1 Numerical Solutions to ODEs

Much of physics involves obtaining the solution to ordinary or partial differential equations. We first look ordinary differential equations (ODEs).

We begin our exploration of ODEs by first examining simple ODEs qualitatively using the following example,

$$\dot{y} \equiv \frac{dy}{dt} = -2t^2 y. \tag{1}$$

(1) Consider the plane made up of (t, y) and suppose $y(t)$ is a solution to Eq.(1). Further suppose that $y(t)$ passes through the point, (t_1, y_1) on the (t, y) plane.

   (i) Geometrically, what does Eq. (1) give at the point (t_1, y_1).
   Slope

   (ii) Using figure 1 on the next page, arbitrarily pick several (t_i, y_i) and draw a short line segment at each point that faithfully represents the slope at that point(the slope as given by Eq. (1)). Given an initial point, can you qualitatively obtain a solution to Eq. (1) and if so, describe the process.

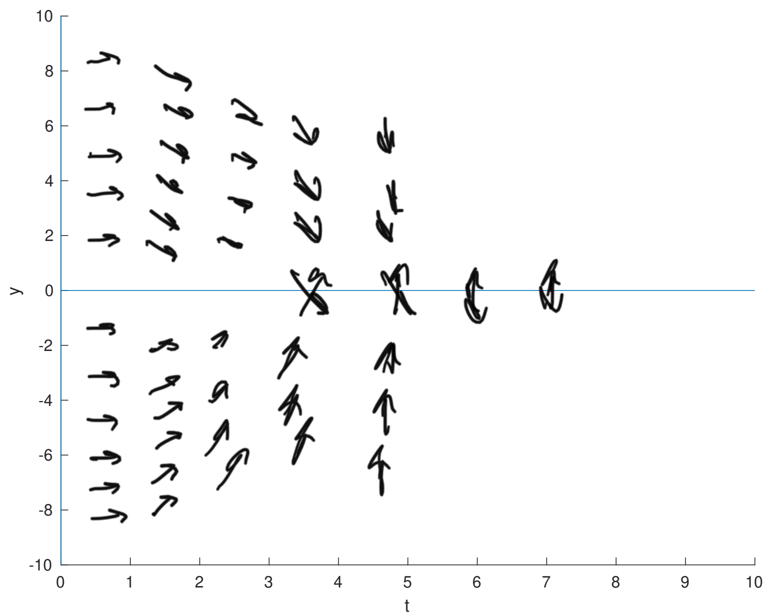

Figure 4.1 in the Book

Figure 1: Sketch qualitative solutions to Eq.(1).

(2) Consider the ODE,

$$\frac{dy}{dt} = \exp(-t)\cos y.$$

Find the first three equilbrium points. Sketch a phase line and determine what kind of equilibrium points they are.

(3) The sketch in (1) is called a *slope field* and is a qualitative solution to the ODE. Notice that the solution merely involves plotting points and so is very simple, and gives the general behavior of the ODE. However, it does not actually give numerical answers to the ODE. That is, given a specific t, one cannot extract an exact value for $y(t)$. To find actual values of $y(t)$ we need to obtain an explicit solution. Eq. (1) is "solvable" in prinicple, how could one solve ODEs of the type exemplified by Eq.(1). Note I'm not asking about numerical techniques, just a general descriptionof how to solve Eq. (1).

$dy/dt = f(t)g(y) \rightarrow$ Separable

$\rightarrow dy/dt = f(t)g(y) \rightarrow dy/g(y) = f(t)dt$

$\rightarrow \int dy/g(y) = \int f(t)\, dt$

(4) Unfortunately, most ODES cannot be integrated and so we must resort to approximation techniques. In question (1), you exploited the fact that geometrically, the first derivative is the slope at a point. We will use this to develop our first algorithm for solving ODEs numerically as follows

(i) We know that that geometrically, the derivative is the slope at a point. To approximate the derivative at a point, x_o, we can replace the derivative by the slope of a secant line between two points close to x_o. Given an ODE of the form,

$$\frac{dy}{dt} = f(y, t)$$

write an expression that approximates this using the slope of a secant line instead.

$\frac{dy}{dt} = Sec(y \cdot t)$

(ii) To actually "solve" for the values of y, what information do you need.

$\frac{dy(t)}{dt} \approx \frac{\Delta y(t)}{\Delta t} = f(t, y)$ $\quad y_1 = y_0 + f(t_0, y_0)\Delta t$

$\vdots$

$y_f = y_0 + f(t_0, y_0)(t_f - t_0)$ $\quad y_n = y_{n-1} + f(t_{n-1}, y_{n-1})\Delta t$

(iii) Using an initial starting point, $(t, y) = (0, 3)$ and a $\Delta t = 0.5$ construct a table that approximateluy "solves" Eq(1) by replacing derivative with slopes of secant lines. The table should contain, at a minimum, columns, t and y.

(5) The plot of the exact solution to Eq. (1), $(t_o, y_o) = (0, 3)$ is shown in figure 2. On this figure, use the table you constructed in (4.iii) to sketch your solution. This is a first crude approximation of the 'true solution'.

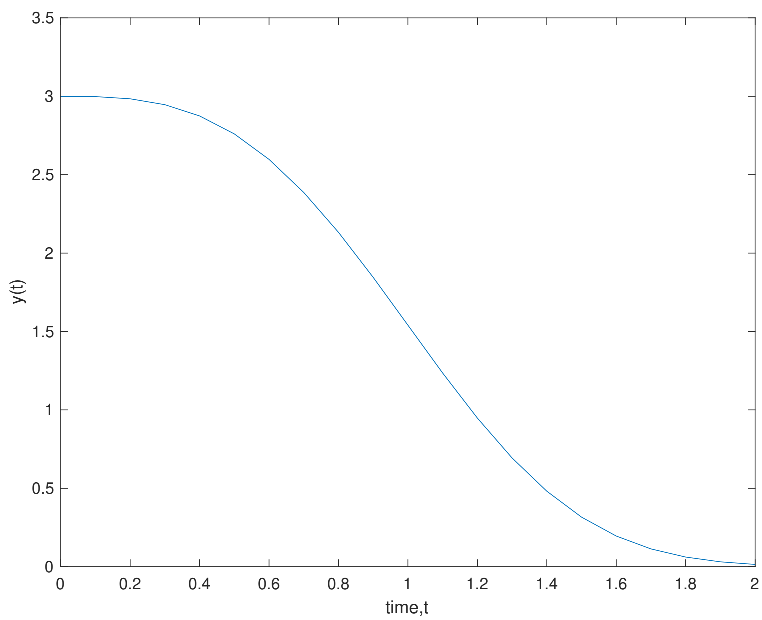

Figure 2: The exact solution to Eq.(1). On this figure, sketch how your method approximates this exact solution

(6) Does the approximation start to deviate greatly from the exact solution and if so, where. Explain why the deviation occurs. How might you make the approximation better, that is, closer to the exact solution.

(7) The code `myodesolver` is located on the Teams page. What is lacking in the code are the external functions `myeuler, myeuler_mid, myeuler_improve`. Write those functions and then solve the ODE given in problem 2 using all three methods. Use the initial condition, $t_o = 0, y = 1$.

(8) Let's take another look at both Euler's and the modified Euler methods.

(i) Figure 3 shows the initial condition, $(t, y) = (t_o, y_o)$ for an arbitrary ODE, $dy/dt = f(t, y)$. Using Euler's method, sketch the location of y_1 at $t = t_o + h$.

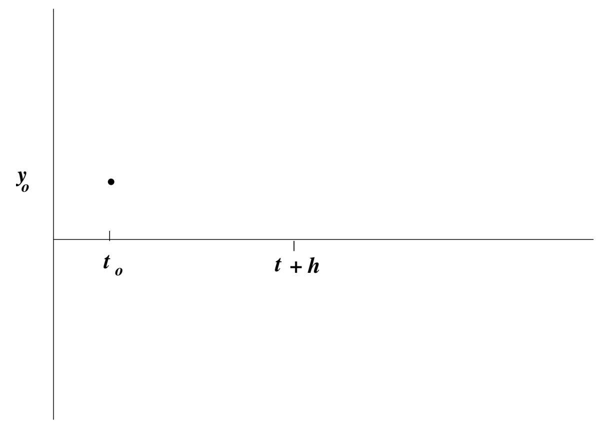

Figure 3: Intial point for solution

(ii) On the same figure, use Euler's modified method to find the location of y_1 at $t = t_o + h$.

(iii) This type of procedure, estimating the next data point by taking the slopes at intermediate points, is an example of general methods called *Runge-Kutta* techniques. Euler's methods are first order Runge-Kutta methods. The most commonly used of these methods is the fourth order Runge-Kutta method (fourth order means that the error in the solution is on the order of h^4). The solution is found by use of intermediate points (as in the modified Euler method).

The intermediate points are give by

$$\begin{aligned} f_o &= f(t_o, y_o) \\ f_1 &= f(t_o + \tfrac{h}{2}, y_o + \tfrac{h}{2} f_o) \\ f_2 &= f(t_o + \tfrac{h}{2}, y_o + \tfrac{h}{2} f_1) \\ f_3 &= f(t_o + h, y_o + h f_2) \end{aligned}$$

The solution is then

$$y(x_o + h) = y(x_o) + \frac{h}{6}\left(f_o + 2f_1 + 2f_2 + f_3\right).$$

Add this algorithm to your `myodesolver` routine.

(9) Consider the ODE,

$$\frac{dy}{dt} = e^t \sin(y)$$

Solve this ODE using Euler's method, the improved Euler method, and the Runge=Kutta 4th order method. Use and $h = 0.05, 0.1, 0.2$ and initial condition $y(0) = 3$. Is there a difference in various methods? Solve the problem for $0 \le t < 5$, then again for $0 \le t \le 10$. Try to explain any differences that arise during the two different time spans. This ODE is separable and has a solution

$$y(t) = 2\cot^{-1}\big(\exp(c_1 - e^t)\big)$$

where c_1 is a constant determined by the initial conditions. Plot both your solution and the analytic solution and comapre the two.