PHY 474/374 **Spring 2021**

Class Summary—Week 3, Day 1—Monday, April 12

Review of the Lane-Emden Equation

In the previous class, we discussed the **Lane-Emden equation**:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

On Question 1 of today's worksheet, we went over the main points in this equation.

- The quantity ξ is a dimensionless measure of the distance to the center of the star, and is obtained from r by writing
$$r = \alpha\xi$$
where
$$\alpha^2 = \frac{(n+1)\,K\,\rho_c^{1/n}}{4\pi G\rho_c}$$

- The quantity θ is a dimensionless measure of the density, ρ, obtained by writing
$$\rho = \rho_c\,\theta^n$$
where ρ_c is the central density, the density at the center of the star.

- The quantity n is the polytropic index. If the polytrope is defined via $P = K\rho^\gamma$ where P is the pressure and K and γ are constants, then we have that
$$n = \frac{1}{\gamma - 1}$$
so that $\gamma = 1 + \dfrac{1}{n}$.

- For all n, we will have
$$\theta = 1 \qquad \text{for} \qquad \xi = 0$$
This is because the density at the center of the star, where $\xi = 0$, is ρ_c. Thus, in the expression
$$\rho = \rho_c\,\theta^n$$
putting $\rho = \rho_c$ gives $\theta = 1$ for all n.

- The surface of the star is defined by $\xi = \xi_1$. At the surface, we must have zero pressure. Thus, the requirement to have $P = 0$ at the star's surface means that $\rho = 0$ there also, and from the expression $\rho = \rho_c\,\theta^n$, we conclude that if $\rho = 0$, we must have $\theta = 0$ also.

Energy Transport in Stellar Interiors

We will now begin learning about the transport of energy in stellar interiors. All energy is generated in the core of the star by nuclear fusion; we will learn more details about energy generation by fusion later in the quarter. This energy is transported to the surface of the star by radiation (energy carried by photons) and convection (energy carried by bulk motions of gas), with the type of transport (radiative, convective, or both) depending on the characteristics of the stellar material, and hence ultimately depending on the mass of the star. For example, Sun-like stars have a convection zone outside of their radiation zone; see the posted video for details. Meanwhile, low-mass stars transport almost all their energy by convection, whereas high mass stars have a small inner convective layer and a large outer radiation zone. The net transport of energy in a star is outward, from the center to the surface. Photons go in a random walk and so it can take energy over a million years to get from the center to the surface.

Today, we will learn about **energy transport by radiation**. Transport of energy by radiation, including interaction of radiation with matter, is a complex topic (e.g., Rybicki & Lightman). In stellar interiors where the mean free path of photons is very short, however, we can get by with a simplified description.

Consider the energy transport in the time interval dt through an area dA orthogonal to the direction to the center of the star, at $r = r_0$, as shown in the figure below. The direction of motion of the photons is specified by the angle θ between the outward directed normal to dA (the black dashed line) and the direction of motion (along the solid blue and then dashed blue line).

If the photons are isotropically distributed, then the fraction of photons with directions between θ and $(\theta + d\theta)$ is given by

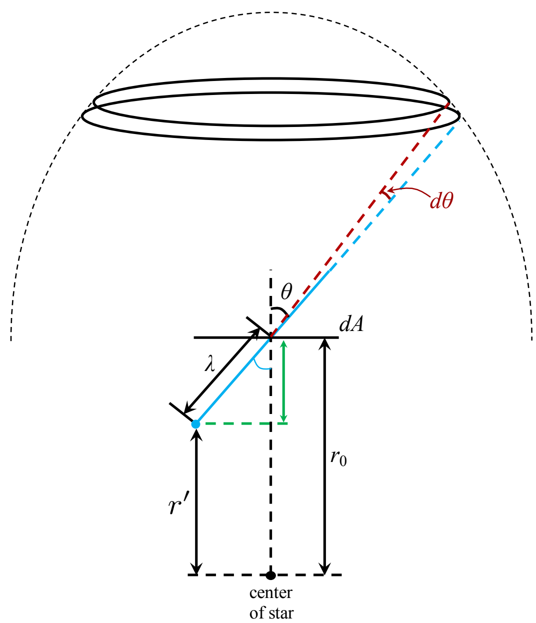
$$\frac{\sin\theta\, d\theta}{2}$$

as you showed in Question 2(a) of today's worksheet.

If λ is the mean free path between interactions (like scattering or absorption) of a photon and gas particle, then photons that go through dA with directions between θ and $(\theta + d\theta)$ on average come from a distance r' from the center. From the figure, we see that

$$r' = r_0 - \lambda \cos\theta$$

as you showed in Question 2(b) of today's worksheet.



Since the photons going through dA with directions between θ and $(\theta + d\theta)$ come from distance r', they correspond to the energy density $u_R(r')$, and their contribution to the energy transport through dA is the product of the quantities in the three square brackets below, which correspond respectively to the contribution to the energy density, the projected area, and the path length.

$$\left[\frac{\sin\theta\, d\theta}{2}\, u_R(r')\right] \left[dA\, \cos\theta\right] \left[c\, dt\right] \tag{5.1}$$

The total energy transport through dA in time dt is obtained by integrating equation (5.1) on the previous page over all directions θ, from 0 to π. Note that the integration from 0 to π in the interior of the star is because of the random walk of photons; thus, because photons at distance r' from dA are passing both upward and downward through dA. Of course, there should in the net be more photons going upward (i.e., outward) than downward (i.e., inward), and we know this because $u = aT^4$, so higher T regions on the inside establish a temperature gradient and hence a flow outward.

To find the energy transported through dA, first do a Taylor expansion of $u_R(r')$ at r_0, retaining terms up to first order. You should get

$$u_R(r') = u_R(r_0) - \Big(\lambda\, \cos\theta\Big)\, \frac{du_R}{dr}$$

as you demonstrated on Question 3(a) of today's worksheet.

Then, integrate equation (5.1) on the previous page from $\theta = 0$ to π, *as you did on Question 3(b) of today's worksheet*:

$$dE = \int_0^\pi \left[\frac{\sin\theta\, d\theta}{2}\, u_R(r')\right] \Big[dA\, \cos\theta\Big] \Big[c\, dt\Big]$$

Putting $u_R(r')$ from above, this becomes

$$dE = \frac{c}{2} \int_0^\pi \left[u_R(r_0) - \lambda\, \cos\theta\, \frac{du_R}{dr}\right] \cos\theta\, \sin\theta\, d\theta\, dA\, dt$$

so that

$$dE = \left[\frac{c}{2}\, u_R(r_0) \int_0^\pi \cos\theta\, \sin\theta\, d\theta - \frac{\lambda c}{2}\, \frac{du_R}{dr} \int_0^\pi \cos^2\theta\, \sin\theta\, d\theta\right] dA\, dt$$

Putting

$$\cos\theta = x \qquad \text{so that} \qquad -\sin\theta d\theta = dx$$

we get that the first integral above integrates to $-x^2/2$ and the second integral to $-x^3/3$. Thus

$$dE = \left\{\frac{c}{2}\, u_R(r_0) \left[-\frac{\cos^2\theta}{2}\right]_0^\pi - \frac{\lambda c}{2}\, \frac{du_R}{dr} \left[-\frac{\cos^3\theta}{3}\right]_0^\pi\right\} dA\, dt$$

so that

$$dE = \left\{-\frac{c}{2}\, u_R(r_0) \left(\frac{\cos^2\pi - \cos^2 0}{2}\right) + \frac{\lambda c}{2}\, \frac{du_R}{dr} \left(\frac{\cos^3\pi - \cos^3 0}{3}\right)\right\} dA\, dt$$

and thus

$$dE = \left\{-\frac{c}{2}\, u_R(r_0) \left(\frac{0 - 0}{2}\right) + \frac{\lambda c}{2}\, \frac{du_R}{dr} \left(\frac{\{-1\}^3 - \{1\}^3}{3}\right)\right\} dA\, dt$$

or

$$dE = \left\{0 + \frac{\lambda c}{2}\, \frac{du_R}{dr} \left(-\frac{2}{3}\right)\right\} dA\, dt$$

Therefore, the total energy transport through dA in time dt is

$$dE = -\frac{\lambda c}{3}\, \frac{du_R}{dr}\, dA\, dt \tag{5.2}$$

Our result on the previous page is consistent with the direction of energy flow from the interior to the surface of the sphere, because equation (5.2) tells us that the uniform part of the energy density, $u_R(r_0)$, makes no contribution to the energy transport. Instead, the energy transport is determined by the change in the energy density with position, du_R/dr.

We will now derive an equation for the **temperature gradient**, dT/dr, in a star.

By convention, one uses the **opacity** κ to describe the interaction between radiation and matter instead of using λ. It is defined such that the mean free path λ is related to the opacity κ by

$$\lambda = (\kappa\rho)^{-1} \tag{5.5}$$

We will also define the **radiative flux**, F_R, as the energy transported through dA per unit area per unit time, so that

$$dE = F_R \, dA \, dt \tag{5.3}$$

From equation (5.2), we can then write the radiative flux as

$$F_R = \frac{dE}{dA\,dt} = -\frac{\lambda c}{3}\frac{du_R}{dr}$$

Putting $\lambda = (\kappa\rho)^{-1}$ from equation (5.5), we get that the radiative flux is then given by

$$F_R = -\frac{c}{3\kappa\rho}\frac{du_R}{dr}$$

as you showed in Question 4(a) on today's worksheet

Next, the energy density of radiation is

$$u_R = aT^4$$

where a is called the radiation constant. Putting this into the equation for F_R, we get

$$F_R = -\frac{c}{3\kappa\rho}\frac{d}{dr}\left[aT^4\right] = -\frac{ca}{3\kappa\rho}\left(4T^3\frac{dT}{dr}\right)$$

so that

$$F_R = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr} \tag{5.6}$$

Now, the total amount of energy transported by radiation through a sphere of radius r is given by

$$L(r) = 4\pi r^2 F_R \tag{5.7}$$

Putting equation (5.6) in equation (5.7), we get

$$L(r) = 4\pi r^2\left[-\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}\right]$$

so that, *as you showed in Question 4(b) on today's worksheet*

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi ac\,r^2T^3} \tag{5.8}$$

which is one of the fundamental equations of stellar structure.

Radiation from the Stellar Surface

Near the surface of the star where the density is lower compared to the interior, the mean free path becomes very large, so the analysis carried out so far does not apply. To estimate the energy radiated from the stellar surface, we can still use equation (5.1) with the following modifications.

In the interior of the star, the number of photons directed outward and inward at any given location is very near balance as we saw in the analysis on the preceding pages, even though the net flow of energy is outward. This is not the case on the stellar surface, and the number of photons directed outward and inward does not need to be near balance at the stellar surface. This has two consequences:

- We don't need to take the r-dependence of u_R into account at the surface.
- Only photons directed outward with $\theta \leq \pi/2$ will contribute to the energy radiated from the stellar surface.

Modifying equation (5.1) by writing u_R as a constant instead of being r-dependent, and integrating from 0 to $\pi/2$, we get

$$dE = \int_0^{\pi/2} \left[\frac{\sin\theta\, d\theta}{2}\, u_R\right] \left[dA\, \cos\theta\right] \left[c\, dt\right]$$

so that

$$dE = \frac{c}{2}\, u_R\, dA\, dt \int_0^{\pi/2} \cos\theta\, \sin\theta\, d\theta$$

Again, putting $\cos\theta = x$ so that $-\sin\theta d\theta = dx$ in the integral above, we get that it integrates to $-x^2/2$, and so

$$dE = \frac{c}{2}\, u_R\, dA\, dt \left[-\frac{\cos^2\theta}{2}\right]_0^{\pi/2}$$

Thus

$$dE = -\frac{c}{2}\, u_R\, dA\, dt \left[\frac{\cos^2(\pi/2) - \cos^2 0}{2}\right] = -\frac{c}{2}\, u_R\, dA\, dt \left[\frac{0-1}{2}\right]$$

Therefore

$$dE = \frac{c}{4}\, u_R\, dA\, dt$$

Using equation (5.3), the radiative flux is then

$$F_R = \frac{dE}{dA\, dt} = \frac{c}{4}\, u_R$$

and if we put $u_R = aT^4$, we obtain

$$F_R = \left(\frac{ac}{4}\right) T^4 = \sigma\, T^4$$

where σ is the Stefan-Boltzmann constant.

If T_{eff} is the effective temperature of the surface of the star, then the surface luminosity, L_s, of the star is given by

$$L_s = 4\pi R^2 \sigma T_{\text{eff}}^4 \tag{5.12}$$

where R is the radius of the star.