Upcoming deadlines:

- Homework 1 is due on Thursday at the beginning of class.
- Warm-up quiz 2 (section 1.11) is due next Tuesday

Plan for today

- Go over questions from the reading
- Constraints, degrees of freedom, holonomic versus non-holonomic.
- Virtual work.
- Worksheets 2 and 3

Reading Questions

If your generalized coordinates are angles, what will be the units of the generalized velocity and force?

Generalized velocity: $\dot{q} = \frac{dq}{dt}$. If $q = \theta$ then $\dot{q} = \dot{\theta}$ is angular velocity

Generalized force: $Q = F\frac{\partial x}{\partial q}$. If $q = \theta$ (radians) then $F\frac{\partial x}{\partial q}$ has the units of Nm, which is the unit of torque.

Let’s do an example together:

Exercise 1.11 A particle is acted upon by a force with components F_x and F_y. Determine the generalized forces in polar coordinates. Answer: $Q_r = F_x \cos\theta + F_y \sin\theta$ and $Q_\theta = -F_x r \sin\theta + F_y r \cos\theta$.

Two identical particles are constrained to move on the surface of a sphere of radius R, as shown above. The particles are also connected by a rigid rod of length L.

- How many constraints are there? Write down the equations for the constraints.
- Are the constraints holonomic?
- How many degrees of freedom does this system have?

Two free particles would have six degrees of freedom. There are three constraints:

Both particles are constraint to the surface of the sphere:

$$x_1^2 + y_1^2 + z_1^2 - R^2 = 0$$
$$x_2^2 + y_2^2 + z_2^2 - R_2 = 0$$

Distance between particles is constant

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - L^2 = 0$$

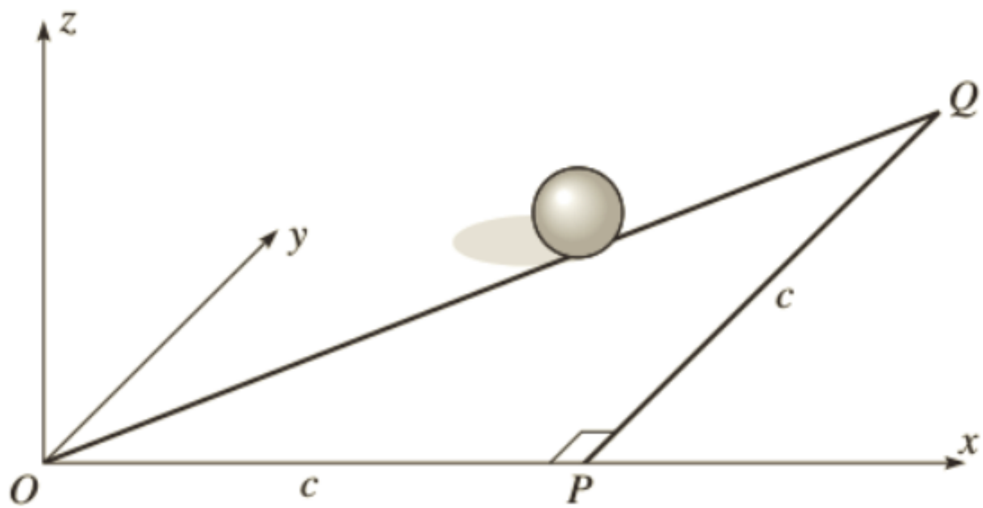
The constraints are all holonomic because they have the form $f(q_1, q_2 \dots, t) = 0$.

So the system has 6-3=3 degrees of freedom. For example, we could express the configuration of the system by specifying three angles: θ and ϕ of one particle as well as the angle between the rod and a line of constant θ or of constant ϕ.

Rolling without slipping in one dimension is holonomic

- Two coordinates describe the configuration of the ball: x and θ.
- Rolling without slipping: $x - R\theta = 0$.
  This is a holonomic constraint. We can use it to eliminate x or θ.
- Only one independent variable (either x or θ)

Rolling without slipping in two dimensions in a nonholonomic constraint



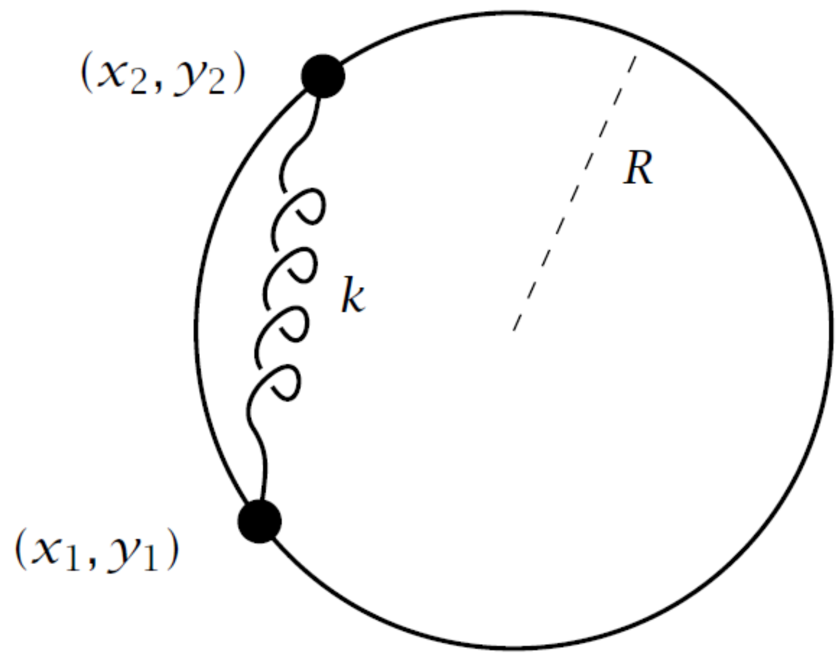
From Taylor, Classical Mechanics

Unlike in the one-dimensional case, the rolling without slipping constraint doesn't allow us to eliminate a rotation angle. For example:

1. Mark the top of the ball
2. Roll the ball by one circumference *c* from O to P -> mark is back on top
3. Roll the ball by one circumference *c* from P to Q -> mark is back on top
4. Roll the ball from Q to back to O, distance $\sqrt{2}\,c$ - > mark is NOT on top

We can have different rotation angles at the same position, so we can't use the rolling without slipping contraint to eliminate any of the five variables.

Worksheet 2



- Derive the total energy in both Cartesian and polar coordinates
- For both horizontal and vertical hoops

Worksheets 3&4: Principle of Virtual Work

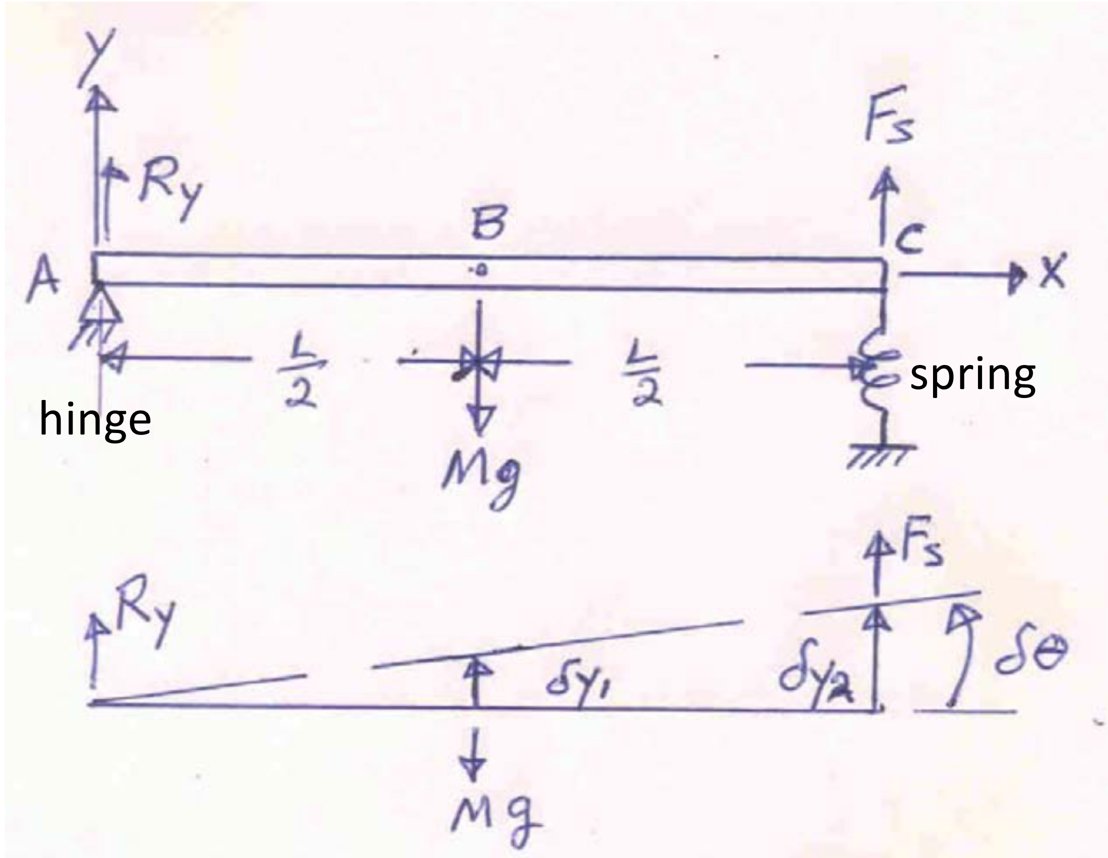
If a system is in equilibrium then virtual work (the work done by a virtual displacement) is 0:

$$\delta W = \sum_{\alpha} Q_{\alpha} \delta q_{\alpha} = 0$$

With generalized force

$$Q_{\alpha} = \sum_{i} F_i \frac{\partial x_i}{\partial q_{\alpha}}$$

Example of the Principle of Virtual Work



Generalized coordinate is θ $\quad y_0 = 0 \;\; y_1 = \frac{L}{2}\,\theta \;\; y_2 = L\theta$

$$Q_\theta = \sum_i F_i \frac{\partial y_i}{\partial \theta} = R_y \frac{\partial y_0}{\partial \theta} + Mg \frac{\partial y_1}{\partial \theta} + F_s \frac{\partial y_2}{\partial \theta} = 0 - Mg\frac{L}{2} + F_s L$$

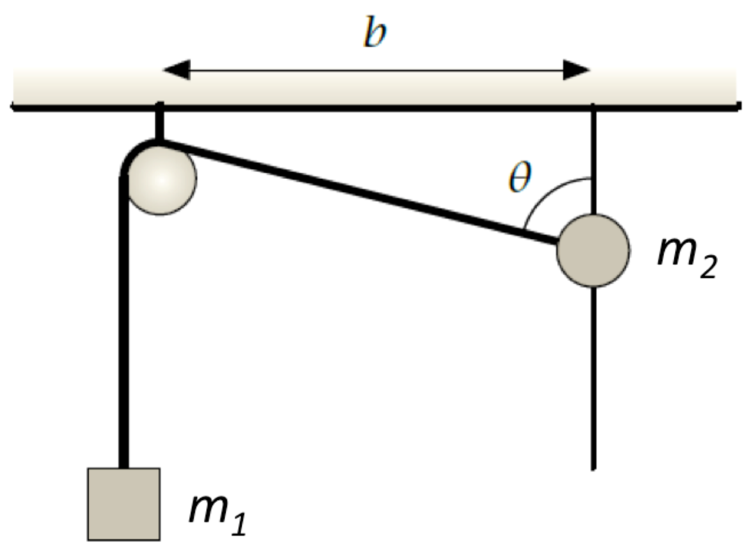
$\delta W = (-Mg\frac{L}{2} + F_s L)\delta\theta$ $\quad$ Since $\delta\theta \neq 0$: $F_s L = Mg\frac{L}{2} \Rightarrow F_s = \frac{Mg}{2}$

Another example of the principle of virtual work: Hamill Exercise 1.12

(handout)

1. Identify the generalized coordinate
2. Compute the coordinate transformations
3. Express the forces on both masses in cartesian coordinates
4. Compute the generalized forces
5. Set $\delta W = 0$

Worksheet 3



- Find the equilibrium value for θ using the principle of virtual work.