PHY 420 Spring 2021

Class Summary—Week 8, Day 1—Tuesday, May 18

Matrix Representation of Lorentz Transformations

Physical laws must be **covariant**, meaning that they must have the same form in different coordinate systems. In particular:

- Physical laws must be covariant under translations in space and time since space is homogenous.
- Physical laws must be covariant under rotations in 3-dimensional space since space is isotropic.
- Physical laws must be covariant under Lorentz transformations to conform to Special Relativity.

Recall that we are in the process of identifying a group that will represent the covariance of physical laws under the transformations specified above. Together, these transformations form part of the inhomogenous Lorentz group, or Poincare group.

Recall also that we introduced a matrix representation in the previous class to make the manipulation explicit and less abstract. In this representation, the components x^0, x^1, x^2, x^3 of a contravariant 4-vector form the elements of a column vector, and the metric tensor $g_{\alpha\beta}$ is represented by a square 4×4 matrix

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

with

$$g^2 = I$$

where I is the 4×4 unit matrix.

Moreover, the covariant 4-vector x_α can be obtained from the contravariant x^β by contraction with the metric tensor $g_{\alpha\beta}$, as we saw in equation (11.72): $x_\alpha = g_{\alpha\beta}\, x^\beta$, which means that by matrix multiplication of g on x above, we find the covariant coordinate vector to be

$$gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix} \longrightarrow \text{ thus, } gx \text{ is } x_\alpha = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}; \tag{11.82}$$

We have already learned that matrix **scalar products** (a, b) of 4-vectors a and b are defined in the usual way by summing over the products of the elements of a and b, or equivalently, by the matrix multiplication:

$$(a, b) \equiv \tilde{a}b \tag{11.80}$$

where $\tilde{a}$ is the transpose of a (and hence a row vector). In this compact notation, the scalar product of two 4-vectors could also be written as

$$a \cdot b = (a, gb) = \tilde{a}gb \tag{11.83}$$

or, if we write out the elements explicitly

$$a \cdot b = \tilde{a}gb = a^\alpha g_{\alpha\beta} b^\beta = a^\alpha b_\alpha$$

We seek a group of linear transformations that leaves $x \cdot x = (x, gx)$ invariant, so that we can use it to represent the invariance of physical laws in all inertial frames, as postulated by Special Relativity. Since $x \cdot x$ or (x, gx) is the norm of a 4-vector, we are effectively seeking a group of transformations that preserves the "length" in the 4-dimensional metric.

Our quest to find a group of linear transformations that leaves $x \cdot x = (x, gx)$ invariant is equivalent to finding all square 4×4 matrices A which, when they transform the coordinates as

$$x' = Ax \tag{11.84}$$

then they will leave the norm (x, gx) invariant:

$$x' \cdot x' = \tilde{x}' g x' = \tilde{x} g x = x \cdot x \tag{11.85}$$

which is nothing other than just a mathematical statement of Special Relativity. The notation we're using in this equation is defined in equation (11.83) on the previous page.

From equation (11.84), we get the equation for the transpose

$$\tilde{x}' = \tilde{x}\tilde{A}$$

and substituting this, and x' from equation (11.84) into equation (11.85), we get

$$\tilde{x}\tilde{A}gAx = \tilde{x}gx$$

Since this must hold for all coordinate vectors x, A must satisfy the matrix equation

$$\tilde{A}gA = g \tag{11.86}$$

Certain properties of the transformation matrix A can be deduced immediately from equation (11.86). Taking the determinant of both sides of equation (11.86) gives

$$\det(\tilde{A}gA) = \det g$$

and since the left hand side can be written as

$$\det(\tilde{A}gA) = \det g \, \det(\tilde{A}\, A) = \det g \, (\det A)^2$$

the equation above that $\det(\tilde{A}gA) = \det g$ becomes

$$\det g \, (\det A)^2 = \det g$$

But $\det g = -1 \neq 0$, so we get

$$\det A = \pm 1$$

which is a constraint on the allowed matrices A, and hence the allowed transformations.

There are two classes of transformations. The language can be confusing, so let's be clear!

- **Proper** Lorentz transformations, with $\det A = +1$.
- **Improper** Lorentz transformations can have either sign of $\det A$.

We prefer proper Lorentz transformations because they involve rotations and (velocity) boosts, which can be built out of infinitesimal rotations and boosts. On the other hand, improper transformations involve space inversion or space and time inversion.

Henceforth, we will focus only on proper transformations, with $\det A = +1$. It turns out that we can then build A by writing

$$A = e^L \tag{11.87}$$

where L is a 4×4 matrix, and therefore we need to figure the form of L. Procedures which we won't consider here illustrate that L, and therefore A, can be built with only six free parameters: **three rotations** and **three velocity boosts**. Therefore, we will now figure out how to generate rotations, starting with the infinitesimal generator for rotations around the x^3 axis.

You may remember that the matrix for rotations about the x^3 axis by an angle ϕ_3 is

$$R_3(\phi_3) = \begin{pmatrix} \cos\phi_3 & -\sin\phi_3 & 0 \\ \sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The corresponding infinitesimal generator X_3 is then

$$X_3 = \left.\frac{dR_3}{d\phi_3}\right|_{\phi_3=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In our 4-dimensional space-time, we then have for finite rotations about the x^3 axis by angle ϕ_3 that

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & \cos\phi_3 & -\sin\phi_3 & 0 \\ 0 & \sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

so that the correponding infinitesimal generator, which we'll call S_3 for rotation around the x^3 axis, is

$$S_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Similarly, we can write two other matrices S_1 and S_2 to generate rotations around x^1 and x^2 respectively, where

$$S_1 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad S_2 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

Next, we need to generate the **velocity boosts**. We will do so with infinitesimal Lorentz transforms.

First, recall that when we started discussing relativity, we said that the Lorentz transforms could be parametrized by the alternative parametrization in which

$$\left.\begin{aligned} \beta &= \tanh\zeta \\ \gamma &= \cosh\zeta \\ \gamma\beta &= \sinh\zeta \end{aligned}\right\} \tag{11.20}$$

where the parameter ζ is known as the **boost parameter** or **rapidity**. So, to "boost" between frames means to carry out a Lorentz transformation from one frame to another. We then rewrote the Lorentz transforms for frame K' moving along the x_1 direction of frame K in terms of the boost parameter ζ:

$$\begin{aligned} x_0' &= +x_0\,\cosh\zeta - x_1\,\sinh\zeta \\ x_1' &= -x_0\,\sinh\zeta + x_1\,\cosh\zeta \end{aligned} \tag{11.21}$$

and, writing now in matrix form, we have

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

where I've also included ${x'}^2 = x^2$, and ${x'}^3 = x^3$.

For an infinitesimal boost, we set ζ to be small ($\Delta\zeta$), so that

$$\cosh(\Delta\zeta) = 1, \qquad \sinh(\Delta\zeta) = \Delta\zeta$$

and we can write

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & -\Delta\zeta & 0 & 0 \\ -\Delta\zeta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

We can write the 4×4 matrix as the sum of two matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Big(-\Delta\zeta \Big)$$

and call the second matrix K_1, the matrix for an infinitesimal boost along the x^1 direction. Likewise, we can generate K_2 and K_3, matrices for an infinitesimal boost along the x^2 and x^3 directions respectively.

The net result of the above processes is that we get **six fundamental matrices** defined by

$$
\begin{aligned}
S_1 &= \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad
S_2 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \qquad
S_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
K_1 &= \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad
K_2 = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad
K_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)
\end{aligned}
\tag{11.91}
$$

The matrices S_i generate rotations in three dimensions, while the matrices K_i produce boosts. The six matrices in this equation are a representation of the infinitesimal generators of the Lorentz group.

The matrices in equation (11.91) have some very interesting (and useful) properties. Note that the squares of these six matrices are all diagonal, and of the form

$$
\begin{aligned}
S_1^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad
S_2^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad
S_3^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
K_1^2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad
K_2^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad
K_3^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
\tag{11.92}
$$

Moreover, you *showed on the worksheet* that

$$
S_1^3 = -S_1 \qquad \text{and} \qquad K_1^3 = K_1
$$

and likewise for the other matrices. This implies that any power of one of the matrices can be expressed as a multiple of the matrix or its square. This is a very useful property that we will be using shortly, so let us write it down in general form:

$$
\left(\hat{\epsilon}\cdot\vec{S}\right)^3 = -\hat{\epsilon}\cdot\vec{S} \qquad \text{and} \qquad \left(\hat{\epsilon}\,'\cdot\vec{K}\right)^3 = \hat{\epsilon}\,'\cdot\vec{K}
$$

where $\hat{\epsilon}_i$ is a unit vector along the x^i axis; thus, $\hat{\epsilon}$ and $\hat{\epsilon}\,'$ are unit 3-vectors. Although Jackson associates a primed unit vector with $\vec{K}$ at this stage just to keep the boost frame distinct from the rotating frame, we shall soon see that we can use unprimed coordinates for both frames when the context is clear (see, e.g., Case 1 discussed on the next page).

From equation (11.91), we have much better justification for what Jackson is doing. Having written $A = e^L$ in equation (11.87), he writes the determinant of A as

$$\det A = \det\left(e^L\right) = e^{\mathrm{Tr}\, L}$$

If L is a real matrix, $\det A = -1$ is excluded. Furthermore, if L is traceless, then $\det A = +1$. Thus, for proper Lorentz transformations, L is a real, traceless 4×4 matrix.

Next, L just encodes the 3 rotations and 3 translations which we've specified using the matrices S_i and K_i in equation (11.91), and Jackson claims that we can write

$$\begin{aligned} L &= -\vec{\omega} \cdot \vec{S} - \vec{\zeta} \cdot \vec{K} \\ A &= e^{-\vec{\omega}\cdot\vec{S} - \vec{\zeta}\cdot\vec{K}} \end{aligned} \tag{11.93}$$

where $\vec{\omega}$ and $\vec{\zeta}$ are constant 3-vectors. The three components each of $\vec{\omega}$ and $\vec{\zeta}$ correspond to the six parameters of the transformation.

Rather than prove this in general, we discussed that it was valid for two examples:

- **Case 1:** For no rotation, and a boost along the x^1 axis, we have

$$\vec{\omega} = 0, \qquad \vec{\zeta} = \zeta \hat{\epsilon}_1$$

- **Case 2:** For rotation along the x^3 axis, with no boost

$$\vec{\omega} = \omega \hat{\epsilon}_3, \qquad \vec{\zeta} = 0$$

Also, for a boost (without rotation) in an arbitrary direction, we have

$$A = e^{-\vec{\zeta}\cdot\vec{K}}$$

The boost vector can be written in terms of the relative velocity $\vec{\beta}$ as

$$\vec{\zeta} = \hat{\beta}\, \tanh^{-1}\, \beta$$

where $\hat{\beta}$ is a unit vector in the direction of the relative velocity of the two inertial frames. The pure boost is then

$$A_{\mathrm{boost}}(\vec{\beta}) = e^{-\hat{\beta}\cdot\vec{K}\, \tanh^{-1}\, \beta} \tag{11.97}$$

Finally, the matrices in equation (11.91) that are a representation of the infinitesimal generators of the Lorentz group satisfy the following commutation relations:

$$\begin{aligned} [S_i, S_j] &= \epsilon_{ijk}\, S_k \\ [S_i, K_j] &= \epsilon_{ijk}\, K_k \\ [K_i, K_j] &= -\epsilon_{ijk}\, S_k \end{aligned} \tag{11.99}$$

where the commutator is $[A, B] = AB - BA$. The first implies that two rotations are equivalent to a rotation in the third coordinate, the second implies that a boost following a rotation is equivalent to a boost in the perpendicular direction, and the third implies that two successive boosts are equivalent to a rotation about a perpendicular axis.