Homework 3

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Problem 1

A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces of the cylinder is zero, while the potential on the cylindrical surface is $V(\phi, z)$.

The Laplace equation in cylindrical coordinates is gave by

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

The first step in approaching partial differential equations is to separate the variables. The variables for a cylinder are

$$\vec{\Phi}(\vec{x}) = R(\rho)Q(\phi)Z(z)$$

and plugging these variables into the Laplace equation gives

$$\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right)\frac{1}{R} + \frac{1}{\rho^2}\frac{d^2Q}{d\rho^2}\frac{1}{Q} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

The boundary conditions are at the bottom of the cylinder $z = 0$ and the top of the cylinder $z = L$, where the length of the cylinder is some length L. With all boundary conditions applied to the problem, the new potential can be expressed in the form

$$\Phi(\rho, \phi, z) = \sum_{m=0}\sum_{n=1} J_m\Big(\frac{in\pi\rho}{L}\Big)(C_{m,n}e^{im\phi} + D_{m,n}e^{-im\phi})sin\Big(\frac{n\pi z}{L}\Big).$$

The Bessel function of the first kind is gave from the equation I_ν. Therefore, $J_\nu(ix) = i^\nu I_\nu(x)$ which can be substituted into the equation above and gives

$$\Phi(\rho, \phi, z) = \sum_{m=0}\sum_{n=1} i^m I_m\Big(\frac{in\pi\rho}{L}\Big)(C_{m,n}e^{im\phi} + D_{m,n}e^{-im\phi})sin\Big(\frac{n\pi z}{L}\Big).$$

From here the final boundary condition can be applied. The potential on the cylindrical surface is $V(\phi, z)$. Therefore, we set $\Phi(\phi, z) \to V(\phi, z)$ and since ρ is not included in the new potential, then set $\rho = a$. The new potential can now be expressed as

$$V(\phi, z) = \sum_{m=0}\sum_{n=1} i^m I_m\Big(\frac{in\pi a}{L}\Big)(C_{m,n}e^{im\phi} + D_{m,n}e^{-im\phi})sin\Big(\frac{n\pi z}{L}\Big).$$

The potential needs to be multiplied by sin and integrated such that

$$\int_0^L dz \int_0^{2\pi} V(\phi, z)e^{-im\phi} sin\Big(\frac{n'\pi z}{L}\Big)d\phi$$

and now the coefficients can be solved for. Where

$$C_{m,n} = \Bigg[L\pi i^m I_m\Big(\frac{n\pi a}{L}\Big)\Bigg]^{-1} \int_0^L dz \int_0^{2\pi} V(\phi, z)e^{-im\phi} sin\Big(\frac{n'\pi z}{L}\Big)d\phi$$

and $D_{m,n}$ is just the complex conjugate of $C_{m,n}$, therefore, we have $D_{m,n} = C^*_{m,n}$. Thus, we get the final expression

$$\Phi(\rho, \phi, z) = \frac{1}{L\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{I_m(n\pi\rho/L)}{I_m(n\pi a/L)}(C_{m,n}e^{im\phi} + C^*_{m,n}e^{-im\phi})sin\Big(\frac{n\pi z}{L}\Big).$$

Problem 2

Much of the previous problem can be used to solve this problem. Starting from the integral of $C_{m,n}$, there are now two integrals needed. One of those integrals will be from $-\pi/2$ to $\pi/2$. The other integral will be from $\pi/$ to $3\pi/2$. This set up as the constant looks like

$$C_{m,n} = \int_0^L dz \int_{-\pi/2}^{\pi/2} Ve^{-im\phi} sin\Big(\frac{n\pi z}{L}\Big) d\phi - \int_0^L dz \int_{-\pi/2}^{\pi/2} Ve^{-im\phi} sin\Big(\frac{n\pi z}{L}\Big) d\phi$$

Evaluating the integrals above we get

$$C_{m,n} = \frac{8LV}{mn\pi}(-1)^{(m-1)/2}.$$

Substituting the constant $C_{m,n}$ into the potential from problem 1 we get

$$\Phi(\rho,\phi,z) = \frac{16V}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{I_m(n\pi\rho/L)}{I_m(n\pi a/L)} \frac{(-1)^{(m-1)/2}}{mn} cos(\phi m) sin\Big(\frac{n\pi z}{L}\Big).$$

Problem 3

In Jackson chapter 3, Jackson gives the solutions to $J_\nu(x)$ in equation 3.82 and $J_{-\nu}(x)$ in equation 3.83. To show that $J_{-m}(x) = (-1)^m J_m(x)$, equation 3.83 can be used. Equation 3.83 is gave as

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

If ν is not an integer, then the solutions are linearly independent. If ν is an integer, then we have $\nu = m$. The equation above then can be wrote as

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j-m+1)} \left(\frac{x}{2}\right)^{2j}.$$

When the condition $j < m$ is satisfied, the denominator component $(j-m+1)!$ is infinite. The starting term for the summation is $j = m$, so setting $j = k+m$ gives the result

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{k=0}^{\infty} \frac{(-1)^{(k+m)}}{(k+m)!\,\Gamma(k+m-m+1)} \left(\frac{x}{2}\right)^{2(k+m)} = \left(\frac{x}{2}\right)^{-m} \sum_{k=0}^{\infty} \frac{(-1)^{(k+m)}}{(k+m)!\,k!} \left(\frac{x}{2}\right)^{2(k+m)}$$

Therefore, we get

$$J_{-m}(x) = (-1)^m J_m(x)$$

Problem 4

The figures were generated using the program in the Appendix

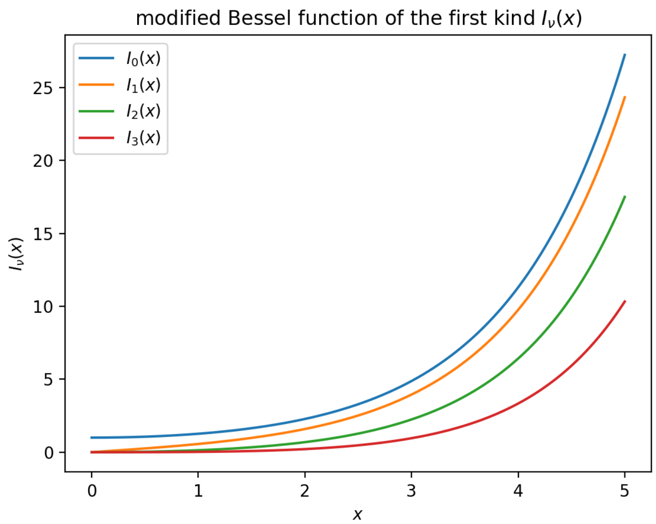

Figure 1: modified Bessel function of the first kind $I_\nu(x)$.

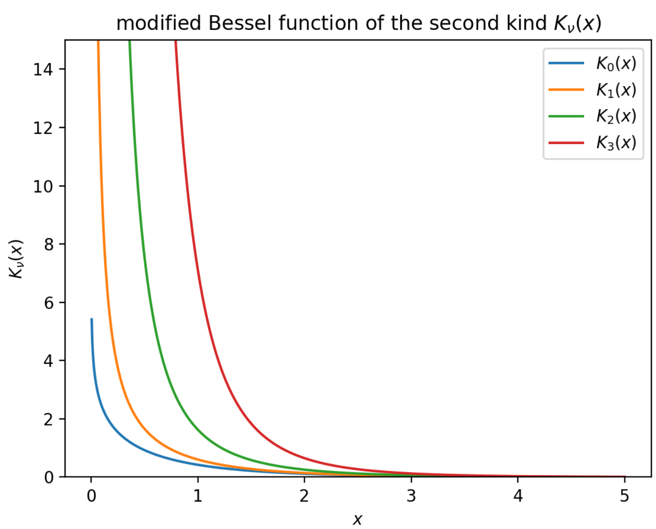

Figure 2: modified Bessel function of the second kind $K_\nu(x)$.

Appendix

```python
import numpy as np
import matplotlib.pyplot as plt
import scipy.special as sp


def I_nu(x, nu):
    I = ((1j) ** (-nu)) * sp.jv(nu, 1j * x)

    return I


def K_nu(x, nu):
    hv = sp.jv(nu, 1j * x) + 1j * sp.yv(nu, 1j * x)
    K = (np.pi/2) * (1j) ** (nu + 1) * hv

    return K


x  = np.linspace(0, 5, 1000)
nu = [0, 1, 2, 3]
I = []
K = []
if __name__ == "__main__":
    for i in range(0, len(nu)):
        I.append(I_nu(x, nu[i]))
        K.append(K_nu(x, nu[i]))


for i in range(0, len(nu)):
    plt.plot(x, I[i], label=r'$I_{}(x)$'.format(i))

plt.title(r'modified Bessel function of the first kind $I_{\nu}(x)$')
plt.xlabel(r'$x$')
plt.ylabel(r'$I_{\nu}(x)$')
plt.legend(loc='best')
plt.show()


for i in range(0, len(nu)):
    plt.plot(x, K[i], label=r'$K_{}(x)$'.format(i))

plt.title(r'modified Bessel function of the second kind $K_{\nu}(x)$')
plt.xlabel(r'$x$')
plt.ylabel(r'$K_{\nu}(x)$')
plt.legend(loc='best')
plt.ylim((0, 15))
```

```
plt.show()
```

Julia modified Bessel program

```
#!/usr/bin/env julia
using SpecialFunctions
using PyPlot

x = range(0, stop=5, length=100)
for nu in range(0, stop=5, length=6)
        I_nu = besseli.(nu , x)
        plot(x, I_nu, label="\$I_{$nu}(x)\$")
end
title(L"modified Bessel function of the first kind $I_{\nu}(x)$")
xlabel(L"$x$")
ylabel(L"$I_{\nu}(x)$")
legend(loc="best")
show()
#PyPlot.svg(true)

for nu in range(0, stop=5, length=6)
        K_nu = besselk.(nu , x)
        plot(x, K_nu, label="\$K_{$nu}(x)\$")
end
title(L"modified Bessel function of the second kind $K_{\nu}(x)$")
xlabel(L"$x$")
ylabel(L"$K_{\nu}(x)$")
legend(loc="best")
ylim((0, 15))
show()
#PyPlot.svg(true)
```