PHY 474 Spring 2021

Homework 5 solutions

1. In class, we calculated the Main Sequence lifetime using a general relation between luminosity and mass. Let's try a different approach. The Main Sequence star α Centauri A has luminosity $L = 1.5\ L_\odot$, and mass $M = 1.1\ M_\odot$. Assume α Centauri A was initially made of pure hydrogen, and that about 10% of its mass is converted to helium during its Main Sequence lifetime.

Useful Information: $M_\odot = 1.99 \times 10^{30}$ kg, $L_\odot = 3.828 \times 10^{26}$ W.

(a) How many helium nuclei are produced during the Main Sequence lifetime of α Centauri A?

Solution: We know that 10% of the 1.1 $M_\odot$ star will be converted to helium during its Main Sequence lifetime. Since the mass of each helium atom is $4m_p$, the number N_{tot} of helium nuclei produced during its Main Sequence lifetime is given by

$$N_{\text{tot}} = \frac{0.1(1.1M_\odot)}{4\,m_p}$$

Putting in the numbers, we get

$$N_{\text{tot}} = \frac{0.1\,(1.1)(1.99 \times 10^{30}\ \text{kg})}{4(1.67 \times 10^{-27}\ \text{kg})} = \boxed{3.3 \times 10^{55}}$$

(b) How much energy does α Centauri A produce during its Main Sequence lifetime? Recall that the energy released when four hydrogen nuclei are fused into one helium nucleus in the pp-chain is 26.7 MeV. Express your answer in J.

Solution: This is the answer in part (a) times 26.7 MeV, thus

$$E_{\text{tot}} = (26.7\ \text{MeV})\,N_{\text{tot}} = (26.7\ \text{MeV})\,(3.3 \times 10^{55}) = 8.8 \times 10^{56}\ \text{MeV}$$

We need to convert this to J, so

$$E_{\text{tot}} = 8.8 \times 10^{56}\ \text{MeV}\left(\frac{10^6\ \text{eV}}{\text{MeV}}\right)\left(\frac{1.602 \times 10^{-19}\ \text{J}}{\text{eV}}\right) = \boxed{1.4 \times 10^{44}\ \text{J}}$$

(c) Compute the Main Sequence lifetime of α Centauri A in Gyr, where 1 Gyr = 10^9 yr.

Solution: Note that the idea was not to do it the way we learned in class. Instead, we can compute the Main Sequence lifetime using our result in part (b). Since E_{tot} is the energy produced during the lifetime of α Centauri A, and this is radiated away at 1.5 $L_\odot$, we can find the lifetime on the Main Sequence by doing

$$t_{\text{MS}} = \frac{E_{\text{tot}}}{L} = \frac{1.4 \times 10^{44}\ \text{J}}{1.5\,(3.828 \times 10^{26}\ \text{W})} = \frac{2.438 \times 10^{17}\ \text{s}}{(3600)(24)(365)\ \text{s/yr}} = 7.7 \times 10^9\ \text{yr}$$

Therefore, the Main Sequence lifetime of α Centauri A is $\boxed{7.7\ \text{Gyr}}$.

2. The kinetic energy of protons at which quantum mechanical tunneling (through the Coulomb barrier) has a significant probability is

$$E \approx \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{2m_p}{h^2}$$

(a) Use this equation to show that for helium to fuse into carbon on the horizontal branch, the kinetic energy per particle at the star's core (and hence T_c) has to be 64 times higher than is required for hydrogen fusion on the Main Sequence.

Solution: It is worth noting first that the expression above is ***for protons only***. Thus, if we were to apply it to helium, we would need to replace e by $2e$, since a helium nucleus has two protons. Meanwhile, a helium nucleus also has two neutrons, so we would need to replace m_p by $4m_p$.

Since we have $(e^2)^2$, replacing e by $2e$ would mean a factor of $(2^2)^2$, equal to 4^2 or 16. This would be multiplied by a factor of 4 in going from m_p to $4m_p$, and thus the kinetic energy per particle, and hence T_c, would be (16) 4 or $\boxed{\text{64 times}}$ higher for helium fusion than is required for hydrogen fusion.

(b) Use the equation for E above, together with $E = (3/2)\,kT$, to compute the temperature for hydrogen fusion and helium fusion.

Solution: Setting

$$\frac{3}{2}\,kT_c \approx \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{2m_p}{h^2}$$

we get

$$T_c \approx \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{4m_p}{3kh^2}$$

where I've subscripted T_c to indicate we are finding the core temperature. Putting in the numbers, we get

$$T_c \approx \left[\frac{(1.602\times10^{-19}\ \text{C})^2}{4\pi\,(8.85\times10^{-12}\ \text{F/m})}\right]^2 \frac{4\,(1.67\times10^{-27}\ \text{kg})}{3(1.38\times10^{-23}\ \text{J/K})\,(6.626\times10^{-34}\ \text{m}^2\,\text{kg/s})^2} = \boxed{1.96\times10^{7}\ \text{K}}$$

For helium fusion, a factor of 64 gives $\boxed{1.2\times10^{9}\ \text{K}}$.

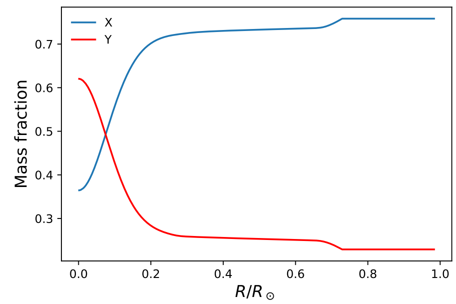
Note: The answer of 20 million K for H-fusion comes out to be somewhat higher than the current temperature in the Sun's core, and certainly higher than the core temperature at ZAMS. We get better results if we use μm_p instead of m_p, where μ is the mean mass per particle; recall that $\mu = 4/(3+5X-Z)$, so that at ZAMS, $\mu = 0.6$. Then, we would get $T_c = 1.1\times10^{7}$ K, or 11 million K.

For helium fusion, the answer we obtained above is about a factor of 10 times higher than the 100 million K that is used as a rule of thumb for the onset of He fusion. With the lower value of H-fusion, we would still get 704 million K, still about a factor of 7 times too high.

3. In class, we learned about the Standard Solar Model. Plots of this model are very useful for understanding what is going on in the Sun. The file provided was downloaded from `sns.ias.edu`; the first row is self-explanatory. Read the data into your software of choice and plot the following.

(a) Plot X *vs.* $R/R_\odot$ and Y *vs.* $R/R_\odot$, where X is the hydrogen mass fraction and Y is the helium mass fraction. Put both on the same plot. Submit *only* this program.

Solution: The plot is shown on the right.

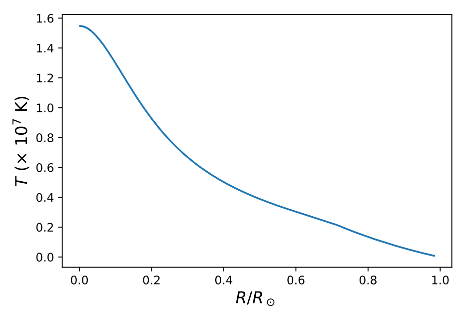


(b) Comment on your plot in part (a), especially any interesting features that stand out.

Solution: Two features are of interest: (i) we see that since nuclear fusion has been going on in the core of the Sun for the last 4.5 billion years, the hydrogen mass fraction (X) at the center is now smaller compared to its initial value (which would have been the value X now has near the surface), and consequently the helium mass fraction (Y) has increased; since they constitute the overwhelming proportion of the Sun's contents, the plot of X is almost a mirror symmetry of Y, and (ii) there is a slight kink in both X and Y near $R = 0.7\ R_\odot$ where the Sun changes from a radiative to a convective zone, reflecting how evenly convection mixes the contents in the outer parts of the Sun.

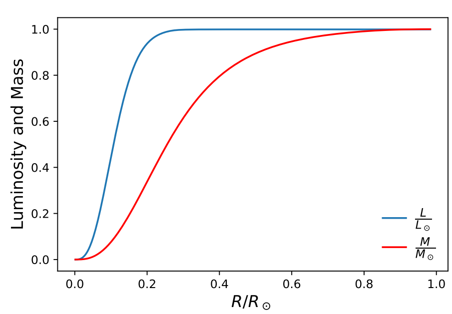
(c) Plot T *vs.* $R/R_\odot$.

Solution: The plot is shown on the right.



(d) Plot $L/L_\odot$ *vs.* $R/R_\odot$ and also $M/M_\odot$ *vs.* $R/R_\odot$, putting both on the same.

Solution: The plot is shown below.



4. In class we learned that all stars lose mass. There are various empirical relationships for such mass loss. One of them is Reimer's law for the mass loss rate on the Asymptotic Giant Branch, and it is given by

$$\frac{dM}{dt} = -\frac{c\eta LR}{M}$$

where L is the luminosity, R is the radius, and M is the mass of the AGB star in solar units, and dM/dt is the mass loss rate of the AGB star in $M_\odot$ yr^{-1}. Meanwhile, $c = 4 \times 10^{-13}$ units is a constant, and η is a free parameter $\sim$1; for this problem, assume $\eta = 1$.

Integrate Reimer's law to derive an expression for the mass $M(t)$ of the AGB star as a function of time. Use boundary conditions M_0 at time $t = 0$, and mass equal to M at time t. You can make the simplifying assumption that L and R of the AGB star do not change as it loses mass.

Solution: We just need to integrate. Separating variables, we get

$$M\,dM = -c\eta LR\,dt$$

Integrating with the given boundary conditions, we get

$$\int_{M_0}^{M} M\,dM = -\int_0^t c\eta LR\,dt$$

As directed, make the simplifying assumption that L and R of the AGB star do not change as it loses mass, to get

$$\int_{M_0}^{M} M\,dM = -c\eta LR\int_0^t dt$$

so that

$$\left[\frac{M^2}{2}\right]_{M_0}^{M} = -c\eta LR\left[t\right]_0^t$$

Thus

$$M^2 - M_0^2 = -2c\eta LR\,t$$

Therefore, we get finally that the mass $M(t)$ of the AGB star as a function of time is given by

$$\boxed{M(t) = \left[M_0^2 - 2c\eta LR\,t\right]^{1/2}}$$