PHY 474/374 Spring 2021

Week 4—Monday, April 19—Discussion Worksheet

Convection

Models of energy transport involving only radiation are unstable. If temperature decreases too rapidly as we go outward from the star, we can have instability. This leads to convection as gas parcels are transported outward due to density differences, which then becomes an alternative source of energy transport.

1. Consider a parcel of gas moved outward by a distance Δr. Before the motion, the pressure and density inside this parcel are P_1^* and ρ_1^* respectively, whereas outside the parcel, they are P_1 and ρ_1. After the motion, they are P_2^* and ρ_2^* inside the parcel, and P_2 and ρ_2 outside.

(a) Show that the buoyant force per unit volume at location 2 is given by

$$f_{\text{buoy}} = -g\,\Delta\rho$$

where $g = Gm/r^2$ is the gravitational acceleration, and $\Delta\rho = \rho_2^* - \rho_2$.

$P_1^* = P_1$, $\rho_1^* = \rho_1^*$

- Difference in density between the element ρ_2^* and its surroundings ρ_2.

$$f_{buoy} = -g(\rho_2^* - \rho_2) = -g\Delta\rho$$

(b) If $f_{\text{buoy}} > 0$, in which direction is the force on the parcel of gas, upward (outward) or downward (inward)? Thus, which would lead to instability, $f_{\text{buoy}} > 0$ or $f_{\text{buoy}} < 0$?

$f_{bouy} > 0 \rightarrow$ upward (outward)

$f_{bouy} < 0 \rightarrow$ downward (inward)

$\rightarrow$ would cause instability, parcels are returning to the star

In order to determine $\Delta\rho$, we make two assumptions: (1) that the motion is slow enough that there is pressure balance between the element and the surroundings, so that we have $P_2^* = P_2$; and (2) the motion is fast enough that there is no heat loss to the surroundings.

2. The second assumption at the bottom of the previous page tells us that the motion must proceed *adiabatically*, so that

$$\frac{d\rho^*}{\rho^*} = \frac{1}{\Gamma_1}\frac{dP^*}{P^*} \qquad \text{where} \qquad \Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_s \tag{1}$$

and Γ_1 is known as the adiabatic exponent.

(a) Show that

$$\Delta\rho = \left[\left(\frac{d\rho}{dr}\right)_{\rm ad} - \frac{d\rho}{dr}\right]\Delta r \qquad \text{where} \qquad \left(\frac{d\rho}{dr}\right)_{\rm ad} \equiv \frac{1}{\Gamma_1}\frac{\rho}{P}\frac{dP}{dr}$$

where $(d\rho/dr)_{\rm ad}$ is the density gradient from adiabatic motion in the given pressure gradient.

$$\Delta\rho = \rho_2^* - \rho_2 = \rho_2^* - \rho_1 - (\rho_2 - \rho_1)$$

$$= \underbrace{\rho_1 \frac{1}{\Gamma_1}\frac{1}{P_1}\frac{dP}{dr}}_{\left(\frac{d\rho}{dr}\right)_{ad}}\Delta r - \frac{d\rho}{dr}\Delta r$$

$$= \left(\frac{d\rho}{dr}\right)_{ad}\Delta r - \frac{d\rho}{dr}\Delta r$$

$$= \left[\left(\frac{d\rho}{dr}\right)_{ad} - \frac{d\rho}{dr}\right]\Delta r$$

(b) Explain why $\Delta\rho < 0$ is the criterion for instability, and hence write down the instability criterion in terms of the density gradients in the expression for $\Delta\rho$.

$$\rightarrow \frac{\Delta\rho}{\Delta r} < \left(\frac{d\rho}{dr}\right)_{ad} - \frac{d\rho}{dr}$$

$$\rightarrow \cancel{(-c)}\frac{d\rho}{dr} > \left(\frac{d\rho}{dr}\right)_{ad}$$

$$\rightarrow \frac{d\rho}{dr} > \left(\frac{d\rho}{dr}\right)_{ad}$$

3. The instability criterion is usually expressed in terms of the temperature gradient.

(a) Using the ideal gas law in the form $P = \rho kT/\mu m_p$, so that $\rho = \mu m_p P/kT$, and assuming that the chemical composition is independent of position and also that the gas is fully ionized so that μ is constant, show by direct differentiation that

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{P}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr} \tag{2}$$

$$\frac{d\rho}{dr} = \frac{\mu m_p}{k}\left[\frac{1}{T}\frac{dP}{dr} + P\left(-\frac{1}{T^2}\right)\frac{dT}{dr}\right]$$

$$= \frac{\mu m_p}{kT}\left[\frac{dP}{dr} + P/T\,\frac{dT}{dr}\right]$$

mult by $1/\rho$

$$\rightarrow \frac{1}{\rho}\frac{d\rho}{dr} = \left(\frac{k\cancel{T}}{\cancel{\mu m_p}\rho}\right)\frac{\mu \cancel{m_p}}{\cancel{kT}}\left[\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr}\right]$$

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{\rho}\frac{d\rho}{dr} - \frac{1}{T}\frac{dT}{dr}$$

(b) Then, show that

$$\left(\frac{d\rho}{dr}\right)_{\text{ad}} - \frac{d\rho}{dr} = -\frac{\Gamma_1 - 1}{\Gamma_1}\frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr} \tag{3}$$

$$= \frac{1}{\Gamma_1}\frac{\rho}{P}\frac{d\rho}{dr} - \rho\left[\frac{1}{\rho}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr}\right]$$

$$= \frac{\rho}{P}\left[\frac{1}{\Gamma_1} - 1\right]\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr}$$

$$= \frac{\rho}{P}\left(\frac{1-\Gamma_1}{\Gamma_1}\right)\frac{d\rho}{dr} + \frac{\rho}{T}\frac{dT}{dr} \Rightarrow \left(\frac{d\rho}{dr}\right)_{ad} - \frac{d\rho}{dr}$$

$$= -\left(\frac{\Gamma_1 - 1}{\Gamma_1}\right)\frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr}$$

4. A correct thermodynamical treatment that takes into account partial ionization and departures from the ideal gas law will show that Γ_1 in equation (3) must be replaced by Γ_2, the adiabatic exponent for the relation between P and T, defined in equation (3.19) as

$$\left(\frac{\partial \ln P}{\partial \ln T}\right)_s = \frac{\Gamma_2}{\Gamma_2 - 1}$$

(a) Show that the instability condition becomes

$$\left(\frac{dT}{dr}\right)_{\rm ad} > \frac{dT}{dr} \qquad \text{where} \qquad \left(\frac{dT}{dr}\right)_{\rm ad} \equiv \frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}\frac{dP}{dr} \tag{4}$$

is the adiabatic temperature gradient.

$$\frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}\frac{dP}{\cancel{dr}} > \frac{dT}{\cancel{dr}} \rightarrow \frac{T\,dP}{P\,dT} = \frac{\Gamma_2}{\Gamma_2 - 1}$$

$$\rightarrow \left(\frac{\partial \ln P}{\partial \ln T}\right)_s = \frac{\Gamma_2}{\Gamma_2 - 1}$$

(b) Show that we can also write the expression *on the left above* in equation (4) as

$$\frac{d\ln T}{d\ln P} > \frac{\Gamma_2 - 1}{\Gamma_2} \tag{5}$$

Note: Remember that the inequality must be reversed if you multiply or divide by a negative number, and the pressure gradient dP/dr is always negative.

$$\frac{d\ln T}{d\ln P} > \frac{\Gamma_2 - 1}{\Gamma_2}$$

$$\frac{1/T\,dT}{1/P\,dP} > \frac{\Gamma_2 - 1}{\Gamma_2}$$

$$\frac{P}{T}\frac{dT/dr}{dP/dr} > \frac{\Gamma_2 - 1}{\Gamma_2}$$

mult by dP/dr

$$\frac{P}{T}\frac{dT}{dr} < \frac{\Gamma_2 - 1}{\Gamma_2}\frac{dP}{dr}$$

$$\frac{dT}{dr} < \frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}\frac{dP}{dr}$$

$$\left(\frac{dT}{dr}\right)_{ad} > \frac{dT}{dr}$$

Equation (5) shows that instability occurs if the temperature decreases too rapidly out through the star. It is convenient to introduce the notation $\nabla = d\ln T/d\ln P$ and $\nabla_{\rm ad} \equiv \Gamma_2 - 1/\Gamma_2 = (d\ln T/d\ln P)_{\rm ad}$ so that the instability condition in equation (4) can now be written as $\nabla > \nabla_{\rm ad}$. The usual terminology for this condition is that *the temperature gradient is superadiabatic.*

5. In order to determine the circumstances under which to expect convection to occur, *Dalsgaard* writes the expression for the temperature gradient that we derived last week as

$$\nabla \equiv \nabla_R \equiv \frac{d\ln T}{d\ln P} = \frac{3k_B}{16\pi acGm_p}\,\frac{\kappa}{\mu}\,\frac{L(r)}{m(r)}\,\frac{\rho}{T^3}$$

(a) How does the ratio $L(r)/m(r)$, the average rate of energy generation per unit mass within the radius r, impact the instability; that is, should L/m be large or small for instability to occur? What implication does this have for convective zones in higher mass stars?

If $L(r)/m(r)$ is large then $\nabla_R > \nabla_{ad}$

(b) How does the opacity κ, and the quantity ρ/T^3, impact the instability? In which kinds of stars will these terms have the most impact, and what implication does this have for convective zones in these stars?

If κ is large or $\rho/T3$ is large

then $\nabla_R > \nabla_{ad}$