Problem 2-9

$$f(x,y,z) = x^2 + 2y^2 + 3z^2 + 2xy + 2xz$$

constraint

$$x^2 + y^2 + z^2 = 1$$

$$\Rightarrow g(x,y,z) = x^2 + y^2 + z^2 - 1$$

So $\delta(f + \lambda g) = 0$

or $\sum_{i=1}^{3} \frac{\partial f}{\partial q_i} + \frac{\partial \lambda g}{\partial q_i} = 0$

or $\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \Rightarrow 2x + 2y + 2z + 2\lambda x = 0$ ①

$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \Rightarrow 4y + 2x + 2\lambda y = 0$ ②

$\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0 \Rightarrow 6z + 2x + 2\lambda z = 0$ ③

From ② $y = -\frac{x}{\lambda+2}$ From ③ $z = -\frac{x}{\lambda+3}$

∴ ① becomes

$$x - \frac{x}{\lambda+2} - \frac{x}{\lambda+3} = \lambda x$$

$$1 - \frac{1}{\lambda+2} - \frac{1}{\lambda+3} = \lambda.$$

$$(\lambda+2)(\lambda+3) - (\lambda+3) - (\lambda+2) = \lambda(\lambda+2)(\lambda+3)$$

or $\lambda^3 + 4\lambda^2 + 3\lambda - 1 = 0$

Using an internet cubic equation solver I got the 3 roots

$$\lambda = 0.247,\ -2.80,\ -1.45$$

Just using the first one

$$y = -\frac{x}{\lambda+2} = -\frac{x}{2.25} \quad \text{and} \quad z = -\frac{x}{\lambda+3} = -\frac{x}{3.25}$$

So $x^2 + y^2 + z^2 = 1 \Rightarrow x^2\left(1 + \frac{1}{(2.25)^2} + \frac{1}{(3.25)^2}\right) = 1 \Rightarrow x = \sqrt{1/1.29} = \underline{0.88}$

So $y = -0.88/2.25 = -0.39$

$z = -0.88/3.25 = -0.27$

So minimum value of f is $(0.88)^2 + 2(-0.39)^2 + 3(0.27)^2 - 2(.88)(.39) - 2(.88)(.2$

$$= \underline{0.136}$$