PHY 420 Spring 2021

Homework 7 solutions—due by 9:00 PM, Friday, May 28

1. In class, we learned that finding a group of linear transformations that leaves $x \cdot x$ invariant is equivalent to finding all square 4×4 matrices A which, when they transform the coordinates as $x' = Ax$, will leave the norm (x, gx) invariant, that is, they will ensure that $x' \cdot x' = x \cdot x$. Using the six fundamental matrices S_i and K_i $(i = 1, 2, 3)$ in equation (11.91), Jackson constructed the matrix A as

$$A = e^{-\vec{\omega}\cdot\vec{S}-\vec{\zeta}\cdot\vec{K}}$$

where $\vec{\omega}$ and $\vec{\zeta}$ are constant 3-vectors whose components correspond to the six parameters of the transformation. In class, you constructed A for the case of no rotation and a boost along the x^1 axis. In this problem, you will do another example.

For the case of **rotation about the x^3 axis without any boost**, we have: $\vec{\omega} = \omega\hat{\epsilon}_3, \vec{\zeta} = 0$. By running through steps similar to those on Questions 5 and 6 in the Discussion Worksheet for Week 8—Tue, May 18, show that you get the expected matrix for A; that is, show that you get the matrix you wrote in Question 2(b) for rotations about the x^3 axis on that worksheet.

Solution: With $\vec{\omega} = \omega\hat{\epsilon}_3,\ \vec{\zeta} = 0$, we get

$$A = e^{-\vec{\omega}\cdot\vec{S}-\vec{\zeta}\cdot\vec{K}} = e^{-\vec{\omega}\cdot\vec{S}-0} = e^{-\omega\hat{\epsilon}_3\cdot\vec{S}} = e^{-\omega S_3}$$

since $\hat{\epsilon}_3 \cdot \vec{S} = S_3$. Expanding the exponential, we get

$$A = e^{-\omega S_3} = I - \omega S_3 + \frac{\omega^2}{2!}\,S_3^2 - \frac{\omega^3}{3!}\,S_3^3 + \ldots$$

where I is the 4×4 unit matrix. Now, since $S_3^3 = -S_3$, this becomes

$$A = I - \omega S_3 + \frac{\omega^2}{2}\,S_3^2 - \frac{\omega^3}{6}\left(-S_3\right) + \ldots$$

Gathering terms

$$A = I - S_3\left(\omega - \frac{\omega^3}{6} + \ldots\right) - S_3^2\left(-\frac{\omega^2}{2} + \ldots\right)$$

The quantity in the first parentheses is the expansion for $\sin\omega$, and let's add 1 and subtract 1 in the second parentheses to get

$$A = I - S_3\left(\sin\omega\right) - S_3^2\left(-1 + 1 - \frac{\omega^2}{2} + \ldots\right)$$

Next, since $\cos\omega = 1 - \omega^2/2 + \ldots$, we get

$$A = I - S_3\left(\sin\omega\right) + S_3^2 - S_3^2\left(\cos\omega\right)$$

Inserting the explicit forms of the matrices for S_3 and S_3^2 calculated in the previous class, then multiplying by $\sin\omega$ or $\cos\omega$ as appropriate and adding all of the above, we should end up with

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv R_3$$

where I've put $\omega \to -\omega$ to match the matrix written in Question 2(b) on the worksheet for Week 8—Tue, May 18, because the rotation implied by the matrix on the left above is clockwise, whereas the rotation matrix we wrote in Question 2(b) on the May 18 worksheet was written for counterclockwise rotations.

2. Express the Lorentz scalar $F^{\alpha\beta}F_{\alpha\beta}$ in terms of $\vec{E}$ and $\vec{B}$, where $F^{\alpha\beta}$ is given by

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{1}$$

and $F_{\alpha\beta}$ can be obtained from $F^{\alpha\beta}$ by the procedure you worked out on the class worksheet (i.e., by putting $E_i \to -E_i$, and leaving B_i unchanged).

Solution: The quantity $F^{\alpha\beta}F_{\alpha\beta}$ is just the scalar product, so all we need to do is multiply the matrices corresponding to $F^{\alpha\beta}$ and $F_{\alpha\beta}$ term by term and write the sum. Since

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2}$$

To find $F^{\alpha\beta}F_{\alpha\beta}$, keep in mind that we need to use the repeated index summation. Use the matrices in equation (1) and equation (2) to sum over α first, to get

$$F^{\alpha\beta}F_{\alpha\beta} = F^{0\beta}F_{0\beta} + F^{1\beta}F_{1\beta} + F^{2\beta}F_{2\beta} + F^{3\beta}F_{3\beta} \tag{3}$$

Next, for each value of α, carry out the sum over β:

$$\begin{aligned} F^{\alpha\beta}F_{\alpha\beta} = F^{00}F_{00} + F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03} & \qquad \text{(summing over } \beta \text{ for } \alpha = 0) \\ + F^{10}F_{10} + F^{11}F_{11} + F^{12}F_{12} + F^{13}F_{13} & \qquad \sum_\beta \text{ for } \alpha = 1 \\ + F^{20}F_{20} + F^{21}F_{21} + F^{22}F_{22} + F^{23}F_{23} & \qquad \sum_\beta \text{ for } \alpha = 2 \\ + F^{30}F_{30} + F^{31}F_{31} + F^{32}F_{32} + F^{33}F_{33} & \qquad \sum_\beta \text{ for } \alpha = 3 \end{aligned} \tag{4}$$

Read off the individual elements in equation (4) from equations (1) and (2):

$$\begin{aligned} F^{\alpha\beta}F_{\alpha\beta} &= (0)(0) + (-E_x)(E_x) + (-E_y)(E_y) + (-E_z)(E_z) \\ &\quad (E_x)(-E_x) + (0)(0) + (-B_z)(-B_z) + (B_y)(B_y) \\ &\quad (E_y)(-E_y) + (B_z)(B_z) + (0)(0) + (-B_x)(-B_x) \\ &\quad (E_z)(-E_z) + (-B_y)(-B_y) + (B_x)(B_x) + (0)(0) \\ &= -2E_x^2 - 2E_y^2 - 2E_z^2 + 2B_x^2 + 2B_y^2 + 2B_z^2 \\ &= -2|\vec{E}|^2 + 2|\vec{B}|^2 \end{aligned}$$

Therefore

$$F^{\alpha\beta}F_{\alpha\beta} = 2\Big(|\vec{B}|^2 - |\vec{E}|^2\Big)$$

3. Consider the fundamental matrices $S_1, S_2, S_3, K_1, K_2, K_3$ written in equation (11.91) in Jackson. By explicit matrix multiplication, find the commutators

$$[S_2, S_3], \qquad [S_2, K_3], \qquad \text{and} \qquad [K_2, K_3]$$

Solution: We know the results from the commutation relations; this is just an opportunity to sharpen matrix multiplication skills. For reference, the matrices S_1, S_2, and so on, are written on page 5 of the Class Summary for Week 8—Tue, May 18. For the first one, we have

$$[S_2, S_3] = S_2 S_3 - S_3 S_2$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = S_1}$$

which we know is correct, since $[S_i, S_j] = \epsilon_{ijk}\, S_k$, so $[S_2, S_3] = \epsilon_{231}\, S_1 = +S_1$. Next

$$[S_2, K_3] = S_2 K_3 - K_3 S_2$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = K_1}$$

which we know is correct, since $[S_i, K_j] = \epsilon_{ijk}\, K_k$, so $[S_2, S_3] = \epsilon_{231}\, K_1 = +K_1$. Finally

$$[K_2, K_3] = K_2 K_3 - K_3 K_2$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -S_1}$$

which we know is correct, since $[K_i, K_j] = -\epsilon_{ijk}\, S_k$, so $[K_2, K_3] = -\epsilon_{231}\, S_1 = -S_1$.

4. In class, we wrote the field-strength tensor $F^{\alpha\beta}$ starting from the Maxwell equations. You will now derive the elements of $F^{\alpha\beta}$ by writing $\vec{E}$ and $\vec{B}$ in terms of Φ and $\vec{A}$.

(a) Write down ***all the components*** of $\vec{E}$ and $\vec{B}$ using the ∂^α notation.

Solution: Let's begin by reminding ourselves about notation, which is the most confusing part of this problem. The **contravariant** ∂^α is such that we have

$$\partial^0 = \frac{\partial}{\partial x^0}, \qquad \partial^1 = -\frac{\partial}{\partial x^1}, \qquad \partial^2 = -\frac{\partial}{\partial x^2}, \qquad \partial^3 = -\frac{\partial}{\partial x^3} \tag{5}$$

and the 4-vector potential is $A^\alpha = (\Phi, \vec{A})$, which means that $\Phi = A^0$.

Recall that the fields $\vec{E}$ and $\vec{B}$ can be expressed in terms of the potentials as

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi \qquad \text{and} \qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{6}$$

From equation (6) above, we get that

$$E_1 = -\frac{1}{c}\frac{\partial A^1}{\partial t} - \frac{\partial \Phi}{\partial x^1} = -\frac{\partial A^1}{\partial (ct)} - \frac{\partial A^0}{\partial x^1} = -\frac{\partial A^1}{\partial x^0} - \frac{\partial A^0}{\partial x^1} \tag{7}$$

Writing the right hand side of equation (7) in terms of the expressions in equation (5), we get

$$E_1 = -\partial^0 A^1 - (-\partial^1 A^0)$$

so that

$$\boxed{E_1 = -\left(\partial^0 A^1 - \partial^1 A^0\right)} \tag{8}$$

It looks like E_1 involves the time component x_0 and the component x_1. So, by analogy

$$\boxed{E_2 = -\left(\partial^0 A^2 - \partial^2 A^0\right)} \qquad \text{and} \qquad \boxed{E_3 = -\left(\partial^0 A^3 - \partial^3 A^0\right)} \tag{9}$$

Meanwhile, B's involve a cross product, $\vec{\nabla} \times \vec{A}$, so we have

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{\epsilon}_1 & \hat{\epsilon}_2 & \hat{\epsilon}_3 \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ A^1 & A^2 & A^3 \end{vmatrix} = \hat{\epsilon}_1 \left(\frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}\right) + \hat{\epsilon}_2 \left(\frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1}\right) + \hat{\epsilon}_3 \left(\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\right)$$

From this, we have

$$B_1 = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = -\partial^2 A^3 - (-\partial^3 A^2) \tag{10}$$

where I've used the expressions from equation (5). Therefore

$$\boxed{B_1 = -\left(\partial^2 A^3 - \partial^3 A^2\right)} \tag{11}$$

By inspection, we get

$$\boxed{B_2 = -\left(\partial^3 A^1 - \partial^1 A^3\right)} \qquad \text{and} \qquad \boxed{B_3 = -\left(\partial^1 A^2 - \partial^2 A^1\right)} \tag{12}$$

(b) Show that the components of $\vec{E}$ and $\vec{B}$ you obtained above are the elements of the field tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{13}$$

by using equation (13) to explicitly generate ***all*** elements of $F^{\alpha\beta}$, and comparing your results to the expressions you obtained in part (a) and referencing equation (1) written in Question 2.

Solution: It is straightforward to write equation (8) for E_1 in terms of $F^{\alpha\beta}$ given in equation (13):

$$E_1 = -\left(\partial^0 A^1 - \partial^1 A^0\right) = -F^{01}$$

Moreover, we can also write from this same equation (8) that

$$E_1 = -\left(\partial^0 A^1 - \partial^1 A^0\right) = -\partial^0 A^1 + \partial^1 A^0 = \partial^1 A^0 - \partial^0 A^1 = F^{10}$$

Therefore

$$F^{01} = -E_1 \qquad \text{and} \qquad F^{10} = E_1 \tag{14}$$

Likewise, from equation (9) for E_2 and E_3, we get that

$$E_2 = -\left(\partial^0 A^2 - \partial^2 A^0\right) = -F^{02} = F^{20} \qquad \text{and} \qquad E_3 = -\left(\partial^0 A^3 - \partial^3 A^0\right) = -F^{03} = F^{30}$$

so that

$$F^{02} = -E_2 \qquad F^{20} = E_2 \qquad F^{03} = -E_3 \qquad F^{30} = E_3 \tag{15}$$

Finally, by inspection of equations (11) and (12) written above, we get that

$$B_1 = -F^{23} = F^{32}; \qquad B_2 = -F^{31} = F^{13}; \qquad B_3 = -F^{12} = F^{21} \tag{16}$$

Also, it is clear from equation (13) that $F^{\alpha\alpha} = 0$; thus, all diagonal terms in the matrix for $F^{\alpha\beta}$ will be zero. Finally, therefore, we can use equations (14), (15), and (16) to construct the $F^{\alpha\beta}$ matrix term by term

$$F^{\alpha\beta} = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

which is identical to the matrix for $F^{\alpha\beta}$ written in equation (1) written in Question 2, except that here I've replaced $E_x = E_1$, and so on.