Learning Goals

1. Matlab’s implementation of FFT and Spectral decomposition

As we've discussed previously in this section of the course, it is critical that when you using a *black-box FFT*, that one understand the various conventions, storing of coefficients, etc.

Today we'll use the MATLAB built in *FFT* routine as an example of the kinds of factors you have to understand.

Recall that the notation we use for the discrete Fourier transform is $g(n\Delta\omega) = \sum_{m=0}^{N-1} f(m\Delta t)e^{-i2\pi mn/N}$

where *n, m = 0, 1, …,N-1*, and $\Delta\omega = \frac{2\pi}{T}$ *T* being the total time the signal is sampled and Δ t the sampling time

MATLAB uses the notation for Fourier transforms, $y_{k+1} = \sum_{j=0}^{n-1} \omega^{ij}\ x_{j+1}$ where *x* is a uniformly sampled data set.

Relate or translate the notation we have used to MATLAB's notation. Make sure you discuss the *+1* in the subscripts

$$\begin{aligned} n\Delta\omega &\rightarrow k \\ m\Delta t &\rightarrow j \\ g &\rightarrow y \\ f &\rightarrow x \\ e^{-i2\pi/N} &\rightarrow \omega \end{aligned}$$

And notice that exponential is negative, that there isn't a factor in front of the sums, etc. Before you use any *black box*, these are the first things you need to understand.

Okay, now that have some notation down and have understood the conventions MATLAB uses, lets start exploring.

1. Fire up MATLAB and enter the following commands. In your group, make sure you discuss what each of the commands you will enter does.
   a) `t= 0:1/50: 10 – 1/50;`
   b) `x = sin (2 *pi *15*t);`
2. Plot *t vs x* and make sure you understand the plot
3. The MATLAB command to perform the fast Fourier transform is, surprisingly enough, `fft(signal)`. Take the Fourier transform of *x*, and store it in a variable, *y*. Make sure you suppress the output
4. Compare the lengths of the variable *x* and *y.* Then display the contents of the variable *y,* and do a quick inspection, what do you notice about *y*
5. Let's look at the coefficients a little closer. Display the value of the first coefficient, *y(1)*. Can you say anything about this coefficient?
6. Change the original data vector *x* to *x = 2 +* sin (2 * pi* 15 *t), take the *FFT* of this variable, and again store it in *y.* Again display the value of *y(1)* and try to understand what this value might be.
7. Just for fun, take the mean of *x.* Now evaluate *y(1)/length(y)* and compare. What might you hypothesize the first component of the *FFT* is now.
8. Confirm your suspicions by adding different constants to the *sin* term.

Let's return to our original function. Enter the command `x = sin (2 *pi *15*t);`

1. Take the *FFT* of *x* and store it in *y.*
2. Enter the commands:
   a) `f = (0:length(y)-1)*50/length(y);`
   b) `figure`
   c) `plot(f,abs(y))`
3. Physically, what is the variable *f*. Explain the plot you see.
4. Now change the function to `x = 2 +sin (2 *pi *15*t);` and repeat steps 1 – 3. Keep the figure in 2-c, do not overwrite it.
5. Compare the plots in 2-c to those in (4). Do they make sense.
6. Now consider the function `x = sin (2 *pi *15*t) + sin (2*pi*20*t).` Before doing any MATLAB stuff, predict what you expect to see after you perform steps 1 and 2 for this function. Repeat steps 1 and 2 for this new function.
7. Now, consider the following commands. At your table discuss why one would want to do this.
   a) `fnew=f(2:length(f)/2);`
   b) `ynew=y(2:length(y)/2);`
   c) `plot(fnew,abs(ynew))`

In your group recap all that you've learned so far about MATLAB's *fft* command

Now that you've been introduced to MATLAB's *fft,* let's start considering more realistic situations.

First some terminology. The plot of the Fourier coefficients squared vs frequency is called

1. The periodogram
2. The power spectrum density
3. The power spectrum
4. The energy spectral density
5. Probably others as well, this can be frustrating and confusing

I will generally refer to this quantity as the *power spectrum.*

Okay, now enter the following commands and discuss at your table what each line does and what each variable means, physically (if it has a physical interpretation)

Okay, lets explore more. Make sure you understand what's happening at every step

```
1. Fs = 1000;
2. t = 0:1/Fs:1-1/Fs;
3. x = cos (2*pi*100*t) + randn(size(t));
4. plot(t,x);
5. N = length(x);
6. xdft = fft(x);
7. xdft = xdft(1:N/2+1);
8. mean(x)
9. xdft(1)/N;
10.freq = 0:Fs/length(x):Fs/2;
11.plot(freq,abs(xdft)). %%% discuss in your group what results
```

Now lets continue

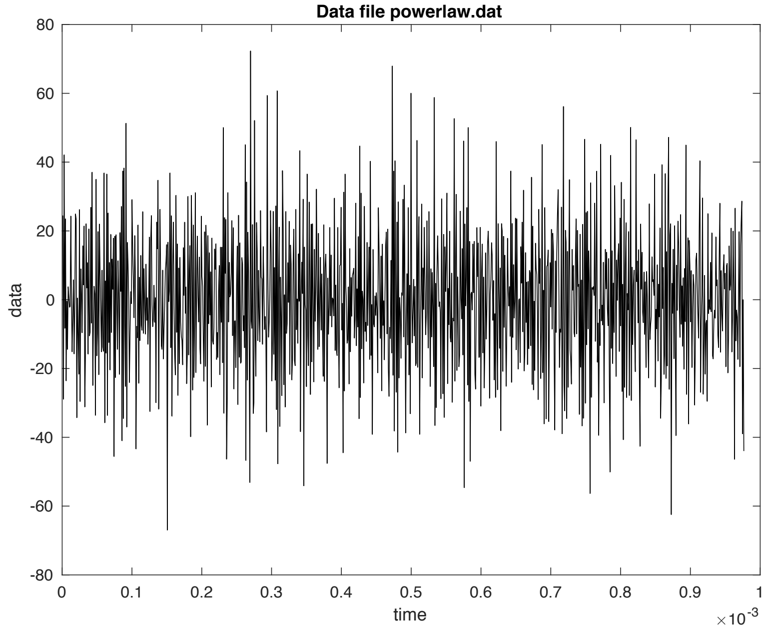
```
12.figure;
13.plot(freq,abs(xdft).^2). %% this is often done so that this quantity is power
14.psdx = (1/(Fs*N)) * abs(xdft).^2;   %% discuss at your table what this quantity is
15.psdx(2:end-1) = 2*psdx(2:end-1); %% oh what the hell now!
16.figure;
17.plot(freq,psdx) %% well at least this makes since, kind of
```

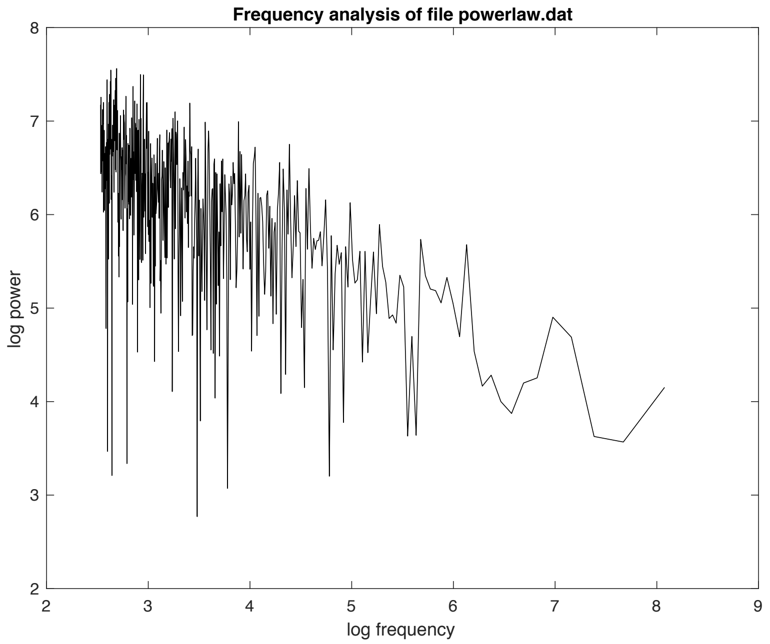
Be aware that in electrical engineering, the power spectrum is often shown on a log scale, that is, line 14 would be, `plot(freq,10*log10(psdx))` and/or if one has a signal characterized by a power law, one can extract that power by `plot(log(freq,)log(psdx))`

Do questions (1) and (2)on the worksheet where you asked to analyze an unknown signals using spectral methods.

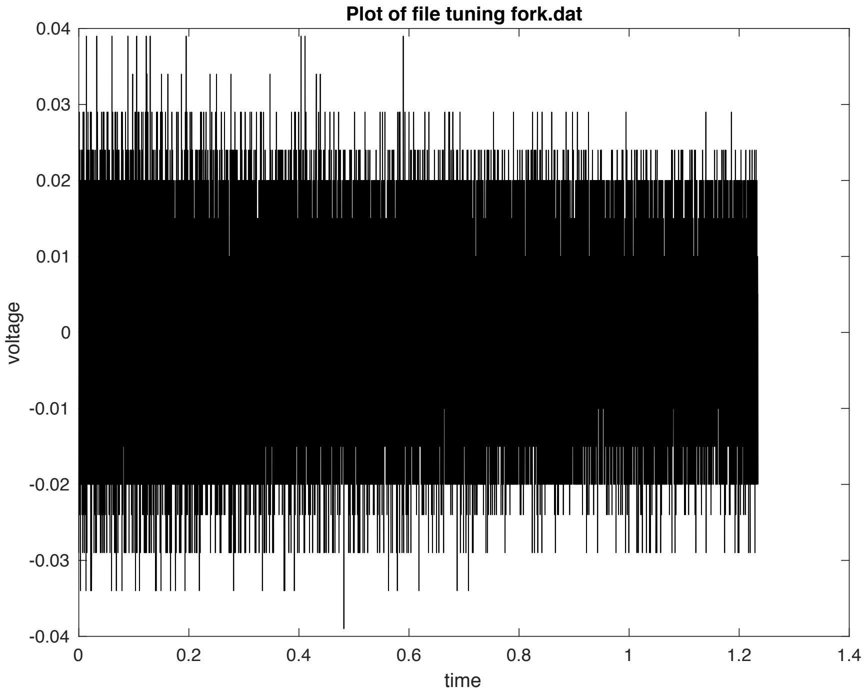
Fourier-05

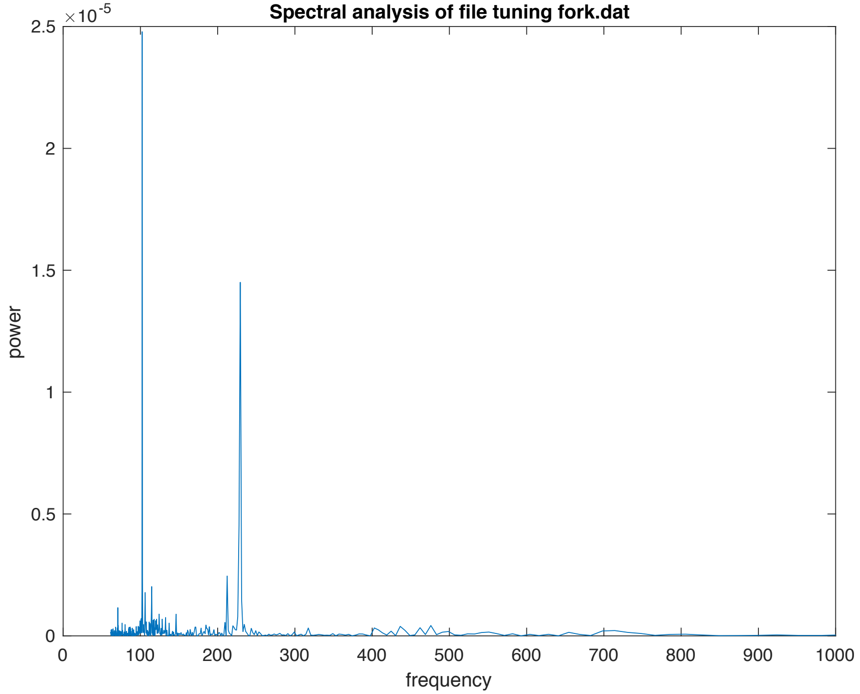
(1)





(2)





One last thing about the power spectrum. It has another interpretation. To see what this is consider the following.

For a set of discrete the data, the mean and variance of the data (often called the *population*) are defined as

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i; \quad \sigma_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

where *n* is the total number of points in the data set

For a continuous distribution, we can similarly define a mean and variance as

$$\overline{f(t)} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt; \quad \sigma^2(x) = \frac{1}{T}\int_{-T/2}^{T/2}\left(f(t) - \overline{f(t)}\right)^2 dt = \frac{1}{T}\int_{-T/2}^{T/2}\left(f^2(t) - \overline{f(t)}^2\right)\,dt$$

Now lets look at the Fourier series of f^2:

$$\begin{aligned} \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)\,dt &= \frac{1}{T}\int_{-T/2}^{T/2}\left(\sum_n c_n e^{-i\omega_n t}\right)\left(\sum_m c_m^* e^{i\omega_m t}\right) \\ &= \sum_n\sum_m \frac{c_n c_m^*}{T}\int_{-T/2}^{T/2} e^{i(\omega_m - \omega_n)}dt \\ &= \sum_n c_n c_n^* = \sum_n |c_n|^2 \end{aligned}$$

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&= \sum_n \sum_m \frac{c_n c_m^*}{T}\int_{-T/2}^{T/2} e^{i(\omega_m-\omega_n)}dt \\
&= \sum_n c_n c_n^* = \sum_n |c_n|^2
\end{aligned}$$

Also note that $\overline{f(t)} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = c_o$ so that

$$\mathrm{var}(f_n) \equiv \frac{1}{T}\int_{-T/2}^{T/2}\left(f^2(t) - \overline{f(t)}\right)^2 dt = \sum_n |c_n|^2 - c_o \Rightarrow \sum_{n\neq 0} |c_n|^2$$

That is, the power spectrum is the variance as a function of frequency of the function