PHY 411 **Winter 2021**

Week 8—Thursday, Feb 25—Discussion Worksheet

Magnetostatics

We will cover selected topics in this chapter, more so because we've already seen many of the results in our discussions. I'll write most of the details into the class summary and have you take a look.

Some of the key points to remember:

- Since there are no "free magnetic charges" the basic entity in magnetostatics is the magnetic dipole.
- In the presence of magnetic materials, the dipole tends to align itself in a certain direction. That direction is, by definition, the direction of the $\vec{B}$, provided the dipole is sufficiently small and weak that it does not perturb the existing field.
- It is worth remembering that the charge density ρ at any point in space is related to the current density J in that neighborhood by a continuity equation.

1. Let's begin by looking at the continuity equation.

(a) Write down the continuity equation.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$$

(b) Explain the continuity equation you wrote above in words.

Due to the law of energy conservation, the charge density at any point in space must be related to the charge density in that neighborhood.

(c) What form does the continuity equation take in magnetostatics?
Hint: Steady-state magnetic phenomena are characterized by no change in the net charge density anywhere in space, that is, there is no growth or depletion of charge at any one point.

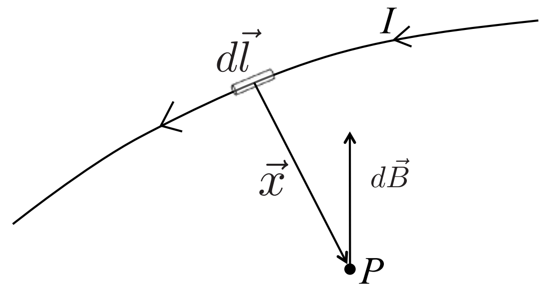
<u>Magnetostatics</u>

$$\nabla \cdot J = 0$$

Biot-Savart Law

If $d\vec{l}$ is an element of length, pointing in the direction of current flow, of a filamentary wire that carries a current I and $\vec{x}$ is the coordinate vector from the element of length to an observation point P, as shown in the figure (taken from Figure 5.1 on page 175 in Jackson), then $d\vec{B}$ at the point P is given in magnitude and direction by

$$d\vec{B} = kI\,\frac{d\vec{l} \times \vec{x}}{|\vec{x}|^3} \tag{5.4}$$



In SI units, $k = \mu_0/4\pi = 10^{-7}$ N A^{-2}.

2. With $I d\vec{l}$ in equation (5.4) written as $\vec{J}(\vec{x}\,')\,d^3x'$, we have

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}\,') \times \frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3}\,d^3x' \tag{5.14}$$

(a) Show that equation (5.14) can be written as

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\vec{\nabla} \times \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x' \tag{5.16}$$

Hint: You will have to use the relation (from *Jackson*, page 29): $\dfrac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3} = -\vec{\nabla}\left(\dfrac{1}{|\vec{x}-\vec{x}\,'|}\right)$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}\,') \times \frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3}\,d^3x' \quad \text{where}$$

$$\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) \quad \text{and} \quad \vec{\nabla}\times(a\vec{A}) = \vec{\nabla}a \times \vec{A} + a\vec{\nabla}\times\vec{A}$$

$$\vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\times\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)d^3x' = \frac{\mu_0}{4\pi}\int d^3x'\,\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\times\vec{J}(\vec{x}')$$

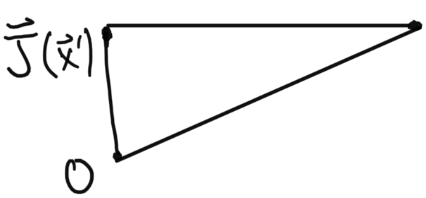
$$= \frac{\mu_0}{4\pi}\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$$

(b) Use equation (5.16) to show that

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{5.17}$$

using $\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a})}_{=0}$

$$= \vec{\nabla}\cdot\vec{\nabla}\times(\ldots) = 0$$



Ampere's Law

From equation (5.16), we also get

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{5.22}$$

The integral equivalent of equation (5.22) is called *Ampere's law.*

3. By integrating the normal component of equation (5.22) over a surface S bounded by a closed curve C, and applying Stokes' theorem (inside front cover of Jackson), show that we get Ampere's law

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I \tag{5.23}$$

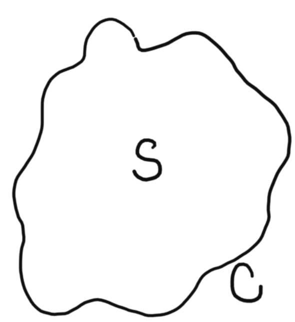
$$\rightarrow \int_S \underbrace{\vec{\nabla} \times \vec{B} \cdot \hat{n}\, da}_{\text{Stoke's theorem}} = \mu_0 \int_S \vec{J} \cdot \hat{n}\, da$$

Curl theorem: $\int (\nabla \times A) \cdot da = \oint A \cdot dl$

$$\rightarrow \oint_C \vec{B} \cdot dl = \mu_0 \int_S J \cdot \hat{n}\, da$$

$$\rightarrow \oint_C \vec{B} \cdot dl = \mu_0 I$$

(the integral is over
Closed Curve C
that is the bounding
Curve of Surface S)



Vector Potential

Since $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{B}$ must be the curl of some vector field $\vec{A}(\vec{x})$, called the **vector potential**; we learned this earlier in the quarter for the full electromagnetic field. So, we can write

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}) \tag{5.27}$$

Now, we've already written $\vec{B}$ in this form in equation (5.16), so from it, we get the general form of $\vec{A}$ to be

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x' + \vec{\nabla}\Psi(\vec{x}) \tag{5.28}$$

The added gradient of an arbitrary scalar function Ψ shows that for a given magnetic induction $\vec{B}$, the vector potential can be freely transformed according to

$$\vec{A} \to \vec{A} + \vec{\nabla}\Psi \tag{5.29}$$

Recall from earlier this quarter that such a transformation is called a *gauge transformation*, and such a gauge transformation gives us the freedom to make $\vec{\nabla} \cdot \vec{A}$ have any convenient functional form we wish.

4. Consider equation (5.27) written about that relates $\vec{B}$ and the vector potential $\vec{A}$.

(a) Write down the expression you get when you substitute equation (5.27) into $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}(\vec{x})) = \mu_0 \vec{J}$$

(b) Show that in the Coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$), the expression in part (a) reduces to the "Poisson" equation

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \tag{5.31}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}(\vec{x})) = \mu_0 \vec{J}$$

Second Derivative $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

If $\vec{\nabla} \cdot \vec{A} = 0$ then $\cancel{\vec{\nabla}(\vec{\nabla} \cdot \vec{A})}^{\,0} - \nabla^2 \vec{A} = \mu_0 \vec{J}$

$\rightarrow -\nabla^2 \vec{A} = \mu_0 \vec{J} \rightarrow \boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$