Learning outcomes

1. Fourier Series Coefficients
2. Fourier Transform
3. Discrete Fourier Transform

Last time we saw that a large class of functions that appear in physics, the *piecewise continuous functions*, form a vector space.

In addition, we saw that this space has a basis composed of $\{1, \cos nx, \sin mx\}$ where n, m are positive integers.

We can therefore write a general *piecewise continuous function, f(x),* as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

For later convenience

The orthogonal relationships are:

$$\int_{-\pi}^{\pi} \sin mt \sin nt \;\; dt = \begin{cases} \pi\delta_{m,n} & m \neq 0 \\ 0 & m = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mt \cos nt \;\; dt = \begin{cases} 2\pi\delta_{m,n} & m \neq 0 \\ 2\pi & m = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \cos nt \;\; dt = 0, \quad \text{all integer } m, n$$

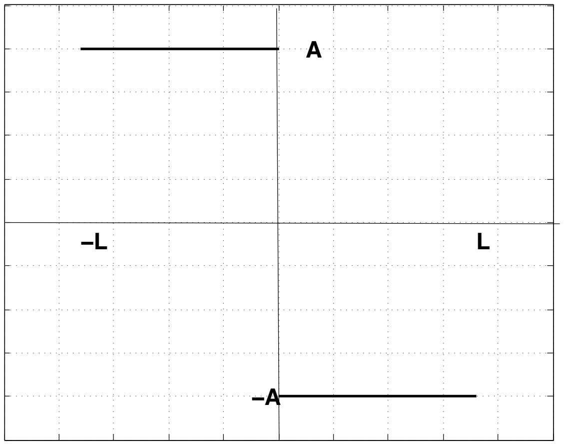
To find the coefficients, a_n, b_n we use the orthogonality relations to find that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \;\; dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \;\; dx$$

Let's work through an example of how this all works. Consider the function

$$f(x) = \begin{cases} A & \text{if } \; -L < x < 0, \\ -A & \text{if } \; 0 < x < L. \end{cases}$$



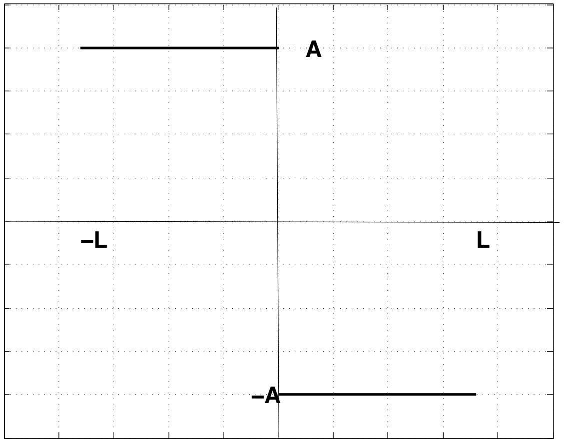
$$\vec{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

average

how much function looks like cos (nx)

how much function looks like sin (nx)



a_o = 0   odd function, so no cosines

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} A\sin\frac{k\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} -A\sin\frac{k\pi x}{L}\,dx$$

$$= \frac{A}{L}\left[-\frac{L}{k\pi}\cos\frac{k\pi x}{L}\bigg|_{-L}^{0} + \frac{L}{k\pi}\cos\frac{k\pi x}{L}\bigg|_{0}^{L}\right] = \frac{A(-1+\cos k\pi + \cos k\pi - 1)}{k\pi}$$

$$= -\frac{2A}{k\pi}\left[1-\cos k\pi\right].$$

k is an integer. Note that only the odd terms survive.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = -\frac{4A}{\pi}\left[\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \cdots\right].$$

Do question (1) on the worksheet

Fourier series are powerful tools with which to extract information from functions. However, they are limited to *periodic functions in finite intervals.*

We now generalize the series so that *non-periodic functions over infinite intervals* can be analyzed.

To begin, let's change the interval given by *[-π, π]* to a more general *[-T, T]* by letting *t→ πt/T*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

then

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{in\pi t/T}$$

with

$$c_n = \frac{1}{2T}\int_{-T}^{T} f(t) e^{-in\pi t/T}$$

Notice the we can identify the discrete frequencies in the summation as $\omega = \frac{n\pi}{T}$

Note that n is an integer

Next note the difference between successive frequencies is,

$$\Delta\omega = \frac{(n+1)\pi}{T} - \frac{n\pi}{T} = \frac{\pi}{T}$$

And using this, we can write the Fourier series of function on [*-T, T*] as

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{in\Delta\omega t}$$ where

$$c_n = \frac{\Delta\omega}{2\pi}\int_{-T}^{T} f(t)e^{-in\Delta\omega t}\,dt$$

Now, let's define a function $g(n\Delta\omega) = \frac{1}{\sqrt{2\pi}}\int_{-T}^{T} f(t)e^{-in\Delta\omega t}\,dt$ so that

$$c_n = \frac{\Delta\omega}{\sqrt{2\pi}} g(n\Delta\omega)$$ and

$$f(t) = \frac{1}{\sqrt{2\pi}}\sum_{-\infty}^{\infty} \underbrace{\Delta\omega\; g(n\Delta\omega)}_{c_n}\; e^{in\Delta\omega t}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \Delta\omega\ g(n\Delta\omega)\ e^{in\Delta\omega t}.$$

Take the limit as $T \rightarrow \infty$. In doing that, $n\Delta\omega$ becomes continuous so that (recall that $\omega \approx \frac{1}{T}$)

Inverse Fourier Transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega \tag{1}$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{2}$$

Fourier Transform

Stuff:

1. ω is a continuous variable. So now frequency takes on a more interesting meaning
2. The normalization $1/\sqrt{2\ \pi}$ is not universal
3. The Fourier transform is a linear operation, that is, the Fourier transform of two functions, $f_1 + f_2 = g_1 + g_2$ where g_1 , g_2 are the respective Fourier transforms
4. See http://www.thefouriertransform.com/transform/properties.php for other properties.

Do questions 2 – 5 on the worksheet

The *Discrete Fourier Transform*

1. In physical applications, the underlying function is only *sampled*
   i. No instrument has *infinite resolution*
   ii. No experiment has *infinite time to run*
2. We will cover the evenly sampled case. To arrive at the *discrete Fourier transform*
   i. Begin with complex version of *Fourier Series* on an interval [0,*T*]

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{in\pi t/T}$$

with c_n now

$$c_n = \frac{1}{T}\int_0^T f(t) e^{-in\pi t/T}$$

Since time begins at 0

Note that a key difference here is that our lower limit is 0, so we have to sort of *derive* the Fourier transform again.

Now recall that $g(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$

Let's suppose that our finite resolution is characterized as a finite sampling of time, Δt, sampled *N* times. That is, we sample the function *m* Δt times, with *m = 0, 1,2,…N-1*.

Recalling that $\Delta\omega = \dfrac{2\pi}{T}$ We can approximate the integral in the Fourier transform as

$$g(n\Delta\omega) = \sum_{m=0}^{N-1} f(m\Delta t) e^{-in\Delta\omega m\Delta t} = \sum_{m=0}^{N-1} f(m\Delta t) e^{-i2\pi mn/N} \quad \leftarrow \textit{DFT}$$

The *nth* frequency

The *sum* of the *m* data points

The inverse discrete Fourier transform is found to be

$$f(m\Delta t) = \frac{1}{N} \sum_{n=0}^{N-1} g(n\Delta\omega)\, e^{i2\pi mn/N} \quad \leftarrow \textit{DFT}^{-1}$$

There is a relationship between Fourier transform and discrete Fourier Transform

$$
\begin{aligned}
\mathcal{F}\left[f(t)\right] &= \frac{\Delta t}{\sqrt{2\pi}}\mathcal{DFT}\left[f(t)\right] \\
\mathcal{F}^{-1}\left[g(\omega)\right] &= \frac{\sqrt{2\pi}}{\Delta t} i\, \mathcal{DFT}\left[g(\omega)\right]
\end{aligned}
$$

* careful!

Finish the worksheet