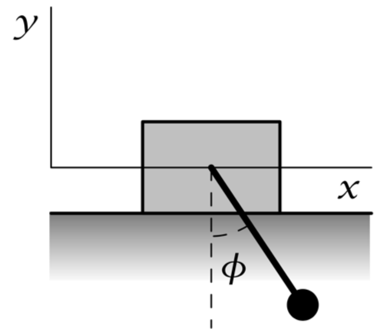
Activity 6: Horizontal Block and Pendulum

1. A block of mass m_1 slides along a frictionless horizontal surface as shown below. Hanging from the block is a pendulum of length l and mass m_2.



a. Using the $x - y$ axes shown, let the position of the block be (x_1, y_1) and the position of the pendulum bob be (x_2, y_2). Write down the kinetic and potential energy for the block and the pendulum.
b. Expressing the position of the pendulum in terms of the angle shown, find the Lagrangian and the equations of motion for the system.
c. You should find that there is a cyclic coordinate. Find the corresponding constant of the motion. What does this constant represent in terms of the physics of the system?
d. Let the constant from part (3) be called P. Use P to eliminate $\dot{x}$ from the equation of motion and show that the angle of the pendulum motion is determined by the equation
$\ddot{\phi} = \left(\frac{g}{l}\sin\phi + \frac{\alpha}{2}\sin(2\phi)\,\dot{\phi}^2\right)/(1 - \alpha\cos^2\phi)$
where $\alpha = \frac{m_2}{m_2+m_1}$
e. What happens with the equation of motion for ϕ if $m_2 \ll m_1$? Provide a physical explanation for this.

a. $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2), V = m_2 g y_2$
b. Generalized coordinates x_1, ϕ

$x_2 = x_1 + l\sin\phi \qquad y_2 = -l\cos\phi$

$\dot{x}_2 = \dot{x}_1 + l\dot{\phi}\cos\phi \quad \dot{y}_2 = l\dot{\phi}\sin\phi$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left[\left(\dot{x}_1 + l\dot{\phi}\cos\phi\right)^2 + \left(l\dot{\phi}\sin\phi\right)^2\right] + m_2 gl\cos\phi$$
$$= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left[\dot{x}_1^2 + l^2\dot{\phi}^2\cos^2\phi + l^2\dot{\phi}^2\sin^2\phi + 2\dot{x}_1 l\dot{\phi}\cos\phi\right] + m_2 gl\cos\phi$$

$$= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left[\dot{x}_1^2 + l^2\dot{\phi}^2 + 2\dot{x}_1 l\dot{\phi}\cos\phi\right] + m_2 gl\cos\phi$$
$$= \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2\dot{x}_1 l\,\dot{\phi}\cos\phi + m_2 gl\cos\phi$$

Lagrange's equations for x_1 and ϕ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}\left[(m_1 + m_2)\dot{x}_1 + ml\dot{\phi}\cos\phi\right] = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(ml^2\dot{\phi} + ml\dot{x}_1\cos\phi\right) = -ml\dot{x}_1\dot{\phi}\sin\phi - mgl\sin\phi$$

c. x_1 does not appear in the Lagragian, so it is cyclic, and $P = \frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1 + ml\dot{\phi}\cos\phi$ is a constant of motion. This is the total linear momentum of the system in x direction.

d. $P = (m_1 + m_2)\dot{x}_1 + ml\dot{\phi}\cos\phi \Rightarrow \dot{x}_1 = \frac{1}{m_1+m_2}(P - ml\dot{\phi}\cos\phi)$

$$\frac{d}{dt}\left(m_2 l^2\dot{\phi} + \frac{m_2 l\cos\phi}{m_1 + m_2}(P - m_2 l\dot{\phi}\cos\phi)\right)$$

$$= -m_2 l\sin\phi\frac{\dot{\phi}}{m_1 + m_2}(P - m_2 l\dot{\phi}\cos\phi) - m_2 gl\sin\phi$$

$$m_2 l^2\ddot{\phi} - \frac{m_2 l\dot{\phi}\sin\phi}{m_1 + m_2}\left(P - m_2 l\dot{\phi}\cos\phi\right) - \frac{m_2^2 l^2\,\ddot{\phi}\cos\phi}{m_1 + m_2} + \frac{m_2^2 l^2\dot{\phi}^2\cos\phi\sin\phi}{m_1 + m_2}$$

$$= \frac{-m_2 l\dot{\phi}\sin\phi}{m_1 + m_2}(P - m_2 l\dot{\phi}\cos\phi) - m_2 gl\sin\phi$$

$$\ddot{\phi} - \frac{m_2\,\ddot{\phi}\cos\phi}{m_1 + m_2} = \frac{-\,m_2\dot{\phi}^2\sin\phi\cos\phi}{m_1 + m_2} - \frac{g}{l}\sin\phi$$

substituting $\alpha = \frac{m_2}{m_1+m_2}$ and $\sin\phi\cos\phi = \frac{1}{2}\sin 2\phi$ we get

$$\ddot{\phi}(1 - \alpha\cos\phi) = -\frac{\alpha}{2}\dot{\phi}^2\sin 2\phi - \frac{g}{l}\sin\phi$$

and

$$\ddot{\phi} = \frac{-\frac{\alpha}{2}\dot{\phi}^2\sin 2\phi - \frac{g}{l}\sin\phi}{1 - \alpha\cos\phi}$$

e. for $m_1 \gg m_2 \Rightarrow \alpha \to 0$

$$\ddot{\phi} = -\frac{g}{l}\sin\phi$$

This is the equation of motion for a simple pendulum. If the mass of the pendulum is much smaller than the mass of the block it does not affect the motion of the block. The block therefore moves with constant velocity, and the pendulum simply swings back and forth.