PHY 411 **Winter 2021**

Class Summary—Week 2, Day 2—Thursday, Jan 14

Plane Wave Solutions to Maxwell's Equations

In the previous class, we wrote down the *Helmholtz wave equation*:

$$\begin{aligned} \left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{E} &= 0 \\ \left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{B} &= 0 \end{aligned} \tag{7.3}$$

and learned that plane wave solutions to the Helmholtz wave equation can be written in the form

$$\begin{aligned} \vec{E}(\vec{x},t) &= \vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \\ \vec{B}(\vec{x},t) &= \vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \end{aligned} \tag{7.8.a}$$

where $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are *constant* vectors, and $\hat{n}$ is a unit vector along the direction of propagation.

The above has been mostly mathematics following from the wave equation. But now, let us apply some physics from Maxwell's equations that we wrote in equation (7.1) of the previous lecture, and see what it tells us about electromagnetic waves in particular.

From Gauss' Law $\vec{\nabla}\cdot\vec{E} = 0$ in a source-free region (i.e., the right hand side is zero because $\rho = 0$), we get from equation (7.8.a) that

$$\vec{\nabla}\cdot\vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = 0$$

or, because $\vec{\mathcal{E}}$ is a constant vector

$$\vec{\mathcal{E}}\cdot\vec{\nabla}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = 0$$

as you proved in the Discussion Worksheet for today. The expression above simplifies to

$$\vec{\mathcal{E}}\cdot ik\hat{n}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = 0$$

as, again, *you proved in the Discussion Worksheet for today.*

Since we're now just dealing with vectors and $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$, we can write the expression above as

$$ik\hat{n}\cdot\vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = 0$$

Dividing out by the nonzero terms, we get finally that

$$\hat{n}\cdot\vec{\mathcal{E}} = 0$$

But what does it mean when the dot product of two vectors is equal to zero? As long as both vectors have non-zero magnitude (which is certainly true in this case), it means that the two vectors are perpendicular to each other. That means $\vec{E}$ is perpendicular to the direction of propagation $\hat{n}$.

Repeating the above procedure for $\vec{B}$, we get that $\hat{n}\cdot\vec{\mathcal{B}} = 0$, which tells us that $\vec{B}$ is also perpendicular to the direction of propagation $\hat{n}$.

So, what have we found? Writing both the equations from the previous page together:

$$\hat{n}\cdot\vec{\mathcal{E}}=0 \qquad \text{and} \qquad \hat{n}\cdot\vec{\mathcal{B}}=0 \tag{7.10}$$

we conclude that *both* $\vec{E}$ and $\vec{B}$ are each perpendicular to the direction of propagation $\hat{n}$. Such a wave is called a **transverse wave**.

What else can we learn about $\vec{E}$ and $\vec{B}$?

Putting equation (7.8.a) into the curl equations, say, Faraday's law $\vec{\nabla}\times\vec{E}=-\partial\vec{B}/\partial t$, we get

$$\vec{\nabla}\times\vec{\mathcal{E}}\,e^{i(k\hat{n}\cdot\vec{x}-\omega t)}=-\frac{\partial\vec{B}}{\partial t}$$

Borrowing the result to switch $\vec{\nabla}$ and $\vec{\mathcal{E}}$ from the dot product procedure written on the previous page (that you proved on today's Discussion Worksheet), but now switching the sign since we're working with the cross product, the above expression modifies to

$$-\vec{\mathcal{E}}\times\vec{\nabla}\,e^{i(k\hat{n}\cdot\vec{x}-\omega t)}=-\frac{\partial}{\partial t}\left[\vec{\mathcal{B}}\,e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right]$$

On Homework 2, you will show that this leads to

$$\vec{\mathcal{B}}=\sqrt{\mu\epsilon}\,\left(\hat{n}\times\vec{\mathcal{E}}\right) \tag{7.11}$$

except that in Homework 2, I've written $\hat{n}=\vec{k}/k$ (and the full $\vec{E}$ and $\vec{B}$, instead of just $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$).

Equation (7.11) tells us that $\vec{B}$ is perpendicular to $\vec{E}$. Moreover, we've already shown in equation (7.10) that $\vec{E}$ and $\vec{B}$ are each *separately* perpendicular to the direction of propagation $\hat{n}$. Therefore, we have demonstrated that **plane electromagnetic waves are transverse waves**, in which the electric and magnetic fields are perpendicular to each other, and each is separately perpendicular to the direction of propagation.

Since $\sqrt{\mu\epsilon}=n/c$ from equation (7.5), where n is the index of refraction, equation (7.11) can be written as

$$\vec{\mathcal{B}}=\frac{n}{c}\,\left(\hat{n}\times\vec{\mathcal{E}}\right)$$

Careful! There are two different uses of the symbol in the same formula: n is used in this expression for the index of refraction and $\hat{n}$ is a unit vector specifying the direction of $\vec{k}$.

Multiplying on both sides by c, we get

$$c\,\vec{\mathcal{B}}=n\,\left(\hat{n}\times\vec{\mathcal{E}}\right)$$

which tells us that $c\vec{B}$ and $\vec{E}$ have the same magnitude for electromagnetic waves in free space and differ by the index of refraction in a medium.

Since $\vec{B}=\mu\,\vec{H}$, equation (7.11) may also be written as

$$\vec{\mathcal{H}}=\frac{\sqrt{\mu\epsilon}}{\mu}\,\left(\hat{n}\times\vec{\mathcal{E}}\right)$$

or

$$\vec{\mathcal{H}}=\frac{\hat{n}\times\vec{\mathcal{E}}}{Z} \tag{7.11$'$}$$

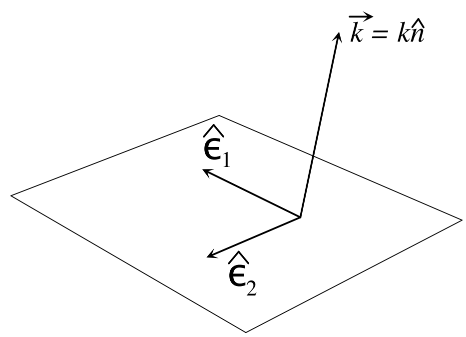
where $Z=\sqrt{\mu/\epsilon}$ is called the impedance.

Moreover, if $\hat{n}$ is real ... Hold on a minute, you say — what do you mean ***if*** $\hat{n}$ is real? Doesn't it have to be real — it's a unit vector — *no, it doesn't, and we'll shortly examine the implications and consequences of this.*

If $\hat{n}$ is real, though, then equation (7.11) implies that $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ have the same phase.

Note that this is what Jackson says, but I think it is better to say that the phase factor $e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$ does not contribute any additional terms when $\hat{n}$ is real, so $\mathcal{E}$ and $\mathcal{B}$ are related simply by equation (7.11), implying $\mathcal{B}$ is perpendicular to the plane of $\hat{n}$ and $\mathcal{E}$, so $\mathcal{B}$ and $\mathcal{E}$ are perpendicular to each other.

No matter how we put it, equation (7.11) implies that we can introduce a set of real mutually orthogonal unit vectors $(\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{n})$, as shown in Figure 7.1 of Jackson (page 297), which is reproduced below.



We can then use this set of unit vectors to express the field strengths. One possibility is to write

$$\vec{\mathcal{E}} = \hat{\epsilon}_1 E_0 \qquad \text{and} \qquad \vec{\mathcal{B}} = \hat{\epsilon}_2 \sqrt{\mu\epsilon}\, E_0 \tag{7.12}$$

where E_0 is a constant that may be complex.

Another option is

$$\vec{\mathcal{E}} = \hat{\epsilon}_2 E_0' \qquad \text{and} \qquad \vec{\mathcal{B}} = -\hat{\epsilon}_1 \sqrt{\mu\epsilon}\, E_0' \tag{7.12$'$}$$

where, again, E_0' is a constant that may be complex, and the minus sign for $\vec{\mathcal{B}}$ arises from the right-handed nature of the coordinate system (i.e., fingers of the right hand curled from $\vec{E}$ to $\vec{B}$ must advance the thumb in the direction of $\hat{n}$).

Equations (7.8) and (7.12) above describe a transverse wave propagating in the direction $\hat{n} = \hat{\epsilon}_1 \times \hat{\epsilon}_2$. This can be shown by writing the time-averaged Poynting vector

$$\vec{S} = \frac{1}{2}\,\vec{E} \times \vec{H}^*$$

which, for the choice of $\vec{E}$ in equation (7.12), becomes

$$\vec{S} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,\Big|E_0\Big|^2\,\hat{n} \tag{7.13}$$

as you showed on today's Discussion Worksheet. Note that $\vec{S}$ here gives the *time-averaged energy flow* — the energy transmitted per unit area per unit time. The Poynting vector itself is written in equation (6.109) as $\vec{S} = \vec{E} \times \vec{H}$, so don't get confused by Jackson using the same symbol here.

Complex unit vector $\hat{n}$

In the discussion above, we've assumed that $\hat{n}$ is a real unit vector. But, in general, $\hat{n}$ could be complex, that is, $\vec{k} = k\hat{n}$ is complex, while the wave number k remains real. But remember that the equations (7.10) and (7.11) must still hold. That is, we only assumed $\hat{n}$ is real in the discussion after equation (7.11), in order to get equation (7.12), and thereafter.

If $\hat{n}$ is complex, and written as

$$\hat{n} = \hat{n}_R + i\hat{n}_I$$

then the requirement that $\hat{n} \cdot \hat{n} = 1$ has the real and imaginary parts that satisfy the relations (*as you showed in today Discussion Worksheet*):

$$\begin{aligned} n_R^2 - n_I^2 &= 1 \\ \hat{n}_R \cdot \hat{n}_I &= 0 \end{aligned} \tag{7.15}$$

The second of these conditions shows that $\hat{n}_R$ and $\hat{n}_I$ are perpendicular to each other.

If $\hat{n}$ is complex, with $\hat{n} = \hat{n}_R + i\hat{n}_I$, then the exponential part of the fields in equation (7.8) becomes

$$e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = e^{-k\hat{n}_I\cdot\vec{x}}\, e^{i(k\hat{n}_R\cdot\vec{x}-\omega t)} \tag{7.15.a}$$

Such a wave is called an **inhomogenous plane wave**. It exhibits exponential growth or decay (due to the $e^{-k\hat{n}_I\cdot\vec{x}}$ term) in some directions, while remaining a "plane wave" in directions perpendicular to them (due to the $e^{i(k\hat{n}_R\cdot\vec{x}-\omega t)}$ term); the "perpendicular" part comes from the second of the two conditions in equation (7.15).

Note on Jackson's comment that "the surfaces of constant amplitude and constant phase are still planes, but they are no longer parallel" — taken from M. Destrade, Journal of Elasticity, 1999: "In certain physical contexts, such as gravity waves, surface waves, or reflection and refraction of waves, an attenuation of the amplitude occurs in a direction distinct from the direction of propagation. Thus arises the need to find 'inhomogeneous plane wave' solutions to the wave equation."

"A simple form for the displacement is that of a vector field $\vec{u}(\vec{x}, t)$ which varies sinusoidally with frequency ω in the direction of a vector $\vec{S}^+$ and is attenuated exponentially in the direction of another vector $\vec{S}^-$, so that $\vec{u}(\vec{x}, t)$ is the real part of the complex quantity $e^{-\omega\vec{S}^-\cdot\vec{x}}\,[\vec{A}e^{i\omega(\vec{S}^+\cdot\vec{x}-t)}]$, where $\vec{A}$ is the amplitude of the wave." Compare to equation (7.15.a) above, where $\vec{S}^+$ is $\hat{n}_R$, and $\vec{S}^-$ is $\hat{n}_I$.

"The complex vector $\vec{S} = \vec{S}^+ + i\vec{S}^-$ is called the 'slowness bivector' and its real and imaginary parts describe the 'planes of constant phase' ($\vec{S}^+\cdot\vec{x} =$ constant) and the 'planes of constant amplitude' ($\vec{S}^-\cdot\vec{x} =$ constant). When $\vec{S}^+$ and $\vec{S}^-$ are parallel, the plane wave is said to be homogeneous; otherwise, it is inhomogeneous."

For now, we won't discuss complex $\hat{n}$ any more, because nature provides us with few examples of imaginary $\hat{n}$. We will come across examples of inhomogenous plane waves when we study, e.g., total internal reflection, but there the inhomogeneity arises from a complex wave number rather than a complex unit vector $\hat{n}$.

Polarization of Plane Waves

In equations (7.8) and (7.12), we wrote the expression for a wave with its *electric field vector always pointing in the direction* $\hat{\epsilon}_1$. Such a wave is said to be **linearly polarized** with polarization vector $\hat{\epsilon}_1$ (for reference, see the figure on page 3).

Likewise, the wave described in equation (7.12′) is linearly polarized with polarization vector $\hat{\epsilon}_2$.

These ($\hat{\epsilon}_1$ and $\hat{\epsilon}_2$) are evidently two (linearly) independent polarizations of a transverse plane wave.

We will now explore further the subject of *polarization*.

Let us start by writing the most general homogenous wave propagating in the direction $\vec{k} = k\hat{n}$. In Figure (7.1) from Jackson (reproduced on page 3), we defined unit vectors $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ that are perpendicular to each other and each perpendicular to the direction of propagation $\hat{n}$ respectively. Let us write electric fields along these directions as in Jackson's equation (7.18):

$$\vec{E}_1 = \hat{\epsilon}_1\, E_1\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{7.18}$$

and

$$\vec{E}_2 = \hat{\epsilon}_2\, E_2\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

with the magnetic field written from equation (7.11) as

$$\vec{B}_1 = \sqrt{\mu\epsilon}\,\left[\frac{\vec{k}}{k}\times\vec{E}_1\right]$$

and

$$\vec{B}_2 = \sqrt{\mu\epsilon}\,\left[\frac{\vec{k}}{k}\times\vec{E}_2\right]$$

Note that instead of writing $\hat{n}\times\ldots$, we've written $\dfrac{\vec{k}}{k}\times\ldots$, which is the same thing, because $\vec{k} = k\hat{n}$.

Now combine $\vec{E}_1$ and $\vec{E}_2$ to get the most general homogenous plane wave propagating in the direction $\vec{k} = k\hat{n}$:

$$\vec{E}(\vec{x},t) = \Big(\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2\Big)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{7.19}$$

where E_1 and E_2 are, as usual, complex amplitudes to allow for the possibility that fields that are linearly polarized in different directions have different phases, as we will need below.

The polarization of the plane wave is then specified by the relative *direction, magnitude, and phase of the electric field component* of the wave. Some well-known cases are described on the next page.

Linear Polarization

If E_1 and E_2 have the *same phase* (although not necessarily the same magnitude), then equation (7.19) represents a **linearly polarized** wave.

The sketch of such a wave is shown in Figure 7.2 in Jackson (page 299). The magnitude of the polarization vector is

$$E = \sqrt{E_1^2 + E_2^2}$$

and the polarization vector makes an angle θ with the direction of $\hat{\epsilon}_1$ given by

$$\theta = \arctan\left(\frac{E_2}{E_1}\right)$$

as shown in Figure 7.2 in Jackson (page 299).

Elliptical Polarization

If E_1 and E_2 have *different phases and different amplitudes*, then equation (7.19) represents an **elliptically polarized** wave. The polarization vector (i.e., the electric field vector $\vec{E}$) traces out an ellipse in the plane defined by $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$.

Circular Polarization — a special case of elliptical polarization

To understand elliptical polarization, let us look at the simplest case: **circular polarization**. We get a circularly polarized wave when E_1 and E_2 have the same magnitude but are out of phase by $\pi/2$. Since $e^{\pm i\pi/2} = \pm i$, equation (7.19) becomes

$$\vec{E}(\vec{x}, t) = E_0 \left(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2\right) e^{i(\vec{k}\cdot\vec{x} - \omega t)} \tag{7.20}$$

where we have put $E_1 = E_2 = E_0$ because the amplitudes are equal for a circularly polarized wave, and we've accounted for the $\pi/2$ difference in phase between E_1 and E_2 by inserting the $\pm i$ because, as we noted above, $e^{\pm i\pi/2} = \pm i$.

For example, we could choose $\hat{\epsilon}_1$ along the x-direction (i.e., $\hat{\epsilon}_1 \equiv \hat{x}$) and $\hat{\epsilon}_2$ along the y-direction. Then, the circularly polarized wave in equation (7.20) would be traveling along the positive z-direction.

At a *fixed point in space*, the electric field vector is constant in magnitude, but sweeps around in a circle at frequency ω — see Figure 7.3 in Jackson (on page 300). Alternatively and equivalently, if the wave is frozen in time, the electric field vector traces out a helix along the direction of propagation — see, e.g., the figure from hyper physics at this link (*but note that their definition of right and left circular polarization is the reverse of Jackson's given below*).

For the upper sign ($\hat{\epsilon}_1 + i\hat{\epsilon}_2$) in equation (7.20), the rotation of the electric vector is counterclockwise when an observer is looking toward an oncoming wave. Such a wave is called *left circularly polarized* (in optics) or said to have *positive helicity* (in modern physics) — although contradictory definitions do exist in the literature.

For the lower sign ($\hat{\epsilon}_1 - i\hat{\epsilon}_2$) in equation (7.20), the rotation of the electric vector is clockwise when an observer is looking toward an oncoming wave. Such a wave is called *right circularly polarized* (in optics) or said to have *negative helicity* (in modern physics).

Another set of basis vectors

Another general representation of the polarization is in terms of the complex orthogonal unit vectors

$$\hat{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}\left(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2\right) \tag{7.22}$$

with properties

$$\begin{aligned} \hat{\epsilon}^*_{\pm} \cdot \hat{\epsilon}_{\mp} &= 0 \\ \hat{\epsilon}^*_{\pm} \cdot \hat{\epsilon}_{3} &= 0 \\ \hat{\epsilon}^*_{\pm} \cdot \hat{\epsilon}_{\pm} &= 1 \end{aligned} \tag{7.23}$$

based on which, we get another general representation equivalent to equation (7.19), given by:

$$\vec{E}(\vec{x}, t) = \left(E_+\hat{\epsilon}_+ + E_-\hat{\epsilon}_-\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{7.24}$$

where, as always, E_+ and E_- are *complex* amplitudes.

The polarization of a plane electromagnetic wave is known if it can be written in the form of either equation (7.19) with known coefficients (E_1, E_2), or equation (7.24), with known coefficients (E_+, E_-).

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Stokes Parameters

We learned above how the polarization of a plane electromagnetic wave is known if it can be written ***either*** with coefficients (E_1, E_2) in the form of equation (7.19), ***or*** with known coefficients (E_+, E_-) in the form of equation (7.24). In practice, though, we are usually confronted with the converse of this — we are given an electromagnetic wave of the form in equation (7.8), and we need to determine its state of polarization. To do this, we use the Stokes parameters.

Stokes parameters can be written either in the linear polarization basis, or circular polarization basis. I'll update later with the details but in the interests of posting this immediately after class, I'm going to put the slides from class here for now (and update later with details).

Update: I had planned to update the following, but I like the feel of the slides better, and you can just look up the equations on the pages referenced below, so I'm going to keep it as it is currently.

Stokes Parameters

Linear polarization basis:

$$\vec{E}(\vec{x}, t) = \left(\hat{\epsilon}_1 \underset{\substack{\downarrow \\ a_1 e^{i\delta_1}}}{E_1} + \hat{\epsilon}_2 \underset{\substack{\downarrow \\ a_2 e^{i\delta_2}}}{E_2}\right) e^{i(\vec{k}\cdot\vec{x} - \omega t)} \qquad (7.19)$$

Can then write Stokes parameters s_0, s_1, s_2, s_3, in linear polarization basis

- See eq. (7.27) on page 301; will use on Homework 2

Circular polarization basis:

$$\vec{E}(\vec{x}, t) = \left(\underset{\substack{\downarrow \\ a_+ e^{i\delta_+}}}{E_+} \hat{\epsilon}_+ + \underset{\substack{\downarrow \\ a_- e^{i\delta_-}}}{E_-} \hat{\epsilon}_-\right) e^{i(\vec{k}\cdot\vec{x} - \omega t)} \qquad (7.24)$$

Can then write Stokes parameters s_0, s_1, s_2, s_3, in circular polarization basis

- See eq. (7.28) on page 301; will use on Homework 2