Physics 412—Quantum Mice Mixed Ensembles — Winter 2020

One use for mixed ensembles is to describe the results of multiple measurements on a pure ensemble. If a measurement has multiple possible results, making the measurement on a pure ensemble of systems will convert the pure ensemble into a mixed ensemble. We'll explore this in the context of our quantum mice.

The four properties of our quantum mice are attitude (operator A, eigenstates $|h\rangle$ and $|u\rangle$), behavior (operator B, eigenstates $|p\rangle$ and $|a\rangle$), energy (operator H, eigenstates $|4\rangle$ and $|2\rangle$), and size (operator W, eigenstates $|s\rangle$ and $|l\rangle$). If we use the attitude states as our basis, we can represent the operators as matrices and the states as column vectors as follows:

$$
\begin{aligned}
A &\leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, & |h\rangle &\leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & |u\rangle &\leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
B &\leftrightarrow \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, & |p\rangle &\leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}, & |a\rangle &\leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix}, \\
H &\leftrightarrow \frac{2}{5} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}, & |4\rangle &\leftrightarrow \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, & |2\rangle &\leftrightarrow \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \\
W &\leftrightarrow \frac{2}{5} \begin{bmatrix} 21 & -8 \\ -8 & 9 \end{bmatrix}, & |s\rangle &\leftrightarrow \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, & |l\rangle &\leftrightarrow \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.
\end{aligned}
$$

Note that H and W share eigenstates because they commute. None of the other operators commute.

You can answer all of these working in the attitude basis, using the representation given above.

(1) Suppose that you have a pure ensemble of mice in the state $|\Psi\rangle = |h\rangle$.

(a) You measure the weight of the mice. What is the mixed ensemble that results from this measurement? Find the density matrix that describes this mixed ensemble.

(b) Use your density matrix to find the ensemble average of the behavior of the mice in the mixed ensemble from part (a). Compare your results to what you would calculate from the probabilities of measuring the behavior of the mice immediately after you measured their weight.

(c) Use your density matrix to find the ensemble average of the attitude of the mice in the mixed ensemble from part (a). Compare your results to what you would calculate from the probabilities of measuring the attitude of the mice immediately after you measured their weight.

(d) Use your density matrix to find the ensemble average of the energy of the mice in the mixed ensemble from part (a). Compare your results to what you would calculate from the probabilities of measuring the energy of the mice immediately after you measured their weight.

(2) Suppose that you wanted to make a mixed ensemble of mice with a "random" attitude (similar to the unpolarized beam of particles discussed in the course notes).

(a) What density operator (and density matrix) would describe this ensemble?

(b) What would be the ensemble averages for the attitude, behavior, energy, and weight for your random attitude mixed ensemble of mice? Interpret your results in light of the eigenvalues for the quantities and the fact that this is supposed to be a "random" ensemble of mice.

(c) Is there a pure ensemble that has the same average for measurements of attitude, behavior, energy, and weight as you found in part (b)? Explain.

1 (a) Initially, $|\psi\rangle = |h\rangle$

If we measure weight,

$$P(s) = |\langle s|h\rangle|^2 = \left|\frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right|^2 = \left|\frac{1}{\sqrt{5}}\cdot 1\right|^2 = \frac{1}{5}$$

$$P(\ell) = |\langle \ell|h\rangle|^2 = \left|\frac{1}{5}\begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right|^2 = \left|\frac{1}{\sqrt{5}}(-2)\right|^2 = \frac{4}{5}$$

After the measurement we have a mixed ensemble of $\frac{1}{5}$ small mice & $\frac{4}{5}$ large mice. The density operator for this mixed ensemble is

$$\rho = \frac{1}{5}|s\rangle\langle s| + \frac{4}{5}|\ell\rangle\langle \ell|$$

The density matrix is

$$\rho \leftrightarrow \frac{1}{5}\left(\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}\right)\left(\frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\end{bmatrix}\right) + \frac{4}{5}\left(\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\end{bmatrix}\right)\left(\frac{1}{\sqrt{5}}\begin{bmatrix}-2 & 1\end{bmatrix}\right)$$

$$\leftrightarrow \frac{1}{25}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} + \frac{4}{25}\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix} = \frac{1}{25}\begin{bmatrix}17 & -6\\ -6 & 8\end{bmatrix}$$

(b)

$$\mathrm{Tr}(\rho B) = \mathrm{Tr}\left(\frac{1}{25}\begin{bmatrix}17 & -6\\ -6 & 8\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\right) = \mathrm{Tr}\left(\frac{1}{25}\begin{bmatrix}-6i & -17i\\ 8i & 6i\end{bmatrix}\right)$$

$$= \frac{1}{25}(-6i + 6i) = 0 = \langle B\rangle$$

Compare to

$$\begin{tikzcd}
 & & {|p\rangle} \\
 & {|s\rangle} \arrow[ur] \arrow[r] & {|a\rangle} \\
{|h\rangle} \arrow[ur, "\frac{1}{5}"] \arrow[dr, "4/5"'] & & \\
 & {|e\rangle} \arrow[r] \arrow[dr] & {|p\rangle} \\
 & & {|a\rangle}
\end{tikzcd}$$

$|p\rangle$ prob $= |\langle p|s\rangle|^2 = \left|\frac{1}{\sqrt{10}}(1-2i)\right|^2 = \frac{1}{2}$

$|a\rangle$ prob $= |\langle a|s\rangle|^2 = \left|\frac{1}{\sqrt{10}}(1+2i)\right|^2 = \frac{1}{2}$

$|p\rangle$ prob $= |\langle p|e\rangle|^2 = \left|\frac{1}{\sqrt{10}}(-2-i)\right|^2 = \frac{1}{2}$

$|a\rangle$ prob $= |\langle a|e\rangle|^2 = \left|\frac{1}{\sqrt{10}}(-2+i)\right|^2 = \frac{1}{2}$

Overall $P(p) = \frac{1}{5}\cdot\frac{1}{2} + \frac{4}{5}\cdot\frac{1}{2} = \frac{1}{2}$

$P(a) = \frac{4}{5}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1}{2} = \frac{1}{2}$

$p \Rightarrow +1$, $a \Rightarrow -1$

$\Rightarrow \langle B\rangle = \frac{1}{2} - \frac{1}{2} = 0$ ✓

(c)

$$\mathrm{Tr}(\rho A) = \mathrm{Tr}\left(\frac{1}{25}\begin{bmatrix} 17 & -6 \\ -6 & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$

$$= \mathrm{Tr}\left(\frac{1}{25}\begin{bmatrix} 17 & 6 \\ -6 & -8 \end{bmatrix}\right) = \frac{1}{25}(17-8) = \frac{9}{25}$$

$$\overline{\langle A\rangle} = \frac{9}{25}$$

Compare to

$$|h\rangle \xrightarrow{\frac{1}{5}} |s\rangle \to |h\rangle \quad \text{prob} = |\langle s|h\rangle|^2 = \left|\frac{1}{\sqrt{5}}\, 1\right|^2 = \frac{1}{5}$$

$$|s\rangle \to |u\rangle \quad \text{prob} = |\langle s|u\rangle|^2 = \left|\frac{1}{\sqrt{5}}\, 2\right|^2 = \frac{4}{5}$$

$$|h\rangle \xrightarrow{4/5} |\ell\rangle \to |h\rangle \quad \text{prob} = |\langle \ell|h\rangle|^2 = \left|\frac{1}{\sqrt{5}}(-2)\right|^2 = \frac{4}{5}$$

$$|\ell\rangle \to |u\rangle \quad \text{prob} = |\langle \ell|u\rangle|^2 = \left|\frac{1}{\sqrt{5}}\, 1\right|^2 = \frac{1}{5}$$

Overall,

$$P(h) = \frac{1}{5}\cdot\frac{1}{5} + \frac{4}{5}\cdot\frac{4}{5} = \frac{17}{25}$$

$$P(u) = \frac{1}{5}\cdot\frac{4}{5} + \frac{4}{5}\cdot\frac{1}{5} = \frac{8}{25}$$

$h \longrightarrow +1$, $u \longrightarrow -1$ $\qquad \langle A\rangle = \frac{17}{25} - \frac{8}{25} = \frac{9}{25}$

2 a "Random" mixed ensemble:

$$\rho = \frac{1}{2}|h\rangle\langle h| + \frac{1}{2}|u\rangle\langle u|$$

$$\rho \leftrightarrow \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$

$$\leftrightarrow \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

b

$$Tr(\rho B) = Tr\left(\frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\right) = Tr\left(\frac{1}{2}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\right) = 0$$

$$\overline{\langle B\rangle} = 0$$

$$Tr(\rho A) = Tr\left(\frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right) = Tr\left(\frac{1}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right) = 0$$

$$\overline{\langle A\rangle} = 0$$

$$Tr(\rho H) = Tr\left(\frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\frac{2}{5}\begin{bmatrix}6 & 2\\2 & 9\end{bmatrix}\right) = Tr\left(\frac{1}{5}\begin{bmatrix}6 & 2\\2 & 9\end{bmatrix}\right) = 3$$

$\overline{\langle H\rangle} = 3 =$ average of 4 & 2

$$Tr(\rho W) = Tr\left(\frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\frac{2}{5}\begin{bmatrix}21 & -8\\-8 & 9\end{bmatrix}\right) = Tr\left(\frac{1}{5}\begin{bmatrix}21 & -8\\-8 & 9\end{bmatrix}\right) = 6$$

$\overline{\langle W\rangle} = 6 =$ average of 2 & 10

(c) $|\psi\rangle \leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix}$ In general a & b are both complex.

Pick $a = \text{real}$

let $b = re^{i\theta}$

$|\psi\rangle \leftrightarrow \begin{bmatrix} a \\ re^{i\theta} \end{bmatrix}$ a & r are real & positive

also $\boxed{a^2 + r^2 = 1}$

$$\langle\psi|A|\psi\rangle = \begin{bmatrix} a & re^{-i\theta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ re^{i\theta} \end{bmatrix}$$

$$= \begin{bmatrix} a & re^{-i\theta} \end{bmatrix} \begin{bmatrix} a \\ -re^{i\theta} \end{bmatrix} = a^2 - r^2 = 0$$

$$\boxed{a = r}$$

Check behavior:

$$\langle\psi|B|\psi\rangle = \begin{bmatrix} a & re^{-i\theta} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ re^{i\theta} \end{bmatrix}$$

$$= \begin{bmatrix} a & re^{-i\theta} \end{bmatrix} \begin{bmatrix} -ire^{i\theta} \\ ia \end{bmatrix} = iar(-e^{i\theta} + e^{-i\theta})$$

$$= 2ar\sin\theta$$

$\langle B\rangle = 0$ if $\theta = 0$ or $\theta = \pi$

So the state must be $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

in order for $\langle A\rangle = \langle B\rangle = 0$

Now check weight:

$$\langle w \rangle = \frac{1}{4}\frac{2}{5}\begin{bmatrix} 1 & \pm 1 \end{bmatrix}\begin{bmatrix} 21 & -8 \\ -8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 1 & \pm 1 \end{bmatrix}\begin{bmatrix} 21 \mp 8 \\ -8 \pm 9 \end{bmatrix}$$

so, with $+1$, $\langle w \rangle = \frac{1}{10}(13 + 1) = 1.4$

with -1, $\langle w \rangle = \frac{1}{10}(29 + 17) = 4.6$

Neither match mixed ensemble