PHY 420 **Spring 2021**

Homework 4 solutions

1. In this problem, you will apply the Green function technique discussed in class to the wave equation for the vector potential $\vec{A}$, which is

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = -\mu_0 \vec{J} \tag{1}$$

(a) Write down the ***Green function equation*** corresponding to equation (1) above.

Solution: Recall from class that *corresponding to the differential equation*

$$\mathcal{D}\Psi(\vec{x}) = f(\vec{x})$$

where $\mathcal{D}$ is a differential operator, the ***Green function equation*** is

$$\mathcal{D}G(\vec{x}, \vec{x}\,') = \delta(\vec{x} - \vec{x}\,')$$

From equation (1), the differential operator is

$$\mathcal{D} \equiv \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$$

and since we now have not only an $\vec{x}$-dependence, but also a t-dependence, the Green function corresponding to equation (1) satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_k^{(\pm)}(\vec{x}, t; \vec{x}\,', t') = -\mu_0\, \delta(\vec{x} - \vec{x}\,')\, \delta(t - t') \tag{6.41.a}$$

Notice that I've replaced 4π from equation (6.41) with μ_0, choosing instead to insert the factor 4π into the Green function itself in part (b). Or you could put it in equation (6.41.a) instead, which means that ***another acceptable way*** to write the Green function equation is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_k^{(\pm)}(\vec{x}, t; \vec{x}\,', t') = -\frac{\mu_0}{4\pi}\, \delta(\vec{x} - \vec{x}\,')\, \delta(t - t') \tag{6.41.b}$$

What matters is that $\vec{A}(\vec{x}, t)$ comes out the same in the end, no matter which of the above you've written as the Green function equation.

Of course, depending on which of the above you wrote as the Green function equation, your Green function itself will be different.

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(b) Write down the Green function, which is the solution to the equation you wrote in part (a).

Solution: Proceeding as on pages 6-7 of the Class Summary for Week 3—Thu (Apr 15), the solution to equation (6.41.a) is the Green function given by

$$G^{(\pm)}(\vec{x}, t; \vec{x}\,', t') = -\frac{\delta\left(t' - \left[t \mp \frac{|\vec{x} - \vec{x}\,'|}{c}\right]\right)}{4\pi\, |\vec{x} - \vec{x}\,'|} \tag{6.44.a}$$

Notice that I've put the factor 4π in the Green function rather than in equation (6.41.a).

Alternatively, if you decided to put the factor of 4π in the Green function equation itself as in equation (6.41.b), then your Green function will be given by

$$G^{(\pm)}(\vec{x}, t; \vec{x}\,', t') = -\frac{\delta\left(t' - \left[t \mp \frac{|\vec{x} - \vec{x}\,'|}{c}\right]\right)}{|\vec{x} - \vec{x}\,'|} \tag{6.44.b}$$

Notice that now the denominator ***does not have*** the factor 4π, because I put it in the Green function equation (6.41.b).

(c) Use the Green function in part (b) to write down the solution to equation (1) on the previous page.

Solution: Since we're interested in only the outward-going solution, we choose the retarded Green function G^+ (which corresponds to the minus sign in the square brackets), so that equation (6.44.a) gives

$$G^{+}(\vec{x}, t; \vec{x}\,', t') = -\frac{\delta\left(t' - \left[t - \frac{|\vec{x} - \vec{x}\,'|}{c}\right]\right)}{4\pi\, |\vec{x} - \vec{x}\,'|}$$

The solution to equation (1) is then

$$\vec{A}(\vec{x}, t) = \int d^3x' \int G^+ \Big[-\mu_0 J\Big]\, dt' \;=\; \frac{\mu_0}{4\pi} \int d^3x' \,\frac{\vec{J}(\vec{x}\,', t')}{|\vec{x} - \vec{x}\,'|} \int \delta\left(t' - \left[t - \frac{|\vec{x} - \vec{x}\,'|}{c}\right]\right) dt'$$

We can leave it like this, or use the property of the δ-function to carry out the t'-integration. To do that, we must insert explicitly the harmonic time dependence:

$$\vec{J}(\vec{x}\,', t) = \vec{J}(\vec{x}\,')\, e^{-i\omega t'} \qquad \text{and} \qquad \vec{A}(\vec{x}, t) = \vec{A}(\vec{x})\, e^{-i\omega t}$$

so that the equation above becomes

$$\vec{A}(\vec{x}) e^{-i\omega t} = \frac{\mu_0}{4\pi} \int d^3x' \,\frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \int e^{-i\omega t'}\, \delta\left(t' - \left[t - \frac{|\vec{x} - \vec{x}\,'|}{c}\right]\right) dt'$$

and use the property of the δ-function: $\displaystyle\int f(t)\delta(t-a)dt = f(a)$, to carry out the t'-integration, so that we get finally

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \,\frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, e^{i\omega|\vec{x} - \vec{x}\,'|/c}$$

Notice that I've canceled the $e^{-i\omega t}$ factor that is common to both sides (after the t-integration on the right hand side).

2. The vector potential $\vec{A}$ of an oscillating electric dipole is given by equation (9.16):

$$\vec{A}(\vec{x}) = -\frac{i\mu_0\omega}{4\pi}\,\vec{p}\,\frac{e^{ikr}}{r}$$

where $\vec{p}$ is the dipole moment, and $k = \omega/c$ is the wave number.

Show that its magnetic field is given by

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

where $\hat{n}$ is a unit vector in the direction of $\vec{x}$, so that $\vec{x} = r\hat{n}$

Solution: We have

$$\vec{H} = \frac{1}{\mu_0}\,\vec{\nabla}\times\vec{A} = -\frac{i\omega}{4\pi}\,\vec{\nabla}\times\left[\vec{p}\,\frac{e^{ikr}}{r}\right]$$

$$= \frac{i\omega}{4\pi}\,\vec{p}\times\vec{\nabla}\left(\frac{e^{ikr}}{r}\right)$$

Now, e^{ikr}/r is a function of r only, and $\hat{n}$ is a unit vector along $\vec{x}$, so the above becomes

$$\vec{H} = \frac{i\omega}{4\pi}\,\vec{p}\times\hat{n}\,\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right)$$

$$= \frac{i\omega}{4\pi}\,\vec{p}\times\hat{n}\left[\frac{ike^{ikr}}{r} + e^{ikr}\left(-\frac{1}{r^2}\right)\right]$$

$$= \frac{i\omega}{4\pi}\,\vec{p}\times\hat{n}\,\frac{ike^{ikr}}{r}\left[1 - \frac{1}{ikr}\right]$$

$$= -\frac{\omega k}{4\pi}\,\vec{p}\times\hat{n}\,\frac{e^{ikr}}{r}\left[1 - \frac{1}{ikr}\right]$$

Finally, since $\omega = ck$, this becomes

$$\vec{H} = -\frac{(ck)k}{4\pi}\,\vec{p}\times\hat{n}\,\frac{e^{ikr}}{r}\left[1 - \frac{1}{ikr}\right]$$

and therefore

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

which is the expression that we were asked to derive.

3. Meanwhile, it can be shown that the electric field of an oscillating electric dipole is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2\left(\hat{n}\times\vec{p}\right)\times\hat{n}\,\frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}\right]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}$$

Show that, in the near zone, the $\vec{H}$ you found in Question 2, and the $\vec{E}$ written above take the form

$$\vec{H} \simeq \frac{i\omega}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{r^2}$$

and

$$\vec{E} \simeq \frac{1}{4\pi\epsilon_0}\left[3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}\right]\frac{1}{r^3}$$

Solution: Starting from the equation we derived in Question 2:

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

we get, by taking $1/ikr$ outside the parentheses that

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{ikr^2}\left(ikr - 1\right)$$

Now, expand e^{ikr} in a Taylor series to get

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{ikr^2}\left[1 + ikr + \ldots\right]\left(ikr - 1\right)$$

Then, multiply the term in square brackets with the term in parentheses to get

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{ikr^2}\left[(ikr - 1) + (i^2k^2r^2 - ikr) + \ldots\right]$$

In the expression above:

- the first and fourth terms in the square brackets cancel,
- in the near zone, $kr \ll 1$, so the third term which is of the order $(kr)^2$ can be dropped, as can all other terms in the sum other than the second (i.e., -1).

After applying the above, we get

$$\vec{H} \simeq \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{ikr^2}\left[-1\right]$$

Then, after canceling a k in the numerator with the k in the denominator, we get

$$\vec{H} \simeq \frac{ck}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{ir^2}\left[-1\right]$$

Since $k = \omega/c$, we will now replace ck in the numerator by ω.

At the bottom of the previous page, we obtained

$$\vec{H} \simeq \frac{ck}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{ir^2}\left[-1\right]$$

Replacing ck in the numerator by ω, we get

$$\vec{H} \simeq \frac{\omega}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{ir^2}\left[-1\right]$$

Now, move the i in the denominator into the square brackets:

$$\vec{H} \simeq \frac{\omega}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{r^2}\left[-\frac{1}{i}\right]$$

Finally, since $-1/i = i$, we get

$$\vec{H} \simeq \frac{i\omega}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{1}{r^2}$$

which is the desired relation.

Meanwhile, starting from

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2\left(\hat{n}\times\vec{p}\right)\times\hat{n}\,\frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}$$

Multiply and divide the first term inside the curly brackets by r^2 to get

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2r^2\left(\hat{n}\times\vec{p}\right)\times\hat{n}\,\frac{e^{ikr}}{r^3} + \left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}$$

Then, multiply and divide the last term inside the parentheses by r to get

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2r^2\left(\hat{n}\times\vec{p}\right)\times\hat{n}\,\frac{e^{ikr}}{r^3} + \left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ikr}{r^3}\right)e^{ikr}\right\}$$

The first term in curly brackets is then zero because $kr \ll 1$, and so is the last term inside the parentheses. We are left with

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{0 + \left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-0\right)e^{ikr}\right\}$$

and since $kr \ll 1$, we have also that

$$\frac{e^{ikr}}{r^3} \approx \frac{1}{r^3}$$

Therefore, we obtain

$$\vec{E} \simeq \frac{1}{4\pi\epsilon_0}\left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\frac{1}{r^3}$$

which is the desired relation.

4. The time-averaged power radiated per unit solid angle by an oscillating dipole is given by

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[r^2\,\hat{n}\cdot\vec{E}\times\vec{H}^*\right] \tag{9.21}$$

where the fields $\vec{E}$ and $\vec{H}$ in the far zone are given by equation (9.19):

$$\begin{aligned}\vec{H} &= \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\\ \vec{E} &= \frac{k^2}{4\pi\epsilon_0}\left[(\hat{n}\times\vec{p})\times\hat{n}\right]\frac{e^{ikr}}{r}\end{aligned} \tag{9.19}$$

Show that

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}\,k^4\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2 \tag{9.22}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space.

Solution: Let us begin by doing $\vec{E}\times\vec{H}^*$.

$$\begin{aligned}\vec{E}\times\vec{H}^* &= \left\{\frac{k^2}{4\pi\epsilon_0}\left[(\hat{n}\times\vec{p})\times\hat{n}\right]\frac{e^{ikr}}{r}\right\}\times\frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{-ikr}}{r}\\ &= \frac{ck^4}{16\pi^2\epsilon_0\,r^2}\left\{(\hat{n}\times\vec{p})\times\hat{n}\right\}\times\left(\hat{n}\times\vec{p}\right)\end{aligned}$$

Then

$$\hat{n}\cdot\left[\vec{E}\times\vec{H}^*\right] = \frac{ck^4}{16\pi^2\epsilon_0\,r^2}\,\hat{n}\cdot\left[\left\{(\hat{n}\times\vec{p})\times\hat{n}\right\}\times\left(\hat{n}\times\vec{p}\right)\right] \tag{H4.4}$$

If we set

$$\hat{n} = \vec{a}, \qquad (\hat{n}\times\vec{p})\times\hat{n} = \vec{b}, \qquad \text{and} \qquad \hat{n}\times\vec{p} = \vec{c}$$

then the vector relation from the inside front cover of Jackson $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a})$ lets us write equation (H4.4) as

$$\begin{aligned}\hat{n}\cdot\left[\vec{E}\times\vec{H}^*\right] &= \frac{ck^4}{16\pi^2\epsilon_0\,r^2}\left[\left\{(\hat{n}\times\vec{p})\times\hat{n}\right\}\cdot\left(\hat{n}\times\vec{p}\right)\times\hat{n}\right]\\ &= \frac{ck^4}{16\pi^2\epsilon_0\,r^2}\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2\end{aligned}$$

Finally, therefore

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[r^2\,\hat{n}\cdot\vec{E}\times\vec{H}^*\right] = \frac{ck^4}{32\pi^2\epsilon_0}\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2 = \frac{c}{c}\,\frac{ck^4}{32\pi^2\epsilon_0}\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2$$

To get this in the form in equation (9.22), put $1/c = \sqrt{\mu_0\epsilon_0}$, so that we get

$$\frac{dP}{d\Omega} = c\sqrt{\mu_0\epsilon_0}\,\frac{ck^4}{32\pi^2\epsilon_0}\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2 = \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2 = \frac{c^2 Z_0}{32\pi^2}\,k^4\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2$$

which is equation (9.22).