Mid-Term, Next Wednesday, Feb. 05; 3 – 4:30 . Please note time.

If you can’t make it at that time, please see me to make other arrangements

Last time we looked at *toy universes* with a *single component*. While we gained insight into some of the behavior of the universe, we'd now like to get more realistic by looking at *multi-component universes.*

We will first get some general results for multi-component universes. Then, much like we did with single components, we'll explore different combinations.

We begin as usual with the Friedmann equations written in terms of the Hubble parameter:

$$H(t)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{kc^2}{R_o^2 a(t)^2} \qquad (1)$$

It turns out to be convenient to write Eq. (1) in a different form. Do question (1) on the worksheet to arrive at this form.

(1a) $$H(t)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{H_o^2}{a(t)^2}(\Omega_o - 1)$$

(1b) $$\frac{H(t)^2}{H_o^2} = \frac{\epsilon(t)}{\epsilon_{c,o}} - \frac{\Omega_o - 1}{a(t)^2}$$

(1c) $$\frac{H(t)^2}{H_o^2} = \frac{\Omega_{r,o}}{a^4} + \frac{\Omega_{m,o}}{a^3} + \Omega_{\Lambda,o} + \frac{1-\Omega_o}{a^2}$$

Note that $\Omega_o = \Omega_{r,o} + \Omega_{m,o} + \Omega_\Lambda = \underbrace{1}_{\text{current observations}}$ which means the last term in (1c) is 0. But we'll carry it along in case observations are incorrect.

We've re-written the Friedmann equations as

$$\frac{H(t)^2}{H_o^2} = \frac{\Omega_{r,o}}{a^4} + \frac{\Omega_{m,o}}{a^3} + \Omega_{\Lambda,o} + \frac{1-\Omega_o}{a^2} \qquad (1)$$

We can make *some* progress now by recalling that $H = \frac{\dot{a}}{a}$ and multiplying the Friedmann equation by a^2 gives

$$\frac{\dot{a}}{H_o} = \left[\frac{\Omega_{r,o}}{a^2} + \frac{\Omega_{m,o}}{a} + \Omega_{\Lambda,o}a^2 + (1-\Omega_o)\right]^{1/2}$$

Which can be formally integrated to give $t = \frac{1}{H_o}\int_0^a \frac{da'}{\left[\Omega_{r,o}/a^2 + \Omega_{m,o}/a + \Omega_{\Lambda,o}a^2 + (1-\Omega_o)\right]^{1/2}}$ (2)

This integral has no simple analytic solution and normally has to be found numerically.

You'll now go through a series of universes containing pairs of different components. In each of these, you will use Eqs (1) and (2) to gain insight into how these universes behave.

Do question 2a only on the worksheet and S T O P

(2a) $\Omega_{r,o} = 0, \Omega_{\Lambda,o} = 0$ so $\frac{H^2(t)}{H_o^2} = \frac{\Omega_{m,o}}{a^3} + \frac{1-\Omega_o}{a^2}$

Do question (2b) and (2c) on the worksheet and S T O P

(2b) For a universe initially expanding to then stop expanding, it must be true that at some point, $H_o = 0$. The Friedmann equation then becomes

$$0 = \underbrace{\frac{\Omega_{m,o}}{a_{max}^3}}_{\text{positive}} + \underbrace{\frac{1-\Omega_o}{a_{max}^2}}_{\text{negative}} \quad ; \quad \Omega_o > 1 \Rightarrow k = 1$$

So this type of universe will have positive curvature.

(2c) $$a_{max} = \frac{\Omega_{m,o}}{1-\Omega_o}$$

So, a universe consisting of matter and curvature that is initially expanding, will at some point stop and then begin contracting.

The contraction will be symmetrical in time because of H^2 appearing in the Friedmann equations. Does this mean that *history will repeat itself?*

Do questions (2d) and (2e) on the worksheet and S T O P

(2d) $$\frac{H^2(t)}{H_o^2} = \underbrace{\frac{\Omega_{m,o}}{a_{max}^3}}_{\text{positive}} + \underbrace{\frac{1-\Omega_o}{a_{max}^2}}_{\text{positive}} \Rightarrow$$ Universe will expand forever

(2e) $$t = \frac{1}{H_o}\int_0^a \frac{da'}{[\Omega_{m,o}/a + (1-\Omega_o)]^{1/2}}$$

The integral, $t = \dfrac{1}{H_o}\displaystyle\int_0^a \dfrac{da'}{[\Omega_{m,o}/a + (1-\Omega_o)]^{1/2}}$ is best solved *parametrically*. For $\Omega_o > 1$, we obtain:

$$\begin{aligned} a(\theta) &= \frac{\Omega_{m,o}}{2(\Omega_o - 1)}\,(1-\cos\theta) \\ t(\theta) &= \frac{\Omega_{m,o}}{2H_o(\Omega_o-1)^{3/2}}\,(\theta - \sin\theta) \end{aligned}$$

$$0 \geq \theta \geq 2\pi$$

For this universe, when $\theta = 0$ we have a *Big Bang*. But $a(\theta)$ returns to 0 when $\theta = 2\pi$. This is called the *Big Crunch*.

The time to the big crunch can be calculated using the expression for time above. We obtain:

$$t_{\rm crunch} = \frac{\pi\,\Omega_{m,o}}{H_o(\Omega_o-1)^{3/2}}$$

For $\Omega_o < 1$, we obtain:

$$\begin{aligned} a(\eta) &= \frac{\Omega_{m,o}}{2(\Omega_o - 1)}\,(\cosh\eta - 1) \\ t(\eta) &= \frac{\Omega_{m,o}}{2H_o(\Omega_o-1)^{3/2}}\,(\sinh\eta - \eta) \end{aligned}$$

$$0 \geq \eta \geq \infty$$

Finish question (2) on the worksheet and S T O P

We now study a 2-component universe that is very close to the one we seem to be living in. Specifically, we now consider a spatially flat universe consisting of Lambda + Matter.

We'll begin our study by reminding ourselves that for a spatially flat universe,

$$\Omega_{r,o} + \Omega_{m,o} + \Omega_\Lambda = 1.$$

Do question (3a) and (3b) on the worksheet and S T O P

(3a) $\Omega_\Lambda = 1 - \Omega_{m,o}.$

(3b) $$\frac{H^2(t)}{H_o} = \underbrace{\frac{\Omega_{m,o}}{a^3}}_{\text{matter \& positive}} + \underbrace{(1 - \Omega_{m,o})}_{\text{Lambda}}$$

$$\underbrace{(1 - \Omega_{m,o})}_{\text{Lambda}} > 0 \text{ if } \Omega_{m,o} < 1 \Rightarrow \Omega_\Lambda > 0.$$

$$\underbrace{(1 - \Omega_{m,o})}_{\text{Lambda}} < 0 \text{ if } \Omega_{m,o} > 1 \Rightarrow \Omega_\Lambda > 0.$$

Do questions (3c) and (3d) on the worksheet and S T O P

(3c,d) The universe will eventually start to contract when *H = 0*. At that point $a_{\max} = \left(\frac{\Omega_{m,o}}{\Omega_{m,o} - 1}\right)^{1/3}$

As before one can solve for the time when *a = 0* again and find that $t_{\rm crunch} = \dfrac{2\pi}{3H_o\sqrt{\Omega_{m,o}-1}}$

For this universe, the Friedmann equations can be directly integrated so that

$$H_o t = \frac{2}{3\sqrt{\Omega_{m,o}-1}}\ \sin^{-1}\left[\left(\frac{a}{a_{\rm max}}\right)^{3/2}\right]$$

For the case in which $\Omega_{m,o} < 1; \Omega_\Lambda > 0$, $H_o t = \dfrac{2}{3\sqrt{\Omega_{m,o}-1}}\ \ln\left[\left(\dfrac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1+\left(\dfrac{a}{a_{m\Lambda}}\right)^3}\right]$ where $a_{m\Lambda} = \left(\dfrac{\Omega_{m,o}}{1-\Omega_{m,o}}\right)^{1/3}$ $(\epsilon_m = \epsilon_\Lambda)$

Finish question (3) on the worksheet

(3e) $t_o = \dfrac{2}{3H_o\sqrt{1-\Omega_{m,o}}}\ln\left[\dfrac{\sqrt{1-\Omega_{m,o}}+1}{\sqrt{\Omega_{m,o}}}\right]$

(3f) $t_o = 0.955\ H_o^{-1} = 13.74\ \mathrm{Gyr}$

(3g) $t_{m\Lambda} = \dfrac{2}{3H_o\sqrt{1-\Omega_{m,o}}}\ln\left[1+\sqrt{2}\right] = 0.707\,H_o^{-1} = 10.17\ \mathrm{Gyr}$

Do question (4) on the worksheet and work on homework