Homework

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Problem 1

The inverse Lorentz transformation equations for a frame K' traveling at velocity v along the positive x-direction of a frame K are given by

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \qquad x = \gamma(x' + vt') \qquad y = y' \qquad z = z'$$

The x and t equations in differential form are gave as

$$dx = \gamma(dx' - v\ dt') \qquad \text{and} \qquad dt = \gamma\left(dt' - \frac{v}{c^2}dx'\right)$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. Now, u_x can be found by

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' - v\ dt')}{\gamma\left(dt' - \frac{v}{c^2}dx'\right)}.$$

This can also be expressed as

$$u_x = \frac{\frac{dx'}{dt'} - v}{1 - \frac{v(dx'/dt')}{c^2}}$$

which can be reduced down to

$$u_x = \frac{u'_x - v}{1 - \frac{vu'_x}{c^2}}$$

The same approach is used to find u_y. The differential form for y is $dy = dy'$. Then, u_y is gave as

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' - \frac{v}{c^2}dy'\right)}.$$

This can then be written as

$$\frac{\frac{dy'}{dt'}}{\gamma\left(1 - \frac{v(dy'/dt')}{c^2}\right)}$$

and reduced to

$$\frac{u'_y}{\gamma\left(1 - \frac{vu'_y}{c^2}\right)}.$$

Problem 2

(a)

For u' parallel to $\vec{v}$, we start with

$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}}$$

The angle between u' and v is zero when parallel, this can also be shown by

$$u = \frac{u'_{\parallel} + v}{1 + \frac{|v||u|cos(0)}{c^2}}$$

and reduced to

$$u = \frac{u' + v}{1 + \frac{vu'}{c^2}}$$

(b)

If $u' = c$, then

$$u = \frac{u' + v}{1 + \frac{vu'}{c^2}} = \frac{c + v}{1 + \frac{v}{c}} = c.$$

From the equation above we find that $u = c$, just as we have $u' = c$. Therefore, this tells us that c is the maximum limit of the speed of light, this is the upper bound of speed in the universe. This limit is a key aspect and a postulate of Special Relativity.

(c)

Speeds u' and v both small compared to c. From part (a) we found the equation for u. Since u' and v are small compared to c, then the part of the equation $u'v/c^2 = 0$ since c will be much larger than $u \cdot v$. Therefore, our new equation

$$u = \frac{u' + v}{1 + 0} = \frac{u' + v}{1} = u' + v.$$

Problem 3

The 4-velocity equation is gave by

$$U = (\gamma_u c, \gamma_u \vec{u})$$

where $\gamma_u = (1 - u^2/c^2)^{-1/2}$, and $\vec{u} = d\vec{x}/dt$ is the 3-dimensional velocity.

(a)

From the equation for U above, U can also be rewrote as

$$U = \gamma_u(c - \vec{u}) \qquad \text{or} \qquad U^2 = \gamma_u^2(c^2 - \vec{u} \cdot \vec{u})$$

Therefore, U^2 can be wrote and reduced to

$$\begin{aligned} U^2 &= \gamma_u^2(c^2 - \vec{u} \cdot \vec{u}) \\ U^2 &= \frac{(c^2 - u^2)}{(1 - u^2/c^2)} \\ U^2 &= c^2 \end{aligned}$$

(b)

We know that

$$U = \frac{dx}{d\tau}$$

Therefore, the 4-acceleration

$$A = \frac{dU}{dt}\frac{dt}{d\tau}$$
$$A = \gamma\frac{d}{dt}(c, u\gamma)$$
$$A = \gamma\left(c\frac{d\gamma}{dt}, \frac{d\gamma}{dt}u + \gamma\frac{du}{dt}\right)$$

where $du/dt = a$.

(c)

Find the scalar product $U \cdot A$ of the 4-velocity and the 4-acceleration. After taking the derivatives in the equation for A, we find that

$$U \cdot A = 0$$

Problem 4

The Lorentz transformation equations are given by

$$x_0' = \gamma(x_0 - \beta x_1)$$
$$x_1' = \gamma(x_1 - \beta x_0)$$
$$x_2' = x_2$$
$$x_3' = x_3$$

where

$$\beta = \frac{v}{c} \qquad \text{and} \qquad \gamma = (1 - \beta^2)^{-1/2}$$

(a)

If we have $\beta = tanh\zeta$, then we can use the equation for γ to find that

$$\gamma = (1 - \beta^2)^{-1/2} = (1 - tanh^2\zeta)^{-1/2}.$$

The alternative form for this equation is then

$$\gamma = cosh\zeta.$$

Now, if we know that $\gamma = cosh\zeta$, then

$$\gamma\beta = cosh\zeta tanh\zeta$$

where the identity for $cosh(x)tanh(x)$ is

$$\gamma\beta = sinh\zeta.$$

(b)

The Lorentz transformation equations can now be wrote as

$$
\begin{aligned}
x_0' &= \gamma x_0 - \gamma\beta x_1 = cosh\zeta x_0 - sinh\zeta x_1 \\
x_1' &= \gamma x_1 - \gamma\beta x_0 = cosh\zeta x_1 - sinh\zeta x_0 \\
x_2' &= x_2 \\
x_3' &= x_3
\end{aligned}
$$

and in matrix for this is

$$
\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} cosh\zeta & -sinh\zeta & 0 & 0 \\ -sinh\zeta & cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}
$$