PHY 420 – Electrodynamics II

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In Preparation for Midterm Exam

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Warning!

This is *not a substitute* for class summaries and homework solutions. It is only a list of ideas and/or equations I want you to remember. If this is the only thing you look at, expect to do very badly on the exam!

Caveat: In the following slides, I've mentioned several relations that will be supplied on formula sheets. To avoid any confusion, I'd like to add here that *an exception would be if I asked you to derive any of these relations on the test, in which case, they would obviously not be on the Formula Sheet.*

Electrostatic Energy

(Week 1 – Day 1 Class Summary & Worksheet)

Several slides will refer to the day of the Class Summary here

- Recall that the scalar potential has a physical interpretation when we consider the work done on a test charge q in transporting it from a point A to another point B in the presence of an electric field $\vec{E}(\vec{x})$: $q\Phi$ can be interpreted as the potential energy of the test charge in the electrostatic field.

- So, if a point charge q_i is brought from infinity to a point $\vec{x}_i$ in a region of localized electric fields described by the scalar potential Φ (which vanishes at infinity), the work done on the charge (and hence its potential energy) is given by

$$W_i = q_i \Phi(\vec{x}_i) \tag{1.47}$$

- If the potential Φ is produced by an array of $(n-1)$ charges q_j at positions $\vec{x}_j$, where $j = 1, 2, 3, \ldots, (n-1)$, then

$$\Phi(\vec{x}_i) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|} \tag{1.48}$$

Electrostatic Energy
(Week 1 – Day 1 Class Summary, page 2)

- By adding each charge in succession, the total potential energy of all the charges due to all the forces acting between them, written in a symmetric form with *i* and *j* unrestricted, and then dividing by 2, is

$$W = \frac{1}{8\pi\epsilon_0} \sum_i \sum_{\substack{j \\ i \neq j}} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \tag{1.51}$$

Both these equations will be on the Formula Sheet

- For a continuous charge distribution

$$W = \frac{1}{8\pi\epsilon_0} \int \int \frac{\rho(\vec{x})\, \rho(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x\, d^3x' \tag{1.52}$$

Since this is just $\Phi(\vec{x})/2$

Electrostatic potential energy is in terms of the positions of the charges

$$\boxed{W = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x} \tag{1.53}$$

Energy of a system of charges in free space

Electrostatic Energy

(Week 1 – Day 1 Class Summary, page 3)

- Can also interpret the energy as being stored in the electric field.

This expression will be provided on the Formula Sheet

$$W = \frac{\epsilon_0}{2} \int \left| \vec{E} \right|^2 d^3x \tag{1.54}$$

- Interpret integrand as the energy density

This will ***not*** be provided, since you can work it out from (1.54) above

$$w = \frac{\epsilon_0}{2} \left| \vec{E} \right|^2 \tag{1.55}$$

- Positive definite energy is due to contribution by self-energy terms.

Capacitance

(Week 1 – Day 1 Class Summary, page 4)

- For a system of n conductors, each with potential V_i and total charge Q_i, where i= 1, 2, 3,…, n, in otherwise empty space, the electrostatic potential energy can be expressed in terms of the potentials alone and certain geometrical quantities called *coefficients of capacity*.

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j \qquad (i = 1, 2, 3, \ldots, n) \tag{1.61}$$

- C_{ii} are called *capacitances*, C_{ij} $(i \neq j)$ are called *coefficients of induction*

- The potential energy for the system of conductors is

$$W = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} V_i V_j \tag{1.62}$$

Both (1.61) and (1.62) above will be provided on the Formula Sheet

Electrostatic Energy in Dielectric Media

(Week 1 – Day 1 Class Summary, pages 5-6)

- To be general, we won't make any assumptions initially about linearity or uniformity in the response of a dielectric to an applied field.

- Instead, consider a small change in the energy due to some sort of change in the macroscopic charge density that exists in all space.

$$\delta W = \int \delta\rho(\vec{x})\,\underbrace{\Phi(\vec{x})}_{\substack{\text{potential due to the}\\ \text{charge density } \rho(\vec{x}) \text{ already present}}}\,d^3x \qquad (4.84)$$

- We can then derive a formal expression for the total electrostatic energy:

$$W = \int d^3x \int_0^D \vec{E}\cdot\delta\vec{D}$$

- If the medium is linear, we get the total electrostatic energy:

$$W = \frac{1}{2}\int \vec{E}\cdot\vec{D}\,d^3x \qquad (4.89)$$

Electrostatic Energy in Dielectric Media

(Week 1 – Day 1 Class Summary, pages 6-7)

- A problem of considerable interest is the change in energy when a dielectric object is placed in an electric field whose sources are fixed.

- Suppose initially that there exists an electric field $\vec{E}_0$ due to a distribution of charges $\rho_0(\vec{x})$ in a medium with ϵ_0, which may be a function of position. The initial electrostatic energy is

$$W_0 = \frac{1}{2}\int \vec{E}_0 \cdot \vec{D}_0 \, d^3x$$

- Then, with the sources fixed in position, a dielectric object of volume V_1 is introduced into the field, changing the field from $\vec{E}_0$ to $\vec{E}$. Then $\epsilon(\vec{x})$ has the value ϵ_1 inside V_1 and ϵ_0 outside V_1. To avoid mathematical difficulties, we imagine $\epsilon(\vec{x})$ to be a smoothly varying function of position that falls rapidly but continuously from ϵ_1 to ϵ_0 at the edge of the volume V. The electrostatic energy is now

$$W_1 = \frac{1}{2}\int \vec{E} \cdot \vec{D} \, d^3x$$

- We then derive that

$$W = -\frac{1}{2}\int_{V_1} \vec{P} \cdot \vec{E}_0 \, d^3x$$

This expression will be provided on the Formula Sheet

where $\vec{P}$ is the (induced) polarization of the dielectric that has been introduced into the field. $\vec{P} = (\epsilon_1 - \epsilon_0)\,\vec{E}$

A dielectric object will tend to move toward regions of increasing $\vec{E}_0$, provided $\epsilon_1 > \epsilon_0$.

Electrostatic Energy in Dielectric Media

(Week 1 – Day 1 Class Summary, pages 7- 8)

- What, however, if the *potentials are kept fixed*?
- This is usually true in practical situations involving the motion of dielectrics, where the electric fields are often produced by a configuration of electrodes held at fixed potentials by connecting to an external source such as a battery.
- To maintain the potentials constant as the distribution of dielectric varies, charge will flow to or from the battery to the electrodes.
- This means that energy is being supplied from the external source, and we're now going to compare the energy supplied in that way with the energy change we calculated above for fixed sources of the field.
- The process of altering the dielectric properties in some way (by moving the dielectric bodies, by changing their susceptibilities, etc.) in the presence of electrodes at fixed potentials can be viewed as a two-step process.

Electrostatic Energy in Dielectric Media

(Week 1 – Day 1 Class Summary, page 8)

Two-step process for altering the dielectric properties at fixed potentials:

- In the first step, the electrodes are disconnected from their batteries and the charges on them held fixed $(\delta\rho = 0)$, so
$$\delta W_1 = \frac{1}{2}\int \rho\,\delta\Phi_1\,d^3x$$
- In the second step, the batteries are connected again to the electrodes to restore their potentials to the original values. There will be a flow of charge $\delta\rho_2$ from the batteries accompanying the change in potential $\delta\Phi_2 = -\delta\Phi_1$, which is necessary to re-establish the original potentials. So, the energy change in the second step is
$$\delta W_2 = \frac{1}{2}\int \Big(\rho\,\delta\Phi_2 + \Phi\,\delta\rho_2\Big)\,d^3x$$
- We showed that
$$\delta W_2 = -2\Big(\delta W_1\Big)$$
- So that, finally:
$$\delta W_V = -\delta W_Q$$

where the subscript denotes the quantity being held fixed. In other words, *the change in energy at fixed potentials is the negative of the energy change at fixed charges.*

Energy in the Magnetic Field

(Week 1 – Day 1 Class Summary, pages 9-12)

- The creation of a steady-state configuration of currents and associated magnetic fields involves an initial transient period during which the currents and fields are brought from zero to the final values. Since we have time-varying fields, and hence *dB*/*dt*, there are induced electromotive forces that cause the sources of current to do work. Since the energy in the field is, by definition, the total work done to establish the fields from a state of zero field, we must consider these contributions.

- We find that

$$\delta W = \int \vec{H} \cdot \delta \vec{B} \, d^3x \tag{5.147}$$

This relation is applicable to *all magnetic media*, including ferromagnetic substances.

- If we assume that the medium is paramagnetic or diamagnetic, so that a linear relation exists between $\vec{B}$ and $\vec{H}$, then

$$W = \frac{1}{2} \int \vec{H} \cdot \vec{B} \, d^3x \tag{5.148}$$

magnetic analog of (4.89): $W = \frac{1}{2} \int \vec{E} \cdot \vec{D} \, d^3x$

- Also

$$W = \frac{1}{2} \int \vec{J} \cdot \vec{A} \, d^3x \tag{5.149}$$

magnetic analog of (1.53): $W = \frac{1}{2} \int \rho(\vec{x}) \, \Phi(\vec{x}) \, d^3x$

Energy in the Magnetic Field

(Week 1 – Day 1 Class Summary, pages 9-12)

- When an object of permeability μ_1 is introduced into a region which has an existing magnetic induction $\vec{B}_0$ (and magnetic field $\vec{H}_0$), and permeability μ_0, then it can be verified that for fixed sources of the field

$$W = \frac{1}{2}\int_{V_1}\left(\vec{B}\cdot\vec{H}_0 - \vec{H}\cdot\vec{B}_0\right)d^3x$$

where $\vec{B}$ and $\vec{H}$ are the fields after the object is in place, and the integration is over the volume V_1 of the object.

- If the object is in otherwise free space, the change in energy can be expressed in terms of the magnetization as

This expression will be provided on the Formula Sheet

$$W = \frac{1}{2}\int_{V_1}\vec{M}\cdot\vec{B}_0\,d^3x \tag{5.150}$$

- This is equivalent to the electrostatic result in (4.93): $W = -\frac{1}{2}\int_{V_1}\vec{P}\cdot\vec{E}_0\,d^3x$

except for the sign. The sign is different because the energy W consists of the total energy change occurring when the permeable body is introduced into the field, including the work done by the sources against the induced electromotive forces. In this respect, the magnetic problem with fixed currents is analogous to the electrostatic problem with fixed potentials on the surfaces that determine the fields.

Energy conservation in the electromagnetic field
Poynting's Theorem

(Week 1 – Day 2 Class Summary & Worksheet)

Discussion begins from next slide.

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 1)

- Consider the force on a single charge q:

Must know: $$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

velocity of the charge

- Can use this to show that (as you did on worksheet):

$$\frac{dE_{\text{kin}}}{dt} = \vec{F} \cdot \vec{v} = q\vec{E} \cdot \vec{v}$$

- Can then write the rate of work done by the fields (see next slide).

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 1)

- Total rate of doing work by the fields in a finite volume:

You must know this:

$$\int_V \vec{J} \cdot \vec{E}\, d^3x \qquad (6.103)$$

- This power represents a conversion of electromagnetic energy into mechanical or thermal energy.

- It must be balanced by a corresponding rate of decrease of energy in the electromagnetic field within the volume V.

- To exhibit this conservation law explicitly, use the Maxwell equations to express (6.103) in other terms.

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 2)

$$\int_V \vec{J}\cdot\vec{E}\,d^3x = -\int_V \left[\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right] d^3x \qquad (6.105)$$

- Equation (6.105) represents the rate of decrease of energy in the electromagnetic field within the volume V, and this goes into increasing the mechanical or thermal energy of the moving charges.

You are not expected to memorize equation (6.105), but you must be able to write what it means.

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 3)

To proceed, we will make two assumptions:

- Assume that the macroscopic medium is linear in its electric properties, with negligible dispersion or losses, so that

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} \left(\vec{E} \cdot \vec{D} \right)$$

  and likewise, linear in its magnetic properties, again with negligible dispersion or losses, so that

$$\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} \left(\vec{B} \cdot \vec{H} \right)$$

- Assume also that the total electromagnetic energy, even for time-varying fields, is the sum of (4.89) and (5.148): $W = \frac{1}{2} \int \vec{E} \cdot \vec{D} \, d^3x$

$$W = \frac{1}{2} \int \vec{H} \cdot \vec{B} \, d^3x$$

  so that the total energy density is given by

$$u = \frac{1}{2} \left(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H} \right) \qquad (6.106)$$

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 3)

$$\int_V \vec{J}\cdot\vec{E}\,d^3x = -\int_V \left[\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right] d^3x \qquad (6.105)$$

$$-\int_V \vec{J}\cdot\vec{E}\,d^3x = \int_V \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\underbrace{\left(\vec{E}\times\vec{H}\right)}_{\vec{S}}\right] d^3x \qquad (6.107)$$

- Since the volume V is arbitrary, the integrand can be written in the form of a differential continuity equation or conservation law

Must know:

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\vec{S}\right) = -\vec{J}\cdot\vec{E} \qquad (6.108)$$

You must know the Poynting vector

$$\text{where } \vec{S} = \vec{E}\times\vec{H} \qquad (6.109)$$

The vector $\vec{S}$ represents energy flow, and is called the *Poynting vector*

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 4)

$$-\int_V \vec{J}\cdot\vec{E}\,d^3x = \int_V \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right)\right] d^3x \tag{6.107}$$

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\vec{S}\right) = -\vec{J}\cdot\vec{E} \tag{6.108}$$

Poynting vector

Consider the physical meaning of either (6.107) or (6.108).

- Either equation tells us that the time rate of change of electromagnetic energy within a certain volume, plus the energy per unit time flowing out through the boundary surfaces of the volume, is equal to the negative of the total work done by the fields on the sources within the volume.

  This is a statement of the *conservation of energy*.

Conservation of Linear Momentum

(Week 2 – Day 1 Class Summary, page 1)

- Consider again the force on a single charge q:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Newton's 2nd Law

$$\frac{d\vec{P}_{\mathrm{mech}}}{dt} \qquad\qquad \rho\, d^3x \qquad\qquad \vec{J}\, d^3x$$

$$\frac{d\vec{P}_{\mathrm{mech}}}{dt} = \int_V \left(\rho\vec{E} + \vec{J} \times \vec{B}\right) d^3x \tag{6.114}$$

Conservation of Linear Momentum

(Week 2 – Day 1 Class Summary, pages 2-3)

- Equations are messier for this part, but when we assign the following volume integral as the total *electromagnetic field momentum*

$$\vec{P}_{\text{field}} = \epsilon_0 \int\limits_V \vec{E} \times \vec{B}\, d^3x = \mu_0 \epsilon_0 \int\limits_V \vec{E} \times \vec{H}\, d^3x \tag{6.117}$$

with the integrand interpreted as *electromagnetic momentum density*

$$\vec{g} = \frac{1}{c^2}\left(\vec{E} \times \vec{H}\right) \tag{6.118}$$

we obtain a cleaner-looking equation for the α^{th} component of the change in momentum:

$$\frac{d}{dt}\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_\alpha = \sum_\beta \int\limits_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\, d^3x \tag{6.121}$$

although, of course, some of the messiness has been hidden in the $T_{\alpha\beta}$ term, the so-called *Maxwell Stress Tensor*

All expressions on this page will be provided on the Formula Sheet if needed

Maxwell Stress Tensor

(Week 2 – Day 1 Class Summary, pages 4–5)

In the α^{th} component of the change in momentum:

$$\frac{d}{dt}\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_\alpha = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\, d^3x \qquad (6.121)$$

we have defined the *Maxwell Stress Tensor*

$$\boxed{T_{\alpha\beta} = \epsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2 \vec{B}\cdot\vec{B}\right)\delta_{\alpha\beta}\right]} \qquad (6.120)$$

We can also write the α^{th} component of the change in momentum as

$$\frac{d}{dt}\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_\alpha = \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta\, da \qquad (6.122)$$

where $\hat{n}$ is the outward normal to the closed surface S

All expressions on this page will be provided on the Formula Sheet if needed

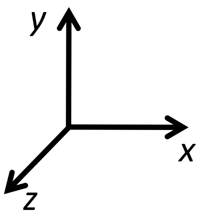
Maxwell Stress Tensor

- In the stress tensor, α and β refer to coordinates x, y, z, and so $T_{\alpha\beta}$ has 9 components.

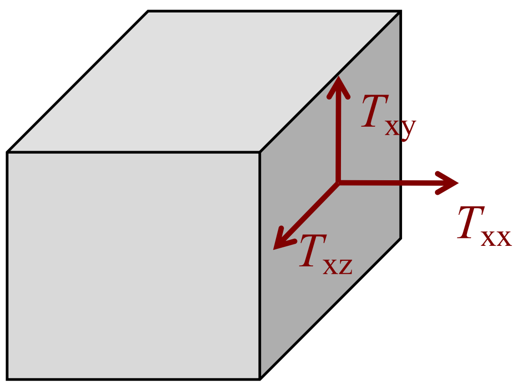
- From

$$\frac{d}{dt}\left(\vec{P}_{\mathrm{mech}} + \vec{P}_{\mathrm{field}}\right)_{\alpha} = \oint_S \sum_{\beta} T_{\alpha\beta}\, n_{\beta}\, da$$

  we see that the stress tensor tells us the force per unit area (or stress) acting on the surface.



- More precisely, $T_{\alpha\beta}$ is the force per unit area in the α^{th} direction acting on element of the surface oriented in the β^{th} direction.



- Diagonal elements are pressures, and off-diagonal elements are shears.

Conservation of Linear Momentum

$$\frac{d}{dt}\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_\alpha = \oint_S \boxed{\sum_\beta T_{\alpha\beta}\, n_\beta}\, da \qquad (6.122)$$

α^{th} component of the flow per unit area of momentum across the surface S into the volume V

In other words, it is the force per unit area transmitted across the surface S and acting on the combined system of particles and fields inside V

Laplace Equation in Cylindrical Coordinates

(Week 2 – Day 2 & Week 3 – Day 1

Class Summary & Worksheet)

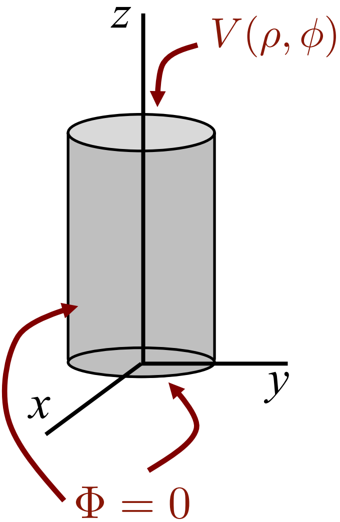
Laplace Equation in Cylindrical Coordinates

(3.71) will be provided, if needed

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \qquad (3.71)$$

You must be able to write everything on this slide below this point.

$$\Phi(\rho,\phi,z)=R(\rho)\,Q(\phi)\,Z(z) \qquad (3.72)$$



Type A problem (In-class)

$$\frac{d^2Z}{dz^2}-k^2Z=0$$

solution

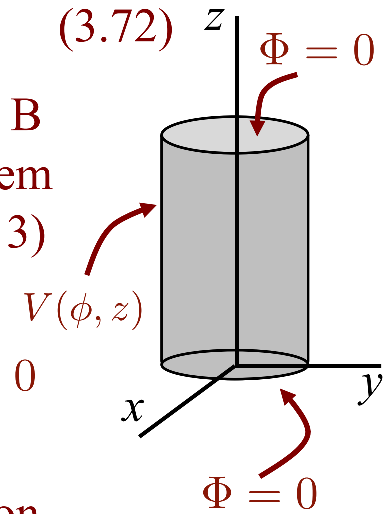
$$Z(z)=e^{\pm kz}$$

Type B problem (HW 3)

$$\frac{d^2Z}{dz^2}+k^2\,Z=0$$

solution

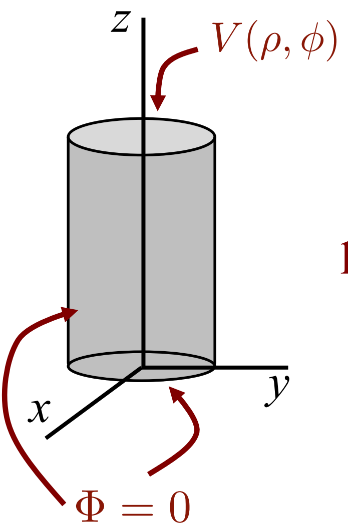
$$Z(z)=e^{\pm ikz}$$



Recall that there are *two kinds* of such problems; I've called them Type A and Type B here to help you remember.

Type A problem

(Week 2 – Day 2 & Week 3 – Day 1 Class Summary & Worksheet)



$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad (3.71)$$

$$\Phi(\rho, \phi, z) = R(\rho)\, Q(\phi)\, Z(z) \qquad (3.72)$$

Type A problem (In-class)

Other than (3.71), (3.75), & (3.77), which will be provided if needed, you must be able to write everything else on this slide.

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \xrightarrow{\text{solution}} Z(z) = e^{\pm kz} \qquad (3.73)$$

$$\frac{d^2 Q}{d\phi^2} + \nu^2 Q = 0 \xrightarrow{\text{solution}} Q(\phi) = e^{\pm i\nu\phi} \qquad (3.74)$$

$\nu = m$ (i.e., integer)

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0 \qquad (3.75)$$

change form

$$\frac{d^2 R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0 \xrightarrow{\text{solution}} R(\rho) = \boxed{C\, J_m(k\rho)} + D N_m(k\rho)$$

Bessel equation

(3.77)

Use only this inside cylinder

Be sure you can solve for *C*, *D*

Solution is in terms of **Bessel functions**. We discussed their properties, including recursion relations, etc.

Type B problem

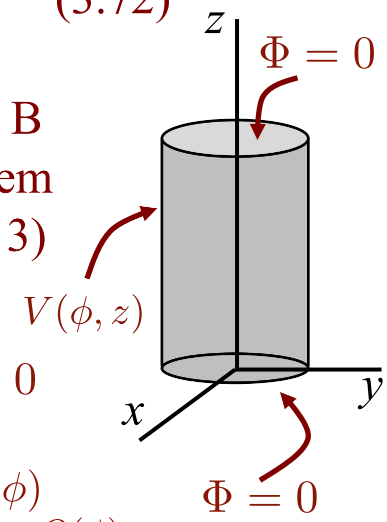
(Homework 3 – Problems 1 & 2)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \qquad (3.71)$$

$$\Phi(\rho,\phi,z)=R(\rho)\,Q(\phi)\,Z(z) \qquad (3.72)$$

Other than (3.71), you must be able to write everything else on this slide.

Type B problem (HW 3)



$$Z(z)=e^{\pm ikz} \xleftarrow{\text{solution}} \frac{d^2Z}{dz^2}+k^2Z=0$$

$$Q(\phi)=\sin m\phi+\cos m\phi \xleftarrow{\text{solution}} Q(\phi)$$

Same equation for $Q(\phi)$

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}-\left(1+\frac{\nu^2}{x^2}\right)R=0$$

Modified Bessel equation

solution

$$R(\rho)=C\,I_m(k\rho)+D\,K_m(k\rho)$$

Solution is in terms of **Modified Bessel functions**

Be sure you can solve for *C, D*

Green Functions

(Week 3 – Day 2 Class Summary & Worksheet)

- Green functions provide a technique for solving differential equations

$$\mathcal{D}\Psi(\vec{x}) = f(\vec{x}) \qquad \text{(W3.1)}$$

Any differential operator

e.g., $\nabla^2 + k^2$

solution

$$\Psi(\vec{x}) = \Psi_h + \Psi_{\text{part}} \qquad \text{(W3.2)}$$

Green Functions

Wish to solve:

$$\mathcal{D}\Psi(\vec{x}) = f(\vec{x}) \quad \text{(W3.1)}$$

Solution

Associated inhomogenous equation

$$\mathcal{D}G(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}') \quad \text{(W3.2)}$$

Solve for $G(\vec{x},\vec{x}')$

$$\Psi(\vec{x}) = \Psi_h + \int_V G(\vec{x},\vec{x}')\, f(\vec{x}')\, d^3x'$$

Green Functions in Electrostatics

Poisson equation

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

Green function equation

$$\nabla^2 G(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}')$$

Showed in previous class that

$$\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = -4\pi\delta(\vec{x} - \vec{x}')$$

$$\Downarrow$$

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|}$$

Constructing the solution:

$$\Psi(\vec{x}) = \Psi_h + \int_V G(\vec{x}, \vec{x}')\, f(\vec{x}')\, d^3x'$$

$$\Phi(\vec{x}) = \Phi_h(\vec{x}) + \int_V G(\vec{x}, \vec{x}') \left[\frac{\rho(\vec{x}')}{\epsilon_0} \right] d^3x'$$

Ignore Φ_h, assuming no boundary surfaces in problem

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x'$$

Green Functions for the Wave Equation

Wave equation

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\vec{x},t)$$

Green function equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_k^{(\pm)}(\vec{x},t;\vec{x}',t') = -4\pi\delta(\vec{x}-\vec{x}')\delta(t-t')$$

Given the wave equation, you must be able to write the Green function equation, and also the Green function.

$G^{(+)}$ is the retarded Green function, and $G^{(-)}$ is the advanced Green function

$$G^{(\pm)}(\vec{x},t;\vec{x}',t') = \frac{\delta\left(t' - \left[t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right]\right)}{|\vec{x}-\vec{x}'|}$$

Constructing the particular solution:

$$\Psi(\vec{x},t) = \Psi_{\text{in}}(\vec{x},t) + \int\int G^{(+)}(\vec{x},t;\vec{x}',t')\, f(\vec{x}',t')\, d^3x'dt' \quad (6.45)$$

Chapter 9: Radiation

(Week 4 – Day 1 Class Summary & Worksheet)

- We'll assume that the sources are radiating in otherwise empty space, i.e., no boundaries or materials present.

- Assume harmonic time dependence:

$$\begin{aligned} \rho(\vec{x},t) &= \rho(\vec{x})\, e^{-i\omega t} \\ \vec{J}(\vec{x},t) &= \vec{J}(\vec{x})\, e^{-i\omega t} \end{aligned} \tag{9.1}$$

- Electromagnetic fields and potentials also have the same time dependence.

Chapter 9: Radiation

(Week 4 – Day 1 Class Summary, pages 1-4)

- We used a Green function technique to solve the wave equation for $\vec{A}$ and obtained

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \, d^3x' \qquad (9.3)$$

Equation (9.3) will be supplied, if needed

- Equation (9.3) is useful because we can use it to find the fields:

Must know: $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} \qquad (9.4)$$

Should be able to work this out: $$\vec{E} = \frac{iZ_0}{k} \vec{\nabla} \times \vec{H} \qquad (9.5)$$

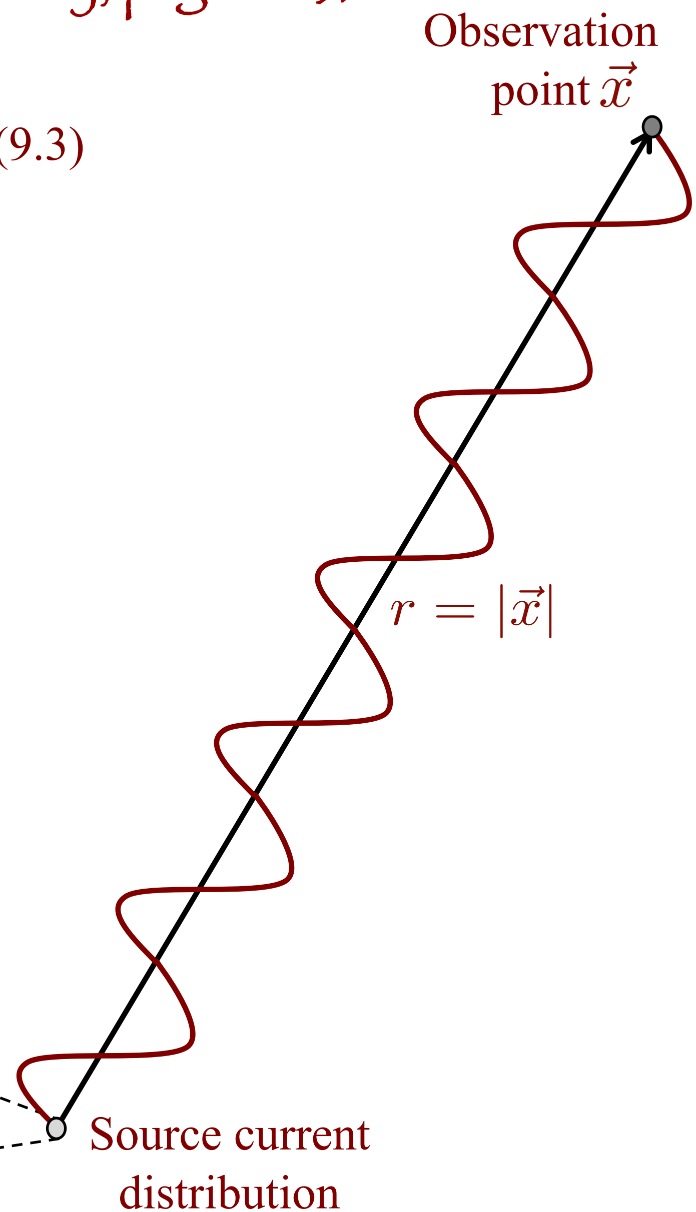
Three zones of interest

(Week 4 – Day 1 Class Summary, pages 4-5)

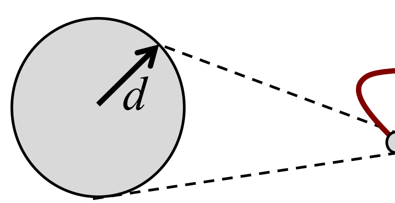
$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\,\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x' \qquad (9.3)$$

- Won't integrate (9.3) directly. Instead ...

- The near (static) zone: $d \ll r \ll \lambda$
- The intermediate zone: $d \ll r \sim \lambda$
- The far (radiation) zone: $d \ll \lambda \ll r$



Long wavelength approximation: $d \ll \lambda$



Consequences .....

(Week 4 – Day 1 Class Summary, page 6)

- After appropriate expansions, we get

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int \vec{J}(\vec{x}')\,(\hat{n}\cdot\vec{x}')^n\, d^3x' \qquad (9.9)$$

$\downarrow$

magnitude of n^{th} term

$\downarrow$

$$\frac{1}{n!}\int \vec{J}(\vec{x}')\,(k\hat{n}\cdot\vec{x}')^n\, d^3x'$$

order of magnitude is d

$\downarrow$

$$kd \ll 1$$

successive terms fall off rapidly with n

Radiation emitted from the source will come mainly from the first non vanishing term in this expansion

The beginning of this chapter is also summarized in the Class Summary for Week 4, Day 2

Electric Dipole Radiation

(Week 5 – Day 1 Class Summary & Worksheet)

- Keeping only the 1st term in (9.9), get

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')\, d^3x' \qquad (9.13)$$

- You showed on the worksheet that this can be written as

$$\vec{A}(\vec{x}) = -\frac{i\omega\mu_0}{4\pi}\,\vec{p}\,\frac{e^{ikr}}{r} \qquad (9.16)$$

$$\vec{p} = \int x'\rho(\vec{x}')\, d^3x' \qquad (9.17)$$

Dipole moment

All equations on this page will be supplied, if needed

Electric Dipole Fields

(Week 5 – Day 1 Class Summary, pages 2-3)

$$\vec{A}(\vec{x}) = -\frac{i\omega\mu_0}{4\pi}\,\vec{p}\,\frac{e^{ikr}}{r} \qquad (9.16)$$

- Starting from (9.16), can write magnetic and electric fields (see Homework 4)

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$

Magnetic field is always *perpendicular* to the radial vector

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \qquad (9.18)$$

Electric field has components *perpendicular and parallel* to the radial vector

All equations on this page will be supplied, if needed

Electric Dipole Fields in the Near Zone

(Week 5 – Day 1 Class Summary, page 5)

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r}+\left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \qquad (9.18)$$

$\downarrow$

In the near zone

(Homework 4)

$\downarrow$

$$\vec{H} = \frac{i\omega}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{r^2}$$
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\frac{1}{r^3} \qquad (9.20)$$

Apart from its oscillations in time, this is just the static electric dipole field

Details are on the last page of the Class Summary for Week 5, Day 1.

Electric Dipole Fields in the Far Zone

(Week 5 – Day 1 Class Summary, page 3)

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r}+\left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \qquad (9.18)$$

$\downarrow$

In the far zone $kr \gg 1$

$\downarrow$

Magnetic field perpendicular to $\hat{n}$

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}$$

$$\vec{E} = \frac{k^2}{4\pi\epsilon_0}\left[(\hat{n}\times\vec{p})\times\hat{n}\right]\frac{e^{ikr}}{r} \qquad (9.19)$$

Electric field perpendicular to magnetic field and to $\hat{n}$

Fields in the far zone behave as transverse waves carrying energy away from the source.

Electric Dipole Radiation

(Week 5 – Day 1 Class Summary, page 4)

- At any given point in space, S gives the energy per unit area per unit time flowing past that point, i.e., it is the time-averaged power radiated per unit surface area.

- In the current scenario, though, we are more interested in time-averaged power per unit solid angle.

- So, must write relation between solid angle $d\Omega$ and surface area perpendicular to the direction of energy flow *dA*, easy to do:

$$dA = r^2\, d\Omega$$

- The amount of energy flowing through would be $\hat{n} \cdot \vec{S}$

- So, time-averaged power radiated per unit solid angle by the oscillating dipole is

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[r^2 \hat{n} \cdot \vec{E} \times \vec{H}^*\right] \qquad (9.21)$$

where *E* and *H* are as in (9.19).

(9.21) will be supplied, if needed

Electric Dipole Radiation

(Week 5 – Day 1 Class Summary, page 4)

- On Homework 4, found that for electric dipole radiation

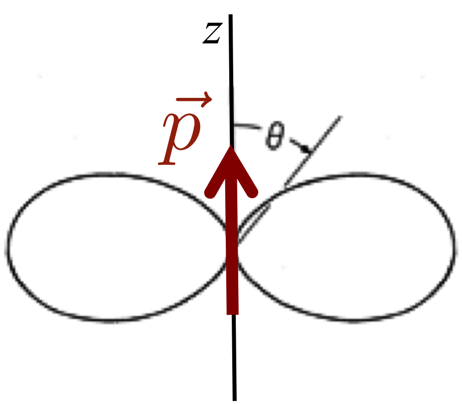
$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left| (\hat{n} \times \vec{p}) \times \hat{n} \right|^2 \qquad (9.22)$$

(9.22) will be supplied, if needed

- If all the components of $\vec{p}$ have the same phase, then the angular distribution of the radiation is a typical dipole pattern

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left| \vec{p} \right|^2 \sin^2\theta \qquad (9.23)$$

- Eq. (9.23) tells us that if, e.g., the dipole is oriented along the z-axis (which passes through the north and south pole of the sphere), then the radiation is zero on the poles of the sphere, but peaks at the equator, perpendicular to the orientation of the dipole.



Material not on the Midterm Exam

Magnetic Dipole and Electric Quadrupole radiation covered on Week 5 – Thursday, Apr 29 will not be on the Midterm Exam.

Midterm Exam

- You must be present on the Zoom session in order for your Midterm to be graded, otherwise you will be considered absent and will need to get an excused absence approved by the Dean of Students to take a makeup exam.

- Please get on to the zoom session on time! You must finish when the class finishes, no matter what time you get to the test.

- Please read all questions carefully, and make sure you've answered what was asked of you.

- Remember that makeups are only allowed for excused absences (which require a note from the Dean of Students).