Last time we found the following set of equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{R_o^2 a^2} \qquad (1)$$

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0 \qquad (2)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) \qquad (3)$$

However the equations are not independent, Eq. (3) is a combination of (1) and (2). The problem is that we have three time dependent unknown quantities:

$$a(t); \quad \epsilon(t); \quad P(t)$$

So we need another equation. Once again, thermodynamics comes to rescue by the use of *equations of state.*

*Equations of state*, are equations that relate the *state variables* of a physical system.

A *state variable* is a quantity which can be determined by the state of the system at a *single moment in time*. The temperature, volume, and pressure of a thermodynamic system are examples of this - you can simply measure them, and you don't need any information about precisely *how* the system arrived in its current state. *Work*, for example, is *not* a state variable.

Most famous equation of state: $PV = nk\,T$, but there are many (often very complicated) others. Luckily in cosmology we are dealing with diffuse gases which simplifies things.

Note that $kT$ is an energy term, and we can write the ideal gas law as $P = \dfrac{nkT}{V} = \dfrac{nE}{V} = w\epsilon$

Equation of state $P = w\epsilon$

- Non-relativistic matter: $P_{\mathrm{nonrel}} = w\epsilon_{\mathrm{nonrel}}; \quad w \approx 0$
- Relativistic gas (photons, etc): $P_{\mathrm{nonrel}} = w\epsilon_{\mathrm{nonrel}}; \quad w \approx 1/3$
- In cosmology, when a component, $\varepsilon_i$, has $w \approx 0$ it is called *matter*, when $w \approx 1/3$ we call it *radiation,* when $w < -1/3$, we call it dark energy

While $w$ is dimensionless, there are the restrictions as to what values $w$ can have. To see this, first recall $P = w\epsilon$
Now recall that a perturbation in pressure is a *sound wave* and it can be shown that speed of that wave is

$$c_s^2 = c^2 \left(\frac{dP}{d\epsilon}\right); \quad c \equiv \text{speed of light}$$ But $$\left(\frac{dP}{d\epsilon}\right) = w$$ so $$c_s^2 = c^2 w$$

Do question (1) on the worksheet and S T O P

| $w$ | Form |
|---|---|
| 0 | Matter |
| +1/3 | Radiation |
| + 1 | Free massless scalar fields |
| -1/3 | Curvature Energy |
| -2/3 | Domain Walls |
| -1 | Cosmological constant |

We derived the Friedmann equations using Newtonian mechanics + just a couple of results from GR.

It turns out we can use the same Newtonian mechanics to show that a *static universe is unstable*.

To show this, let's begin with Poisson equation:

$$\nabla^2\Phi = 4\pi G\rho; \quad a = -\nabla\Phi$$

where *a* here is the acceleration. You'll now use these facts to show the consequences. Do question (2) on the worksheet and S T O P

(2a) $a = 0$

(2b) $\nabla\Phi = 0 \Rightarrow \Phi = \text{constant}$

(2c) $\nabla\Phi^2 = 0 \Rightarrow \rho = 0$

(2d) Only a universe with nothing in it can be stable!

Einstein's dilemma. Evidence points to a static universe.

Yet his equations, which he believes are correct, imply a dynamic universe.

So Einstein tries to *fix* his equations

## Lessons from E & M

- Recall that $\vec{B} = \nabla \times \vec{A};$ where $\vec{A}$ is the vector potential
- But now suppose we modify $\vec{A}$ $\vec{A} = A(\vec{r}) + \nabla\chi(\vec{r});$ where $\chi$ is an arbitrary scalar function
- Then
$$\begin{aligned} \vec{B} &= \nabla \times (A + \nabla\chi) \\ &= \nabla \times A + \underbrace{(\nabla \times \nabla\chi)}_{\substack{=0; \\ \text{standard vector relation}}} \\ \vec{B} &= \nabla \times A \end{aligned}$$

Do question (3) on the worksheet and S T O P

Einstein use the idea of gauge transformations (i.e. a change in a part of the expression that does not affect an observable) to modify the Friedmann equations. Einstein found that could add a constant, $\Lambda$, that would affect any observable.

Specifically, he modified the Friedmann equations as
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{R_o^2 a^2} + \underbrace{\frac{\Lambda}{3}}_{\text{new term}}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left(\epsilon + \underbrace{\frac{c^2}{8\pi G}\Lambda}_{\epsilon_\Lambda}\right) - \frac{kc^2}{R_o^2 a^2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{R_o^2 a^2} + \underbrace{\frac{\Lambda}{3}}_{\text{new term}}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left(\epsilon + \underbrace{\frac{c^2}{8\pi G}\Lambda}_{\epsilon_\Lambda}\right) - \frac{kc^2}{R_o^2 a^2}$$

Consequences:

Fluid equation unchanged: $\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0 \Rightarrow \dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0$

Acceleration equation: $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) \Rightarrow \boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) + \frac{\Lambda}{3}}$

If , $\Lambda$, remains constant in time, then so does its associated energy density. The fluid equation for this component is the

$$P_\Lambda = -\epsilon_\Lambda = -\frac{c^2}{8\pi G}\Lambda$$

To explore some of the other consequences, do question (4) on the worksheet and S T O P

(4a) $$0 = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3} \Rightarrow \boxed{\Lambda = 4\pi G\rho}$$

(4b) $$\begin{aligned} 0 &= \frac{8\pi G}{3}\rho - \frac{kc^2}{R_o^2} + \frac{\Lambda}{3} \\ k &= \frac{4\pi G\rho R_o^2}{c^2} > 0 \Rightarrow \boxed{k = 1} \end{aligned}$$

(4c) $$R_o = \frac{c}{2\sqrt{\pi G\rho}} = \frac{c}{\sqrt{\Lambda}}$$

Start homework problem 4.5--Hints

- Start with fluid equation and set $P = w\,\epsilon$
- Write time derivative as $\frac{d}{dt} = \frac{da}{dt}\frac{d}{da}$
- Plug away

For now, we are going to ignore the cosmological constant so that we are working with

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{R_o^2 a^2} \qquad (1)$$

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0 \qquad (2)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) \qquad (3)$$

with $P = w\,\epsilon$

Issues to making progress:

- $w$ is different for different kinds of energy densities
- $w$ may have changed with time
- It's possible that each component evolved differently so that the dominant component will change
- Components may interact with each other.

So long as components only interact with each other for short periods of times, we can write that

$$\epsilon = \sum_w \epsilon_w; \quad P = \sum_w w\epsilon_w$$

and each component can be separated and we'll have a fluid equation for each component.

Do question (5) on the worksheet

(5a and b)

$$\frac{d\epsilon_w}{dt} = \frac{-3}{a}\frac{da}{dt}(\epsilon_w + P_w) \quad \text{now use } P_w = w\epsilon_w$$

$$\frac{d\epsilon_w}{dt} = \frac{-3}{a}\frac{da}{dt}\epsilon(1+w)$$

$$\frac{d\epsilon}{\epsilon} = -3(1+w)\frac{da}{a}$$

$$\epsilon_w = \epsilon_{w_o} a^{-3(1+w)} \text{ where I have set } a(t_o) = 1$$

(5c)

**Matter:**

$$w = 0; \epsilon_m = \frac{\epsilon_{0,m}}{a^3}$$

**Radiation**

$$w = \frac{1}{3}; \epsilon_r = \frac{\epsilon_{0,r}}{a^4}$$

**Dark Energy**

$$w = -1, \epsilon_\Lambda = \epsilon_{\Lambda,0}$$

Do question (6) on the worksheet