PHY 420 Spring 2021

Week 7—Tuesday, May 11—Discussion Worksheet

4-vectors

Physical quantities that transform under Lorentz transformations in the same manner as the time and space coordinates of a point are known as *4-vectors*. The arbitrary 4-vector is (A_0, A_1, A_2, A_3), where A_1, A_2, A_3 are the components of a 3-vector $\vec{A}$. Note: a **4-vector is a special 4-dimensional vector that obeys the Lorentz transformations**.

Now, recall that the Lorentz transformations were based on the invariance of the squared quantity

$$c^2t^2 - (x^2 + y^2 + z^2) = c^2t'^2 - (x'^2 + y'^2 + z'^2) \tag{11.15}$$

Similarly, we can write for a general 4-vector the invariant quantity

$$\Delta A^2 = A_0'^2 - \left|\vec{A}'\right|^2 = A_0^2 - \left|\vec{A}\right|^2 \tag{11.23}$$

where the components $(A_0', \vec{A}')$ and $(A_0, \vec{A})$ refer to any two inertial reference frames, and hence the Lorentz transformation law equivalent to equation (11.16) for an arbitrary 4-vector is

$$A_0' = \gamma\,(A_0 - \vec{\beta}\cdot\vec{A}) \qquad A_\parallel' = \gamma\,(A_\parallel - \beta A_0) \qquad A_\perp' = A_\perp \tag{11.22}$$

where the parallel and perpendicular signs indicate components relative to the velocity $\vec{v} = c\vec{\beta}$.

The structure of equation (11.31) in *Jackson* for the addition of velocities makes it clear that the law of transformation of velocities is not that of 4-vectors. In order to find a 4-vector related to ordinary velocity, we will need to define the proper time $d\tau$. Recall that **proper time** is the ***time interval measured by a clock in a frame in which the clock is at rest***.

1. If a clock is at rest in frame K' where it measures the proper time interval $\Delta\tau$, then show that this is related to the time interval Δt measured in frame K by $\Delta t = \gamma\Delta\tau$.

Proper time must be measured by the same clock at the same position in space t'.

$$\left.\begin{aligned} ct_1 &= \gamma(ct_1' + \beta x') \\ ct_2 &= \gamma(ct_2' + \beta x') \end{aligned}\right\} \quad \cancel{c}(t_2 - t_1) = \gamma\,\cancel{c}(t_2' - t_1')$$

$$\Delta t = \gamma \Delta \tau$$

2. We will now write an expression for the **4-velocity**.

(a) The 4-velocity is obtained by dividing (cdt, dx, dy, dz) by $d\tau$. Show that the 4-velocity, with $\gamma_u = (1 - u^2/c^2)^{-1/2}$, is then

$$U = (U_0, \vec{U}) = (\gamma_u c, \gamma_u \vec{u})$$

$U_0 = \frac{c\,dt}{d\tau}$ where $\Delta t = \gamma_u (\Delta \tau)$

$U_0 = c(\gamma_u)$ $\Rightarrow \frac{dt}{d\tau} = \gamma_u$

$U_x = \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{dx}{dt}(\gamma_u) = u_x(\gamma_u)$

$U_y = u_y(\gamma_u)$, $U_z = u_z(\gamma_u)$

$\vec{U} = \hat{x} U_x + \hat{y} U_y + \hat{z} U_z = \hat{x} u_x \gamma_u + \hat{y} u_y \gamma_u + \hat{z} u_z \gamma_u$

$= \gamma_u [\hat{x} u_x + \hat{y} u_y + \hat{z} u_z] = \gamma_u \vec{u}$

4-vector

$U = (U_0, \vec{U}) = (\gamma_u c, \gamma_u \vec{u})$

(b) On the next page, we will also write the 4-momentum. In order to do so, recall that the relativistic momentum and energy of a particle are given by

$$\vec{p} = \gamma_u m \vec{u} \qquad \text{and} \qquad E = \gamma_u m c^2$$

where m is the mass of the particle, $\vec{u}$ is its velocity, and $\gamma_u = (1 - u^2/c^2)^{-1/2}$. Show that the velocity of the particle can be expressed in terms of its momentum and energy as

$$\vec{u} = \frac{c^2 \vec{p}}{E}$$

$\frac{\vec{p}}{E} = \frac{\gamma_u m \vec{u}}{\gamma_u m c^2} \; : \; \vec{u} = c^2 \frac{\vec{p}}{E}$

3. We will now write an expression for the **4-momentum**.

(a) Starting from the 4-velocity $U = (\gamma_u c, \gamma_u \vec{u})$, where $\gamma_u = (1 - u^2/c^2)^{-1/2}$, show that the 4-momentum $mU = m(\gamma_u c, \gamma_u \vec{u})$ is given by $(E/c, \vec{p})$.

$$U = (\gamma_u c, \gamma_u \vec{u})$$

4-momentum

$$P = mU = m(\gamma_u c, \gamma_u \vec{u}) = (\gamma_u m c, \gamma_u m \vec{u})$$

$$E = \gamma_u m c^2 \qquad = \left(\frac{E}{c}, \vec{P}\right)$$

$$E/c = \gamma_u m c$$

(b) Show that the invariant for the 4-momentum (also known as the energy-momentum 4-vector) $(p_0 = E/c, \vec{p})$ is

$$p_0^2 - \vec{p} \cdot \vec{p} = (mc)^2$$

$$P_0^2 - \vec{P} \cdot \vec{P} = E^2/c^2 - (\gamma_u m \vec{u}) \cdot (\gamma_u m \vec{u})$$

$$= \gamma_u^2 m^2 c^2 - \gamma_u^2 m^2 u^2 = \gamma_u^2 m^2 c^2 \left[1 - \frac{u^2}{c^2}\right]$$

$$= \cancel{\gamma_u^2} m^2 c^2 \left[\cancel{\gamma_u^{-2}}\right]$$

$$\gamma_u = (1 - u^2/c^2)^{-1/2}$$

$$\gamma_u^2 = (1 - u^2/c^2)^{-1}$$

$$\gamma_u^{-2} = 1 - u^2/c^2$$

Thus,

$$P_0^2 - \vec{P} \cdot \vec{P} = (mc)^2$$

(c) Show that $$E = \sqrt{p^2c^2 + m^2c^4}$$

$$P_0^2 - \vec{P} \cdot \vec{P} = (mc)^2$$

$$E^2/c^2 - p^2 = m^2c^2 \Rightarrow \frac{E^2 - p^2c^2}{c^2} = m^2c^2$$

Thus, $$E^2 - p^2c^2 = m^2c^4$$

$$E = (p^2c^2 + m^2c^4)^{1/2}$$

Mathematical properties of space-time: Tensors

We will begin looking at tensors by way of Griffiths (although with symbols modified to ours).

Recall that the dot (or scalar) product of a 3-dimensional vector

$$\vec{A} \cdot \vec{B} \equiv A_x B_x + A_y B_y + A_z B_z$$

is invariant (unchanged) under rotations. The analog to this in 4-dimensions is

$$c^2 t^2 - (x^2 + y^2 + z^2)$$

because no matter what frame of reference we are in, this quantity is invariant, i.e.,

$$c^2 t'^2 - (x'^2 + y'^2 + z'^2) = c^2 t^2 - (x^2 + y^2 + z^2)$$

This motivates us to establish a set of rules for the scalar product of any two 4-vectors to be invariant under Lorentz transformations.

So, think of designating a 4-vector B_α as a **covariant vector**, where ***covariant implies that its dot (or scalar) product is invariant under Lorentz transformations***. We'll see later how this can be written as a row vector.

Then, to form a dot product, we need a column vector. So, we'll define such a vector A^α and designate it as a **contravariant vector** so that the ***dot (or scalar) product of two vectors is defined as the product of the components of a covariant and a contravariant vector***:

$$B \cdot A \equiv \sum_{\alpha=0}^{3} B_\alpha A^\alpha$$

Starting with this equation, we will begin using a convention developed by Einstein himself. Known as **Einsteinian summation**, it invokes the rule that ***repeated indices indicate summation***, so that we write

$$B \cdot A \equiv B_\alpha A^\alpha \tag{11.66}$$

where $\alpha = 0, 1, 2, 3$. The summation is implied by the repeated index α. We will frequently use this summation convention for repeated indices.

4. To get some practice in this notation, consider

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta$$

(a) Which index (or indices) is (are) being summed over? Answer: β, with $\beta = 0, 1, 2, 3$

(b) Write down below all the terms in this sum.

$$(A')^\alpha = \frac{\partial x'^\alpha}{\partial x^0} A^0 + \frac{\partial x'^\alpha}{\partial x^1} A^1 + \frac{\partial x'^\alpha}{\partial x^2} A^2 + \frac{\partial x'^\alpha}{\partial x^3} A^3$$

Now, recall that the space-time continuum is defined in terms of a 4-dimensional space for which we've designated the coordinates as

$$(ct, z, x, y) \equiv (x^0, x^1, x^2, x^3)$$

Previously, we had written the indices as subscripts, but henceforth, we will write them as superscripts to emphasize that they constitute a contravariant vector.

We suppose that there is a well-defined transformation that yields new coordinates x'^0, x'^1, x'^2, x'^3 according to some rule:

$$x'^\alpha = x'^\alpha\left(x^0, x^1, x^2, x^3\right) \qquad \text{where } \alpha = 0, 1, 2, 3 \tag{11.60}$$

For example, the rule could be the Lorentz transformations; writing these for a general contravariant vector (A^0, A^1, A^2, A^3), we have

$$A'^0 = \gamma\left(A^0 - \beta A^1\right) \qquad A'^2 = A^2$$
$$A'^1 = \gamma\left(A^1 - \beta A^0\right) \qquad A'^3 = A^3$$

We say that the *contravariant vector* A^α has four components A^0, A^1, A^2, A^3. The **transformation rule for a *contravariant* vector** is then

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta \tag{11.61}$$

where the repeated index β implies a summation over $\beta = 0, 1, 2, 3$ as you found in Question 4(a).

5. Let's do an **example**. Recall that one example of A^α are the space-time coordinates

$$x'^0 = \gamma\left(x^0 - \beta x^1\right) \qquad x'^2 = x^2$$
$$x'^1 = \gamma\left(x^1 - \beta x^0\right) \qquad x'^3 = x^3$$

(a) Use the transformations written above to find

$$\frac{\partial x'^0}{\partial x^0} = \underline{\ \gamma\ }, \quad \frac{\partial x'^0}{\partial x^1} = -\underline{\ \beta\gamma\ }, \quad \frac{\partial x'^0}{\partial x^2} = \underline{\ 0\ }, \quad \frac{\partial x'^0}{\partial x^3} = \underline{\ 0\ }$$

(b) By writing the component $\alpha = 0$ in equation (11.61), use the results in part (a) to show that

$$A'^0 = \gamma(A^0 - \beta A^1)$$

which is the Lorentz transformation relation for the component A'^0.

$\alpha = 0$

$$(A')^0 = \frac{\partial x'^0}{\partial x^0} A^0 + \frac{\partial x'^0}{\partial x^1} A^1 + \frac{\partial x'^0}{\partial x^2} A^2 + \frac{\partial x'^0}{\partial x^3} A^3$$

$$A'^0 = \gamma A^0 - \gamma \beta A^1 + 0 + 0$$
$$= \gamma (A^0 - \beta A^1)$$

Having defined equation (11.61) as the transformation for a contravariant vector, we can now define the **transformation for *covariant* vectors**:

$$B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} \tag{11.62}$$

And so, we have started our discussion of tensors. From equation (11.61) and equation (11.62), we see that tensors are defined by their transformation properties relative to a transformation of the underlying coordinate system $x \to x'$. The rank of a tensor is determined by the number of indices it possesses. You've seen how the vectors we have discussed are designated as A^{α} or B_{β}, meaning that they are tensors with rank equal to one. Meanwhile, a scalar is a tensor of rank zero, an ordinary number (which may be real or complex) whose value is unchanged by such a transformation (of the underlying coordinate system). Examples of scalars are numbers like π, and the inner product $\vec{A} \cdot \vec{B}$.

For vectors, or tensors of rank 1, we have identified two types. *Contravariant vectors* $A^{\alpha} \equiv (A^0, A^1, A^2, A^3)$ transform according to equation (11.61), where all the indices are in the superscript, as is the new primed coordinate. *Covariant vectors* $B_{\alpha} \equiv (B_0, B_1, B_2, B_3)$ transform according to equation (11.62).

6. We will now consider **tensors of rank two** (i.e., with two indices).

(a) Write down the transformation rule for a contravariant tensor ($F^{\alpha\beta}$) of rank two (of which there are 16 such quantities):

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}$$

(b) Write down the transformation rule for a covariant tensor ($G_{\alpha\beta}$) of rank two:

$$G'_{\alpha\beta} = \frac{\partial x^{\delta}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} G_{\gamma\delta}$$

(c) Write down the transformation rule for the mixed second-rank tensor H^{α}_{β}:

$$H'^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} H^{\gamma}_{\delta}$$

Contravariant Covariant

The generalization to contravariant, covariant, and mixed tensors of arbitrary rank should be obvious from the previous examples. We can also form higher rank tensors by doing an outer product, in which we just take two tensors of some rank and multiply them component by component. For example, we can form the contravariant tensor of rank two, $F^{\alpha\beta}$, by doing

$$F^{\alpha\beta} = A^{\alpha} B^{\beta}$$

Going the other way, we can also reduce the rank of tensors by a process known as *contraction.* To contract two tensors, we set two of the indices to be equal and carry out the Einstein summation over the common range of indices. One index is contravariant and the other covariant. An example of such a contraction is the dot (or scalar) product of two vectors; recall that we defined the dot product as the product of the components of a covariant and a contravariant vector:

$$B \cdot A \equiv B_{\alpha} A^{\alpha} \tag{11.63}$$

where A is a contravariant vector defined in equation (11.61), and B is a covariant vector defined in equation (11.62), and the Einsteinian summation over repeated indices is implied.