Homework 2

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Problem 1

An infinitely long straight wire along the z-axis carries a uniform current I (moving upward toward the positive z-direction). A spherical shell of radius R with a total charge Q uniformly distributed over its surface is centered at the origin (through which the wire also passes, since the wire is along the z-axis).

(a)

The electric field for this problem can be expressed as

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}\hat{r}$$

this is just a point charge.

(b)

The magnetic field for this problem can be expressed as

$$\int \vec{B}\cdot\ell \rightarrow \vec{B} = \frac{\mu_0 I}{2\pi R}\hat{\phi}$$

(c)

The Poynting vector is expressed as

$$\vec{S} = (\vec{E} \times \vec{H}).$$

From the prior two sections of this problem, both $\vec{E}$ and $\vec{B}$ were found. However, the Poynting vector requires the $\vec{H}$. The only real change for $\vec{B}$ is by a factor of μ_0. Where $\vec{B} = \mu_0\vec{H} + \vec{M}$ but since this is a linear media $\vec{B} = \mu_0\vec{H}$ or $\vec{H} = \vec{B}/\mu_0$. Therefore,

$$\vec{H} = \frac{I}{2\pi R}\hat{\phi}$$

and then the Poynting vector becomes

$$\vec{S} = (\frac{Q}{4\pi\epsilon_0 R^2}\hat{r} \times \frac{I}{2\pi R}\hat{\phi}) = -\frac{QI}{8\pi^2\epsilon_0 R^3}\hat{\theta}$$

(d)

We can express the spherical vectors as Cartesian vectors by

$$\hat{r} = sin(\theta)cos(\phi)\hat{x} + cos(\theta)sin(\phi)\hat{y} + cos(\theta)\hat{z} \qquad \hat{\phi} = -sin(\phi)\hat{x} + cos(\phi)\hat{y}$$

Therefore, $\vec{E}$ can be wrote as

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2} sin(\theta)cos(\phi)\hat{x} + cos(\theta)sin(\phi)\hat{y} + cos(\theta)\hat{z}$$

Since $\vec{J}$ only depends on th $\hat{z}$ component, the $\hat{x}$ and $\hat{y}$ components drop. The current density is gave as

$$\vec{J} = \frac{I}{2\pi R}\hat{z}.$$

Therefore,

$$\vec{J} \cdot \vec{E} = \frac{I}{2\pi R}\hat{z} \cdot \frac{Q}{4\pi\epsilon_0 R^2} cos(\theta)\hat{z} = \frac{QI}{8\pi^2\epsilon_0 R^3} cos(\theta).$$

Problem 2

Let's assume that our wave is traveling along the $\hat{z}$ direction. This would put the screen on the xy-plane. Meaning that

$$\vec{E} = \vec{E}_x\hat{x} \qquad\qquad \vec{B} = \vec{B}_y\hat{y}.$$

The Maxwell stress tensor is gave as

$$T_{\alpha\beta} = \epsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})\delta_{\alpha\beta}\right]$$

Therefore, the similar components can be wrote as

$$T_{xx} = -\frac{\epsilon_0}{2}\left[E_x^2 + c^2 B_y^2\right] \qquad T_{yy} = \frac{\epsilon_0}{2}\left[E_x^2 - c^2 B_y^2\right] \qquad T_{zz} = \frac{\epsilon_0}{2}\left[-E_x^2 + c^2 B_y^2\right]$$

Equation 6.121 is expressed in the course notes as

$$\frac{d}{dt}\left[\left(\vec{P}_{mech} + \vec{P}_{field}\right)_\alpha\right] = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta} d^3x$$

With our values this can be expressed as

$$\frac{d}{dt}\left[\left(\vec{P}_{mech} + \vec{P}_{field}\right)_z\right] = \sum_z \int_V \frac{\partial}{\partial x_z} T_{zz} d^3x$$

Therefore, the pressure can be found by

$$P = \frac{\epsilon_0}{2}\left[\vec{E}_x^2 + c^2\vec{B}_y^2\right]$$

and

$$u = \frac{\epsilon_0}{2}\left[\vec{E}_x^2 + c^2\vec{B}_y^2\right]$$

Thus,

$$P = u$$

Problem 3

The pressure can be found by

$$P = \frac{S}{c} = \frac{1.4 \times 10^3 \; Wm^{-2}}{3 \times 10^8 \; ms^{-1}} = 5 \times 10^{-6} \; Nm^{-2}$$

We have a pressure and pressure is force over and area or simply $P = F \cdot A^{-1}$. But this problem requires us to find the acceleration and acceleration is gave by $F = m{\cdot}a$ or $a = F{\cdot}m^{-1}$. Substitution the equation with pressure into the equation with acceleration we get

$$a = \frac{P \cdot A}{m} = \frac{5 \times 10^{-6} \; Nm^{-2}}{1 \times 10^{-3} \; kg \cdot m^2} = 5 \times 10^{-3} \; m \cdot s^{-2}.$$

Problem 4

The momentum associated with the electromagnetic field is gave by

$$\vec{P}_{field} = \epsilon_0 \int_V \vec{E} \times \vec{H} d^3x.$$

In order to find $\vec{P}_{field}$ both the magnetic and electric filed have to be found first. The magnetic field of a toroidal coil is expressed as

$$\vec{B} = \pm\frac{\mu_0 NI}{2\pi a}\hat{\phi} \qquad \vec{H} = \pm\frac{NI}{2\pi a}\hat{\phi}$$

while the electric field is due to a point charge which is gave by

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 a^2}\hat{r}.$$

We can express the spherical vectors as Cartesian vectors by

$$\hat{r} = sin(\theta)cos(\phi)\hat{x} + cos(\theta)sin(\phi)\hat{y} + cos(\theta)\hat{z} \qquad \hat{\phi} = -sin(\phi)\hat{x} + cos(\phi)\hat{y}$$

The cross product of $\vec{E} \times \vec{H}$ is

$$\vec{E} \times \vec{H} = \pm\frac{QIN}{8\pi^2\epsilon_0 a^3}\hat{z}$$

Integrating over the volume for the z component yields $2\pi aA$ and the result is

$$\vec{P}_{field} = \pm\frac{\mu_0 QINA}{4\pi\epsilon_0 a^2}.$$

The components for x and y are then

$$(\vec{P}_{field})_x = 0 \qquad (\vec{P}_{field})_y = 0$$

Appendix