PHY 420 Spring 2021

Homework 6—due by 9:00 PM, Thursday, May 20

Submit late homework into the late D2L dropbox for reduced credit (see syllabus); late submissions will be accepted until 8 AM on Saturday (May 22); also see the next page.

1. The inverse Lorentz transformation equations for a frame K' traveling at velocity v along the positive x-direction of a frame K are given by

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \qquad x = \gamma\left(x' + vt'\right) \qquad y = y' \qquad z = z'$$

By explicit differentiation, derive the Lorentz transformation law for velocities:

$$u_x = \frac{u'_x + v}{1 + \dfrac{vu'_x}{c^2}} \qquad \text{and} \qquad u_y = \frac{u'_y}{\gamma\left(1 + \dfrac{vu'_x}{c^2}\right)}$$

2. Now consider the more general case of the frame K' moving with velocity $\vec{v} = c\vec{\beta}$ with respect to the frame K. Then, the components of velocity transform according to

$$u_\parallel = \frac{u'_\parallel + v}{1 + \dfrac{\vec{v}\cdot\vec{u}'}{c^2}} \qquad \vec{u}_\perp = \frac{\vec{u}'_\perp}{\gamma_v\left(1 + \dfrac{\vec{v}\cdot\vec{u}'}{c^2}\right)} \tag{1}$$

where $u_\parallel$ and $\vec{u}_\perp$ refer to components of velocity parallel and perpendicular, respectively, to $\vec{v}$, and the subscript on γ_v explicitly identifies the relationship to be $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) For the simple case where u' is parallel to the direction of v, use equation (1) to find an expression for u.

(b) If $u' = c$, then use the expression you derived in part (a) to find u. Comment on how this result reflects a key aspect of Special Relativity.

(c) For speeds u' and v both small compared to c, show that the velocity addition law reduces to the Galilean result: $u = u' + v$.

3. In class, we derived the 4-velocity

$$U = \left(\gamma_u c, \gamma_u \vec{u}\right)$$

where $\gamma_u = (1 - u^2/c^2)^{-1/2}$, and $\vec{u} = d\vec{x}/dt$ is the usual 3-dimensional velocity.

(a) Find the norm or invariant length U^2 of this 4-velocity. Reduce to the simplest possible form.

(b) Starting from U, write down an expression for the 4-acceleration A.

(c) Find the scalar product $U \cdot A$ of the 4-velocity and the 4-acceleration.

4. The Lorentz transformation equations are given by

$$x'_0 = \gamma\,(x_0 - \beta x_1)$$

$$x'_1 = \gamma\,(x_1 - \beta x_0)$$

$$x'_2 = x_2$$

$$x'_3 = x_3$$

where

$$\beta = \frac{v}{c} \qquad \text{and} \qquad \gamma = (1-\beta^2)^{-1/2}$$

(a) If we introduce the parametrization $\beta = \tanh\zeta$, then **show that** the relations above imply that

$$\gamma = \cosh\,\zeta \qquad \text{and} \qquad \gamma\beta = \sinh\zeta$$

where ζ is known as the *boost parameter* or *rapidity.*

Note: This is different from what's written in the class summary, so please make sure you do what's asked here. In this question, you are asked to start from β and derive the expressions for γ and $\gamma\beta$, whereas the class summary is the other way around.

(b) Using the parametrization in part (a), show that the Lorentz transformation equations written above can be put in the form

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

You may write by hand and scan as a single PDF, or write in latex (using the template file provided) or Word, and generate PDF. Please submit one PDF file only. Only questions and sub-parts that are numbered clearly, with numbers corresponding to those in this document, will be graded. See the syllabus for more detailed rules.

Emailed or paper copies of homework are never accepted; in particular, do not attach homework to email to make an end-run around the D2L deadline or late deadline; such emails are automatically deleted and do not count as submissions.