1 Partial Differential Equations

Let's continue from where we left off last lecture. **The Heat Equation.** We now look at an example of a PDE that is 2-D in spatial variables and in which time evolution plays no role. We study the *heat equation* in the *steady state* case.

The general heat equation for a homogeneous body is given by

$$\nabla^2 u(x,y,z,t) = \frac{1}{c^2}u_t, \quad c^2 = \frac{K}{\sigma\rho} \tag{1}$$

where u is the temperature of the body at position (x,y) and time, t, K is the thermal conductivity, σ the specific heat of the body, and ρ the density of the body.

In the steady state case, $u_t = 0$ so that heat equation reduces to the two-dimensional Laplace equation,

$$u_{xx} + u_{yy} = 0. \tag{2}$$

We will explore numerical solutions to Eq.(2). The finite differencing of Eq.(2) proceeds as usual. First lets look at the case in which the body is rectangular in shape with dimensions $a \times b$. The grid we introduce is

$$\begin{aligned} x_i &= ih_x, \quad i = 0, 1, \ldots, N_x \\ y_i &= jh_y, \quad j = 0, 1, \ldots, N_y \end{aligned}$$

Using the notation,

$$u_{i,j} = u(x_i, y_j)$$

we have that

$$\underbrace{\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2}}_{u_{xx}} + \underbrace{\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2}}_{u_{yy}} = 0.$$

Solving for $u_{i,j}$ yields,

$$u_{i,j} = \frac{h_x^2 h_y^2}{2h_x^2 + 2h_y^2}\left[\frac{u_{i+1,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} + u_{i,j-1}}{h_y^2}\right]. \tag{3}$$

(1) Download the code `MyHeatEq.m` from the course *D2L* page. Identify by line number, where the code is doing what the algorithm you developed needs done. For a heat equation on a Cartesian grid, it can be shown that the optimal α for use in the SOR process is

$$\alpha = \frac{4}{2 + \sqrt{4 - \left[\cos\frac{\pi}{N_x} + \cos\frac{\pi}{N_y}\right]^2}} - 1.$$

Make sure you identify exactly where the *SOR* process is occuring. Also try to understand what lines 25 – 28 do.

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(2) Explore some of the consequences of wave equation by starting problem 2 from the homework.

(3) Investigate the stability issue by starting homework problem 3.

(4) Thus far you've looked at the string equation with the ends fixed. Now look at what happens when one end is not constrained by starting homework problem 4.