Homework 4

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Problem 1

The wave equation for the vector potential $\vec{A}$ is gave by

$$\nabla^2 \vec{A} - \frac{1}{C^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

(a)

The Green function equation for the equation above can be expressed as

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_k^{(\pm)}(\vec{x}, t; \vec{x}', t') = \mu_0 \vec{J} \delta(\vec{x} - \vec{x}')\delta(t - t')$$

(b)

$$G^{(\pm)}(\vec{x}, t; \vec{x}', t') = \frac{1}{|\vec{x} - \vec{x}'|}\delta\left[t' - t + \frac{|\vec{x} - \vec{x}'|}{c}\right]$$

(c)

We first need

$$\psi^{(\pm)}(\vec{x}, t) = \int\int G^{(\pm)}(\vec{x}, t; \vec{x}, t') f(\vec{x}, t) d^3x' dt'$$

then this can be expressed as

$$\vec{A}(\vec{x},t) = \int d^3x' \int \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\delta\left(t' - \left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right)$$

Problem 2

The vector potential $\vec{A}$ of an oscillating electric dipole is given by

$$\vec{A}(\vec{x}) = -\frac{i\mu_0\omega}{4\pi}\,\vec{p}\,\frac{e^{ikr}}{r}.$$

To find its magnetic field we first have to write it as the curl of $\vec{A}$ such that

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}.$$

$\vec{A}$ will only have the radial dependence, therefore, we only have to express this equation in terms of x. This is expressed as

$$\vec{H} = \frac{1}{\mu_0}\frac{\partial}{\partial x}\hat{n}\times\left(-\frac{i\mu_0\omega}{4\pi}\,\vec{p}\,\frac{e^{ikx}}{x}\right).$$

Taking the partial derivative of $\vec{A}$ with respect to x gives the result

$$\vec{H} = -\frac{i\omega}{4\pi}(\hat{n}\times\vec{p})\left(-\frac{e^{ikx}}{x^2}+\frac{ike^{ikx}}{x}\right)$$
$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikx}}{x}\left(1-\frac{1}{ikx}\right)$$

Therefore, the final expression when $r = x$ is

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$

Problem 3

The electric field of an oscillating electric dipole is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\frac{e^{ikr}}{r} + [3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}$$

and in the prior question the magnetic field was found to be

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right).$$

To find these fields in the near zone, the factor $kr << 1$ must remain true. If this holds then the magnetic field is

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{i\cdot 0}}{r}\left(1 - \frac{1}{i\cdot 0}\right).$$

Then $exp(i\cdot 0) = 1$ and $1/0 = 0$, and multiply by i/kr our equation becomes

$$\vec{H} = \frac{ck}{4\pi}(\hat{n}\times\vec{p})\frac{1}{r^2}.$$

The same logic can be applied to the electric field. The electric field is gave again by

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\frac{e^{ikr}}{r} + [3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}$$

The first part of the equation will be zero because of the denominator. Therefore, we are left with

$$\begin{aligned}\vec{E} &= \frac{1}{4\pi\epsilon_0}\left\{[3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}]\left(\frac{i}{kr^4} + \frac{1}{r^3}\right)e^{ikr}\right\}\\ &= \frac{1}{4\pi\epsilon_0}\left\{[3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}]\left(0 + \frac{1}{r^3}\right)e^{0}\right\}\end{aligned}$$

Therefore, we are just left with

$$\vec{E} = \frac{1}{4\pi\epsilon_0}[3\hat{n}(\hat{n} \cdot \vec{p}) - \vec{p}]\frac{1}{r^3}$$

Problem 4

The time-averaged power radiated per unit solid angle by an oscillating dipole is given by

$$\frac{dP}{d\omega} = \frac{1}{2} Re[r^2 \hat{n} \cdot \vec{E} \times \vec{H}^*]$$

where the fields $\vec{E}$ and $\vec{H}$ in the far zone are given by

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n} \times \vec{p})\frac{e^{ikr}}{r} \qquad \vec{E} = \frac{k^2}{4\pi\epsilon_0}[(\hat{n} \times \vec{p}) \times \hat{n}]\frac{e^{ikr}}{r}$$

Substituting $\vec{H}$ and $\vec{E}$ into the equation above gives

$$\frac{dP}{d\omega} = \frac{1}{2} Re[r^2 \hat{n} \cdot \frac{k^2}{4\pi\epsilon_0}\left[(\hat{n} \times \vec{p}) \times \hat{n}]\frac{e^{ikr}}{r} \times \frac{ck^2}{4\pi}(\hat{n} \times \vec{p})\frac{e^{ikr}}{r}\right]$$

so

$$\begin{aligned} \vec{E} \times \vec{H}^* &= \frac{c}{\epsilon_0}\left(\frac{k^2}{4\pi r}\right)^2 [\hat{n} \times (\vec{p} \times n] \times (\hat{n} \times \vec{p})^* \\ &= \frac{c}{\epsilon_0}\hat{n} \times (\hat{n} \times \vec{p}) \cdot (\hat{n} \times \vec{p})^*. \end{aligned}$$

This can reduce to

$$\frac{dP}{d\omega} = \frac{1}{2}\frac{c}{\epsilon_0}\left(\frac{k^2}{4\pi r}\right)^2 |(\hat{n} \times \vec{p}) \times \hat{n}|^2$$

and further reduced to

$$\frac{dP}{d\omega} = \frac{c^2 Z^0}{32\pi^2} k^4 |(\hat{n} \times \vec{p}) \times \hat{n}|^2$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$.