PHY 474/374 Spring 2021

Week 3—Monday, April 12—Discussion Worksheet

The Lane-Emden Equation

In the previous class, we derived the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

1. Answer the following questions. You should try to answer first by discussing in your group, but without looking at your textbook.

(a) What quantity does ξ parametrize in this equation?

$\xi = r/\alpha$ where $\alpha^2 = \frac{(n+1) K \rho_c^{1/n - 1}}{4\pi G}$

distance to the center of the star

(b) What quantity does θ parametrize in this equation?

quantity θ, a dimensionless measure of ρ, density.

given by $\rho = \rho_c \theta^n$, where ρ_c is central density

(c) What is n?

Polytropic is defined via $P = K\rho^\gamma$

Polytropic index $n = 1/\gamma - 1$, so that $\gamma = 1 + 1/n$

(d) What is the value of θ for $\xi = 0$ for all n? Why?

For all n, $\theta = 1$ for $\xi = 0$

(e) The surface of the model is defined by the point $\xi = \xi_1$. What is the value of θ at this point?

$\rho = 0 \rightarrow \theta = 0$

Please STOP here! Do not proceed until told to do so.

Energy Transport in Stellar Interiors

The net movement of radiation in a star is outward, from the center to the surface. Photons go in a random walk and so it can take energy over a million years to get from the center to the surface.

2. Consider the energy transport in the time interval dt through an area dA orthogonal to the direction to the center of the star, at $r = r_0$ (see figure below). The direction of motion of the photons is specified by the angle θ between the outward directed normal to dA and the direction of motion.

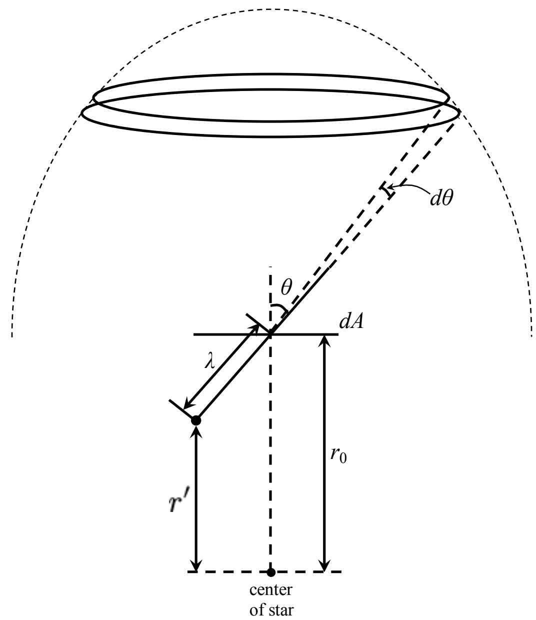
(a) If the photons are isotropically distributed, show that the fraction of photons with directions between θ and $(\theta + d\theta)$ is given by

$$\frac{\sin\theta \, d\theta}{2}$$

$$[\theta, (\theta + d\theta)]$$

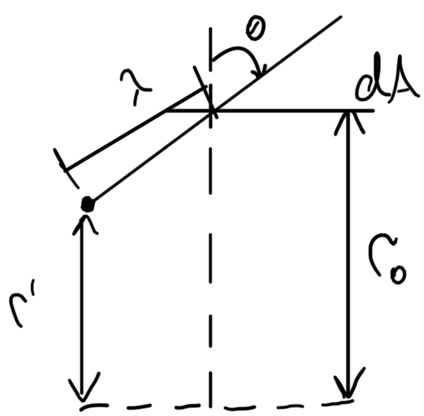
$$\rightarrow \frac{2\pi \sin\theta \, d\theta}{4\pi}$$

$$\rightarrow \frac{\sin\theta \, d\theta}{2}$$



(b) If λ is the mean free path between interactions (like scattering or absorption) of a photon and gas particle, then photons that go through dA with directions between θ and $(\theta + d\theta)$ on average come from a distance r' from the center. Use the figure above to show that

$$r' = r_0 - \lambda \cos\theta$$



From the figure

$$r' = r_0 - \lambda \cos\theta$$

Since the photons going through dA with directions between θ and $(\theta + d\theta)$ come from distance r', they correspond to the energy density $u_R(r')$, and their contribution to the energy transport through dA is the product of the quantities in the three square brackets below, which correspond respectively to the contribution to the energy density, the projected area, and the path length.

$$\left[\frac{\sin\theta \, d\theta}{2} u_R(r')\right] \left[dA \cos\theta\right] \left[c \, dt\right] \tag{1}$$

3. The total energy transport through dA in time dt is obtained by integrating equation (1) on the previous page over all directions θ, from 0 to π.

(a) First, do a Taylor expansion of $u_R(r')$ at r_0, retaining terms up to first order. You should get

$$u_R(r') = u_R(r_0) - \Big(\lambda \cos\theta\Big) \frac{du_R}{dr}$$

Note: The Taylor expansion of $f(x)$ at $x = a$ is

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \ldots$$

$f(a) = U_R(r_0)$

$f(x) = U_R(r') = U_R(r_0) - (\lambda \cos\theta)\frac{dU_R}{dr}$

$f(a)(x-a) = -(\lambda \cos\theta)\frac{dU_R}{dr}$

(b) Then, integrate equation (1) from $\theta = 0$ to π after incorporating your result in part (a) above, and show that the total energy transport through dA in time dt is

$$dE = -\frac{\lambda c}{3}\,\frac{du_R}{dr}\,dA\,dt$$

$$dE = \int_0^\pi \left[\frac{\sin\theta}{2}\, U_R(r')\right]\left[dA\cos\theta\right]\left[c\,dt\right]$$

$$= \frac{c}{2}\int_0^\pi \left[U_R(r_0) - \lambda\cos\theta\,\frac{dU_R}{dr}\right]\cos\theta\,\sin\theta\,dA\,dt$$

$$= \left[\frac{c}{2}U_R(r_0)\int_0^\pi \cos\theta\sin\theta\,d\theta - \frac{\lambda c}{2}\frac{dU_R}{dr}\int_0^\pi \cos^2\theta\sin\theta\,d\theta\right]dA\,dt$$

$\rightarrow \cos\theta = x,\ -\sin\theta\,d\theta = dx \Rightarrow -\int x\,dx = -x^2/2 \rightarrow -\int x^2 dx = -x^3/3$

$$dE = \left\{\frac{c}{2}U_R(r_0)\left[-\frac{\cos^2\theta}{2}\right]_0^\pi - \frac{\lambda c}{2}\frac{dU_R}{dr}\left[-\frac{\cos^3\theta}{3}\right]_0^\pi\right\}dA\,dt$$

$$= \left\{-\frac{c}{2}U_R(r_0)\left[\frac{\cos^2\pi - \cos^2 0}{2}\right] + \frac{\lambda c}{2}\frac{dU_R}{dr}\left[\frac{\cos^3\pi - \cos^3 0}{3}\right]\right\}dA\,dt$$

$$= \left\{-\frac{c}{2}U_R r_0\left[\frac{1-1}{2}\right] + \frac{\lambda c}{2}\frac{dU_R}{dr}\left[\frac{-1-1}{3}\right]\right\}dA\,dt$$

$$= 0 - \frac{\lambda c}{2}\frac{dU_R}{dr}\left(\frac{2}{3}\right)dA\,dt \Rightarrow dE = -\frac{\lambda c}{3}\frac{dU_R}{dr}\,dA\,dt$$

4. We will now derive an equation for the temperature gradient dT/dr in a star.

(a) By convention, one uses the opacity κ to describe the interaction between radiation and matter instead of using λ. It is defined such that $\lambda = (\kappa\rho)^{-1}$.

Show that the radiative flux F_R, defined by $dE = F_R\, dAdt$, is then given by

$$F_R = -\frac{c}{3\kappa\rho}\frac{du_R}{dr} = -\frac{\lambda c}{3}\frac{du_R}{dr}$$

$$F_R = \frac{dE}{dAdt} = -\frac{\lambda c}{3}\frac{du_R}{dr}\frac{\cancel{dAdt}}{\cancel{dAdt}} = -\frac{\lambda c}{3}\frac{du_R}{dr}$$

$\lambda = 1/\kappa\rho$, so

$$F_R = -\left(\frac{1}{\kappa\rho}\right)\frac{c}{3}\frac{du_R}{dr}$$

$$F_R = -\frac{c}{3\kappa\rho}\frac{du_R}{dr}$$

(b) The energy density of radiation is $u_R = aT^4$, where a is the radiation constant. Moreover, the total amount of energy transported by radiation through a sphere of radius r is given by $L(r) = 4\pi r^2 F_R$. Use these, together with the expression you derived in part (a) to obtain one of the fundamental equations of stellar structure

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi a c r^2 T^3}$$

$$F_R = -\frac{c}{3\kappa\rho}\frac{du_R}{dr}$$

$$= -\frac{c}{3\kappa\rho}\frac{d}{dr}\left[aT^4\right] = -\frac{c}{3\kappa\rho}\left(a4T^3\frac{dT}{dr}\right) = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$$

Now,

$$L(r) = 4\pi r^2 F_R = 4\pi r^2\left[-\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}\right]$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi a c r^2 T^3}$$

Radiation from the Stellar Surface

Near the surface of the star, the mean free path becomes very large, so the analysis carried out so far does not apply. To estimate the energy radiated from the stellar surface, we can still use equation (1) with the following modifications.

5. Since the number of photons directed outward and inward is not near balance at the stellar surface, like it is in the interior of the star, we don't need to take the r-dependence of u_R into account at the surface. Also, only photons directed outward with $\theta \leq \pi/2$ will contribute to the energy radiated from the stellar surface.

(a) Modifying equation (1) by writing u_R as a constant instead of being r-dependent, and integrating from 0 to $\pi/2$, show that the radiative flux at the stellar surface is given by

$$F_R = \sigma T^4$$

where $\sigma = ac/4$ is the Stefan-Boltzmann constant.

$$dE = \int_0^{\pi/2} \left[\frac{\sin\theta\, d\theta}{2} u_R\right]\left[dA\cos\theta\right]\left[c\,dt\right]$$

$$= \frac{c}{2} u_R\, dA\, dt \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta$$

$$= \frac{c}{2} u_R\, dA\, dt \left[-\frac{\cos^2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{c}{2} u_R\, dA\, dt \left[-\frac{(\cos^2 \pi/2 - \cos^2 0)}{2}\right] = -\frac{c}{2} u_R\, dA\, dt\left[\frac{0-1}{2}\right] = \frac{c}{4} u_R\, dA\, dt$$

$$F_R = \frac{dE}{dA\,dt} = \frac{c}{4} u_R \frac{dA\,dt}{dA\,dt} = \frac{c}{4}\left[a T^4\right]$$

$$F_R = \left(\frac{ac}{4}\right) T^4 = \sigma T^4 = \sigma T^4 \text{ where } \sigma = \frac{ac}{4}$$

(b) If T_{eff} is the effective temperature of the surface of the star, show that the surface luminosity of the star, L_s, is given by

$$L_s = 4\pi R^2 \sigma T_{\text{eff}}^4$$

where R is the radius of the star.

$$L_s = L(r=R) = 4\pi R^2 F_R$$

$$= 4\pi R^2 (\sigma T_{eff}^4)$$

$$L_s = 4\pi R^2 \sigma T_{eff}^4$$