Legendre Transformations

- A Legendre transformation switches from a function of one set of variables to another function of a *conjugate* set of variables. Both functions will have the same dimensions.
- We are interested in this because we can apply Legendre transformations to switching from Lagrangian to Hamiltonian mechanics. In this case the generalized velocity $\dot{q}$ and the linear momentum *p* are conjugate variables, and both L and H have the dimension of energy.
- Another area where Legendre transformations play a role is thermodynamics, where it connects internal energy, enthalpy, and Gibbs and Helmholtz free energies.

$$f = f(u_i) \quad v_i = \frac{\partial f}{\partial u_i} \quad g = g(v_i)$$

$f \longrightarrow L$
$g \longrightarrow H$
$u_i \longrightarrow \dot{q}_i$
$v_i \longrightarrow p_i$
q_i are passive variables

Legendre transformation of f to g

$$g = \sum_{i=1}^{n} u_i v_i - f$$

$$L = L(q_i; \dot{q}_i; t) \qquad p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad H = H(q_i, p_i t)$$

$\dot{q}_i$ are the “active” variables
q_i and t are the passive variables

Legendre transformation of L to H

$$H(q_i, p_i, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q_i, \dot{q}_i t)$$

active variables

Example 1: Free particle

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad p_y = m\dot{y} \quad p_z = m\dot{z}$$

$$H(q_i, p_i, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q_i, \dot{q}_i t)$$

$$H(q_i, p_i, t) = m\dot{x}\dot{x} + m\dot{y}\dot{y} + m\dot{z}\dot{z} - \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\dot{y}^2 - \frac{1}{2} m\dot{z}^2$$

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2 + \frac{1}{2} m\dot{z}^2 = \frac{1}{2}\left[\frac{(m\dot{x})^2}{m} + \frac{(m\dot{y})^2}{m} + \frac{(m\dot{z})^2}{m}\right]$$

$$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

Example 2: Disk rolling down an inclined plane

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{4}mR^2\dot{\theta}^2 + mg(y - l)\sin\alpha$$

$$p_y = m\dot{y} \quad p_\theta = \frac{1}{2}mR^2\dot{\theta} \quad p_\theta^2 = \frac{1}{4}m^2R^4\dot{\theta}^2$$

$$H(q_i, p_i, t) = \sum_{i=1}^{n} p_i\dot{q}_i - L(q_i, \dot{q}_i t)$$

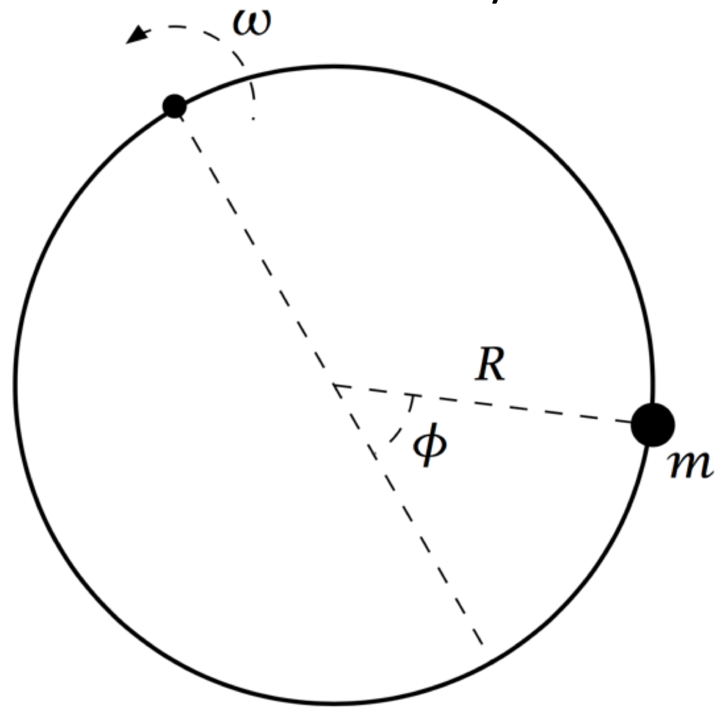
$$= p_y\dot{y} + p_\theta\dot{\theta} - \frac{1}{2}m\dot{y}^2 - \frac{1}{4}mR^2\dot{\theta}^2 - mg(y - l)\sin\alpha$$

$$= m\dot{y}^2 + \frac{1}{2}mR^2\dot{\theta}^2 - \frac{1}{2}m\dot{y}^2 - \frac{1}{4}mR^2\dot{\theta}^2 - mg(y - l)\sin\alpha$$

$$= \frac{1}{2}m\dot{y}^2 + \frac{1}{4}mR^2\dot{\theta}^2 - mg(y - l)\sin\alpha$$

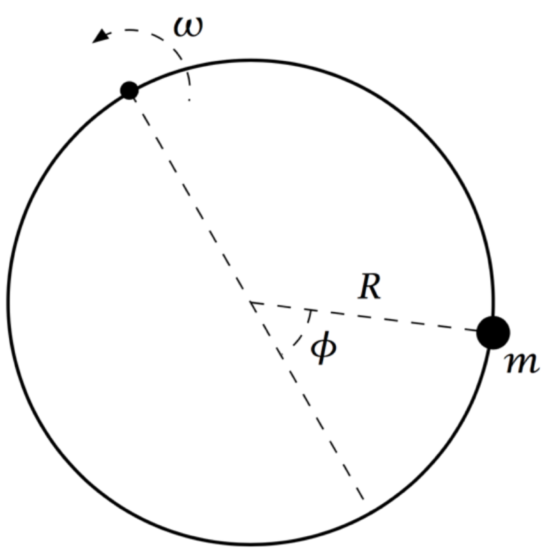
$$= \frac{p_y^2}{2m} + \frac{p_\theta^2}{mR^2} - mg(y - l)\sin\alpha$$

Remember activity 10?



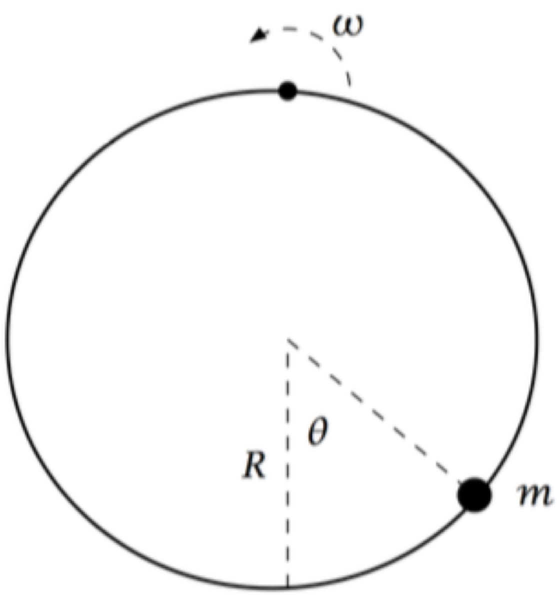
- Is the Hamiltonian H a constant of motion?
- Is the energy of the bead conserved?
- Is Hequal to the the total energy of the system?

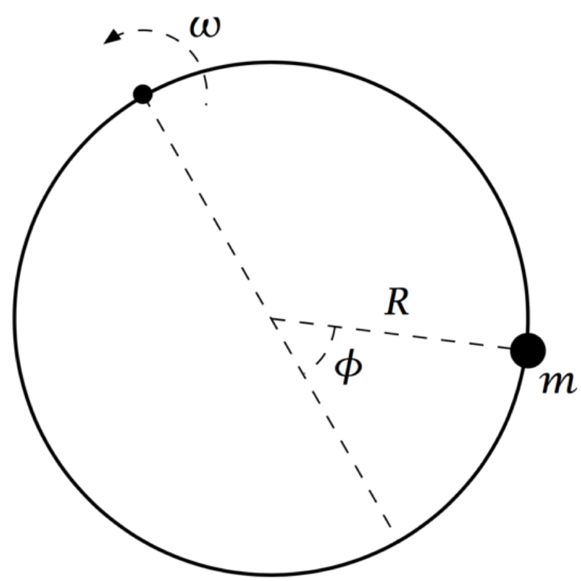
Remember activity 10?



- The energy of the bead is not conserved because the hoop rotates and can do work on the bead, and the bead can do work on the hoop.
- The Hamiltonian is a constant of motion because $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$ and the Lagrangian does not depend explicitly on time. (Equations 4.18 and 4.19)
- The is an example of a system where $H \neq T + V$ but the Hamiltonian is still a constant of motion.
- Using the reasoning on p. 100, $H \neq T + V$ because the transformation equation from cartesian to generalized coordinates depend explicitly on time.

But wait! Didn't we have a version of the Lagrangian that did have an explicit time dependence?





$$L = \frac{mR^2}{2}\left[\omega^2 + \dot{\theta}^2 + 2\omega\dot{\theta}\cos(\theta - \omega t)\right]$$

$$L = \frac{mR^2}{2}\left[\omega^2 + (\dot{\phi} + \omega)^2 + 2\omega(\dot{\phi} + \omega)\cos(\phi)\right]$$

θ is not an appropriate generalized coordinate because to know the value of θ we need to know the position of the hoop, which means we need to know the time. We can't calculate the kinetic energy of the bead from θ without also knowing t. θ is not sufficient to completely specify the configuration of the system.