Physics 412—Quantum Mice Evolution Winter 2020

Now that we know about the Schrödinger equation, we can let our mice evolve in time. They're not static anymore, they can evolve!

If we use the attitude states as our basis, we can represent the operators as matrices and the states as column vectors as follows:

$$A \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad |h\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |u\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B \leftrightarrow \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad |p\rangle \leftrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}, \qquad |a\rangle \leftrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix},$$

$$H \leftrightarrow \frac{2}{5}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}, \qquad |4\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad |2\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

$$W \leftrightarrow \frac{2}{5}\begin{bmatrix} 21 & -8 \\ -8 & 9 \end{bmatrix}, \qquad |s\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad |l\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

You can assume that the energy eigenvalues of 4 and 2 are expressed in units where $\hbar = 1$, so you can drop the factor of $\hbar$ from the exponents involving time. (Alternatively, you can assume that the energy eigenvalues are actually $4\hbar$ and $2\hbar$.)

You can work in the attitude basis if you like, or your can switch to the energy basis; your choice.

(1) We'll first consider the fate of happy mice.

- (a) In you want to work in the energy basis, express A and the states $|h\rangle$ and $|u\rangle$ in that basis.
- (b) Suppose at time $t_0 = 0$ you have a happy mouse. Find the state of the mouse at a later time t.
- (c) What is the probability that you will have a happy mouse at a later time t? Is there any later time t at which the mouse will be guaranteed to be unhappy?
- (d) How does the average happiness of a pure ensemble of happy mice change over time? In other words, calculate $\langle A\rangle$. Does your result agree with Ehrenfest's theorem?

(2) Next, the fate of passive mice.

- (a) In you want to work in the energy basis, express B and the states $|p\rangle$ and $|a\rangle$ in that basis.
- (b) Suppose at time $t_0 = 0$ you have a passive mouse. Find the state of the mouse at a later time t.
- (c) What is the probability that you will have a passive mouse at a later time t? Is there any later time t at which the mouse will be guaranteed to be aggressive?
- (d) How does the average behavior of a pure ensemble of passive mice change over time? In other words, calculate $\langle B\rangle$. Does your result agree with Ehrenfest's theorem?

(3) Lastly, the fate of small mice.

- (a) In you want to work in the energy basis, express W and the states $|s\rangle$ and $|l\rangle$ in that basis.
- (b) Suppose at time $t_0 = 0$ you have a small mouse. Find the state of the mouse at a later time t.
- (c) What is the probability that you will have a small mouse at a later time t? Is there any later time t at which the mouse will be guaranteed to be large?
- (d) How does the average size of a pure ensemble of small mice change over time? In other words, calculate $\langle W\rangle$. Does your result agree with Ehrenfest's theorem?

(1) I will work in the attitude basis for this so I will skip part (a).

(b) I will make the propagator:

$$U(t) = e^{-4it}|4\rangle\langle 4| + e^{-2it}|2\rangle\langle 2|$$

$$\leftrightarrow \frac{e^{-4it}}{5}\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix} + \frac{e^{-2it}}{5}\begin{bmatrix}-2\\1\end{bmatrix}\begin{bmatrix}-2 & 1\end{bmatrix}$$

$$\leftrightarrow \frac{e^{-4it}}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} + \frac{e^{-2it}}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$$

$$U(t) \leftrightarrow \frac{1}{5}\begin{bmatrix}e^{-4it}+4e^{-2it} & 2e^{-4it}-2e^{-2it}\\2e^{-4it}-2e^{-2it} & 4e^{-4it}+e^{-2it}\end{bmatrix}$$

Then

$$|h(t)\rangle = U(t)|h\rangle \leftrightarrow \frac{1}{5}\begin{bmatrix}e^{-4it}+4e^{-2it}\\2e^{-4it}-2e^{-2it}\end{bmatrix}$$

(c) Probability of being happy at a later time

$$P(h) = \frac{1}{25}\left|e^{-4it}+4e^{-2it}\right|^2 = \frac{1}{25}\left(1+4e^{-2it}+4e^{2it}+16\right)$$

$$\boxed{P(h) = \frac{1}{25}\left(17+8\cos 2t\right)}$$

$$P(u) = \frac{1}{25}\left|2e^{-4it}-2e^{-2it}\right|^2 = \frac{1}{25}\left(4-4e^{-2it}-4e^{+2it}+4\right)$$

$$\boxed{P(u) = \frac{1}{25}\left(8-8\cos 2t\right)}$$

minimum of $\mathcal{P}(h)$ is $\frac{9}{25}$ $\rightarrow$ mouse is never unhappy

(d) $\langle A \rangle = \mathcal{P}(h) - \mathcal{P}(u)$

$= \frac{1}{25}\left(9 + 16\cos 2t\right)$ oscillates between 1 and $-7/25$

Compare to Ehrenfest

$$\frac{d}{dt}\langle A \rangle = -\frac{32}{25}\sin 2t$$

$$[A,H] = \frac{2}{5}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{2}{5}\begin{bmatrix} 6 & 2 \\ -2 & -9 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 6 & -2 \\ 2 & -9 \end{bmatrix} = \frac{2}{5}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

$$\langle [A,H] \rangle = \frac{2}{125}\begin{bmatrix} e^{+4it} + 4e^{+2it}, & 2e^{+4it} - 2e^{+2it} \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} e^{-4it} + 4e^{-2it} \\ 2e^{-4it} - 2e^{-2it} \end{bmatrix}$$

$$= \frac{2}{125}\begin{bmatrix} e^{+4it} + 4e^{+2it}, & 2e^{+4it} - 2e^{+2it} \end{bmatrix}\begin{bmatrix} 8e^{-4it} - 8e^{-2it} \\ -4e^{-4it} - 16e^{-2it} \end{bmatrix}$$

$$= \frac{2}{125}\left(\cancel{8} - \cancel{32} + 32e^{-2it} - 8e^{2it} - \cancel{8} + 8e^{-2it} - 32e^{2it} + \cancel{32}\right)$$

$$= \frac{2}{125}\left(40e^{-2it} - 40e^{2it}\right) \overset{?}{=} -\frac{32i}{25}\sin 2t$$

From Eq. (4.9) $|h(t)\rangle = e^{-4it}|4\rangle\langle 4|h\rangle + e^{-2it}|2\rangle\langle 2|h\rangle$

$$\langle 4|h\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \qquad \langle 2|h\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{2}{\sqrt{5}}$$

$$|h(t)\rangle = \frac{e^{-4it}}{\sqrt{5}}|4\rangle - \frac{2e^{-2it}}{\sqrt{5}}|2\rangle$$

$$\leftrightarrow \frac{e^{-4it}}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2e^{-2it}}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$|h(t)\rangle \leftrightarrow \frac{1}{5}\begin{bmatrix} e^{-4it} + 4e^{-2it} \\ 2e^{-4it} - 2e^{-2it} \end{bmatrix}$$

2 We can reuse the propagator:

$$U(t) \leftrightarrow \frac{1}{5}\begin{bmatrix} e^{-4it} + 4e^{-2it} & 2e^{-4it} - 2e^{-2it} \\ 2e^{-4it} - 2e^{-2it} & 4e^{-4it} + e^{-2it} \end{bmatrix}$$

Then $|p(t)\rangle = U(t)|p\rangle$, where $|p\rangle \leftrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

$$|p(t)\rangle \leftrightarrow \frac{1}{5\sqrt{2}}\begin{bmatrix} e^{-4it} + 4e^{-2it} & 2e^{-4it} - 2e^{-2it} \\ 2e^{-4it} - 2e^{-2it} & 4e^{-4it} + e^{-2it} \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$|p(t)\rangle \leftrightarrow \frac{1}{5\sqrt{2}}\begin{bmatrix}(1+2i)e^{-4it}+(4-2i)e^{-2it}\\(2+4i)e^{-4it}-(2-i)e^{-2it}\end{bmatrix}$$

$$\mathcal{P}(p) = |\langle p|p(t)\rangle|^2 = \frac{1}{100}\left|\begin{bmatrix}1 & -i\end{bmatrix}\begin{bmatrix}(1+2i)e^{-4it}+(4-2i)e^{-2it}\\(2+4i)e^{-4it}-(2-i)e^{-2it}\end{bmatrix}\right|^2$$

$$= \frac{1}{100}\left|5e^{-4it}+5e^{-2it}\right|^2 = \frac{1}{4}\left|e^{-4it}+e^{-2it}\right|^2$$

$$= \frac{1}{4}\left(1+1+e^{-2it}+e^{2it}\right) = \frac{1}{2}(1+\cos 2t)$$

So the mouse will sometimes be aggresive with 100% probability — when $\cos 2t = -1$

$$\mathcal{P}(a) = 1-\mathcal{P}(p) = \frac{1}{2}(1-\cos 2t)$$

$$\langle B\rangle = (+1)\mathcal{P}(p)+(-1)\mathcal{P}(a)$$
$$= \frac{1}{2}(1+\cos 2t)-\frac{1}{2}(1-\cos 2t) = \cos 2t$$

oscillates between +1 and -1