Practice Assessment 3

The normalized energy eigenstates for the particle in a cube are

$$\psi_{n_x,n_y,n_z}(x,y,z) = \sqrt{\frac{8}{L^3}} \sin\frac{n_x \pi L x}{L} \sin\frac{n_y \pi L}{L} \sin\frac{n_z \pi z}{L}$$

With energies

$$E_{n_x,n_y,n_z} = \frac{\hbar^2 \pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

(1) A particle is prepared in the State

$$\psi(x,y,z) = A\left[\psi_{111}(x,y,z) - 2i\,\psi_{122}(x,y,z)\right]$$

(a) Find A and explain why you don't have to evaluate any integrals to do so.

We can rewrite the State as

$$|\psi\rangle = \left[\psi_{111}(x,y,z) - 2i\,\psi_{122}(x,y,z)\right]$$

Since we write this in Bracket notation we can then Solve is such that $A = 1/\sqrt{\langle\psi|\psi\rangle}$. Therefor, we have

$$1 = \langle\psi|\psi\rangle = A^2\left[\left(\langle 1,1,1| + 2i\langle 1,2,2|\right)\left(|1,1,1\rangle - 2i|1,2,2\rangle\right)\right]$$

$$1 = A^2\left[\cancel{\langle 1,1,1|1,1,1\rangle}^{\,1} + 4\cancel{\langle 1,2,2|1,2,2\rangle}^{\,1}\right] \Rightarrow 1 = A^2[5]$$

$$A^2 = 1/5 \rightarrow \boxed{A = 1/\sqrt{5}}$$

$A = 1/5 \quad \rightarrow \quad \boxed{A = 1/\sqrt{5}}$

$$|\psi\rangle = \frac{1}{\sqrt{5}}\Big[\,|1,1,1\rangle - 2i\,|1,2,2\rangle\Big]$$

(B)

$$E_{111} = \frac{\hbar^2\pi^2}{2mL^2}\left(1^2+1^2+1^2\right) = \boxed{\frac{3\hbar^2\pi^2}{2mL^2}}$$

$$E_{221} = \frac{\hbar^2\pi^2}{2mL^2}\left(2^2+2^2+1^2\right) = \boxed{\frac{9\hbar^2\pi^2}{2mL^2}}$$

$$P = |\langle\psi_{n_x,n_y,n_z}|\psi\rangle|$$

$$P_{111} = \frac{1}{5}\left|\langle 111|111\rangle\right| = \boxed{\frac{1}{5}}$$

$$P_{221} = \frac{1}{5}\left|4\langle 221|221\rangle\right| = \boxed{\frac{4}{5}}$$

$\Big\}\; 1 \checkmark$

Not Required (For Practice)

$$\langle H\rangle = \sum E_n P_n = \left(\frac{1}{5}\right)\frac{3\hbar^2\pi^2}{2mL^2} + \left(\frac{4}{5}\right)\frac{9\hbar^2\pi^2}{2mL^2}$$

$$= \frac{3\hbar^2\pi^2}{10mL^2} + \frac{36\hbar^2\pi^2}{10mL^2} = \boxed{\frac{39\hbar^2\pi^2}{10mL^2}}$$

(C) $$\int_0^{L/2} \psi^*(x,y,z)\,\psi(x,y,z)$$

$$= \int_0^{L/2}\int_0^{L/2}\int_0^{L/2} \frac{1}{5}\Bigg[\Bigg[\sqrt{\frac{8}{L^3}}\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\sin\frac{\pi z}{L} +$$

$$2i\sqrt{\frac{8}{L^3}}\sin\frac{\pi x}{L}\sin\frac{2\pi y}{L}\sin\frac{2\pi z}{L}\Bigg]\Bigg[\sqrt{\frac{8}{L^3}}\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\sin\frac{\pi z}{L}$$

$$+ 2i \int \frac{8}{L^3} \sin\frac{\pi x}{L} \sin\frac{2\pi y}{L} \sin\frac{2\pi z}{L} \Big] \Big] dx\,dy\,dz$$

Cross terms will cancel

$$\int_0^{L/2}\int_0^{L/2}\int_0^{L/2} \frac{1}{5}\left[\frac{8}{L^3}\sin^2\frac{\pi x}{L}\sin^2\frac{\pi y}{L}\sin^2\frac{\pi z}{L}\right.$$

$$\left. + 4\,\frac{8}{L^3}\sin^2\frac{\pi x}{L}\sin^2\frac{2\pi y}{L}\sin^2\frac{2\pi z}{L}\right]$$

$$\rightarrow \frac{8}{5L^3}\left[\left(\int_0^{L/2}\sin^2\frac{\pi x}{L}dx\right)^3 + 4\int_0^{L/2}\sin^2\frac{\pi x}{L}\sin^2\frac{2\pi y}{L}\sin^2\frac{2\pi z}{L}\right]$$

$$\rightarrow \frac{8}{5L^3}\left[\frac{L^3}{64} + 4\left[\frac{L^3}{64}\right]\right] \rightarrow \frac{8}{\cancel{5}\cancel{L^3}}\left[\frac{\cancel{5}\cancel{L^3}}{64}\right] = \frac{8}{64}$$

$$\boxed{= \frac{1}{8}}$$