The physics. Three terms you need to be aware of: *recombination, decoupling, last scattering surface*.

- Recombination—the epoch during which previously ionized baryonic matter becomes neutral
- Decoupling—the time at which photons stop interacting with baryonic matter
- Last scattering– the time at which a CMB photon undergoes its last scattering from an electron

Some preliminaries

- H is neutral hydrogen
- *p* is ionized hydrogen (i.e. a proton)
- *e* is electrons

Is all baryonic matter ionized. No. The fraction that is given by $X \equiv \frac{n_p}{n_p + n_{\rm H}} = \frac{n_p}{n_{\rm bary}} = \frac{n_e}{n_{\rm bary}}$

Hydrogen is ionized when *E = 13.6* eV so a photon with much energy can cause the reaction

$$\begin{aligned} {\rm H} + \gamma &\rightarrow p + e \\ p + e &\rightarrow {\rm H} + \gamma \end{aligned}$$

An *X* depends on the details of these two reactions

Now let's play with the physics a bit. We'll go back to a moment, $a = 10^{-5}$, $z = 10^5$ and the universe is about 70 years old (using the benchmark model).

At that moment, $T = 3 \times 10^5\,\mathrm{K};\ E_\gamma = 60\,\mathrm{eV}$ and since there are so many photons than baryons, very little chance that neutral hydrogen is present, and $X = 1$.

With the universe fully ionized, the dominant reaction is $\gamma + e \rightarrow \gamma + e$

Three facts: $\sigma_e = 6.65 \times 10^{-29}\,\mathrm{m}^2$; $\lambda = \dfrac{1}{n_e \sigma_e}$; $n_e = n_{\mathrm{baryons}} = \dfrac{n_{\mathrm{baryons,o}}}{a^3} = \dfrac{0.25\,\mathrm{m}^{-3}}{a^3}$

Do question (1) on the worksheet

(1a) $\Gamma = \frac{c}{\lambda} = n_e \, \sigma_e \, c$

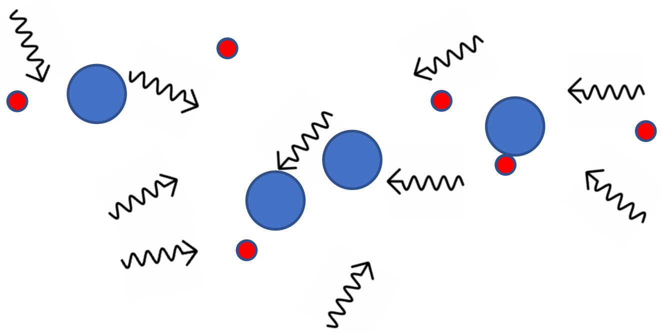
(1b) $\Gamma = n_e \, \sigma_e \, c = 5.0 \times 10^{-6}\,\mathrm{s}^{-1}$

(1c) $\frac{H^2}{H_o^2} = \frac{\Omega_{r,o}}{a^4} \Rightarrow H = \frac{2.1 \times 10^{-20}\mathrm{s}^{-1}}{a^2}$

(1d) $H = 2.1 \times 10^{-10}\,\mathrm{s}^{-1}$

The expansion is much slower than the collision rate, so photons are well-coupled at this time.

Recall that we are in the midst of trying to understand the physics of recombination.



Consider the figure above in which the blue circles represent protons, the red electrons, and the waves, photons. Do question (2) on the worksheet and S T O P

(2) The number of *photons, protons, electrons, and Hydrogen*, along with the ionization energy, *Q.*

We need the statistics of the various species. Because these species obey quantum rules, we need quantum statistics

- Fermions: $n_x(p)dp = g_x \frac{4\pi}{h^3} \frac{p^2 dp}{\exp\left([E-\mu_x]/kT\right)+1}$

- Bosons: $n_x(p)dp = g_x \frac{4\pi}{h^3} \frac{p^2 dp}{\exp\left([E-\mu_x]/kT\right)-1}$

g_x is called the statistical weight
μ_x is the chemical potential

- Fermions, particles with *n/2* spin (*n* an integer) and have $g_x = 2$.
- Bosons, particles with integer spin and have $g_x = 2$.

Now substituting in for *E* and integrating over all momentum we have

$$n_\gamma = 0.2436 \left(\frac{kT}{\hbar c}\right)^3$$

$$n_x = g_x \left(\frac{m_x kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(\frac{-m_x c^2 + \mu_x}{kT}\right)$$

Do question (3) on the worksheet and S T O P

$$n_\gamma = 0.2436 \left(\frac{kT}{\hbar c}\right)^3$$

$$n_x = g_x \left(\frac{m_x kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(\frac{-m_x c^2 + \mu_x}{kT}\right)$$

These can be further simplified as follows:

- Equilibrium of $\mathrm{H} + \gamma \rightleftharpoons p + e$ implies that
  $$\mu_\gamma = 0, \ \mu_\mathrm{H} = \mu_p + \mu_e$$
- The binding energy is $Q = 13.6\,\mathrm{eV} = (m_p + m_e - m_\mathrm{H})\,c^2$
- The statistical weights are $g_p = g_e = 2; \ g_\mathrm{H} = 4$

$$\frac{n_\mathrm{H}}{n_p\, n_e} = \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{Q}{kt}\right)$$

Saha equation

This is real progress, but we'd like convert this into a relationship between $X; T; \eta \left(= \frac{n_{\mathrm{bary,o}}}{n_{\gamma,o}}\right)$

Do question (4) on the worksheet and S T O P

(4)
$$n_{\mathrm{H}} = \frac{1-X}{X} n_p$$

$$\frac{1-X}{X} = n_p \left(\frac{m_i kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{Q}{kT}\right)$$

$\Longrightarrow$

Using η we find that

$$n_p = 0.2436\, X\, \eta \left(\frac{kT}{\hbar c}\right)^3$$

and finally

$$\frac{1-X}{X^2} = 3.48\eta \left(\frac{kT}{m_e c^2}\right)^{3/2} \exp\left(\frac{Q}{kT}\right)$$

which yields

$$X = \frac{-1+\sqrt{1+4S}}{2S} \quad \text{where}$$

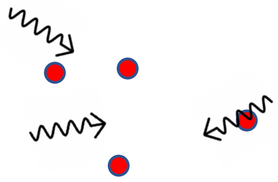
$$S(T,\eta) = 3.84\eta \left(\frac{kT}{m_e c^2}\right)^{3/2} \exp\left(\frac{Q}{kT}\right)$$

Using this, we find that at $X = 1/2;\ \ T_{\mathrm{rec}} = 3760\ \mathrm{K};\ \ z_{\mathrm{rec}} = 1380;\ \ t_{\mathrm{rec}} = 250,000\ \mathrm{yr}$

Do question (5) on the worksheet and S T O P

Decoupling

The scattering rate, Γ , of photons will depend on the number of free electrons

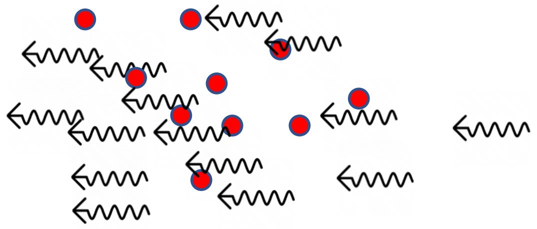


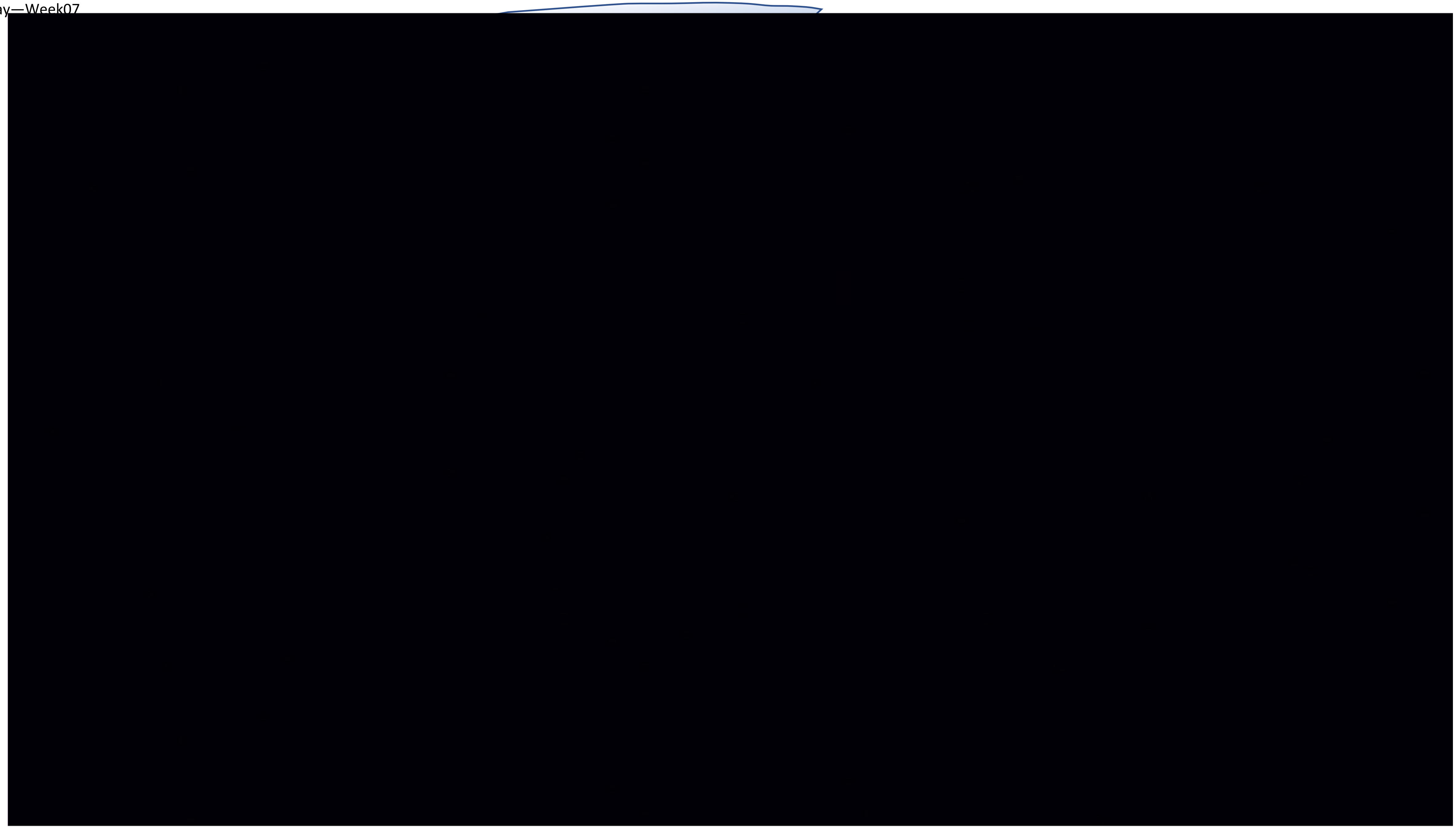
$$T_{\text{dec}} = 2870\,\text{K}; \quad z_{\text{dec}} = 1090; \quad t_{\text{dec}} = 371,000\,\text{yr}$$

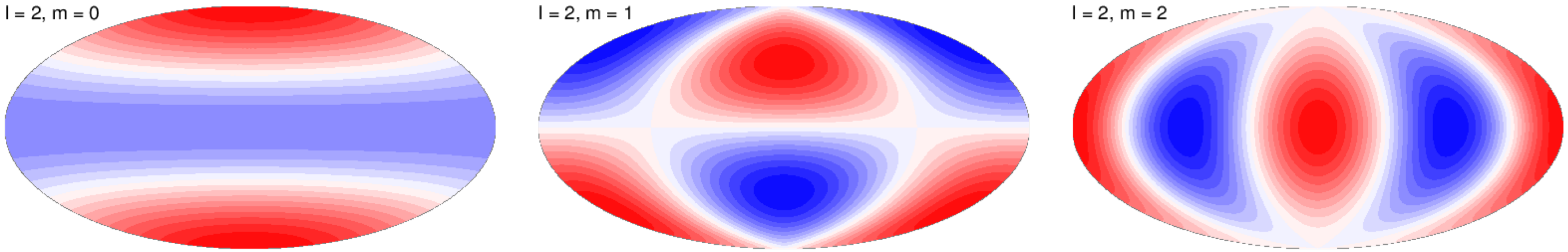
A CMB photon detected today, would have undergone $\tau = \int_t^{t_o} \Gamma(t)\,dt$ scatterings. The time for which $\tau = 1$ is called the time of last scattering and is the time elapsed since a photon last scattered from an electron.

Decoupling happens very soon after recombination

Not all the photons decouple a the same time, and the time given above is just an (pretty good) approximation







$$\frac{\delta T}{T}(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m}\, Y_{l,m}(\theta,\phi)$$