Learning Goals

1. Implicit schemes to solve PDEs
2. Crank-Nicolson Method

The finite difference schemes we've used so far are called *explicit schemes.*

They are called this because the PDE is solved at each point in the spatial grid one at a time.

These methods have some disadvantages, especially stability and that these schemes are only accurate to first order in the time. This means small time steps and many of them.

An alternative set of schemes exist. One is called *finite element* schemes which are in wide use in structural engineering because unusual shaped boundary conditions are easily handled (there are other advantages as well)

Another set of schemes still based on *finite differences* are called *implicit schemes.* We'll now introduce these schemes using the 1- D heat equation as the example

$$u_t = k u_{xx}$$

where *u* is the temperature at position *x* and time *t* and *k* is the thermal conductivity.

Notation: $u(x_i, t_m) \equiv u_i^m.$

Time step

Grid point

Following our usual procedure we obtain

$$u_{xx} \approx \frac{u_{i+1}^{m+1} - 2u_i^{m+1} + u_{i-1}^{m+1}}{h_x^2}$$

$$u_t \approx \frac{u_i^{m+1} - u_i^m}{\delta_t}$$

Substituting into heat equation gives,

$$-\lambda u_{i-1}^{m+1} + (1+2\lambda)u_i^{m+1} - \lambda u_{i+1}^{m+1} = u_i^m \text{ where } \lambda = k\frac{\delta_t}{h_x^2}. \qquad (1)$$

To apply (1) we need two things

- Initial conditions for the case *m = 0*
- Boundary conditions at *i = 0, i = N-1*

Implicit schemes involve solving for *all* points simultaneously at the same time.
Implicit schemes are unconditionally stable
Implicit schemes are harder to apply to 2 and 3 – D spatial problems.

Let’s see how implicit schemes work. To begin do question (1) on the worksheet and S T O P

(1) At first interior point

$$
\begin{aligned}
-\lambda u_0^1 + (1+2\lambda)u_1^1 - \lambda u_2^1 &= u_1^0 \\
(1+2\lambda)u_1^1 - \lambda u_2^1 &= u_1^0 + \lambda u_0^1 \\
1.8u_1^1 - 0.4u_2^1 &= 0 + .4(100) \\
1.8u_1^1 - 0.4u_2^1 &= 40
\end{aligned}
$$

Interior point 1
Time step 1

Interior point 2
Time step 1

Now do question (2) on the worksheet and S T O P

(2) For interior points *i =2,3:*

$$
\begin{aligned}
-\lambda u_{i-1}^1 + (1+2\lambda)u_i^1 - \lambda u_{i+1}^1 &= u_i^0 \\
\text{for } i &= 2: \\
-0.4u_1^1 + 1.8u_2^1 - 0.4u_3^1 &= 0 \\
\text{for } i &= 3: \\
-0.4u_2^1 + 1.8u_3^1 - 0.4u_4^1 &= 0
\end{aligned}
$$

Do question (3) on the worksheet and S T O P

(3) For *i = 4*

$$-0.4u_3^1 + 1.8u_4^1 = 20$$

We can write these results as

$$
\begin{pmatrix} 1.8 & -0.4 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 \\ 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & -0.4 & 1.8 \end{pmatrix}
\begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \end{pmatrix}
=
\begin{pmatrix} 40 \\ 0 \\ 0 \\ 20 \end{pmatrix}
$$

Solving for the *u's* yields:

$$\begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \end{pmatrix} = \begin{pmatrix} 23.6 \\ 6.14 \\ 4.03 \\ 12.0 \end{pmatrix}$$

In general we have that

At first interior point: $$(1+2\lambda)u_1^{m+1} - \lambda u_2^{m+1} = u_1^m + \overbrace{\lambda u_0^{m+1}}^{\text{known}}$$

Away from the boundary: $$-\lambda u_{i-1}^{m+1} + (1+2\lambda)u_i^{m+1} - \lambda u_{i+1}^{m+1} = \overbrace{u_i^m}^{\text{known}}$$

At last interior point: $$(1+2\lambda)u_i^{m+1} - \lambda u_{i-1}^{m+1} = u_i^m + \underbrace{\lambda u_{i+1}^{m+1}}_{\text{known}}$$

A couple of things

- The matrix of coefficients *does not change* at each time step
- All we need to do is update the R.H.S
- The matrix is *tri-diagonal.*

For the next time step we have

$$(1+2\lambda)u_1^{m+1} - \lambda u_2^{m+1} = u_1^m + \lambda u_0^{m+1}$$

$$-\lambda u_{i-1}^{m+1} + (1+2\lambda)u_i^{m+1} - \lambda u_{i+1}^{m+1} = u_i^m$$

$$(1+2\lambda)u_i^{m+1} - \lambda u_{i-1}^{m+1} = u_i^m + \lambda u_{i+1}^{m+1}$$

$$\underbrace{\begin{pmatrix} 1.8 & -0.4 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 \\ 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & -0.4 & 1.9 \end{pmatrix}} \begin{pmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \\ u_4^2 \end{pmatrix} = \begin{pmatrix} 23.6 + 40 \\ 6.14 \\ 4.03 \\ 12.0 + 20 \end{pmatrix}$$

The state of the art in implicit schemes are the *Crank-Nicolson methods*

Crank-Nicolson methods just up the order of accuracy in the spatial dimensions.

The partials are now approximated as

$$
\begin{aligned}
u_{xx} &= \frac{1}{2}\left[\frac{u_{i-1}^{m} - 2u_{i}^{m} + u_{i+1}^{m}}{h_x^2} + \frac{u_{i-1}^{m+1} - 2u_{i}^{m+1} + u_{i+1}^{m+1}}{h_x^2}\right] \\
u_{tt} &= \frac{u_i^{m+1} - u_i^{m}}{\delta_t}
\end{aligned}
$$

and after substituting into the heat equation, we have that

$$
-\lambda u_{i-1}^{m+1} + 2(1+\lambda)u_i^{m+1} - \lambda_{i+1}^{m+1} = \lambda u_{i-1}^{m} + 2(1+\lambda)u_i^{m} - \lambda_{i+1}^{m}
$$

The resulting matrix equation will be

$$\begin{pmatrix} 2+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 2+2\lambda & -\lambda & \dots & 0 \\ 0 & \vdots & \vdots & \vdots & \\ 0 & \vdots & -\lambda & 2+2\lambda & -\lambda \\ 0 & \dots & 0 & -\lambda & 2+2\lambda \end{pmatrix} \begin{pmatrix} u_1^{m+1} \\ u_2^{m+1} \\ \vdots \\ \vdots \\ u_{N-1}^{m+1} \end{pmatrix} =$$

$$\begin{pmatrix} \lambda u_o^m + (2-2\lambda)u_1^m + \lambda u_2^m + \lambda u_o^{m+1} \\ \lambda u_1^m + (2-2\lambda)u_2^m + \lambda u_3^m \\ \vdots \\ \vdots \\ \lambda u_{N-2}^m + (2-2\lambda)u_{N-1}^m + \lambda_N^m + \lambda u_N^{m+1} \end{pmatrix}$$

Do questions (4) and (5) on the worksheet.