Cosmic Dynamics–II

(1) In the lecture we have seen that

$$c_s^2 = c^2 w$$

where c_s is the speed of sound, c the speed of light, and w the equation of state constant. Using this equation, determine what restrictions exist on w.

(2) In the lecture, you've been reminded the Poisson equation applied to gravity is

$$\nabla^2 \Phi = 4\pi G \rho; \quad a = -\nabla \Phi \tag{1}$$

where Φ is the gravitational potential, ρ the mass density, and a the acceleration.

(a) The evidence that Einstein had when he developed GR was that the universe was static. What must be true of the acceleration if the universe is to be static?

(b) What does your answer for (a) imply about the gravitational potenial, Φ?

(c) What does your answer for (b) imply about the mass density of the universe?

(d) What does your answer for (c) tell you must be the condition to have a static universe?

(3) In the lecture we showed that replacing $\vec{A}$ by $\vec{A} + \nabla\chi$ in the expression $\vec{B} = \nabla \times \vec{A}$ does not change the equation for $\vec{B}$.

(a) Since $\vec{B}$ is unchanged, is it true that $\vec{A} = \vec{A} + \nabla\chi$?

(b) How might you interpret the fact that the vector potential can be changed as shown. As an aside, the change in the vector potential by the addition of a scalar function that does not affect the magnetic field is called a *gauge transformation*. There are many such transformations in physics that play important roles.

(4) Recall that for a static universe we must have $\dot{a}$ and $\ddot{a} = 0$.

(a) Setting $\ddot{a} = 0$ in the fluid equation, find the constant, Λ. (Recall that $P = 0$ for matter)

(b) Setting $\dot{a} = 0$ in the Friedmann equation, determine the curvature of the universe in Einstein's static universe.

(c) Find the radius of curvature.

(5) We assume that the different energy densities are separable so that the fluid equation is

$$\dot{\epsilon}_w = -3\frac{\dot{a}}{a}\left(\epsilon_w + P_w\right)$$

where w is the different components.

(a) Using that

$$P_w = w\epsilon_w$$

rewrite the fluid equation as a first order, separable differential equation.

(b) Integrate (a) to find the dependence of ϵ_w on the scale factor a and the parameter w.

(c) Use your answer in (b) to find how the following energy densities evolve with scale factor a.

Energy Density	w
Matter	0
Radiation	$\frac{1}{3}$
Cosmological Constnt	-1

(6) Recap the important topics covered today. Compare and contrast your results with others at your table.

Homework 02–Due Friday Jan 31

1. Problem 4.4
2. Problem 4.5
3. Problem 5.1
4. Problem 5.3

Additional Grad Student Problem(s)

5. Consider a flat universe universe with a single component characterized by the equation of state parameter, $w = -1$.

   (a) Show that in such a universe the Friedmann equation takes the form
   $$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon_\Lambda a^2 \tag{2}$$

   (b) Show that the Hubble constant in a Λ-dominated universe is given by
   $$H_o \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_o} = \left(\frac{8\pi G\epsilon_\Lambda}{3c^2}\right)^{1/2} \tag{3}$$

   (c) Solve Eq. (1) and show that the dependence of the scale factor on time is given by
   $$a(t) = e^{H_o(t-t_o)}$$