PHY 420 Spring 2021

Homework 3 solutions

1. A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces of the cylinder is zero, while the potential on the cylindrical surface is $V(\phi, z)$.

Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.

Solution: Write the potential $\Phi(\rho, \phi, z)$ at any point inside the cylinder as the product of solutions

$$\Phi(\rho, \phi, z) = R(\rho)\, Q(\phi)\, Z(z)$$

Substituting into the Laplace equation and after separating variables, we get

$$Q(\phi) = (\dots)\ \sin m\phi + (\dots)\ \cos m\phi \tag{1}$$

where m is an integer in order to allow single-valued potentials when the full azimuthal span is allowed. I won't write any constants for now, but insert them at the end in the consolidated solution.

Next, we seek a periodic solution for $Z(z)$ due to the nature of the boundary conditions, so

$$Z(z) = e^{\pm ikz} = (\dots)\ \sin kz + (\dots)\ \cos kz$$

Since $\Phi = 0$ at $z = 0$, this becomes

$$Z(z) = (\dots)\ \sin kz$$

Now apply the boundary condition $\Phi = 0$ at $z = L$. We know that $\sin(n\pi) = 0$, for $n = 0, 1, 2, 3, \dots$, so this gives

$$kL = n\pi, \qquad n = 0, 1, 2, 3, \dots$$

so that the solution is

$$Z(z) = (\dots)\ \sin\left(\frac{n\pi z}{L}\right) \tag{2}$$

Finally, recall that when we choose periodic boundary conditions in z, we get the radial solution in terms of modified Bessel functions. Therefore, we have in general

$$R(\rho) = I_\nu(k\rho) + K_\nu(k\rho)$$

However, we seek the solution inside the cylinder and K_ν diverges at $\rho = 0$. Therefore, we cannot have K_ν in our expression, and so the solution reduces to

$$R(\rho) = I_\nu(k\rho)$$

But from the $Q(\phi)$ solution, we know that $\nu = m$, an integer. And from the Z solution, we know that $k = n\pi/L$, where $n = 0, 1, 2, 3, \dots$, therefore we get

$$R(\rho) = I_m\left(\frac{n\pi\rho}{L}\right) \tag{3}$$

We can now combine equation (1), (2), and (3) to build the complete solution

$$\Phi(\rho, \phi, z) = \sum_m \sum_n I_m\left(\frac{n\pi\rho}{L}\right)\Big[C_{mn}\ \sin m\phi + D_{mn}\ \cos m\phi\Big]\ \sin\left(\frac{n\pi z}{L}\right) \tag{4}$$

but we still need to evaluate the constants C_{mn} and D_{mn}.

On the previous page, we wrote the solution

$$\Phi(\rho, \phi, z) = \sum_m \sum_n I_m \left(\frac{n\pi\rho}{L}\right) \Big[C_{mn} \sin m\phi + D_{mn} \cos m\phi\Big] \sin\left(\frac{n\pi z}{L}\right)$$

We still have to evaluate the constants C_{mn} and D_{mn}. To do so, apply the third boundary condition at $\rho = b$:

$$V(\phi, z) = \sum_m \sum_n I_m \left(\frac{n\pi b}{L}\right) \Big[C_{mn} \sin m\phi + D_{mn} \cos m\phi\Big] \sin\left(\frac{n\pi z}{L}\right) \tag{5}$$

Use orthogonality of sines and cosines to solve for C_{mn} and D_{mn} respectively.

For example, multiply both sides of equation (5) by $\sin(n'\pi z/L)$, and evaluate the following integral on the right hand side

$$\int \sin\left(\frac{n'\pi z}{L}\right) \sin\left(\frac{n\pi z}{L}\right) dz = \delta_{n'n}$$

and for $n' = n$, we get

$$\int_0^L \sin^2\left(\frac{n'\pi z}{L}\right) dz = \frac{1}{2}\int_0^L \left[1 - \cos\left(\frac{2n\pi z}{L}\right)\right] dz = \frac{L}{2} \tag{6}$$

Likewise, multiplying both sides of equation (5) by $\sin(m'\phi)$, we get for the integral on the right hand side for $m' = m$ that

$$\int_0^{2\pi} \sin^2(m\phi)\, d\phi = \pi \tag{7}$$

Similar steps apply for D_{mn}, except with cosines for the ϕ part, so I won't repeat them here.

Finally, therefore, the constants evaluate to

$$C_{mn} = \frac{2}{\pi L\, I_m\left(\frac{n\pi b}{L}\right)} \int_0^L \int_0^{2\pi} V(\phi, z) \sin(m\phi) \sin\left(\frac{n\pi z}{L}\right) d\phi\, dz \tag{8}$$

and

$$D_{mn} = \frac{2}{\pi L\, I_m\left(\frac{n\pi b}{L}\right)} \int_0^L \int_0^{2\pi} V(\phi, z) \cos(m\phi) \sin\left(\frac{n\pi z}{L}\right) d\phi\, dz \tag{9}$$

2. For the same arrangement as in Question 1 above, suppose the cylindrical surface is made up of two equal half-cylinders so that

$$V(\phi, z) = \begin{cases} V & \text{for } -\dfrac{\pi}{2} < \phi < \dfrac{\pi}{2} \\ -V & \text{for } \dfrac{\pi}{2} < \phi < \dfrac{3\pi}{2} \end{cases}$$

Find the potential inside the cylinder.

Note: You may use your results from Question 1; there is no need to derive them again.

Solution: Substituting for $V(\phi, z)$ in equation (8), we get

$$C_{mn} = \frac{2}{\pi L\, I_m\left(\frac{n\pi b}{L}\right)} \int_0^L \sin\left(\frac{n\pi z}{L}\right) dz \left[V \int_0^{\pi} \sin(m\phi)\, d\phi - V \int_{\pi}^{2\pi} \sin(m\phi)\, d\phi\right] \tag{10}$$

Note that the limits are given from $-\pi/2$ to $3\pi/2$, but should work out to be the same if you take 0 to 2π. If this is confusing, set $\alpha = \phi - \pi/2$ to get the limits from $-\pi/2$ to $3\pi/2$, and then $\beta = \alpha + \pi/2$ to restore the limits from 0 to 2π. Do the integrations in equation (10):

$$C_{mn} = (\ldots)(-)\left.\frac{\cos\left(\frac{n\pi z}{L}\right)}{n\pi/L}\right|_0^L V\left[(-)\left.\frac{\cos m\phi}{m}\right|_0^{\pi} - (-)\left.\frac{\cos m\phi}{m}\right|_{\pi}^{2\pi}\right]$$

where I've written the constant terms at the front as $(\ldots)$ for brevity. Continue simplifying:

$$C_{mn} = (\ldots)\left(\frac{L}{n\pi}\right)(-)\Big[\cos(n\pi) - \cos 0\Big]\left(\frac{V}{m}\right)\Big[(-)\Big\{\cos(m\pi) - \cos 0\Big\} + \Big\{\cos(2m\pi) - \cos(m\pi)\Big\}\Big]$$

so that

$$C_{mn} = (\ldots)\left(\frac{VL}{mn\pi}\right)\Big[1 - (-1)^n\Big]\Big[1 - 2\cos(m\pi) + \cos(2m\pi)\Big]$$

We see that

$$C_{mn} = 0 \qquad \text{if } m = \text{even, or } n = \text{even}$$

otherwise, if m = odd and n = odd,

$$C_{mn} = (\ldots)\left(\frac{VL}{mn\pi}\right)(2)\Big[1 - 2(-1) + 2\Big] = (\ldots)\left(\frac{8VL}{mn\pi}\right)$$

Putting in the other terms, we get

$$C_{mn} = \frac{2}{\pi L\, I_m\left(\frac{n\pi b}{L}\right)}\left(\frac{8VL}{mn\pi}\right)$$

so that finally

$$C_{mn} = \frac{16V}{mn\pi^2\, I_m\left(\frac{n\pi b}{L}\right)}$$

Meanwhile, you should be able to show that $D_{mn} = 0$.

Therefore, after substituting into equation (4) from Question 1, the complete solution is

$$\Phi(\rho, \phi, z) = \frac{16V}{\pi^2} \sum_{m=\text{odd}} \sum_{n=\text{odd}} \frac{1}{mn} \frac{I_m\left(\dfrac{n\pi\rho}{L}\right)}{I_m\left(\dfrac{n\pi b}{L}\right)} \sin(m\phi) \sin\left(\frac{n\pi z}{L}\right)$$

3. The solutions $J_{\pm\nu}(x)$ are called *Bessel functions of the first kind* of order $\pm\nu$. If ν is an integer, the solutions are linearly dependent. For $\nu = m$, an integer, show that we get

$$J_{-m}(x) = (-1)^m \, J_m(x)$$

Solution: Putting $\nu = m$ in equation (3.83) in Jackson, we get

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \; \Gamma(j - m + 1)} \left(\frac{x}{2}\right)^{2j} \tag{11}$$

Since $\Gamma(p) = (p-1)!$, equation (11) can be written as

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \; (j - m)!} \left(\frac{x}{2}\right)^{2j} \tag{12}$$

Put $(j - m) = l$, so that $j = (l + m)$; equation (12) then becomes

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{l=0}^{\infty} \frac{(-1)^{(l+m)}}{(l + m)! \; l!} \left(\frac{x}{2}\right)^{2(l+m)} \tag{13}$$

Note that the summation must still start at $l = 0$, because $(j - m)! = \infty$, for $j < m$. In other words, the series begins at $j = m$. Equation (13) then gives

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{l=0}^{\infty} \frac{(-1)^l \, (-1)^m}{(l + m)! \; l!} \left(\frac{x}{2}\right)^{2l} \left(\frac{x}{2}\right)^{2m}$$

Terms with m can be brought outside the summation, so that

$$J_{-m}(x) = (-1)^m \left(\frac{x}{2}\right)^{2m} \left(\frac{x}{2}\right)^{-m} \sum_{l=0}^{\infty} \frac{(-1)^l}{(l + m)! \; l!} \left(\frac{x}{2}\right)^{2l}$$

or

$$J_{-m}(x) = (-1)^m \left(\frac{x}{2}\right)^{m} \sum_{l=0}^{\infty} \frac{(-1)^l}{l! \; (l + m)!} \left(\frac{x}{2}\right)^{2l}$$

Except for $(-1)^m$, everything on the right hand side is just $J_m(x)$, see equation (3.82), except that the series is written with l instead of j. That is, since l is just a counter now, we could replace it with counter j so that

$$J_{-m}(x) = (-1)^m \left(\frac{x}{2}\right)^{m} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \; (j + m)!} \left(\frac{x}{2}\right)^{2j}$$

or, restoring the Γ function

$$J_{-m}(x) = (-1)^m \left(\frac{x}{2}\right)^{m} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \; \Gamma(j + m + 1)} \left(\frac{x}{2}\right)^{2j}$$

Therefore, we have shown that

$$J_{-m}(x) = (-1)^m \, J_m(x)$$

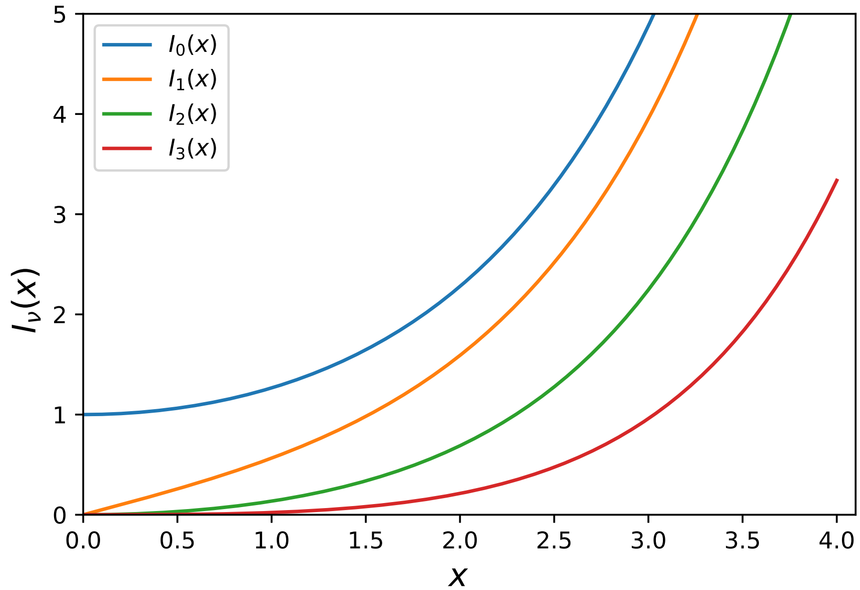
4. Explore plots of the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ in Matlab (or Python) by plotting the following:

$$I_0(x), I_1(x), I_2(x), I_3(x) \qquad \text{and} \qquad K_0(x), K_1(x), K_2(x), K_3(x)$$

Submit your code. Plots generated using an online calculator like Desmos will get zero points.

Note: Both functions go to infinity on one end, and some may be zero on the other end, so you will need to work with the axes limits until you get a reasonably good visual representation.

Solution: The plots are shown below. First, we have $I_\nu(x)$:



Next, we have $K_\nu(x)$: