Cosmic Dynamics–I

(1) In the lecture a figure is shown in which a test particle of mass say $m_p$ is on the surface of a spherically symmetric sphere of lets say mass $M_s$. You were also given *Birkhoff's theorem* which states that for a spherically symmetric system, the force due to gravity at radius, $r$ is determined only by the mass interior to that radius.

(a) Recall that Newton's gravitational law is $F_G = -Gm_1m_2/r^2$. Use this law in Newton's second law ($F = ma$) but express the masses as densities and use $r(t) = a(t)r_o$.

(b) In the lecture we showed that we could write,

$$\ddot{a} = -\frac{4\pi}{3}\frac{G\rho_o}{a^2}.$$

We know that $\rho_o \neq 0$, what does this say about the time development of $a(t)$?

(c) In the lecture we've wrote that

$$\ddot{a} + \frac{4\pi}{3}\frac{G\rho_o}{a^2} = 0.$$

What follow are a set of *tricks* that will allow us to integrate this equation. The first step is to multiply both sides by $\dot{a}$ and use the fact that $d(\dot{a})/dt = 2\dot{a}\ddot{a}$. Do so and write down the resulting expression.

(d) Almost there. We have arrived at

$$\frac{1}{2}\frac{d\,(\dot{a})}{dt} + \frac{4\pi}{3}\frac{G\rho_o}{a^2}\frac{da}{dt} = 0$$

Now apply the fact that

$$\frac{1}{a^2}\frac{da}{dt} = -\frac{d\,(1/a)}{dt}$$

to the result and write the expression as a single time derivative.

(e) At your table, go over the derivation of Friedmann's equation, making sure you are clear on all the critical steps.

(2) We've shown that the Friedmann equation is

$$H(t)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{kc^2}{R_o^2 a(t)^2}.$$

(a) Isolate the curvature term, $k$.

(b) Set

$$\epsilon(t) = \frac{3c^2}{8\pi G}H(t)^2$$

What do find $k$ to be at this value of $\epsilon(t)$.

(c) Suppose $\epsilon(t) > (3c^2/8\pi G)H(t)^2$ is $k$ positive, negative, or zero?

(d) Suppose $\epsilon(t) < (3c^2/8\pi G)H(t)^2$ is $k$ positive, negative, or zero?

(e) Because of the behavior of $\epsilon$ at this value, this value is called the *critical densisty* and defined as

$$\epsilon_c(t) = \frac{3c^2}{8\pi G}H(t)^2.$$

Define the *density parameter* as

$$\Omega(t) = \frac{\epsilon(t)}{\epsilon_c(t)}$$

and rewrite the Friedmann equation using the density parameter.

(3) In the lecture we stated that the two key equations could be combined to form the *acceleration equation*,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) \tag{3}$$

You'll now probe this equation a bit.

(a) Look at the right hand side of the equation. Considering only the physics you've learned to this point (i.e., ignore anything you may have read in the *popular press*) what sign must $\epsilon$ and $P$ be?

(b) Given your result in (a), what sign should the acceleration of the expansion of the universe be?

(c) You have heard that the universe is positively accelerating. Suppose we assert that $\epsilon > 0$, what value(s) of $P$ make the acceleration positive. At your table, discuss this result

(4) Recap the important topics covered today. Compare and constrast your results with others at your table

Homework 01–Due Friday Jan 17

1. Problem 2.2
2. Problem 3.3
3. Problem 3.5

**Additional Grad Student Problem(s)**

4. Problem 2.4
5. The critical energy density at the present time is,

$$\epsilon_{c,o} = \frac{3c^2}{8\pi G} H_o^2.$$

   (a) Calculate $\epsilon_{c,o}$ in SI units using $H_o = 70$ km s$^{-1}$ Mpc$^{-1}$. Since $H_o$ is known to within 10%, express your answer for $\epsilon_{c,o}$ to with in 20% limits.

   (b) Convert your answer to units more often used in cosmology, GeV m$^{-3}$.

   (c) Find the equivalent mass density, $\rho_{c,o} = \epsilon_{c,o}/c^2$ in SI units.

   (d) Convert your answer to solar masses per Mpc$^{-3}$. A solar mass, $M_\odot = 1.99 \times 10^{30}$ kg

Homework 02–Due Friday Jan 31

1. Problem 4.4
2. Problem 4.5
3. Problem 5.1
4. Problem 5.3

**Additional Grad Student Problem(s)**

5. Consider a flat universe universe with a single component characterized by the equation of state parameter, $w = -1$.

   (a) Sho that in such a universe the Friedmann equation takes the form

   $$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon_\Lambda a^2 \tag{1}$$

   (b) Show that the Hubble constant in a $\Lambda$-dominated universe is give by

   $$H_o \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_o} = \left(\frac{8\pi G\epsilon_\Lambda}{3c^2}\right)^{1/2} \tag{2}$$

   (c) Solve Eq. (1) and show that the dependence of the scale factor on time is given by

   $$a(t) = e^{H_o(t-t_o)}$$