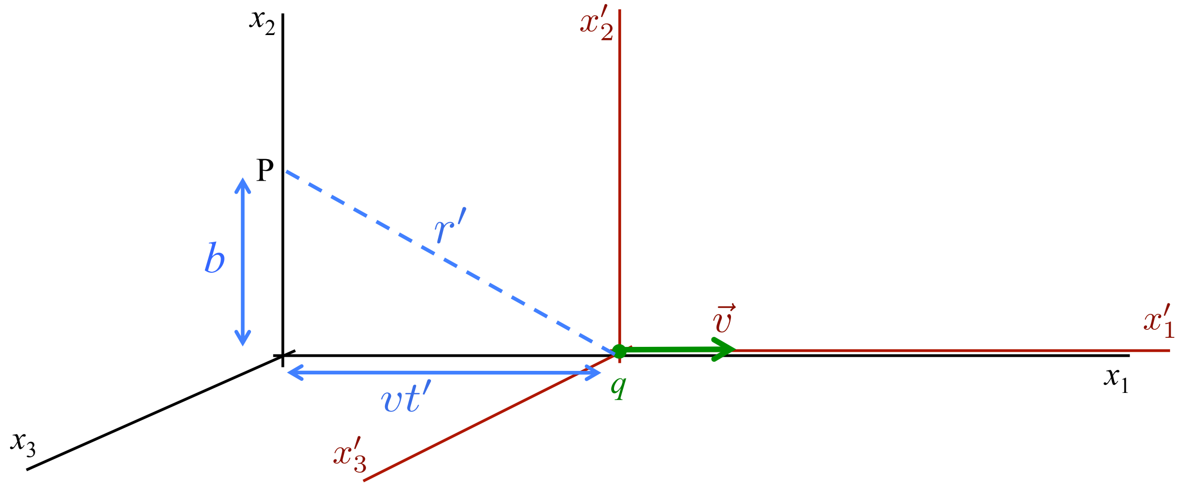
PHY 420 **Spring 2021**

Class Summary—Week 10, Day 1—Tuesday, June 1

In the previous class, we began discussing the transformation of electromagnetic fields. We will now consider an **example of the transformation of fields** to illustrate some important points.



Consider the example shown in the figure above (modified Figure 11.8 on page 559 in Jackson), in which a point charge q at rest in frame K' is traveling along the x_1 direction of frame K with velocity v. An observer is located at point P on the x_2 axis, at a distance b from the origin of frame K. Since the origins of K and K' coincide at $t' = t = 0$, b is thus the distance of closest approach of the charge to the observer located at point P.

From visual inspection of the figure above, the coordinates in frame K' of the point P are

$$x'_1 = -vt', \qquad x'_2 = b, \qquad x'_3 = 0$$

as you wrote in Question 1(a) on today's worksheet.

Since the charge q is at rest in frame K', there is only an electric field in K'. Thus, the electric and magnetic fields at the observation point P in K' are

$$\vec{E}\,' = \frac{q\hat{r}\,'}{r'^{\,2}} = \frac{q\vec{r}\,'}{r'^{\,3}} \qquad\qquad \vec{B}\,' = 0$$

If we wrote out $\vec{E}\,'$ in terms of its components, we would get

$$\vec{E}\,' = \frac{q}{r'^{\,3}}\left[x'_1\hat{x}'_1 + x'_2\hat{x}'_2 + x'_3\hat{x}'_3\right]$$

This means that

$$E'_1 = \frac{qx'_1}{r'^{\,3}} = -\frac{qvt'}{r'^3} \qquad \text{and} \qquad E'_2 = \frac{qx'_2}{r'^{\,3}} = -\frac{qb}{r'^3}$$

where I've put $x'_1 = -vt'$ and $x'_2 = b$, as written above. Therefore, writing the components separately (*as you did in Question 1(b) on today's worksheet*), we get

$$\begin{aligned} E'_1 &= -\frac{qvt'}{r'^3} \qquad & E'_2 &= \frac{qb}{r'^3} \qquad & E'_3 &= 0 \\ B'_1 &= 0 & B'_2 &= 0 & B'_3 &= 0 \end{aligned} \tag{11.151}$$

where I've also written each component of $\vec{B}\,'$ equal to zero.

Next, we want to write the **transformed fields in frame K**.

The first step is to express r', the distance from the origin of K' to the observer at point P (see figure on the previous page) in terms of quantities in frame K; we can use the triangle marked in blue in the figure on the previous page to write

$$r' = \sqrt{(vt')^2 + b^2}$$

All quantities are already in terms of what can be used for frame K except for t'. So, all we need to do is use the Lorentz transformation

$$x_0' = \gamma(x_0 - \beta x_1)$$

taking note that

$$x_0 = ct, \qquad x_0' = ct' \qquad \text{and} \qquad x_1 = 0$$

the latter (i.e., x_1) obtained from the space coordinates of point P in frame K, which are $(0, b, 0)$. Substituting these into the Lorentz transformation equation for x_0' written above, we get that

$$ct' \ = \gamma \Big[ct - \beta(0)\Big]$$

and thus

$$t' \ = \gamma\, t$$

Thus, in terms of quantities in frame K, we get

$$r' = \sqrt{(vt')^2 + b^2} = \sqrt{(v\gamma t)^2 + b^2}$$

Swapping the two terms, we obtain

$$r' = \sqrt{b^2 + \gamma^2 v^2 t^2}$$

Inserting this into the inverse transformation equations that we obtained from equation (11.148) in Jackson in the worksheet from the previous class, we get the equations in frame K. Let's do an example. For E_1, the transformation equation is simply $E_1 = E_1'$, and substituting the expression for E_1' written in equation (11.151) on the previous page, we get

$$E_1 = E_1' = -\frac{qvt'}{(r')^3} = -\frac{qv\,(\gamma t)}{\left(\sqrt{b^2 + \gamma^2 v^2 t^2}\right)^3} = -\frac{q\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

Let's do another example. The transformation equation for E_2 is

$$E_2 = \gamma\Big(E_2' + \beta B_3'\Big) = \gamma\Big(E_2' + \beta[0]\Big) = \gamma\, E_2' = \gamma\,\frac{qb}{r'^3} = \frac{q\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

where I've put $B_3' = 0$, and $E_2' = qb/(r')^3$ from equation (11.151) on the previous page.

Finally, the transformation equation for B_3 is

$$B_3 = \gamma\Big(B_3' + \beta E_2'\Big) = \gamma\Big(0 + \beta E_2'\Big) = \gamma\,\beta E_2' = \beta\Big[\gamma E_2'\Big] = \beta\Big[E_2\Big]$$

It is easy to show that the remaining components are zero ($E_3 = B_1 = B_2 = 0$) *as you derived in Question 2 on today's worksheet.*

Thus, *you showed in Question 2 on today's worksheet* that the transformed fields in frame K are

$$E_1 = E_1' = -\frac{q\gamma vt}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \tag{11.152.a}$$

$$E_2 = \gamma E_2' = \frac{\gamma qb}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \tag{11.152.b}$$

$$B_3 = \gamma\beta E_2' = \beta E_2 \tag{11.152.c}$$

with the other components vanishing; that is, $E_3 = 0; B_1 = 0; B_2 = 0$.

Fields such as those in equation (11.152) exhibit interesting behavior when the velocity of the charge approaches that of light. We will now discuss this in more detail.

First, though, consider the information that is being conveyed by the relations in equation (11.152) above. They are telling us the fields at a point $(0, b, 0)$ in frame K when a charge q moves along the x_1 axis with speed v, passing the origin at time $t = 0$.

- At large values of t, the second term in the denominator dominates for all three field components E_1, E_2, and B_3. Then
$$E_1 \sim \frac{t}{(t^2)^{3/2}} = \frac{t}{t^3}$$
and thus $E_1 \sim 1/t^2$. Meanwhile
$$E_2 \sim \frac{1}{(t^2)^{3/2}} = \frac{1}{t^3}$$
and thus E_2 and B_3 both go as $1/t^3$.

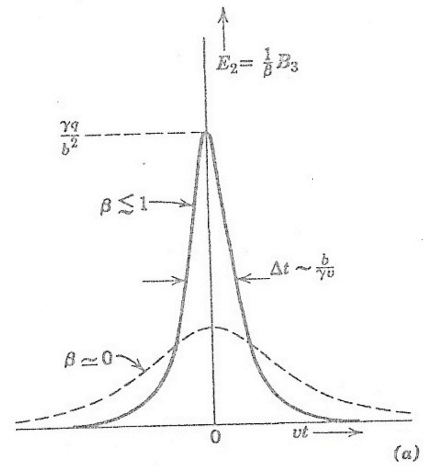
  If you count the past as negative time, this means that **all the field components (E_1, E_2, and B_3) go to zero at large negative times (the past) and large positive times (the future)**, *as you wrote in Question 3(a) on today's worksheet.*

- Then consider first the **behavior of E_2**. *As you showed in Question 3(b) on today's worksheet*, as t approaches zero, we get
$$E_2 = \frac{\gamma qb}{(b^2+\gamma^2 v^2 t^2)^{3/2}} = \frac{\gamma qb}{(b^2+\gamma^2 v^2 [0]^2)^{3/2}} = \frac{\gamma qb}{(b^2)^{3/2}} = \frac{\gamma qb}{(b)^3} = \frac{\gamma q}{b^2}$$
Thus, as t approaches zero from negative values, E_2 rises to a **maximum value** of $\gamma q/b^2$ at $t = 0$, and then falls back down to zero at positive values of t.

- Next, to obtain a **characteristic time scale**, consider equation (11.152.b) when $t = b/\gamma v$. Then
$$E_2 \sim \frac{\gamma qb}{(b^2+\gamma^2 v^2\{b/\gamma v\}^2)^{3/2}} = \frac{\gamma qb}{(2b^2)^{3/2}} = \frac{1}{2\sqrt{2}}\frac{\gamma q}{b^2}$$
Thus, at $t = b/\gamma v$, E_2 falls to $1/2\sqrt{2}$ of its peak value (of $\gamma q/b^2$ at $t = 0$). Therefore, $\Delta t = b/\gamma v$ gives a **characteristic width** (in time) for E_2, *as you found in Question 3(c) on today's worksheet.*

- All of the above can be applied to draw graphs for the field components E_1, E_2, and B_3; they are shown and discussed on the next page.

The **graph of E_2** is plotted on the top left of Figure 11.9 in Jackson, and a scanned copy is shown on the *right*. *You plotted this graph in Question 4(a) on today's worksheet.*



The dashed line shows E_2 for a charge moving slowly compared to c (i.e., $\beta \simeq 0$), and the solid line shows E_2 for a charge moving much faster, although slower than light speed (i.e., $\beta \lesssim 1$).

Clearly, E_2 **narrows and its peak pushes higher** as the speed of the charge increases. It is easy to work this out from the points discussed on the previous page (*as you did in Question 4(b) on today's worksheet*); as β approaches 1, meaning the velocity of the charge approaches that of light, the factor $\gamma \gg 1$. The width in time of the field E_2 has γ in the denominator, so larger γ values lead to narrowing in the width of E_2. Meanwhile, the peak value of E_2 is $\gamma q/b^2$ at $t = 0$ as we discussed on the previous page, so larger γ values will boost the peak and make it higher.

Next, let's consider the **behavior of B_3**.

- From equation (11.152.c) on the previous page, we know that $B_3 = \beta E_2$.
- This means that as the velocity of the charge approaches that of light, so that $\beta \to 1$, the magnetic field B_3 becomes almost equal to the transverse electric field, *as you found in Question 4(c) on today's worksheet.*

In fact, it is because $B_3 = \beta E_2$ that the graph shown in the figure above has its y-axis marked as both E_2 and B_3/β.

- Thus, the same considerations as for E_2 hold true for B_3 and the graph for B_3 has a similar shape to that of E_2, except that it is multiplied everywhere by β, *as you concluded in Question 5(a) on today's worksheet.*
- This means that the maximum value of B_3 at $t = 0$ is $\beta(\gamma q/b^2)$.

The fact that $B_3 = \beta E_2$ has another interesting consequence. If frame K' is moving very slowly with respect to K, so that v is small compared to c, then we must have $\beta \approx 0$. Meanwhile, since

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

this implies that $\gamma \approx 1$. In that case

$$E_2 = \gamma E_2' \approx E_1'$$

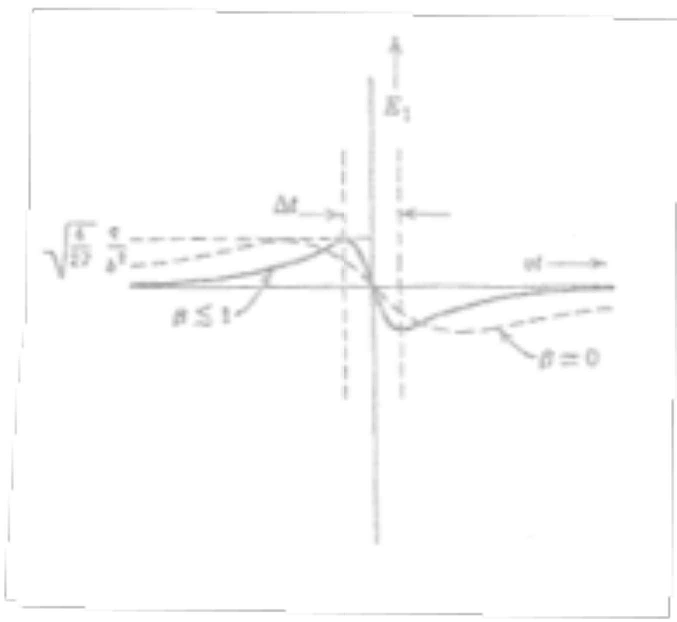
meaning that both frames K and K' see roughly the same electric fields. Also, since $B_3 = \beta E_2$, we would get $B_3 \approx 0$ for $\beta \approx 0$. All of this makes perfect sense; if K is at rest with respect to K', then we would also have only an electric field in frame K as we do in frame K' in which the charge is at rest. Also, if K' is moving very slowly with respect to K, then charge q is also almost at rest in frame K and both frames see similar electric fields, and K should see a tiny magnetic field very near a value of zero. It is only when K' is moving very fast compared to c that the magnitude of B_3 picks up, approaching that of E_2 as v approaches c (i.e., $\beta \to 1$).

Finally, let's consider the **behavior of E_1**.

As you showed in Question 5(b) on today's worksheet, $E_1 = 0$ at $t = 0$.

Meanwhile, *in Question 5(c) on today's worksheet*, you discussed by working with the signs how we would expect E_1 to be positive at negative values of t (i.e., in the past), and E_1 to be negative at positive values of t.

Since we know that $E_1 \to 0$ at large $\pm t$ (both past and future), we conclude that starting from zero in the past (negative t), E_1 should rise to a maximum at some value of $-t$, go to zero at $t = 0$, then dip to a minimum at some value of $+t$, before going to zero again at large values of $+t$. This is shown in the graph below.



By setting $dE_1/dt = 0$, we can find the extremum values for E_1 that are marked in the figure. See the video for the expression for dE_1/dt. Setting the numerator of this expression to zero, we get

$$q\gamma v b^2 + q\gamma^3 v^2 t^2 - 3\, q\gamma^3 v^3 t^2 = 0$$

Canceling a common factor of $q\gamma v$, we get

$$b^2 = 3\gamma^2 v^2 t^2 - \gamma^2 v^2 t^2$$

so that

$$b^2 = 2\gamma^2 v^2 t^2$$

Thus we get

$$t = \pm\frac{1}{\sqrt{2}}\,\frac{b}{\gamma v}$$

and substituting this in the expression for E_1, we get

$$E_1 = \mp\sqrt{\frac{4}{27}}\,\frac{q}{b^2}$$

So far, we've discussed the time dependence of the fields at a fixed observation point. An alternative description can be obtained by looking at the **spatial distribution of the fields at fixed time**. From equation (11.152), we see that $E_1/E_2 = -vt/b$. The figure on the first page (taken from Figure 11.8 in Jackson) then shows that the electric field is directed along $\hat{n}$, a unit radial vector from the present position of the charge to the observation point P (which is just like it is for a static Coulomb field). On Homework 8, you showed that the electric field in terms of the present position of the charge is given by

$$\vec{E} = \frac{q\vec{r}}{r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}} \tag{11.154}$$

where r is the radial distance from the present position of the charge to the observer (see Figure 11.8, reproduced on the first page of this class summary), and the angle $\psi = \cos^{-1}(\hat{n}\cdot\hat{v})$ is between the direction of $\hat{n}$ and $\vec{v}$, where $\hat{n}$ is a unit radial vector from the present position of the charge to the observation point P (i.e., a unit vector along the direction of $\vec{r}$), and $\vec{v}$ is along the positive x_1-axis.

- We see from equation (11.154) that the electric field is radial, but that the lines of force are isotropically distributed only for $\beta = 0$; otherwise, their distribution is anisotropic.
- Notice that for $\psi = 0, \pi$, we get

  $$E(\psi = 0, \pi) = \frac{qr}{r^3\gamma^2} = \gamma^{-2}\left(\frac{q}{r^2}\right)$$

  and since $\gamma = 1/(1-\beta^2)^{1/2}$, we get for $\psi = \pi/2$ that

  $$E(\psi = \pi/2) = \frac{qr}{r^3\gamma^2(1-\beta^2)^{3/2}} = \frac{\gamma^3 qr}{r^3\gamma^2} = \gamma\left(\frac{q}{r^2}\right)$$

  Thus, along the direction of motion ($\psi = 0, \pi$), the field strength is *reduced* by a factor of γ^{-2} relative to isotropy (for which it would be q/r^2), whereas in the transverse direction ($\psi = \pi/2$), the peak electric field strength is *larger* by a factor of γ relative to isotropy.

Therefore, we obtain what Jackson calls a whiskbroom pattern of lines of force, as shown in the figure on the right below (taken from Figure 11.9(b) in Jackson); it is the spatial snapshot equivalent of the temporal behavior shown in the figure on the previous page (taken from Figure 11.9(a) in Jackson).