PHY 411 Winter 2021

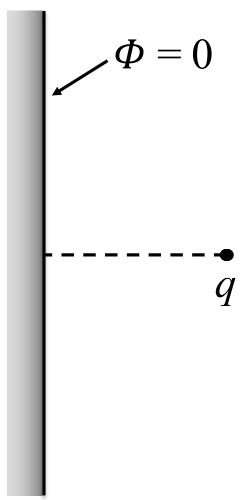
Week 9—Thursday, Mar 4—Discussion Worksheet

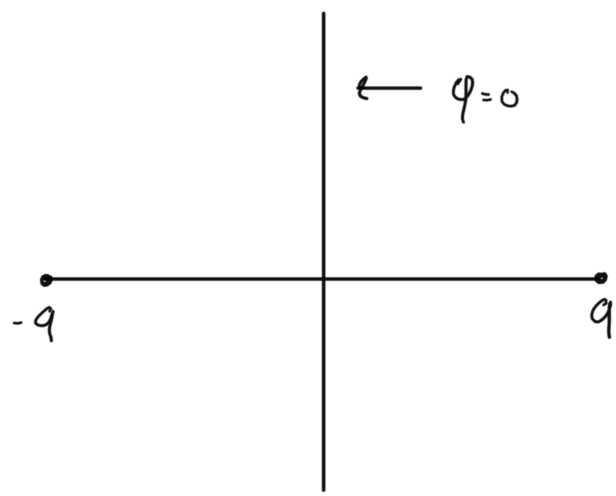
Method of Images

In principle, all boundary value problems in electrostatics can be solved using Green functions. In practice, however, finding a Green function may be difficult. One way around this is expansion in orthogonal functions that we've already learned. Another is the method of images, which we'll learn today.

The method of images can be applied when we have one or more point charges in the presence of boundary surfaces, e.g., conductors that are grounded or held at fixed potentials. If the geometry is favorable, we can place *charges of appropriate magnitudes in suitable locations external to the region of interest* to simulate the required boundary conditions. These charges are known as **image charges**, and the substitution of the actual problem with boundaries by a larger region with image charges but no boundaries is known as the **method of images**.

1. A simple example of the method of images is that of a point charge located in front of an infinite plane conductor at zero potential, as shown in the figure below.





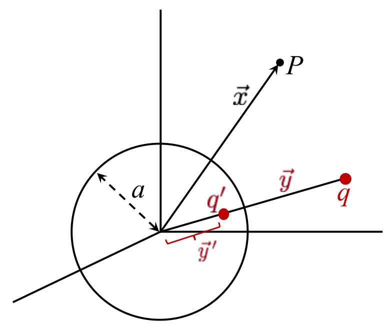
Discuss what the equivalent (image charge) problem will be for this situation in the space below, and draw a sketch of your answer in the space *above on the right*.

Within the conductor $\varphi = 0$ everywhere,

Point Charge Outside a Grounded Conducting Sphere

2. Consider a point charge q located at position $\vec{y}$ relative to the origin, as shown in the figure below. Centered on the origin, we have a *grounded* conducting sphere of radius a. By symmetry, the image charge (assuming only one is needed) will lie on the line from the origin to q (see figure). If q is *outside* the sphere, the image charge q' will be *inside* the sphere at $\vec{y}'$.

(a) Write down the potential $\Phi(\vec{x})$ at P due to q and q'.



$$\varphi(x) = \frac{q/4\pi\epsilon_0}{|x-y|} + \frac{q'/4\pi\epsilon_0}{|x-y'|}$$

(b) The object of the exercise is to choose q' and $\vec{y}'$ so that the potential vanishes on the surface of the sphere. Do this, and show that the magnitude and position of the image charge are

$$q' = -\frac{a}{y}\,q \qquad \text{and} \qquad y' = \frac{a^2}{y}$$

Make $\varphi(x=a)=0$

$$\frac{q}{|x-y|} = \frac{q'}{|x-y'|} \Rightarrow \left(\frac{q'}{q}\right)^2 = \frac{(x-y)^2}{(x-y')^2} = \frac{x^2+y'^2-2xy'}{x^2+y^2-2xy}$$

$$\left(\frac{q'}{q}\right)^2(a^2+y^2-2ay\cos\theta) - (a^2+y'^2-2ay'\cos\theta) = 0$$

$$\left(\left(\frac{q'}{q}\right)^2(-2ay+2ay')\cos\theta + \left(\frac{q'}{q}\right)^2(a^2+y^2) - (a^2+y'^2) = 0$$

$$\cos\theta = 0$$

$$\left(q'/q\right)^2 = y'/y \quad \text{and} \quad q' = -q\sqrt{\frac{y'}{y}}$$

$$\left(\frac{y'}{y}\right)(a^2+y^2) - (a^2+y'^2) = 0 \Rightarrow y'^2 - \left(\frac{a^2+y^2}{y}\right) + a^2 = 0$$

$$\left(y' - \frac{a^2}{y}\right)(y'-y) = 0 \qquad \boxed{y' = \frac{a^2}{y}, \quad q' = -q\frac{a}{y}}$$

3. Now that you have the position of the image charge, evaluate the normal derivative on the surface of the sphere and show that the surface charge density induced on the surface of the sphere is given by

$$\sigma = -\epsilon_0 \left.\frac{\partial \Phi}{\partial x}\right|_{x=a} = -\frac{q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1-\frac{a^2}{y^2}}{\left(1+\frac{a^2}{y^2}-2\frac{a}{y}\,cos\gamma\right)^{3/2}}$$

where γ is the angle between $\vec{x}$ and $\vec{y}$.

$$\varphi(x) = \frac{q/4\pi\epsilon_0}{|x-y|} + \frac{q'/4\pi\epsilon_0}{|x-y'|} \qquad y' = \frac{a^2}{y}, \quad q' = -q\frac{a}{y}$$

$$\varphi(x) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|x-y|} - \frac{q(a/y)}{|x-(a^2/y)|}\right)$$

$$\frac{\partial}{\partial x}\frac{q}{|x-y|} = \boxed{-\frac{q}{(x-y)^2}} \qquad \frac{\partial}{\partial x}\frac{q(a/y)}{|x-(a^2/y)|} = \boxed{-\frac{y^2 q\left(\frac{a}{y}\right)}{(a^2-xy)^2}}$$

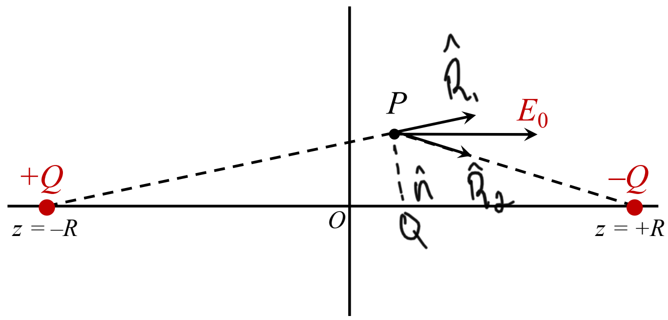
$$\sigma = -\left(\frac{1}{4\pi}\right)\left(-\frac{q}{(x-y)^2} + \frac{y^2 q(a/y)}{(a^2-\cos\gamma)^2}\right)$$

σ

We can now apply the method of images to additional situations of interest.

4. Consider the problem of a **conducting sphere in a uniform electric field**, E_0. The sphere has radius a, and E_0 can be considered to be produced by placing appropriate positive and negative charges at infinity.

(a) First, **show** that if two charges $\pm Q$ are placed at positions $z = \mp R$ as shown in the figure below, then in a region near the origin whose dimensions are small compared to R, there is an approximately constant electric field $E_0 \simeq 2Q/4\pi\epsilon_0 R^2$ parallel to the z-axis. In the limit $R, Q \to \infty$, with Q/R^2 constant, this approximation becomes exact.



$|\vec{E}|$ at P due to $+Q$ is $Q/4\pi\epsilon_0(R^2+n^2)|$ in direction of arrow $\hat{R}_1$, $|\vec{E}^*|$ at P due to $-Q$ is $|-Q/4\pi\epsilon_0(R^2+n^2)]|$ in the direction of $\hat{R}_2$

Taking both of these along $+\hat{z}$, and since PQ is small, we can approximate on z-axis

$$E_0 = \frac{Q}{4\pi\epsilon_0 R^2} + \frac{Q}{4\pi\epsilon_0 R^2} = \frac{2Q}{4\pi\epsilon_0 R^2}$$

(b) Suppose now that a conducting sphere of radius a is placed at the origin. The potential $\Phi(x)$ will then be due to charge $+Q$ at $z = -R$, charge $-Q$ at $z = +R$, and their image charges $-Qa/R$ at $z = -a^2/R$ and $+Qa/R$ at $z = +a^2/R$. By writing the potentials due to each of these four charges at an observation point P which is located at distance r from the origin and at angle θ to the z-axis, write down an expression for $\Phi(x)$ below.

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|r-R|} + \frac{q/a}{|r-R|} + \frac{-q/a}{|r-R|} + \frac{-Q}{|r-R|}\right]$$

5. Consider the problem of a conducting sphere of radius a in a uniform electric field E_0.

(a) Starting from the potential you wrote on part (b) of the previous page, show that in the limit where $2Q/4\pi\epsilon_0 R^2$ becomes the applied uniform electric field, the potential is given by

$$\Phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{R\left(1+r^2/R^2+2r/R\cos\theta\right)^{1/2}} - \frac{Q}{R\left(1+r^2/R^2-2r/R\cos\theta\right)^{1/2}}\right.$$

$$\left. - \frac{Qa/R}{r\left(1+a^4/r^2R^2+2a^2/rR\cos\theta\right)^{1/2}} + \frac{Qa/R}{r\left(1+a^4/r^2R^2-2a^2/rR\cos\theta\right)^{1/2}}\right]$$

Expand $(1+n)^x = (1+nx+\dots)$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{R}\left(-\frac{2r}{R}\cos\theta\right) - \frac{Qa}{Rr}\left(-\frac{2a^2}{rR}\cos\theta\right)\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{2Q}{R^2}\left[r\cos\theta - \frac{a^3}{r^2}\cos\theta\right] \Rightarrow \Phi(\vec{x}) = -E_0\left[r-\frac{a^3}{r^2}\right]\cos\theta$$

(b) Interpret your result. What causes each term in the expression for Φ?

$$\Phi(\vec{x}) = \underbrace{-E_0 r\cos\theta}_{= -E_0 z} + \underbrace{E_0 a^3/r^2\cos\theta}$$

Thus, first term is the potential of a uniform field E_0

Potential due to the image charges and thus, equivalently, the potential due to the induced surface-charge density. Note that the image charges form a dipole