## Physics 460—Spin Correlations Spring 2020

When working with two-particle systems, we often talk about the correlations between measurements on the two particles. At the most basic level, the correlation between two measurements tells us if one of the measurements gives us any information about the other measurement. One way to judge this is to look at the probabilities of the joint measurements. For example, if the state of a spin-1/2 system is $|+;+\rangle$, then measurements of the spin of both particles always yields $+\hbar/2$ for both spins, and the results are correlated.

A second way to judge correlations is through a correlation coefficient. Let measurements that result in either spin up or spin down for both particles be assigned a value of $+1$, and measurements that result in spin up for one particle and spin down for the other particle be assigned a value of $-1$. Then the correlation coefficient for this measurement is

$$C = \mathcal{P}(+;+) + \mathcal{P}(-;-) - \mathcal{P}(+;-) - \mathcal{P}(-;+),$$

where $\mathcal{P}(+;+)$ is the probability that the measurement will result in spin up for both particles, *etc.* This coefficient will always have a value in the range $C = [-1, +1]$.

A correlation coefficient of $C = 1$ means the measured values are always aligned (both spin up or both spin down), and we would say that the spin are positively correlated. A correlation coefficient of $C = -1$ means the measured values are always opposite (one spin up and the other spin down), and we would say that the spin are negatively correlated or anti-correlated. A correlation coefficient of $C = 0$ means there is no relationship between the two measurements and we would say that the spins are uncorrelated. Intermediate values between zero and $\pm 1$ are also possible, and are evidence of partial correlation.

Note that there is no reason the two measurements have to be along the same axis, or that the person making one measurement even knows what axis is being used for the other measurement. By comparing results of the measurements, we can calculate a correlation coefficient and determine the amount of correlation between the two measurements.

(1) Consider the two two-particle states for spin-1/2 particles

$$|\Psi_A\rangle = \frac{1}{\sqrt{2}}\Big(|+;-\rangle - |-;+\rangle\Big) \quad \text{and} \quad |\Psi_B\rangle = \frac{1}{\sqrt{2}}\Big(|+;+\rangle - |-;+\rangle\Big). \quad = \frac{1}{\sqrt{2}}\Big(|+\rangle - |-\rangle\Big)|+\rangle$$

These states are expressed in the $z$ basis for both particles.

(a) Find the probabilities $\mathcal{P}(+;+)$, $\mathcal{P}(+;-)$, $\mathcal{P}(-;+)$, and $\mathcal{P}(-;-)$ if the spin of both particles is measured along the $z$ axis. Are these results correlated for state $|\Psi_A\rangle$? Are they correlated for state $|\Psi_B\rangle$?

(b) Find the probabilities $\mathcal{P}(+;+)$, $\mathcal{P}(+;-)$, $\mathcal{P}(-;+)$, and $\mathcal{P}(-;-)$ if the spin of both particles is measured along the $x$ axis. Are these results correlated for state $|\Psi_A\rangle$? Are they correlated for state $|\Psi_B\rangle$?

(c) Find the probabilities $\mathcal{P}(+;+)$, $\mathcal{P}(+;-)$, $\mathcal{P}(-;+)$, and $\mathcal{P}(-;-)$ if the spin of the first particle is measured along the $x$ axis and the spin of the second particle is measured along the $z$ axis. Are these results correlated for state $|\Psi_A\rangle$? Are they correlated for state $|\Psi_B\rangle$?

(a) for state A:

$P(+,+) = 0$

$P(+,-) = \frac{1}{2}$

$P(-,+) = \frac{1}{2}$

$P(-,-) = 0$

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{1} \\ & & + & - \\ \hline 2 & + & 0 & \frac{1}{2} \\ \hline & - & \frac{1}{2} & 0 \\ \hline \end{array}$$

anti-correlated

for state B:

$P(+,+) = \frac{1}{2}$

$P(+,-) = 0$

$P(-,+) = \frac{1}{2}$

$P(-,-) = 0$

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{1} \\ & & + & - \\ \hline 2 & + & \frac{1}{2} & \frac{1}{2} \\ \hline & - & 0 & 0 \\ \hline \end{array}$$

uncorrelated

$$C_A = 0 + 0 - \frac{1}{2} - \frac{1}{2} = -1 \quad \checkmark$$

$$C_B = \frac{1}{2} + 0 - 0 - \frac{1}{2} = 0 \quad \checkmark$$

(b) Write the states in terms of

$$|+_x\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right), \quad |-_x\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

$$\Rightarrow |+\rangle = \frac{1}{\sqrt{2}}\left(|+_x\rangle + |-_x\rangle\right), \quad |-\rangle = \frac{1}{\sqrt{2}}\left(|+_x\rangle - |-_x\rangle\right)$$

$$|\psi_A\rangle = \frac{1}{2\sqrt{2}}\left(\cancel{|+_x;+_x\rangle} + |-_x;+_x\rangle - |+_x;-_x\rangle - \cancel{|-_x;-_x\rangle}\right)$$
$$- \frac{1}{2\sqrt{2}}\left(\cancel{|+_x;+_x\rangle} + |+_x;-_x\rangle - |-_x;+_x\rangle - \cancel{|-_x;-_x\rangle}\right)$$

$$|\psi_A\rangle = -\frac{1}{\sqrt{2}}\left(|+_x;-_x\rangle - |-_x;+_x\rangle\right)$$

$$|\psi_B\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) \otimes |+\rangle = |-_x\rangle \otimes \frac{1}{\sqrt{2}}\left(|+_x\rangle + |-_x\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left(|-_x;+_x\rangle + |-_x;-_x\rangle\right) = \frac{1}{\sqrt{2}} |-_x\rangle \left(|+_x\rangle + |-_x\rangle\right)$$

state A:

$P(+,+) = 0$
$P(+,-) = \frac{1}{2}$
$P(-,+) = \frac{1}{2}$
$P(-,-) = 0$

| 2 \ 1 | + | - |
|---|---|---|
| + | 0 | 1/2 |
| - | 1/2 | 0 |

anti-correlated

$$C_A = 0 + 0 - \frac{1}{2} - \frac{1}{2} = -1$$

state B

$P(+,+) = 0$
$P(+,-) = 0$
$P(-,+) = \frac{1}{2}$
$P(-,-) = 1/2$

| 2 \ 1 | + | - |
|---|---|---|
| + | 0 | 1/2 |
| - | 0 | 1/2 |

uncorrelated

$$C_B = 0 + \frac{1}{2} - 0 - \frac{1}{2} = 0$$

(c) Write the state of the first particle in terms of $|+_x\rangle$ and $|-_x\rangle$

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|+_x\rangle + |-_x\rangle\right)\otimes|-\rangle - \frac{1}{\sqrt{2}}\left(|+_x\rangle - |-_x\rangle\right)\otimes|+\rangle\right)$$

$$|\psi_A\rangle = \frac{1}{2}\left(|+_x;-\rangle + |-_x;-\rangle - |+_x;+\rangle + |-_x;+\rangle\right)$$

$$|\psi_B\rangle = |-_x\rangle\otimes|+\rangle = |-_x;+\rangle$$

state A:

$P(+,+) = \frac{1}{4}$

$P(+,-) = \frac{1}{4}$

$P(-,+) = \frac{1}{4}$

$P(-,-) = \frac{1}{4}$

| 2 \ 1 | + | − |
|---|---|---|
| + | 1/4 | 1/4 |
| − | 1/4 | 1/4 |

uncorrelated

$C_A = \frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} = 0$

state B:

$P(+,+) = 0$

$P(+,-) = 0$

$P(-,+) = 1$

$P(-,-) = 0$

| 2 \ 1 | + | − |
|---|---|---|
| + | 0 | 1 |
| − | 0 | 0 |

anticorrelated

$C_B = 0 + 0 - 1 - 0 = -1$