Introduction to structure formation

1. In the lecture, we've seen how a sphere with a slight over density will have it's surface evolve in time as,

$$\frac{\ddot{R}}{R} = -\frac{4\pi\, G\, \bar{\rho}}{3}\delta(t) \tag{1}$$

   (a) Suppose $\delta > 0$, what happens to the sphere?

   (b) How many unknowns appear in Eq. (1). What principle might you use to find another relation involving the unknowns.

   (c) In the lecture we've seen that an auxiliary condition we can use is the conservation of mass which leads to

$$R(t) = R_o\,[1 + \delta(t)]^{-1/3}$$

   where

$$R_o \equiv \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3} = \text{constant}$$

   Using the fact that for $\delta \ll 1$, we can write

$$R(t) \approx R_o\left[1 - \frac{1}{3}\delta(t)\right]$$

   take the second derivative of $R(t)$ with respect to time and substitute the result into Eq. (1) to form a new differential equation for $\delta(t)$. (Note make the substitution that $R_o \approx R$ after taking the second derivative.)

(d) Solve the differential equation and interpret the results.

2. In the lecture we've seen that

$$t_{\text{dyn}} = \frac{9.6\,\text{hours}}{\sqrt{\bar{\rho}}}.$$

Suppose the air around you had a density perturbation, $\delta(\vec{r}, t) \ll 1$, a not unreasonable supposition. Air has a mean density of $\bar{\rho} = 1$ kg m^{-3}.

(a) What is the dynamical time for gravitational collapse for the air around you.

(b) Do witness the collapse of air on the time scale you calculated in part (a)?

(c) Speculate on why one doesn't see the collapse calculated in (a).

3. In the lecture we've just seen that time it takes for pressure to build up in a region of radius R is

$$t_{\text{P}} = \frac{R}{c_s}$$

where c_s is the speed of sound in the region.

(a) At your table, discuss how t_{P} must compare to t_{dyn} in order for gravitational collapse to be stopped by the pressure. Make sure you justify your claim.

(b) We've seen that in order for pressure to stop a gravitational collapse due to a small over-density, we must have $t_{\mathrm{P}} < t_{\mathrm{dyn}}$. What length scale does this condition on t_{P} correspond to.

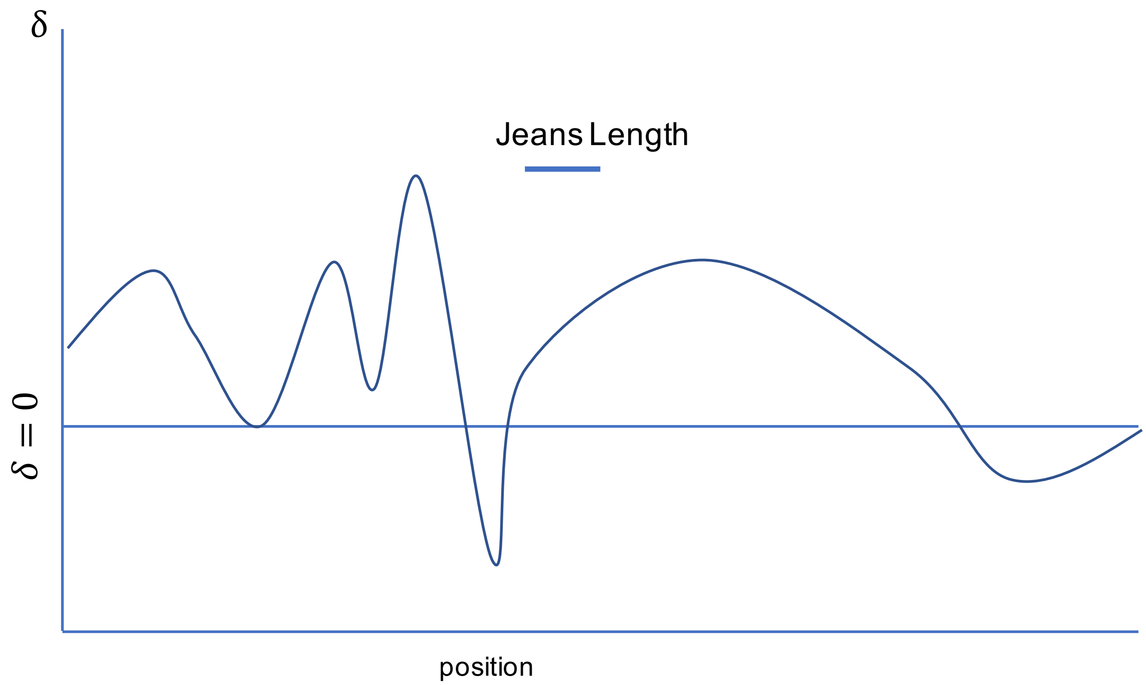
(c) Recalling that

$$t_{\mathrm{dyn}} = \frac{1}{\sqrt{4\pi G\bar{\rho}}},$$

rewrite the expression for the length scale. This length scale is called the *Jean's length*, λ_J.

(d) Discuss at your table what happens to over-dense regions that are larger that λ_J, and what happens to over-dense regions that are smaller than λ_J

(e) The figure below shows a 1-D universe density fluctuation field. The Jean's length is shown as a solid line and labeled as such. Find regions on the graph where gravitational collapse would occur, and label regions of under density and over density.



4. In the lecture, we've seen that at cosmic scales, the Jeans length is

$$\lambda_j = 2\pi \left(\frac{2}{3}\right)^{1/2} \frac{c_s}{H} = 2\pi \left(\frac{2}{3}\right)^{1/2} \sqrt{w}\frac{c}{H}.$$

   (a) For radiation, $w = 1/3$. Find the Jeans length for the time when the universe was radiation dominated and discuss at your table some of the consequences of this result.

(b) At decoupling, there are essentially two *gases*, a photon gas and a baryonic gas. For the baryonic gas, the sound speed is given by

$$c_s(\text{baryon}) = \left(\frac{kT}{mc^2}\right)^{1/2} c$$

Compare the Jeans length of the universe immediately before and immediately after decoupling. (Note $kT_{\text{dec}} \approx 0.26$ eV; $mc^2 \approx 1140$ MeV)

(c) At your table, discuss the evolution of structure formation before and after the era of decoupling.

5. For the case of a static universe, we obtained the expression,

$$\ddot{\delta} = 4\pi G\bar{\rho}\delta.$$

When we do the same analysis for an expanding universe we get

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}\delta$$

as we've just seen in the lecture.

(a) At your table discuss the differences on density fluctuations between the two expression above.

(b) The expression we derived above was done in the context of Newtonian gravity. The full GR derivation gives,

$$\ddot{\delta} + 2H\dot{\delta} = \frac{4\pi G}{c^2}\bar{\epsilon}_m\delta$$

Rewrite this equation in terms of the density parameter for matter,

$$\Omega_m = \frac{\bar{\epsilon}_m}{\epsilon_c} = \frac{8\pi G\bar{\epsilon}_m}{3c^2H^2}$$

(c) Is the expression in (b) generally applicable. If not, when does it apply?

(d) Consider the epoch in which the universe was radiation dominated. In this case $\Omega_m \ll 1, H = 1/(2t)$. Write your expression in (5b) for this era.

(e) Do the same for a universe that is dominated by a cosmological constant, in which case $H = H_\Lambda$.

Homework 04–Due Monday,March 9

1. Problem 7.3
2. Problem 7.4
3. Suppose $\Omega = 0.5$ in the early universe when the energy density is $\epsilon = 10^{16}$ GeV m^{-3}. At his time, suppose all the matter in the universe obeys $P = -\epsilon$ (i.e., single component universe).
   (a) After the scale factor increases by 60 e-foldings, what is the new value of Ω
   (b) Suppose at the end of the expansion described in part (a), all the energy density is instantly transformed into radiation (so the value of ϵ does not change, but the equation of state does). Assuming that the matter in the universe is composed *entirely* of radiation, what is the value of Ω when $T = 10^4 K$. The starting value of Ω you start here is the value you got in part **a.).**
4. Problem 11.4
5. **Grad Problem.** In this problem, you will carry out a very simple version of the parameter space process that cosmologists use to determine cosmological parameters in the Benchmark model. Please use a plotting software package to do this assignment, I do not want hand-drawn figures.
   (a) Draw a graph in which the x-axis is $\Omega_{m,o}$ and the y-axis is Ω_Λ. Each axis should go from 0 to 1. As you know the best shows that $\Omega_{m,o} + \Omega_\Lambda = 1$. Plot this line on the graph.
   (b) Observations of supernovae show that $\Omega_{m,o} - \Omega_\Lambda = -0.4$. Plot this line on the graph
   (c) If the CMB and supernovae results are both correct, what can you conclude about the values of $\Omega_{m,o}$ and Ω_Λ
   (d) What does the quantity $\Omega_{m,o} - \Omega_\Lambda$ tell you about the universe? For example, if the value $\Omega_{m,o} - \Omega_\Lambda = +0.4$ instead of what you plotted on the graph, what would be different about the universe? Explain in detail.