PHY 474/374 Spring 2021

Week 2—Wednesday, April 7—Discussion Worksheet

Hydrostatic Equilibrium

Today, we will discuss the balance of forces inside a star, known as ***hydrostatic equilibrium***. We already know from our discussion of the dynamical timescale that the forces inside a star must be perfectly balanced. *Dalsgaard* considers a spherical shell of radius r and thickness dr in the star. By considering that this shell is subject to the gravitational force and pressure, *Dalsgaard* obtains the equation of motion

$$\rho \frac{d^2r}{dt^2} = -\rho \frac{Gm}{r^2} - \frac{dP}{dr} \tag{4.3}$$

where ρ is the density, and $m = m(r)$ is the mass interior to the shell. Note that equations are numbered to match those in *Dalsgaard*, for easy reference.

1. Using equation (4.3), together with the Ideal Gas Law from the previous class, we can show that the central pressure and temperature of a star with mass M and radius R are given by

$$P_c \simeq \frac{GM^2}{R^4} \qquad \text{and} \qquad T_c \simeq \frac{\mu_c m_p}{k}\frac{GM}{R}$$

where the mean molecular weight μ_c at the center of the star can be found by using the expression from the previous class for the mean molecular weight: $\mu = 4/(3 + 5X - Z)$.

(a) Find P_c and T_c at the center of our Sun. Use $X = 0.35, Z = 0.02$ (because nuclear fusion has reduced the hydrogen mass fraction at the center to 0.35 from 0.73 near the surface).

Useful Information: $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_\odot = 1.99 \times 10^{30}$ kg, $R_\odot = 6.96 \times 10^8$ m, $m_p = 1.67 \times 10^{-27}$ kg, $k = 1.38 \times 10^{-23}$ J/K

$\mu = 4/(3+5X-Z) = 4/(3+5(0.35)-0.02) = 0.846$

$$P_c \simeq \frac{(6.67\times10^{-11}\,\mathrm{Nm^2kg^{-2}})(1.99\times10^{30}\,\mathrm{kg})^2}{(6.96\times10^{8}\,\mathrm{m})^4} = 1.13\times10^{15}\,\mathrm{N/m^2}$$

$$T_c \simeq \frac{(0.846)(1.67\times10^{-27}\,\mathrm{kg})}{(1.38\times10^{-23}\,\mathrm{J/K})}\frac{(6.67\times10^{-11}\,\mathrm{Nm^2kg^{-2}})(1.99\times10^{30}\,\mathrm{kg})}{(6.96\times10^{8}\,\mathrm{m})}$$

$$= 1.95\times10^{7}\,\mathrm{K}$$

(b) How well did this order of magnitude estimate do? More realistic models of the Sun find $P_c = 2.4 \times 10^{16}$ N/m^2, and $T_c = 15$ million K.

Close on orders of magnitude

2. **Hydrostatic equilibrium** requires that the left hand side of equation (4.3) be equal to zero, so that

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}$$

(a) Use the expression for hydrostatic equilibrium written above, together with the expression $dm/dr = 4\pi r^2 \rho$, to show that

$$\frac{d}{dr}\left[P + \frac{Gm^2}{8\pi r^4}\right] = -\frac{Gm^2}{2\pi r^5}$$

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2} \rightarrow -\frac{Gm}{4\pi r^4} 4\pi r^2 \rho \rightarrow -\frac{Gm}{4\pi r^4}\frac{dm}{dr}$$

$$\rightarrow -\frac{d}{dr}\left(\frac{Gm^2}{8\pi r^4}\right) - \frac{Gm^2}{2\pi r^5}$$

$$\frac{d}{dr}\left(P + \frac{Gm^2}{8\pi r^4}\right) = -\frac{Gm^2}{2\pi r^5}$$

(b) The expression you derived above shows that the quantity $\Psi(r) = P + Gm^2/8\pi r^4$ is a decreasing function of r. Explain why $\Psi(0) = P_c$, that is, show that the 2nd term of $\Psi(r)$ goes to zero at small r.

$\Psi(r) = P + \frac{GM^2}{8\pi r^4}$ $\quad \rho \simeq \frac{m}{r^3}$, $V = \frac{4\pi}{3} r^3$, thus $m \propto r^3$

$\frac{Gm^2}{8\pi r^4} \propto \frac{(r^3)^2}{r^4} = r^2$ Thus, $\frac{Gm^2}{8\pi r^4} \propto r^4 \rightarrow 0$, at small r

$\Psi(0) = P_c$

(c) Meanwhile, at the surface of the star, P is essentially zero. Use your result in part (b), together with the fact that $\Psi(r)$ is a decreasing function of r, to show that the limit to the central pressure is

$$P_c > \frac{GM^2}{8\pi R^4}$$

$\Psi(0) = P_c$

$\Psi(R) = 0 + \frac{GM^2}{8\pi R^4}$

Surface — zero at Surface — at $r = R$, $m = M$

Since $\Psi(r)$ is a decreasing function of r, we must have

$\Psi(0) > \Psi(R)$

$P_c > \frac{GM^2}{8\pi R^4}$

3. The **Virial Theorem** tells us that

$$\Omega + 2U = 0$$

where Ω is the gravitational potential energy and U is the (total) internal energy.

The Virial Theorem is very powerful. Here is one example.

(a) Use the Virial Theorem to find an expression the *average temperature* of the Sun.

Note: For a spherical object, $\Omega = -\frac{3}{5}\frac{GM^2}{R}$. Meanwhile, $U = \frac{3}{2}NkT$. From the previous class, we can write $N = nV$, and so $N = M/\mu m_p$.

$$2U = -\Omega \Rightarrow 2\left(\frac{3}{2}NkT\right) = -\left(-3/5\ \frac{GM^2}{R}\right)$$

$$T = \frac{1}{5}\frac{1}{Nk}\frac{GM^2}{R}$$

$$= \frac{1}{5}\frac{1}{(M/\mu m_p)}\frac{GM}{R}$$

$$= \frac{1}{5}\frac{\mu m_p}{k}\frac{GM}{R}$$

(b) Use the expression you derived above to calculate the average temperature of the Sun. Assume that the Sun is pure, ionized hydrogen, so that $X = 1$ (thus $Z = 0$) in the expression for μ.

$$\mu = 4/(3+5X-Z) = 4/(3+5(1)-0) = 1/2 = 0.5$$

$$T = \frac{(6.67\times10^{-11}\,\mathrm{Nm^2\,kg^{-2}})(1.99\times10^{30}\,\mathrm{kg})(0.5)(1.67\times10^{-27}\,\mathrm{kg})}{5\,(1.38\times10^{-23}\,\mathrm{J/K})(6.96\times10^{8}\,\mathrm{m})}$$

$$= 2.31\times10^{6}\,\mathrm{K}$$

(c) Discuss what your result in part (b) tells you about the temperature at the Sun's core, given that the temperature at the solar surface can be found by measurement to be about 5800 K.

Temperature, T_c, at the Sun's core must be very high for the average to be 6000 K and T_c = 2.3 million K.

4. Although solutions to the equation of hydrostatic equilibrium require more material to be discussed later in the quarter, there are two exceptions, one when ρ is a known function of r (an example of which you'll do on the homework), and the other when ρ is a known function of P, a particular example of which is a relation of the form

$$P(r) = K\Big[\rho(r)\Big]^{\gamma} \tag{4.36}$$

where K and *gamma* are constants. This is called a ***polytropic relation*** and the resulting models are called ***polytropic models***.

(a) To obtain the equation satisfied by polytropic models, use the hydrostatic equilibrium equation together with the expression for dm/dr, both of which are given in Question 2, to show that

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho r^2$$

$$\rightarrow \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr}$$

$$\rightarrow \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho r^2$$

(b) Then, substitute equation (4.36) in the expression you obtained in part (a) above to show that

$$K\gamma\frac{d}{dr}\left(r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right) = -4\pi G\rho r^2$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\overbrace{\frac{dP}{dr}}^{(4.36)}\right) = -4\pi G\rho r^2$$

$$\rightarrow \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}K\big[\rho(r)\big]^{\gamma}\right) = -4\pi G\rho r^2$$

$$\rightarrow K\frac{d}{dr}\left(\frac{r^2}{\rho}\rho^{\gamma-1}\frac{d\rho}{dr}\right) = -4\pi G\rho r^2$$

$$\rightarrow K\frac{d}{dr}\left(r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right) = -4\pi G\rho r^2$$

5. We are in the process of obtaining the equation satisfied by polytropic models.

(a) Introduce a dimensionless measure θ of the density, so that $\rho = \rho_c \theta^n$, where ρ_c is the central density, and the polytropic index $n = 1/(\gamma - 1)$, so that $\gamma = 1 + 1/n$. With these replacements, show that the expression you obtained in Question 4(b) becomes

$$\frac{(n+1)\, K\, \rho_c^{1/n}}{4\pi G \rho_c} \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) = -\theta^n$$

$1/n - 1 = \gamma$

$$K\gamma \, d/dr\left(r^2 \rho^{\gamma-2}\, d\rho/dr\right) = -4\pi G \rho r^2$$

$$\rightarrow K\left(1+\frac{1}{n}\right)\frac{d}{dr}\rho^{((1+1/n)-2)}\left(r^2 \frac{d\rho_c\theta^n}{dr}\right) = -4\pi G\rho r^2$$

$$\rightarrow \frac{K(1+1/n)\, d/dr\, \rho^{((1+1/n)-2)} \cdot \rho_c\left(r^2\, d\theta/dr\right)}{-4\pi G r^2} = -\rho_c\theta^n$$

$$\rightarrow \frac{(n+1)K\rho_c^{1/n}}{4\pi G\rho_c}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

(b) Simplify the equation further by introducing a dimensionless measure ξ of the distance to the center given by $r = \alpha\xi$, where $\alpha^2 = (n+1)\, K\, \rho_c^{1/n}/4\pi G \rho_c$, and show that you get

$$\frac{1}{\xi^2} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n$$

which is known as the **Lane-Emden equation**.

$$\underbrace{\frac{(n+1)K\rho_c^{1/n}}{4\pi G\rho_c}}_{\alpha^2}\underbrace{\frac{1}{r^2}}_{1/\alpha^2\xi^2}\frac{d}{dr}\left(\underbrace{r^2}_{\alpha^2\xi^2}\frac{d\theta}{dr}\right) = -\theta^n$$

$$\rightarrow \frac{\cancel{\alpha^2}}{\cancel{\alpha^2}\xi^2}\frac{d}{\underbrace{dr}_{\cancel{\alpha}\xi}}\left(\cancel{\alpha^2}\xi^2\frac{d\theta}{\underbrace{dr}_{\cancel{\alpha}\xi}}\right) = -\theta^n \rightarrow \boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n}$$