PHY 420 Spring 2021

Homework 4—due by 9:00 PM, Monday, May 3

There is no late deadline on this homework due to the Midterm Exam.

You may write by hand and scan as a single PDF, or write in latex (using the template file provided) or Word, and generate PDF. Please submit one PDF file only. Only questions and sub-parts that are numbered clearly, with numbers corresponding to those in this document, will be graded. See the syllabus for more detailed rules.

1. In this problem, you will apply the Green function technique discussed in class to the wave equation for the vector potential $\vec{A}$, which is

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \tag{1}$$

(a) Write down the ***Green function equation*** corresponding to equation (1) above.

(b) Write down the Green function, which is the solution to the equation you wrote in part (a).

(c) Use the Green function in part (b) to write down the solution to equation (1) above.

2. The vector potential $\vec{A}$ of an oscillating electric dipole is given by equation (9.16):

$$\vec{A}(\vec{x}) = -\frac{i\mu_0\omega}{4\pi}\,\vec{p}\,\frac{e^{ikr}}{r}$$

where $\vec{p}$ is the dipole moment, and $k = \omega/c$ is the wave number.

Show that its magnetic field is given by

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

where $\hat{n}$ is a unit vector in the direction of $\vec{x}$, so that $\vec{x} = r\hat{n}$

3. Meanwhile, it can be shown that the electric field of an oscillating electric dipole is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2\left(\hat{n}\times\vec{p}\right)\times\hat{n}\,\frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}\right]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}$$

Show that, in the near zone, the $\vec{H}$ you found in Question 2, and the $\vec{E}$ written above take the form

$$\vec{H} \simeq \frac{i\omega}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{r^2}$$

and

$$\vec{E} \simeq \frac{1}{4\pi\epsilon_0}\left[3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}\right]\frac{1}{r^3}$$

Question 4 is on the next page.

4. The time-averaged power radiated per unit solid angle by an oscillating dipole is given by

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[r^2\,\hat{n}\cdot\vec{E}\times\vec{H}^*\right] \tag{9.21}$$

where the fields $\vec{E}$ and $\vec{H}$ in the far zone are given by equation (9.19):

$$\begin{aligned} \vec{H} &= \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r} \\ \vec{E} &= \frac{k^2}{4\pi\epsilon_0}\left[(\hat{n}\times\vec{p})\times\hat{n}\right]\frac{e^{ikr}}{r} \end{aligned} \tag{9.19}$$

Show that

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}\,k^4\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2 \tag{9.22}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space.