We've seen a general tour of cosmology, and now it's time to get to work to understanding cosmology.

The first point. A list of forces

- The strong force, F_{strong}, the force that keeps protons together
- The weak force, F_{weak}, the force responsible for nuclear decay
- The Electromagnetic force, $F_{E\&m}$, the force experienced by charged particles
- The gravitational force, F_G, the force generated and experienced by objects
- $F_{strong} >> F_{weak} >> F_{E\&m} >> F_G$
- However on cosmological scales, only F_G matters—the weak inherit the universe!

Gravity

- Newton $F_G = -\frac{GM_g m_g}{r^2}$ where M_g m_g are the masses of objects and r the distance separating them.

- Do question (1) on the worksheet and S T O P

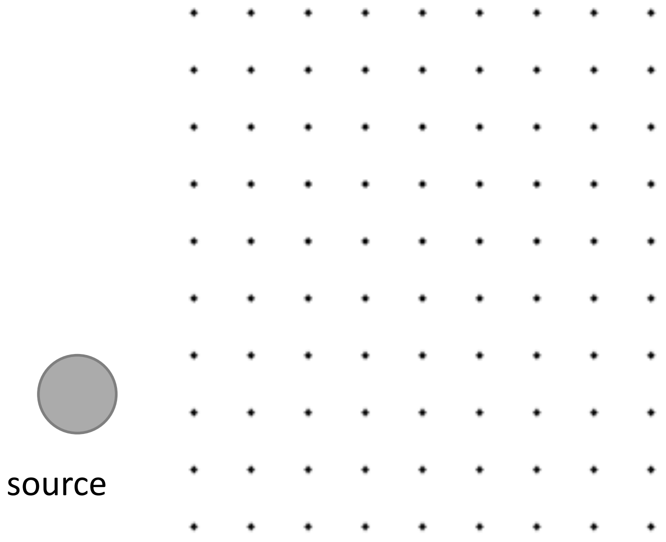
(1 d)

$$-\frac{GM_g m_g}{r^2} = m_I a$$

$$a = -\frac{GM_g m_g}{r^2}\left(\frac{m_g}{m_I}\right)$$

This term appears to be 1, i.e., $m_g = m_I$. This is really what is being tested when we drop different massed objects from the same height. Latest tests show this ratio is = 1 to within one part in 10^{13}. This is called the *equivalence principle*.

The equivalence principle implies that at every point, *r*, there is a unique gravitational acceleration, *a(r).*



This is what physicists mean by *field*

To find *a(r):*

(1) Define gravitational potential:

$$\nabla^2\Phi(r) = 4\pi G\rho(r)$$

(2) To solve for potential (at least in theory):

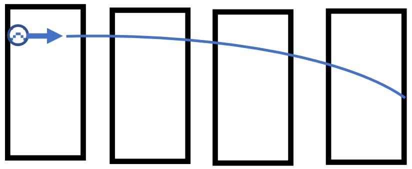
$$\Phi(r) = -G\int \frac{\rho(x)}{|x-r|}d^3x$$

(3) To find *a(r):*

$$a(r) = -\nabla\Phi(r)$$

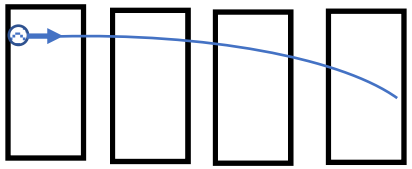
Newton's path to gravity: Mass tells gravity how to exert a force, $F_G = -\dfrac{GM_g m_g}{r^2}$.
Force tells mass how to accelerate, *F = m a*

Einstein's path to gravity. Do question (2) on the worksheet and S T O P



$a = 9.8\ m/s^2$

Absent other information, it is impossible to determine the difference between being accelerated or being in a gravitational field



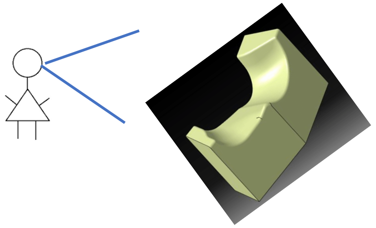
$g = 9.8\ m/s^2$

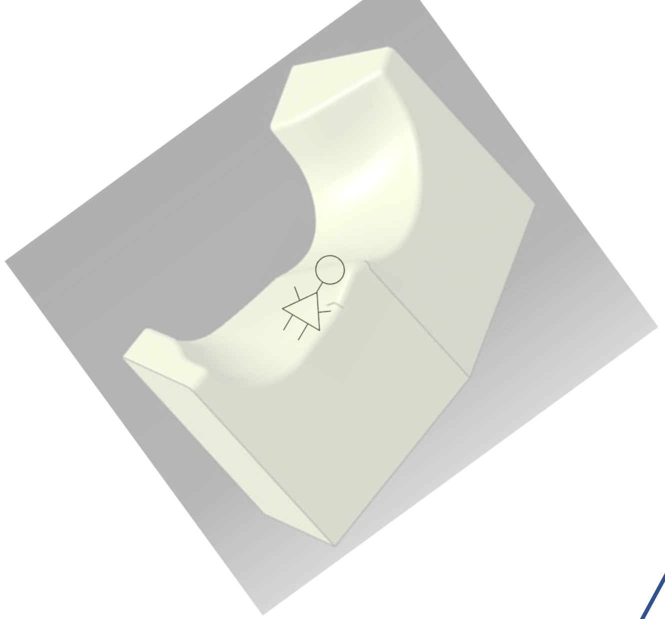
And the object could be a ray of light. So gravity *bends light*, even though it has no mass!

But how, light has mass = 0 and Newton's gravity requires mass for there to be a gravitational force?

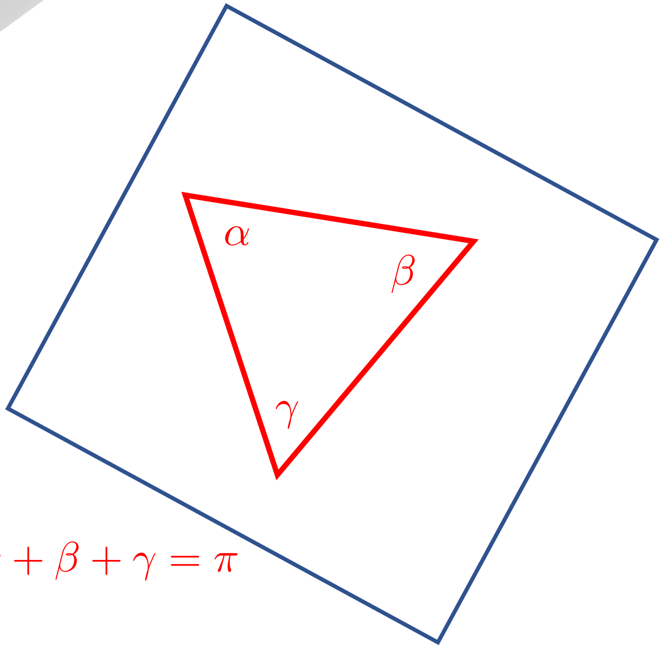
Einstein reasoned that there is no gravitational force, instead, space-time is curved in the presence of mass!

So to understand Einstein path to gravity, we must be able to describe curvature on a surface.





Measuring curvature



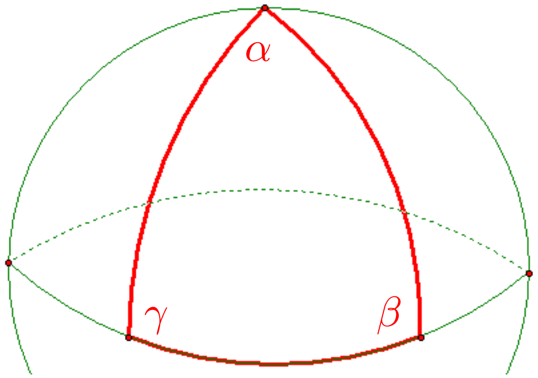
(1) 2-D Euclidean plane with an embedded triangle and $\alpha + \beta + \gamma = \pi$

(2) The distance between points (*x,y*) and (*x +dx, y + dy*) is

$$dl^2 = dx^2 + dy^2 \text{ or more conveniently } dl^2 = dr^2 + r^2 d\theta^2$$

What happens if we try to draw a triangle on a sphere?

Positive curvature

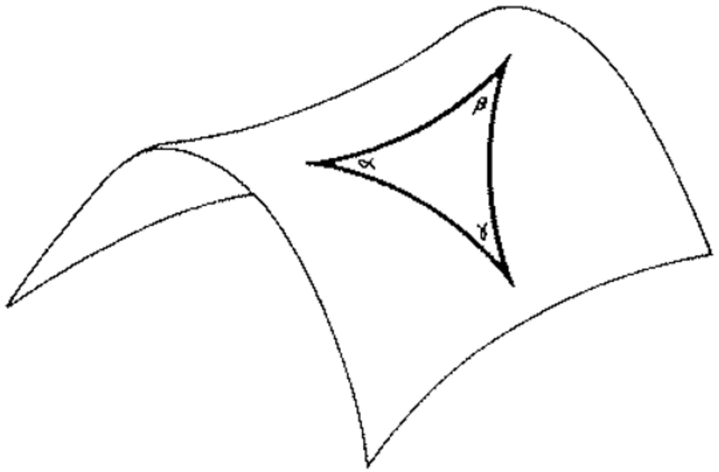


(1) The interior angles now add as $\alpha + \beta + \gamma = \pi + A/R^2$ where A is the area of the triangle and R the radius of sphere

(2) The distance between points (r, θ) and $(r + dr, \theta + d\,\theta)$ is

$$dl^2 = dr^2 + R^2 \sin^2(r/R)\, d\theta^2$$

Negative curvature



(1) The interior angles now add as $\alpha + \beta + \gamma = \pi - A/R^2$

(2) The distance between two points is

$$dl^2 = dr^2 + R^2 \sinh^2(r/R) d\theta^2$$

Metric: The expression that gives the distant (along a geodesic) between nearby points.
Geodesic: The shortest path between two points on a surface
Curvature constant, κ: $\kappa = 0$, for *flat* space, $\kappa = 1$, for *positive curved space,* $\kappa = -1$ for *negative curved space*

Metric: $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$

$$dl^2 = dr^2 + S_\kappa(r)^2 d\Omega^2$$

where $S_\kappa(r) = \begin{cases} R\sin(r/R) & \kappa = +1 \\ r & \kappa = 0 \\ R\sinh(r/R) & \kappa = -1 \end{cases}$

Begin homework problem 3.5

More about metrics. Do question (3) parts (a) –(c) on the worksheet and S T O P

(3a) $(A_x, A_y, A_z)\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = A_x B_x + A_y B_y + A_z B_z$

(3b) $\vec{A} = A_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + A_y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + A_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(3c) $\vec{A} \cdot \vec{B} = A_x\,(1, 0, 0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} B_y + \cdots$

Denote row vector component by A_μ

Denote column vector component by A^μ

Then (Einstein summation convention)

$$\vec{A} \cdot \vec{B} = (A_\nu e^\nu) \cdot (B^\mu e_\mu) = A_\nu\,(e^\nu \cdot e_\mu)\,B^\mu = A_\mu B^\mu$$

Do part (d) of question (3) and S T O P

Do question (3d) on the worksheet and S T O P

(3d) $\hat{e}_1 \cdot \hat{e}_1 = 1;\quad \hat{e}_1 \cdot \hat{e}_1 = 0;$
$\hat{e}_2 \cdot \hat{e}_1 = 0;\quad \hat{e}_2 \cdot \hat{e}_2 = 1$

We define a new quantity, $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Do question (3e) on the worksheet and S T O P

This stuff is interesting, but what does all this have to do with metrics?

Recall that for a 2-D Euclidean geometry we have $dl^2 = dx^2 + dy^2;$ or $dl^2 = dr^2 + r^2 d\theta^2$

Consider the Cartesian case first. In that case, we have $x^i = (x, y)$, $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $dl^2 = \underbrace{g_{ij}}_{\text{metric}}\, dx^i\, dx^j$

For the polar coordinate case, we have $x^i = (r, \theta)$, $g_{i,j} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ then $dl^2 = \underbrace{g_{ij}}_{\text{metric}}\, dx^i\, dx^j$

This demonstrates that the metric depends not only the kind of surface, but also on the coordinate system used. It is ***always*** true that $dl^2 = \underbrace{g_{ij}}_{\text{metric}}\, dx^i\, dx^j$. A space is Euclidean if a coordinate system exists such that $g_{ij} = \delta_{ij}$

Finish problem 3

So far we've only looked at Euclidean spaces. However, Euclidean spaces do no accurately embed the universe we occupy. For example, in inertial systems, *special relativity* has the following:

$$x^{\mu} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \; ; \quad g_{\mu\nu} = \begin{pmatrix} \boxed{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In cosmology, things get a bit more complicated because even in an empty universe, space-time itself changes as a function of time. But we saw that we can describe the expansion by use of the scale factor *a(t)*.

Robertson and Walker (and at least two others) worked out that the metric for a homogeneous and isotropic expanding universe is

$$x^{\mu} = \begin{pmatrix} ct \\ r \\ \theta \\ \phi \end{pmatrix} \; ; \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix}$$

So distances in an expanding universe which is both isotropic and homogeneous are found using

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_k(r)^2 d\Omega^2 \right]$$