Homework 1

(1) Using the properties of the angular momentum operators and their eigenstates, evaluate

$$J_x|j,m\rangle \text{ and } J_y|j,m\rangle$$

$$J_+|j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\,|j,m+1\rangle$$

$$J_-|j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle$$

$$J_x|j,m\rangle = \frac{1}{2}(J_+ + J_-)|j,m\rangle$$

$$= \frac{\hbar}{2}\left[\sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle + \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle\right.$$

$$J_y|j,m\rangle = \frac{1}{2i}(J_+ + J_-)|j,m\rangle$$

$$= \frac{\hbar}{2i}\left[\sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle + \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle\right.$$

(2) Show $\langle J_x\rangle = \langle J_y\rangle = 0$ where $|\psi\rangle = |j,m\rangle$

$$\langle J_x\rangle = \langle\psi|J_x|\psi\rangle = \langle m,j|J_x|j,m\rangle$$

$$\frac{1}{i\hbar}\langle m,j|[J_y, J_z]|j,m\rangle$$

$$= \frac{1}{i\hbar}\langle m,j|J_yJ_z - J_zJ_y|j,m\rangle$$

$$= \frac{1}{i\hbar}\left[\langle m,j|J_yJ_z|j,m\rangle - \langle m,j|J_zJ_y|j,m\rangle\right]$$

$$= \frac{1}{i\hbar}\left[\hbar m\langle m,j|J_y|j,m\rangle - \hbar m\langle m,j|J_y|j,m\rangle\right]$$

$$= 0$$

Same example for $\langle J_y\rangle$

(3) $\langle J_x^2 \rangle = \langle J_y^2 \rangle = \hbar^2 [j(j+1) - m^2]/2$

$\langle J_x \rangle = \langle \psi | J_x^2 | \psi \rangle = \langle m, j | J_x^2 | j, m \rangle$

$$J_x^2 = \frac{1}{4}(J_+^2 + J_+ J_- + J_- J_+ + J_-^2) = J_y^2$$

$$\langle J_x^2 \rangle = \langle J_y^2 \rangle$$

$$= \frac{1}{2}\Big[\langle j, m | J^2 | j, m \rangle$$

$$- \langle j, m | J_z | j, m \rangle\Big] = \frac{\hbar^2}{2}\left[j(j+1) - m^2\right]$$

(4) Spin: $|+3/2\rangle, |+1/2\rangle, |-1/2\rangle, |-3/2\rangle$

$$J_x = \begin{bmatrix} \langle +3/2 | J_x | +3/2 \rangle & \langle +1/2 | J_x | +3/2 \rangle & \langle +1/2 | J_x | +3/2 \rangle & \langle +1/2 | J_x | +3/2 \rangle \\ \langle +3/2 | J_x | +1/2 \rangle & \langle +1/2 | J_x | +1/2 \rangle & \langle +1/2 | J_x | +1/2 \rangle & \langle +1/2 | J_x | +1/2 \rangle \\ \langle +3/2 | J_x | -1/2 \rangle & \langle +1/2 | J_x | -1/2 \rangle & \langle +1/2 | J_x | -1/2 \rangle & \langle +1/2 | J_x | -1/2 \rangle \\ \langle +3/2 | J_x | -3/2 \rangle & \langle +1/2 | J_x | -3/2 \rangle & \langle +1/2 | J_x | -3/2 \rangle & \langle +1/2 | J_x | -3/2 \rangle \end{bmatrix}$$

$\vdots$

$$\hbar \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \end{bmatrix} \quad = \quad \hbar \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$J_x = \frac{\hbar}{2}\begin{bmatrix} \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \qquad J_z = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$J_y = \frac{i\hbar}{2}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \qquad J^2 = \frac{\hbar^2}{2}\begin{bmatrix} -3 & 0 & 2\sqrt{3} & 0 \\ 0 & -1 & 0 & 2\sqrt{3} \\ 2\sqrt{3} & 0 & 1 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{bmatrix}$$