PHY 420 – Electrodynamics II

Spring 2021

In Preparation for Final Examination

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Warning!

This is *not a substitute* for class summaries and homework solutions. It is only a list of ideas and/or equations I want you to remember. If this is the only thing you look at, expect to do very badly on the exam!

Caveat: In the following slides, I've mentioned several relations that will be supplied on formula sheets. To avoid any confusion, I'd like to add here that *an exception would be if I asked you to derive any of these relations on the test, in which case, they would obviously not be on the Formula Sheet*.

Electrostatic Energy

(Week 1 – Day 1 Class Summary & Worksheet)

- We covered Electrostatic and Magnetic Energy on Week 1 – Day 1. You should *look over* the Class Summary and Worksheet for that day to remind yourself of all the terms associated with these topics.

- However, there will be no direct questions on this material (from Week 1 – Day 1).

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 1)

- Consider the force on a single charge q:

Must know: $$\vec{F} = q\vec{E} + q\underbrace{\vec{v}}_{\text{velocity of the charge}} \times \vec{B}$$

- Can use this to show that (as you did on worksheet):

$$\frac{dE_{\text{kin}}}{dt} = \vec{F} \cdot \vec{v} = q\vec{E} \cdot \vec{v}$$

- Can then write the rate of work done by the fields (see next slide).

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 1)

- Total rate of doing work by the fields in a finite volume:

You must know this:

$$\int_V \vec{J} \cdot \vec{E} \, d^3x \qquad (6.103)$$

- This power represents a conversion of electromagnetic energy into mechanical or thermal energy.

- It must be balanced by a corresponding rate of decrease of energy in the electromagnetic field within the volume V.

- To exhibit this conservation law explicitly, use the Maxwell equations to express (6.103) in other terms.

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 2)

$$\int_V \vec{J}\cdot\vec{E}\,d^3x = -\int_V \left[\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right] d^3x \qquad (6.105)$$

- Equation (6.105) represents the rate of decrease of energy in the electromagnetic field within the volume V, and this goes into increasing the mechanical or thermal energy of the moving charges.

You are not expected to memorize equation (6.105), but you must be able to state what it means.

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 3)

To proceed, we will make two assumptions:

- Assume that the macroscopic medium is linear in its electric properties, with negligible dispersion or losses, so that

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} \left(\vec{E} \cdot \vec{D} \right)$$

  and likewise, linear in its magnetic properties, again with negligible dispersion or losses, so that

$$\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} \left(\vec{B} \cdot \vec{H} \right)$$

No need to memorize these; You should just remember the expression for u below.

- Assume also that the total electromagnetic energy, even for time-varying fields, is the sum of (4.89) and (5.148): $W = \frac{1}{2} \int \vec{E} \cdot \vec{D}\, d^3x$

$$W = \frac{1}{2} \int \vec{H} \cdot \vec{B}\, d^3x$$

  so that the total energy density is given by

You must know this: $$u = \frac{1}{2} \left(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H} \right) \qquad (6.106)$$

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 3)

$$\int_V \vec{J}\cdot\vec{E}\,d^3x = -\int_V \left[\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right] d^3x \qquad (6.105)$$

No need to memorize; these are just intermediate steps in the development of the expression

$$-\int_V \vec{J}\cdot\vec{E}\,d^3x = \int_V \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\underbrace{\left(\vec{E}\times\vec{H}\right)}_{\vec{S}}\right] d^3x \qquad (6.107)$$

- Since the volume V is arbitrary, the integrand can be written in the form of a differential continuity equation or conservation law

No need to memorize, but you must know what each term means:

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\vec{S}\right) = -\vec{J}\cdot\vec{E} \qquad (6.108)$$

$$\text{where } \vec{S} = \vec{E}\times\vec{H} \qquad (6.109)$$

You must know the Poynting vector

The vector $\vec{S}$ represents energy flow, and is called the *Poynting vector*

Energy conservation in the electromagnetic field

(Week 1 – Day 2 Class Summary, page 4)

$$-\int_V \vec{J}\cdot\vec{E}\,d^3x = \int_V \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right)\right] d^3x \qquad (6.107)$$

No need to memorize either of these, but you must know what each term means.

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\vec{S}\right) = -\vec{J}\cdot\vec{E} \qquad (6.108)$$

Poynting vector

Consider the physical meaning of either (6.107) or (6.108).

- Either equation tells us that the time rate of change of electromagnetic energy within a certain volume, plus the energy per unit time flowing out through the boundary surfaces of the volume, is equal to the negative of the total work done by the fields on the sources within the volume.

  This is a statement of the *conservation of energy*.

Conservation of Linear Momentum

(Week 2 – Day 1 Class Summary, page 1)

- Consider again the force on a single charge q:

Must know: $$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Newton's 2nd Law

$$\frac{d\vec{P}_{\text{mech}}}{dt} \qquad \rho\, d^3x \qquad \vec{J}\, d^3x$$

Knowing expression for F above, should be able to write this too:

$$\frac{d\vec{P}_{\text{mech}}}{dt} = \int_V \left(\rho\vec{E} + \vec{J} \times \vec{B}\right) d^3x \tag{6.114}$$

Conservation of Linear Momentum

(Week 2 – Day 1 Class Summary, pages 2-3)

- Equations are messier for this part, but when we assign the following volume integral as the total *electromagnetic field momentum*

No need to memorize; all expressions on this page will be provided if needed

$$\vec{P}_{\text{field}} = \epsilon_0 \int_V \vec{E} \times \vec{B} \, d^3x = \mu_0 \epsilon_0 \int_V \vec{E} \times \vec{H} \, d^3x \tag{6.117}$$

with the integrand interpreted as *electromagnetic momentum density*

$$\vec{g} = \frac{1}{c^2} \left(\vec{E} \times \vec{H} \right) \tag{6.118}$$

we obtain a cleaner-looking equation for the α^{th} component of the change in momentum:

$$\frac{d}{dt} \left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}} \right)_\alpha = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta} \, d^3x \tag{6.121}$$

although, of course, some of the messiness has been hidden in the $T_{\alpha\beta}$ term, the so-called *Maxwell Stress Tensor*

All expressions on this page will be provided on the Formula Sheet if needed

Maxwell Stress Tensor

(Week 2 – Day 1 Class Summary, pages 4–5)

No need to memorize; all expressions on this page will be provided if needed

In the α^{th} component of the change in momentum:

$$\frac{d}{dt}\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_{\alpha} = \sum_{\beta} \int_{V} \frac{\partial}{\partial x_{\beta}} T_{\alpha\beta}\, d^3x \qquad (6.121)$$

we have defined the *Maxwell Stress Tensor*

$$\boxed{T_{\alpha\beta} = \epsilon_0 \left[E_{\alpha}E_{\beta} + c^2 B_{\alpha}B_{\beta} - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right)\delta_{\alpha\beta}\right]} \qquad (6.120)$$

We can also write the α^{th} component of the change in momentum as

$$\frac{d}{dt}\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_{\alpha} = \oint_{S} \sum_{\beta} T_{\alpha\beta}\, n_{\beta}\, da \qquad (6.122)$$

where $\hat{n}$ is the outward normal to the closed surface S

All expressions on this page will be provided on the Formula Sheet if needed

Maxwell Stress Tensor

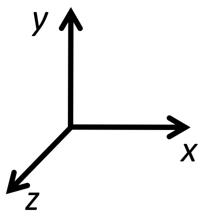
- In the stress tensor, α and β refer to coordinates x, y, z, and so $T_{\alpha\beta}$ has 9 components.

- From

$$\frac{d}{dt}\left(\vec{P}_{\mathrm{mech}} + \vec{P}_{\mathrm{field}}\right)_\alpha = \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta\, da$$

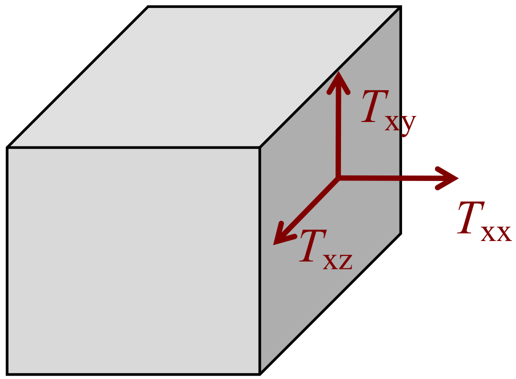
As noted on the previous slide, there is no need to memorize this expression.

we see that the stress tensor tells us the force per unit area (or stress) acting on the surface.



- More precisely, $T_{\alpha\beta}$ is the force per unit area in the α^{th} direction acting on element of the surface oriented in the β^{th} direction.

- Diagonal elements are pressures, and off-diagonal elements are shears.



Conservation of Linear Momentum

$$\frac{d}{dt}\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_\alpha = \oint_S \boxed{\sum_\beta T_{\alpha\beta}\, n_\beta}\, da \qquad (6.122)$$

No need to memorize the equation as noted before, but be sure you understand and remember the interpretations written below it

α^{th} component of the flow per unit area of momentum across the surface S into the volume V

In other words, it is the force per unit area transmitted across the surface S and acting on the combined system of particles and fields inside V

Laplace Equation in Cylindrical Coordinates

(Week 2 – Day 2 & Week 3 – Day 1

Class Summary & Worksheet)

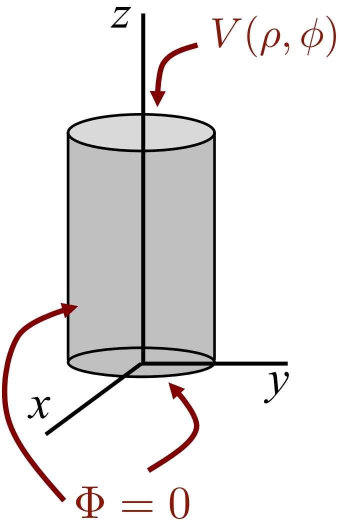
Laplace Equation in Cylindrical Coordinates

(3.71) will be provided, if needed

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \qquad (3.71)$$

You must be able to write everything on this slide below this point.

$$\Phi(\rho,\phi,z)=R(\rho)\,Q(\phi)\,Z(z) \qquad (3.72)$$

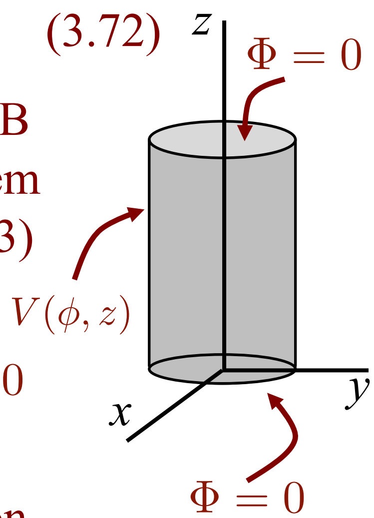


Type A problem (In-class)

$$\frac{d^2Z}{dz^2}-k^2Z=0$$

solution

$$Z(z)=e^{\pm kz}$$



Type B problem (HW 3)

$$\frac{d^2Z}{dz^2}+k^2Z=0$$

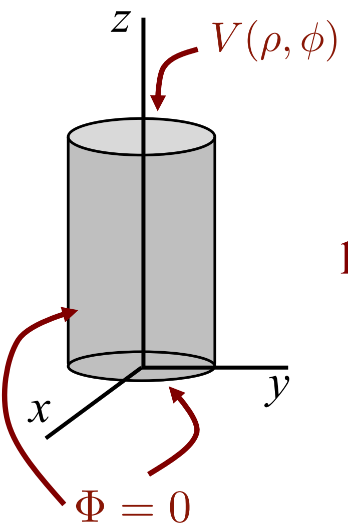
solution

$$Z(z)=e^{\pm ikz}$$

Recall that there are *two kinds* of such problems; I've called them Type A and Type B here to help you remember.

Type A problem

(Week 2 – Day 2 & Week 3 – Day 1 Class Summary & Worksheet)



$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \tag{3.71}$$

$$\Phi(\rho,\phi,z)=R(\rho)\,Q(\phi)\,Z(z) \tag{3.72}$$

Type A problem (In-class)

Other than (3.71), (3.75), & (3.77), which will be provided if needed, you must be able to write everything else on this slide.

$$\frac{d^2Z}{dz^2}-k^2Z=0 \xrightarrow{\text{solution}} Z(z)=e^{\pm kz} \tag{3.73}$$

;

$$\frac{d^2Q}{d\phi^2}+\nu^2Q=0 \xrightarrow{\text{solution}} Q(\phi)=e^{\pm i\nu\phi} \tag{3.74}$$

$\nu = m$ (i.e., integer)

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+\left(k^2-\frac{\nu^2}{\rho^2}\right)R=0 \tag{3.75}$$

change form

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{\nu^2}{x^2}\right)R=0 \xrightarrow{\text{solution}} R(\rho)=\boxed{C\,J_m(k\rho)}+D N_m(k\rho)$$

Bessel equation (3.77)

Use only this inside cylinder

Be sure you can solve for *C*, *D*

Solution is in terms of **Bessel functions**. We discussed their properties, including recursion relations, etc.

Type B problem

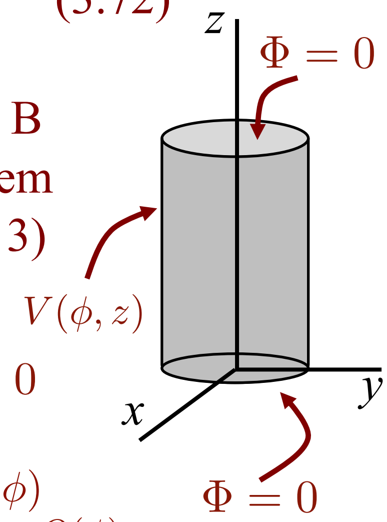
(Homework 3 – Problems 1 & 2)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \qquad (3.71)$$

$$\Phi(\rho,\phi,z)=R(\rho)\,Q(\phi)\,Z(z) \qquad (3.72)$$

Other than (3.71), you must be able to write everything else on this slide.

Type B problem (HW 3)



$$Z(z)=e^{\pm ikz} \xleftarrow{\text{solution}} \frac{d^2Z}{dz^2}+k^2Z=0$$

$$Q(\phi)=\sin m\phi+\cos m\phi \xleftarrow{\text{solution}} Q(\phi)$$

Same equation for $Q(\phi)$

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}-\left(1+\frac{\nu^2}{x^2}\right)R=0$$

Modified Bessel equation

solution

$$R(\rho)=C\,I_m(k\rho)+D\,K_m(k\rho)$$

Solution is in terms of **Modified Bessel functions**

Be sure you can solve for *C, D*

Green Functions

(Week 3 – Day 2 Class Summary & Worksheet)

- Green functions are very important for procedures in the chapter on radiation, but there will not be any direct questions on Green functions, other than in the context of the chapter.

- Green functions provide a technique for solving differential equations

$$\mathcal{D}\Psi(\vec{x}) = f(\vec{x}) \qquad \text{(W3.1)}$$

Any differential operator

e.g., $\nabla^2 + k^2$

solution

$$\Psi(\vec{x}) = \Psi_h + \Psi_{\text{part}} \qquad \text{(W3.2)}$$

Green Functions

(Week 3 – Day 2 Class Summary, page 1)

Wish to solve:

$$\mathcal{D}\Psi(\vec{x}) = f(\vec{x}) \quad \text{(W3.1)}$$

Solution

Associated inhomogenous equation

$$\mathcal{D}G(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}') \quad \text{(W3.2)}$$

Solve for $G(\vec{x},\vec{x}')$

$$\Psi(\vec{x}) = \Psi_h + \int_V G(\vec{x},\vec{x}')\, f(\vec{x}')\, d^3x'$$

- Again, Green functions are very important for procedures in the chapter on radiation, but there will not be any direct questions on Green functions, other than in the context of the chapter.

Green Functions in Electrostatics

(Week 3 – Day 2 Class Summary, pages 2-3)

Poisson equation

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

Green function equation

$$\nabla^2 G(\vec{x}, \vec{x}\,') = -4\pi\delta(\vec{x} - \vec{x}\,')$$

Showed in previous class that

$$\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}\,'|} \right) = -4\pi\delta(\vec{x} - \vec{x}\,')$$

$$\Downarrow$$

$$G(\vec{x}, \vec{x}\,') = \frac{1}{|\vec{x} - \vec{x}\,'|}$$

Constructing the solution:

$$\Psi(\vec{x}) = \Psi_h + \int_V G(\vec{x}, \vec{x}\,')\, f(\vec{x}\,')\, d^3x'$$

$$\Phi(\vec{x}) = \Phi_h(\vec{x}) + \int_V G(\vec{x}, \vec{x}\,') \left[\frac{\rho(\vec{x}\,')}{\epsilon_0} \right] d^3x'$$

Ignore Φ_h, assuming no boundary surfaces in problem

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x'$$

Green Functions for the Wave Equation

(Week 3 – Day 2 Class Summary, pages 4-7)

Wave equation

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\vec{x},t)$$

Green function equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_k^{(\pm)}(\vec{x},t;\vec{x}\,',t') = -4\pi\delta(\vec{x}-\vec{x}\,')\delta(t-t')$$

Although no direct questions on Green functions, this slide is here because you need to know all this for the radiation chapter.

$G^{(+)}$ is the retarded Green function, and $G^{(-)}$ is the advanced Green function

$$G^{(\pm)}(\vec{x},t;\vec{x}\,',t') = \frac{\delta\left(t' - \left[t \mp \frac{|\vec{x}-\vec{x}\,'|}{c}\right]\right)}{|\vec{x}-\vec{x}\,'|}$$

Constructing the particular solution:

$$\Psi(\vec{x},t) = \Psi_{\text{in}}(\vec{x},t) + \int\int G^{(+)}(\vec{x},t;\vec{x}\,',t')\, f(\vec{x}\,',t')\, d^3x'dt' \quad (6.45)$$

Chapter 9: Radiation

(Week 4 – Day 1 Class Summary & Worksheet)

- We'll assume that the sources are radiating in otherwise empty space, i.e., no boundaries or materials present.

- Assume harmonic time dependence:

$$\begin{aligned} \rho(\vec{x},t) &= \rho(\vec{x})\,e^{-i\omega t} \\ \vec{J}(\vec{x},t) &= \vec{J}(\vec{x})\,e^{-i\omega t} \end{aligned} \tag{9.1}$$

- Electromagnetic fields and potentials also have the same time dependence.

Chapter 9: Radiation

(Week 4 – Day 1 Class Summary, pages 1-4)

- We used a Green function technique to solve the wave equation for $\vec{A}$ and obtained

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\,\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x' \qquad (9.3)$$

Equation (9.3) will be supplied, if needed

- Equation (9.3) is useful because we can use it to find the fields:

Must know: $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} \qquad (9.4)$$

Should be able to work this out: $\vec{E} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H}$ (outside source) (9.5)

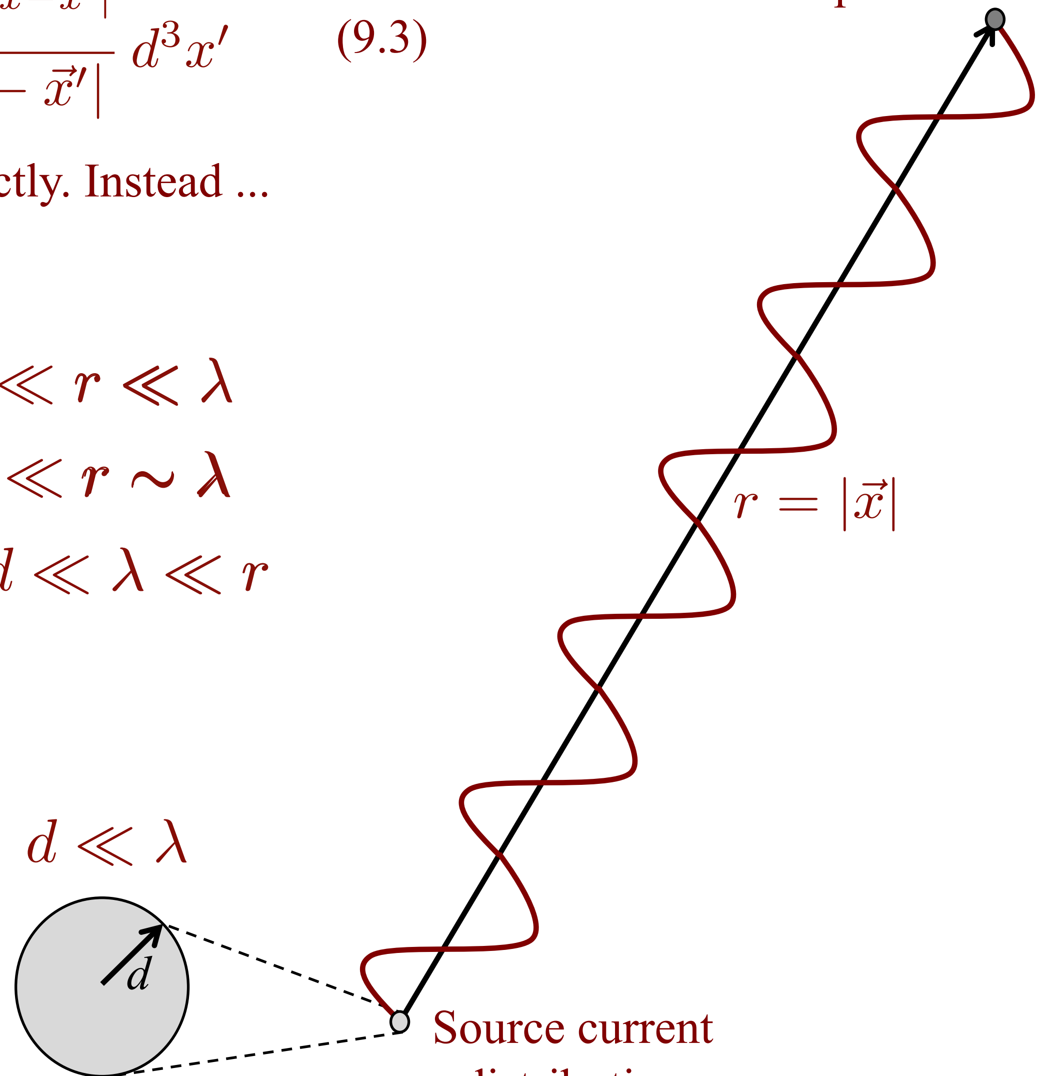
Three zones of interest

(Week 4 – Day 1 Class Summary, pages 4-5)

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x' \qquad (9.3)$$

- Won't integrate (9.3) directly. Instead ...

- The near (static) zone: $d \ll r \ll \lambda$
- The intermediate zone: $d \ll r \sim \lambda$
- The far (radiation) zone: $d \ll \lambda \ll r$



Consequences .....

(Week 4 – Day 1 Class Summary, page 6)

- After appropriate expansions, we get

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\int \vec{J}(\vec{x}')\,(\hat{n}\cdot\vec{x}')^n\,d^3x' \qquad (9.9)$$

No need to memorize (9.9), but be aware of what we're doing in this intermediate step

magnitude of n^{th} term

$$\frac{1}{n!}\int \vec{J}(\vec{x}')\,(k\hat{n}\cdot\vec{x}')^n\,d^3x'$$

order of magnitude is d

$$kd \ll 1$$

successive terms fall off rapidly with n

Radiation emitted from the source will come mainly from the first non vanishing term in this expansion

The beginning of this chapter is also summarized in the Class Summary for Week 4, Day 2

Electric Dipole Radiation

(Week 5 – Day 1 Class Summary & Worksheet)

No need to memorize; all equations on this page will be provided if needed

- Keeping only the 1st term in (9.9), get

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}\,')\,d^3x' \qquad (9.13)$$

- You showed on the worksheet that this can be written as

$$\vec{A}(\vec{x}) = -\frac{i\omega\mu_0}{4\pi}\,\vec{p}\,\frac{e^{ikr}}{r} \qquad (9.16)$$

$$\vec{p} = \int x'\,\rho(\vec{x}\,')\,d^3x' \qquad (9.17)$$

Dipole moment

All equations on this page will be supplied, if needed

Electric Dipole Fields

(Week 5 – Day 1 Class Summary, pages 2-3)

No need to memorize; all equations on this page will be provided if needed

$$\vec{A}(\vec{x}) = -\frac{i\omega\mu_0}{4\pi}\,\vec{p}\,\frac{e^{ikr}}{r} \qquad (9.16)$$

- Starting from (9.16), can write magnetic and electric fields (see Homework 4)

$$\begin{aligned}\vec{H} &= \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) \\ \vec{E} &= \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}\end{aligned} \qquad (9.18)$$

Magnetic field is always *perpendicular* to the radial vector

Electric field has components *perpendicular and parallel* to the radial vector

All equations on this page will be supplied, if needed

Electric Dipole Fields in the Near Zone

(Week 5 – Day 1 Class Summary, page 5)

$$\vec{H} = \frac{ck^2}{4\pi} \left(\hat{n} \times \vec{p}\right) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \left\{k^2 (\hat{n} \times \vec{p}) \times \hat{n} \, \frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n} \cdot \vec{p}) - \vec{p}\right] \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr}\right\}$$

(9.18)

No need to memorize (9.18); will be provided if needed

$\downarrow$

In the near zone

(Homework 4)

$\downarrow$

$$\vec{H} = \frac{i\omega}{4\pi} \left(\hat{n} \times \vec{p}\right) \frac{1}{r^2}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \left[3\hat{n}(\hat{n} \cdot \vec{p}) - \vec{p}\right] \frac{1}{r^3}$$

(9.20)

No need to memorize (9.20)

Apart from its oscillations in time, this is just the static electric dipole field

Details are on the last page of the Class Summary for Week 5, Day 1.

Electric Dipole Fields in the Far Zone

(Week 5 – Day 1 Class Summary, page 3)

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r}+\left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \qquad (9.18)$$

No need to memorize (9.18); will be provided if needed

$\downarrow$

In the far zone $kr \gg 1$

$\downarrow$

Magnetic field perpendicular to $\hat{n}$

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}$$

$$\vec{E} = \frac{k^2}{4\pi\epsilon_0}\left[(\hat{n}\times\vec{p})\times\hat{n}\right]\frac{e^{ikr}}{r} \qquad (9.19)$$

No need to memorize (9.19), but should remember the characteristics of the fields, as listed in the boxes on this page

Electric field perpendicular to magnetic field and to $\hat{n}$

Fields in the far zone behave as transverse waves carrying energy away from the source.

Electric Dipole Radiation

(Week 5 – Day 1 Class Summary, page 4)

- At any given point in space, S gives the energy per unit area per unit time flowing past that point, i.e., it is the time-averaged power radiated per unit surface area.

- In the current scenario, though, we are more interested in time-averaged power per unit solid angle.

- So, must write relation between solid angle $d\Omega$ and surface area perpendicular to the direction of energy flow *dA*, easy to do:

$$dA = r^2\, d\Omega$$

- The amount of energy flowing through would be $\hat{n} \cdot \vec{S}$

- So, time-averaged power radiated per unit solid angle by the oscillating dipole is

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[r^2 \hat{n} \cdot \vec{E} \times \vec{H}^*\right] \qquad (9.21)$$

(9.21) will be supplied, if needed

where *E* and *H* are as in (9.19).

Electric Dipole Radiation

(Week 5 – Day 1 Class Summary, page 4)

- On Homework 4, found that for electric dipole radiation

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left|(\hat{n} \times \vec{p}) \times \hat{n}\right|^2 \qquad (9.22)$$

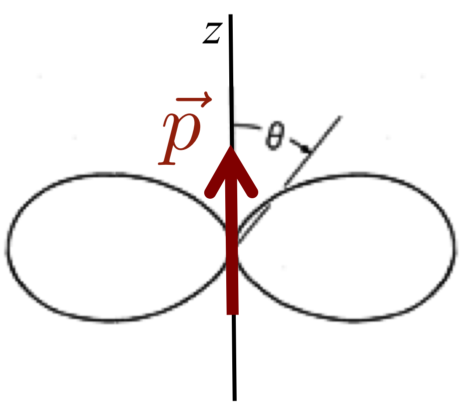
(9.22) will be supplied, if needed

- If all the components of $\vec{p}$ have the same phase, then the angular distribution of the radiation is a typical dipole pattern

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left|\vec{p}\right|^2 \sin^2\theta \qquad (9.23)$$

(9.23) will be supplied, if needed

- Eq. (9.23) tells us that if, e.g., the dipole is oriented along the *z*-axis (which passes through the north and south pole of the sphere), then the radiation is zero on the poles of the sphere, but peaks at the equator, perpendicular to the orientation of the dipole.



Magnetic Dipole and Electric Quadrupole Radiation

(Week 5 – Day 2 Class Summary & Worksheet)

- Starting again from the magnetic vector potential:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}d^3x' \tag{9.3}$$

Expand in powers of 1/r

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = \frac{e^{ikr}}{r}\left[1 + \left\{\frac{1}{r} - ik\right\}\frac{\vec{x}\cdot\vec{x}'}{r} + \dots\right]$$

Electric dipole term

Magnetic dipole & electric quadrupole term

No need to memorize, but be aware of what we are doing in this step.

Magnetic Dipole and Electric Quadrupole Radiation

(Week 5 – Day 2 Class Summary, pages 1-3)

No need to memorize; all equations on this page will be provided if needed

- The (second term) in the magnetic vector potential is given by

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\int \vec{J}(\vec{x}')\,(\hat{n}\cdot\vec{x}')\,d^3x' \qquad (9.30)$$

- Can write the integrand as sum of a symmetric and antisymmetric part:

$$(\hat{n}\cdot\vec{x}')\,\vec{J} = \underbrace{\frac{1}{2}\left[(\hat{n}\cdot\vec{x}')\,\vec{J} + (\hat{n}\cdot\vec{J})\vec{x}'\right]}_{\text{Electric quadrupole term}} + \underbrace{\frac{1}{2}\,(\vec{x}'\times\vec{J})\times\hat{n}}_{\text{Magnetic dipole term}}$$

- The magnetic dipole vector potential is given by

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\,(\hat{n}\times\vec{m})\,\frac{e^{ikr}}{r}\left[1 - \frac{1}{ikr}\right] \qquad (9.33)$$

All equations on this page will be supplied, if needed

Magnetic Dipole Fields
(Week 5 – Day 2 Class Summary, page 4)

No need to memorize; all equations on this page will be provided if needed

- By inspection (as you showed on the worksheet), get the magnetic dipole fields

$$\vec{H} = \frac{1}{4\pi}\left\{k^2(\hat{n}\times\vec{m})\times\hat{n}\frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n}\cdot\vec{m}) - m\right]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\} \qquad (9.35)$$

$$\vec{E} = -\frac{Z_0}{4\pi}\,k^2(\hat{n}\times\vec{m})\,\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right) \qquad (9.36)$$

- Equations (9.35) and (9.36) show that for magnetic dipoles, all arguments concerning the behavior of fields in the near and far zone are the same as for the electric dipole, with replacements:

$$\vec{E}_{\text{elec dipole}} \to cZ_0\vec{H},\ \vec{H}_{\text{elec dipole}} \to -c\vec{E}/Z_0,\ \text{and}\ \vec{p} \to \vec{m}$$

- The radiation pattern and total power radiated are the same for the two kinds of dipoles.

- For electric dipoles, recall that $\vec{E}$ lies in the plane defined by $\hat{n}$ and $\vec{p}$, whereas for magnetic dipoles, $\vec{E}$ is perpendicular to the plane defined by $\hat{n}$ and $\vec{m}$.

All equations on this page will be supplied, if needed

Electric Quadrupole Fields

(Week 5 – Day 2 Class Summary, pages 5-6)

- The electric quadrupole vector potential is given by

$$\vec{A}(\vec{x}) = -\frac{\mu_0 c k^2}{8\pi}\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\int \vec{x}'\,(\hat{n}\cdot\vec{x}')\,\rho(\vec{x}')\,d^3x' \qquad (9.38)$$

(9.38) will be provided if needed

- Too difficult to find *E* and *H* in general for electric quadrupole radiation (in any case, better methods like vector spherical multipoles exist).

- Fields in the far (radiation) zone are given by

$$\vec{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\,\hat{n}\times\vec{Q}(\hat{n}) \qquad (9.44)$$

No need to memorize (9.44); will be provided if needed

$$\vec{E} = -\frac{ic^2k^3\mu_0}{24\pi}\frac{e^{ikr}}{r}\left[\hat{n}\times\vec{Q}(\hat{n})\right]\times\hat{n}$$

components of this vector that involve elements of the quadrupole moment tensor are defined in the Week 5 – Day 2 class summary (page 5)

Electric Quadrupole Radiation

(Week 5 – Day 2 Class Summary, page 6)

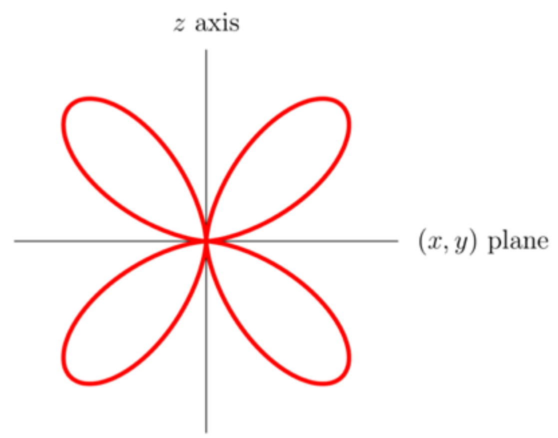
- The time-averaged power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2} k^6 \left| \left\{ \hat{n} \times \vec{Q}(\hat{n}) \right\} \times \hat{n} \right|^2 \qquad (9.45)$$

(9.45) will be supplied, if needed

- The general angular distribution is complicated, but we see that the radiated power varies as the 6th power of the frequency, a much stronger dependence compared to the fourth power for dipole radiation.

- Example angular distribution: spheroidal charge distribution (elongated cigar) along the *z*-axis. The radiated power pattern would be a four-lobed pattern (source:~utexas.edu).



Relativistic Electrodynamics

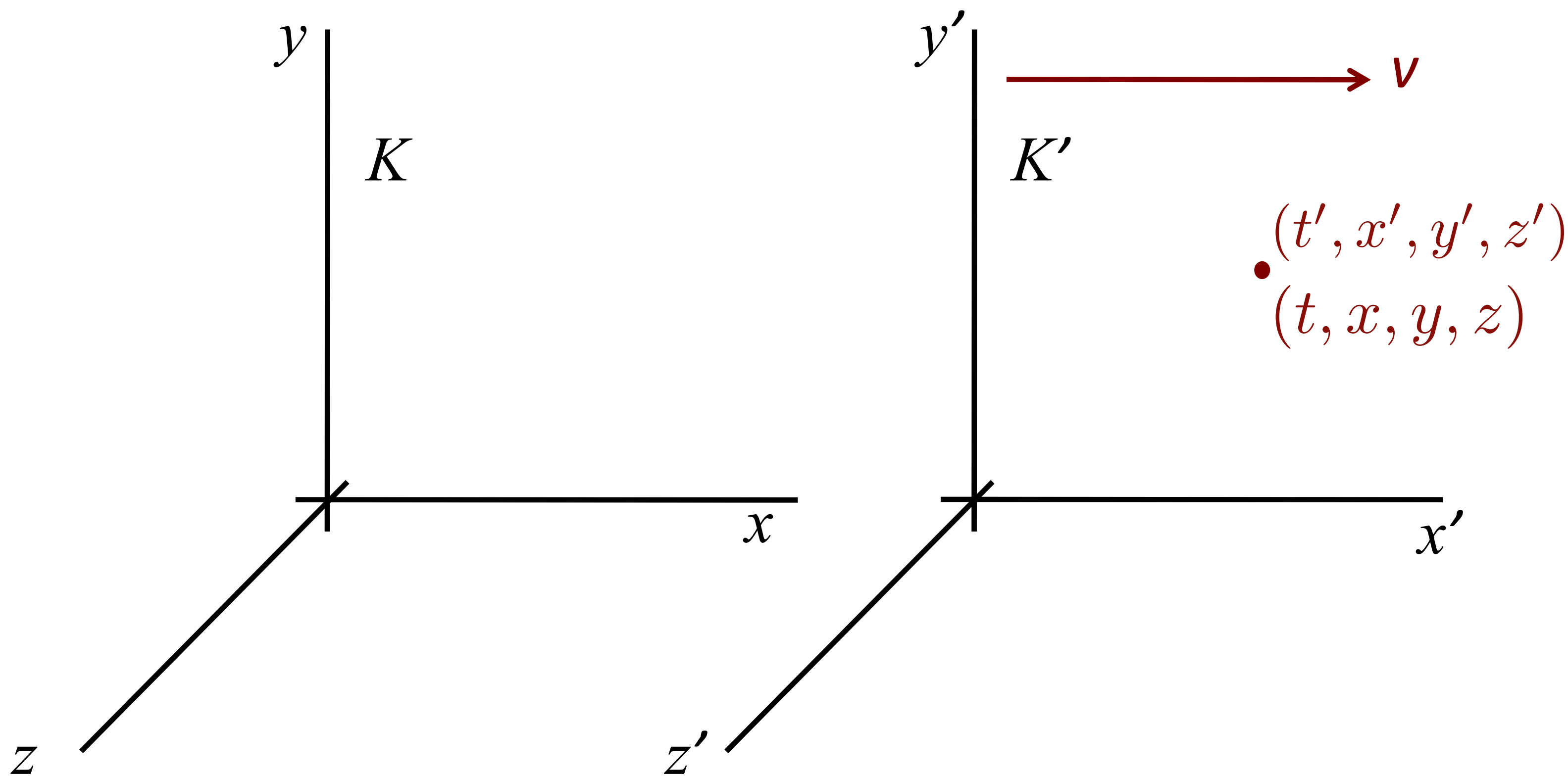
- Begin with discussion of the *Special Theory of Relativity*, starting from Week 6, Day 1 class summary; read pages 1-3 for background.

The Special Theory of Relativity

- Note: In some (earlier) parts of this chapter, Jackson uses

$$x_0 = ct, \qquad x_1 = z, \qquad x_2 = x, \qquad x_3 = y$$

whereas in other parts, Jackson uses $x_1 = x$, etc. I'll try to be as clear about this as possible.



Lorentz Transformations

(Week 6 – Day 1 Class Summary & Worksheet)

$$x_0 = ct, \qquad x_1 = z, \qquad x_2 = x, \qquad x_3 = y$$

To work out how coordinates and derivatives transform, e.g., $\partial\psi/\partial x'$, see page 2 of Week 6–Day 1 class summary.

$$\left.\begin{aligned} x'_0 &= \gamma\left(x_0 - \beta x_1\right) \\ x'_1 &= \gamma\left(x_1 - \beta x_0\right) \\ x'_2 &= x_2 \\ x'_3 &= x_3 \end{aligned}\right\} \qquad (11.16)$$

Do not memorize (11.16). Will be provided if needed, unless asked to derive.

Inverse transformations are in (11.18)

where

Generalized versions of (11.16) are in (11.19) on page 4 of Week 6–Day 1. Parametrized versions in terms of hyperbolic functions are written in (11.20)-(11.21), page 5 of Week 6-Day 1.

Must know β and γ:

$$\beta = \frac{v}{c}, \qquad \text{also remember: } \vec{\beta} = \frac{\vec{v}}{c}$$

$$\gamma = \frac{1}{\sqrt{(1-\beta^2)}}$$

Written for general 4-vectors $(A_0, \vec{A})$ in (11.22)

Addition of velocities

(Week 6 – Day 1 Class Summary, page 6)

$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + \dfrac{\vec{v} \cdot \vec{u}\,'}{c^2}}$$

$$\vec{u}_{\perp} = \frac{\vec{u}\,'_{\perp}}{\gamma_v \left(1 + \dfrac{\vec{v} \cdot \vec{u}\,'}{c^2}\right)} \qquad (11.31)$$

No need to memorize;
will be provided if needed,
unless asked to derive

For the specific case of parallel velocities:

$$u = \frac{u' + v}{1 + \dfrac{vu'}{c^2}} \qquad (11.33)$$

No need to memorize;
will be provided if needed,
unless asked to derive

Relativistic Energy and Momentum

(Week 6 – Day 1 Class Summary, page 7)

Must know these:

$$\boxed{\begin{gathered}\vec{p} = \gamma m\vec{u} \\ E = \gamma mc^2 \\ E = \sqrt{p^2c^2 + m^2c^4}\end{gathered}}$$

$$\vec{p} = \gamma m\vec{u} = \frac{m\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad (11.46)$$

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad (11.51)$$

$$E = \sqrt{p^2c^2 + m^2c^4}$$

Mathematical properties: 4-vectors

(Week 7 – Day 1 Class Summary, pages 1-3)

- 4-vectors $A^\alpha = (A^0, \vec{A})$ are 4-dimensional vectors that transform according to Lorentz transformations; written out in full: $A^\alpha = (A^0, A^1, A^2, A^3)$.

- Lorentz transformation for general 4-vector is in (11.22), invariant "span" (scalar product) is in (11.23).

- Useful relation: $dt = \gamma_u d\tau$

- Examples:

4-velocity: $U = (U_0, \vec{U}) = \left(\gamma_u\, c, \gamma_u\, \vec{u}\right)$

4-momentum: $P = \left(\gamma_u mc, \gamma_u m\vec{u}\right) = \left(\dfrac{E}{c}, \vec{p}\right)$

$$\gamma_u = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$$

You must be able to write these down from memory if asked to do so.

See the posted PowerPoint slides from the Class Summary for Week 10 – Day 2 for a more physically intuitive approach to the mathematics of relativity.

Mathematical Properties: Tensors

(Week 7 – Day 1 Class Summary, pages 4-5)

- Groups are discussed on page 4. You may read them for context, but there will be *no direct questions* on properties of groups.

- Invariants and scalar products are discussed on page 5. Be sure you understand them; you'll be applying them a lot.

Mathematical Properties: Tensors
(Week 7 – Day 1 Class Summary, page 6)

- Space-time continuum is defined in terms of a 4-dimensional space:

$$(ct, z, x, y) \equiv (x^0, x^1, x^2, x^3)$$

- We suppose existence of a transformation rule for new coordinates:

$$x'^\alpha = x'^\alpha\left(x^0, x^1, x^2, x^3\right)$$

- e.g., Lorentz transformations for a general contravariant vector:

Lorentz transformations for x^α will be provided on the Formula Sheet; should be able to write these by analogy.

$$A'^0 = \gamma\left(A^0 - \beta A^1\right) \qquad A'^2 = A^2$$
$$A'^1 = \gamma\left(A^1 - \beta A^0\right) \qquad A'^3 = A^3$$

- We say that the contravariant vector A^α has 4 components that transform according to the rule:

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta \qquad (11.61)$$

repeated index summation

Mathematical Properties: Tensors

(Week 7 – Day 1 Class Summary, pages 6-8)

- Contravariant vectors:

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} \qquad (11.61)$$

- Write explicitly the repeated index summation:

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{0}} A^{0} + \frac{\partial x'^{\alpha}}{\partial x^{1}} A^{1} + \frac{\partial x'^{\alpha}}{\partial x^{2}} A^{2} + \frac{\partial x'^{\alpha}}{\partial x^{3}} A^{3}$$

- Covariant vectors:

$$B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} \qquad (11.62)$$

- Again, write explicitly: $$B'_{\alpha} = \frac{\partial x^{0}}{\partial x'^{\alpha}} B_{0} + \frac{\partial x^{1}}{\partial x'^{\alpha}} B_{1} + \frac{\partial x^{2}}{\partial x'^{\alpha}} B_{2} + \frac{\partial x^{3}}{\partial x'^{\alpha}} B_{3}$$

Number of indices is rank of tensor

Vector is tensor of rank **one**
Scalar is tensor of rank **zero**

- Scalar Product: $B \cdot A = B_{\alpha} A^{\alpha}$

Also see page 1, Week 7–Day 2 class summary

The Metric Tensor

(Week 7 – Day 2 Class Summary, pages 2-3)

$$(ds)^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{11.68}$$

Must know: $$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{11.81}$$

Raising & lowering indices: $$x_\alpha = g_{\alpha\beta}\, x^\beta \tag{11.72}$$

$$x^\alpha = g^{\alpha\beta}\, x_\beta \tag{11.73}$$

Contravariant

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Covariant

$$gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

Useful Procedures

(Week 7 – Day 2 Class Summary, pages 4-6)

- Useful notation for the 4-derivative:

Must know both:

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right)$$

$$\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x_0}, -\vec{\nabla} \right) \tag{11.76}$$

- Allows us to write 4-divergence; eq. (11.77) on page 4
- Laplacian operator: bottom of page 4 of Week 7–Day 2 class summary
- Matrix representation of Lorentz Transformations on pages 5-6 of Week 7–Day 2 class summary; also see page 1 of Week 8–Day 1 class summary.

Seeking Covariance .....

(Week 8 – Day 1 Class Summary, page 2)

- Physical laws must be covariant (i.e., invariant in form) under:
  - Translations in space and time
  - Rotations in 3-dimensional space
  - Lorentz transformations
- We seek a group of linear transformations that leaves $(x, gx) = x \cdot x$ invariant.
- Since (x, gx) or $x \cdot x$ is the norm of a 4-vector, we are effectively seeking a group of transformations that preserves the "*length*" in the 4-dimensional metric.
- In other words, we seek all square 4×4 matrices A on the coordinates

$$x' = Ax$$

that leave the norm (x, gx) invariant.

$\downarrow$

Get det A = ± 1

Will consider only transformations with det A = +1, called proper Lorentz transformations

Matrix Representation for A

(Week 8 – Day 1 Class Summary, page 3)

- Consider only proper transformations with det $A = +1$.

- We can then build $A = e^{L}$ with six free parameters: 3 rotations, 3 velocity boosts

$$\downarrow$$

Use
infinitesimal generators

Rotation and boost matrices

(Week 8 – Day 1 Class Summary, pages 4–6)

$$S_1 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad S_2 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \quad S_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

No need to memorize; will be provided if needed

$$K_1 = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad K_2 = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad K_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

$$A = e^{-\vec{\omega}\cdot\vec{S} - \vec{\zeta}\cdot\vec{K}}$$

$$x' = Ax$$

No need to memorize;
will be provided if needed

We did two examples: boost only, no rotation (Week 8–Day 1 worksheet), and rotation only, no boost (Homework 7)

Covariance of Electrodynamics

(Week 8 – Day 2 Class Summary & Worksheet)

- The *invariance in form* of the equations of electrodynamics under Lorentz transformations was shown by Lorentz and Poincare before the formulation of the Special Theory of Relativity.

- A more precise word here is *covariance*, meaning that the *form* of the equations does not change.

- Not only are Maxwell's equations *invariant in form* (*covariant*) under Lorentz transformations, but also the Lorentz force law and the continuity equation.

- To demonstrate this covariance, we have to write our equations using 4-tensors. That is because 4-tensors are invariant in form (covariant) under Lorentz transformations by definition, so expressing our equations in terms of 4-tensors will render the equations themselves covariant under Lorentz transformations.

Covariance of Electrodynamics

(Week 8 – Day 2 Class Summary, page 1)

- Covariance: Form of the equations does not change under Lorentz transformations.

- Recall useful notation:

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right)$$

$$\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x_0}, -\vec{\nabla} \right) \qquad (11.76)$$

- Continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \Longrightarrow \quad \frac{\partial (c\rho)}{\partial (ct)} + \vec{\nabla} \cdot \vec{J} = 0$$

$$\frac{\partial (c\rho)}{\partial x^0} + \vec{\nabla} \cdot \vec{J} = 0 \qquad J^\alpha = (c\rho, \vec{J})$$

Continuity equation in covariant form: $\boxed{\partial_\alpha J^\alpha = 0}$

Given the continuity equation (on the Formula Sheet), you must be able to write this covariant form.

The Field Strength Tensor

(Week 8 – Day 2 Class Summary, page 2)

- By writing the two inhomogenous Maxwell equations in explicit matrix form, can derive the field strength tensor:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad (11.137)$$

(11.137) will be provided on the Formula Sheet

- Covariant form of the two inhomogenous Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

Given the Maxwell equations on the Formula Sheet, must be able to write this (best to memorize):

$$\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta} \qquad (11.141)$$

- A useful form of the field tensor in terms of the vector potential is (Homework 7):

Must know: $$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \qquad (11.136)$$

- We can also write the field tensor with covariant indices by putting $\vec{E} \to -\vec{E}$

The Dual Field Strength Tensor

(Week 8 – Day 2 Class Summary, page 3)

- By definition, the dual field-strength tensor is

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\,\epsilon^{\alpha\beta\gamma\delta}\,F_{\gamma\delta}$$

where
- $\epsilon^{\alpha\beta\gamma\delta} = +1$, for $\alpha = 0, \beta = 1, \gamma = 2, \delta = 3$, or any even permutation.
- $\epsilon^{\alpha\beta\gamma\delta} = -1$, for any odd permutation of $\alpha, \beta, \gamma, \delta$.
- $\epsilon^{\alpha\beta\gamma\delta} = 0$, if any two indices are equal.

The field-strength tensor $F^{\alpha\beta}$ in (11.137) will be supplied, but you must be able to build the dual field-strength tensor in (11.140) by yourself, using any of the methods on this page.

$$\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \qquad (11.140)$$

- Can get this from (11.137) by putting $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$

This is probably the easiest way to remember how to build the dual field-strength tensor in (11.140).

- Covariant form of the two homogenous Maxwell equations:

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

Given the Maxwell equations on the Formula Sheet, must be able to write this (best to memorize):

$$\boxed{\partial_\alpha \mathcal{F}^{\alpha\beta} = 0} \qquad (11.142)$$

More Covariant Equations
(Week 8 – Day 2 Class Summary, page 4)

- Wave Equations:

$$\left.\begin{aligned}\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}-\nabla^2\vec{A}&=\frac{4\pi}{c}\vec{J}\\ \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}-\nabla^2\Phi&=4\pi\,\rho\end{aligned}\right\}\longrightarrow \boxed{\Box A^\alpha=\frac{4\pi}{c}J^\alpha}$$

No need to memorize the wave equations; will be provided if needed, but must then be able to write the covariant form.

Another useful operator:

$$\Box\equiv\partial_\alpha\partial^\alpha=\frac{\partial^2}{\partial x^{02}}-\nabla^2 \qquad (11.78)$$

Must know Laplacian $\Box$,
or, be able to write it, knowing (11.76)

Covariant form of wave equation (11.133)

- Lorenz condition:

$$\frac{1}{c}\frac{\partial\Phi}{\partial t}-\vec{\nabla}\cdot\vec{A}=0 \longrightarrow \boxed{\partial_\alpha A^\alpha=0}$$

Lorenz condition will be provided if needed, but must then be able to write the covariant form.

Must know A^α:

where the 4-vector potential is $A^\alpha=(\Phi,\vec{A})$

- Macroscopic Equations in covariant form (must know):

$G^{\alpha\beta}=(\vec{D},\vec{H})$

Must know

Must know:

$$\boxed{\partial_\alpha G^{\alpha\beta}=\frac{4\pi}{c}J^\beta, \qquad \partial_\alpha \mathcal{F}^{\alpha\beta}=0}$$

Covariant Equation for Force Equation

(Week 9 – Day 1 Class Summary, pages 1-2)

- Lorentz Force Equation:

(Must know)

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B}\right)$$

- Covariant form of Lorentz Force Equation:

$$\boxed{\frac{dp^\alpha}{d\tau} = m\,\frac{dU^\alpha}{d\tau} = \frac{q}{c}\,F^{\alpha\beta}\,U_\beta}$$

(Must know, including the form
of the energy-momentum 4-vector)

Transformation of Electromagnetic Fields

(Week 9 – Day 1 Class Summary, page 3 & Worksheet, pages 3-6)

- Recall the field strength tensor:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad (11.137)$$

(11.137) will be provided on the Formula Sheet

- Transformation from this frame K to a new frame K' can be done by

$$F'^{\,\alpha\beta} = \frac{\partial x'^{\,\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\,\beta}}{\partial x^{\delta}} F^{\gamma\delta} \qquad (11.146)$$

Must know these procedures

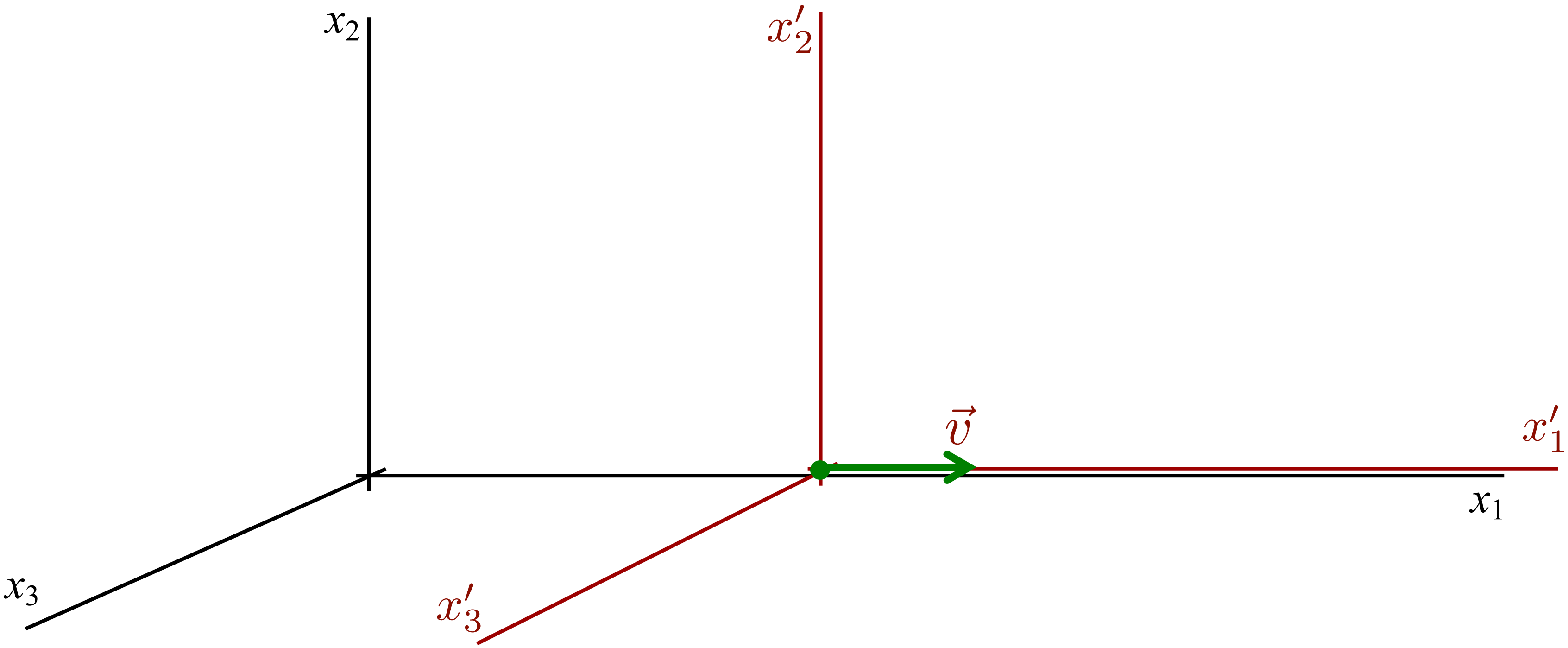
which can be written in matrix form as

$$F' = A\, F\, \tilde{A}$$

where A is the matrix we built with 3 elements for rotations and 3 elements for Lorentz boosts.

Transformation of Electromagnetic Fields

(Week 9 – Day 1 Class Summary, page 3)



- For example, when we have a Lorentz boost along x_1 axis (but no rotation)

No need to memorize;
will be provided if needed

$$A = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma = \cosh\zeta$$

$$\gamma\beta = \sinh\zeta$$

No need to memorize;
will be provided if needed

Transformation of Electromagnetic Fields

(Week 9 – Day 2 Class Summary, pages 1-2)

- For boost along x^1 axis (see previous slide), transformation equations are

$$
\begin{aligned}
E_1' &= E_1 & \qquad B_1' &= B_1 \\
E_2' &= \gamma\left(E_2 - \beta B_3\right) & \qquad B_2' &= \gamma\left(B_2 + \beta E_3\right) \\
E_3' &= \gamma\left(E_3 + \beta B_2\right) & \qquad B_3' &= \gamma\left(B_3 - \beta E_2\right)
\end{aligned}
\qquad (11.148)
$$

will be provided on the Formula Sheet

- No independent existence of fields; a pure electric field in one frame becomes a mixture of electric and magnetic fields in another.

- For example, if frame K has *only* an E-field and *zero* B-field (so that B_1, B_2, B_3 are all zero in frame K), we will still have *non-zero* B_2' and B_3' in frame K'. Also see example worked out on page 2 of the Week 9 – Day 2 class summary.

- Note exception that a pure electric field in one frame cannot become a pure magnetic field in another (*Homework 8*).

Transformation of Electromagnetic Fields

(Week 9 – Day 2 Class Summary, page 1)

- General case:
  (for K' moving in any direction with $\vec{\beta}$ relative to K)

$$\begin{aligned}\vec{E}\,' &= \gamma\left(\vec{E}+\vec{\beta}\times\vec{B}\right)-\frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)\\ \vec{B}' &= \gamma\left(\vec{B}-\vec{\beta}\times\vec{E}\right)-\frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right)\end{aligned} \qquad (11.149)$$

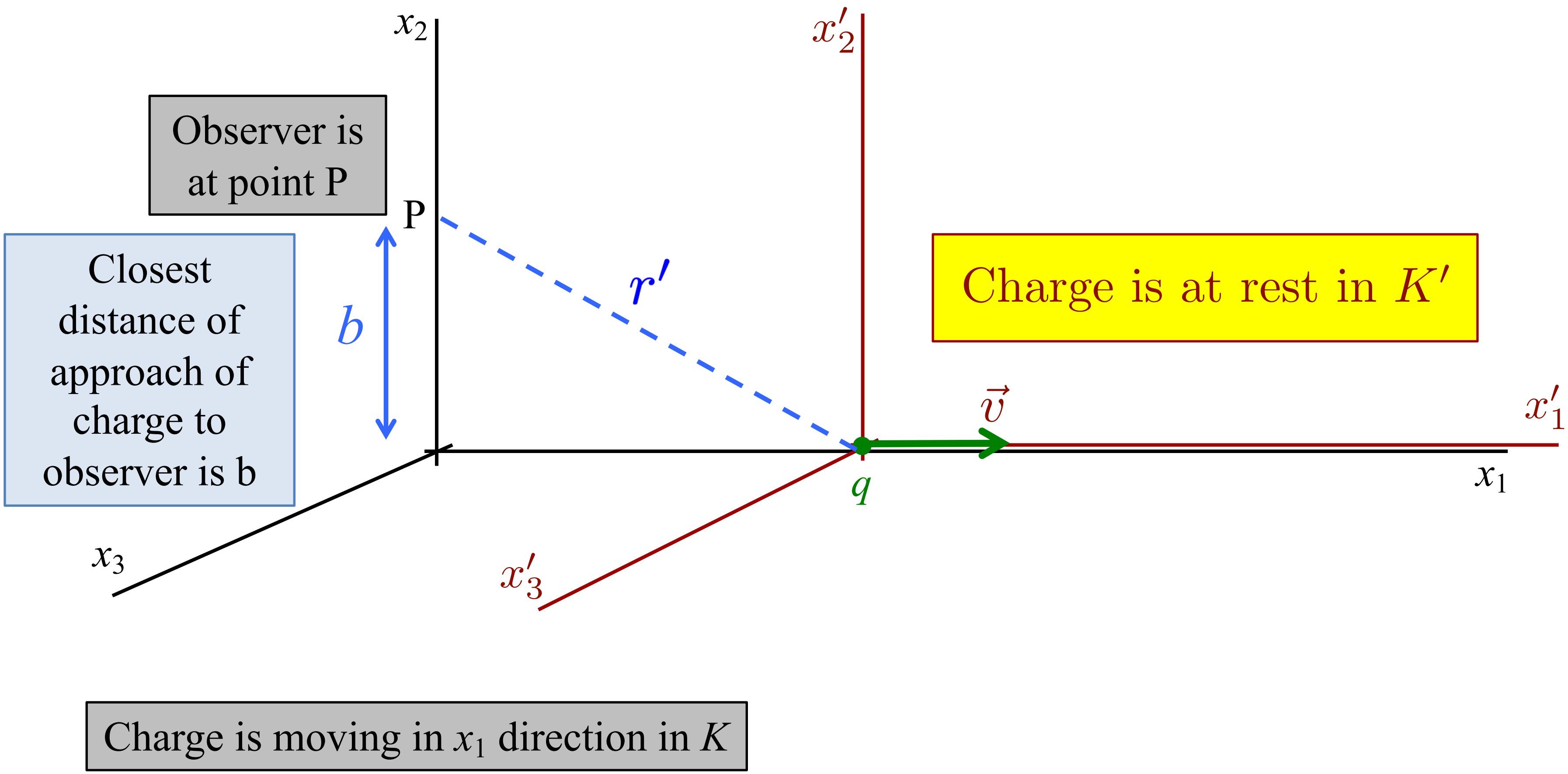
No need to memorize these equations; will be provided if needed, but must be able to work with these in different situations. For an example, see *page 4* of the Class Summary for Week 8–Day 2, where these equations were used to derive a relation between fields in frame *K* when no magnetic field exists in the primed frame. Also see Homework 7, Problem 2.

- Inverse transformation equations, both for (11.148) and the general case (11.149) above, are written on page 1 of the Week 9 – Day 2 class summary.

Example Transformation of Fields

(Week 10 – Day 1 Class Summary & Worksheet)

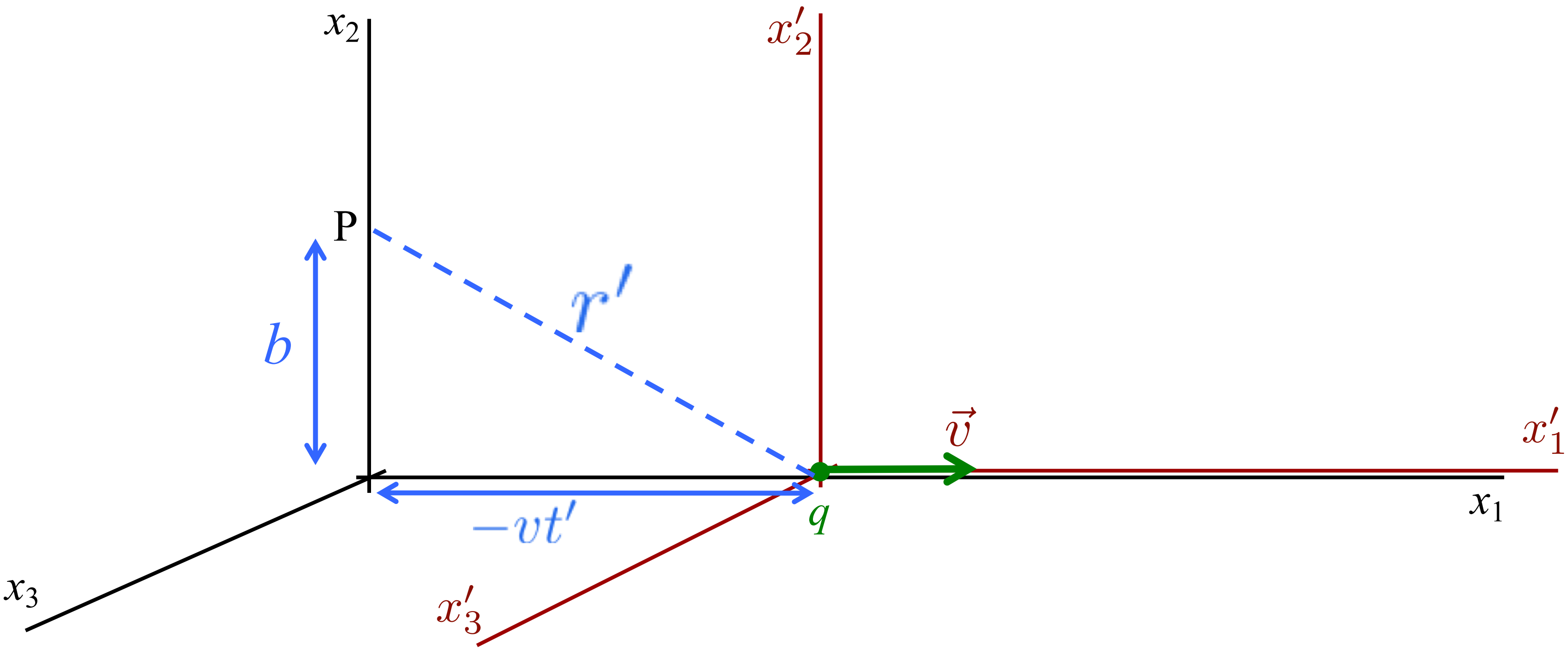
- Point charge q moving in a straight-line path with velocity $\vec{v}$



At $t = t' = 0$, origins of the two coordinate systems coincide, and q is at its closest distance to observer at P.

Example Transformation of Fields

(Week 10 – Day 1 Class Summary, page 1)



- In frame K', the observer's point P has coordinates

$$x_1' = -vt', \qquad x_2' = b, \qquad x_3' = 0$$

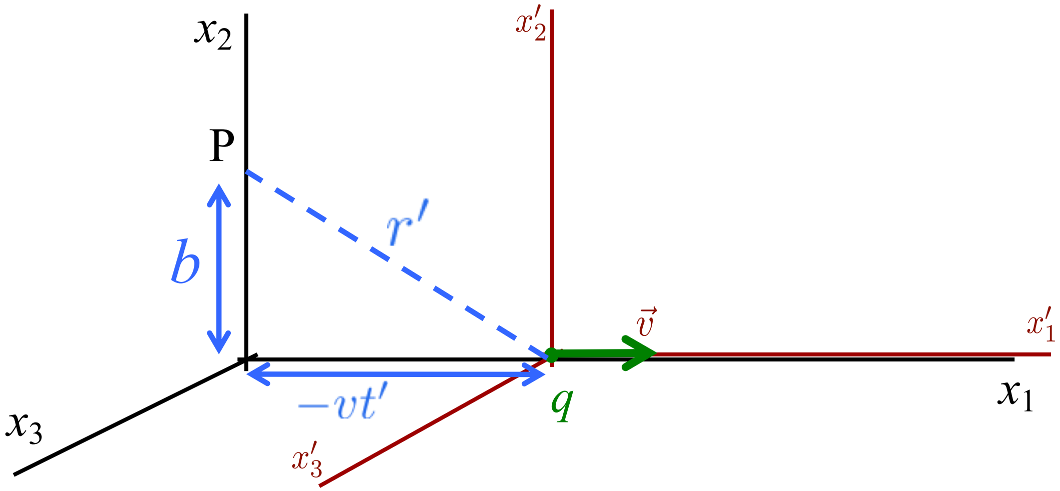
and is at distance from q given by

$$r' = \sqrt{b^2 + (vt')^2}$$

Must be able to write all these on your own

Example Transformation of Fields

- In frame K', rest frame of charge q, we only have the electric field, so $\vec{B}' = 0$



* So, electric and magnetic fields in K' at observation pt. P are

$$\vec{E}' = \frac{q}{|r'|^2} = \frac{q\vec{r}'}{|r'|^3} = \frac{q\,x_1'(\hat{x}_1') + q\,x_2'(\hat{x}_2') + q\,x_3'(\hat{x}_3')}{|r'|^3}$$

or, in terms of components, with $x_1' = -vt', x_2' = b, x_3' = 0$

Fields in frame K'

Must be able to derive these:

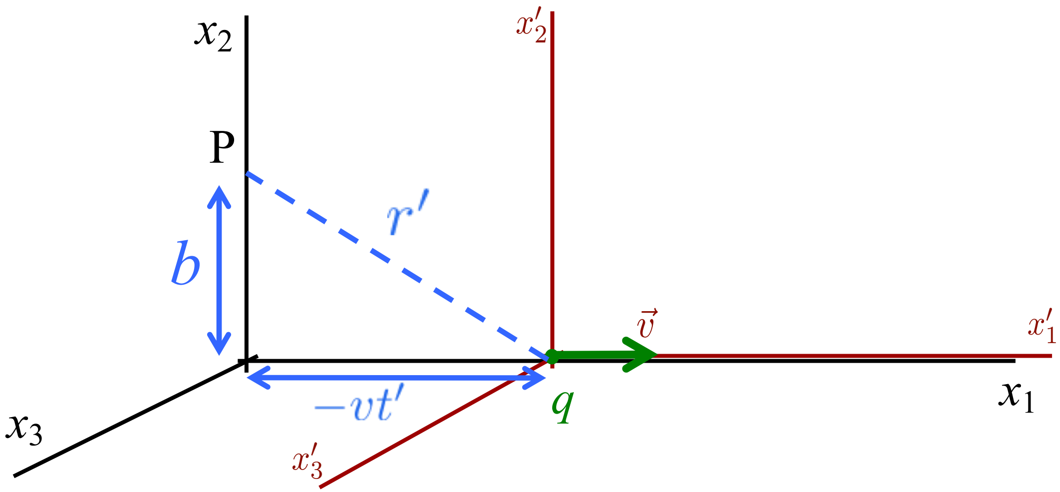
$$E_1' = -\frac{qvt'}{|r|'^3} \qquad E_2' = \frac{qb}{|r|'^3} \qquad E_3' = 0 \qquad (11.151)$$

$$B_1' = 0 \qquad B_2' = 0 \qquad B_3' = 0$$

Example Transformation of Fields

(Week 10 – Day 1 Discussion Worksheet, page 2)

- To get the fields in K, we will need to express r' in coordinates of K



- We already know that

$$r' = \sqrt{b^2 + (vt')^2}$$

- So the only transformation needed is

$$t' = \gamma\left[t - \left(\frac{v}{c^2}\right) x_1\right] = \gamma t$$

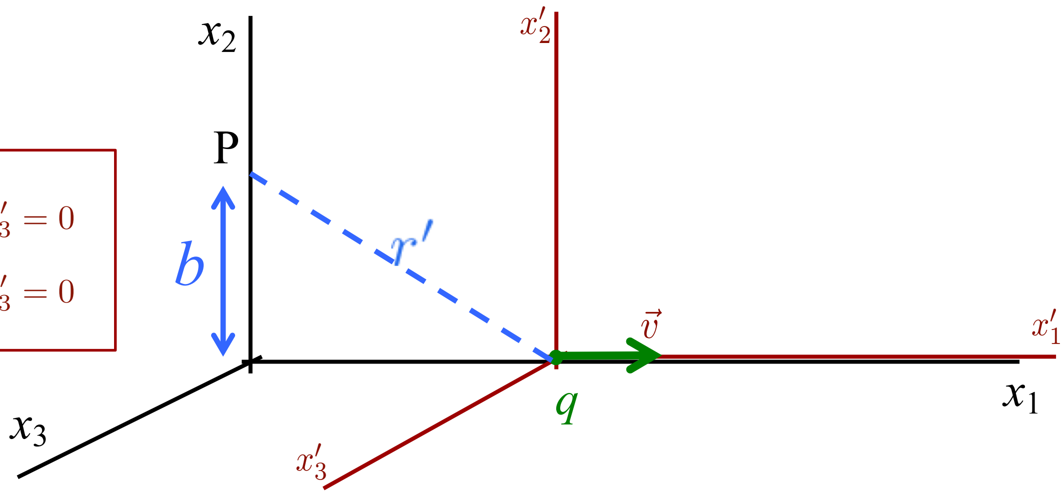
Must be able to work out from transformation for *ct* on Formula Sheet

Example Transformation of Fields

(Week 10 – Day 1 Class Summary, pages 2-3)

- Recall that the fields in K' are

$$\boxed{\begin{array}{lll} E_1' = -\dfrac{qvt'}{|r|'^3} & E_2' = \dfrac{qb}{|r|'^3} & E_3' = 0 \\ B_1' = 0 & B_2' = 0 & B_3' = 0 \end{array}}$$



- The transformed fields in frame K are

$$E_1 = E_1' = -\frac{\gamma q v t}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

$$E_2 = \gamma E_2' = \frac{q\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \qquad (11.152)$$

$$B_3 = \gamma\beta E_2' = \beta E_2$$

- All other components vanish ($E_3 = B_1 = B_2 = 0$).

(11.152) are the end points in a problem, and you *must be able to work them out* on your own.

Notice that even though there is only an E-field in K', there is also a B-field in the frame K (B_3 in 11.152)

Example Transformation of Fields

(Week 10 – Day 1 Class Summary, page 3)

- All fields go to zero at large "negative" and "positive" times
- E_2 has its maximum value of $\gamma q/b^2$ at t=0, so this peak increases in proportion to γ
- Can define a characteristic timescale

$$\Delta t = \frac{b}{\gamma v} \tag{11.153}$$

- Can use these to draw graphs.

Example Transformation of Fields

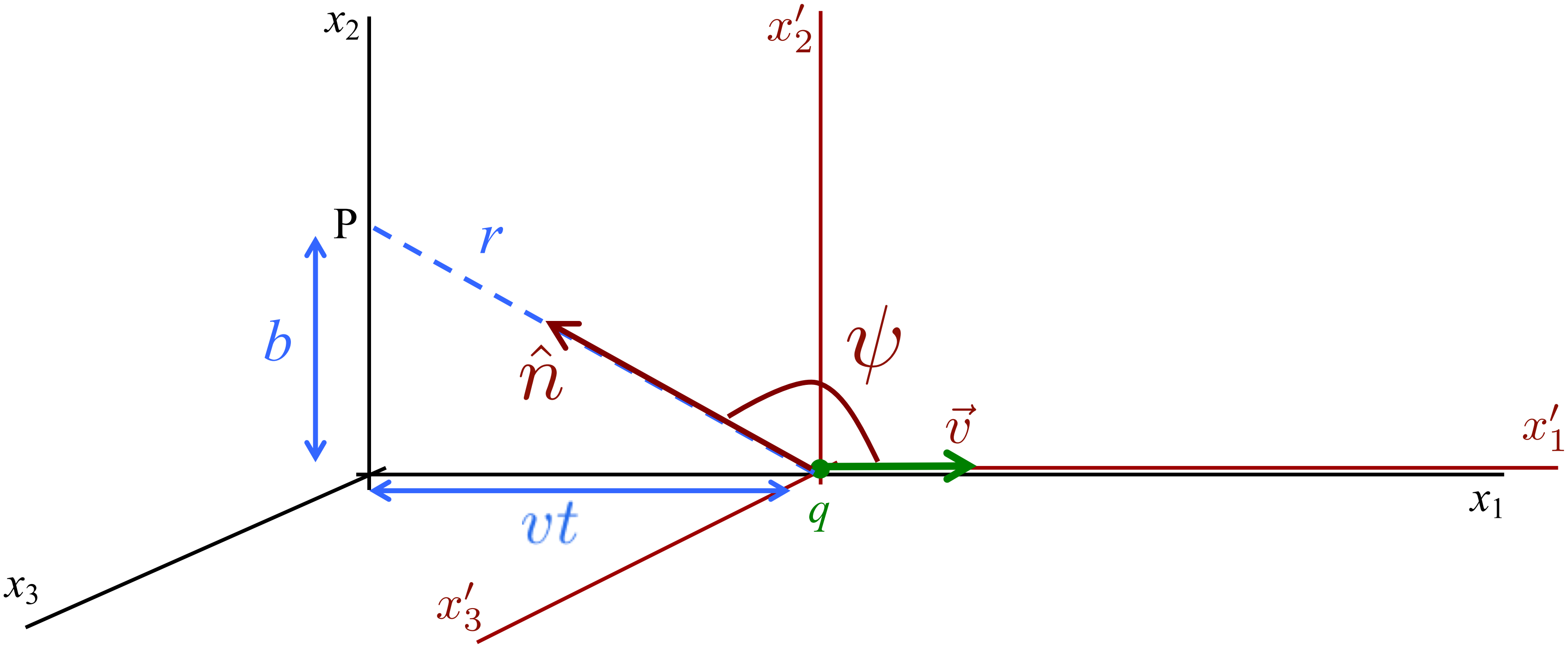
(Week 10 – Day 1 Class Summary, pages 4-5)

- A graph of E_2 *vs.* t is drawn and discussed at the top of page 4 (Week 10 – Day 1).
- The behavior of B_3 is discussed on page 4 (Week 10 – Day 1).
- The behavior of E_1 is discussed on page 5 (Week 10– Day 1), its maximum and minimum values and the corresponding values of t are also worked out, and a graph of E_1 *vs.* t is also drawn and discussion on that page.

Transformed Fields: Spatial Distribution

(Week 10 – Day 1 Class Summary, page 6)

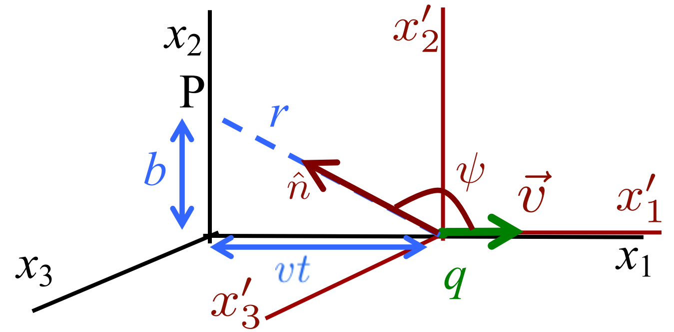
- Figure 11.9 (a), that we viewed moments ago, emphasizes time dependence of the fields at a fixed observation point.
- Another way to view the result is to look at the spatial distribution of the field as a function of time.



Transformed Fields: Spatial Distribution

(Week 10 – Day 1 Class Summary, page 6)

- On Homework 8–Problem 2, you showed that



$$\vec{E} = \frac{q\vec{r}}{r^3\gamma^2(1-\beta^2\,\sin^2\psi)^{3/2}}$$

where the angle ψ is shown in the figure.

- This expression tells us that the electric field is radial, but the lines of force are isotropically distributed only for $\beta = 0$.

- The magnitude of E is

$$E = \frac{qr}{r^3\gamma^2\,(1-\beta^2\,\sin^2\psi)^{3/2}} = \frac{q}{r^2\gamma^2\,(1-\beta^2\,\sin^2\psi)^{3/2}}$$

- For $\beta = 0$ (and hence, $\gamma = 1$): $\quad E = \dfrac{q}{r^2}$

because $\gamma = (1-\beta^2)^{-1/2}$

Example Transformation of Fields

(Week 10 – Day 1 Class Summary, page 6)

- For $\beta = 0$:

$$E = \frac{q}{r^2}$$

The electric field is radial, and the lines of force are isotropically distributed

- For $\beta \neq 0$:

$$E\Big|_{\psi=0,\pi} = \frac{1}{\gamma^2}\left(\frac{q}{r^2}\right)$$

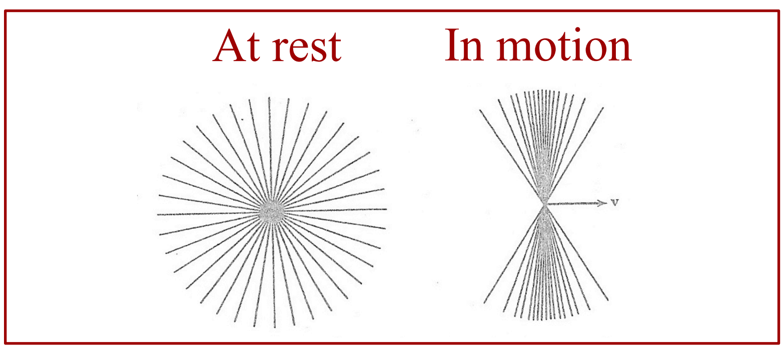
The electric field is still radial, but along the direction of motion ($\psi = 0, \pi$), the electric field strength is ***down*** by a factor of γ^2 relative to isotropy (where it would be q/r^2).

$$E\Big|_{\psi=\pi/2} = \frac{q}{r^2\gamma^2\left(1-\beta^2\right)^{3/2}}$$

But $\gamma = (1-\beta^2)^{-1/2}$, so $(1-\beta^2)^{-3/2} = \gamma^3$

$$E\Big|_{\psi=\pi/2} = \frac{q\gamma^3}{r^2\gamma^2} = \gamma\left(\frac{q}{r^2}\right)$$

The electric field is still radial, but along the transverse direction ($\psi = \pi/2$), the field strength is ***larger*** by a factor of γ relative to isotropy.



Appears like a "Lorentz contraction"

Final Examination

- You must be present in the Zoom session in order for your Final Exam to be graded, otherwise you will be considered absent and will need to get an excused absence approved by the Dean of Students to take a makeup exam.

- Please get on to the zoom session on time! You must finish when the class finishes, no matter what time you get to the test.

- Please read all questions carefully, and make sure you've answered what was asked of you.

- As this is a Final Exam, please make sure you number all questions, including sub-parts. I will not go hunting around trying to find answers!

- Remember that makeups are only allowed for excused absences (which require a note from the Dean of Students).