PHY 420 Spring 2021

Class Summary—Week 5, Day 1—Tuesday, Apr 27

Electric Dipole Fields

In order to learn about how electromagnetic waves are generated, we wrote that for a localized system of charges and currents that vary in time as $e^{-i\omega t}$, the vector potential $\vec{A}(\vec{x})$ is given by

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}\,') \frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} \, d^3x' \tag{9.3}$$

In the **far (radiation) zone**, the observation point r is very far from the source and much larger than the wavelength of the light. Thus, since $(r/\lambda) \gg 1$ and $k = 2\pi/\lambda$, we have $kr \gg 1$ in the far zone, and we can write

$$\left|\vec{x}-\vec{x}\,'\right| \simeq r - \hat{n}\cdot\vec{x}\,' \tag{9.7}$$

where $\hat{n}$ is a unit vector in the direction of $\vec{x}$. In fact, *from your derivation of this approximation on the worksheet from last week*, you know that equation (9.7) is valid for $r \gg d$ (independent of kr), so it is a reasonable approximation even in the near zone.

With the approximation in equation (9.7), the vector potential can be written as

$$\lim_{kr\to\infty} \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}\,')\, e^{-ik\hat{n}\cdot\vec{x}\,'} \, d^3x' \tag{9.8}$$

where I've used the limit to signify that we're in the far zone. Notice that e^{ikr}/r is just an outgoing spherical wave, so equation (9.8) tells us that in the far zone the vector potential behaves as an outgoing spherical wave times a coefficient that depends on an integral over the source.

On the worksheet last week, you showed by writing the exponential term in as a series expansion and taking the summation and some terms outside the integral in equation (9.8), that

$$\lim_{kr\to\infty} \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int \vec{J}(\vec{x}\,')\, (\hat{n}\cdot\vec{x}\,')^n \, d^3x' \tag{9.9}$$

The magnitude of the n^{th} term in equation (9.9) above is given by

$$\frac{1}{n!} \int \vec{J}(\vec{x}\,')\, (k\hat{n}\cdot\vec{x}\,')^n \, d^3x' \tag{9.10}$$

Since the order of magnitude of $\vec{x}\,'$ is d and $kd \ll 1$, the successive terms in the expansion of $\vec{A}(\vec{x})$ written in equation (9.9) fall off rapidly with n. Consequently, the radiation emitted from the source will come mainly from the first non vanishing term in the expansion of equation (9.9).

If we **keep only the first term** (corresponding to n=0) in equation (9.9), then we get

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}\,')\, d^3x' \tag{9.13}$$

Before proceeding, it is worth noting that since equation (9.7) is valid for $r \gg d$, independent of the value of kr, therefore equation (9.13) is a reasonable approximation everywhere outside the source, not just in the far zone.

We will now go through a series of steps to enable us to write $\vec{A}(\vec{x})$ in terms of the electric dipole moment $\vec{p}$.

Let's begin by applying the following identity (*which you proved in Question 1 and 2 today's worksheet*):

$$\int \vec{J}(\vec{x}\,')\,d^3x' = -\int \vec{x}\,'(\vec{\nabla}'\cdot\vec{J})\,d^3x'$$

together with the continuity equation

$$\vec{\nabla}'\cdot\vec{J}(\vec{x}\,',t') = -\frac{\partial\rho(\vec{x}\,',t')}{\partial t'}$$

which I've written in primed coordinates $\vec{x}\,'$ and t' to make it easier to follow when we substitute inside the integral. With the assumed harmonic dependence of

$$\rho(\vec{x}\,',t') = \rho(\vec{x}\,')\,e^{-i\omega t'} \qquad \text{and} \qquad \vec{J}(\vec{x}\,',t') = \vec{J}(\vec{x}\,')\,e^{-i\omega t'}$$

from equation (9.1), the continuity equation above can be written as

$$\vec{\nabla}'\cdot\vec{J}(\vec{x}\,')\,e^{-i\omega t'} = -\frac{\partial}{\partial t'}\left[\rho(\vec{x}\,')\,e^{-i\omega t'}\right]$$

which, after carrying out the differentiation on the right hand side, becomes

$$\vec{\nabla}'\cdot\vec{J}(\vec{x}\,')\,e^{-i\omega t'} = -(-i\omega)\rho(\vec{x}\,')\,e^{-i\omega t'}$$

or, after canceling $e^{-i\omega t'}$ from both sides, we get

$$\vec{\nabla}'\cdot\vec{J}(\vec{x}\,') = -\left[-i\omega\rho(\vec{x}\,')\right]$$

This, and what follows below to the end of the page, is essentially the solution to Question 3(a) today's worksheet.

Substituting the expression above into the identity at the top of this page (that you derived in Question 2(b) of today's worksheet), we get

$$\int \vec{J}(\vec{x}\,')\,d^3x' = -\int \vec{x}\,'(\vec{\nabla}'\cdot\vec{J})\,d^3x' = -i\omega\int x'\rho(\vec{x}\,')\,d^3x' \tag{9.14}$$

Putting equation (9.14) into equation (9.13), we get

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}\,')\,d^3x' = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left[-i\omega\int x'\rho(\vec{x}\,')\,d^3x'\right]$$

We can then write the vector potential as

$$\vec{A}(\vec{x}) = -\frac{i\omega\mu_0}{4\pi}\,\vec{p}\,\frac{e^{ikr}}{r} \tag{9.16}$$

where

$$\vec{p} = \int x'\rho(\vec{x}\,')\,d^3x' \tag{9.17}$$

is the **electric dipole moment**, as defined in equation (4.8) by Jackson, *and this is the expression for $\vec{p}$ you were asked to write in Question 3(b) of today's worksheet.*

Having determined the vector potential, we can write the fields from equation (9.4) and equation (9.5), *as you will do on the homework*:

$$\begin{aligned}
\vec{H} &= \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\\
\vec{E} &= \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r}+\left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}
\end{aligned}\tag{9.18}$$

Remember that $\hat{n}$ in equation (9.18) above is a unit vector in the direction of $\vec{x}$, the observation point, i.e., it is the radial vector; but since we have a tiny source at the origin and the waves are propagating radially outward, the radial vector and the wave vector $\vec{k}$ along which the wave is propagating point in the same direction; thus, $\hat{n}$ is also along the same direction as $\vec{k}$.

As you determined in Questions 4(a) and 4(b) of today's worksheet, we see from equation (9.18) that the magnetic field is always perpendicular to $\hat{n}$ and hence perpendicular to the direction of propagation of the wave, but the electric field has components parallel and perpendicular to $\hat{n}$ (don't worry, though, only the transverse component will be present in the far zone, as we'll see below).

In the far zone, where $kr \gg 1$, the fields take on the limiting forms

$$\begin{aligned}
\vec{H} &= \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\\
\vec{E} &= \frac{k^2}{4\pi\epsilon_0}\left[(\hat{n}\times\vec{p})\times\hat{n}\right]\frac{e^{ikr}}{r}
\end{aligned}\tag{9.19}$$

as you showed in Questions 4(c) and 5 of today's worksheet. We are back in familiar territory! In the electromagnetic wave emitted by an electric dipole oscillating harmonically, we note that:

- The factor $\hat{n}\times\hat{p}$, where $\hat{p}$ is the direction of the dipole moment, gives the direction of the magnetic field in this electromagnetic wave. Clearly, the magnetic field direction is then perpendicular to the direction of propagation of the wave. The electric field is perpendicular to the magnetic field and also perpendicular to the direction of propagation $\hat{n}$. We conclude that we have transverse waves carrying energy away from the oscillating source. Notice that $\vec{B}$ oscillates in a plane normal to the dipole $\hat{p}$, whereas $\vec{E}$ oscillates in a plane parallel to the dipole $\vec{p}$.

- The $\hat{n}\times\hat{p}$ factor also modulates the field in the angular direction, $|\hat{n}\times\hat{p}| = \sin\theta$, where θ is the angle between the two vectors. Note that the propagation direction along the axis of the dipole has $\theta = 0°$, and, therefore, there are no fields propagating in this direction. In summary, waves are not radiated along the axis of an electric dipole and are most strongly radiated perpendicular to the dipole.

- The factor $ck^2p/4\pi$, where p is the magnitude of the dipole moment, gives the overall strength of the magnetic field, whereas $k^2p/4\pi\epsilon_0$ gives the overall strength of the electric field in the wave. Clearly, increasing the strength of the dipole increases the magnetic field (and the electric field). Somewhat surprisingly, increasing the wave number k (or, equivalently, the frequency) also increases the magnetic field (and the electric field).

- The factor e^{ikr}/r modulates the overall field strength, so that we have a wave that travels radially outward and decreases in strength as it travels and spreads out. Notice that both fields fall off as $1/r$.

So, what have we written in equation (9.18) and equation (9.19)?

If we take an electric dipole with dipole moment $\vec{p}$ and let it oscillate harmonically, it will emit electromagnetic waves. In the long wavelength limit ($r \gg d$), the electric and magnetic fields of these electromagnetic waves are given by equation (9.18), whereas the electric and magnetic fields of the electromagnetic waves in the far zone are given by equation (9.19).

Next, we would like to obtain an expression for the power radiated. To do so, one can begin with the Poynting vector

$$\vec{S} = \frac{1}{2}\,\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right]$$

where the factor $1/2$ comes from the time average. One can then show that the **time-averaged power radiated per unit solid angle** by the oscillating electric dipole is given by

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[r^2\,\hat{n}\cdot\vec{E} \times \vec{H}^*\right] \tag{9.21}$$

as you demonstrated in Question 6 of today's worksheet, where $\hat{n}$ is effectively a unit vector in the direction of propagation.

On *Homework 4*, you will show that this modifies to

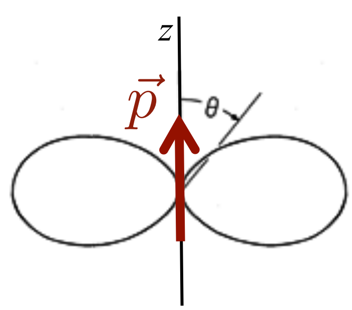
$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}\,k^4\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2 \tag{9.22}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space.

If all the components of $\vec{p}$ (in some chosen coordinate system) have the same phase, then the dipole moment $\vec{p}$ lies along a line, and the angular distribution of radiation is that of a (linearly polarized) dipole pattern:

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}\,k^4\left|\vec{p}\right|^2\,\sin^2\theta \tag{9.23}$$

where the angle θ is measured from the direction of $\vec{p}$, as shown in the figure below.



As predicted by equation (9.19), if the dipole is oriented along the z direction, the radiation is zero along the poles but peaks at the equator perpendicular to the orientation of the dipole.

Finally, let us discuss briefly the **near zone**. As noted before, equation (9.13) is valid for $r \gg d$ independent of the value of kr, and is therefore a reasonable approximation even in the near zone. Thus in the near zone where $kr \ll 1$, the fields in equation (9.19) take on the limiting forms

$$\begin{aligned} \vec{H} &= \frac{i\omega}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{r^2} \\ \vec{E} &= \frac{1}{4\pi\epsilon_0}\left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\frac{1}{r^3} \end{aligned} \tag{9.20}$$

Apart from its oscillations in time, the second of the two equations written above is just the electric field of a (static) dipole. To see this, recall that in PHY 411 last quarter, we wrote a multipole expansion for the potential in terms of spherical harmonics:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\,q_{lm}\,\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \tag{4.1}$$

where the coefficients q_{lm} are known as the *multipole moments*. You may recall demonstrating that the multipoles corresponding to $l=1$ (i.e., q_{11}, q_{10}) are proportional to the components of the electric dipole moment $\vec{p}$, where $\vec{p}$ is given by

$$\vec{p} = \int \vec{x}\,'\,\rho(\vec{x}\,')\,d^3x' \tag{4.8}$$

Since $\vec{E} = -\vec{\nabla}\Phi$, we can show by direct differentiation of equation (4.1) that the coordinates of the electric field are

$$\begin{aligned} E_r &= \frac{(l+1)}{(2l+1)\,\epsilon_0}\,q_{lm}\,\frac{Y_{lm}(\theta,\phi)}{r^{l+2}} \\ E_\theta &= -\frac{1}{(2l+1)\,\epsilon_0}\,q_{lm}\,\frac{1}{r^{l+2}}\,\frac{\partial}{\partial\theta}\,Y_{lm}(\theta,\phi) \\ E_\phi &= \frac{1}{(2l+1)\,\epsilon_0}\,q_{lm}\,\frac{1}{r^{l+2}}\,\frac{im}{\sin\theta}\,Y_{lm}(\theta,\phi) \end{aligned} \tag{4.9}$$

For a dipole $\vec{p}$ along the z-axis, the fields in equation (4.9) reduce to:

$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \qquad E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \qquad E_\phi = 0 \tag{4.10}$$

We can write this in vector form, so that the field at the observation point $\vec{x}$ due to a dipole $\vec{p}$ at the point $\vec{x}_0$ is

$$\vec{E}(\vec{x}) = \frac{3\hat{n}(\vec{p}\cdot\hat{n})-\vec{p}}{4\pi\epsilon_0\,|\vec{x}-\vec{x}_0|^3} \tag{4.11}$$

where $\hat{n}$ is a unit vector directed from $\vec{x}_0$ to $\vec{x}$.

Equation (9.20) tells us that:

- The magnetic field in the near zone is a factor kr smaller, and since $kr \ll 1$, it is negligible in the near zone; therefore, the fields in the near zone are dominantly electric in nature.
- In the static limit $k \to 0$, the magnetic field vanishes, meaning that the near zone extends to infinity; another way of saying this is that in the static limit, an electric dipole only has an electric field around it, and no magnetic field.