PHY 411 **Winter 2021**

Reading Assignment for Week 1—Tuesday, Jan 5

Math You Should Know for Electrodynamics

Before we begin discussing the math you should know, it is important to realize that Electrodynamics is not just math! It might seem that way, especially as you struggle through page after page on a problem, but learning Electrodynamics involves more than just using math. I mention this now, because not realizing this can lead to immense frustration — one tries to solve a problem from only a mathematical perspective and finds they are unable to do so. That being said, there are some mathematical ideas that are worth remembering; some of the following you might have encountered in your undergrad math courses, the rest we will learn as we go along.

Complex Numbers

Refreshing your memory about complex number algebra is highly recommended. Hopefully, you remember the definition of the imaginary number:

$$i^2 = -1 \tag{W1.1}$$

so that the complex number z can be written as

$$z = x + iy \tag{W1.2}$$

where x and y are real numbers. **Note:** Equation numbers ***not from*** Jackson's text will be in this format, with the week number followed by equation number. The **complex conjugate** is defined to be

$$z^* = x - iy \tag{W1.3}$$

You should be able to demonstrate that

$$|z|^2 = z^* z = zz^* = x^2 + y^2$$

To visualize complex numbers, we usually represent them with their real part along a horizontal axis and their imaginary part along a vertical axis. Then, $|z|$ is the polar distance of the complex number z from the origin. Since then $x = |z| \cos\theta$ and $y = |z| \sin\theta$, this gives us a useful representation in trigonometric form:

$$z = x + iy$$

Putting $x = |z| \cos\theta$ and $y = |z| \sin\theta$, this becomes

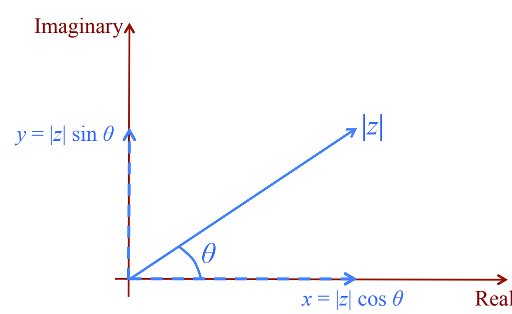
$$z = |z| \Big(\cos\theta + i \sin\theta\Big)$$

and hence

$$z = |z|\, e^{i\theta} \tag{W1.4}$$

where

$$|z| = \sqrt{x^2 + y^2}, \text{and} \;\; \theta = \arctan\left(\frac{y}{x}\right) \tag{W1.5}$$



Vectors

Vectors, as you know, have both magnitude and direction. It is best to begin with their Cartesian representation:

$$\vec{A} = A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z} \tag{W1.6}$$

where A_x, A_y and A_z are the components along the x, y and z axes respectively, and $\hat{x}, \hat{y}$ and $\hat{z}$ are unit vectors along those axes (sometimes, we will use $\hat{\imath}, \hat{\jmath}, \hat{k}$ instead).

Addition of vectors is carried out by adding components along each direction independently. Given another vector $\vec{B} = B_x\,\hat{x} + B_y\,\hat{y} + B_z\,\hat{z}$, the **sum $(\vec{A}+\vec{B})$** is found by doing

$$\vec{A}+\vec{B} = (A_x + B_x)\,\hat{x} + (A_y + B_y)\,\hat{y} + (A_z + B_z)\,\hat{z} \tag{W1.7}$$

whereas the **difference of two vectors** is given by

$$\vec{A}-\vec{B} = (A_x - B_x)\,\hat{x} + (A_y - B_y)\,\hat{y} + (A_z - B_z)\,\hat{z} \tag{W1.8}$$

Multiplication of a vector $\vec{A}$ by a scalar g simply scales the vector, so that we get a new vector $g\vec{A}$ whose magnitude is equal to $g|\vec{A}|$, and direction is the same as that of $\vec{A}$.

For two vectors, $\vec{A} = A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}$, and $\vec{B} = B_x\,\hat{x} + B_y\,\hat{y} + B_z\,\hat{z}$, we define two ways to multiply:

- The **Scalar Product** or **Dot Product**, defined by

  $$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\,\cos\theta = A_xB_x + A_yB_y + A_zB_z \tag{W1.9}$$

  where θ is the angle between $\vec{A}$ and $\vec{B}$, and

- The **Vector Product** or **Cross Product**, defined by

  $$\vec{A}\times\vec{B} = \Big(|\vec{A}||\vec{B}|\,\sin\theta\Big)\,\hat{n} \tag{W1.10}$$

  where $\hat{n}$ is a unit vector in a direction perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$, can be written out in full as

  $$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{W1.11}$$

  $$= \Big(A_yB_z - A_zB_y\Big)\,\hat{x} + \Big(A_zB_x - A_xB_z\Big)\,\hat{y} + \Big(A_xB_y - A_yB_x\Big)\,\hat{z} \tag{W1.12}$$

These expressions are a result of the following relations satisfied by the unit vectors (which you will need):

$$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1 \qquad \text{and} \qquad \hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0 \tag{W1.13}$$

whereas

$$\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0 \tag{W1.14}$$

along with

$$\hat{x}\times\hat{y} = \hat{z};\quad \hat{y}\times\hat{z} = \hat{x};\quad \hat{z}\times\hat{x} = \hat{y} \tag{W1.15}$$

and the anticommutation

$$\hat{y}\times\hat{x} = -\hat{z};\quad \hat{z}\times\hat{y} = -\hat{x};\quad \hat{x}\times\hat{z} = -\hat{y} \tag{W1.16}$$

We will also need vector triple products, but as you can see already from the cross product, writing out components can get cumbersome. Therefore, we introduce simplifying notation (although, it does take getting used to); below, we will first define the **Einstein summation convention** and then the **Levi-Civita tensor**.

To define the Einstein summation convention, notice that the dot product in expanded form

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

can be written in the compact form

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i \tag{W1.17}$$

where we take $i = 1$ to be x, $i = 2$ to be y, and $i = 3$ to be z. We can make this even more compact if we drop the summation sign and simply write

$$\vec{A} \cdot \vec{B} = A_i B_i \tag{W1.18}$$

where we **imply by repeated indices that we need to sum over all the orthogonal Cartesian components**. This is known as the **Einstein summation convention**.

The **Levi-Civita tensor** is defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ are cyclic permutations of 1, 2, 3} \\ -1 & \text{if } i, j, k \text{ are cyclic permutations of 3, 2, 1} \\ 0 & \text{otherwise (i.e., if any pair of indices are repeated)} \end{cases} \tag{W1.19}$$

For example, we can use the Levi-Civita tensor to write the cross product as

$$(\vec{A} \times \vec{B})_k = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \epsilon_{ijk} \tag{W1.20}$$

where $k = 1, 2, 3$, that is, $k = 1$ is x, $k = 2$ is y, and $k = 3$ is z.

Using the Einstein summation convention, we can make the expression for the cross product even more compact:

$$(\vec{A} \times \vec{B})_k = A_i B_j \epsilon_{ijk} \tag{W1.21}$$

where we imply from the repeated indices i and j that we wish to sum over them (as shown in the explicit summation written above).

We will also need the **Kronecker delta function**:

$$\delta_{ij} = \begin{cases} 1 & \text{if } \; i = j \\ 0 & \text{if } \; i \neq j \end{cases} \tag{W1.22}$$

Notice how much they simplify the expressions below:

- The **scalar triple product**

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = A_i \delta_{ij} \left(\epsilon_{mnj} \, B_m C_n \right) \tag{W1.23}$$

- The **vector triple product**

$$\left[\vec{A} \times (\vec{B} \times \vec{C}) \right]_k = A_i \left(B_m C_n \, \epsilon_{mnj} \right) \epsilon_{ijk} \tag{W1.24}$$

Having defined several operations for vectors, it is well worth remembering the following as we move forward.

The scalar product of two vectors is commutative:

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

but the cross product anti-commutes:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Both are, however, distributive:

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

and

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

provided, of course, for the latter we maintain the order of the product (since the cross product is not commutative).

There is a very useful identity for the vector triple product:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{C}\,(\vec{A} \cdot \vec{B}) \tag{W1.25}$$

often called the **(BAC-CAB) rule**. Proving it is tedious algebra, but can be simplified by use of the so-called *epsilon-delta* identity:

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km} \tag{W1.26}$$

The next logical step would be to discuss tensors, but we will postpone this discussion until the point when we need to make use of them.

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Differentiation

It might seem trivial to bring up differentiation; after all, you should know it very well by now. I do so because there are a few simple procedures worth recalling because they will show up frequently in problems.

The Chain Rule: If $f(x)$ is a function and $x(t)$ is itself a function of t (I use t in the sense of time here since that is the chain we usually encounter in physics), then

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} \tag{W1.27}$$

The Product Rule: If we have two functions $f(x)$ and $g(x)$, then

$$\frac{d}{dx}\Big(fg\Big) = \left(\frac{df}{dx}\right) g + f\left(\frac{dg}{dx}\right) \tag{W1.28}$$

Frequently, we will apply these rules as differentials, rather than as derivatives with respect to a specific variable; that is

$$df = \frac{df}{dx}\,dx \tag{W1.29}$$

and

$$d(fg) = g\,df + f\,dg \tag{W1.30}$$

The above procedures dealt with *ordinary derivatives*, in which $f(x)$ is a function of a single variable x; that is, we did

$$\frac{df}{dx} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{W1.31}$$

In Electrodynamics, though, we will more often be dealing with vectors which are functions of three spatial dimensions (and often even a fourth—time), and hence we must work with partial derivatives. Let us begin by defining a partial derivative of the function $f(x, y, z)$ in analogy to the total derivative of the function of a single variable:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x} \tag{W1.32}$$

Thus, the partial derivative is just like taking an ordinary (scalar) derivative, keeping the other variables as constants.

The total differential is then given by

$$df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z}\right) dz \tag{W1.33}$$

Stare long enough at this expression and you'll see that it looks like the dot product of two vectors. Let us write it out explicitly.

Writing the total differential in equation (W1.33) on the previous page explicitly as a product of two vectors:

$$df = \left[\left(\frac{\partial f}{\partial x}\right)\hat{x} + \left(\frac{\partial f}{\partial y}\right)\hat{y} + \left(\frac{\partial f}{\partial z}\right)\hat{z}\right] \cdot \left[dx\hat{x} + dy\hat{y} + dz\hat{z}\right] \tag{W1.34}$$

The quantity in the first square bracket on the right hand side is known as the gradient of the function f, and is usually written in differential operator form (in Cartesian coordinates) as

$$\vec{\nabla} = \hat{x}\left(\frac{\partial}{\partial x}\right) + \hat{y}\left(\frac{\partial}{\partial y}\right) + \hat{z}\left(\frac{\partial}{\partial z}\right) \tag{W1.35}$$

I've moved the unit vectors to the left of the differentials to avoid any confusion. By analogy with df/dx, you can see that the gradient of a function $f(x, y, z)$ gives you the directed slope of the function.

Two additional operations can be defined using $\vec{\nabla}$, the divergence of a vector $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$:

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{W1.36}$$

which is a measure of how much the vector field $\vec{A}$ is diverging from or converging to a point.

We also define the curl of a vector $\vec{A}$ as

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{W1.37}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} \tag{W1.38}$$

and the curl is thus a measure of the rotation of a vector field about a point.

The distributive properties over sums are straightforward, so I won't write them here. The distribution over products is another story, however! First, for the gradient — since there are two ways to make a scalar, fg and $\vec{A} \cdot \vec{B}$, we have two distributive properties for the gradient:

$$\vec{\nabla}(fg) = f(\vec{\nabla}g) + g(\vec{\nabla}f) \tag{W1.39}$$

and

$$\vec{\nabla}(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \vec{\nabla})\,\vec{B} + (\vec{B} \cdot \vec{\nabla})\,\vec{A} + \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) \tag{W1.40}$$

Clearly, the latter is more tedious to prove than the former, but nevertheless it is an important relation! It is well worth remembering the important operator

$$\vec{A} \cdot \vec{\nabla} = A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z} \tag{W1.41}$$

Likewise, since $f\vec{A}$ and $\vec{A} \times \vec{B}$ are both vectors, we have two divergence rules:

$$\vec{\nabla} \cdot (f\vec{A}) = (\vec{A} \cdot \vec{\nabla})\,f + f(\vec{\nabla} \cdot \vec{A}) \tag{W1.42}$$

and

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \tag{W1.43}$$

Finally, we have two rules for the curl:

$$\vec{\nabla} \times (f\vec{A}) = \vec{\nabla} f \times \vec{A} + f\,\vec{\nabla} \times \vec{A} \tag{W1.44}$$

and

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A}) + (\vec{B} \cdot \vec{\nabla})\,\vec{A} - (\vec{A} \cdot \vec{\nabla})\,\vec{B} \tag{W1.45}$$

Equations (W1.40)-(W1.45) are also in the inside front cover of Jackson (3rd ed.), provided you purchased the U.S. edition (but with ψ instead of f).

Next, we have second derivatives, and there are five ways to define them, and two of them are very important. Consider first the second derivative defined by doing

$$\vec{\nabla} \cdot \vec{\nabla} f = \nabla^2 f$$

The ∇^2 operator is known as the **Laplacian** and it will play an important role in Electrodynamics. Writing it out explicitly:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{W1.46}$$

Of the next three second derivatives for the gradient, divergence, and curl respectively, the first and third are in the inside front cover of Jackson, and the second has no simpler form:

$$\vec{\nabla} \times (\vec{\nabla} f) = 0 \tag{W1.47}$$

As mentioned above, the following has no simpler form:

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) \tag{W1.48}$$

and the third is again in the inside front cover of Jackson:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \tag{W1.49}$$

Finally, the fifth of the second derivatives plays an important role in Electrodynamics; as you will see later in this course, it enables a key step in deriving the wave equation from Maxwell's equations:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \tag{W1.50}$$

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Integration

As with differentiation, there are a number of ways to integrate vectors. One could define a line integral along some specified path C (which may be curvilinear in general):

$$\int_C \vec{A} \cdot d\vec{\ell}$$

or around a closed loop:

$$\oint_C \vec{A} \cdot d\vec{\ell}$$

No doubt, you will remember these kinds of integrals from your work with potential and potential energy in your undergrad courses.

The next kind are surface integrals, for which there are two common notations:

$$\int_S \vec{A} \cdot \hat{n}\, da \equiv \int_S \vec{A} \cdot d\vec{a}$$

and we'll follow Jackson and use the first (note that Griffiths prefers the second type). Here S is an *open* surface bounded by a closed curve C, and $\hat{n}$ is a unit normal to the surface S; da is some element of area on surface S. Although, in principle, the direction of $\hat{n}$ is arbitrary (i.e., perpendicular to S in either direction), the direction of $\hat{n}$ will usually be specified by the characteristics of the problem, e.g., the electric field will be so directed that $\hat{n}$ will be picked as an outward normal to the surface. In analogy to the second kind of line integral around a closed loop, one could also define a surface integral

$$\oint_S \vec{A} \cdot \hat{n}\, da \equiv \oint_S \vec{A} \cdot d\vec{a}$$

in which case S is a closed surface that encloses some volume V, e.g., like a soap bubble.

Finally, we have volume integrals, and the most popular ways of writing them are

$$\int_V A\, dV \equiv \int_V A\, d^3r \equiv \int_V A\, d\tau$$

Jackson, however, writes the volume element as d^3x, and although this could be potentially confusing, we will have to get used to it!

You might be wondering why we're integrating only scalar quantities. One doesn't have to, but these are what is relevant for Electrodynamics. Integrals over vector quantities like $\int_C \vec{A}\, d\ell$ are permissible mathematically.

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Next, we will write a couple of important theorems, but before doing that, consider the *fundamental theorem of calculus* that defines the integral:

$$\int_a^b df = \int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a) \tag{W1.51}$$

From equation (W1.34), we can write

$$df = \vec{\nabla} f \cdot d\vec{\ell} \tag{W1.52}$$

where $d\vec{\ell} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$. Putting this in equation (W1.51), we get

$$\int_a^b df = \int_a^b \vec{\nabla} f \cdot d\vec{\ell} = f(b) - f(a) \tag{W1.53}$$

implying that the integration is independent of the path and depends only on the end points a and b. Whenever we come across this, we know that integration around a closed path will give zero, and so

$$\oint_C \vec{\nabla} f \cdot d\vec{\ell} = 0$$

No doubt, you remember this integral from your definition of work in mechanics; if you put the force $F = -\vec{\nabla} U$ in equation (W1.53), you get the work done from point a to b. The equation above has generalized this — any (vector) force that can be written as the (negative) gradient of a (smooth, differentiable) potential energy function is a conservative force. Now, on to two important theorems.

The Divergence Theorem

The divergence theorem connects a volume integral to a surface integral, and using Jackson's notation (inside front cover), we can write it as

$$\int_V (\vec{\nabla} \cdot \vec{A})\, d^3x = \int_S \vec{A} \cdot \hat{n}\, da \tag{W1.54}$$

where V is a (three-dimensional) volume with volume element d^3x (I don't like this and would rather write it as $d\tau$ or d^3r, but I'll retain Jackson's notation to avoid confusion); S is a ***closed*** (two-dimensional) surface bounding V; da is an area element on this surface and $\hat{n}$ is a unit outward normal to the surface S (at da). In words, this reads: *the divergence of a vector field $\vec{A}$ in some volume V bounded by a closed surface S is equal to the flux of the vector field $\vec{A}$ outward through the closed surface S.*

Two useful corollaries (also written in the inside front cover of Jackson) are:

$$\int_V \vec{\nabla} f\, d^3x = \int_S f\, \hat{n}\, da \tag{W1.55}$$

where I've notated the scalar function as $f \equiv f(x, y, z)$ instead of Jackson's ψ, and

$$\int_V (\vec{\nabla} \times \vec{A})\, d^3x = \int_S (\hat{n} \times \vec{A})\, da \tag{W1.56}$$

Stokes' Theorem

Stokes' theorem connects a surface integral to a line integral and following Jackson's notation, we will write it as

$$\int_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, da = \oint_C \vec{A} \cdot d\vec{\ell} \tag{W1.57}$$

where S is an ***open*** surface and C is the contour bounding this surface; da is an area element in S, and $d\vec{\ell}$ is a line element along C. Jackson says (on the inside front cover) that S is "defined by the right-hand-screw rule in relation to the sense of the line integral around C." This additional part is required because S is an open surface; in principle, $\hat{n}$ could point along one of two directions, and you need to choose one of these two possible directions: if you curl the fingers of your right hand around C along the direction in which you wish to do the integration, your thumb will point along the direction of $\hat{n}$.

A corollary is:

$$\int_S (\hat{n} \times \vec{\nabla} f)\, da = \oint_C f\, d\vec{\ell} \tag{W1.58}$$

where, again, I've used $f \equiv f(x, y, z)$ instead of Jackson's ψ.

Integration by Parts

You will frequently use integration by parts in solving Electrodynamics problems, so let us begin with a simple scalar definition. Starting from the product rule for the differential

$$d(fg) = f\, dg + g\, df$$

integrate both sides

$$\int_a^b d(fg) = fg\Big|_a^b = \int_a^b f\, dg + \int_a^b g\, df$$

and rearrange

$$\int_a^b f\, dg = fg\Big|_a^b - \int_a^b g\, df \tag{W1.59}$$

Using the chain rule, we can also write this as

$$\int_a^b f \left(\frac{dg}{dx}\right) dx = fg\Big|_a^b - \int_a^b g \left(\frac{df}{dx}\right) dx \tag{W1.60}$$

The situation is not so clear-cut when we get to vectors. There are lots of ways to do vector integration by parts, and we will encounter them as we progress through the course. Let us do one example on the next page to illustrate.

Vector Integration by Parts

We will now do one example of vector integration by parts, noting that there are many ways to put such integrals together.

Consider the product rule

$$\vec{\nabla}\cdot(f\vec{A}) = (\vec{A}.\vec{\nabla})\,f + f(\vec{\nabla}\cdot\vec{A})$$

Integrate this over a volume bounded by a closed surface

$$\int_V \vec{\nabla}\cdot(f\vec{A})\,dV = \int_V (\vec{A}.\vec{\nabla})\,f\,dV + \int_V f(\vec{\nabla}\cdot\vec{A})\,dV$$

Apply the divergence theorem in equation (W1.54) to the left hand side to get

$$\int_S (f\vec{A})\cdot\hat{n}\,da = \int_V (\vec{A}.\vec{\nabla})\,f\,dV + \int_V f(\vec{\nabla}\cdot\vec{A})\,dV$$

and rearrange

$$\int_V (\vec{A}.\vec{\nabla})\,f\,dV = \int_S (f\vec{A})\cdot\hat{n}\,da - \int_V f(\vec{\nabla}\cdot\vec{A})\,dV$$

In order to proceed (i.e., for the expression above to be useful), we note that one of two cases could apply in Electrodynamics problems:

- ***Either*** f or $\vec{A}$ vanishes at infinity, and since V is usually taken over all space, the surface integral is then zero (since the surface is now at infinity where either f or $\vec{A}$ vanishes), and we get

$$\int_V (\vec{A}.\vec{\nabla})\,f\,dV = -\int_V f(\vec{\nabla}\cdot\vec{A})\,dV$$

- ***Or*** we have $\vec{\nabla}\cdot\vec{A} = 0$, meaning that we have a vector field with zero divergence, so that we get

$$\int_V (\vec{A}.\vec{\nabla})\,f\,dV = \int_S (f\vec{A})\cdot\hat{n}\,da$$

Depending on which of the above cases is valid for a given problem, we can proceed with simplifying steps.

As I noted above, we don't have a unique rule for vector integration by parts, unlike equation (W1.59) for scalars. For example, we could repeat the above procedure for $\vec{\nabla}\times(f\vec{A})$, or $\vec{\nabla}\cdot(\vec{A}\times\vec{B})$. The key point is to find some version of the product rule for the vector expression, integrate both sides, and then apply useful theorems like the divergence theorem or Stokes theorem as the case may be. We will certainly encounter examples as we work through the problems, and therefore, defer additional discussion until we reach those examples in this course.

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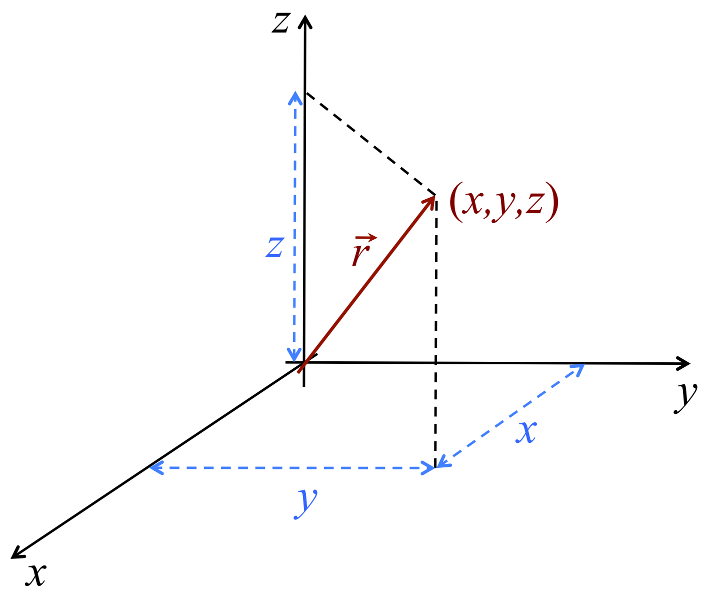
Rectangular, Spherical, and Cylindrical Coordinates

The Rectangular (Cartesian), Spherical (polar), and Cylindrical coordinates are the three coordinate systems you will use in Electrodynamics. The choice of coordinate system, as you should know by now, is driven by the geometry of a problem.

Rectangular (Cartesian) Coordinates

So far in this document, we've been working mostly with the **rectangular Cartesian coordinate system**. In this system, the coordinates of a point are represented by (x, y, z), each of these three being scalar lengths along three orthogonal directions (see figure below).



The **position vector** from the origin to the location specified by the coordinates (x, y, z) is given by

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \tag{W1.61}$$

whereas the **element of length** $d\vec{\ell}$ is given by

$$d\vec{\ell} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz \tag{W1.62}$$

and the **volume element** $dV \equiv d\tau \equiv d^3r \equiv d^3x$ is

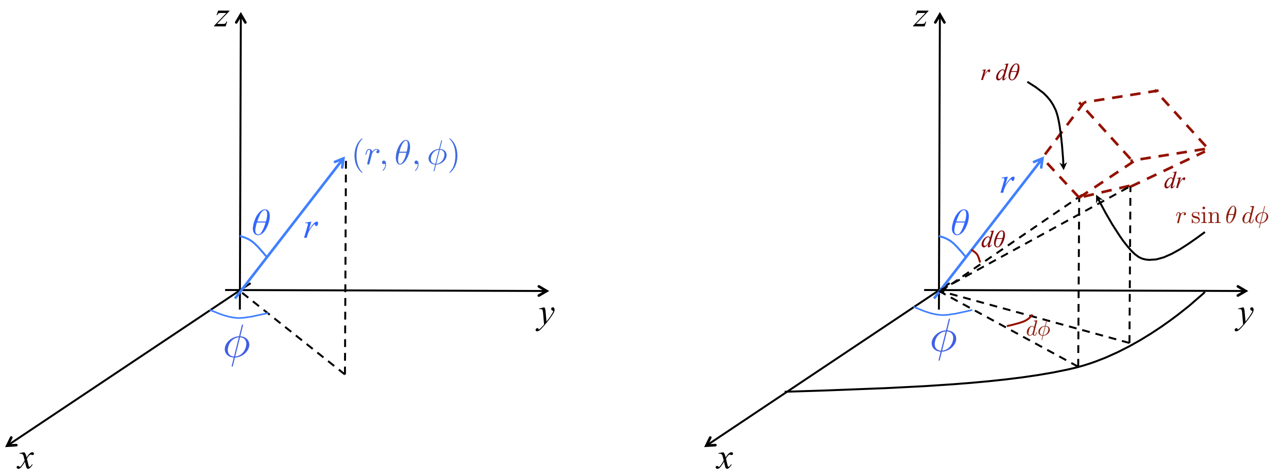
$$d^3x = dx\,dy\,dz \tag{W1.63}$$

where I've used d^3x to be consistent with Jackson, even though there is scope for confusion.

I've already written the gradient, divergence, curl, and Laplacian in the rectangular coordinate system in equations (W1.35), (W1.36), (W1.37), and (W1.46) respectively, and so won't repeat them here.

Spherical Polar Coordinates

In spherical polar coordinates, the coordinates of a point are represented by (r, θ, ϕ), as shown in the figure below *on the left*. **Caution:** Some authors interchange θ and ϕ!



From the figure above on the left, we see that the spherical coordinates are related to the Cartesian coordinates by

$$x = r\,\sin\theta\,\cos\phi \qquad y = r\,\sin\theta\,\sin\phi \qquad z = r\,\cos\theta \tag{W1.64}$$

You will need the unit vectors in this system relative to the Cartesian $\hat{x}, \hat{y}, \hat{z}$:

$$\hat{r} = \sin\theta\,\cos\phi\,\hat{x} + \sin\theta\,\sin\phi\,\hat{y} + \cos\,\theta\,\hat{z} \tag{W1.65}$$

$$\hat{\theta} = \cos\theta\,\cos\phi\,\hat{x} + \cos\theta\,\sin\phi\,\hat{y} - \sin\,\theta\,\hat{z} \tag{W1.66}$$

$$\hat{z} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \tag{W1.67}$$

The ***element of length*** $d\vec{\ell}$ in spherical coordinates is given by

$$d\vec{\ell} = \hat{r}\,dr + \hat{\theta}\,rd\theta + \hat{\phi}\,r\sin\theta d\phi \tag{W1.68}$$

and the ***volume element*** $dV \equiv d\tau \equiv d^3r \equiv d^3x$ is

$$d^3x = dr\,(rd\theta)\,(r\sin\theta d\phi) = r^2\sin\theta\,drd\theta d\phi \tag{W1.69}$$

I've shown in the figure above *on the right* how the volume element is constructed; note that the cuboid is somewhat deformed because I'm following the lines from the origin, but remember that the shape shown in dark red is indeed a cuboid since we're dealing with infinitesimal lengths here. Of the three dimensions, the length of the edge of the infinitesimal cuboid along r is just dr, the length of the edge that spans the angle $d\theta$ at the origin is $r\,d\theta$, and the length of the third edge is $r\sin\theta\,d\phi$; all of this should be evident from the figure above *on the right*.

The gradient, divergence, and curl in spherical coordinates are on the next page.

Although you won't have to memorize the following for this class, it is well worth remembering if you ever do have to memorize them that all angular relations must have a dimension of length multiplying them so that the relation is dimensionally correct overall. Thus, the **gradient** of a spatial function ψ in spherical coordinates is given by

$$\vec{\nabla}\psi = \hat{r}\,\frac{\partial \psi}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial \psi}{\partial \theta} + \hat{\phi}\,\frac{1}{r\sin\,\theta}\,\frac{\partial \psi}{\partial \phi}$$

Notice that I've changed from using f for the scalar function to ψ in order to be consistent with Jackson's relations written on the inside back cover.

The **divergence** in spherical coordinates for a vector $\vec{A} \equiv (A_r, A_\theta, A_\phi)$ is given by

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\,\frac{\partial}{\partial r}\,(r^2 A_r) + \frac{1}{r\sin\,\theta}\,\frac{\partial}{\partial \theta}\,(\sin\,\theta\,A_\theta) + \frac{1}{r\sin\,\theta}\,\frac{\partial A_\phi}{\partial \theta}$$

The **curl** in spherical coordinates for a vector $\vec{A} \equiv (A_r, A_\theta, A_\phi)$ is given by

$$\vec{\nabla}\times\vec{A} = \hat{r}\,\frac{1}{r\sin\,\theta}\,\left[\frac{\partial}{\partial \theta}(\sin\,\theta\,A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\theta}\,\left[\frac{1}{r\sin\,\theta}\,\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\,\frac{\partial}{\partial r}(rA_\phi)\right] + \hat{\phi}\,\frac{1}{r}\,\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]$$

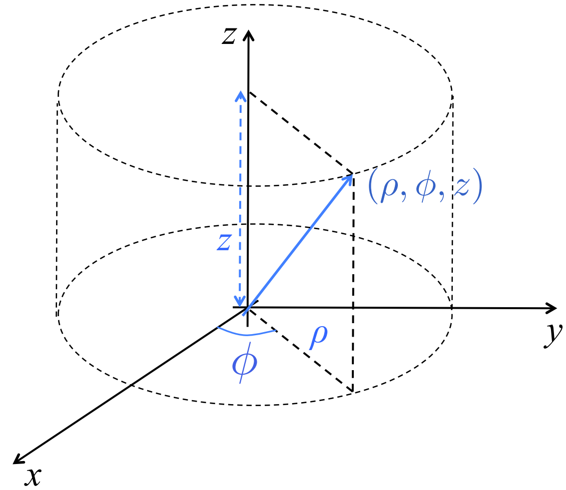
The **Laplacian** in spherical coordinates for a spatial function ψ is given by

$$\nabla^2\psi = \frac{1}{r^2}\,\frac{\partial}{\partial r}\,\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\,\sin\theta}\,\frac{\partial}{\partial \theta}\,\left(\sin\theta\,\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2\,\sin^2\theta}\,\frac{\partial^2 \psi}{\partial \phi^2}$$

Space left blank for student notes; class summary continues on next page.

Cylindrical Coordinates

Different texts use different symbols for cylindrical coordinates; we will follow Jackson and use (ρ, ϕ, z), as shown in the figure below.



From the figure above, we see that the cylindrical coordinates are related to the Cartesian coordinates by

$$x = \rho\,\cos\phi \qquad y = \rho\,\sin\phi \qquad z = z \tag{W1.70}$$

Again, you will need the unit vectors in this system relative to the Cartesian $\hat{x}, \hat{y}, \hat{z}$:

$$\hat{\rho} = \hat{x}\,\cos\phi + \hat{y}\,\sin\phi \tag{W1.71}$$

$$\hat{\phi} = -\hat{x}\,\sin\phi + \hat{y}\,\cos\phi \tag{W1.72}$$

$$\hat{z} = \hat{z} \tag{W1.73}$$

The ***element of length*** $d\vec{\ell}$ in cylindrical coordinates is given by

$$d\vec{\ell} = \hat{\rho}\,d\rho + \hat{\phi}\,\rho d\phi + \hat{z}\,dz \tag{W1.74}$$

and the ***volume element*** $dV \equiv d\tau \equiv d^3r \equiv d^3x$ is

$$d^3x = d\rho\,(\rho d\phi)\,(dz) = \rho\,d\rho\,d\phi\,dz \tag{W1.75}$$

The volume element is easier to construct compared to the spherical case; imagine an elemental cuboid in the figure above with dimensions $d\rho$ along the ρ-axis, and dz along the z-axis, and the third edge that subtends an angle ϕ at the origin will have length $\rho d\phi$. The volume of the elemental cuboid can then be constructed by multiplying these three lengths, as I've done above.

The gradient, divergence, and curl in cylindrical coordinates are on the next page.

Although you don't need to memorize the following for this class, but it is well worth remembering that all angular relations must have a dimension of length multiplying them so that the relation is dimensionally correct overall. Thus, the **gradient** of a spatial function ψ in cylindrical coordinates is given by

$$\vec{\nabla}\psi = \hat{\rho}\,\frac{\partial\psi}{\partial\rho} + \hat{\phi}\,\frac{1}{\rho}\,\frac{\partial\psi}{\partial\phi} + \hat{z}\,\frac{\partial\psi}{\partial z}$$

Again, notice that I've changed from using f for the scalar function to ψ in order to be consistent with Jackson's relations written on the inside back cover.

The **divergence** in cylindrical coordinates for a vector $\vec{A} \equiv (A_\rho, A_\phi, A_z)$ is given by

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\,\frac{\partial}{\partial\rho}\,(\rho A_\rho) + \frac{1}{\rho}\,\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

The **curl** in in cylindrical coordinates for a vector $\vec{A} \equiv (A_\rho, A_\phi, A_z)$ is given by

$$\vec{\nabla}\times\vec{A} = \hat{\rho}\,\left[\frac{1}{\rho}\,\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right] + \hat{\phi}\,\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right] + \hat{z}\,\frac{1}{\rho}\,\left[\frac{\partial}{\partial\rho}\,(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]$$

The **Laplacian** in cylindrical coordinates for a spatial function ψ is given by

$$\nabla^2\psi = \frac{1}{\rho}\,\frac{\partial}{\partial\rho}\,\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\,\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

The Dirac δ-Function

The Dirac δ-function is very useful in many situations in Electrodynamics, as you will find later in this course. Jackson has done a great job discussing it on page 26, so I will limit my comments here and point you to it. The Dirac δ-function is defined through two key properties:

$$\delta(x-a) = 0 \quad \text{for } x \neq a$$

and

$$\int \delta(x-a)\,dx = 1$$

if the region of integration includes $x = a$, but is zero otherwise.

Thus, the Dirac δ-function may be thought of as the limit of a peaked curve such as a Gaussian that becomes narrower and narrower but higher and higher in such a way that the area under the curve is always constant, and the distribution is normalized so that its integral is equal to one. Thus, the δ-function is not so much a function, but rather the limit of a distribution. In and of itself, it doesn't have much use; its primary purpose in Electrodynamics is to have it multiplied by a function and integrated, whereupon it can draw out the value of the function at a particular point, e.g.,

$$\int f(x)\,\delta(x-a)\,dx = f(a)$$

Additional properties of the Dirac δ-function are listed on pages 26-27 of Jackson, and you should familiarize yourself with them.