Physics 412—Practice S–8 (Due Mar. 9, 11 am) Name:

S–8: I can explain the relationships between the energy eigenstates of the simple harmonic oscillator and use the eigenstates to make predictions about measurements.

Unsatisfactory   Progressing   Acceptable   Polished

(1) The harmonic oscillator Hamiltonian is

$$H = \frac{P^2}{2m} + \frac{m{\omega_0}^2 X^2}{2},$$

and the raising and lowering operators are

$$a^\dagger = \frac{1}{\sqrt{2}d_0}\left(X - \frac{\mathrm{i}P}{m\omega_0}\right), \quad a = \frac{1}{\sqrt{2}d_0}\left(X + \frac{\mathrm{i}P}{m\omega_0}\right), \quad X = \frac{d_0}{\sqrt{2}}\left(a^\dagger + a\right), \quad P = \frac{\mathrm{i}\hbar}{\sqrt{2}d_0}\left(a^\dagger - a\right),$$

where $d_0 = \sqrt{\hbar/m\omega_0}$. For the rest of this question, assume that we live in a universe where X and P commute, so that $\left[X, P\right] = 0$. You're going to investigate how this changes the quantum harmonic oscillator.

(a) Express the Hamiltonian in terms of the raising and lowering operators, a and $a^\dagger$.

(b) Does the Hamiltonian commute with a and/or $a^\dagger$? Find $\left[H, a\right]$ and $\left[H, a^\dagger\right]$.

(c) If $|E\rangle$ is an energy eigenstate of the harmonic oscillator, what are $a|E\rangle$ and $a^\dagger|E\rangle$? (Hint: evaluate $H(a|E\rangle)$ and $H(a^\dagger|E\rangle)$.)

(d) Are $a^\dagger$ and a still raising and lowering operators? Explain.

(2) For this question we are back in the real world where X and P do not commute and the raising and lowering operators have the actions

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad \text{and} \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

when applied to energy eigenstate $|n\rangle$ with energy $E_n = \hbar\omega(n+1/2)$. (Note that the definitions of a and $a^\dagger$ on the other side still apply.)

At time $t = 0$, a quantum harmonic oscillator is in the state $|\Psi(0)\rangle = A\Big[|0\rangle + 2|1\rangle + 2\mathrm{i}|3\rangle\Big]$, where A is a normalization constant.

(a) If you measured the energy of the oscillator, what values could you obtain and with what probabilities?

(b) Calculate $\langle X\rangle$ for this state.