PHY 411 Winter 2021

Week 2—Tuesday, Jan 12—Discussion Worksheet

Chapter 7: Electromagnetic Waves and Propagation

The genius of Maxwell becomes evident as soon as we begin playing around with his equations and realize that they lead naturally to traveling wave solutions that transport energy from one point to another. We will now study the simplest and most fundamental electromagnetic waves: transverse plane waves. To keep things simple, let us study the properties of fields **in free space where there are no sources** (i.e., with $\rho = 0$ and $\vec{J} = 0$). In such regions, **Maxwell's equations are**

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} \tag{7.1}$$

Let us assume solutions with harmonic time dependence

$$\vec{E}(\vec{x},t) = \vec{E}(\vec{x})\,e^{-i\omega t}, \qquad \vec{B}(\vec{x},t) = \vec{B}(\vec{x})\,e^{-i\omega t}, \qquad \vec{D}(\vec{x},t) = \vec{D}(\vec{x})\,e^{-i\omega t}, \qquad \vec{H}(\vec{x},t) = \vec{H}(\vec{x})\,e^{-i\omega t}$$

To avoid crowding, we will just write $\vec{E}$ instead of $\vec{E}(\vec{x})$, even though it could be confused with $\vec{E}(\vec{x},t)$, but it is consistent with Jackson's notation. *You've already shown in the previous class* that with this choice of time dependence, you can derive the **Helmholtz wave equation**:

$$\left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{E} = 0 \qquad \text{and} \qquad \left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{B} = 0 \tag{7.3}$$

Let us write equation (7.3) as

$$\left(\nabla^2 + k^2\right)\vec{u} = 0 \tag{7.3.a}$$

where $\vec{u}$ stands for $\vec{E}$ or $\vec{B}$ and k is called the wave number — we'll see why shortly.

$k = \mu\epsilon\omega^2$

1. Answer the following.

(a) By comparing equation (7.3) and equation (7.3.a), find k^2, and then show that this leads to

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$$

$k^2 = \mu\epsilon\omega^2$

$\Rightarrow k = \sqrt{\mu\epsilon}\,\omega \Rightarrow 1 = \frac{\sqrt{\mu\epsilon}\,\omega}{k} \Rightarrow \boxed{\frac{1}{\sqrt{\mu\epsilon}} = \frac{\omega}{k}}$

(b) Calculate the value of $\dfrac{1}{\sqrt{\mu_0\epsilon_0}}$

$\mu_0 = 1.256637 \times 10^{-6}\ H\,m^{-1}$

$\epsilon_0 = 8.85419 \times 10^{-12}\ F\,m^{-1}$

$c = \dfrac{1}{\sqrt{(1.25\times10^{-6}\ Hm^{-1})(8.85\times10^{-12}\ Fm^{-1})}} = 2.99 \times 10^{8}\ m/s$

No doubt, you found in part (b) on the previous page that $1/\sqrt{\mu_0\epsilon_0}$ is equal to c, the speed of light in vacuum — one of the original spectacular results of Maxwell's work!

Thus, the quantity ω/k has the dimension of speed and is called the ***phase velocity*** of the wave.

2. By defining the index of refraction n (which we'll discuss in much greater detail when we get to § 7.3) of a medium with permittivity ϵ and permeability μ as

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

derive the following expression for the phase velocity:

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n} \tag{7.5}$$

Yes, this is yet another use of the symbol n!!!!! — get used to it, it happens a lot.

$C = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ $\quad n = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$

$\Rightarrow \frac{\cancel{(\sqrt{\mu_0\epsilon_0})^{-1}}\ \cancel{(\sqrt{\mu_0\epsilon_0})}}{(\sqrt{\mu\epsilon})} \Rightarrow \frac{1}{\sqrt{\mu\epsilon}}$

$\frac{1}{\sqrt{\mu\epsilon}} = \frac{\omega}{k} \Rightarrow \frac{\omega}{k} = v$

Units (m/s)

therefore,

$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{C}{n}$

Plane Wave Solutions: One-dimensional Case

We will now look at plane wave solutions to the Helmholtz wave equation. Before looking at the general plane wave solution of equation (7.3.a) in three dimensions, though, let us follow Jackson and consider a solution for a plane wave propagating along the x-direction.

Presumably, you remember that whenever you worked with plane waves propagating along one dimension, you used $\sin(kx - \omega t)$ or $\cos(kx - \omega t)$ to represent such waves. To be more general, and also because it makes the math easier to manipulate, we'll use $e^{i(kx-\omega t)}$ as a possible solution, recognizing of course that we've put back the t-dependence in the wave equation. Such a plane wave, as you must know, is traveling in the positive x-direction. Moreover, a plane wave propagating in the negative x-direction, given by $e^{-i(kx-\omega t)}$ is also a solution. Before proceeding, let us verify that this is indeed the case by explicit substitution of these solutions into the wave equation.

3. In this problem, you will verify the plane wave solution to the Helmholtz wave equation for the one-dimensional case. $(\nabla^2 + k^2)u^2 = 0$ 7.3a $\vec{u}(\vec{x},t) = \vec{u}(\vec{x})e^{-i\omega t}$

(a) With reference to equation (7.3.a), write down the 1-D Helmholtz wave equation (along x).

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right)\vec{u}(\vec{x},t) = 0$$

(b) Consider a plane wave traveling in the positive x-direction so that, e.g., the electric field goes as $e^{i(kx-\omega t)}$ (without worrying about any constants to specify the amplitude for now). Show that this plane wave satisfies the 1-D Helmholtz wave equation.

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right)\vec{u}(\vec{x})e^{i(kx-\omega t)} = 0$$

$$\frac{\partial^2}{\partial x^2} = k^2\left(-e^{i(kx-t\omega)}\right)$$

$$\left(-u(\vec{x})k^2 e^{i(kx-t\omega)} + \vec{u}(\vec{x})k^2 e^{i(kx-t\omega)}\right) = 0$$

(c) Consider a plane wave traveling in the negative x-direction so that, e.g., the electric field goes as $e^{-i(kx-\omega t)}$. Show that this plane wave satisfies the 1-D Helmholtz wave equation.

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right)\vec{u}(\vec{x})e^{-i(kx-\omega t)} = 0$$

$$\frac{\partial^2}{\partial x^2} = -k^2 e^{i(kx-t\omega)}$$

$$\left(-u(\vec{x})k^2 e^{i(kx-t\omega)} + \vec{u}(\vec{x})k^2 e^{i(kx-t\omega)}\right) = 0$$

Based on (b) and (c) above, the most general solution to equation (7.3.a) is therefore

$$u(x,t) = a\,e^{i(kx-\omega t)} + b\,e^{-i(kx+\omega t)} \tag{7.6}$$

where u is standing in for $\vec{E}$ or $\vec{B}$, and a and b are arbitrary constants.

By pulling out k, we can write equation (7.6) as

$$u_k(x,t) = a\,e^{ik(x-\{\frac{\omega}{k}\}t)} + b\,e^{-ik(x+\{\frac{\omega}{k}\}t)}$$

We will now look at this equation in more detail.

4. Consider the equation we wrote at the bottom of the previous page:

$$u_k(x,t) = a\, e^{ik(x-\{\frac{\omega}{k}\}t)} + b\, e^{-ik(x+\{\frac{\omega}{k}\}t)}$$

Show that (ω/k) must have dimensions of speed.

$$\left(x - \frac{\omega}{k}\cdot t\right) \Rightarrow \left(L - \left(\frac{\omega}{k}\right)\cdot T\right)$$

For this to work the time in t has to cancel with k or ω and Length has to be subtracted witen ω/k so

$$\left(L - \left(\frac{L}{\cancel{T}}\right)\cdot \cancel{T}\right) = (L - L) = 0$$ so ω is a length and k is a time or $V = \omega/k$ is a velocity

By Fourier superposition of all the different $u_k(x,t)$, the most general solution of the wave equation in one-dimension is

$$u(x,t) = f(x - vt) + g(x + vt) \tag{7.7}$$

where $f(\ldots)$ and $g(\ldots)$ are arbitrary functions, and v is the speed with which the wave travels along the x-axis, known as the ***phase velocity*** of the wave. As is clear from the equation above, the phase velocity is $v = \omega/k$. Also, we see now why k is called the wave number, because $kv = \omega$.

Plane Wave Solutions: General Case

Consider now the general case of a three-dimensional wave by considering a general electromagnetic plane wave of frequency ω and wave vector $\vec{k} = k\vec{n}$, where k is the wave number and $\vec{n}$ is the direction of propagation of the wave. We will demonstrate below that $\vec{n}$ must be a unit vector $\hat{n}$. You might be wondering at this stage why we don't just apportion magnitudes so that $\vec{k} = k\hat{n}$, but we need to be sure that the wave vector $\vec{k}$ can indeed be split up like this (i.e., all of $\vec{k}$ is along the direction of propagation defined by $\hat{n}$), with the wave number k being its magnitude.

We must make this three-dimensional plane wave satisfy equation (7.3), but note that $\vec{E}$ and $\vec{B}$ are not independent, they are linked via Maxwell's equations, so even if the wave is traveling in the direction indicated by $\vec{n}$, the amplitudes of $\vec{E}$ and $\vec{B}$ are linked; we cannot choose them separately. Let us write the plane waves as

$$\begin{aligned} \vec{E}(\vec{x},t) &= \vec{\mathcal{E}}\, e^{i(k\vec{n}\cdot\vec{x}-\omega t)} \\ \vec{B}(\vec{x},t) &= \vec{\mathcal{B}}\, e^{i(k\vec{n}\cdot\vec{x}-\omega t)} \end{aligned} \tag{7.8}$$

Remember that this isn't the final expression — we will demonstrate shortly that $\vec{n}$ is a unit vector. Notice also that $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ in equation (7.8) are defined as ***constant vectors***; that is, the spatial and time dependencies are in the exponential term.

Each component of $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ from equation (7.8), e.g., $E_x(\vec{x},t), B_x(\vec{x},t), E_y(\vec{x},t)$, etc., satisfies the Helmholtz equation (7.3) provided

$$k^2\, \vec{n}\cdot\vec{n} = \mu\epsilon\omega^2 \tag{7.9}$$

You will now demonstrate this explicitly for $E_x(\vec{x},t)$.

5. On the previous page, it was stated that each component of $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ from equation (7.8) satisfies the Helmholtz equation (7.3) provided

$$k^2\,\vec{n}\cdot\vec{n} = \mu\epsilon\omega^2$$

(a) Prove this explicitly for $E_x(\vec{x},t)$, the x-component of the electric field. That is, prove that $E_x(\vec{x},t)$ satisfies the Helmholtz equation (7.3) provided equation (7.9), reproduced above, holds.

$$\vec{E}(\vec{x},t) = \vec{\mathcal{E}}\, e^{i(k\vec{n}\cdot\vec{x}-\omega t)}$$

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) E(\vec{x},t) = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + k^2\right) \vec{\mathcal{E}}\, e^{i(k\vec{n}\cdot\vec{x}-\omega t)} = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} \vec{\mathcal{E}}\, e^{i(k\vec{n}\cdot\vec{x}-\omega t)} + k^2 \vec{\mathcal{E}}\, e^{i(k\vec{n}\cdot\vec{x}-\omega t)}\right) = 0$$

$$\Rightarrow \vec{\mathcal{E}}\, k^2\, \vec{n}\cdot\vec{n}\left(-e^{i(k\vec{n}\cdot\vec{x}-\omega t)}\right) + k^2 \vec{\mathcal{E}}\, e^{i(k\vec{n}\cdot\vec{x}-\omega t)} = 0$$

$$\Rightarrow \cancel{\vec{\mathcal{E}}}\, k^2\, \cancel{e^{i(k\vec{n}\cdot\vec{x}-\omega t)}} = \cancel{\vec{\mathcal{E}}}\, k^2\, \vec{n}\cdot\vec{n}\left(\cancel{e^{i(k\vec{n}\cdot\vec{x}-\omega t)}}\right)$$

$$\Rightarrow k^2 = k^2\, \vec{n}\cdot\vec{n} \quad \text{and} \quad k^2 = \mu\epsilon\omega^2$$

thus

$$\boxed{\mu\epsilon\omega^2 = k^2\,\vec{n}\cdot\vec{n}}$$

Equation (7.9) reproduced at the top of the page above also imposes the constraint that $\vec{n}$ must be a unit vector, such that

$$\hat{n}\cdot\hat{n} = 1$$

You will demonstrate this on the next page.

You learned today that if $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ are written as three-dimensional plane waves, as in equation (7.8), then each component of these fields, e.g., $E_x(\vec{x},t), B_x(\vec{x},t), E_y(\vec{x},t)$, etc., satisfies the Helmholtz equation (7.3) provided

$$k^2\,\vec{n}\cdot\vec{n} = \mu\epsilon\omega^2$$

(b) Show that the equation above, equation (7.9) in Jackson, is only true if

$$\vec{n}\cdot\vec{n} = 1$$

implying that what we've written in the equation as $\vec{n}$ is actually the unit vector $\hat{n}$.
Hint: Look at the expression for k^2 that you wrote in Question 1 (a).

$$(\nabla^2 + \mu\epsilon\omega^2)\,E_x(\vec{x},t) = 0$$

$$\Rightarrow \left[\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{y^2} + \frac{\partial^2 E_x}{\partial z^2} + \mu\epsilon\omega^2 E_x\right] e^{i(k\vec{n}\cdot\vec{x}-\omega t)} = 0$$

Thus,

$$-k(n_x^2 + n_y^2 + n_z^2)E_x + \mu\epsilon\omega^2 E_x = 0$$

$$k^2\,\vec{n}\cdot\vec{n} = \mu\epsilon\omega^2$$

$$\vec{n}\cdot\vec{n} = 1$$

only posible if $\vec{n}\cdot\vec{n}$, a unit vector

Given the result you derived above, let us write equation (7.8) again, but now with the unit vector $\hat{n}$:

$$\vec{E}(\vec{x},t) = \vec{\mathcal{E}}\,e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$
$$\vec{B}(\vec{x},t) = \vec{\mathcal{B}}\,e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \tag{7.8.a}$$

Remember that $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are *constant* vectors; in other words, the spatial and time dependencies of $\vec{E}(\vec{x},t)$ are in the exponential term.

The above has been mostly mathematics following from the wave equation. In the next class, we will apply physics from Maxwell's equations (7.1) and see what that tells us about electromagnetic waves in particular.