PHY 420 Spring 2021

Class Summary—Week 1, Day 2—Thursday, Apr 1

Poynting's Theorem

Today, we will learn about the conservation of energy in the electromagnetic field, often called **Poynting's theorem**. To do so, we will begin with the force on a single moving charge.

From your undergraduate days, you will no doubt remember that the force on a single charge q traveling at velocity v is given by

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

where the first term on the right hand side is the force due to the electric field, and the second term is the force due to the magnetic field.

If E_{kin} is the kinetic energy of the charges, then the rate of doing work by the external electromagnetic fields $\vec{E}$ and $\vec{B}$ is

$$\frac{dE_{\text{kin}}}{dt} = \vec{F} \cdot \vec{v} = q\vec{E} \cdot \vec{v}$$

as you showed in Question 1(a) on the worksheet in class today. The magnetic field does no work, since the magnetic force is perpendicular to the velocity.

For a continuous distribution of charge and current, we replace $q\vec{v}$ by $\vec{J}\,d^3x$ and then integrate over the volume V of the charge distribution to get the **total rate of work done by the fields** in a finite volume:

$$\int_V \vec{J} \cdot \vec{E}\, d^3x \tag{6.103}$$

This power represents a conversion of *electromagnetic energy into mechanical or thermal energy*. It must be balanced by *a corresponding rate of decrease of energy in the electromagnetic field* within the volume V. We will now write this conservation law explicitly. To do so, first use the Ampere-Maxwell law:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

to write

$$\vec{J} = \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t}$$

and then use the right hand side to replace $\vec{J}$ in equation (6.103):

$$\int_V \vec{J} \cdot \vec{E}\, d^3x = \int_V \left[\left(\vec{\nabla} \times \vec{H} \right) - \frac{\partial \vec{D}}{\partial t} \right] \cdot \vec{E}\ d^3x$$

Distributing and flipping the order in which we take the dot product, we get

$$\int_V \vec{J} \cdot \vec{E}\, d^3x = \int_V \left[\vec{E} \cdot \left(\vec{\nabla} \times \vec{H} \right) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right]\, d^3x \tag{6.104}$$

as you showed in Question 1(b) on today's worksheet.

On the previous page, we obtained equation (6.104):

$$\int_V \vec{J}\cdot\vec{E}\,d^3x = \int_V \left[\vec{E}\cdot\left(\vec{\nabla}\times\vec{H}\right) - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right] d^3x$$

Now, consider the vector identity

$$\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) = \vec{H}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}\cdot\left(\vec{\nabla}\times\vec{H}\right)$$

which, upon rearranging, can be written as

$$\vec{E}\cdot\left(\vec{\nabla}\times\vec{H}\right) = \vec{H}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right)$$

Use this expression to replace $\vec{E}\cdot\left(\vec{\nabla}\times\vec{H}\right)$ on the right hand side of equation (6.104):

$$\int_V \vec{J}\cdot\vec{E}\,d^3x = \int_V \left[\vec{H}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right] d^3x$$

Then use Faraday's law, $\vec{\nabla}\times\vec{E} = -\partial B/\partial t$, in the equation above

$$\int_V \vec{J}\cdot\vec{E}\,d^3x = \int_V \left[\vec{H}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) - \vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right] d^3x$$

and rearrange, *as you did in Question 2 on today's worksheet*, to get

$$\int_V \vec{J}\cdot\vec{E}\,d^3x = -\int_V \left[\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right] d^3x \qquad (6.105)$$

Equation (6.105) represents the rate of decrease of energy in the electromagnetic field within the volume V, and this goes into increasing the mechanical (or thermal) energy of the moving charges.

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To proceed, we will make **two assumptions**:

- Assume that the macroscopic medium is linear in its electric properties (i.e., $\vec{D} = \epsilon\vec{E}$), with negligible dispersion or losses, so that

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E} \cdot \vec{D}\right)$$

  and, likewise, linear in its magnetic properties ($\vec{B} = \mu\vec{H}$), again with negligible dispersion or losses, so that

$$\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\vec{B} \cdot \vec{H}\right)$$

- We assume also that the total electromagnetic energy, even for time-varying fields, is the sum of equation (4.89):

$$W = \frac{1}{2}\int \vec{E} \cdot \vec{D}\, d^3x$$

  and equation (5.148):

$$W = \frac{1}{2}\int \vec{H} \cdot \vec{B}\, d^3x$$

  so that the total energy density is given by

$$u = \frac{1}{2}\left(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}\right) \tag{6.106}$$

Combining both of the above, we get that

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}\right) = \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}\right)\right] = \frac{\partial u}{\partial t}$$

as you showed in Question 3(a) on today's worksheet.

Substituting this expression in equation (6.105) and moving the minus sign to the left hand side, we get

$$-\int_V \vec{J} \cdot \vec{E}\, d^3x \;= \int_V \left[\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right)\right] d^3x \tag{6.107}$$

as you showed in Question 3(b) on today's worksheet.

Since the volume V in equation (6.107) is arbitrary, the integrand in that equation can be written in the form of a differential continuity equation or conservation law

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) = -\vec{J} \cdot \vec{E}$$

or

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \left(\vec{S}\right) = -\vec{J} \cdot \vec{E} \tag{6.108}$$

where

$$\vec{S} = \vec{E} \times \vec{H} \tag{6.109}$$

The vector $\vec{S}$ represents energy flow, and is called the **Poynting vector**.

The *Poynting vector*, $\vec{S} = \vec{E} \times \vec{H}$, has the dimensions of **energy/(area $\times$ time)**, *as you verified in Question 4 of today's worksheet.*

Let us now consider the **physical meaning** of the integral form in equation (6.107) or the differential form in equation (6.108). Either equation tells us that the time rate of change of electromagnetic energy within a certain volume, plus the energy per unit time flowing out through the boundary surfaces of the volume, is equal to the negative of the total work done by the fields on the sources within the volume. This is a statement of the *conservation of energy*.

Note that the **applicability** of the simple version of Poynting's theorem written in equation (6.107) or equation (6.108) is to *vacuum* macroscopic or microscopic fields. Even for linear media, there is always dispersion (with accompanying losses). In that case, the right hand side of equation (6.105) does not have the simple interpretation of equation (6.107). The more realistic situation for linear dispersive media is discussed by Jackson in Section 6.8.

So far, we've emphasized on the energy of the electromagnetic fields. The work done per unit time per unit volume by the fields ($\vec{J} \cdot \vec{E}$) is a conversion of electromagnetic energy into mechanical or heat energy (e.g., for ohmic conductors, $\vec{J} = \sigma\vec{E}$, and $\vec{J} \cdot \vec{E}$ is converted to heat via the resistance of the material). So, since matter is ultimately composed of charged particles (electrons and atomic nuclei), we can think of this rate of conversion as a rate of increase of energy of the charged particles per unit volume. Then, we can interpret Poynting's theorem for the *microscopic* fields ($\vec{E}$ and $\vec{B}$) as a statement of conservation of energy of the combined system of particles and fields.

If we denote the total energy of the particles within the volume V as E_{mech} and assume that no particles move out of the volume, we have

$$\frac{dE_{\text{mech}}}{dt} = \int_V \vec{J} \cdot \vec{E}\, d^3x \tag{6.110}$$

This allows us to write Poynting's theorem in terms of the energy in the field and the energy of the particles.

In Question 5 of today's worksheet, you showed that Poynting's theorem expresses the conservation of energy for the combined system (of the **energy of the particles** and the **energy in the field**) as

$$\frac{dE}{dt} = \frac{d}{dt}\Big(E_{\text{mech}} + E_{\text{field}}\Big) = -\oint_S \hat{n} \cdot \vec{S}\, da \tag{6.111}$$

where the total field energy within V is

$$E_{\text{field}} = \int_V u\, d^3x = \frac{\epsilon_0}{2} \int_V \Big(\vec{E}^2 + c^2\vec{B}^2\Big)\, d^3x \tag{6.112}$$

In the next class, we will discuss the *conservation of linear momentum.*