PHY 420 Spring 2021

Class Summary—Week 8, Day 2—Thursday, May 20

Covariance of Electrodynamics

The invariance in form of the equations of electrodynamics under Lorentz transformations was shown by Lorentz and Poincare before the formulation of the Special Theory of Relativity. A more precise word for it would be **covariance**, meaning that the form of the equations does not change. Not only are Maxwell's equations invariant under Lorentz transformations, but also the Lorentz force law and the continuity equation. That is, $\rho, \vec{J}, \vec{E}$ and $\vec{B}$ transform in well defined ways under Lorentz transformations. However, we have not demonstrated any of this yet. In the last few classes, we've developed the mathematics that will express that invariance (covariance). Now, we will recast our basic equations in 4-tensor form, so that we can demonstrate that invariance. That is, **since 4-tensors are invariant (covariant) under Lorentz transformations by definition, expressing equations** like (11.127) below **in terms of such quantities will ensure that the equations themselves are covariant under Lorentz transformations**. I won't proceed along Jackson's route, however, at least not initially, preferring instead a more direct approach.

Let us begin with the **continuity equation**:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{11.127}$$

If we rewrite this as

$$\frac{\partial (c\rho)}{\partial (ct)} + \vec{\nabla} \cdot \vec{J} = 0$$

then because $ct = x^0$, it becomes

$$\frac{\partial (c\rho)}{\partial x^0} + \vec{\nabla} \cdot \vec{J} = 0$$

Compare the *left hand side* to equation (11.77), in which we wrote the 4-divergence for a 4-vector $A^\alpha = (A^0, \vec{A})$:

$$\partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A}$$

Such a comparison tells us that $c\rho$ and $\vec{J}$ together form a 4-vector J^α:

$$J^\alpha = (c\rho, \vec{J}) \tag{11.128}$$

as you found on Question 1 of today's worksheet. Using the notation from equation (11.76) that

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right)$$

the continuity equation (11.127) takes the obviously covariant form

$$\partial_\alpha J^\alpha = 0 \tag{11.129}$$

It is a wondrous sight to behold; the expression brings to mind John Keats' "*Beauty is Truth, Truth Beauty.*"

Next, let us see what we get from **Maxwell's equations**. To keep matters simple, we will work with the microscopic Maxwell equations for now, without $\vec{D}$ and $\vec{H}$. Consider first the two Maxwell equations with source terms:

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \text{and} \qquad \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

Note that these may look different from last quarter, because we're writing them now in gaussian units (see page 781 in Jackson for details).

On Question 2 of today's worksheet, you wrote these two equations in component form, using the notation from equation (11.76) that

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)$$

Jackson uses $x_1 = x, x_2 = y, x_3 = z$ here, and you should too, so that our expressions are comparable to his. With $\rho = J^0/c$, you should have obtained

$$\begin{aligned}
\partial_x E_x + \partial_y E_y + \partial_z E_z &= \frac{4\pi}{c} J^0 \\
-\partial_0 E_x \qquad\quad + \partial_y B_z - \partial_z B_y &= \frac{4\pi}{c} J^1 \\
-\partial_0 E_y - \partial_x B_z \qquad\quad + \partial_z B_x &= \frac{4\pi}{c} J^2 \\
-\partial_0 E_z + \partial_x B_y - \partial_y B_x \qquad\quad &= \frac{4\pi}{c} J^3
\end{aligned}$$

The left hand side can be written as the product of a row vector and a 4×4 matrix:

$$\begin{pmatrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} = \frac{4\pi}{c} \begin{pmatrix} J^0 \\ J^1 \\ J^2 \\ J^3 \end{pmatrix}$$

The 4×4 matrix is known as the **field-strength tensor** $F^{\alpha\beta}$, and its matrix form is

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{11.137}$$

The two **inhomogenous Maxwell equations** can then be put into a remarkably simple form:

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \tag{11.141}$$

where $F^{\alpha\beta}$ is the field-strength tensor, and J^α is the 4-current.

We can also write the field-strength tensor $F_{\alpha\beta}$ with two covariant indices, by using the raising and lowering property of the metric tensor $g_{\alpha\beta}$. In other words, by doing $F_{\alpha\beta} = g_{\alpha\gamma}F^{\gamma\delta}g_{\delta\beta}$, *as you did in Question 4 of today's worksheet*, we get

$$F_{\alpha\beta} = g_{\alpha\gamma}F^{\gamma\delta}g_{\delta\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{11.138}$$

Clearly, the elements of $F_{\alpha\beta}$ are obtained from $F^{\alpha\beta}$ by putting $\vec{E} \to -\vec{E}$ (but keeping $\vec{B}$ unchanged).

Consider next the two homogenous Maxwell equations:

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{and} \qquad \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

Using a procedure similar to that shown on the previous page for the inhomogenous Maxwell equations (but taking care to write the components of the three equations involving the curl of $\vec{E}$ with negative signs), we get the **dual field-strength tensor** $\mathcal{F}^{\alpha\beta}$, *as you did in Question 3 on today's worksheet*, given by

$$\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{11.140}$$

In other words, the elements of the dual tensor are obtained by putting $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ in equation (11.137).

An alternative way to get the dual field-strength tensor $\mathcal{F}^{\alpha\beta}$, as done in Jackson, is to define the **totally antisymmetric 4th rank tensor** $\epsilon^{\alpha\beta\gamma\delta}$, *as you did in Question 5 on today's worksheet* where

- $\epsilon^{\alpha\beta\gamma\delta} = +1$, for $\alpha = 0, \beta = 1, \gamma = 2, \delta = 3$, or any even permutation of these.
- $\epsilon^{\alpha\beta\gamma\delta} = -1$, for any odd permutation of $\alpha, \beta, \gamma, \delta$.
- $\epsilon^{\alpha\beta\gamma\delta} = 0$, if any two indices are equal.

Note that $\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}$.

The dual field-strength tensor $\mathcal{F}^{\alpha\beta}$ can then be obtained by doing

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\,\epsilon^{\alpha\beta\gamma\delta}\,F_{\gamma\delta}$$

where summation over γ and δ is implied.

The two **homogenous Maxwell equations** can be written in the form

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \tag{11.142}$$

where $\mathcal{F}^{\alpha\beta}$ is the dual field-strength tensor defined in equation (11.140).

Next, consider the wave equation for the vector potential $\vec{A}$ and the scalar potential Φ in the Lorenz gauge, *as you did in Question 6 of today's worksheet*:

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} &= \frac{4\pi}{c}\,\vec{J} \\ \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi &= 4\pi\,\rho \end{aligned} \tag{11.130}$$

with the Lorenz condition

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0 \tag{11.131}$$

The differential operators on the left hand side in equation (11.130) are the **4-dimensional Laplacian operators** we defined in equation (11.78):

$$\Box \equiv \partial_\alpha\partial^\alpha = \frac{\partial^2}{\partial(x^0)^2} - \nabla^2$$

while the right hand sides are the component of the 4-vector J^α, since we can write the right hand side of the second equation as $(4\pi/c)\,c\rho = (4\pi/c)\,J^0$. With a **4-vector potential** A^α formed by Φ and $\vec{A}$:

$$A^\alpha = (\Phi, \vec{A}) \tag{11.132}$$

the **wave equation** and the **Lorenz condition** takes on the covariant forms

$$\begin{aligned} \Box A^\alpha &= \frac{4\pi}{c}\,J^\alpha \\ \partial_\alpha A^\alpha &= 0 \end{aligned} \tag{11.133}$$

With the definitions of J^α in equation (11.128), A^α in equation (11.132), and $F^{\alpha\beta}$ in equation (11.137), together with the wave equations in equation (11.133), or the Maxwell equations in equation (11.141) and equation (11.142), **the covariance of electrodynamics is established**.

Finally, for the macroscopic Maxwell equations, we must write two different field strength tensors

$$F^{\alpha\beta} = (\vec{E}, \vec{B}) \qquad \text{and} \qquad G^{\alpha\beta} = (\vec{D}, \vec{H})$$

where $F^{\alpha\beta}$ is given by equation (11.137) and $G^{\alpha\beta}$ is obtained from equation (11.137) by substituting $\vec{E} \to \vec{D}$ and $\vec{B} \to \vec{H}$.

The **covariant form of the macroscopic Maxwell equations** is then

$$\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c}\,J^\beta, \qquad \partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \tag{11.134}$$

All we have left is to put the **Lorentz force** equation into covariant form, which we will do in the next class. Note that Jackson starts with this, but then has to digress to putting the Maxwell equations into covariant form before returning to the Lorentz force equation. In the approach described in this class summary, there is no need to do that, and the Lorentz force equation falls into place naturally after we develop a covariant description of the Maxwell equations.