PHY 411 – Electrodynamics I

Winter 2021

In Preparation for the Midterm Examination

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Warning!

This is *not a substitute* for class summaries and homework solutions. It is only a list of ideas and/or equations I want you to remember. If this is the only thing you look at, expect to do very badly on the exam!

Caveat: In the following slides, I've mentioned several relations that will be supplied on formula sheets. To avoid any confusion, I'd like to add here that *an exception would be if I asked you to derive any of these relations on the test, in which case, they would obviously not be on the Formula Sheet*.

Maxwell's Equations

- $\vec{\nabla} \cdot \vec{D} = \rho$
- $\vec{\nabla} \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}$
- $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$
- $\vec{\nabla} \cdot \vec{B} = 0$

I will supply on Formula Sheet for Midterm, but expect you to learn them eventually.

Expect you to remember the constitutive relations:

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

I expect this is not a big deal this year, since the exam is open book. Still, I'd advise knowing these, since time could be a factor otherwise.

See Class Summaries for more details

Maxwell's Equations in source-free space

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} - i\omega\vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \vec{\nabla} \times \vec{B} + i\omega\mu\epsilon\vec{E} = 0$$

You should know how to derive these from the supplied Maxwell equations.

See Class Summaries and Worksheets for detailed discussions, especially on deriving these.

Helmholtz Wave Equation

$$\left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{E} = 0$$

$$\left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{B} = 0 \qquad (7.3)$$

I will supply as

$$\left(\nabla^2 + k^2\right)\vec{u} = 0$$

Expect you to know

$$k = \omega\sqrt{\mu\epsilon}$$

Jackson, equation (7.5)

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}, \qquad \text{where } n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

This is a key set of relationships. Make sure you learn it, and remember it well!

How do you remember? It's enough to remember that $k = \omega\sqrt{\mu\epsilon}$

Plane Electromagnetic Waves

Plane Wave Solutions:

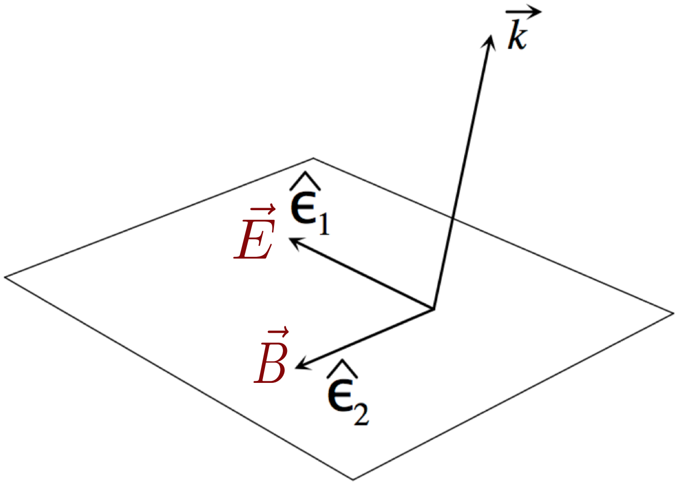
$$\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\vec{B} = \sqrt{\mu\epsilon}\, \frac{\vec{k} \times \vec{E}}{k}$$

$\vec{B}$ will be on the Formula Sheet, but you must be able to write $\vec{E}$ ***by yourself***, especially in 1-D: see HW 4

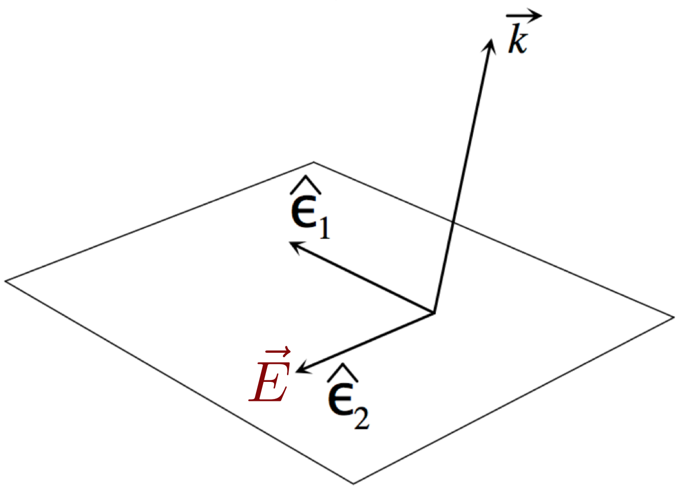
Polarization of Waves

$$\vec{E}_1 = \hat{\epsilon}_1\, E_1\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$\vec{E}_2 = \hat{\epsilon}_2\, E_2\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.18)$$



most general homogenous plane wave

$$\vec{E}(\vec{x},t) = \left(\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.19)$$



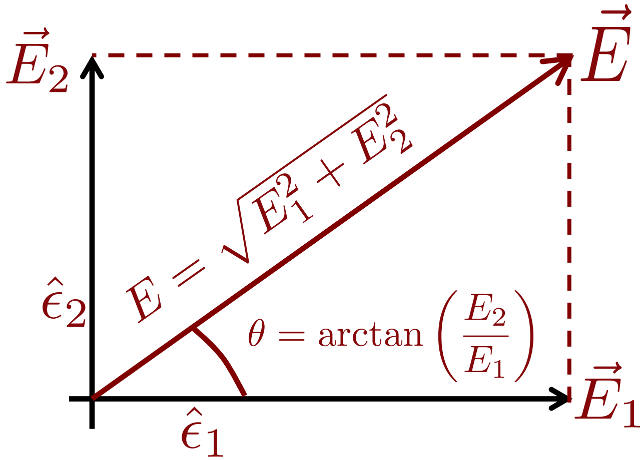
You must know how to write these – there is nothing new to memorize here since you'd be able to write these if you knew how to write a plane wave. However, you must understand why you can represent waves in this manner.

Polarization of Waves

$$\vec{E}_1 = \hat{\epsilon}_1 \, E_1 \, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$\vec{E}_2 = \hat{\epsilon}_2 \, E_2 \, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad (7.18)$$

$$\vec{E}(\vec{x},t) = \Big(\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2\Big) \, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad (7.19)$$

Linear Polarization:



Again, there is nothing new to memorize here. However, you must understand what is a linearly polarized wave and how it is represented in this formalism.

Likewise for elliptically and circularly polarized waves on the next slide.

For details, see Class Summaries.

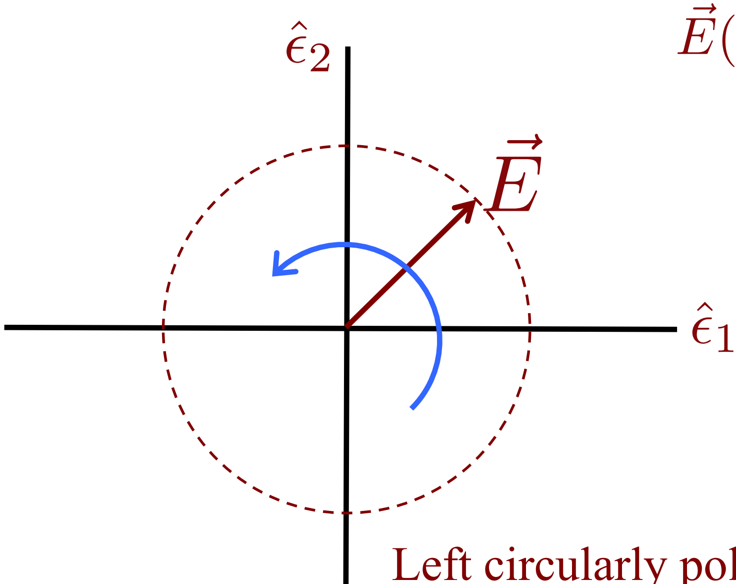
Polarization of Waves

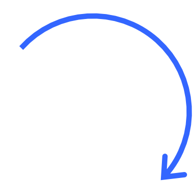
Must know everything on this page!

Elliptical Polarization: E_1 and E_2 have different phases

Circular Polarization: E_1 and E_2 are equal in magnitude but differ in phase by π/2

$$\vec{E}(\vec{x}, t) = E_0 \left(\hat{\epsilon}_1 \pm i \hat{\epsilon}_2 \right) e^{i(\vec{k}\cdot\vec{x} - \omega t)} \qquad (7.20)$$



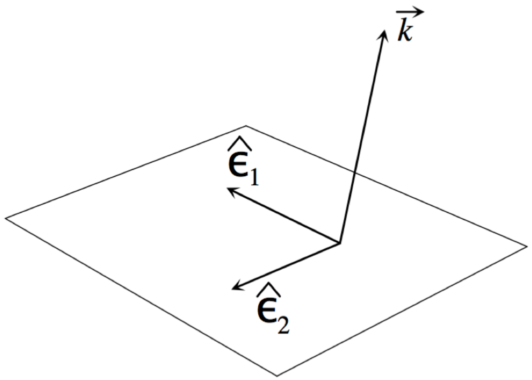


Right circularly polarized $\left(\hat{\epsilon}_1 - i \hat{\epsilon}_2 \right)$

Left circularly polarized $\left(\hat{\epsilon}_1 + i \hat{\epsilon}_2 \right)$

Polarization: Another set of basis vectors

$$\hat{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}\left(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2\right) \qquad (7.22)$$



$$\vec{E}(\vec{x},t) = \left(E_+\hat{\epsilon}_+ + E_-\hat{\epsilon}_-\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.24)$$

$$\vec{E}(\vec{x},t) = \left(\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.19)$$

I don't expect you to memorize how the new set of basis vectors is constructed – I'll provide that on the test, if there is a problem/question on it. But I do expect you to be able to appreciate that they are a completely equivalent set of basis vectors to describe polarization.

Stokes Parameters

Linear polarization basis:

$$\vec{E}(\vec{x},t) = \left(\hat{\epsilon}_1 \underset{\downarrow}{\boxed{E_1}} + \hat{\epsilon}_2 \underset{\downarrow}{\boxed{E_2}}\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.19)$$

$$E_1 \rightarrow a_1 e^{i\delta_1} \qquad E_2 \rightarrow a_2 e^{i\delta_2}$$

Can then write Stokes parameters s_0, s_1, s_2, s_3, in linear polarization basis

- See eq. (7.27) on page 301

Circular polarization basis:

$$\vec{E}(\vec{x},t) = \left(\boxed{E_+}\hat{\epsilon}_+ + \boxed{E_-}\hat{\epsilon}_-\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (7.24)$$

$$E_+ \rightarrow a_+ e^{i\delta_+} \qquad E_- \rightarrow a_- e^{i\delta_-}$$

Can then write Stokes parameters s_0, s_1, s_2, s_3, in circular polarization basis

- See eq. (7.28) on page 301

Reflection and Refraction: Kinematic Properties

Incident wave:

$$\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$\vec{B} = \sqrt{\mu\epsilon}\,\frac{\vec{k}\times\vec{E}}{k}$$

Refracted wave:

$$\vec{E}' = \vec{E}'_0\, e^{i(\vec{k}'\cdot\vec{x}-\omega t)}$$

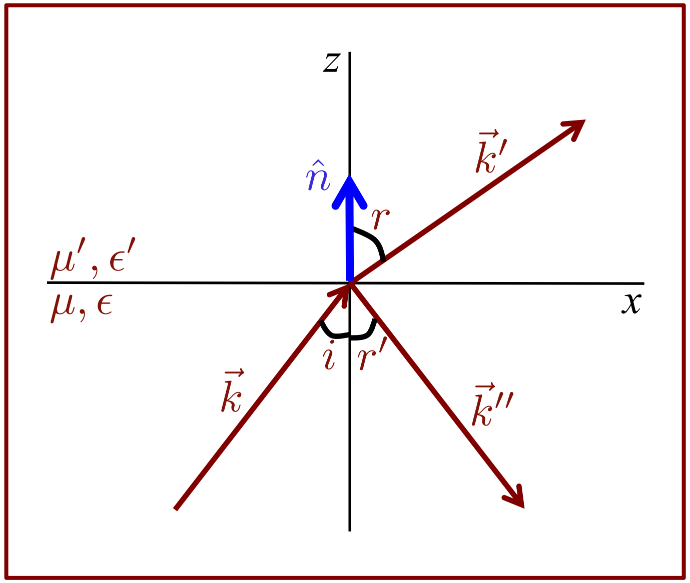
$$\vec{B}' = \sqrt{\mu'\epsilon'}\,\frac{\vec{k}'\times\vec{E}'}{k'}$$

Reflected wave:

$$\vec{E}'' = \vec{E}''_0\, e^{i(\vec{k}''\cdot\vec{x}-\omega t)}$$

$$\vec{B}'' = \sqrt{\mu\epsilon}\,\frac{\vec{k}''\times\vec{E}''}{k}$$

You must remember and be able to write E and B for incident, refracted and reflected waves (especially in 1-D): see HW 4.



Boundary conditions must be satisfied at all points on the interface $z = 0$ at all times

$$(\vec{k}\cdot\vec{x})_{z=0} = (\vec{k}'\cdot\vec{x})_{z=0} = (\vec{k}''\cdot\vec{x})_{z=0} \qquad (7.34)$$

Must know everything else on this page.

$$k \sin\, i = k' \sin\, r = k'' \sin\, r' \qquad (7.35)$$

- All three wave vectors lie in the same plane
- Law of reflection: angle i = angle r'
- Snell's Law: $n \sin\, i = n' \sin\, r$

Reflection and Refraction: Dynamic Properties

Boundary conditions:

$$\left[\epsilon\left(\vec{E}_0 + \vec{E}_0''\right) - \epsilon'\,\vec{E}_0'\right]\cdot\hat{n} = 0 \longrightarrow \text{(7.37.a)}$$

$$\left[\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0'' - \vec{k}'\times\vec{E}_0'\right]\cdot\hat{n} = 0 \longrightarrow \text{(7.37.b)}$$

Must know:

Normal components of $\vec{D}$ and $\vec{B}$ are continuous

$$\left[\vec{E}_0 + \vec{E}_0'' - \vec{E}_0'\right]\times\hat{n} = 0 \longrightarrow \text{(7.37.c)}$$

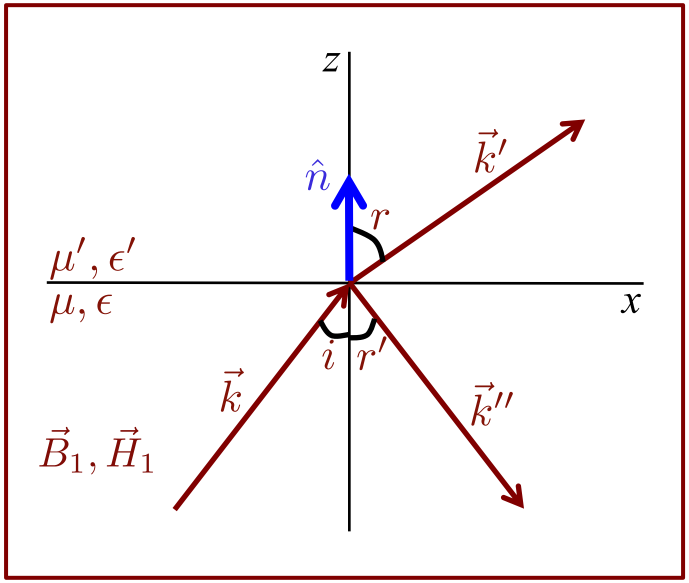
$$\left[\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0''\right) - \frac{1}{\mu'}\left(\vec{k}'\times\vec{E}_0'\right)\right]\times\hat{n} = 0 \longrightarrow \text{(7.37.d)}$$

Tangential components of $\vec{E}$ and $\vec{H}$ are continuous

I don't expect you to memorize the equations above (it would be useless), but I do expect that you will be able to *construct/derive* them as needed by remembering whose normal components are continuous, and whose tangential components are continuous. See Class Summaries and HW 3.

The way ahead: Break into 2 problems

- $\vec{E}$ perpendicular to plane of incidence
- $\vec{E}$ parallel to plane of incidence



Jackson Figure 7.6 (a) on page 305

Reflection and Transmission Coefficients

$$\boxed{\vec{S}\cdot\hat{n} == \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left|E_0\right|^2 \cos\, i}$$

$$\vec{S}'\cdot\hat{n} = \frac{1}{2}\sqrt{\frac{\epsilon'}{\mu'}}\left|E_0'\right|^2 \cos\, r$$

$$\vec{S}''\cdot\hat{n} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left|E_0''\right|^2 \cos\, r'$$

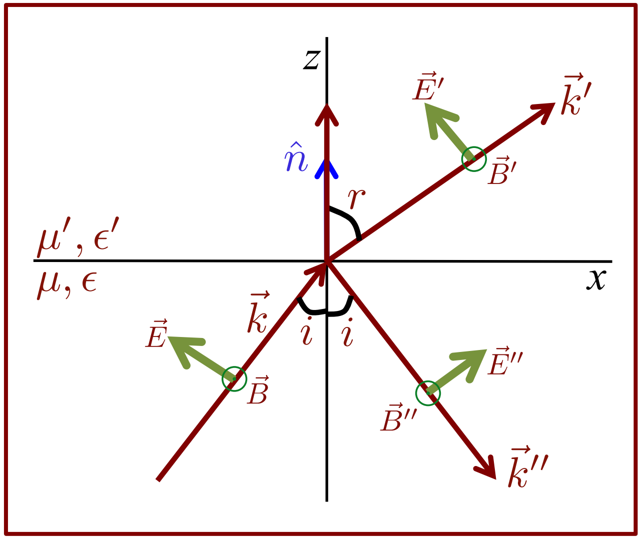
$$T = \frac{\vec{S}'\cdot\hat{n}}{\vec{S}\cdot\hat{n}}$$

$$R = \frac{\vec{S}''\cdot\hat{n}}{\vec{S}\cdot\hat{n}}$$

You must remember and be able to write these. See Class Summaries and HW 3 for how to use them.

This generic version will be supplied on Formula Sheet.
You must be able to build the other two by analogy.

Do not memorize derived transmission and reflection coefficients for perpendicular and parallel cases. You must obtain them, if there is such a problem, by starting from scratch (i.e., by writing E and B fields).



Brewster's Angle

For $\vec{E}$ parallel to the plane of incidence

$$\frac{E_0''}{E_0} = \frac{\boxed{\frac{\mu}{\mu'} n'^{\,2} \cos i - n\sqrt{n'^{\,2} - n^2 \sin^2 i}}}{\frac{\mu}{\mu'} n'^{\,2} \cos i + n\sqrt{n'^{\,2} - n^2 \sin^2 i}} \qquad = 0$$

I don't expect you to memorize the equation above (it would be useless), but I do expect that you will know about the Brewster angle and how it is calculated (see below), and understand how the reflection coefficient curves will look (see HW 3).

The amplitude of the reflected wave will be zero

The angle of incidence for which the amplitude of the reflected wave will be zero is given by:

$$i_B = \arctan\left(\frac{n'}{n}\right)$$

Total Internal Reflection

If $n > n'$, then Snell's law tells us there is an i_0 for which $r = 90°$.

$$\sin\, i_0 = \frac{n'}{n} \quad \Rightarrow \quad i_0 = \arcsin\left(\frac{n'}{n}\right) \qquad (7.43)$$

What happens if $i > i_0$?

- $\sin\, r = \left(\frac{n}{n'}\right)\sin\, i = \frac{\sin\, i}{\sin\, i_0} \quad \Rightarrow \quad \sin\, r > 1$

- $\cos\, r = i\sqrt{\left(\frac{\sin\, i}{\sin\, i_0}\right)^2 - 1}$

  Caution! i = imaginary (if in blue-colored font)

- $e^{i\vec{k}'\cdot\vec{x}} = e^{ik'(x\sin\, r + z\cos\, r)} = \underbrace{e^{-k'\left(\sqrt{(\sin\, i/\sin\, i_0)^2-1}\right)z}}_{\text{The refracted wave is attenuated exponentially beyond the interface } z = 0} e^{ik'(\sin\, i/\sin\, i_0)\,x}$

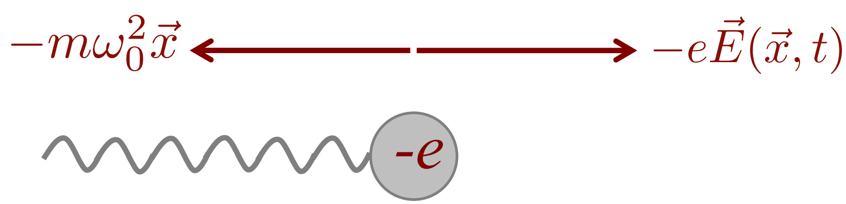
I expect you to remember and be able to apply Snell's law to find the critical angle of incidence for total internal reflection, and derive what happens to the refracted wave, as done on this page

For $i > i_0$, get Total Internal Reflection

Dispersion
Model for time-varying fields

$$m\left[\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}\right] = -e\vec{E}(\vec{x},t) \qquad (7.49)$$

I expect you to understand how this, and the static model before it, are set up. See Class Summaries for details.



$$\sum F \equiv -e\vec{E} - m\omega_0^2\vec{x} = \quad m\ddot{\vec{x}}$$

Electron interacts with other electrons, losing energy; so account for this by introducing a damping term

$$-m\gamma\dot{\vec{x}}$$

To be supplied, if needed:

$$\epsilon = \epsilon_0\left(1 + \chi_e\right)$$

$$\vec{P} = \epsilon_0\,\chi_e\,\vec{E}$$

$$\vec{p}_{\rm mol} = e\vec{x}$$

$$\vec{P} = N\left\langle \vec{p}_{\rm mol}\right\rangle$$

Dispersion

Complex Dielectric Constant:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\boxed{N} e^2}{\epsilon_0 m} \sum_j \frac{\boxed{f_j}}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \tag{7.51}$$

Number of molecules per unit volume

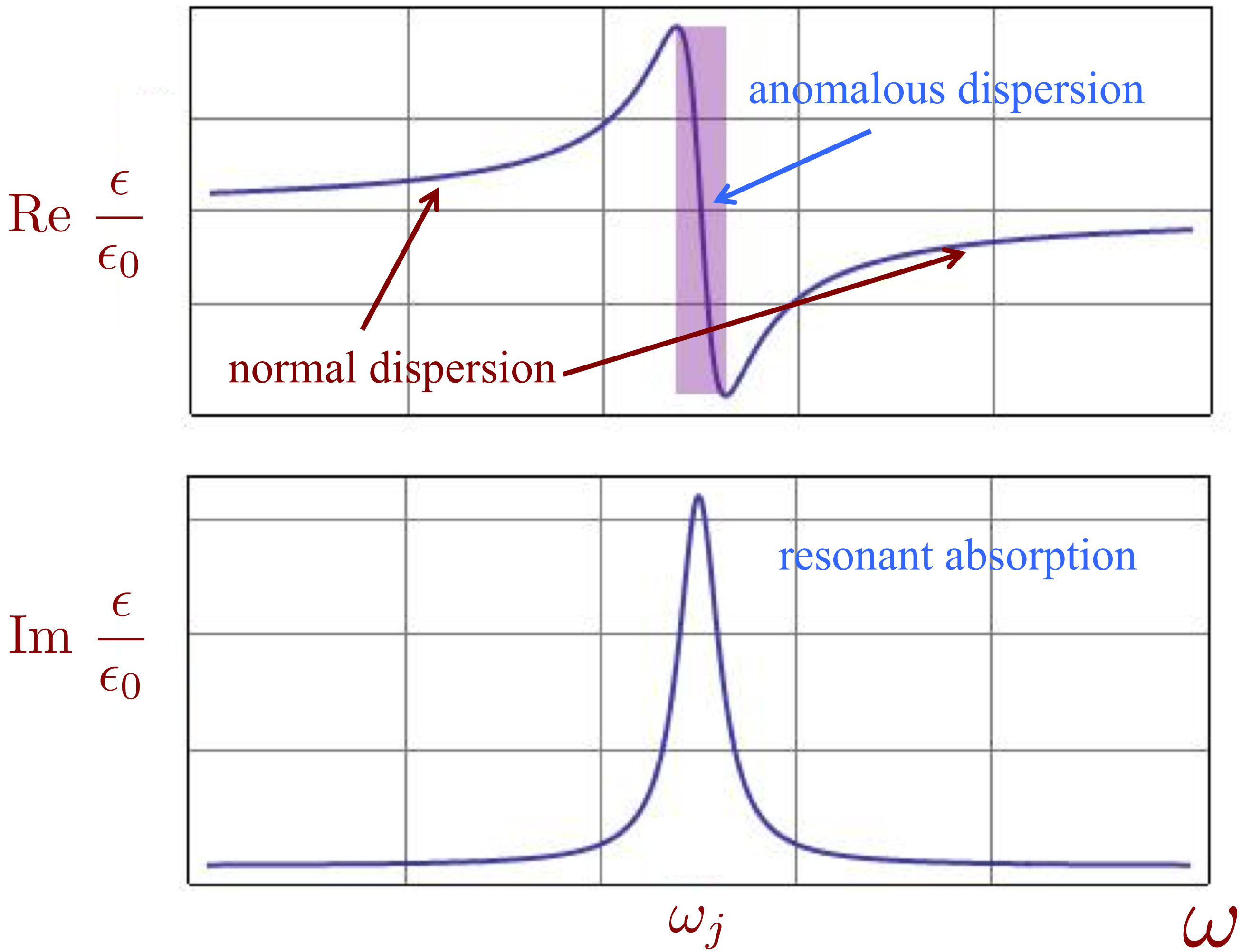
Number of electrons per molecule with binding frequency ω_j

I don't expect you to memorize this equation, but you must understand it enough to be able to explain what all the terms mean.

You must also understand the consequences of a complex dielectric constant, one example being the graph on the next page, another being the attenuation of a plane wave as it propagates.

Read the Class Summaries and refer to the Worksheets for details.

Normal and Anomalous Dispersion



Source: *Wolfram*

Attenuation of a plane wave

What is the most appropriate quantity to pick in order to describe the attenuation of a plane wave?

Complex $\epsilon(\omega)$ ⇨ Complex $n(\omega)$ ⇨ Complex k

Be able to express k in this form, and then be aware of what you get, as shown in the expression below, and the boxes in this slide.

$$k = \beta + i\frac{\alpha}{2} \qquad (7.53)$$

$$\vec{E}(\vec{x}) \sim e^{ikz} = e^{-\frac{\alpha}{2}z} e^{i\beta z}$$

intensity of the wave ($|\vec{E}|^2$) falls off as $e^{-\alpha z}$

α is called the *attenuation constant* or *absorption coefficient*

Affects traveling part of the wave; its dependence on frequency means that waves with different frequencies travel at different speeds

Low Frequency Behavior

In the limit $\omega \rightarrow 0$, there is a qualitative difference in the response of the medium depending on whether or not there is a resonance at zero.

- If a resonance does not exist at $\omega_i = 0$ (i.e., the lowest resonant frequency is different from zero), we have a dielectric insulator whose details match what we wrote while discussing the static case.
- If there is a resonance $\omega_0 = 0$, then $\epsilon(\omega)$ has a complex component that attenuates the propagation of electromagnetic energy. We know now that this describes conduction.

Suppose some fraction f_0 of the electrons per molecule have their lowest resonance frequency at $\omega_0 = 0$.

$\Downarrow$

"free" electrons

Low Frequency Behavior

Suppose some fraction f_0 of the electrons per molecule have their lowest resonance frequency at $\omega_0 = 0$.

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \left[\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \right] \tag{7.51}$$

$$\underbrace{\epsilon(\omega) = \epsilon_b(\omega)}_{\text{"bound" dipoles}} + \underbrace{i \left[\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \right]}_{\text{"free" electrons}} \tag{7.56}$$

I don't expect you to memorize (7.56) but you must understand where it came from, and what it means. See the Class Summaries and Worksheets for details of this and the previous slide.

Low Frequency Behavior

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial (\epsilon \vec{E})}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = -i\omega \left(\epsilon_b + i\,\frac{\sigma}{\omega} \right) \vec{E}$$

$$\vec{\nabla} \times \vec{H} = -i\omega \left[\epsilon_b + i \left\{ \frac{N e^2 f_0}{m\omega(\gamma_0 - i\omega)} \right\} \right] \vec{E}$$

Drude model for electrical conductivity

$$\sigma = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)} \qquad (7.58)$$

Tells us that the "conductivity" is closely related to the complex dielectric constant when the lowest resonant frequency is zero.

Nothing to memorize here, but be aware of what you need to do to get to the Drude model (as you did on the Discussion Worksheet).

The High Frequency Limit

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \left[\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \right] \qquad (7.51)$$

- At frequencies >> highest resonant frequency

$$\frac{\epsilon(\omega)}{\epsilon_0} \simeq 1 - \frac{\omega_P^2}{\omega^2} \qquad (7.59)$$

I don't expect you to memorize (7.59), but you must know where it came from and how to obtain it, and hence to find (7.60). You must also understand the consequences of this limit, as described on this slide and in the class summary.

where

$$\omega_P^2 = \frac{NZe^2}{\epsilon_0 m} \qquad (7.60)$$

Plasma frequency

Number of molecules per unit volume

Number of electrons per molecule

NZ = number of electrons per unit volume

The High Frequency Limit

- At frequencies >> highest resonant frequency

$$\frac{\epsilon(\omega)}{\epsilon_0} \simeq 1 - \frac{\omega_P^2}{\omega^2} \qquad (7.59)$$

where

$$\omega_P^2 = \frac{NZe^2}{\epsilon_0 m} \qquad (7.60)$$

In particular, you must understand that starting from (7.59) and (7.60), you can obtain a dispersion relation, as written below.

- Wave number *k* in this limit:

$$ck = \sqrt{\omega^2 - \omega_p^2} \qquad (7.61)$$

sometimes written as

$$\omega^2 = \omega_p^2 + c^2 k^2 \longrightarrow$$ dispersion relation for $\omega = \omega(k)$

The High Frequency Limit

- At frequencies >> highest resonant frequency

$$\frac{\epsilon(\omega)}{\epsilon_0} \simeq 1 - \frac{\omega_P^2}{\omega^2} \qquad (7.59)$$

- In dielectric media, (7.59) holds only for $\omega^2 \gg \omega_p^2$

- In certain situations (ionosphere, tenuous lab plasmas) where all electrons are essentially free so that damping is negligible, (7.59) can hold for a wide range of frequencies, including $\omega < \omega_p$

But if $\omega < \omega_p$, then wave number k is purely imaginary.

$$ck = \sqrt{\omega^2 - \omega_p^2} \qquad (7.61)$$

- Such waves incident on a plasma are reflected and the fields inside fall off exponentially with distance from the surface.

Attenuation constant at $\omega = 0$

$$\alpha_{\text{plasma}} \simeq \frac{2\omega_p}{c} \qquad (7.62)$$

The High Frequency Limit: Metals

- The reflectivity of metals at optical and higher frequencies is caused by essentially the same behavior as for the tenuous plasma

$$\epsilon(\omega) = \epsilon_b(\omega) + i\left[\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}\right] \qquad (7.56)$$

You must know about these results.

- At frequencies for which $\omega \gg \gamma_0$

$$\epsilon(\omega) \simeq \epsilon_b(\omega) - \frac{\omega_p^2}{\omega^2}\epsilon_0 \qquad (7.59.a)$$

$$\Downarrow$$

$k = \omega\sqrt{\mu\epsilon}$ is imaginary

- Same behavior as plasma for $\omega \ll \omega_p$ $\longrightarrow$ Light penetrates only a short distance into the metal and is almost completely reflected.

The High Frequency Limit: Metals

$$\vec{E} \sim e^{i(\underbrace{k}_{\beta + i\alpha/2} z - \omega t)}$$

$$\vec{E} \sim e^{-\underbrace{\boxed{\frac{\alpha}{2}}}_{\boxed{\text{dimensions of 1/length}}} z} e^{\cdots}$$

- Therefore, the penetration into the metal is characterized by a parameter known as the *skin depth* δ, given by

$$\delta = \frac{2}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Don't memorize, but you must be able to derive such an expression for the skin depth.

- However, when the frequency is increased into the domain where the dielectric constant becomes positive, which is typically in the ultraviolet, metals are suddenly able to transmit light, and become transparent.

Propagation through Dispersive Media

- There are no truly monochromatic waves!
- But we can superpose monochromatic plane-wave solutions
- If medium is dispersive (i.e., dielectric constant is a function of frequency), then the phase velocity is no longer the same for each frequency component of the wave. As a result, different components of the wave travel with different speeds and tend to change phase with respect to one another.

Must remember the expressions for phase and group velocity.

$$v_p = \frac{\omega}{k} \qquad v_g = \frac{d\omega}{dk}$$

However, building up superposition by Fourier series is not needed.

Not included on the Midterm

Midterm includes material up to Tuesday (Feb 2).

What is NOT included on the Midterm?

- Vector and Scalar Potentials, and Green functions, discussed in class on Thursday (Feb 4) are NOT included on the Midterm.

Types of Questions

Here are some examples of short questions you might expect to see (taken from past exams).

- What is the name given to ω/k, and what information do you get from it?

- What special quantity can be constructed from the product of the permittivity and permeability in a vacuum?

- Consider an electromagnetic wave with

$$\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

What is $\vec{k} \cdot \vec{E}_0$ equal to, and why?

- For a wave polarized parallel to the plane of incidence, there is a special angle of incidence for which there is no reflected wave. What is the name by which this angle of incidence is known?

Types of Questions

Here are some examples of long questions you might expect to see.

- Homework 2, Question 1

- Shorter version of Homework 4, Question 1 (e.g., I might supply the incident, reflected, and transmitted *E* and *B* fields, and ask you to match boundary conditions and derive *R* or *T*).