PHY 420 **Spring 2021**

Week 5—Thursday, Apr 29—Discussion Worksheet

Magnetic Dipole and Electric Quadrupole Moments

Consider again the vector potential

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}\,')\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|}\,d^3x' \tag{9.3}$$

A more careful job on the expansion than we did in the previous class gives

$$\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} = \frac{e^{ikr}}{r}\left[1+\left\{\frac{1}{r}-ik\right\}\frac{\vec{x}\cdot\vec{x}\,'}{r}+\dots\right]$$

The first term gives the electric dipole radiation that we discussed in the previous class.

The **second term** in the expansion above, after writing $\vec{x}\cdot\vec{x}\,'/r = r\hat{n}\cdot\vec{x}\,'/r = \hat{n}\cdot\vec{x}\,'$, gives

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int \vec{J}(\vec{x}\,')\,(\hat{n}\cdot\vec{x}\,')\,d^3x' \tag{9.30}$$

The integrand in equation (9.30) can be written as the sum of a part symmetric in $\vec{J}$ and $\vec{x}\,'$ and an antisymmetric part:

$$(\hat{n}\cdot\vec{x}\,')\,\vec{J} = \frac{1}{2}\left[(\hat{n}\cdot\vec{x}\,')\,\vec{J}+(\hat{n}\cdot\vec{J})\vec{x}\,'\right]+\frac{1}{2}\,(\vec{x}\,'\times\vec{J})\times\hat{n} \tag{9.31}$$

The first term in square brackets, the symmetric part, is related to the electric quadrupole. The second term is related to the magnetic dipole, as we can tell from the expression for the magnetic moment density:

$$\vec{m} = \frac{1}{2}\int \vec{x}\,'\times\vec{J}(\vec{x}\,')\,d^3x' \tag{5.54}$$

1. Show that the vector potential for the magnetic dipole term is

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\,(\hat{n}\times\vec{m})\,\frac{e^{ikr}}{r}\left[1-\frac{1}{ikr}\right] \tag{9.33}$$

$$A(x) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int \vec{J}(\vec{x}\,')(\hat{n}\cdot\vec{x}\,')\,d^3x'$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int \underbrace{\frac{1}{2}\left[(\hat{n}\cdot\vec{x}\,')\vec{J}+(\hat{n}\cdot\vec{J})\vec{x}\,'\right]}_{\text{Electric Dipole}}+\underbrace{\frac{1}{2}(\vec{x}\,'\times\vec{J})\times\hat{n}}_{\vec{m}}$$

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}(\hat{n}\times\vec{m})\frac{e^{ikr}}{r}\left[1-\frac{1}{ikr}\right]$$

To determine the fields, we can proceed in one of two ways: either directly calculate them from $\vec{A}$, or use the fact that $\vec{A}(\vec{x})$ for the magnetic dipole in equation (9.33) is proportional to $\vec{H}$ for an electric dipole written in equation (9.18):

$$\begin{aligned} \vec{H}_{\text{elec dipole}} &= \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) \\ \vec{E}_{\text{elec dipole}} &= \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r}+\Big[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\Big]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \end{aligned} \tag{9.18}$$

2. By comparing equation (9.33) with (9.18) above, write down $\vec{H}$ for a magnetic dipole, and then write down $\vec{E}$.

Compare 9.33 and 9.18

Note: Neither of these requires an actual derivation, but you will need to multiply and divide to move constants around.

$$\vec{A} = \frac{ik\mu_0}{4\pi}(\hat{n}\times\vec{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) \qquad \vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$

$\vec{H}_{elec}$ becomes $\vec{A}$ for magnetic dipole by replacing $ck^2\vec{p} \to ik\mu_0\vec{m}$

or, if you replace $\vec{p}$ by $\vec{m}$ in (9.18), then $\vec{H}_{elec} \to \vec{A}\left(\frac{ck}{i\mu_0}\right)$

Now, $\vec{E}_{elec} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H}_{elec} \to \frac{1}{Z_0}\vec{E}_{elec} = \frac{i}{k}\left(\vec{\nabla}\times\vec{H}_{elec}\right)$

$$\to \frac{\cancel{i}}{\cancel{k}}\left(\vec{\nabla}\times\vec{A}\right)\frac{c\cancel{k}}{\cancel{i}\mu_0} = \frac{c}{\mu_0}\vec{\nabla}\times\vec{A} \to \text{for } \vec{H} \to \vec{H} = \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{A}\right)$$

So

$$\vec{H} = \frac{\vec{E}_{elec}}{cZ_0} = \sqrt{\mu_0\epsilon_0}\sqrt{\frac{\epsilon_0}{\mu_0}}\left[\frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{m})\times\hat{n}\,\frac{e^{ikr}}{r} + [3\hat{n}(\hat{n}\cdot\vec{m}) \ldots\right.\right.$$

$$\ldots - \vec{m}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}$$

Electric Quadrupole Radiation

It is difficult to write the general solution for the electric quadrupole — even for Jackson!!! So, we will consider only the fields in the radiation zone, where

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{1}{\mu_0}ik\,(\hat{n}\times\vec{A}) = -\frac{ick^3}{8\pi}\frac{e^{ikr}}{r}\int(\hat{n}\times\vec{x}\,')\,(\hat{n}\cdot\vec{x}\,')\rho(\vec{x}\,')\,d^3x' \tag{9.40}$$

This will need to be written in terms of the electric quadrupole tensor $\overleftrightarrow{Q}$, where

$$Q_{ij} = \int\left(3x_i'x_j' - r'^2\,\delta_{ij}\right)\rho(\vec{x}\,')\,d^3x' \tag{9.41}$$

3. By evaluating the x-component of the expression below, verify that the integral in equation (9.40) above can be written as

$$\hat{n}\times\int\vec{x}\,'\,(\hat{n}\cdot\vec{x}\,')\rho(\vec{x}\,')\,d^3x' = \frac{1}{3}\hat{n}\times\vec{Q}(\hat{n}) \tag{9.42}$$

where the vector $\vec{Q}(\hat{n})$ has components

$$Q_i = \sum_j Q_{ij}\,n_j \tag{9.43}$$

This is somewhat weird notation by Jackson, so see the posted class summary for comments.

$$\left[\hat{n}\times\int\vec{x}\,'(\hat{n}\cdot\vec{x}\,')\rho(\vec{x}\,')d^3x'\right]_x = n_y\left[\int\ldots\right]_z - n_z\left[\int\ldots\right]_y$$

$$= n_y\int\left[n_x x'z' + n_y y'z' + n_z(z')^2\right]\rho(\vec{x}\,')d^3x'$$

$$- n_z\int\left[n_x x'y' + n_y(y')^2 + n_z y'z'\right]\rho(x')d^3x'$$

$$\left[\hat{n}\times\vec{Q}(\hat{n})\right]_x = n_y Q_z - n_z Q_y \qquad Q_i = \sum_j Q_{ij}n_j \Rightarrow Q_z = \sum_j Q_{zj}n_j$$

$$= n_y\left[Q_{zx}n_x + Q_{zy}n_y + Q_{zz}n_z\right] - n_z\left[Q_{yx}n_x\right.$$

4. We will now write the fields $\vec{H}$ and $\vec{E}$ in terms of $\vec{Q}(\hat{n})$.

(a) Using the relation written in equation (9.42), show that equation (9.40) becomes

$$\vec{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\,\hat{n}\times\vec{Q}(\hat{n}) \tag{9.44}$$

$$H = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}$$

$$= \frac{1}{\mu_0} ik\,\hat{n}\times\vec{A}$$

$$= -\frac{ick^3}{8\pi}\frac{e^{ikr}}{r}\int d^3r'\,(\hat{n}\times\vec{x}')(\hat{n}\cdot\vec{x}')\,\rho(\vec{x}')$$

$$= -\frac{1}{3}\frac{ick^3}{8\pi}\frac{e^{ikr}}{r}\,\hat{n}\times\vec{Q}(\hat{n})$$

(b) Use $\vec{E} = Z_0\vec{H}\times\hat{n}$ from equation (9.19) to show that

$$\vec{E} = -\frac{ic^2k^3\mu_0}{24\pi}\frac{e^{ikr}}{r}\left[\hat{n}\times\vec{Q}(\hat{n})\right]\times\hat{n}$$

$$\vec{E} = \frac{ic\mu_0}{k}\vec{\nabla}\times\vec{H}$$

$$= \frac{ic\mu_0}{k} ik\,\hat{n}\times\vec{H}$$

$$= ick(\hat{n}\times\hat{A})\times\hat{n}$$

$$= c\mu_0\vec{H}\times\hat{n}$$

$$= -\frac{ic^2k^3\mu_0}{24\pi}\frac{e^{ikr}}{r}\left[\hat{n}\times\vec{Q}(\hat{n})\right]\times\hat{n}$$

The time-averaged power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = \frac{r^2}{2}\,\mathrm{Re}\left\{\hat{n}\cdot\vec{E}\times\vec{H}^*\right\}$$

5. Show that

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2}\,k^6\left|\left\{\hat{n}\times\vec{Q}(\hat{n})\right\}\times\hat{n}\right|^2 \tag{9.45}$$

The general angular distribution is complicated, but we can see that the radiated power varies as the sixth power of the frequency for fixed quadrupole moments, a much stronger dependence compared to the fourth power for dipole radiation.