Learning goals

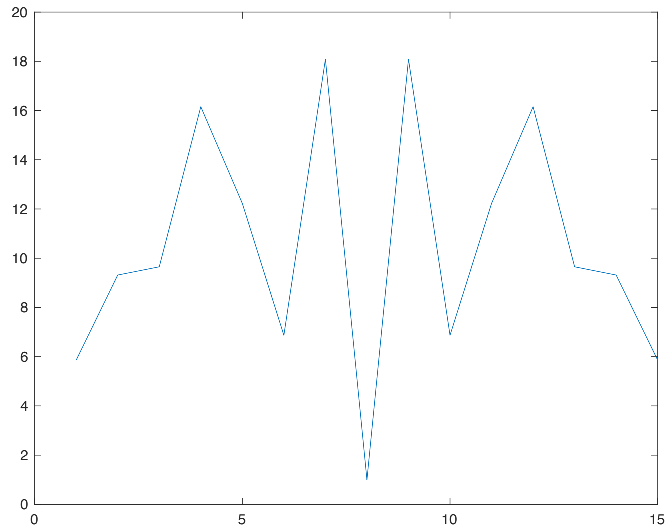
1. Other issues with the discrete Fourier transform and sampling
2. Convolution
3. Correlation

Remaining issue with FFT. What happens when the universe doesn't cooperate and provide you with data set of *2^m length*?

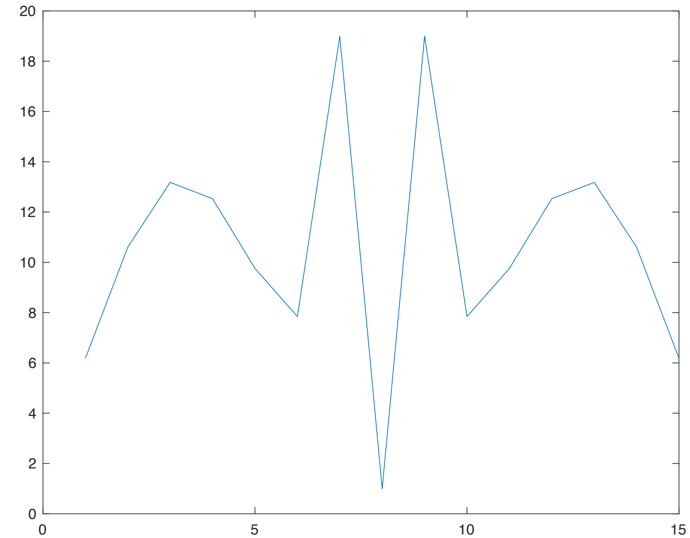
- Use only 2^m points and discard the rest
- Use a periodic extension to extend data set to right length
  - Extend at edge of data by setting point $X_{N+1} = X_0, X_{N+2} = X_1, \ldots.$
  - Extend by symmetrically extending data at both edges. In this scenario: $X_{-1} = X_N, X_{-2} = X_{N-2}, X_{N+1} = X_0, X_{N+2} = X_1, \ldots$
  - Extend by zero padding data until right length is reached.
- Use the *discrete FT*, rather than *the FFT*.

Do problem (1) on the worksheet and S T O P

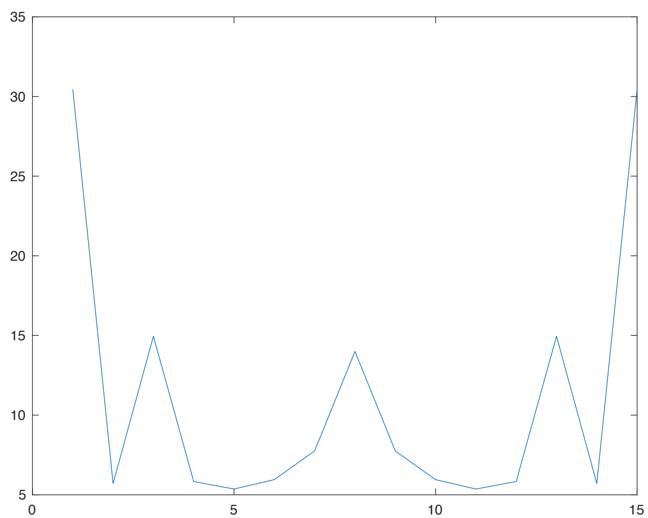
Fourier-04



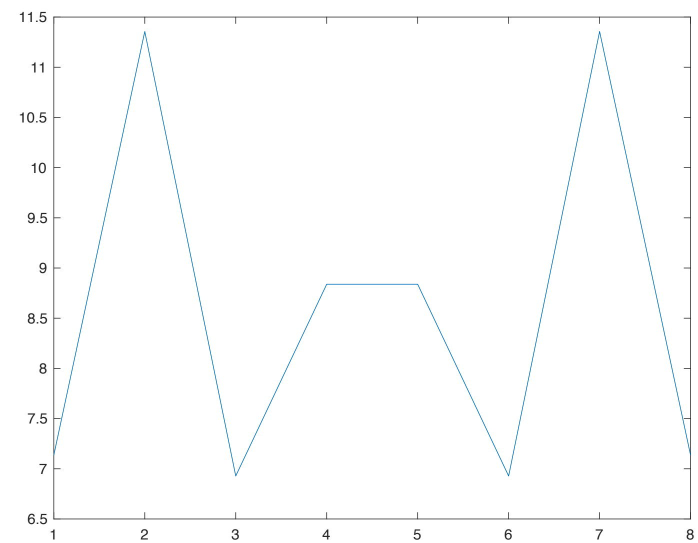
One-way extension



Two-way extension



Zero-padding extension



DFT—no extension

Do problem (2) on the worksheet

Convolution and Correlation—2 useful applications of Fourier transform

Convolution $$p \otimes q \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(\tau) q(t-\tau) d\tau$$

Correlation $$p \odot q \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^*(\tau) q(t+\tau) d\tau$$

Convolution describes how a physical system responds to a given input.

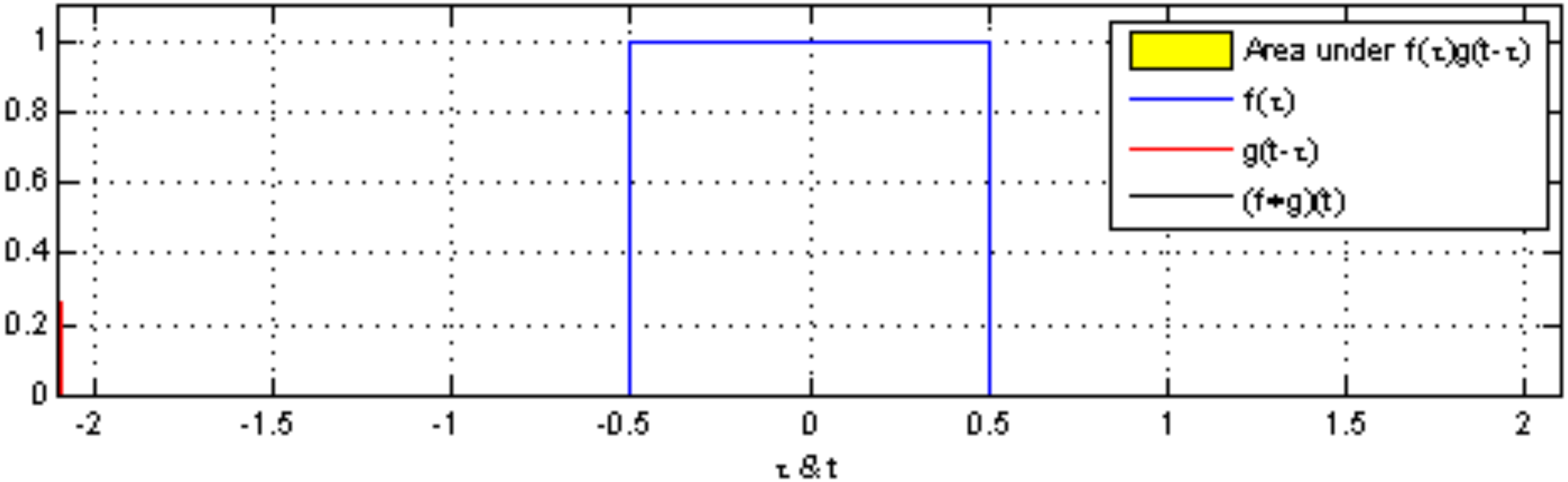
Correlation is useful in extracting a signal from a noisy background.

The usefulness of the quantities comes when we take the Fourier transform to obtain

$$\mathcal{F}(p \otimes q) = \mathcal{F}(p)\mathcal{F}(q)$$

$$\mathcal{F}(p \odot q) = \mathcal{F}(p)^* \mathcal{F}(q)$$

A convolution is an operation to produce a third function that expresses how the shape of one is modified by the other.



By Convolution_of_box_signal_with_itself.gif: Brian Ambergderivative work: Tinos (talk) Convolution_of_box_signal_with_itself.gif, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=11003835

We begin by looking at convolution. *Warning* we are only exploring one aspect of this topic, it has many other uses.

Suppose we have two functions *p, q* for which we can calculate their Fourier transforms. We know that

$$\mathcal{F}(p \otimes q) = \mathcal{F}(p)\mathcal{F}(q)$$

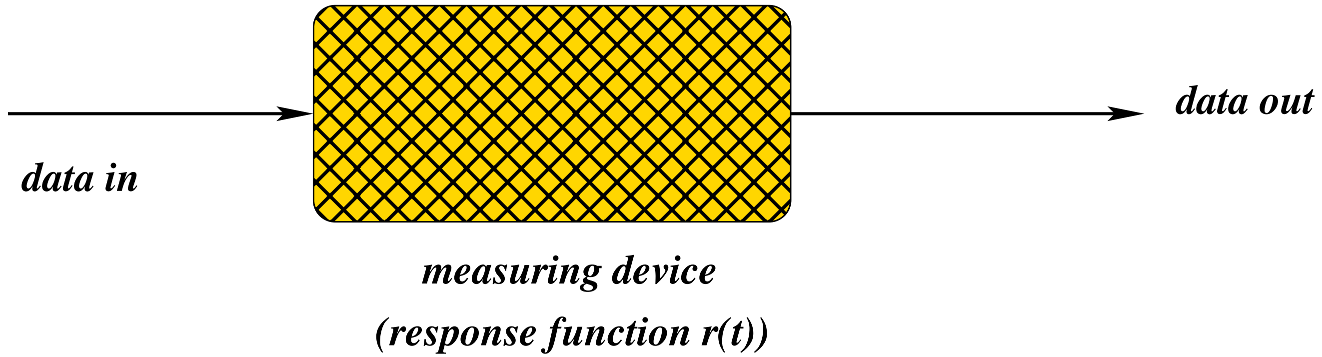
Now re-arranging this we have that

$$\mathcal{F}(p) = \frac{\mathcal{F}(p \otimes q)}{\mathcal{F}(q)}$$

or that

$$p(t) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(p \otimes q)}{\mathcal{F}(q)}\right)$$

So what?

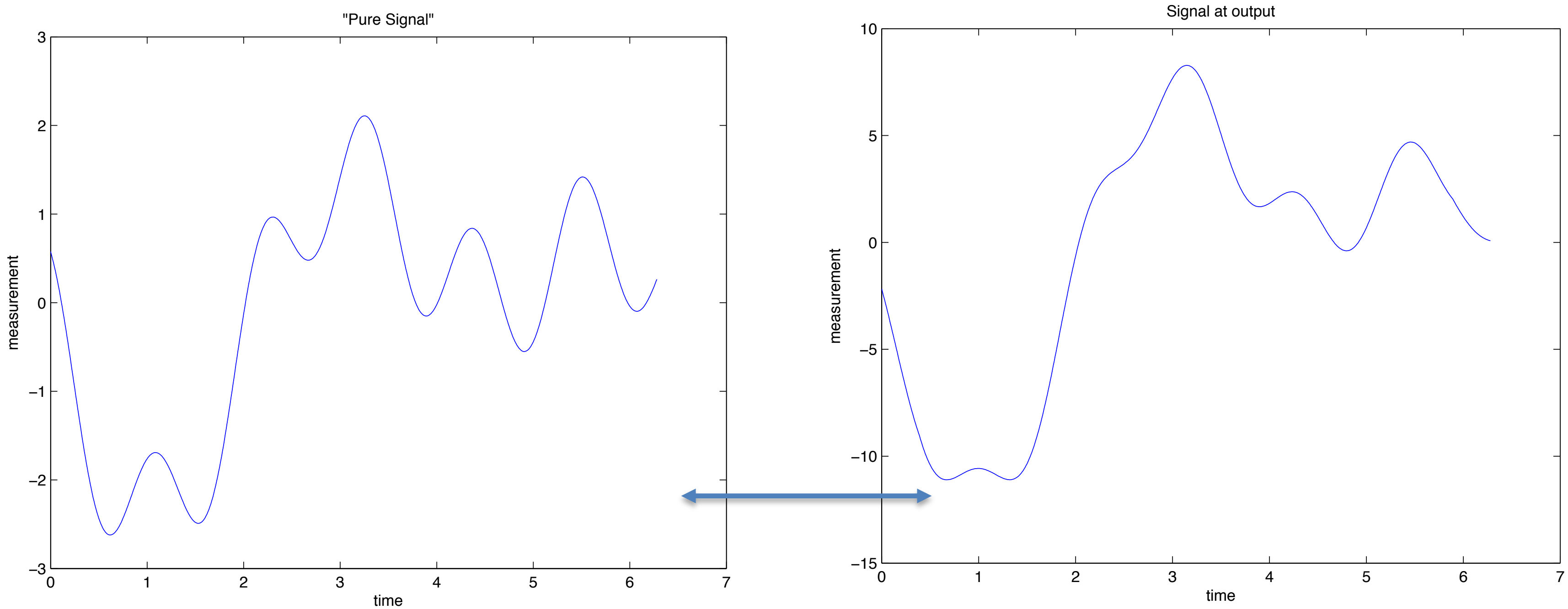


The device modifies the *pure signal* so that what you actually measure is *not* the real signal

This modification is very often *frequency dependent.* The modification caused by the measuring device is called the *response function, r(t).*

The signal your measuring, *p(t)* is then a *convolution* of the real signal *s(t)* and the response function *r(t).* That is

$$p(t) = s(t) \otimes r(t)$$

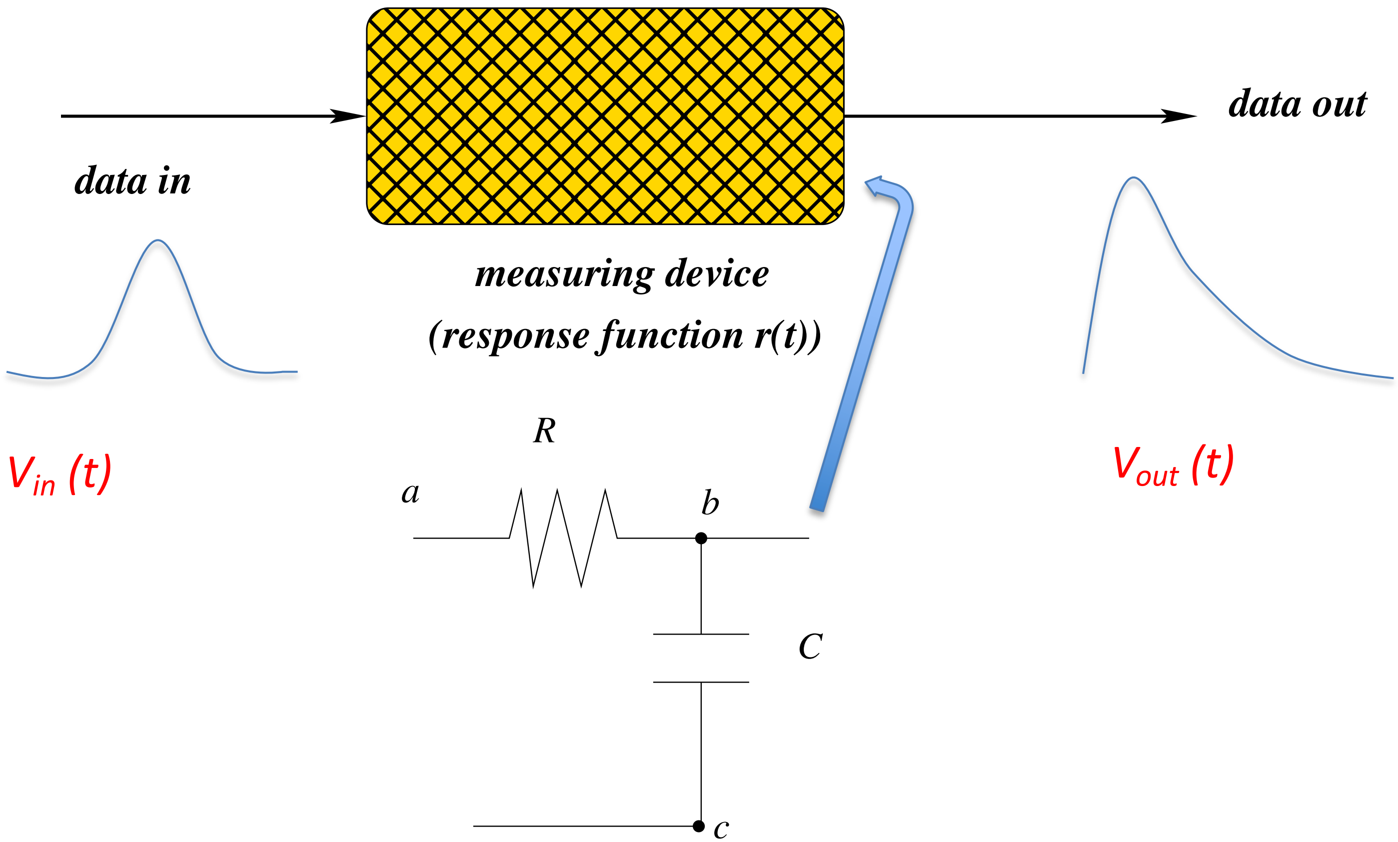


Pure signal

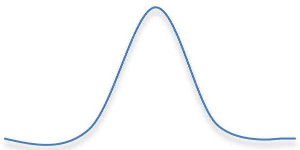
Measured signal

r(t) is exponential like.

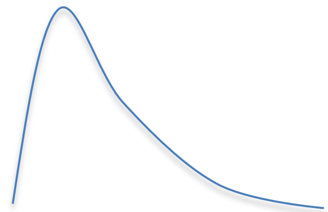
Any real device always has some resistance and capacitance.



$$r(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{RC} e^{-t/RC} & t \geq 0 \end{cases}$$



$V_{in}\ (t)$



$V_{out}\ (t)$

$$V_{out} = V_{in} \otimes r(t)$$

known known

So to find V_{in} : $$V_{in}(t) = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[V_{out}(t)]}{\mathcal{F}[r(t)]}\right]$$

Do question (3) on the worksheet

Correlation. We restrict ourselves to the *autocorrelation function.* Again, this topic is more extensive than we will cover here.

The autocorrelation is $p \odot p = \int p^*(t)\, p(t+\tau)d\tau$ and gives how much a function looks like itself.

What does this have to do with Fourier? Well it can be shown that $\mathcal{F}[p \odot q] = P^*(\omega)Q(\omega)$ Where $P^*(\omega)Q(\omega)$ are the Fourier transforms of *p* and *q* respectively. Thus we have that

$$p \odot q = \mathcal{F}^{-1}\left[P^*(\omega)Q(\omega)\right]$$

So what is all this good for? Consider again a signal of interest, *s(t).* In any realistic signal, what one actually obtains is *p(t) = s(t) + n(t),* where *n(t)* is *noise* in the data.

This noise is not the response function, *r(t).* That is, even if there is no instrument, the data itself will typically contain some noise

If one then performs the autocorrelation of *p(t) = s(t) + n(t),* we get

$$p \odot p = s \odot s + s \odot n + n \odot s + n \odot n.$$

For sufficiently long lag times, the signal and noise are uncorrelated, so

$$s \odot n + n \odot s \to 0$$

so,

$$\underbrace{p \odot p}_{\text{Measured and Includes noise}} = s \odot s + n \odot n = \underbrace{s \odot s}_{\textit{True}\text{ signal}} + |\langle n(t) \rangle|^2.$$

which means that the autocorrelation of *s(t)* is embedded in a *constant noise* term.
This makes extracting the autocorrelation of *s(t)* easier.

To see how this works, do question (4) on the worksheet.