PHY 420 Spring 2021

Class Summary—Week 1, Day 1—Tuesday, Mar 30

Energy and Energy Density

Given the importance of energy underlying all the concepts we have learned, we will begin this quarter with a unifying focus on the idea of energy in the electrostatic, magnetic, and electromagnetic fields, culminating in the Poynting theorem and Maxwell stress tensor. *Since the source material is scattered among different chapters, I will provide the page numbers from Jackson below each major heading, but the class summaries written here are, hopefully, self-sufficient without any need to refer back to Jackson's text, except where noted otherwise*.

Electrostatic Potential Energy and Energy Density

(*Jackson: pages 40-43*)

Recall that the scalar potential has a physical interpretation when we consider the work done on a test charge q in transporting it from a point A to another point B in the presence of an electric field $\vec{E}(\vec{x})$. Since the force acting on the charge at any point is $\vec{F} = q\vec{E}$, the work done in moving the charge from point A to point B is

$$W = -\int_A^B \vec{F} \cdot d\vec{l} = -q\int_A^B \vec{E} \cdot d\vec{l} \tag{1.18}$$

where we've put a minus sign because we are calculating the work done *on* the charge against the action of the field.

Now, since the electric field $\vec{E}(\vec{x})$ is the negative gradient of the potential $\Phi(\vec{x})$, $\vec{E} = -\vec{\nabla}\Phi$, equation (1.18) can be written as

$$\begin{aligned} W &= -q\int_A^B \Big(-\vec{\nabla}\Phi\Big) \cdot d\vec{l} \\ &= q\int_A^B \vec{\nabla}\Phi \cdot d\vec{l} \end{aligned}$$

But the gradient of the potential, evaluated along a line element $d\vec{l}$ is just the change in Φ along that line element, that is,

$$W = q\int_A^B d\Phi$$

Therefore

$$W = q\Big(\Phi_B - \Phi_A\Big) \tag{1.19}$$

The key conclusion from the discussion above is that $q\Phi$ can be interpreted as the **potential energy** of the test charge in the electrostatic field.

You should also remember that we take the zero of potential energy to be when the charge is located at infinite distance from the system under consideration.

More precisely, if a point charge q_i is brought from infinity to a point $\vec{x}_i$ in a region of localized electric fields described by the scalar potential Φ (which vanishes at infinity), the **work done on the charge** (and hence its **potential energy**) is given by

$$W_i = q_i \Phi(\vec{x}_i) \tag{1.47}$$

Now, if the potential Φ is considered to be produced by an array of $(n-1)$ charges q_j at positions $\vec{x}_j$, where $j = 1, 2, 3, \ldots, (n-1)$, then

$$\Phi(\vec{x}_i) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|} \tag{1.48}$$

So, the potential energy of charge q_i that was brought from infinity to $\vec{x}_i$ is

$$W_i = \frac{q_i}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|} \tag{1.49}$$

By adding each charge in succession, the potential energy of all the charges (due to all the forces acting between them) is

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \tag{1.50}$$

To obtain a more symmetric form for the **potential energy**, sum over i and j without restriction, and then divide by 2 (so we don't double count):

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i} \sum_{j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \tag{1.51}$$

where it is understood that the $(i = j)$ terms are omitted in the double sum, because the $(i = j)$ terms would correspond to infinite self-energy of a charge interacting with itself. The content of this page up to here (equations (1.47)-(1.51) above) are the *solution to Question 1 of today's worksheet.*

For a continuous charge distribution, we can put

$$q_i \to \rho(\vec{x}) d^3x \qquad \text{and} \qquad q_j \to \rho(\vec{x}\,') \, d^3x'$$

so that

$$W = \frac{1}{8\pi\epsilon_0} \int \int \frac{\rho(\vec{x}) \, \rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \, d^3x \, d^3x' \tag{1.52}$$

Since one of the integrals is just the scalar potential from equation (1.17):

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \, d^3x'$$

we can write equation (1.52) as

$$W = \frac{1}{2} \int \rho(\vec{x}) \, \Phi(\vec{x}) \, d^3x \tag{1.53}$$

as you showed in Question 2(a) of today's worksheet.

Equations (1.51), (1.52), and (1.53) express the electromagnetic potential energy *in terms of the charges*, emphasizing the interactions between charges via Coulomb forces.

An alternative approach is to emphasize the electric field and to **interpret** the energy as being stored in the electric field surrounding the charges. To obtain this, we use the fact that the potential Φ must satisfy the Poisson equation equation (1.28):

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

so we use this to eliminate the charge density ρ in equation (1.53):

$$W = \frac{1}{2}\int \Big[-\epsilon_0\,\nabla^2\Phi\Big]\,\Phi\,d^3x$$

so that

$$W = -\frac{\epsilon_0}{2}\int \Phi\nabla^2\Phi\,d^3x$$

as you found in Question 2(b) of today's worksheet. Now, use

$$\vec{\nabla}\cdot\left(\Phi\vec{\nabla}\Phi\right) = \left|\vec{\nabla}\Phi\right|^2 + \Phi\nabla^2\Phi$$

or

$$-\Phi\nabla^2\Phi = \left|\vec{\nabla}\Phi\right|^2 - \vec{\nabla}\cdot\left(\Phi\vec{\nabla}\Phi\right)$$

in the expression for W above, $W = -\frac{\epsilon_0}{2}\int \Phi\nabla^2\Phi\,d^3x$, to get

$$W = \frac{\epsilon_0}{2}\left[\int_V \left|\vec{\nabla}\Phi\right|^2 d^3x - \int_V \vec{\nabla}\cdot\left(\Phi\vec{\nabla}\Phi\right)d^3x\right]$$

Apply the Gauss divergence theorem to the second volume integral on the right hand side:

$$W = \frac{\epsilon_0}{2}\left[\int_V \left|\vec{\nabla}\Phi\right|^2 d^3x - \int_S \Phi\vec{\nabla}\Phi\,\cdot\hat{n}\,da\right]$$

Now, since the surface is at infinity, and $\Phi \to 0$ as $r \to \infty$, the surface integral on the right hand side goes to zero, *as you demonstrated in Question 3 of today's worksheet* .

Put $\vec{E} = -\vec{\nabla}\Phi$ in the integrand of the remaining volume integral on the right hand side:

$$W = \frac{\epsilon_0}{2}\int \left|\vec{E}\right|^2 d^3x \tag{1.54}$$

where the integration is over all space, as you no doubt realized when we put the surface integral to zero. In equation (1.54), all explicit reference to charges has gone, and the potential energy is expressed as an integral of the square of the electric field over all space.

This leads naturally to the identification of the integrand as an **energy density** w:

$$w = \frac{\epsilon_0}{2}\left|\vec{E}\right|^2 \tag{1.55}$$

This is intuitively reasonable, since regions of high field must contain considerable energy.

Note one puzzling aspect of equation (1.55), however — the energy density is positive definite. Consequently, its volume integral is nonnegative. This seems to contradict our expression in equation (1.51) that the potential energy of two charges of opposite sign is negative. The reason for this apparent contradiction is that equation (1.54) and equation (1.55) contain "self-energy" contributions to the energy density, whereas the double sum in equation (1.51) does not. Those of you who read the Feynman lectures as an undergraduate would be aware of this, of course; it is a problem with electrodynamics that has never been solved. In his usual lucid style, Feynman states that the idea of locating the energy in the field is inconsistent with the assumption of the existence of point charges. Thus, whenever we deal with interactions, we have to be careful that we discount this self-energy contribution.

Capacitance

For a system of n conductors, each with potential V_i and total charge Q_i, where $i = 1, 2, 3, \ldots, n$, in otherwise empty space, the electrostatic potential energy can be expressed in terms of the potentials alone and certain geometrical quantities called *coefficients of capacity*.

For a given configuration of the conductors, the linear functional dependence of the potential on the charge density implies that the potential of the ith conductor can be written as

$$V_i = \sum_{j=1}^{n} p_{ij} Q_j \qquad (i = 1, 2, 3, \ldots, n)$$

where the p_{ij} depend on the geometry of the conductors. These n equations can be inverted to yield the charge on the ith conductor in terms of all the potentials:

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j \qquad (i = 1, 2, 3, \ldots, n) \tag{1.61}$$

The coefficients C_{ii} are called *capacities* or **capacitances**, while the $C_{ij}, i \neq j$, are called *coefficients of induction*. The capacitance of a conductor is therefore the total charge on the conductor when it is maintained at unit potential, while all other conductors are being held at zero potential.

Sometimes, the capacitance of a system of conductors is also defined. For example, the capacitance of two conductors carrying equal and opposite charges in the presence of other grounded conductors is defined as the ratio of the charge on one conductor to the potential difference between them. The equations in (1.61) can be used to express this capacitance in terms of the coefficients C_{ij}.

From equation (1.53), the potential energy for the system of conductors is

$$W = \frac{1}{2} \sum_{i=1}^{n} Q_i V_i$$

so that, substituting equation (1.61), we get

$$W = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} V_i V_j \tag{1.62}$$

for the potential energy for the system of n conductors, each with potential V_i and total charge Q_i in otherwise empty space.

We will now move on to a discussion of the electrostatic energy in dielectric media.

Electrostatic Energy in Dielectric Media

(*Jackson: pages 165-169*)

On one of the previous pages, we discussed the energy of a system of charges in free space, and obtained in equation (1.53) that

$$W = \frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x \tag{4.83}$$

for the energy due to a charge density $\rho(\vec{x})$ and a potential $\Phi(\vec{x})$; note that Jackson has given it a new number in this chapter.

Equation (4.83) cannot be directly taken over to macroscopic dielectric media, because we assembled the elemental charges bit by bit by bringing each one in from infinitely far away against the action of the then existing electric field. With dielectric media, work is done **not only** to *bring real (macroscopic) charge into position,* **but also** to *produce a certain state of polarization* in the medium.

To find an expression that applies to macroscopic media, consider a small change in the energy δW due to some sort of change $\delta\rho$ in the macroscopic charge density ρ that exists in all space. The work done to accomplish this change is

$$\delta W = \int \delta\rho(\vec{x})\,\Phi(\vec{x})\,d^3x \tag{4.84}$$

where $\Phi(\vec{x})$ is the potential due to the charge density $\rho(\vec{x})$ already present.

Since $\vec{\nabla}\cdot\vec{D} = \rho$, we can relate the change $\delta\rho$ to a change in the displacement of $\delta\vec{D}$:

$$\delta\rho = \vec{\nabla}\cdot(\delta\vec{D}) \tag{4.85}$$

Putting equation (4.85) in equation (4.84), we get

$$\delta W = \int \vec{\nabla}\cdot(\delta\vec{D})\,\Phi(\vec{x})\,d^3x$$

From the inside front cover of Jackson, this can be written as

$$\delta W = \int \left[\vec{\nabla}\cdot\left(\Phi\,\delta\vec{D}\right) - \vec{\nabla}\Phi\cdot\delta\vec{D}\right] d^3x$$

or

$$\delta W = \int_S \vec{\nabla}\left(\Phi\,\delta\vec{D}\right)\cdot\hat{n}\,da - \int \vec{\nabla}\Phi\cdot\delta\vec{D}\,d^3x$$

The first integral on the right hand side is zero because $\Phi \to 0$ at infinity, assuming that $\Phi(\vec{x})$ is a localized charged distribution. Writing $\vec{E} = -\vec{\nabla}\Phi$ in the second term on the right hand side, we get

$$\delta W = \int \vec{E}\cdot\delta\vec{D}\,d^3x \tag{4.86}$$

as you showed in Question 4 on today's worksheet.

Now, by integrating and bringing $\vec{D}$ from an initial value $\vec{D} = 0$ to its final value $\vec{D}$, we can use equation (4.86) to write a formal expression for the total electrostatic energy:

$$W = \int d^3x \int_0^D \vec{E} \cdot \delta \vec{D} \tag{4.87}$$

as you demonstrated in Question 4 on today's worksheet.

If the medium is **linear**, then

$$\vec{E} \cdot \delta \vec{D} = \frac{1}{2}\,\delta\left(\vec{E} \cdot \vec{D}\right) \tag{4.88}$$

and so the total electrostatic energy is

$$W = \frac{1}{2} \int \vec{E} \cdot \vec{D}\, d^3x \tag{4.89}$$

This result can be transformed into equation (4.83) by writing $\vec{E} = -\vec{\nabla}\Phi$ and $\vec{\nabla} \cdot \vec{D} = \rho$, which shows us that equation (4.83) is *valid macroscopically only if the behavior is linear*. Otherwise, the energy of a final configuration must be calculated from equation (4.87) and might conceivably depend on the past history of the system (hysteresis effects).

A problem of considerable interest is the change in energy when a **dielectric object is placed in an electric field whose sources are fixed**. Suppose initially that there exists an electric field $\vec{E}_0$ due to a distribution of charges $\rho_0(\vec{x})$ in a medium with ϵ_0, which may be a function of position. The initial electrostatic energy is

$$W_0 = \frac{1}{2} \int \vec{E}_0 \cdot \vec{D}_0\, d^3x$$

where $\vec{D}_0 = \epsilon_0 \vec{E}_0$.

Then, with the sources fixed in position, a dielectric object of volume V_1 is introduced into the field, changing the field from $\vec{E}_0$ to $\vec{E}$. Then $\epsilon(\vec{x})$ has the value ϵ_1 inside V_1 and ϵ_0 outside V_1. To avoid mathematical difficulties, we imagine $\epsilon(\vec{x})$ to be a smoothly varying function of position that falls rapidly but continuously from ϵ_1 to ϵ_0 at the edge of the volume V. The electrostatic energy is now

$$W_1 = \frac{1}{2} \int \vec{E} \cdot \vec{D}\, d^3x$$

where $\vec{D} = \epsilon_1 \vec{E}$.

The change in the energy is then $W = W_1 - W_0$, given by

$$W = \frac{1}{2} \int \left(\vec{E} \cdot \vec{D} - \vec{E}_0 \cdot \vec{D}_0\right) d^3x \tag{4.90}$$

On Homework 1, you will show that this can be written as

$$W = \frac{1}{2} \int_{V_1} \left(\vec{E} \cdot \vec{D}_0 - \vec{D} \cdot \vec{E}_0\right) d^3x \tag{4.91}$$

where we need to carry out the integration only over the volume V_1 of the object since, outside V_1, we have $\vec{D} = \epsilon_0 \vec{E}$ (meaning that the contribution to the integral is zero outside V_1).

With $\vec{D} = \epsilon_1 \vec{E}$ inside V_1 and $\vec{D}_0 = \epsilon_0 \vec{E}_0$, equation (4.91) can be written as

$$W = \frac{1}{2} \int\limits_{V_1} \Big(\epsilon_0 - \epsilon_1\Big)\, \vec{E} \cdot \vec{E}_0 \, d^3x$$

or

$$W = -\frac{1}{2} \int\limits_{V_1} \Big(\epsilon_1 - \epsilon_0\Big)\, \vec{E} \cdot \vec{E}_0 \, d^3x \tag{4.92}$$

If the medium surrounding the dielectric body is free space, then using the definition of polarization $\vec{P} = (\epsilon_1 - \epsilon_0)\, \vec{E}$ from equation (4.57), we can write equation (4.92) as

$$W = -\frac{1}{2} \int\limits_{V_1} \vec{P} \cdot \vec{E}_0 \, d^3x \tag{4.93}$$

where $\vec{P}$ is the (induced) polarization of the dielectric that has been introduced into the field. *You showed this on Question 5 of today's worksheet.*

Equation (4.93) shows that the **energy density** of a dielectric placed in a field $\vec{E}_0$ whose sources are fixed is given by

$$w = -\frac{1}{2}\, \vec{P} \cdot \vec{E}_0 \tag{4.94}$$

Equations (4.92) and (4.93) show that the energy goes down if the object moves into a region with a larger electric field E_0, provided $\epsilon_1 > \epsilon_0$. That is, a dielectric object will tend to move toward regions of increasing $\vec{E}_0$, provided $\epsilon_1 > \epsilon_0$.

To calculate the **force**, imagine a small generalized displacement $d\xi$ of the object. Then there will be a change in the energy δW. Since the charges are held fixed, there is no external source of energy and the change in field energy can be interpreted as a change in the potential energy of the body. This means that the force acting on the body is

$$F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)_Q \tag{4.95}$$

where we have placed the subscript Q on the partial derivative to indicate that the *sources of the field are kept fixed.*

What, however, if the **potentials are kept fixed**?

Such a situation, where the potentials are kept fixed, usually arises in practical situations involving the motion of dielectrics, where the electric fields are often produced by a configuration of electrodes held at fixed potentials by connecting to an external source such as a battery. For this case, we need to proceed via a *two-step process*, as you will see when you work this problem on the homework.

More details of the problem when the potentials are kept fixed are ***on the next page***.

On the previous two pages, we discussed the instance of a dielectric object placed in an electric field whose sources are fixed. Another situation involves **the potentials being kept fixed**. You will solve this problem on the homework, but here are some details to help you.

The idea of fixed potentials refers to the situation in which the electric fields are produced by a configuration of electrodes held at fixed potentials by connecting to an external source such as a battery. To maintain the potentials constant as the distribution of dielectric varies, charge will flow to or from the battery to the electrodes. This means that energy is being supplied from the external source, and solving the problem involves comparing the energy supplied in this way with the energy change we calculated above for fixed sources of the field. The process of altering the dielectric properties in some way (by moving the dielectric objects, changing their susceptibilities, etc.) in the presence of electrodes at fixed potentials can be viewed as a **two-step process**.

- In the first step, the electrodes are disconnected from their batteries and the charges on them held fixed ($\delta\rho = 0$), so

  $$\delta W_1 = \frac{1}{2}\int \rho\,\delta\Phi_1\,d^3x \tag{4.98}$$

  where $\delta\Phi_1$ is the change in the potential produced.

- In the second step, the batteries are connected again to the electrodes to restore their potentials to the original values. There will be a flow of charge $\delta\rho_2$ from the batteries accompanying the change in potential $\delta\Phi_2 = -\delta\Phi_1$, which is necessary to re-establish the original potentials. So, the energy change in the second step is

  $$\delta W_2 = \frac{1}{2}\int \Big(\rho\,\delta\Phi_2 + \Phi\,\delta\rho_2\Big)\,d^3x$$

  where we should replace $\delta\Phi_2$ with $(-\delta\Phi_1)$.

To proceed further, we limit ourselves to linear media only, for which $W = \frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x$ from equation (4.83), from which we obtain

$$\delta W = \frac{1}{2}\int \Big(\rho\delta\Phi + \Phi\delta\rho\Big)\,d^3x \tag{4.96}$$

But in equation (4.84), we already found that

$$\delta W = \int \Phi\delta\rho\,d^3x$$

Equating the two, we find a useful relation that can then be applied to the expression for δW_2 above, and after some manipulation (*that you will do on the homework*), you should be able to demonstrate that

$$\delta W_2 = -2\Big(\delta W_1\Big) \tag{4.99}$$

After a few more steps (*which you will also do on the homework*), you should find that

$$\delta W_V = -\delta W_Q \tag{4.100}$$

where the subscript denotes the quantity being held fixed. In words, the change in energy at fixed potentials is the negative of the energy change at fixed charges. Meanwhile, at fixed potentials, the mechanical force for a generalized displacement is

$$F_\xi = +\left(\frac{\partial W}{\partial \xi}\right)_V \tag{4.101}$$

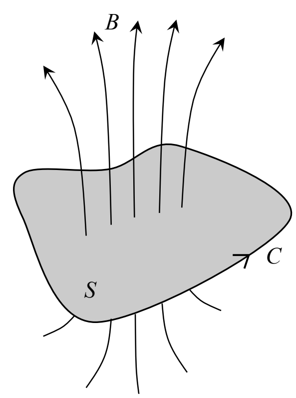
Energy in the Magnetic Field

(*Jackson: pages 212-215*)

In discussing steady-state magnetic fields in Chapter 5 to date, we have avoided the question of field energy and energy density. This is because the creation of a steady-state configuration of currents and associated magnetic fields involves an initial transient period during which the currents and fields are brought from zero to the final values. Time-varying fields, and hence $d\vec{B}/dt$ via Faraday's law, imply the presence of induced electromotive forces that cause the sources of current to do work. Since the energy in the field is, by definition, the total work done to establish the fields from a state of zero field, we must consider these contributions.

For starters, suppose that we have only a single circuit with a constant current I flowing in it.

Imagine the single circuit to be defined by a conducting wire stretched along a stationary closed curve C.



If the flux through the circuit changes, an electromotive force $\mathcal{E}$ is induced around it. Recall that $\mathcal{E}$ is just the integral of the electric field around the curve C:

$$\mathcal{E} = \int_C \vec{E} \cdot d\vec{r}$$

that is, it is the work done on a unit charge moving around the curve C.

Meanwhile, the integral of the magnetic field over the surface S is called the magnetic flux Φ (I generally stick to Jackson's symbols, but it's too much to change this to F, very non-standard use by Jackson!), so that

$$\Phi = \int_S \vec{B} \cdot d\vec{S}$$

From Faraday's law, we have

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

where the minus sign is just Lenz's law.

Now, let's return to our problem. Suppose a constant current I flows along the curve C. We know that this gives rise to a magnetic field and hence a flux $\Phi = \int_S \vec{B} \cdot d\vec{S}$ through the surface S bounded by C.

Now, increase the current I. We know this will increase the flux Φ. But the increase in flux Φ will induce an emf around the curve C. The minus sign in Lenz's law makes this act to resist the change in current. So, to overcome this emf and keep the current constant in the circuit, the sources of that current must do work. The rate of doing that work, or power, from the current source generating the field is then

$$\frac{dW}{dt} = -I\mathcal{E}$$

where we've put a minus sign because the work is being done to overcome the emf and keep the current constant. Combining this with the Faraday law expression for $\mathcal{E}$, we get

$$\frac{dW}{dt} = I\frac{d\Phi}{dt}$$

Thus, if a circuit carrying a current I has a change in flux of $\delta\Phi$, the work done by the sources is

$$\delta W = I\,\delta\Phi$$

Now, consider the work done in establishing a general steady-state distribution of currents and fields. Assume that the buildup occurs at an infinitesimal rate so that $\vec{\nabla}\cdot\vec{J} = 0$ holds to any desired degree of accuracy. Then, we can break up the current distribution into a network of elementary current loops, the typical one being an elemental tube of cross sectional area $\Delta\sigma$ following a closed path C and spanned by a surface S with normal $\hat{n}$. So the elemental current in this loop is $I = J\Delta\sigma$. Meanwhile, the change in flux around the area element $\Delta\sigma$ is

$$\delta\Phi = \int_S \hat{n}\cdot\delta\vec{B}\,da$$

So, using $\delta W = I\,\delta\Phi$, the increment of work done against the induced emf is

$$\Delta(\delta W) = J\Delta\sigma\int_S \delta\vec{B}\cdot\hat{n}\,da$$

where we have an extra Δ on the left hand side because we are considering only one elemental circuit.

Writing $\vec{B}$ in terms of the vector potential $\vec{A}$, we get

$$\Delta(\delta W) = J\Delta\sigma\int_S \left(\vec{\nabla}\times\delta\vec{A}\right)\cdot\hat{n}\,da$$

Apply Stokes' theorem

$$\Delta(\delta W) = J\Delta\sigma\oint_C \delta\vec{A}\cdot d\vec{l}$$

But, by definition,

$$J\Delta\sigma\,d\vec{l} = \vec{J}\,d^3x$$

since $d\vec{l}$ is parallel to $\vec{J}$, which means that the sum over all such elemental loops will be the volume integral.

Therefore, the total increment of work done by the external sources due to a change $\delta\vec{A}(\vec{x})$ in the vector potential is

$$\delta W = \int \delta\vec{A}\cdot\vec{J}\,d^3x \tag{5.144}$$

Let's rewrite this in terms of $\vec{H}$ (and $\vec{B}$). Use Ampere's law

$$\vec{\nabla}\times\vec{H} = \vec{J}$$

to get

$$\delta W = \int \delta\vec{A}\cdot\left(\vec{\nabla}\times\vec{H}\right)d^3x \tag{5.145}$$

Then, use the vector identity

$$\vec{\nabla}\cdot\left(\vec{P}\times\vec{Q}\right)=\vec{Q}\cdot\left(\vec{\nabla}\times\vec{P}\right)-\vec{P}\cdot\left(\vec{\nabla}\times\vec{Q}\right)$$

to transform equation (5.145) to

$$\delta W=\int\left[\vec{H}\cdot\left(\vec{\nabla}\times\delta\vec{A}\right)+\vec{\nabla}\cdot\left(\vec{H}\times\delta\vec{A}\right)\right]\,d^3x \tag{5.146}$$

If the field distribution is localized, then the second integral on the right vanishes. Reintroducing $\vec{\nabla}\times\delta\vec{A}=\delta\vec{B}$, the energy increment becomes

$$\delta W=\int\vec{H}\cdot\delta\vec{B}\,d^3x \tag{5.147}$$

This relation is applicable to *all magnetic media*, including ferromagnetic substances.

It is the magnetic equivalent of the electrostatic equation (4.86): $\delta W=\int\vec{E}\cdot\delta\vec{D}\,d^3x$.

If we assume that the medium is paramagnetic or diamagnetic, so that a linear relation exists between $\vec{B}$ and $\vec{H}$, then

$$\vec{H}\cdot\delta\vec{B}=\frac{1}{2}\,\delta\left(\vec{H}\cdot\vec{B}\right)$$

If we now bring the fields up from zero to their final values, the total magnetic energy will be

$$W=\frac{1}{2}\int\vec{H}\cdot\vec{B}\,d^3x \tag{5.148}$$

and this is the magnetic analog of equation (4.89): $W=\dfrac{1}{2}\int\vec{E}\cdot\vec{D}\,d^3x$.

Meanwhile, from equation (5.144), we can write

$$W=\frac{1}{2}\int\vec{J}\cdot\vec{A}\,d^3x \tag{5.149}$$

assuming a linear relation between $\vec{J}$ and $\vec{A}$.

This is the magnetic analog of equation (1.53): $W=\dfrac{1}{2}\int\rho(\vec{x})\,\Phi(\vec{x})\,d^3x$.

When an object of permeability μ_1 is introduced into a region which has an existing magnetic induction $\vec{B}_0$ (and magnetic field $\vec{H}_0$), and permeability μ_0, then it can be verified that for fixed sources of the field

$$W=\frac{1}{2}\int_{V_1}\left(\vec{B}\cdot\vec{H}_0-\vec{H}\cdot\vec{B}_0\right)d^3x$$

where $\vec{B}$ and $\vec{H}$ are the fields after the object is in place, and the integration is over the volume V_1 of the object.

The equation at the bottom of the previous page can be written in the alternative forms:

$$W = \frac{1}{2}\int_{V_1} \Big(\mu_1 - \mu_0\Big)\,\vec{H}\cdot\vec{H}_0\,d^3x$$

and

$$W = \frac{1}{2}\int_{V_1} \left(\frac{1}{\mu_0} - \frac{1}{\mu_1}\right)\vec{B}\cdot\vec{B}_0\,d^3x$$

Both μ_1 and μ_0 can be functions of position, but they are assumed to be independent of field strength.

If the object is in otherwise free space, the change in energy can be expressed in terms of the magnetization as

$$W = \frac{1}{2}\int_{V_1} \vec{M}\cdot\vec{B}_0\,d^3x \tag{5.150}$$

This is equivalent to the electrostatic result in equation (4.93), except for the sign. The sign is different because the energy W consists of the total energy change occurring when the permeable body is introduced into the field, including the work done by the sources against the induced electromotive forces. In this respect, the magnetic problem with fixed currents is analogous to the electrostatic problem with fixed potentials on the surfaces that determine the fields.

By an analysis equivalent to that of the electrostatic problem done above, we can show that for a small displacement, the work done against the induced emf is twice as large as, and opposite in sign to, the change in potential energy of the body.

Thus, to find the force acting on the body, we consider a generalized displacement ξ and calculate the positive derivative of W with respect to the displacement:

$$F_\xi = \left(\frac{\partial W}{\partial \xi}\right)_J \tag{5.151}$$

where the subscript J implies fixed source currents.