NAME: ______________________________

Exam 1.
Physics 342/442, Fall 2020

There is information attached at the end of the exam that you may find useful. No books or notes allowed. Good Luck !

1.) Answer the following in a clear and concise way. Usually more words means less points.

(a) (*5 points*) **Briefly** describe how adaptive time-step Runge-Kutta methods work to solve ODEs. Please note the emphasis on briefly.

- Runge-Kutta at two adjoining orders are peformed.
- The differences between the two orders is computed
- If the time step is adequate, process move forward. If not a new time step based on the differences of the two orders is computed and the process repeats.

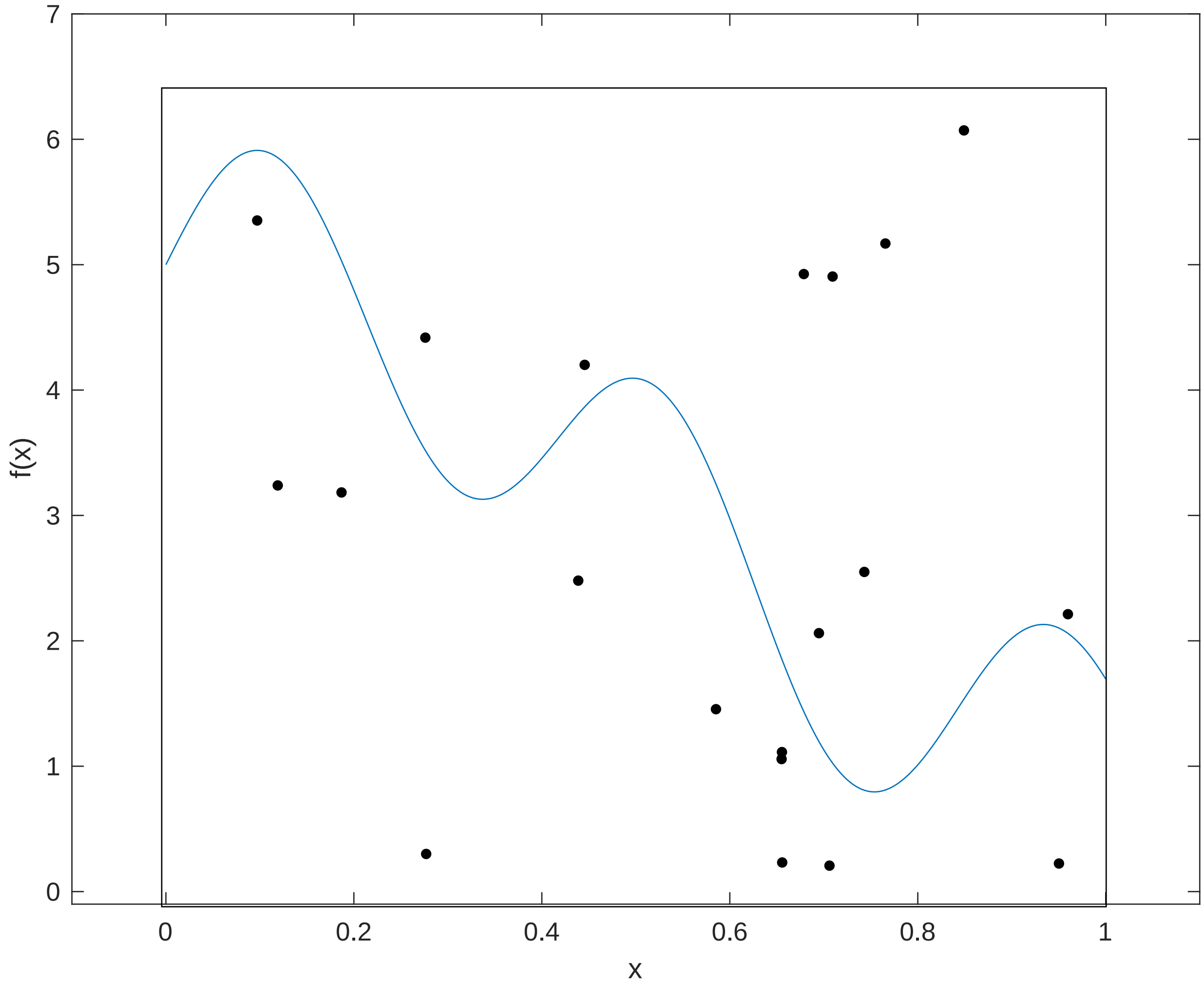
(b) (*5 points*) **Briefly** describe why a very small determinant, say $\sim 10^{-12}$, when using **LU** might cause one to doubt the results. Please note the emphasis on briefly.

- A small determinant means you may have a singular matrix which means the further processing should stop until that possibility is investigated.

(c) (*5 points*) **Briefly** describe two methods for fitting a functions whose normal equations are non-linear. Please note the emphasis on briefly.

- **Method I.** Set some initial values for all parameters. Then choose one to vary until the sum of the squares reaches some minimum. Move to the next parameter and repeat. Then repeat the entire process.
- **Method II.** Set some initial values for all parameters. Taylor expand about these parameters and keep only linear terms. Now system is linear and we can use the methods from linear least squares fits to find the change in parameters. Repeat until parameters do not change very much.

2. (*10 points*) Use the figure below to estimate the integral of the function. The dots have been randomly generated, and the height of the rectangle is 6.5



Solution. There are twenty total points, 11 of which are underneath the curve. The area of the enclosing rectangel is 6.5. Thus the estimate for the are ia

$$A = \frac{11}{20} \cdot 6.5 = 3.575$$

3.) (*10 points*) The figures below show the same experimentally obtained data. On the left panel, sketch a cubic spline appropriate for this data, and on the right panel sketch your estimate of a best fit line for this data. Below each figure describe the conditions that that must be met to obtain a cubic spline and a least-squares best fit line.

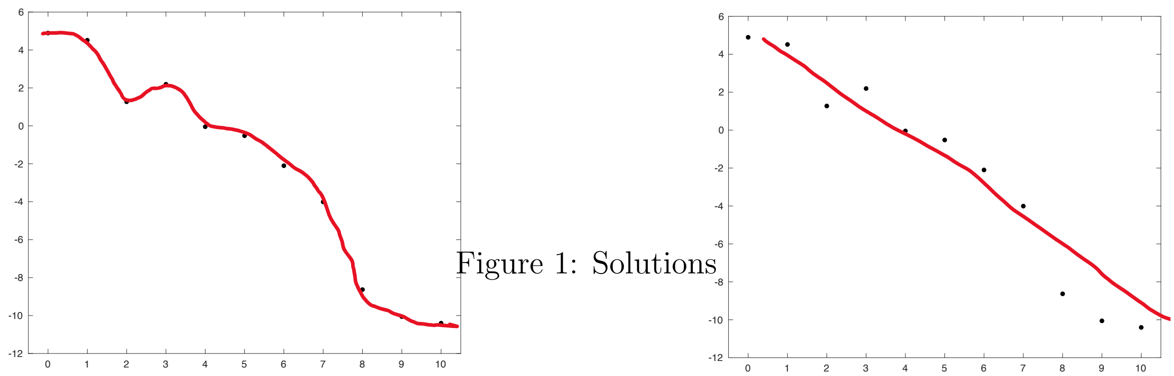


Figure 1: Solutions

Cubic Spline

- Cubic polynomial that passes through every point.
- First and second derivatives at end points of each interval are equal to first and second derivatives as the new interval begins.

Least Squares fit.

- Form least squares fit which has the form,

$$\chi^2 = \sum_{i=1}^{N} \left(y_i - (a_o + a_1 x_i)\right)^2 .$$

- Minimize χ^2 with respect to all the parameters and solve for best fit parameters.

4.)(*15 points*) By explicitly computing the appropriate partial derivatives, determine if the functions require linear or non-linear fits.

(a)

$$f(t) = a_o + a_1 \sin(2\pi t) + a_2 \sin(4\pi t); \quad a_o, a_1, a_2 \text{ parameters}$$

Solution The least squares function is

$$\chi^2 = \sum_i^N [y_i - (a_o + a_1 \sin(2\pi t_i) + a_2 \sin(4\pi t_i)]^2$$

Minimizing with respect to a_1 (the other parameters work similarly)

$$\frac{\partial \chi^2}{\partial a_1} = \sum_i^N 2 \cdot [y_i - (a_o + a_1 \sin(2\pi t_i) + a_2 \sin(4\pi t_i)] (\sin(2\pi t_i))$$

and we see that the expression remains linear in terms of a_1. All other a_i's will also remain linear.

(b)

$$f(x) = a_1 \exp(-a_2 x) + a_3 x; \quad a_1, a_2, a_3 \text{ parameters}$$

I'll just do a_2. We get

$$\frac{\partial \chi^2}{\partial a_2} = \sum_i^N 2 \cdot [y_i - (a_1 \exp(-a_2 x_i) + a_3 x_i] (-a_1 x_i \exp(-a_2 x_i))$$

and we see that for this term we are not linear in terms of the parameters.

(c)

$$f(x) = \exp(-a_1 x) + a_2 x; \quad a_1, a_2 \text{ parameters}$$

Doing just a_1, the others work similarly, we get

$$\frac{\partial \chi^2}{\partial a_2} = \sum_i^N 2 \cdot [y_i - (\exp(-a_1 x_i) + a_2 x_i] (-x_i \exp(-a_1 x_i))$$

5.a) (*5 points*) Convert the following 2nd order ODE to a system of first order ODEs. This requires that a new variable be introduced, call that variable, v.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 1 = 0.$$

Solution

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -2v + 1 \end{aligned}$$

5.b) (*15 points*) Use Euler's and Runge-Kutta fourth order methods on the first order system of ODEs found in part (a) to fill in the following table. The initial conditions are that at $t = 0, y = 0, v = 2$. Use a time step of $\Delta t = h = 1$.

	Euler		Runge-Kutta	
t	y	v	y	v
0	0	1	0	1
1				
2				
3				

Solution. Define:

$$S = \begin{pmatrix} y \\ v \end{pmatrix}; F = \begin{pmatrix} v \\ -2v + 1 \end{pmatrix}$$

then our system can be written as

$$\frac{dS}{dt} = F(v).$$

Euler.

$$\begin{aligned}
\begin{pmatrix} y_f \\ v_f \end{pmatrix} &= \begin{pmatrix} y_o \\ v_o \end{pmatrix} + \begin{pmatrix} v \\ -2v+1 \end{pmatrix} \cdot h \\
\begin{pmatrix} y_f \\ v_f \end{pmatrix} &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \\
\begin{pmatrix} y_f \\ v_f \end{pmatrix} &= \begin{pmatrix} 2 \\ -1 \end{pmatrix}
\end{aligned}$$

The other terms follow similarly

Runge-Kutta.

I will use the same definitions of S and F above. To time-evolve S, we need to find the intermediate functions, $f_o - -f_3$. Because time does not appear in F, I will not include time in evaluating F. We have

$$\begin{aligned}
f_o = F(S_o) &= \begin{pmatrix} 2 \\ -2(2)+1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \\
S = S_o + \tfrac{h}{2} f_o &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \\
f_1 = F(S) &= \begin{pmatrix} 1/2 \\ -2(1/2)+1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \\
S = S_o + \tfrac{h}{2} f_1 &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 2 \end{pmatrix} \\
f_2 = F(S) &= \begin{pmatrix} 2 \\ -2(2)+1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \\
S = S_o + f_2 &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\
f_3 = F(S) &= \begin{pmatrix} -1 \\ -2(-1)+1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}
\end{aligned}$$

So finally, we have that

$$
\begin{aligned}
\begin{pmatrix} y_f \\ v_f \end{pmatrix} &= \\
&\begin{pmatrix} 0 \\ 2 \end{pmatrix} + \frac{1}{6}\left[\begin{pmatrix} 2 \\ -3 \end{pmatrix} + 2\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \left[\begin{pmatrix} 1/3 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 1/6 \\ 0 \end{pmatrix} + \begin{pmatrix} 2/3 \\ -1 \end{pmatrix} + \begin{pmatrix} -1/6 \\ 1/2 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 6/6 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}
$$

Subsequent time steps follow in the same way