\epsilon}{\epsilon_0}\right) \\ \beta\alpha &= \frac{\omega^2}{c^2}\,\mathrm{Im}\left(\frac{\epsilon}{\epsilon_0}\right) \end{aligned} \tag{7.54}$$

Now, divide the imaginary part by the real part:

$$\frac{\dfrac{\omega^2}{c^2}\,\mathrm{Im}\left(\dfrac{\epsilon}{\epsilon_0}\right)}{\dfrac{\omega^2}{c^2}\,\mathrm{Re}\left(\dfrac{\epsilon}{\epsilon_0}\right)} = \frac{\beta\alpha}{\beta^2 - \dfrac{\alpha^2}{4}}$$

If $\beta \gg \alpha$, then $\beta^2 - \alpha^2/4 \approx \beta^2$, and the expression at the bottom of the previous page becomes

$$\frac{\mathrm{Im}\,\epsilon(\omega)}{\mathrm{Re}\,\epsilon(\omega)} \approx \frac{\beta\alpha}{\beta^2}$$

Therefore, as long as $\beta \gg \alpha$, which is true unless the absorption is very strong (or Re ϵ is negative), we can write

$$\frac{\alpha}{\beta} \simeq \frac{\mathrm{Im}\ \epsilon(\omega)}{\mathrm{Re}\ \epsilon(\omega)} \tag{7.55}$$

where

$$\beta \simeq \frac{\omega}{c}\sqrt{\mathrm{Re}\ \left(\frac{\epsilon}{\epsilon_0}\right)}$$

So what does the ratio $\dfrac{\mathrm{Im}\ \epsilon(\omega)}{\mathrm{Re}\ \epsilon(\omega)}$ tell us?

Equation (7.55) tells us that this ratio is α/β, as long as $\beta \gg \alpha$.

But if $\beta \gg \alpha$, then equation (7.53) leads to $k \approx \beta$.

But k is the wave number, so $2\pi/k$ is the wavelength.

Jackson interprets the ratio Im $\epsilon(\omega)$/Re $\epsilon(\omega)$, or α/β as the *fractional decrease in intensity per wavelength* (divided by 2π).

We will now study the behavior in the limit of low frequencies which determines the conduction properties of the material, and in the limit of high frequencies.

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Low Frequency Behavior

In the limit $\omega \to 0$, there is a qualitative difference in the response of the medium depending on whether or not there is a resonance at zero.

- If a resonance does not exist at $\omega_i = 0$ (i.e., the lowest resonant frequency is different from zero), then ϵ is nearly all real and the material is a dielectric insulator whose molecular polarizability is given by equation (4.73) written in the previous class. We covered this aspect in the previous class when we discussed the static case in § 4.6 to motivate the discussion for time-varying fields.
- If there is a resonance $\omega_0 = 0$, then $\epsilon(\omega)$ has a complex component that attenuates the propagation of electromagnetic energy in a nearly static electric field. As we will learn below, this describes conductors.

Suppose some fraction f_0 of the electrons per molecule have their lowest resonance frequency at $\omega_0 = 0$. Let's call these the "free" electrons (in the sense that, since ω_i are the binding frequencies, if they have $\omega_0 = 0$, they may be considered to be "free").

Let's write the contribution of the free electrons separately, ***as you did on the worksheet***; start with equation (7.51):

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m}\sum_j \left[\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}\right]$$

and split out the contribution of the free electrons

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m}\sum_b \left[\frac{f_b}{\omega_b^2 - \omega^2 - i\omega\gamma_b}\right] + \frac{Ne^2}{\epsilon_0 m}\left[\frac{f_0}{0 - \omega^2 - i\omega\gamma_0}\right]$$

Why does the summation go away in the last term for the free dipoles? Don't let the fact that j also subscripts f_j confuse you. Recall that the sum is over frequencies ω_j (and accompanying γ_j) and that f_j is just a number corresponding to each ω_j. So, since we are considering $\omega_0 = 0$, i.e., only one frequency (and its accompanying γ_0), there is no summation here.

Moving on, we get

$$\epsilon(\omega) = \epsilon_0\left(1 + \frac{Ne^2}{\epsilon_0 m}\sum_b \left[\frac{f_b}{\omega_b^2 - \omega^2 - i\omega\gamma_b}\right]\right) + \frac{Ne^2}{m\omega}\left[\frac{f_0}{-\omega - i\gamma_0}\right]$$

Jackson has written ϵ_0 times the term in parentheses as $\epsilon_b(\omega)$, the contribution of all the other dipoles. To get the last term on the right hand side into the form of equation (7.56) in Jackson, multiply and divide it by the imaginary number i:

$$\epsilon(\omega) = \epsilon_b(\omega) + \frac{Ne^2}{m\omega}\left[\frac{if_0}{-i\omega - i^2\gamma_0}\right]$$

from which we obtain

$$\epsilon(\omega) = \epsilon_b(\omega) + i\left[\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}\right] \tag{7.56}$$

Recall that $\epsilon_b(\omega)$ is now the contribution from all the "bound" dipoles.

To understand equation (7.56), let us consider the Maxwell-Ampere equation

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

where, as you may recall, $\vec{J}$ is the current density.

Let's first assume that the medium obeys Ohm's law $\vec{J} = \sigma \vec{E}$, where σ is the electrical conductivity. Let's also assume that the medium has a "normal" dielectric constant ϵ_b.

Note: You might wonder why we're applying Ohm's law here, a law we derived by assuming the electric field was the same everywhere in space, when we are now dealing with traveling electromagnetic fields. For our treatment to be valid, we need to assume that the electric field is roughly constant over some scale relevant to the derivation of Ohm's law. This will be the case if the wavelength of the electric field is greater than the average distance l traveled by the electrons between collisions (the so-called mean free path). In most metals, $l \sim 10^{-7}$ m (see, e.g., Gall, D. 2016, J. App. Phys. 119, 085101), so that Ohm's law should be applicable in cases where $\lambda \geq 100$ nm, and so definitely for the visible spectrum (λ of blue light is $\sim$450 nm).

Then, assuming a harmonic time dependence $\sim e^{-i\omega t}$, Maxwell-Ampere's law written above takes the form

$$\vec{\nabla} \times \vec{H} = -i\omega \left(\epsilon_b + i\,\frac{\sigma}{\omega} \right) \vec{E} \tag{7.57}$$

as you showed on the worksheet for today.

On the other hand, we can set the current contributing to $\vec{J}$ to zero and attribute instead all the properties of the medium to the dielectric constant, that is, we consider everything, including the "currents" present to be the result of the polarization response of the medium, so that by putting $\vec{D} = \epsilon \vec{E}$, we would write

$$\vec{\nabla} \times \vec{H} = \frac{\partial (\epsilon \vec{E})}{\partial t}$$

and with a harmonic time dependence $\sim e^{-i\omega t}$ for $\vec{E}$, we would get

$$\vec{\nabla} \times \vec{H} = -i\omega\epsilon\vec{E}$$

Substituting equation (7.56), this becomes

$$\vec{\nabla} \times \vec{H} = -i\omega \left[\epsilon_b + i \left\{ \frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \right\} \right] \vec{E}$$

We see that the term in parentheses in equation (7.57) can be equated to the quantity in square brackets in the equation above; upon doing this, we get

$$\epsilon_b + i\,\frac{\sigma}{\omega} = \epsilon_b + i \left\{ \frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \right\}$$

Canceling ϵ_b and i from both sides, we obtain

$$\frac{\sigma}{\omega} = \left\{ \frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \right\}$$

from which we get that the conductivity is

$$\sigma = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)} \tag{7.58}$$

This is essentially the Drude model for the electrical conductivity, with $f_0 N$ being the number of free electrons per unit volume in the medium. It establishes that the "conductivity" is closely related to the complex dielectric constant when the lowest resonant frequency is zero. Note that at $\omega = 0$, it is purely real.

So, what does all of this tell us?

In a sense, it shows us that the distinction between dielectrics and conductors is somewhat artificial and a matter of perspective, at least away from the purely static case ($\omega = 0$).

- If the medium possesses free electrons, it is a conductor at low frequencies, otherwise it is an insulator.
- At non zero frequencies, the conductivity contribution to $\epsilon(\omega)$ merely appears as a resonant amplitude.

Therefore, it is purely a matter of choice to describe the dispersive properties of a medium in terms of a complex dielectric constant, or in terms of a frequency-dependent conductivity and a dielectric constant.