Physics 412—Assessment S–8 **Name:** Timothy Holmes

> **S–8:** I can explain the relationships between the energy eigenstates of the simple harmonic oscillator and use the eigenstates to make predictions about measurements.
>
> Unsatisfactory   Progressing   Acceptable   Polished

(1) The harmonic oscillator Hamiltonian can be written in terms of the raising and lowering operators as

$$H = \hbar\omega_0\left(a^\dagger a + \frac{I}{2}\right) = \hbar\omega_0\left(N + \frac{I}{2}\right), \quad \text{where} \quad N = a^\dagger a.$$

The raising and lowering operators have the following properties:

$$\left[a, a^\dagger\right] = I, \quad a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad N|n\rangle = n|n\rangle.$$

Explain, as best you can, how we know the coefficients above are $\sqrt{n}$ and $\sqrt{n+1}$. Where do these come from? Hint: Use the relationship between a, $a^\dagger$, and N, and the fact that the states $|n\rangle$ are eigenstates of N with eigenvalue n ...

$N = aa^\dagger$ $N|n\rangle = n|n\rangle$

$\langle n|N|n\rangle = \langle n|a^\dagger a|n\rangle$

$\langle n|n|n\rangle = (\langle n-1|\alpha^* \alpha|n-1\rangle)$

$n\langle n|n\rangle = |\alpha|^2\langle n-1|n-1\rangle$

$n = |\alpha|^2$

$\alpha = \sqrt{n}$

$a|n\rangle = \sqrt{n}|n-1\rangle$

$aa^\dagger = N+1$ $N|n\rangle = n|n\rangle$

$\langle n|(N+1)|n\rangle = (\langle N|aa^\dagger|n\rangle$

$\langle n|(n+1)|n\rangle = (\langle n+1|\alpha^*)(\alpha|n+1\rangle)$

$(n+1)\langle n|n\rangle = |\alpha|^2\langle n+1|n+1\rangle$

$\sqrt{n+1} = \alpha$

$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

(2) The momentum operator can be written in terms of the raising and lowering operators as

$$P = \frac{i\hbar}{\sqrt{2}d_0}\left(a^\dagger - a\right).$$

$$p^2 = \left[\frac{i\hbar}{\sqrt{2}d_0}(a^\dagger - a)\,\frac{i\hbar}{\sqrt{2}d_0}(a^\dagger - a)\right]$$

Let the state of a simple harmonic oscillator be

$$p^2 = -\frac{\hbar^2}{2d_0^2}\left(a^{\dagger 2} - a^\dagger a - a a^\dagger + a^2\right)$$

$$|\Psi\rangle = \frac{1}{5}\left(3|1\rangle + 4i|3\rangle\right).$$

Calculate the expectation value of the kinetic energy for this state, $\langle T\rangle = \langle P^2\rangle/2m$.

$a^\dagger a - a a^\dagger$ ← will make all go to ∅

$$\langle T\rangle = \langle P^2\rangle/2m$$

$$\langle P^2\rangle = \langle\Psi|P^2|\Psi\rangle = -\frac{\hbar}{2d_0^2}\frac{1}{25}\left[\left(+12i\langle 1|a^2|3\rangle - \right.\right.$$
$$\left.\left.(-12i\langle 3|a^{\dagger 2}|1\rangle\right)\right]$$

$$\langle\Psi|^* = \frac{1}{5}(3\langle 1| - 4i\langle 3|$$

$$-\frac{\hbar^2}{2d_0^2}\frac{1}{25}\frac{1}{2m}\left[12i\langle 1|\overset{a^2}{\cancel{}}|1\rangle + 12i\langle 3|\overset{a}{\cancel{}}|3\rangle\right]$$

$$\langle P^2\rangle = -\frac{\hbar^2\,24i}{100\,m}$$