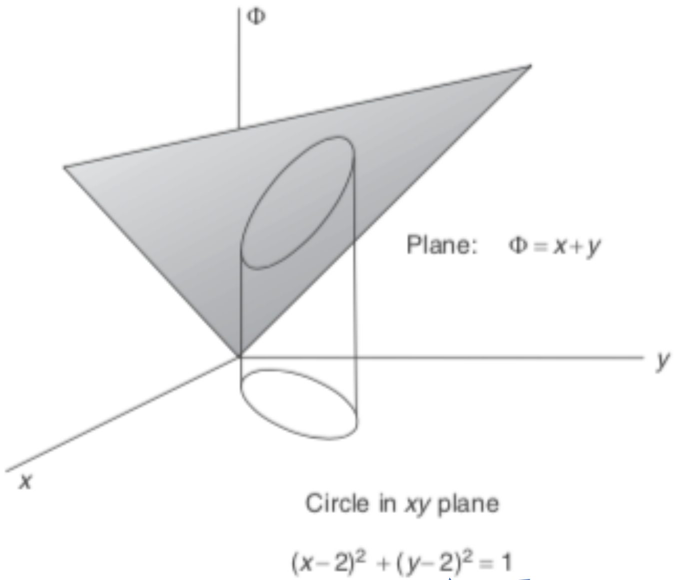
Lagrange Multipliers

Example 2.3



Center at x=2, y=2   radius=1

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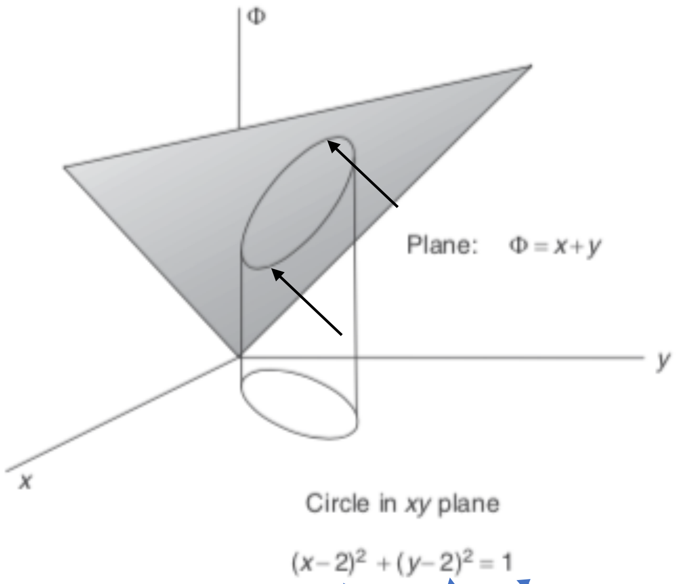

Center at x=2, y=2 radius=1

Our task: Find the x and y for which the Φ has its maximum and its minimum (see black arrows)

Two equations for the two unknowns x and y:

$$\Phi = x + y \quad f = (x-2)^2 + (y-2)^2 - 1 = 0$$

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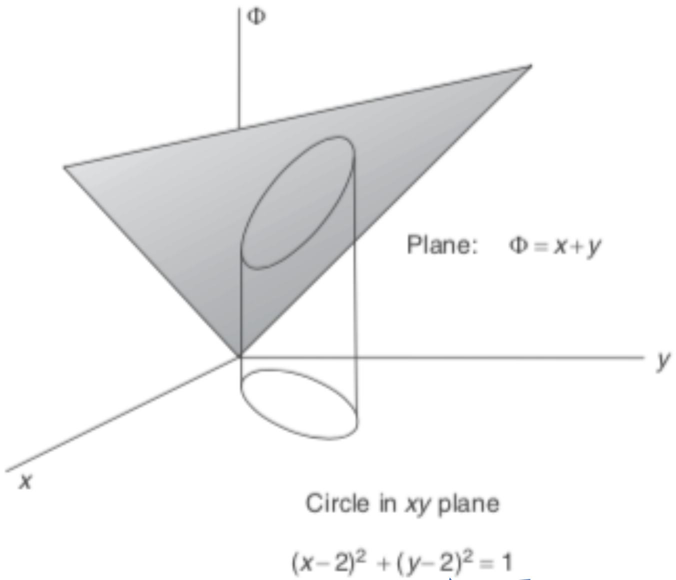

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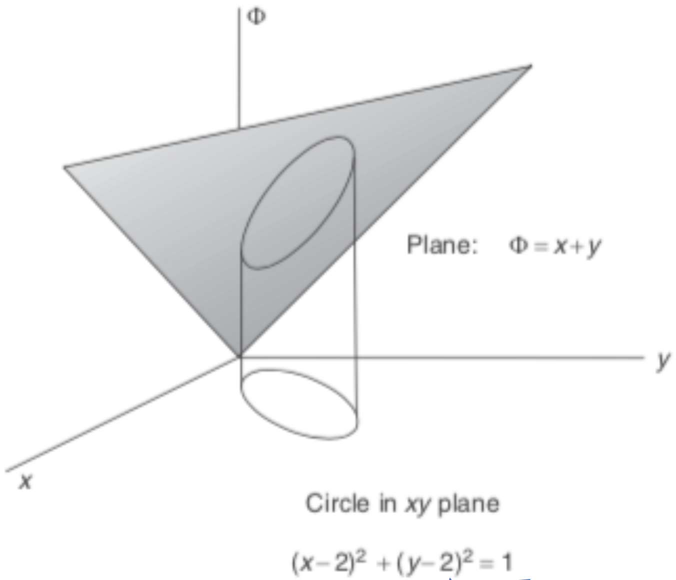

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Using Lagrange multipliers we actually ADD another unknown λ and then determine x, y, λ from the three equations

$$\frac{\partial}{\partial x}(\Phi + \lambda f) = 0 \quad \frac{\partial}{\partial y}(\Phi + \lambda f) = 0 \quad \frac{\partial}{\partial \lambda}(\Phi + \lambda f) = 0$$

Where $\Phi + \lambda f = x + y + (x-2)^2 + (y-2)^2 - 1$

Φ is the function we want to minimize or maximize,
$f = 0$ is the constraint equation

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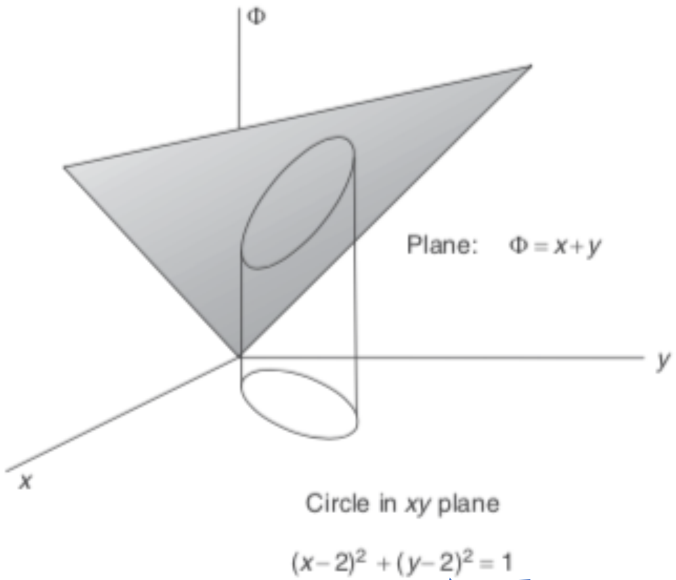

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$$\frac{\partial}{\partial x}(\Phi + \lambda f) = 0 \quad \frac{\partial}{\partial y}(\Phi + \lambda f) = 0 \quad \frac{\partial}{\partial \lambda}(\Phi + \lambda f) = 0$$

Where $\Phi + \lambda f = x + y + \lambda[(x-2)^2 + (y-2)^2 - 1]$

Φ is the function we want to minimize or maximize,
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So:

$1 + \lambda\big(2(x-2)\big) = 0 \quad 1 + \lambda\big(2(y-2)\big) = 0$ and

$(x-2)^2 + (y-2)^2 - 1 = 0$

$(x, y)_{min} = (1.29, 1.29) \; (x, y)_{max} = (2.71, 2.71)$

Example: Find the dimensions of a rectangular box with the largest volume if the total surface area is $64\ cm^2$.

$\Phi = xyz$ Is the function we are trying to maximize (the volume)

$f = 2xy + 2xz + 2yz - 64 = 0$ is the constraint (the given surface area of 64)

$\Phi = xyz \qquad f = xy + xz + yz - 32 = 0$

$$\frac{\partial}{\partial x}(xyz + \lambda[xy + xz + yz - 32] = yz + \lambda y + \lambda z = 0$$

$$\frac{\partial}{\partial y}(xyz + \lambda[xy + xz + yz - 32] = xz + \lambda x + \lambda z = 0$$

$$\frac{\partial}{\partial z}(xyz + \lambda[xy + xz + yz - 32] = xy + \lambda x + \lambda y = 0$$

$$\frac{\partial}{\partial \lambda}(xyz + \lambda[xy + xz + yz - 32] = xy + xz + yz - 32 = 0$$

Four equations to solve for four unknows x, y, z, λ

Example: Minimize the amount of cardboard used to build a rectangular box without lid with volume of $32{,}000\ \mathrm{cm}^2$.

$\Phi = xy + 2xz + 2yz$ Is the function we are trying to minimize (the surface area)

$f = xyz - 32{,}000 = 0$ is the constraint (the given volume)

$$\Phi = xy + 2xz + 2yz \qquad f = xyz - 32{,}000 = 0$$

$$\frac{\partial}{\partial x}(xy + 2xz + 2yz + \lambda[xyz - 32{,}000]) = y + 2z + yz\lambda = 0$$

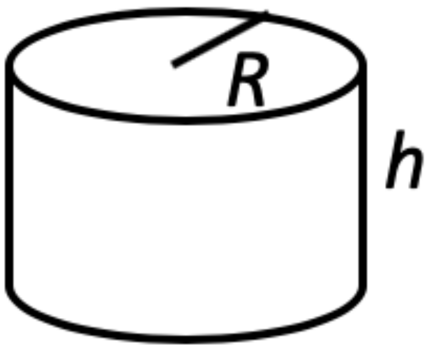
$$\frac{\partial}{\partial y}(xy + 2xz + 2yz + \lambda[xyz - 32{,}000]) = \mathrm{x} + 2z + xz\lambda = 0$$

$$\frac{\partial}{\partial z}(xy + 2xz + 2yz + \lambda[xyz - 32{,}000]) = 2x + 2y + xy\lambda = 0$$

$$\frac{\partial}{\partial \lambda}(xy + 2xz + 2yz + \lambda[xyz - 32{,}000]) = xyz - 32{,}000 = 0$$

Four equations to solve for four unknows x, y, z, λ

Activity 9: Find the relationship between radius and height of a cylinder that minimizes the surface area for a given (fixed) volume. Do this with and without Lagrangian multipliers.



$A = 2\pi Rh + 2\pi R^2$ (the function we want to minimize)

$V_0 = \pi R^2 h$ is fixed (the constraint)

Without Lagrange multipliers

With Lagrange multipliers