Physics 440, Autumn 201 Activity 18: Canonical transformation (free particle)

The Hamiltonian for a free particle in three dimensions is

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

a. Find the canonical transformation that take you from Cartesian coordinates to cylindrical coordinates, and find the new Hamiltonian. Start from the transformation equations $x = r\cos\phi\,, y = r\sin\phi, z = Z$. Use a generating function of the type $F_3(Q, p, t)$.

We want $F_3(r, \phi, Z, p_x, p_y, p_z)$ such that

$$x = -\frac{\partial F_3}{\partial p_x} = r\cos\phi \qquad y = -\frac{\partial F_3}{\partial p_y} = r\sin\phi \qquad z = -\frac{\partial F_3}{\partial p_z} = Z$$

Let's try $F_3 = -p_x(r\cos\phi) - p_y(r\sin\phi) - p_z Z$

Then

$$p_r = -\frac{\partial F_3}{\partial r} = p_x\cos\phi + p_y\sin\phi$$

$$p_\phi = -\frac{\partial F_3}{\partial \phi} = -rp_x\sin\phi + rp_y\cos\phi$$

$$p_Z = -\frac{\partial F_3}{\partial Z} = p_z$$

F_3 is independent of t, so $H = K$

$$p_r^2 = p_x^2\cos^2\phi + p_y^2\sin^2\phi + 2p_xp_y\cos\phi\sin\phi$$

$$\frac{p_\phi^2}{r^2} = p_x^2\sin^2\phi + p_y^2\cos^2\phi - 2p_xp_y\cos\phi\sin\phi$$

$$\Rightarrow p_r^2 + \frac{p_\phi^2}{r^2} = p_x^2 + p_y^2$$

$$\therefore K = H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2} + p_z^2\right)$$

b. Find the canonical transformation that take you from Cartesian coordinates to spherical coordinates, and find the new Hamiltonian. Start from the transformation equations $x = r\sin\theta\cos\phi\,, y = r\sin\theta\sin\phi\,, z = r\cos\theta$. Use a generating function of the type $F_3(Q, p, t)$.

Now we want F_3 such that

$$x = -\frac{\partial F_3}{\partial p_x} = r\sin\theta\cos\phi \qquad y = -\frac{\partial F_3}{\partial p_y} = r\sin\theta\sin\phi \qquad z = -\frac{\partial F_3}{\partial p_z} = r\cos\theta$$

Let's try the following generating function:

$$F_3 = -p_x r\sin\theta\cos\phi - p_y r\sin\theta\sin\phi - p_z r\cos\theta$$

Then:

$$p_r = -\frac{\partial F_3}{\partial r} = p_x\sin\theta\cos\phi + p_y\sin\theta\sin\phi + p_z\cos\theta$$

$$p_\phi = -\frac{\partial F_3}{\partial \phi} = -rp_x\sin\theta\sin\phi + rp_y\sin\theta\cos\phi$$

$$p_\theta = -\frac{\partial F_3}{\partial \theta} = rp_x\cos\theta\cos\phi + rp_y\cos\theta\sin\phi - p_z r\sin\theta$$

$$p_r = (p_x\cos\phi + p_y\sin\phi)\sin\theta + p_z\cos\theta$$

$$\frac{p_\phi}{r\sin\theta} = (p_x\sin\phi + p_y\cos\phi)$$

$$\frac{p_\theta}{r} = (p_x\cos\phi + p_y\sin\phi)\cos\theta - p_z\sin\theta$$

$$p_r^2 + \left(\frac{p_\theta}{r}\right)^2 + \left(\frac{p_\phi}{r\sin\theta}\right)^2 = p_x^2 + p_y^2 + p_z^2$$

$$\therefore H = \frac{1}{2m}\left(p_r^2 + \left(\frac{p_\theta}{r}\right)^2 + \left(\frac{p_\phi}{r\sin\theta}\right)^2\right)$$