Physics 460—Homework Report 8
Due Tuesday, May 26, 1 pm **Name:** ______________________________

Complete all the problems on the accompanying assignment.

List all the problems you worked on in the space below. Circle the ones you fully completed:

Please place the problems into the following categories:

- These problems helped me understand the concepts better: ____________________
- I found these problems fairly easy: ____________________
- I found these problems very challenging: ____________________

In the space below, show your work (even if not complete) for any problems you still have questions about. Indicate where in your work the question(s) arose, and ask specific questions that I can answer.

Use the back of this sheet or attach additional paper, if necessary.

If you have no remaining questions about this homework assignment, use this space for one of the following:

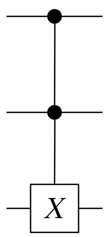
- Write one or two of your solutions here so that I can give you feedback on its clarity.
- Explain how you checked that your work is correct.

(1) Table 5.1 in the course notes lists a number of single-qubit quantum computing gates and their operator representation in the $\{|0\rangle, |1\rangle\}$ basis. (Recall that all quantum computing gates can be represented by a unitary operator U, for which $U^\dagger = U^{-1}$).

(a) Explain where the gate $\sqrt{X}$ gets its name. (Note that sometimes the $\sqrt{X}$ gate is called a V gate, I guess because a V looks like the leading part of the square root symbol.)

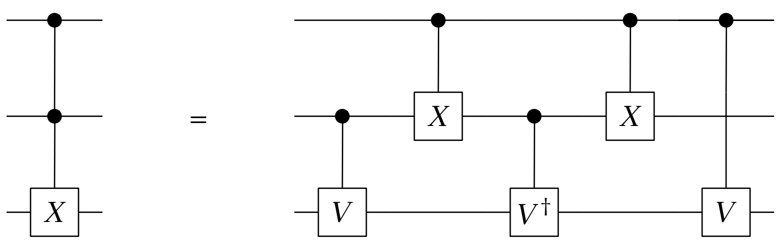
(b) Show that the $\sqrt{X}$ is unitary.

(2) The Toffoli or controlled-controlled-NOT gate is a three-qubit gate that is represented below:



The action of this gate is as follows: the two control qubits are unaffected, while the target bit is reversed only if both control bits are in the state $|1\rangle$.

Show that the Toffoli gate is equivalent to the circuit on the right below:



Here V is the $\sqrt{X}$ gate from question (1) and $V^\dagger$ is the gate that implements the Hermitian conjugate of V.

(3) Modern computers implement error correction, meaning that they check bits to make sure that they haven't accidentally been changed, and they fix any broken bits that they find.

One of the interesting things about quantum computers is that you can't directly check qubits to see if they've changed: if you measure a qubit you will change it! So how can quantum computers implement error correction?

They do so by using redundancy, and using more than one physical qubit to represent each logical qubit. So rather than using a single particle or photon to represent a 1 or a 0, they use more than one. This problem investigates circuits for using three physical qubits to represent each logical qubit to allow for error correction.

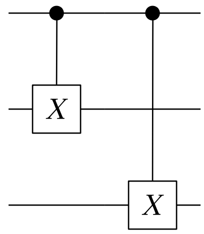
We will call the logical qubits $|0_L\rangle$ and $|1_L\rangle$. An arbitrary qubit is then $|\Psi_L\rangle = a|0_L\rangle + b|1_L\rangle$. We will represent these logical qubits using three physical qubits according to the following:

$$|0_L\rangle \equiv |0;0;0\rangle \quad \text{and} \quad |1_L\rangle \equiv |1;1;1\rangle.$$

So a "zero" in this system is a three-qubit state where all three qubits are $|0\rangle$ and a "one" is a three-qubit state where all three qubits are $|1\rangle$. A general qubit $|\Psi_L\rangle$ is then

$$|\Psi_L\rangle = a|0;0;0\rangle + b|1;1;1\rangle.$$

(a) Show that the circuit below can encode a single incoming qubit $|\Psi\rangle$ into the three-qubit representation.



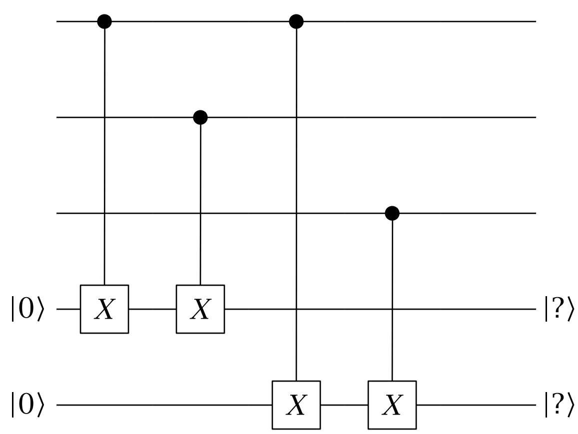
In other words, if the input state is $|\Psi;0;0\rangle = a|0;0;0\rangle + b|1;0;0\rangle$, the output state is $|\Psi_\mathrm{L}\rangle = a|0;0;0\rangle + b|1;1;1\rangle$.

(b) Design the reverse network, that takes the input state $|\Psi_\mathrm{L}\rangle = a|0;0;0\rangle + b|1;1;1\rangle$ back to the state $|\Psi;0;0\rangle = a|0;0;0\rangle + b|1;0;0\rangle$.

(c) We will assume that a single error can occur, that converts the state $|\Psi_\mathrm{L}\rangle$ to one of the following:

$$|\Psi_\mathrm{L}\rangle \longrightarrow \begin{cases} a|0;0;0\rangle + b|1;1;1\rangle & \text{no error,} \\ a|1;0;0\rangle + b|0;1;1\rangle & \text{qubit 1 flipped,} \\ a|0;1;0\rangle + b|1;0;1\rangle & \text{qubit 2 flipped,} \\ a|0;0;1\rangle + b|1;1;0\rangle & \text{qubit 3 flipped.} \end{cases}$$

Show that the network below (which includes two additional qubits that we will call error qubits) can distinguish between these four cases without disturbing the three input qubits. What is the output for the two error qubits for each of these four cases?



(d) Design a circuit that will correct each of these three errors. Your circuit should take the two error qubits as its input and flip the appropriate qubit of the three original qubits to correct the error (without messing up any other qubits, of course). The Toffoli gate is probably useful here.

Homework 8

(1a) Explain where the gate $\sqrt{X}$ gets its name

The X operator is also known as the NOT gate. The $\sqrt{NOT}$ or $\sqrt{X}$ is found by using a unitary matrix multiplied by itself results in $\sqrt{X}$.

$$\sqrt{X} = \sqrt{\neg} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

where

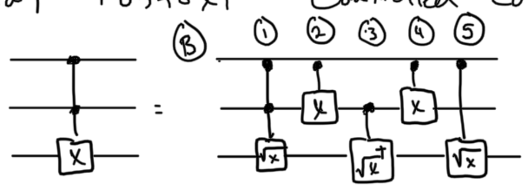
$$\sqrt{X}\sqrt{X} = X \qquad U^{\dagger} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \qquad U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

$$U^{\dagger}U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

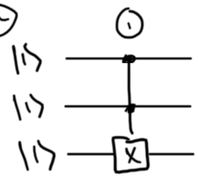
$$UU^{\dagger} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(2a) Toffoli - Controlled-Controlled-NOT gate, three-qubit

(a) (B)



(a)



(a)

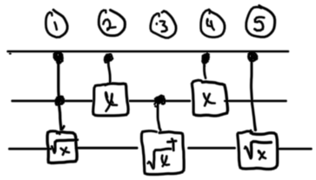
$\alpha\ |1,1,1\rangle \xrightarrow{1} \boxed{|1,1,0\rangle}$, $\epsilon\ |0,0,0\rangle \xrightarrow{1} \boxed{|0,0,1\rangle}$

$\beta\ |1,1,0\rangle \xrightarrow{1} \boxed{|1,1,1\rangle}$, $G\ |0,0,1\rangle \xrightarrow{1} \boxed{|0,0,0\rangle}$

$\gamma\ |1,0,0\rangle \xrightarrow{1} \boxed{|1,0,1\rangle}$, $\eta\ |0,1,1\rangle \xrightarrow{1} \boxed{|0,1,0\rangle}$

$c\ |1,0,1\rangle \xrightarrow{1} \boxed{|1,0,0\rangle}$, $\emptyset\ |0,1,0\rangle \xrightarrow{1} \boxed{|0,1,1\rangle}$

(B)



$$\alpha \quad |1,1,1\rangle \xrightarrow{1} |1,1,0\rangle \xrightarrow{2} |1,0,0\rangle \xrightarrow{3} |1,0,1\rangle \xrightarrow{4} |1,1,1\rangle \xrightarrow{5} \boxed{|1,1,0\rangle}$$

$$\beta \quad |1,1,0\rangle \xrightarrow{1} |1,1,1\rangle \xrightarrow{2} |1,0,1\rangle \xrightarrow{3} |1,0,0\rangle \xrightarrow{4} |1,1,0\rangle \xrightarrow{5} \boxed{|1,1,1\rangle}$$

$$\gamma \quad |1,0,0\rangle \xrightarrow{1} |1,0,1\rangle \xrightarrow{2} |1,1,1\rangle \xrightarrow{3} |1,1,0\rangle \xrightarrow{4} |1,0,0\rangle \xrightarrow{5} \boxed{|1,0,1\rangle}$$

$$\delta \quad |1,0,1\rangle \xrightarrow{1} |1,0,0\rangle \xrightarrow{2} |1,1,0\rangle \xrightarrow{3} |1,1,1\rangle \xrightarrow{4} |1,0,1\rangle \xrightarrow{5} \boxed{|1,0,0\rangle}$$

$$\epsilon \quad |0,0,0\rangle \xrightarrow{1} |0,0,1\rangle \xrightarrow{2} |0,1,1\rangle \xrightarrow{3} |0,1,0\rangle \xrightarrow{4} |0,0,0\rangle \xrightarrow{5} \boxed{|0,0,1\rangle}$$

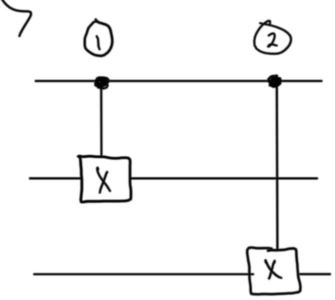
$$\zeta \quad |0,0,1\rangle \xrightarrow{1} |0,0,0\rangle \xrightarrow{2} |0,1,0\rangle \xrightarrow{3} |0,1,1\rangle \xrightarrow{4} |0,0,1\rangle \xrightarrow{5} \boxed{|0,0,0\rangle}$$

$$\eta \quad |0,1,1\rangle \xrightarrow{1} |0,1,0\rangle \xrightarrow{2} |0,0,0\rangle \xrightarrow{3} |0,0,1\rangle \xrightarrow{4} |0,1,1\rangle \xrightarrow{5} \boxed{|0,1,0\rangle}$$

$$\theta \quad |0,1,0\rangle \xrightarrow{1} |0,1,1\rangle \xrightarrow{2} |0,0,1\rangle \xrightarrow{3} |0,0,0\rangle \xrightarrow{4} |0,1,0\rangle \xrightarrow{5} \boxed{|0,1,1\rangle}$$

(3) input $|\psi,0,0\rangle = a|0,0,0\rangle + b|1,0,0\rangle$

output $a|0,0,0\rangle + b|1,1,1\rangle$



$|\psi\rangle \rightarrow a|0,0,0\rangle + b|1,0,0\rangle$

$|0\rangle \rightarrow |0\rangle$

$|0\rangle \rightarrow |0\rangle$

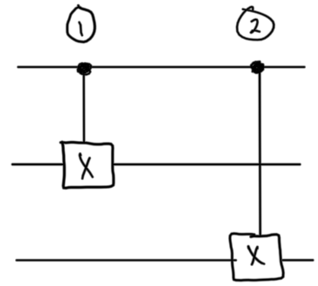
$$|\psi\rangle \xrightarrow{1} a|0,0,0\rangle + b|1,1,0\rangle$$

$$\xrightarrow{2} \boxed{a|0,0,0\rangle + b|1,1,1\rangle}$$

$= a|0,0,0\rangle + b|1,1,1\rangle$

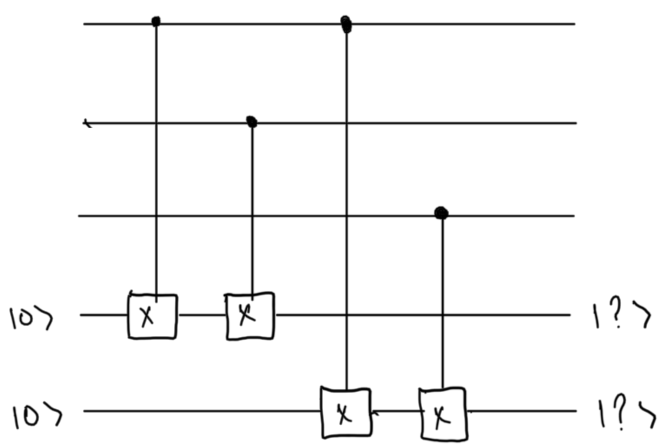
(B) input $|\psi_L\rangle = a|0,0,0\rangle + b|1,1,1\rangle$
output $|\psi,0,0\rangle = a|0,0,0\rangle + b|1,0,0\rangle$

would stay the same



(C)

$$|\psi_L\rangle \rightarrow \begin{cases} a|0,0,0\rangle + b|1,1,1\rangle & \text{No error,} \\ a|1,0,0\rangle + b|0,1,1\rangle & \text{qubit 1 flipped,} \\ a|0,1,0\rangle + b|1,0,1\rangle & \text{qubit 2 flipped,} \\ a|0,0,1\rangle + b|1,1,0\rangle & \text{qubit 3 flipped,} \end{cases}$$



$|\psi_L\rangle_1 = |0,0\rangle$

$|\psi_L\rangle_2 = |1,0\rangle$

$$|\psi_L\rangle_3 = |0,1\rangle$$

$$|\psi_L\rangle_4 = |1,1\rangle$$