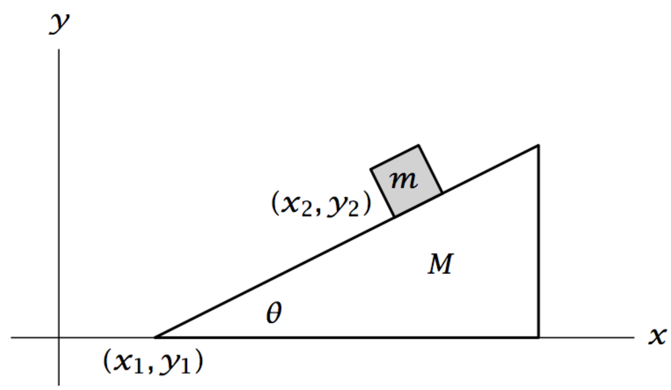
A wedge of angle θ and mass M is free to slide horizontally on a frictionless table. A small block of mass m is placed on the wedge, and can also slide without friction. Using Lagrange multipliers, find the equations of motion and the forces of constraint.



You'll notice that there are now four differential equations, one each for x_1, x_2, y_1 and y_2 (see Hamill equation 3.16) plus two equations of constraint, one constraining the wedge to the horizontal surface, and one constraining the block to the wedge. After deriving these six equations you will be able to solve them for the two Lagrange multipliers $\lambda_1$and λ_2 and then derive the forces of constraint.

$$L = \frac{1}{2}M(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2) - Mgy_1 - mgy_2$$

The two holonomic constraints are that the wedge has to stay on the table:

$$f_1(x_1, y_1, x_2, y_2) = y_1 = 0,$$

and that the block has to stay on the wedge:

$$\frac{y_2 - y_1}{x_2 - x_1} = \tan\theta \Rightarrow f_2(x_1, y_1, x_2, y_2) = y_2 - y_1 - (x_2 - x_1)\tan\theta = 0$$

This gives us the following six equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = \lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1} \quad (i)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = \lambda_1 \frac{\partial f_1}{\partial x_2} + \lambda_2 \frac{\partial f_2}{\partial x_2} \quad (ii)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_1} - \frac{\partial L}{\partial y_1} = \lambda_1 \frac{\partial f_1}{\partial y_1} + \lambda_2 \frac{\partial f_2}{\partial y_1} \quad (iii)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_2} - \frac{\partial L}{\partial y_2} = \lambda_1 \frac{\partial f_1}{\partial y_2} + \lambda_2 \frac{\partial f_2}{\partial y_2} \quad (iv)$$

$$f_1 = y_1 = 0$$

$$f_2 = y_2 - y_1 - (x_2 - x_1)\tan\theta = 0$$

Partial derivatives:

$$\frac{\partial L}{\partial x_1} = 0 \quad \frac{\partial L}{\partial x_2} = 0 \quad \frac{\partial L}{\partial y_1} = -Mg \quad \frac{\partial L}{\partial y_2} = -mg$$

$$\frac{\partial L}{\partial \dot{x}_1} = M\dot{x}_1 \quad \frac{\partial L}{\partial \dot{x}_2} = m\dot{x} \quad \frac{\partial L}{\partial \dot{y}_1} = M\dot{y}_1 \quad \frac{\partial L}{\partial \dot{y}_2} = m\dot{y}_2$$

$$\frac{\partial f_1}{\partial x_1} = 0 \quad \frac{\partial f_1}{\partial x_2} = 0 \quad \frac{\partial f_1}{\partial y_1} = 1 \quad \frac{\partial f_1}{\partial y_2} = 0$$

$$\frac{\partial f_2}{\partial x_1} = \tan\theta \quad \frac{\partial f_2}{\partial x_2} = -\tan\theta \quad \frac{\partial f_2}{\partial y_1} = -1 \quad \frac{\partial f_2}{\partial y_2} = 1$$

Plug these back into the differential equations:

$$(i) \rightarrow \; M\ddot{x}_1 = \lambda_2 \tan\theta$$

$$(ii) \rightarrow \; m\ddot{x}_2 = -\lambda_2 \tan\theta$$

$$(iii) \rightarrow \; M\ddot{y}_1 + Mg = \lambda_1 - \lambda_2 \Rightarrow \lambda_1 - \lambda_2 = Mg \qquad \text{(because } \ddot{f}_1 = \ddot{y}_1 = 0\text{)}$$

$$(iv) \rightarrow m\ddot{y}_2 + mg = \lambda_2 \Rightarrow m(\ddot{x}_2 - \ddot{x}_1)\tan\theta + mg = \lambda_2 \qquad \text{(because } \ddot{y}_2 = (\ddot{x}_2 - \ddot{x}_1)\tan\theta \text{ from } f_2\text{)}$$

$$(ii) - \; (i) \rightarrow \; \ddot{x}_2 - \ddot{x}_1 = -\frac{\lambda_2}{m}\tan\theta - \frac{\lambda_2}{M}\tan\theta = -\lambda_2\tan\theta\left(\frac{1}{m} + \frac{1}{M}\right)$$

$$(iv) \rightarrow \left(-\lambda_2 \tan\theta\left(1 + \frac{m}{M}\right)\right)\tan\theta + mg = \lambda_2 \Leftrightarrow -\lambda_2\left(\tan^2\theta - \frac{m}{M}\tan^2\theta - 1\right) = -mg$$

$$\lambda_2 = \frac{mg}{\left(\tan^2\theta - \frac{m}{M}\tan^2\theta - 1\right)} = \frac{mg}{\left(\left(1 - \frac{m}{M}\right)\tan^2\theta - 1\right)}$$

So the forces of constraint are

$$F_{x_1} = \lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1} = 0 + \lambda_2 \tan\theta = \frac{mg\tan\theta}{\left(\left(1 - \frac{m}{M}\right)\tan^2\theta - 1\right)}$$

$$F_{x_2} = \lambda_1 \frac{\partial f_1}{\partial x_2} + \lambda_2 \frac{\partial f_2}{\partial x_2} = 0 - \lambda_2 \tan\theta = \frac{-mg\tan\theta}{\left(\left(1 - \frac{m}{M}\right)\tan^2\theta - 1\right)}$$

$$F_{y_1} = \lambda_1 \frac{\partial f_1}{\partial y_1} + \lambda_2 \frac{\partial f_2}{\partial y_1} = \lambda_1 - \lambda_2 = Mg$$

$$F_{y_2} = \lambda_1 \frac{\partial f_1}{\partial y_2} + \lambda_2 \frac{\partial f_2}{\partial y_2} = \lambda_2 = \frac{mg}{\left(\left(1 - \frac{m}{M}\right)\tan^2\theta - 1\right)}$$