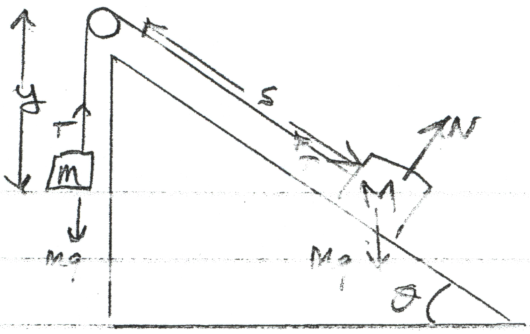
Exercise 1.12



assume equilibrium

$$\text{Virtual work} = \delta W = \sum_{\alpha=1}^{n} Q_\alpha \delta q_\alpha = 0 \text{ for equilibrium}$$

Place origin at pulley. Positions of masses are y and s.

External forces are mg and T acting on m. T = tension in string and Mg, N = Normal force and T acting on M

Generalized coordinates are y and s
The constraint that length of string is constant ($=l$) can be expressed as

$$y + s - l = 0$$

The generalized forces are Q_y and Q_s

The cartesian coordinates can be written as (x_1, y_1) for m and (x_2, y_2) for M, where

$$x_1 = 0 \quad \text{and} \quad y_1 = -y$$

and $\quad x_2 = s\cos\theta \qquad y_2 = -s\sin\theta$

The Forces in Cartesian coordinates are

$$F_{x1} = 0 \qquad F_{y1} = T - mg$$

$$F_{x2} = N\sin\theta - T\cos\theta$$

$$F_{y2} = N\cos\theta + T\sin\theta - Mg$$

So

$$Q_y = \sum F_i \frac{\partial x_i}{\partial q_y}$$

Note $F_i = F_{x1}, F_{y1}, F_{x2}, F_{y2}$

So

$$Q_y = F_{x1}\frac{\partial x_1}{\partial y} + F_{y1}\frac{\partial y_1}{\partial y} + F_{x2}\frac{\partial x_2}{\partial y} + F_{y2}\frac{\partial y_2}{\partial y}$$
$$= 0 \quad + (T - mg)\left(\frac{\partial}{\partial y}(-y)\right) + F_{x2}\underbrace{\frac{\partial}{\partial y}(s\cos\theta)}_{=0} + F_{y2}\underbrace{\frac{\partial}{\partial y}(-s\sin\theta)}_{=0}$$
$$= (T-mg)(-1) = \underline{\underline{mg - T}}$$

Similarly

$$Q_s = F_{x1}\frac{\partial x_1}{\partial s} + F_{y1}\frac{\partial y_1}{\partial s} + F_{x2}\frac{\partial x_2}{\partial s} + F_{y2}\frac{\partial y_2}{\partial s}$$
$$= 0 \quad + (T-mg)\underbrace{\left(-\frac{\partial y}{\partial s}\right)}_{=0} + F_{x2}\frac{\partial}{\partial s}(s\cos\theta) + F_{y2}\frac{\partial}{\partial s}(-s\sin\theta)$$
$$= 0 + 0 + (N\sin\theta - T\cos\theta)\cos\theta + (N\cos\theta + T\sin\theta - Mg)(-\sin\theta)$$
$$= N\sin\theta\cos\theta - T\cos^2\theta - N\sin\theta\cos\theta - T\sin^2\theta + Mg\sin\theta$$
$$= -T(\sin^2\theta + \cos^2\theta) + Mg\sin\theta$$
$$= -T + Mg\sin\theta$$

Therefore

$$\delta W = \sum Q_\alpha \delta q_\alpha = Q_s\delta y + Q_s\delta s$$

Note $\delta y = -\delta s$ from constraint

$\therefore$

$$(mg - T)\delta y + (-T + Mg\sin\theta)\delta s = 0$$
$$(T - mg - T + Mg\sin\theta)\delta s = 0$$
$$-mg + Mg\sin\theta = 0$$
$$\therefore M\sin\theta = m$$

or $$M = \frac{m}{\sin\theta}$$

Q.E.D.