Physics 412—Quantum Mice Inspection Winter 2020

Most universities have an Institutional Animal Care and Use Committee (IACUC), and DePaul is no exception. This committee does periodic inspections of laboratory facilities to make sure the animals are not being mistreated. In particular, the IACUC wants to make sure that any quantum mice kept at DePaul are sufficiently happy and do not exhibit any abnormal behavior.

Although the member of the IACUC are mostly biologists, they do have a rudimentary understanding of quantum mechanics, so they realize they can't expect the mice to be perfectly happy *and* perfectly behaved. They do have high standards, however. When they inspect your lab, they will measure either the attitude or behavior of your mice (but not both), and they expect at least 80% of your mice to be happy upon measurement, or at least 80% of your mice to be passive upon measurement.

How should you prepare for this inspection? Should you put your mice into a pure ensemble, or a mixed ensemble? If a pure ensemble, what state should describe the ensemble? If a mixed ensemble, what states should you use, and with what weights? Or maybe it doesn't matter?!? Or maybe the expectations of the IACUC are too high and this is impossible?

As a reminder, the four properties of our quantum mice are attitude (operator A, eigenstates $|h\rangle$ and $|u\rangle$), behavior (operator B, eigenstates $|p\rangle$ and $|a\rangle$), energy (operator H, eigenstates $|4\rangle$ and $|2\rangle$), and size (operator W, eigenstates $|s\rangle$ and $|l\rangle$). If we use the attitude states as our basis, we can represent the operators as matrices and the states as column vectors as follows:

$$A \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad |h\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |u\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B \leftrightarrow \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad |p\rangle \leftrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}, \qquad |a\rangle \leftrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix},$$

$$H \leftrightarrow \frac{2}{5}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}, \qquad |4\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad |2\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

$$W \leftrightarrow \frac{2}{5}\begin{bmatrix} 21 & -8 \\ -8 & 9 \end{bmatrix}, \qquad |s\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad |l\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Note that H and W share eigenstates because they commute. None of the other operators commute.

(1) What expectation values (or ensemble averages) are the IUCAC expecting, given the 80% requirements for happy or passive mice? Explain.

(2) What ensemble will you create to prepare for the inspection? Explain.

Best of luck!!

My Answer:

(1) If 80% of the mice are happy and 80% are passive, then the relevant expectation values are

$$\langle A\rangle = 0.8 \times (+1) + 0.2 \times (-1) = 0.6, \quad \text{and} \quad \langle B\rangle = 0.8 \times (+1) + 0.2 \times (-1) = 0.6.$$

So those are the values we need to shoot for in the ensemble averages.

(2) The class consensus seemed to be to start with a mixed ensemble, so that's where I will begin. The general form of the density operator for a mixed ensemble is

$$\rho = \sum_{i=1}^{N} w_i |\Psi_i\rangle\langle\Psi_i|,$$

where the ensemble has N different parts, each composed of fraction w_i of the mice, and each described by the state $|\Psi_i\rangle$. There's no law (of quantum mechanics or humanity) that says that the states $|\Psi_i\rangle$ have to be eigenstates of any operators, but starting with the eigenstates of A and B seems reasonable. (I don't see any reason to include the eigenstates of W or H in this analysis.)

With this in mind, I'll pre-calculate some outer products, working as usual in the attitude basis given above:

$$|h\rangle\langle h| \leftrightarrow \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \qquad |u\rangle\langle u| \leftrightarrow \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix},$$

$$|p\rangle\langle p| \leftrightarrow \frac{1}{2}\begin{bmatrix}1\\\mathrm{i}\end{bmatrix}\begin{bmatrix}1 & -\mathrm{i}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -\mathrm{i}\\\mathrm{i} & 1\end{bmatrix}, \qquad |a\rangle\langle a| \leftrightarrow \frac{1}{2}\begin{bmatrix}1\\-\mathrm{i}\end{bmatrix}\begin{bmatrix}1 & \mathrm{i}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & \mathrm{i}\\-\mathrm{i} & 1\end{bmatrix}.$$

From this we see that the terms made from the attitude states only affect the diagonal elements of the density matrix, while those made from the behavior states affect all the elements.

We'll start with a guess for the density matrix; that we want a weight of 0.6 for the happy state and a weight of 0.4 for the passive state. That would give us the following density matrix:

$$\rho = 0.6|h\rangle\langle h| + 0.4|p\rangle\langle p| \leftrightarrow \begin{bmatrix}0.6 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0.2 & -0.2\mathrm{i}\\0.2\mathrm{i} & 0.2\end{bmatrix} = \begin{bmatrix}0.8 & -0.2\mathrm{i}\\0.2\mathrm{i} & 0.2\end{bmatrix}.$$

The ensemble averages are then

$$\overline{\langle A\rangle} = \mathrm{Tr}\,(\rho A) = \mathrm{Tr}\left\{\begin{bmatrix}0.8 & -0.2\mathrm{i}\\0.2\mathrm{i} & 0.2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right\} = \mathrm{Tr}\left\{\begin{bmatrix}0.8 & 0.2\mathrm{i}\\0.2\mathrm{i} & -0.2\end{bmatrix}\right\} = 0.6,$$

$$\overline{\langle B\rangle} = \mathrm{Tr}\,(\rho B) = \mathrm{Tr}\left\{\begin{bmatrix}0.8 & -0.2\mathrm{i}\\0.2\mathrm{i} & 0.2\end{bmatrix}\begin{bmatrix}0 & -\mathrm{i}\\\mathrm{i} & 0\end{bmatrix}\right\} = \mathrm{Tr}\left\{\begin{bmatrix}0.2 & -0.8\mathrm{i}\\0.2\mathrm{i} & 0.2\end{bmatrix}\right\} = 0.4.$$

So that doesn't quite work. The problem here is that the contributions for the ensemble average for B all come from the off-diagonal elements of the density matrix, so the $|h\rangle\langle h|$ doesn't ever increase the number of passive mice.

To investigate this a bit further, I'll pick up on an idea suggested by Noah in class. What properties must the density matrix have in order to satisfy the inspectors? The density matrix must be Hermitian, and its trace must equal 1, so the most general form is

$$\rho \leftrightarrow \begin{bmatrix}a & r\mathrm{e}^{\mathrm{i}\theta}\\r\mathrm{e}^{-\mathrm{i}\theta} & 1-a\end{bmatrix},$$

where a, r, and θ are all real numbers. Using this form, the desired ensemble averages are

$$\overline{\langle A\rangle} = \mathrm{Tr}\,(\rho A) = \mathrm{Tr}\left\{\begin{bmatrix}a & r\mathrm{e}^{\mathrm{i}\theta}\\r\mathrm{e}^{-\mathrm{i}\theta} & 1-a\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right\} = \mathrm{Tr}\left\{\begin{bmatrix}a & -r\mathrm{e}^{\mathrm{i}\theta}\\r\mathrm{e}^{-\mathrm{i}\theta} & a-1\end{bmatrix}\right\} = 2a-1,$$

$$\overline{\langle B\rangle} = \mathrm{Tr}\,(\rho B) = \mathrm{Tr}\left\{\begin{bmatrix}a & r\mathrm{e}^{\mathrm{i}\theta}\\r\mathrm{e}^{-\mathrm{i}\theta} & 1-a\end{bmatrix}\begin{bmatrix}0 & -\mathrm{i}\\\mathrm{i} & 0\end{bmatrix}\right\} = \mathrm{Tr}\left\{\begin{bmatrix}\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta} & -\mathrm{i}a\\\mathrm{i}(1-a) & -\mathrm{i}r\mathrm{e}^{-\mathrm{i}\theta}\end{bmatrix}\right\} = -2r\sin\theta.$$

To get the largest possible value for $\overline{\langle B\rangle}$ we want to take $\theta = 3\pi/2$, so that $\mathrm{e}^{\mathrm{i}\theta} = -\mathrm{i}$. Then the density matrix is

$$\rho \leftrightarrow \begin{bmatrix}a & -\mathrm{i}r\\\mathrm{i}r & 1-a\end{bmatrix},$$

To get $\overline{\langle B\rangle} \geq 0.6$, we then need $r \geq 0.3$, and to get $\overline{\langle A\rangle} \geq 0.6$ we need $a \geq 0.8$. This suggests that we take the coefficient of the $|p\rangle\langle p|$ term to be at least 0.6. That will make the lower right-hand element of the density matrix equal to 0.3, which means the largest the upper right-hand element can be is 0.7, which can be achieved with a coefficient of the $|h\rangle\langle h|$ of 0.4. In other words, the density matrix would be

$$\rho = 0.4|h\rangle\langle h| + 0.6|p\rangle\langle p| \leftrightarrow \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.3 & -0.3\mathrm{i} \\ 0.3\mathrm{i} & 0.3 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.3\mathrm{i} \\ 0.3\mathrm{i} & 0.3 \end{bmatrix}.$$

So this is going to fail, as we can see by finding the ensemble averages:

$$\overline{\langle A\rangle} = \mathrm{Tr}\,(\rho A) = \mathrm{Tr}\left\{\begin{bmatrix} 0.7 & -0.3\mathrm{i} \\ 0.3\mathrm{i} & 0.3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\} = \mathrm{Tr}\left\{\begin{bmatrix} 0.7 & 0.3\mathrm{i} \\ 0.3\mathrm{i} & -0.3 \end{bmatrix}\right\} = 0.4,$$

$$\overline{\langle B\rangle} = \mathrm{Tr}\,(\rho B) = \mathrm{Tr}\left\{\begin{bmatrix} 0.7 & -0.3\mathrm{i} \\ 0.3\mathrm{i} & 0.3 \end{bmatrix}\begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}\right\} = \mathrm{Tr}\left\{\begin{bmatrix} 0.3 & -0.7\mathrm{i} \\ 0.3\mathrm{i} & 0.3 \end{bmatrix}\right\} = 0.6.$$

So we've gotten a sufficient number of passive mice, but have failed on the happy mice.

Rather than continue investigating mixed ensembles, I'll turn to a pure ensemble. Let the state describing the pure ensemble be

$$|\Psi\rangle \leftrightarrow \begin{bmatrix} a \\ r\mathrm{e}^{\mathrm{i}\theta} \end{bmatrix},$$

where a, r, and θ are real. (Of course, these are a different set of a, r, and θ than we used for the density matrix; this is just a convenient form.) Then the expectation values of A and B are

$$\langle A\rangle = \begin{bmatrix} a & r\mathrm{e}^{-\mathrm{i}\theta} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ r\mathrm{e}^{\mathrm{i}\theta} \end{bmatrix} = \begin{bmatrix} a & r\mathrm{e}^{-\mathrm{i}\theta} \end{bmatrix}\begin{bmatrix} a \\ -r\mathrm{e}^{\mathrm{i}\theta} \end{bmatrix} = a^2 - r^2,$$

$$\langle B\rangle = \begin{bmatrix} a & r\mathrm{e}^{-\mathrm{i}\theta} \end{bmatrix}\begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}\begin{bmatrix} a \\ r\mathrm{e}^{\mathrm{i}\theta} \end{bmatrix} = \begin{bmatrix} a & r\mathrm{e}^{-\mathrm{i}\theta} \end{bmatrix}\begin{bmatrix} -\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta} \\ \mathrm{i}a \end{bmatrix} = \mathrm{i}ar\left(-\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right) = 2ar\sin\theta.$$

If we want the largest value of $\langle B\rangle$, we should take $\theta = \pi/2$, so that $\mathrm{e}^{\mathrm{i}\theta} = +\mathrm{i}$, and the state is

$$|\Psi\rangle \leftrightarrow \begin{bmatrix} a \\ \mathrm{i}r \end{bmatrix}.$$

The real numbers a and r need to satisfy

$$a^2 - r^2 \geq 0.6 \quad \text{and} \quad 2ar \geq 0.6.$$

I'm going to guess that these can be satisfied, and even exceeded, and look for a and r to satisfy the following equalities:

$$a^2 - r^2 = \frac{1}{\sqrt{2}} \quad \text{and} \quad 2ar = \frac{1}{\sqrt{2}}.$$

There's a third condition that a and r must satisfy: the state must be normalized, so that $a^2 + r^2 = 1$. Therefore, $r^2 = 1 - a^2$ and

$$a^2 - (1 - a^2) = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad a^2 = \frac{1+\sqrt{2}}{2\sqrt{2}} \quad \text{and} \quad r^2 = \frac{\sqrt{2}-1}{2\sqrt{2}}.$$

It's easy to verify that both conditions above are satisfied with this choice of a and r.

Of course we can use the density matrix formulation for a pure ensemble. The density matrix for this state is

$$\rho = |\Psi\rangle\langle\Psi| \leftrightarrow \begin{bmatrix} a \\ \mathrm{i}r \end{bmatrix}\begin{bmatrix} a & -\mathrm{i}r \end{bmatrix} = \begin{bmatrix} a^2 & -\mathrm{i}ar \\ \mathrm{i}ar & r^2 \end{bmatrix} = \begin{bmatrix} 0.8536 & -0.3536\,\mathrm{i} \\ 0.3536\,\mathrm{i} & 0.1464 \end{bmatrix}.$$

This satisfies the conditions we found for a density matrix to achieve this result. The only unanswered question is can we match or exceed this using a true mixed ensemble by using superposition states for the different pieces.