PHY 411 **Winter 2021**

Week 10—Tuesday, Mar 9—Discussion Worksheet

Boundary value problems with dielectrics

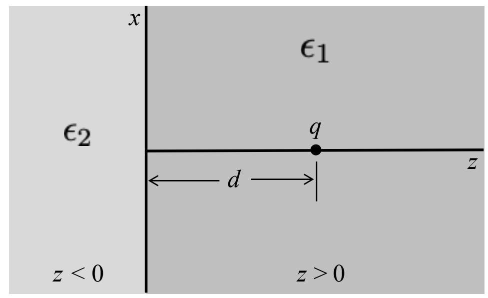
So far, we have discussed the image problems only for conductors. The method of images can be extended easily to handle the presence of dielectrics.

1. Consider a point charge q embedded in a semi-infinite dielectric (of permittivity) ϵ_1 at a distance d from a plane interface that separates the first medium from another semi-infinite dielectric ϵ_2, as shown in the figure below. The interface is taken as the plane $z = 0$. Then we have

$$\epsilon_1 \vec{\nabla} \cdot \vec{E} = \rho, \quad z > 0 \tag{1}$$

$$\epsilon_2 \vec{\nabla} \cdot \vec{E} = 0, \quad z < 0 \tag{2}$$

$$\vec{\nabla} \times \vec{E} = 0, \quad \text{everywhere} \tag{3}$$



where the third equation is Faraday's law for electrostatics.

We also have the boundary conditions

$$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = \sigma = 0 \qquad \text{and} \qquad \hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \tag{4}$$

where $\hat{n}$ is a unit normal to the surface directed from region 1 to region 2.

(a) Show that

$$\epsilon_1 E_z\,(z > 0) = \epsilon_2 E_z\,(z < 0) \tag{5}$$

Since there is no charge at the boundary $z=0$, the normal component of $\vec{D}$ is continuous, and from (4) above, we get $\vec{D}_1 \cdot \hat{n} = \vec{D}_2 \cdot \hat{n}$ where $\hat{n} = \hat{z}$, since it is a unit vector pointing from 1 to 2.

Thus, $D_{1z} = D_{2z} \Rightarrow \underset{(z>0)}{\epsilon_1 E_z} = \underset{(z<0)}{\epsilon_2 E_z}$

(b) Show that

$$E_x\,(z > 0) = E_x\,(z < 0) \qquad \text{and} \qquad E_y\,(z > 0) = E_y\,(z < 0) \tag{6}$$

Since $\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0$, we have that the tangential components of $\vec{E}$ are continuous from 1 to 2.

Thus, E_x is continuous $\Rightarrow E_x(z>0) = E_x(z<0)$

E_y is continuous $\Rightarrow E_y(z>0) = E_y(z<0)$

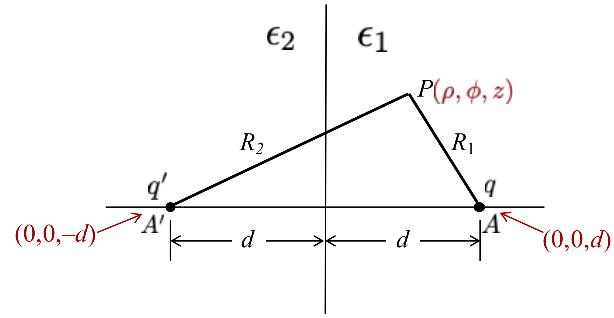
We will now apply the method of images to find the potential Φ.

2. Consider the figure on the right below.

Since q is located in $z > 0$, we'll put an image charge q' at $z = -d$, that is

- q is at A, with coordinates $(0, 0, d)$, and
- q' is at A', with coordinates $(0, 0, -d)$.

as shown in the figure.



Then, for $z > 0$ at a point $P(\rho, \phi, z)$ in cylindrical coordinates, the potential will be given by

$$\Phi_1 = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right), \qquad z > 0 \tag{7}$$

(a) Write down R_1 and R_2 in terms of the coordinates (ρ, ϕ, z) of point P, and the coordinates of points A and A'.

$R_1 = \sqrt{\rho^2 + (z-d)^2}$ $\qquad A = d$

$R_2 = \sqrt{\rho^2 + (z+d)^2}$ $\qquad A' = -d$

(b) Show that the normal component $D_{1\perp}$ of $\vec{D}_1$ at $z = 0$, found by evaluating $\epsilon_1 E_{1\perp} = -\epsilon_1 \partial\Phi_1/\partial z$ at $z = 0$, is given by

$$D_{1\perp}\Big|_{z=0} = -\frac{1}{4\pi}\left[\frac{qd}{(\rho^2+d^2)^{3/2}} - \frac{q'd}{(\rho^2+d^2)^{3/2}}\right]$$

$$\epsilon_1 E_{1\perp} = -\epsilon_1 \frac{\partial \Phi_1}{\partial z}$$

$$= -\cancel{\epsilon_1}\left(\frac{\partial}{\partial z}\left(\frac{1}{4\pi\cancel{\epsilon_1}}\left(\frac{q}{\sqrt{\rho^2+(z-d)^2}} - \frac{q'}{\sqrt{\rho^2+(z+d)^2}}\right)\right)\right)$$

$$\frac{\partial}{\partial z}\left(\frac{q}{\sqrt{\rho^2+(z-d)^2}}\right) = -\frac{q(z-d)}{((z-d)^2+\rho^2)^{3/2}} \quad \Big| \quad \frac{\partial}{\partial z}\left(\frac{q'}{\sqrt{\rho^2+(z+d)^2}}\right) = +\frac{q(z+d)}{((z+d)^2+\rho^2)^{3/2}}$$

evaluate at $z=0$

$$D_{1\perp}\Big|_{z=0} = -\frac{1}{4\pi}\left[\frac{qd}{(\rho^2+d^2)^{3/2}} - \frac{q'd}{(\rho^2+d^2)^{3/2}}\right]$$

So far the problem has been completely analogous to the problem with a conducting material in place of the dielectric ϵ_2 for $z < 0$. But now we must specify the potential for $z < 0$.

3. To specify the potential for $z < 0$, note that we can't have any charges in $z < 0$, and so the simplest assumption is that the potential in $z < 0$ is equivalent to that with a charge q'' at $z = d$, the position of the actual charge q, and thus

$$\Phi_2 = \frac{1}{4\pi\epsilon_2}\frac{q''}{R_1}, \qquad z < 0 \tag{8}$$

(a) Show that the normal component $D_{2\perp}$ of $\vec{D}_2$ at the boundary $z = 0$ is given by

$$D_{2\perp}\Big|_{z=0} = -\epsilon_2 \frac{\partial \Phi_2}{\partial z}\Big|_{z=0} = -\frac{1}{4\pi}\left[\frac{q''\, d}{(\rho^2 + d^2)^{3/2}}\right]$$

$$\epsilon_1 E_{1\perp} = -\epsilon_2 \frac{\partial \varphi_2}{\partial z}$$

$$= -\cancel{\epsilon_2}\left(\frac{\partial}{\partial z}\left(\frac{1}{4\pi \cancel{\epsilon_2}}\left(\frac{q''}{\sqrt{\rho^2 + (z-d)^2}}\right)\right)\right)$$

$$\frac{\partial}{\partial z}\left(\frac{q''}{\sqrt{\rho^2 + (z-d)^2}}\right) = \frac{+\, q''(z-d)}{(\rho^2 + (z-d)^2)^{3/2}} \leftarrow$$

evaluate at $z=0$

$$D_{2\perp}\Big|_{z=0} = -\frac{1}{4\pi}\left[\frac{q''}{(\rho^2 + d^2)^{3/2}}\right]$$

(b) Setting $D_{1\perp}$ from the previous page equal to $D_{2\perp}$ at the boundary $z = 0$, show that

$$q - q' = q''$$

$$D_{1\perp} = D_{2\perp}$$

$$\rightarrow -\cancel{\frac{1}{4\pi}}\left[\frac{q\cancel{d}}{\cancel{(\rho^2+d^2)^{3/2}}} - \frac{q'\cancel{d}}{\cancel{(\rho^2+d^2)^{3/2}}}\right] = -\cancel{\frac{1}{4\pi}}\left[\frac{q''\cancel{d}}{\cancel{(\rho^2+d^2)^{3/2}}}\right]$$

$$\rightarrow \quad q - q' = q''$$

Next, we will write down expressions for the tangential components of $\vec{E}$ to derive a second relation between the charge and its image charges.

4. Consider now the tangential components of $\vec{E}$, given by $(\vec{E})_{\parallel} = (\vec{E})_{\rho} = -\dfrac{\partial \Phi}{\partial \rho}$

(a) Show that the tangential component $(E_1)_{\parallel}$ at $z = 0$ is given by

$$(E_1)_{\parallel}\Big|_{z=0} = -\frac{\partial \Phi_1}{\partial \rho}\Big|_{z=0} = +\frac{1}{4\pi\epsilon_1}\left[\frac{q\rho}{(\rho^2+d^2)^{3/2}} + \frac{q'\rho}{(\rho^2+d^2)^{3/2}}\right]$$

$$-\frac{\partial \varphi}{\partial \rho} = \left(\frac{\partial}{\partial \rho}\left(\frac{1}{4\pi\epsilon_1}\left(\frac{q}{\sqrt{\rho^2+(z-d)^2}} - \frac{q'}{\sqrt{\rho^2+(z+d)^2}}\right)\right)\right)$$

$$-\frac{\partial}{\partial \rho}\left(\frac{q}{(\rho^2+(z-d)^2)}\right) = \frac{q\rho}{(\rho^2+(z-d)^2)^{3/2}} \qquad -\frac{\partial}{\partial \rho}\left(\frac{q}{(\rho^2+(z-d)^2)}\right) = \frac{q\rho}{(\rho^2+(z-d)^2)^{3/2}}$$

$$(E_1)_{\parallel}\Big|_{z=0} = \frac{1}{4\pi\epsilon_1}\left[\frac{q\rho}{(\rho^2+d^2)^{3/2}} + \frac{q'\rho}{(\rho^2+d^2)^{3/2}}\right]$$

(b) Show that the tangential component $(E_2)_{\parallel}$ at $z = 0$ is given by

$$(E_2)_{\parallel} = -\frac{\partial \varphi}{\partial \varphi}$$

$$(E_2)_{\parallel}\Big|_{z=0} = -\frac{\partial \Phi_2}{\partial \rho}\Big|_{z=0} = +\frac{1}{4\pi\epsilon_2}\left[\frac{q''\rho}{(\rho^2+d^2)^{3/2}}\right]$$

$$= \left(\frac{\partial}{\partial \rho}\left(\frac{1}{4\pi}\left(\frac{q''}{\sqrt{\rho^2+(z-d)^2}}\right)\right)\right)$$

$$\frac{\partial}{\partial \rho}\left(\frac{q''}{\sqrt{\rho^2+(z-d)^2}}\right) = \frac{+q''\rho^2}{(\rho^2+(z-d)^2)^{3/2}}$$

evaluate at $z=0$

$$(E_2)_{\parallel}\Big|_{z=0} = \frac{1}{4\pi\epsilon_2}\left[\frac{q''\rho}{(\rho^2+d^2)^{3/2}}\right]$$

Setting $(E_1)_{\parallel}$ equal to $(E_2)_{\parallel}$ at the boundary $z = 0$, you should be able to show from the above that

$$\frac{1}{\epsilon_1}\left(q + q'\right) = \frac{1}{\epsilon_2}q''$$

Similar to above

We now have the system of equations from Question 3(b) and the bottom of the previous page that

$$q - q' = q'' \qquad q' = q - q''$$
$$q + q' = \left(\frac{\epsilon_1}{\epsilon_2}\right) q'' \qquad q' = \left(\left(\frac{\epsilon_1}{\epsilon_2}\right) q''\right) - q$$

5. Solve for q' and q'' in terms of q and show that

$$q' = -\left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\right) q$$
$$q'' = \left(\frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\right) q$$

$$q + q' = \left(\frac{\epsilon_1}{\epsilon_2}\right)(q - q')$$

$$\rightarrow \frac{(q + q')}{\epsilon_1} = \frac{(q - q')}{\epsilon_2}$$

$$\rightarrow \frac{q}{\epsilon_1} + \frac{q'}{\epsilon_1} = \frac{q}{\epsilon_2} - \frac{q'}{\epsilon_2}$$

$$\rightarrow \frac{q'}{\epsilon_1} + \frac{q'}{\epsilon_2} = \frac{q}{\epsilon_2} - \frac{q}{\epsilon_1}$$

$$\rightarrow q'\left(\frac{1}{\epsilon_1 + \epsilon_2}\right) = q\left(\frac{1}{\epsilon_2 - \epsilon_1}\right)$$

$$\rightarrow q' = \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_2 - \epsilon_1}\right) q$$

$$\rightarrow q' = -\left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\right) q$$

$$q - q'' = q''\left(\frac{\epsilon_1}{\epsilon_2}\right) - q$$

$$\rightarrow q + q = q''\left(\frac{\epsilon_1}{\epsilon_2}\right) + q''$$

$$\rightarrow \epsilon_2(q + q) = q''\epsilon_1 + q''\epsilon_2$$

$$\rightarrow 2\epsilon_2 q = q''(\epsilon_1 + \epsilon_2)$$

$$\rightarrow q'' = \left(\frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\right) q$$