Physics 412—Quantum Mice Compatibility Winter 2020

We will continue our studies of the size and attitude of quantum mice, moving onto the compatibility of operators. As we know, quantum mice come in two sizes, small and large. The size property is represented mathematically by the operator W, with two eigenvalues, 2 and 10, and corresponding eigenstates $|s\rangle$ and $|l\rangle$. The attitude property is represented by the operator A, with eigenvalues $+1$ (happy) and -1 (unhappy), and corresponding eigenstates $|h\rangle$ and $|u\rangle$.

If we use the attitude states as our basis, we can represent the operators as matrices and the states as column vectors:

$$A \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad |h\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |u\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$W \leftrightarrow \frac{2}{5}\begin{bmatrix} 21 & -8 \\ -8 & 9 \end{bmatrix}, \qquad |s\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad |l\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Note: You can answer all of these working in the attitude basis, using the representation given above. It's probably easiest that way.

Let's consider two more properties of quantum mice: their behavior and their energy. Just like size and attitude, when we measure these properties, we can only observe one of two values. We will model these using the operators B (for behavior) and H (the Hamiltonian), each with two eigenvalues. In the attitude basis, the operators B and H are represented by the matrices

$$B \leftrightarrow \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \text{and} \quad H \leftrightarrow \frac{2}{5}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}.$$

(1) Calculate the commutators of B and H with A and W: $[A, B]$, $[A, H]$, $[W, B]$, and $[W, H]$. Are either B or H compatible with either A or W? If you find that two operators are incompatible, verify that their commutator can be written in the form of Eq. (3.34), $[A, B] = \mathrm{i}C$, where C is Hermitian.

(2) Find the eigenvalues and eigenvectors of B and H. What basis are these represented in? (Think carefully if you can use the results of question (1) to help with this!)

(3) You should find that B has one positive and one negative eigenvalue. Call the behavior eigenstates $|p\rangle$ (passive, positive eigenvalue) and $|a\rangle$ (aggressive, negative eigenvalue). We can use these as basis states if we like. Eq. (3.10) of the course notes tells us how to construct the unitary operator that will convert from the old basis, $|h\rangle$ and $|u\rangle$, into the new basis, $|p\rangle$ and $|a\rangle$. It is

$$U_{A\to B} = |p\rangle\langle h| + |a\rangle\langle u|.$$

Construct the matrix representation of this operator explicitly, verify that it is unitary, and verify that it does what it's advertised to do (converts $|h\rangle$ and $|u\rangle$ into $|p\rangle$ and $|a\rangle$).

(4) Calculate $U^\dagger_{A\to B} B U_{A\to B}$. Is it what you expected? Explain.

(5) If E_1 and E_2 are the eigenvalues of H, the corresponding states are $|E_1\rangle$ and $|E_2\rangle$. Repeat question (3) for the change of basis from $|h\rangle$ and $|u\rangle$ to $|E_1\rangle$ and $|E_2\rangle$ by constructing

$$U_{A\to H} = |E_1\rangle\langle h| + |E_2\rangle\langle u|,$$

and verify the properties of $U_{A\to H}$ as in questions (3) and (4), calculating $U^\dagger_{A\to H} H U_{A\to H}$.