PHY 420 Spring 2021

Week 8—Tuesday, May 18—Discussion Worksheet

Matrix Representation of Lorentz Transformations

Physical laws must be covariant, meaning that they must have the same form in different coordinate systems. In particular:

- Physical laws must be covariant under translations in space and time since space is homogenous.
- Physical laws must be covariant under rotations in 3-dimensional space since space is isotropic.
- Physical laws must be covariant under Lorentz transformations to conform to Special Relativity.

Recall that we are in the process of identifying a group that will represent the covariance of physical laws under the transformations specified above. Together, these transformations form part of the inhomogenous Lorentz group, or Poincare group.

Recall also that we introduced a matrix representation in the previous class to make the manipulation explicit and less abstract. In this representation, the components x^0, x^1, x^2, x^3 of a contravariant 4-vector form the elements of a column vector, as shown on the extreme left below.

1. Write down the metric tensor $g_{\alpha\beta}$ as a square 4×4 matrix. Also write the covariant 4-vector x_α, which can be obtained from the contravariant x^β by contraction with the metric tensor $g_{\alpha\beta}$, as given in equation (11.72): $x_\alpha = g_{\alpha\beta}\, x^\beta$; written in matrix form, therefore, x_α is just gx:

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

We have already learned that matrix scalar products (a, b) of 4-vectors a and b are defined in the usual way by summing over the products of the elements of a and b, or equivalently, by the matrix multiplication:

$$(a, b) \equiv \tilde{a}b \tag{11.80}$$

where $\tilde{a}$ is the transpose of a (and hence a row vector). In this compact notation, the scalar product of two 4-vectors can also be written as

$$a \cdot b = (a, gb) = \tilde{a}gb \tag{11.83}$$

or, if we write out the elements explicitly

$$a \cdot b = \tilde{a}gb = a^\alpha g_{\alpha\beta} b^\beta = a^\alpha b_\alpha$$

We would like to find a group of linear transformations that leaves $x \cdot x = (x, gx)$ invariant, so that we can use it to represent the invariance of physical laws in all inertial frames, as postulated by Special Relativity. Since $x \cdot x$ or (x, gx) is the norm of a 4-vector, we are effectively seeking a group of transformations that preserves the "length" in the 4-dimensional metric.

Our desire to find a group of linear transformations that leaves $x \cdot x = (x, gx)$ invariant is equivalent to finding all square 4×4 matrices A which, when they transform the coordinates as

$$x' = Ax \tag{11.84}$$

will leave the norm (x, gx) invariant:

$$x' \cdot x' = \tilde{x}' g x' = \tilde{x} g x = x \cdot x \tag{11.85}$$

which is nothing other than just a mathematical statement of Special Relativity. Recall that we proved in the previous class that this leads to the condition on the allowed matrices A that $\det A = \pm 1$. Transformations with $\det A = +1$ are known as Proper Lorentz transformations, and henceforth, we will focus only on them. It turns out in such cases that only 6 parameters are needed to specify A, and they can be conveniently thought of as three rotations and three velocity boosts.

To construct A, we write

$$A = e^L \tag{11.87}$$

where L is a 4×4 matrix. We can justify this choice by expanding infinitesimal rotations in a Taylor series, although the details won't be written here.

2. Let us first figure out how to generate rotations, specifically the infinitesimal generator, first for rotations around the x^3 axis.

(a) You may remember that the matrix for rotations about the x^3 axis by an angle ϕ_3 is

$$R_3(\phi_3) = \begin{pmatrix} \cos\phi_3 & -\sin\phi_3 & 0 \\ \sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Given R_3 above, write down the corresponding infinitesimal generator X_3:

$$X_3 = \left.\frac{dR_3}{d\phi_3}\right|_{\phi_3=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) In our 4-dimensional space-time, we then have for finite rotations about the x^3 axis by angle ϕ_3 that

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & \cos\phi_3 & -\sin\phi_3 & 0 \\ 0 & \sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Write down the corresponding infinitesimal generator, S_3, for rotation around the x^3 axis:

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3. We are in the process of writing matrices for rotations and velocity boosts.

(a) By similar considerations as on the previous page, write two other matrices, S_1 and S_2, to generate rotations around the x^1 and x^2 axes respectively:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Next, we need to generate the velocity boosts. To do so, use infinitesimal Lorentz transforms.

First, note that the Lorentz transforms can be represented by an alternative parametrization in which

$$\beta = \tanh\zeta, \qquad \gamma = \cosh\zeta, \qquad \gamma\beta = \sinh\zeta \tag{11.20}$$

where ζ is known as the *boost parameter* or *rapidity*. So, to "boost" between frames means to carry out a Lorentz transformation from one frame to another. Rewriting the Lorentz transforms for frame K' moving along the x_1 direction of frame K in terms of the boost parameter ζ, we get

$$\begin{aligned} x_0' &= +x_0 \cosh\zeta - x_1 \sinh\zeta \\ x_1' &= -x_0 \sinh\zeta + x_1 \cosh\zeta \end{aligned} \tag{11.21}$$

(b) Write equation (11.21) in matrix form, with ${x'}^2 = x^2$, and ${x'}^3 = x^3$ included.

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

For an infinitesimal boost, we set ζ to be small ($\Delta\zeta$), so that

$$\cosh(\Delta\zeta) = 1, \qquad \sinh(\Delta\zeta) = \Delta\zeta$$

(c) With the infinitesimal boost conditions above applied, write the 4×4 matrix as the sum of two matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \left(-\Delta\zeta\right)$$

and call the second matrix K_1, the matrix for an infinitesimal boost along the x^1 direction. Likewise, we can generate K_2 and K_3, matrices for an infinitesimal boost along the x^2 and x^3 directions respectively.

The net result of the preceding processes is that we get six fundamental matrices defined by

$$
S_1 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad
S_2 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \qquad
S_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)
$$
$$
K_1 = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad
K_2 = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad
K_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \tag{11.91}
$$

The matrices S_i generate rotations in three dimensions, while the matrices K_i produce boosts. The six matrices equation (11.91) represent the ***infinitesimal generators*** of the Lorentz group.

4. In order to build finite rotations in the basis defined by the 6 matrices in equation (11.91) above, let us work out the following:

(a) Find S_1^2.

$$
S_1 \cdot S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
$$

(b) Find S_1^3.

$$
S_1 \cdot S_1 \cdot S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = -S_1
$$

(c) Find K_1^2.

$$
K_1 \cdot K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

(d) Find K_1^3.

$$
K_1 \cdot K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
= K_1
$$

The key conclusion from your calculations on the previous page should be that ***any power of one of the matrices can be expressed as a multiple of the matrix or its square***, that is

$$\left(\hat{\epsilon}\cdot\vec{S}\right)^3 = -\hat{\epsilon}\cdot\vec{S} \qquad \text{and} \qquad \left(\hat{\epsilon}'\cdot\vec{K}\right)^3 = \hat{\epsilon}'\cdot\vec{K}$$

where $\hat{\epsilon}_i$ is a unit vector along the x^i axis. ***This result will be very important for what we're about to do next***. The space below is for you to verify that the expression above is correct.

Since $S_1^3 = -S_1$ then $\left(\hat{\epsilon}\cdot\vec{S}\right)^3 = -\hat{\epsilon}\cdot\vec{S}$

Since $K_1^3 = K_1$ then $\left(\hat{\epsilon}'\cdot\vec{K}'\right)^3 = \hat{\epsilon}'\cdot\vec{K}$

Jackson's claim, as we wrote in equation (11.87), is that $A = e^L$. Based upon the special properties that we wrote above of the basis matrices in equation (11.91), Jackson claims that $L = -\vec{\omega}\cdot\vec{S} - \vec{\zeta}\cdot\vec{K}$, and hence

$$A = e^{-\vec{\omega}\cdot\vec{S}-\vec{\zeta}\cdot\vec{K}} \tag{11.93}$$

where $\vec{\omega}$ and $\vec{\zeta}$ are constant 3-vectors. The three components each of $\vec{\omega}$ and $\vec{\zeta}$ correspond to the six parameters of the transformation. Rather than try to prove this in general, we will look at some specific examples to verify that it is possible to construct A in this manner.

5. Consider first the example of **no rotation, and boost along x^1 axis**.

(a) Write down for this example: $\vec{\omega} =$ 0, $\vec{\zeta} =$ $\zeta\hat{\epsilon}_1$

(b) By expanding the exponential in equation (11.93) in a series, show that we can write

$$A = I - K_1^2 + (\cosh\zeta)\,K_1^2 - (\sinh\zeta)\,K_1$$

where the first term is written as the 4×4 unit matrix I.

$$A = e^{-\vec{\omega}\cdot\vec{S} - \zeta\vec{K}}$$

$$= e^{-\zeta K_1}$$

$$= I - \zeta K_1 + \frac{\zeta^2}{2!}K_1^2 - \frac{\zeta^3 K_1^3}{3!} + \dots$$

$$= I - \zeta K_1 + \frac{\zeta}{2}K_1^2 - \frac{\zeta^3}{6}K_1 + \dots$$

$$= I - K_1\underbrace{\left[\zeta + \frac{\zeta^3}{6} + \dots\right]}_{\sinh\zeta} + K_1^2\left[\frac{\zeta}{2} + \dots\right]$$

On the previous page, you showed that

$$A = I - K_1^2 + (\cosh\zeta)\,K_1^2 - (\sinh\zeta)\,K_1$$

6. By writing the matrices I, K_1, and K_1^2 explicitly, show that A written above is the same as what you obtained in Question 3(b) for boosts along the x^1 axis.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \cosh\zeta & 0 & 0 & 0 \\ 0 & \cosh\zeta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & \sinh\zeta & 0 & 0 \\ \sinh\zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, you should be able to show that the six matrices in equation (11.91) satisfy the following commutation relations:

$$\begin{aligned} [S_i, S_j] &= \epsilon_{ijk}\, S_k \\ [S_i, K_j] &= \epsilon_{ijk}\, K_k \qquad (11.99) \\ [K_i, K_j] &= -\epsilon_{ijk}\, S_k \end{aligned}$$

Space is provided on the next page for you to demonstrate this outside of class.

Important work outside of class: The commutation relations in equation (11.99) written on the previous page are very important. Take some time to work them out explicitly. Use the commutator

$$[A, B] = AB - BA$$

- Using equation (11.99), write down $[S_2, S_1] =$ $S_2 S_1 - S_1 S_2 = \epsilon_{21k} S_k$

  Now, evaluate $[S_2, S_1]$ by explicit matrix multiplication and verify it matches your answer above.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= -S_3$$

- Using equation (11.99), write down $[S_2, K_1] =$ $S_2 K_1 - K_1 S_2$

  Evaluate $[S_2, K_1]$ by explicit matrix multiplication and verify it matches your answer above.

$$= -K_3$$

- Using equation (11.99), write down $[K_2, K_1] =$ $K_2 K_1 - K_1 K_2$

  Evaluate $[K_2, K_1]$ by explicit matrix multiplication and verify it matches your answer above.