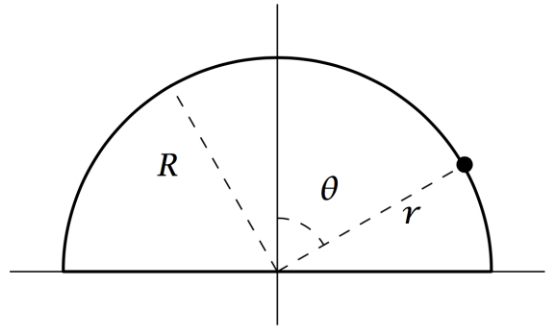
Activity 12: Bead on a Hemisphere

A particle of mass *m* start from rest very near the top of a frictionless hemisphere of radius *R*. Using Lagrange's λ −method and the formula for the constraint forces on p. 80 (which you wrote about in this week's warm-up quiz), find the force of constraint, and determine the angle θ at which the particle leaves the hemisphere. (You can solve this problem using a similar strategy as is used in example 3.1 on page 80-81.)



Because the hemisphere is frictionless, it can only apply a force in positive r direction. We want to find the force of constraint in r −direction, which is given by

$$Q_r = \lambda \frac{\partial f}{\partial r}$$

where λ is the Lagrange multiplier and $f = r - R = 0$ is the equation of constraint. In order to find the force of constraint we need to find an equation for λ. We do this by setting up the following equations:

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{k} \lambda \frac{\partial f_j}{\partial q_i}$ for each generalized coordinate q_i and

$$f_j(q_1, q_2, \dots, q_n) = 0$$

for each constraint j.

The geometry of this problem suggests polar coordinates, so we have two generalized coordinates (r and θ). We have one equation of constraint: $f = r - R = 0$. So we get the following three equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda \frac{\partial f}{\partial r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial f}{\partial \theta}$$

$$f = r - R = 0$$

So our strategy is to (1) derive the Lagrangian, (2) plug it in to the equations above, (3) solve these equations for λ and (4) calculate the force of constraint from $Q_r = \lambda \frac{\partial f}{\partial r}$

(1)

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \;\; V = mgy$$

$$x = r\sin\theta\,, \;\; y = r\cos\theta\,, \dot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta\,, \dot{y} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\frac{1}{2}m(\dot{x}^2+\dot{y}^2)=\frac{1}{2}m\left[\left(\dot{r}\sin\theta+r\dot{\theta}\cos\theta\right)^2+\left(\dot{r}\cos\theta-r\dot{\theta}\sin\theta\right)^2\right]$$

$$=\frac{1}{2}m(\dot{r}^2\sin^2\theta+r^2\dot{\theta}^2\cos^2\theta+\dot{r}^2\cos^2\theta+r^2\dot{\theta}^2\sin^2\theta+\cancel{2\dot{r}\sin\theta\, r\dot{\theta}\cos\theta}-\cancel{2\dot{r}\cos\theta\, r\dot{\theta}\sin\theta})$$

$$=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)$$

$$V=mgr\cos\theta$$

$$L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)-mgr\cos\theta$$

(2)

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{r}}=m\ddot{r},\quad \frac{\partial L}{\partial r}=mr\dot{\theta}^2-mg\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}=\frac{d}{dt}\left(mr^2\dot{\theta}\right)=m2r\dot{r}\dot{\theta}+mr^2\ddot{\theta}=m(2r\dot{r}\dot{\theta}+r^2\ddot{\theta}),\quad \frac{\partial L}{\partial\theta}=mgr\sin\theta$$

$$\frac{\partial f}{\partial r}=1,\ \frac{\partial f}{\partial\theta}=0$$

so

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{r}}-\frac{\partial L}{\partial r}=\lambda\frac{\partial f}{\partial r}\Leftrightarrow m\ddot{r}-mr\dot{\theta}^2+mg\cos\theta=\lambda$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}-\frac{\partial L}{\partial\theta}=\lambda\frac{\partial f}{\partial\theta}\Leftrightarrow m\left(2r\dot{r}\dot{\theta}+r^2\ddot{\theta}\right)-mgr\sin\theta=0$$

The constraint is $r=R$, so $\dot{r}=\ddot{r}=0$, so

$$-mR\dot{\theta}^2+mg\cos\theta=\lambda\quad (i)$$

$$\ddot{\theta}-\frac{g}{R}\sin\theta=0\quad (ii)$$

To solve for λ we integrate equation (ii) to get an expression for $\dot{\theta}^2$ and then plug it into equation (i).

To integrate equation (ii), first multiply it by $\dot{\theta}$:

$$\ddot{\theta}\dot{\theta}-\frac{g}{R}\dot{\theta}\sin\theta=0\Leftrightarrow\frac{d}{dt}\left(\frac{\dot{\theta}^2}{2}\right)+\frac{d}{dt}\left(\frac{g}{R}\cos\theta\right)=0\Leftrightarrow\frac{\dot{\theta}^2}{2}+\frac{g}{R}\cos\theta-C=0$$

We'll assume that at $t=0$ the particle is at rest at the top of the hemisphere ($\theta(t=0)=0,\dot{\theta}(t=0)=0$, so $\frac{g}{R}-C=0\Leftrightarrow\ C=\frac{g}{R}$

$$\frac{\dot{\theta}^2}{2}+\frac{g}{R}\cos\theta-\frac{g}{R}=0\Leftrightarrow\dot{\theta}^2=\frac{2g}{R}(1-\cos\theta)$$

We can now plug this expression for $\dot{\theta}^2$ into equation (i) and then solve for λ.

$$-m2g(1-\cos\theta)\ +mg\cos\theta=\lambda$$

$$\lambda=mg(3\cos\theta-2)$$

Finally, we plug this expression for λ into the equation for the force of constraint:

$$Q_r = \lambda \frac{\partial f}{\partial r} = mg(3\cos\theta - 2)$$

This expression makes sense! At the top of the hemisphere $\theta = 0, \cos\theta = 1$, so $Q_r = mg$, which is the normal force of a horizontal surface. At the bottom of the hemisphere $\theta = \pm\frac{\pi}{2}, \cos\theta = 0$, so the force of constraint is negative, meaning the particle has been lifted off the hemisphere. To find the angle θ at which the particle lifts of we set the force of constraint to 0:

$$mg(3\cos\theta - 2) = 0 \Leftrightarrow \theta = \pm\cos^{-1}\left(\frac{2}{3}\right) = \pm 48.2°$$