PHY 420 Spring 2021

Class Summary—Week 6, Day 1—Tuesday, May 4

The Special Theory of Relativity

Einstein became such a popular figure in the public imagination that the word relativity has become synonymous with him. People forget that relativity has been around since the time of Galileo. Specifically, it had been known since the time of Galileo that the laws of mechanics are the same in different reference frames moving uniformly relative to one another. Einstein's contribution was to extend this principle to all physical laws.

To understand this in context, let us write down first the **equations of Galilean relativity**. Consider two reference frames K and K' moving with velocity $\vec{v}$ relative to each other. Then

$$\begin{aligned} \vec{x}\,' &= \vec{x} - \vec{v}t \\ t' &= t \end{aligned} \tag{11.1}$$

Mathematically, all we mean by relativity is the question: If we write a physical law in frame K, does it take the same form in K'? For example, assuming K' is moving along the z-direction with respect to K, we get

$$F_z' = m\ddot{z}' = m\,\frac{d^2}{dt^2}\Big(z - v_z t\Big) = m\,\frac{d}{dt}\Big(\dot{z} - v_z\Big) = m\ddot{z} = F_z$$

meaning that Newton's second law has the *same* form in frame K' as in frame K. In fact, it can be shown that Newton's law has the same form in every inertial frame. This is an example of Galilean relativity.

So, the next question to ask is: are Maxwell's equations invariant under Galilean relativity?

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We will now demonstrate that the Maxwell equations are **not** invariant under Galilean transformations.

Let us begin by assuming that the Maxwell equations are valid in K', so that the Helmholtz wave equation in K' is

$$\left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right]\psi = 0$$

Consider first the spatial derivative, *as you did in Question 2(a) of today's worksheet.* Begin with

$$\frac{\partial\psi}{\partial x'} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial x'} + \frac{\partial\psi}{\partial t}\frac{\partial t}{\partial x'}$$

The second, third, and fourth terms on the right hand side are zero, because e.g., $\partial y/\partial x' = 0$, etc., and since $x = x' + v_x t$ from equation (11.1), the above equation becomes

$$\frac{\partial\psi}{\partial x'} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial x'} = \frac{\partial\psi}{\partial x}\frac{\partial}{\partial x'}\Big[x' + v_x t\Big] = \frac{\partial\psi}{\partial x}\left[\frac{\partial x'}{\partial x'} + v_x\frac{\partial t}{\partial x'}\right] = \frac{\partial\psi}{\partial x}\Big[1 + v_x(0)\Big]$$

Therefore, we have shown that

$$\frac{\partial\psi}{\partial x'} = \frac{\partial\psi}{\partial x}, \qquad \text{and so likewise} \qquad \frac{\partial\psi}{\partial y'} = \frac{\partial\psi}{\partial y}, \qquad \frac{\partial\psi}{\partial z'} = \frac{\partial\psi}{\partial z}$$

implying that

$$\nabla'\psi = \nabla\psi, \qquad \text{and, by extension} \qquad \nabla'^2\psi = \nabla^2\psi$$

But

$$\frac{\partial\psi}{\partial t'} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial t'} + \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial t'} + \frac{\partial\psi}{\partial t}\frac{\partial t}{\partial t'}$$

and since

$$v_x = \frac{\partial x}{\partial t} = \frac{\partial x}{\partial t'} \qquad \text{and} \qquad \frac{\partial t}{\partial t'} = 1$$

we get

$$\frac{\partial\psi}{\partial t'} = v_x\frac{\partial\psi}{\partial x} + v_y\frac{\partial\psi}{\partial y} + v_z\frac{\partial\psi}{\partial z} + \frac{\partial\psi}{\partial t}$$

or, *as you showed in Question 1 on today's worksheet*

$$\frac{\partial\psi}{\partial t'} = \left(\vec{v}\cdot\vec{\nabla}\right)\psi + \frac{\partial\psi}{\partial t}$$

So, the Helmholtz equation in K' transforms as

$$0 = \left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right]\psi = \nabla^2\psi - \frac{1}{c^2}\left[\vec{v}\cdot\vec{\nabla} + \frac{\partial}{\partial t}\right]\left[\vec{v}\cdot\vec{\nabla} + \frac{\partial}{\partial t}\right]\psi$$

and we get

$$0 = \nabla^2\psi - \frac{1}{c^2}\left(\vec{v}\cdot\vec{\nabla}\right)\left(\vec{v}\cdot\vec{\nabla}\right)\psi - \frac{2}{c^2}\,\vec{v}\cdot\vec{\nabla}\,\frac{\partial\psi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

or

$$0 \neq \left[\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}\right] = \frac{2}{c^2}\,\vec{v}\cdot\vec{\nabla}\,\frac{\partial\psi}{\partial t} + \left(\frac{\vec{v}\cdot\vec{\nabla}}{c}\right)^2\psi$$

proving that the Helmholtz equation, and by extension the Maxwell equations, are **not** invariant under Galilean transformations, *as you demonstrated in Question 2(b) of today's worksheet.*

When Einstein began to think about these matters, there were a few possibilities:

- The Maxwell equations were incorrect. They must be fixed to obey Galilean relativity.
- Our assumption of inertial frames in electrodynamics was wrong. There is a preferred reference frame, in which the medium in which the waves propagate (the so-called ether), is at rest.
- There existed a relativity principle for both classical mechanics and electromagnetism, but it was not Galilean relativity.

The first option was not plausible; there was already enough evidence by Einstein's time to support Maxwell's equations. The second alternative was accepted by most physicists of the time, but every effort to detect the ether had failed.

Einstein chose the third alternative and sought principles of relativity that would encompass not only classical mechanics and electrodynamics, but indeed, all natural phenomena. His Special Theory of Relativity is based on **two postulates**:

- The laws of physics are the same in all inertial references frames (i.e., reference frames moving with constant velocities relative to each other).
- The speed of light (in vacuum) is constant, no matter in what reference frame it is measured.

These postulates can be used to investigate spectacular results in mechanics (e.g., length contraction and time dilation), but since our interest is in electrodynamics, we will instead move on to a discussion of the formal mathematical structure of relativity.

Lorentz Transformations

We will now introduce transformations between inertial frames that are consistent with special relativity, the so-called **Lorentz transformations**.

Consider two inertial reference frames K and K' moving with velocity $\vec{v}$ relative to each other. The time and space coordinates of a point are (t, x, y, z) and (t', x', y', z') in the frames K and K' respectively.

For convenience, we'll consider the two frames to be oriented such that the coordinate axes in the two frames are parallel and K' is moving in the positive z direction with speed v, as viewed from K. Also, for simplicity, we will consider the origins of the coordinates in K and K' to be coincident at $t = t' = 0$.

Consider a light source at rest at the origin in K, and hence moving with a speed v in the negative z-direction as seen from K'. If this light source is flashed on and off rapidly at $t = t' = 0$, then observers in both K and K' will see a spherical shell of radiation expanding outward from their respective origins with speed c. The wave front reaches a point (x, y, z) in the frame K at a time t given by the equation

$$c^2t^2 - \left(x^2 + y^2 + z^2\right) = 0 \tag{11.14}$$

Similarly, in the K' frame:

$$c^2t'^2 - \left(x'^2 + y'^2 + z'^2\right) = 0$$

With the assumption that space-time is homogenous and isotropic, the connection between the two sets of coordinates is linear. The quadratic form above and that in equation (11.14) are then related by

$$c^2t'^2 - \left(x'^2 + y'^2 + z'^2\right) = \lambda^2 \left[c^2t^2 - \left(x^2 + y^2 + z^2\right)\right] \tag{11.15}$$

where $\lambda = \lambda(\vec{v})$ is a possible change of scale between frames. With the axes oriented like we've chosen, we have $\lambda(v) = 1$ for all v.

With the setup above, let us introduce the notation

$$x_0 = ct, \qquad x_1 = z, \qquad x_2 = x, \qquad x_3 = y$$

Then the time and space coordinates in the frames K and K' are related by the **Lorentz transformation**

$$\left.\begin{aligned} x_0' &= \gamma\left(x_0 - \beta x_1\right) \\ x_1' &= \gamma\left(x_1 - \beta x_0\right) \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}\right\} \tag{11.16}$$

where

$$\begin{aligned} \beta &= |\vec{\beta}|, \qquad \text{and: } \vec{\beta} = \frac{\vec{v}}{c} \\ \gamma &= \frac{1}{\sqrt{(1-\beta^2)}} \end{aligned} \tag{11.17}$$

The **inverse transformation**, *as you wrote in Question 3 on today's worksheet*, is

$$\left.\begin{aligned} x_0 &= \gamma\left(x_0' + \beta x_1'\right) \\ x_1 &= \gamma\left(x_1' + \beta x_0'\right) \\ x_2 &= x_2' \\ x_3 &= x_3' \end{aligned}\right\} \tag{11.18}$$

As you are no doubt aware already, equation (11.16) and equation (11.18) reflect that the coordinates perpendicular to the direction of relative motion are unchanged, while the parallel coordinate and the time are transformed.

Moreover, it is interesting to see that the Lorentz transformation and the inverse transformation are perfectly symmetric, with $v \to -v$, which implies that specifying which frame is "at rest" and which frame is "in motion" is, at least mathematically, a matter of perspective.

The Lorentz transformations can be generalized to the case where the velocity $\vec{v}$ of the frame K' in frame K is in an arbitrary direction:

$$\left.\begin{aligned} x_0' &= \gamma\left(x_0 - \vec{\beta}\cdot\vec{x}\right) \\ \vec{x}\,' &= \vec{x} + \frac{\gamma-1}{\beta^2}\left(\vec{\beta}\cdot\vec{x}\right)\vec{\beta} - \gamma\vec{\beta}\,x_0 \end{aligned}\right\} \tag{11.19}$$

To end this section, it is worth noting that in equation (11.17), β and γ are related, and we're using two parameters that are not independent. From equation (11.17), it is clear that

$$\gamma^2 = \frac{1}{1-\beta^2}$$

so that

$$\gamma^2 \Big(1-\beta^2\Big) = 1$$

or

$$\gamma^2 - \gamma^2\,\beta^2 = 1$$

Also, because $\beta = v/c$, it has the range $0 \leq \beta \leq 1$, which means that γ must have the range $1 \leq \gamma \leq \infty$.

All of this allows us an alternative parametrization. The relation $\gamma^2 - \gamma^2\,\beta^2 = 1$ is suggestive of the hyperbolic relation

$$\cosh^2\zeta - \sinh^2\zeta = 1$$

so if we set $\gamma = \cosh\zeta$, and $\gamma\beta = \sinh\zeta$, then we would have

$$\beta = \frac{\gamma\beta}{\gamma} = \frac{\sinh\zeta}{\cosh\zeta} = \tanh\zeta$$

Therefore, we have the alternative parametrization

$$\left.\begin{aligned} \beta &= \tanh\zeta \\ \gamma &= \cosh\zeta \\ \gamma\beta &= \sinh\zeta \end{aligned}\right\} \tag{11.20}$$

where the parameter ζ is known as the *boost parameter* or *rapidity.* So, to "boost" between frames means to carry out a Lorentz transformation from one frame to another. We can rewrite equation (11.16) in terms of the boost parameter ζ:

$$\begin{aligned} x'_0 &= x_0\,\cosh\zeta - x_1\,\sinh\zeta \\ x'_1 &= -x_0\,\sinh\zeta + x_1\,\cosh\zeta \end{aligned} \tag{11.21}$$

The structure of these equations is reminiscent of a rotation of coordinates, except with hyperbolic functions, which comes about because of the relative negative sign between the space and time terms in equation (11.14), more on this later.

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Addition of velocities

We will now move on to the **addition of velocities**. This is an important consideration in Special Relativity, because the Galilean law of addition of velocities produces faster-than-light velocities. So, we need an effective way of adding velocities that does not violate the postulates of Special Relativity.

We won't derive the result here, however. Problem (1.3) in Jackson that *you worked out in Questions 4 and 5 of today's worksheet* can be considered an alternative derivation of the velocity addition theorem, so in a sense you do have a proof of the addition theorem written below.

As we did for the Lorentz transformations, let us consider the frame K' to be moving with velocity $\vec{v} = c\vec{\beta}$ with respect to the frame K. Then, the components of velocity transform according to

$$\begin{aligned} u_{\parallel} &= \frac{u'_{\parallel} + v}{1 + \dfrac{\vec{v} \cdot \vec{u}'}{c^2}} \\ \vec{u}_{\perp} &= \frac{\vec{u}'_{\perp}}{\gamma_v \left(1 + \dfrac{\vec{v} \cdot \vec{u}'}{c^2}\right)} \end{aligned} \tag{11.31}$$

where $u_{\parallel}$ and $\vec{u}_{\perp}$ refer to components of velocity parallel and perpendicular, respectively, to $\vec{v}$, and the subscript on γ_v explicitly identifies the relationship to be $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

For the simple case of parallel velocities, the addition law is

$$u = \frac{u' + v}{1 + \dfrac{vu'}{c^2}} \tag{11.33}$$

For speeds u' and v both small compared to c, the velocity addition law reduces to the Galilean result: $u = u' + v$.

It is impossible to obtain a speed greater than that of light by adding two velocities, even if each is very close to c. In equation (11.33) above, if $u' = c$, then $u = c$ also. That is, if an object is traveling at the speed of light in one frame, it is traveling at the speed of light in all frames. Moreover, this behavior is not unique to light but applies to all objects traveling at the speed of light. This is an explicit example of Einstein's second postulate.

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Relativistic Energy and Momentum

We will now consider the **relativistic generalizations** of the momentum and kinetic energy of a particle.

For a particle with speed small compared to the speed of light, its momentum and energy are known to be

$$\begin{aligned} \vec{p} &= m\vec{u} \\ E &= E(0) + \frac{1}{2}mu^2 \end{aligned} \tag{11.37}$$

where m is the mass of the particle, $\vec{u}$ is its velocity, and $E(0)$ is a constant identified as the rest energy of the particle.

In nonrelativistic considerations, the rest energies can be ignored because they contribute the same additive constant to both sides of an energy balance equation. In special relativity, however, we cannot ignore the rest energy and we will see below that it is the total energy — the sum of the rest energy plus the kinetic energy of a particle that is significant.

We wish to find expressions for the momentum and energy of a particle consistent with the Lorentz transformation law equation (11.31) of velocities that reduces to equation (11.37) for nonrelativistic motion. Let us write

$$\begin{aligned} \vec{p} &= \mathcal{M}(u)\,\vec{u} \\ E &= \mathcal{E}(u) \end{aligned} \tag{11.38}$$

where $\mathcal{M}(u)$ and $\mathcal{E}(u)$ are functions of the magnitude of the velocity $\vec{u}$. With the reasonable assumption that $\mathcal{M}(u)$ and $\mathcal{E}(u)$ are well-behaved monotonic functions of their arguments, we get by comparing with equation (11.37) that

$$\begin{aligned} \mathcal{M}(0) &= m \\ \frac{\partial \mathcal{E}}{\partial u^2} &= \frac{m}{2} \end{aligned} \tag{11.39}$$

To determine the forms of $\mathcal{M}(u)$ and $\mathcal{E}(u)$, one needs to consider elastic collisions in the two frames. After a certain amount of algebra, which we won't get into here (but see pages 533-537 if you're interested), we get that

$$\vec{p} = \gamma m\vec{u} = \frac{m\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{11.46}$$

and

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{11.51}$$

From equation (11.46) and equation (11.51), the velocity of a particle can be expressed in terms of its momentum and energy as

$$\vec{u} = \frac{c^2\vec{p}}{E} \tag{11.53}$$