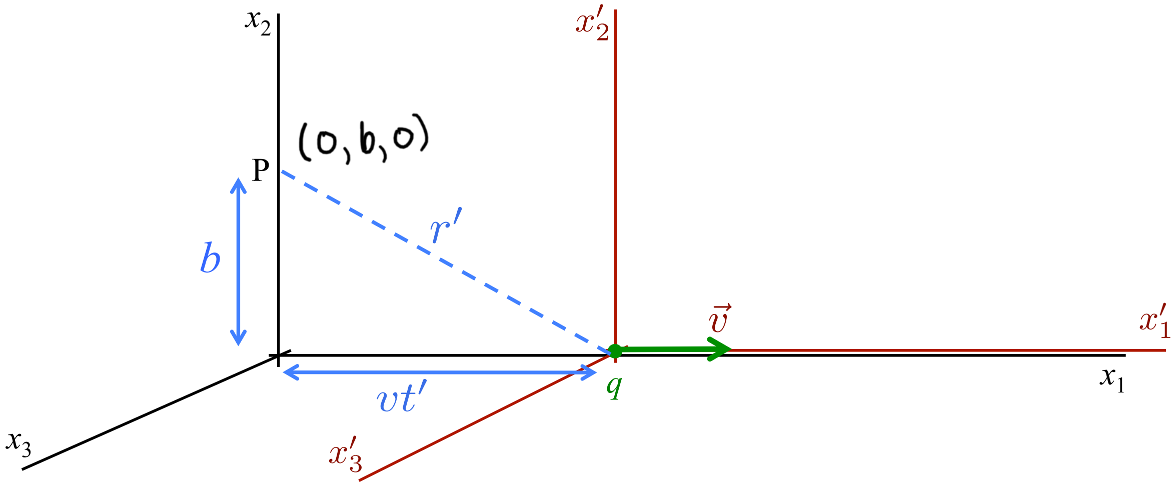
PHY 420 Spring 2021

Week 10—Tuesday, June 1—Discussion Worksheet

In the previous class, we wrote expressions for the transformation of electromagnetic fields. We will now consider an example of the transformation of fields to illustrate some important points.



1. Consider the example shown in the figure above (taken from Figure 11.8 on page 559 in Jackson), in which a point charge q at rest in frame K' is traveling along the x_1 direction of frame K with velocity v. An observer is located at point P, as shown, with b being the closest distance of approach of the charge to the observer, i.e., it is the distance from the origin of K' (and K) to the observation point P at $t' = t = 0$, when the origins of the two reference frames coincide.

(a) What are the coordinates of point P in K'(at the instant shown in the figure)?

$$x_1' = \underline{-vt'}, \qquad x_2' = \underline{\;b\;}, \qquad x_3' = \underline{\;0\;}$$

(b) The electric and magnetic fields at the observation point P in the rest frame K' of the charge are

$$\vec{E}' = \frac{q\hat{r}'}{r'^2} = \frac{q\vec{r}'}{r'^3}, \qquad \vec{B}' = 0$$

Show that in terms of components, the electric and magnetic fields at the observation point P in the rest frame K' of the charge are

$$E_1' = -\frac{qvt'}{r'^3} \qquad E_2' = \frac{qb}{r'^3} \qquad E_3' = 0 \qquad \underbrace{B_1' = 0 \qquad B_2' = 0 \qquad B_3' = 0}_{\text{No magnetic field}}$$

$$E_1' = E' \cdot x_1' = \frac{q\hat{r}'}{r'^2} \cdot \frac{vt'}{r'} = -\frac{qvt'}{r'^3}$$

$$E_2' = E' \cdot x_2' = \frac{q\hat{r}'}{r'^2} \cdot \frac{b}{r'} = \frac{qb}{r'^3}$$

$$E_3' = E' \cdot x_3' = \frac{q\hat{r}'}{r'^2} \cdot 0 = 0$$

2. Show that the transformed fields in frame K are

$$E_1 = -\frac{q\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}} = E_1'$$

$$E_2 = \frac{\gamma qb}{(b^2+\gamma^2v^2t^2)^{3/2}} = \gamma E_2' \tag{11.152}$$

$$B_3 = \beta E_2 = \gamma\beta E_2'$$

Also show that the other components vanish.

$$r' = \sqrt{(vt')^2 + b^2}$$

From Lorentz transformations equations:

$x_0' = \gamma(x_0 - \beta x_1)$ where $x_0 = ct$, $x_0' = ct'$ and $x_1 = 0$

so,

$$ct' = \gamma[ct - \beta(0)] \Rightarrow t' = \gamma t$$

In lh frame coordinates

$$r' = \sqrt{(vt')^2+b^2} = \sqrt{(v\gamma t)^2+b^2} = \sqrt{b^2+\gamma^2v^2t^2}$$

$$E_1 = E_1' \rightarrow E_1 = \frac{-qvt'}{(r')^3} = -\frac{qv\gamma t}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

$$E_2 = \gamma(E_2' + \beta B_3') \rightarrow E_2 = \gamma[E_2' + \beta(0)] = \gamma E_2' = \frac{\gamma qb}{(r')^3}$$

$$= \frac{\gamma qb}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

$$E_3 = \gamma(E_3' - \beta B_2') \rightarrow E_3 = \gamma[E_3' - \beta(0)] = \gamma E_3' = \gamma(0) = 0$$

$$B_1 = B_1' \rightarrow B_1 = 0$$

$$B_2 = \gamma(B_2' - \beta E_3') \rightarrow B_2 = \gamma[0 - \beta(0)] = 0$$

$$B_3 = \gamma(B_3' + \beta E_2') \rightarrow B_3 = \gamma[0 + \beta E_2'] = \gamma\beta E_2'$$

$$= \beta[\gamma E_2'] = \beta E_2$$

Fields such as those in equation (11.152) exhibit interesting behavior when the velocity of the charge approaches that of light. We will now discuss this in more detail.

3. Let's consider first what information is being conveyed by the relations in equation (11.152) on the previous page. These equations are telling us the fields at a point $(0, b, 0)$ in frame K when a charge q moves along the x_1 axis with speed v, passing the origin at time $t = 0$.

(a) If you count the past as negative time, what could you conclude about the fields (E_1, E_2, and B_3) at large negative and large positive times? *Explain.*

At large t, second term in the denominator dominates, the time dependence fore all non-zero fields (E_1, E_2, B_3) goes as either $t/(t^2)^{3/2}$ or $1/(t^2)^{3/2}$, both of which go to zero at large $t \Rightarrow (E_1, E_2, B_3)$ are zero at large t.

(b) How does E_2 behave as t approaches zero? What is its value at $t = 0$?

As $t \to 0$, $E_2 = \dfrac{\gamma q b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$

$$= \frac{\gamma q b}{(b^2)^{3/2}} = \frac{\gamma q b}{b^3} = \frac{\gamma q}{b^2}$$

Thus,

As $t \to 0$, $E_2 \uparrow$

Peak value: $\dfrac{\gamma q}{b^2}$ at $t = 0$

(c) Why would setting $\Delta t = b/\gamma v$ define a characteristic time scale?
Hint: Find the value of E_2 when $t = b/\gamma v$ (in relation to the peak value of E_2).

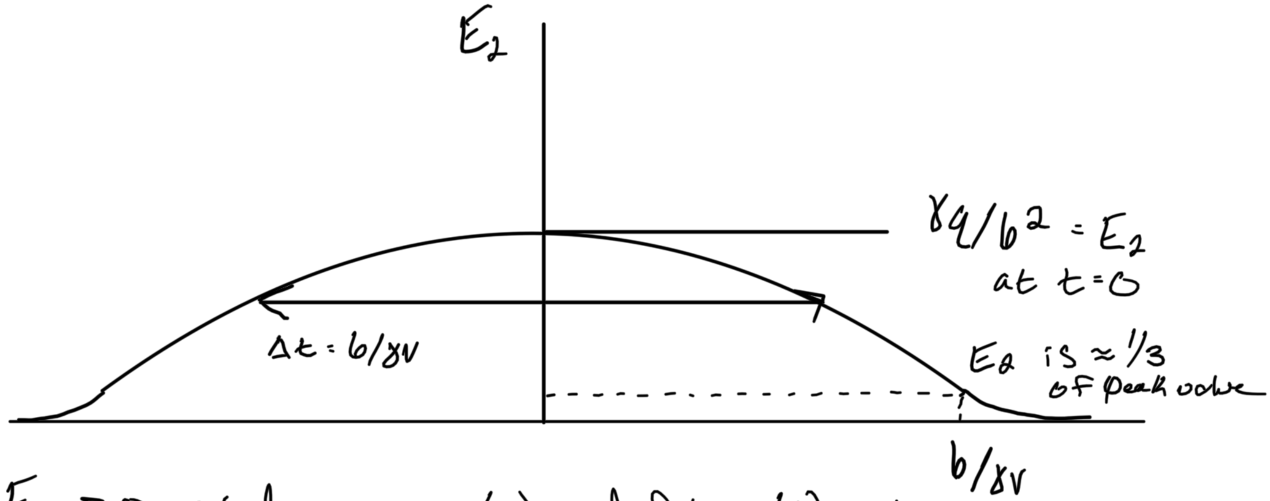
Putting $t = \dfrac{b}{\gamma v}$, get $E_2 = \dfrac{\gamma q b}{\left[b^2 + \gamma^2 v^2 \left(\frac{b^2}{\gamma^2 v^2}\right)\right]^{3/2}}$

$$= \frac{\gamma q b}{(2b^2)^{3/2}} = \frac{\gamma q}{2\sqrt{2}\, b^2}$$

Thus, E_2 decreases to $1/2\sqrt{2}$ of its peak values at $t = b/\gamma v$, making $\Delta t = b/\gamma v$, a characteristic time scale.

4. We will now discuss the behavior of the fields as the speed of q approaches c (so that $\beta \to 1$).

(a) First, sketch a graph of E_2 *vs.* t, based on your answers in Question 3 (don't copy Jackson's!).



$E_2 \to 0$, at large past (-) and future (+) values of t

(b) What would you expect to happen to the graph as $\beta \to 1$, so that $\gamma \gg 1$?

As $\beta \longrightarrow 1$ (Speed of q approaches c), so that $\gamma >> 1$, we would expect E_2 to become a tall narrow spike b/c, peak of $E_2 = \gamma q/b^2 \sim \gamma$, so large γ $\Rightarrow$ higher peak but $\Delta t = b/\gamma v \sim 1/\gamma$, so large $\gamma \Rightarrow$ narrow Δt

(c) Comment on how B_3 is related to E_2. What happens as the speed of the charge approaches that of light, c?

$B_3 = \beta E_2$, as $v \to c$, i.e. $\beta \to 1$

B_3 becomes closer and closer to E_2 in magnitude

5. Continue examining the behavior of the fields.

(a) Use your answer in Question 4(c) to explain how a rough sketch of B_3 *vs.* t would be similar to your graph of E_2 *vs.* t in Question 4(a), and also how it would be different.

$B_3 = \beta E_2$

(b) What is the value of E_1 at $t = 0$?

(c) Predict what a graph of E_1 *vs.* t would look like. Determine quantitatively any peak values in the graph and the value(s) of t at which they are located.