Physics 412—Harmonic Oscillator Dynamics **Winter 2020**

Suppose that the state of a quantum harmonic oscillator at time $t = 0$ is

$$|\Psi_0\rangle = c_m|m\rangle + c_n|n\rangle.$$

Assume that the state is normalized so that $|c_m|^2 + |c_n|^2 = 1$.

(1) Find $|\Psi(t)\rangle$. $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi_0\rangle =$

(2) If you measure the energy at time t, what values can you measure and with what probabilities?

(3) Calculate $\langle X\rangle$ and $\langle P\rangle$ as functions of time for this state. How do your results depend on the relative values of m and n? For this part you'll probably want to use

$$X = \frac{d_0}{\sqrt{2}}\left(a + a^\dagger\right), \quad P = \frac{-i\hbar}{\sqrt{2}d_0}\left(a - a^\dagger\right).$$

(4) Discuss how the values of c_m and c_n affect $\langle X\rangle$ and $\langle P\rangle$ at time $t = 0$. How would you pick c_m and c_n to force $\langle P\rangle$ or $\langle X\rangle$ to be zero when $t = 0$? How would you pick c_m and c_n to force $\langle X\rangle$ or $\langle P\rangle$ to be nonzero at $t = 0$?

(1) $|\psi(t)\rangle = c_m e^{-iE_m t/\hbar}|m\rangle + c_n e^{-iE_n t/\hbar}|n\rangle$

$E_m = \hbar\omega_0(m + 1/2) \leftarrow$ $\quad e^{-i\omega_0 t}e^{-im\omega_0 t} = e^{-in\omega_0 t}$ $\quad n = m+1$

$$|\psi(t)\rangle = e^{-i\omega_0 t/2}\left[c_m e^{-im\omega_0 t}|m\rangle + c_n e^{-in\omega_0 t}|n\rangle\right]$$

(2) measure E_m with probability $|c_m|^2$
E_n with probability $|c_n|^2$ } all times t

(3) $$X|\psi(t)\rangle = e^{-i\omega_0 t/2}\frac{d_0}{\sqrt{2}}\left[c_m e^{-im\omega_0 t}(a + a^\dagger)|m\rangle + c_n e^{-in\omega_0 t}(a + a^\dagger)|n\rangle\right]$$

$$\langle X \rangle = \frac{d_0}{\sqrt{2}} \left(C_m^* e^{imw_0 t} \langle m| + C_n^* e^{inw_0 t} \langle n| \right) \times$$

$$\left(C_m e^{-imw_0 t} (a + a^\dagger) |m\rangle + C_n e^{-inw_0 t} (a + a^\dagger) |n\rangle \right)$$

$$\boxed{\langle x \rangle = 0 \quad \text{unless } n = m \pm 1}$$

Since m & n are arbitrary, assume $m < n$, so $n = m+1$

$$\langle X \rangle = \frac{d_0}{\sqrt{2}} \left(C_m^* \langle m| + C_{m+1}^* e^{iw_0 t} \langle m+1| \right) x$$

$$\left(C_m (a + a^\dagger) |m\rangle + C_{m+1} (a + a^\dagger) e^{-iw_0 t} |m+1\rangle \right)$$

$$\langle m | a | m \rangle = 0, \quad \langle m | a^\dagger | m \rangle = 0$$

$$\langle m+1 | a | m+1 \rangle = 0, \quad \langle m+1 | a^\dagger | m+1 \rangle = 0$$

$$\langle m | a | m+1 \rangle = \sqrt{m+1}, \quad \langle m+1 | a^\dagger | m \rangle = \sqrt{m+1}$$

$$\boxed{\langle X \rangle = \frac{d_0}{\sqrt{2}} \left(C_m^* C_{m+1} e^{-iw_0 t} \sqrt{m+1} + C_m C_{m+1}^* e^{iw_0 t} \sqrt{m+1} \right)}$$

For $\langle p \rangle$, replace $\frac{d_0}{\sqrt{2}}$ by $\frac{-i\hbar}{\sqrt{2} d_0}$ and $+a^\dagger$ by $-a^\dagger$

$$\boxed{\langle p \rangle = \frac{-i\hbar}{\sqrt{2}\, d_0} \left(-C_m^* C_{m+1} e^{-iw_0 t} \sqrt{m+1} + C_m C_{m+1}^* e^{iw_0 t} \sqrt{m+1} \right)}$$

(d) At time $t=0$, we have

$$\langle X \rangle = \frac{d_0}{\sqrt{2}}\sqrt{m+1}\left(c_m^* c_{m+1} + c_m c_{m+1}^*\right)$$

For any complex number z, $z + z^* = 2\,\mathrm{Re}\,z$

$$\therefore \langle X \rangle = \sqrt{2}\, d_0 \sqrt{m+1}\, \mathrm{Re}\left(c_m c_{m+1}^*\right)$$

To make this zero, the product $c_m c_{m+1}^*$ must be imaginary. So we could pick c_m to be real & c_{m+1} imaginary.

To make this non-zero, the product $c_m c_{m+1}^*$ must have a non-zero real part.

At $t=0$,

$$\langle P \rangle = \frac{i\hbar}{\sqrt{2}\, d_0}\sqrt{m+1}\left(c_m^* c_{m+1} - c_m c_{m+1}^*\right)$$

For any complex number z, $z - z^* = 2i\,\mathrm{Im}\,z$

$$\therefore \langle P \rangle = -\frac{\sqrt{2}\,\hbar}{d_0}\sqrt{m+1}\, \mathrm{Im}\left(c_m^* c_{m+1}\right)$$

To make this zero, the product $c_m^* c_{m+1}$ must be real.

To make this non-zero, this product must have an imaginary part.