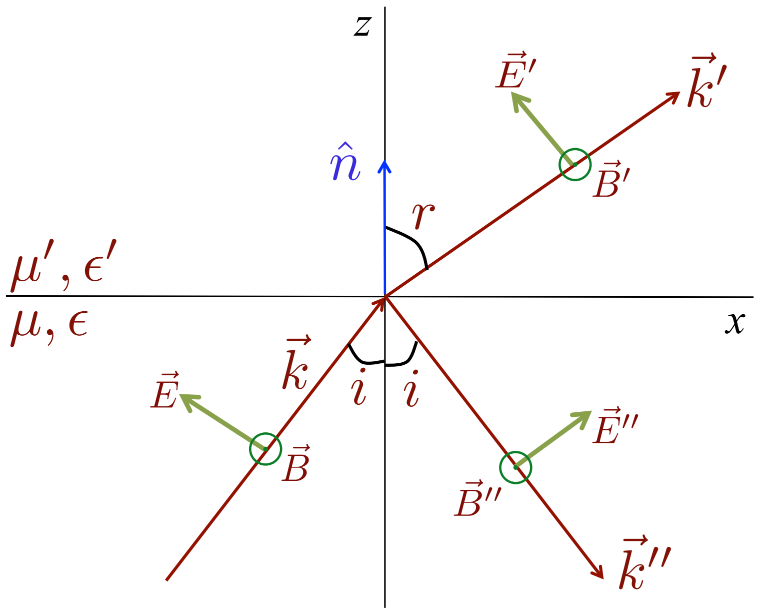
PHY 411 **Winter 2021**

Week 3—Thursday, Jan 21—Discussion Worksheet

$\vec{E}$ parallel to the plane of incidence

In the previous class, we derived expressions for the refracted and reflected amplitudes when the $\vec{E}$-fields were perpendicular to the plane of incidence. The other case to consider is when the $\vec{E}$-fields are parallel to the plane of incidence. The situation is shown in the figure below.



The $\vec{E}$-fields shown in the above figure are in the plane of the page, and the $\vec{B}$-fields are pointing out of the page (so that $\vec{E} \times \vec{B}$ is in the direction of propagation of the waves).

However, we won't run through the exercise of deriving the expressions for the refracted and reflected amplitudes, since the process is very similar to that for the perpendicular case as long as you handle the angles carefully (it might be worth your while to convince yourself you can do it efficiently, however). You should get

$$\begin{aligned} \frac{E_0'}{E_0} &= \frac{2nn' \cos i}{\frac{\mu}{\mu'} n'^{\,2} \cos i + n\sqrt{n'^{\,2} - n^2 \sin^2 i}} \\ \frac{E_0''}{E_0} &= \frac{\frac{\mu}{\mu'} n'^{\,2} \cos i - n\sqrt{n'^{\,2} - n^2 \sin^2 i}}{\frac{\mu}{\mu'} n'^{\,2} \cos i + n\sqrt{n'^{\,2} - n^2 \sin^2 i}} \end{aligned} \tag{7.41}$$

1. Discuss which of the four equations in equation (7.37) you could use to derive equation (7.41).

To derive the equations above, equations 7.37c and 7.37d are needed.

Brewster's Angle

$a^2 - b^2 = (a+b)(a-b)$

For a wave polarized *parallel* to the plane of incidence, *Brewster's angle* is the angle of incidence for which there is no reflected wave.

2. Show that if a wave is incident from a medium with refractive index n and refracted into a medium with refractive index n', then the Brewster angle i_B is given by

$$i_B = \arctan\left(\frac{n'}{n}\right)$$

assuming $\mu = \mu'$ to simplify the algebra (which is true at optical frequencies).

$$\frac{E_0''}{E_0} = \frac{\sqrt{\frac{\mu\epsilon'}{\mu'\epsilon}} - 1}{\sqrt{\frac{\mu\epsilon'}{\mu'\epsilon}} + 1} \rightarrow \frac{n' - n}{n + n} \quad (7.42)$$

$$= (n')^2 \cos i_B - n\sqrt{(n')^2 - n^2 \sin^2 i_B} = 0$$

$$= (n')^2 (1 - \sin^2 i_B) = n\sqrt{(n')^2 - n^2 \sin^2 i_B}$$

$$= \left(\frac{\mu}{\mu'}\right) n'^2 \cos^2(i_B) = n^2 \cos^2(i_B)$$

$$= \frac{\mu n'}{\mu' n} = \frac{\cos(i_B)}{\cos(i_B)}$$

$$i_B = \tan^{-1}\left(\frac{n'}{n}\right)$$

Total Internal Reflection

3. From freshman physics, you know that if the incident ray is in a medium of larger refractive index than the refracted ray ($n > n'$), then Snell's law says there is an angle of incidence i_0 for which the angle of refraction will be 90°. Therefore

(7.36)

$$\sin i_0 = \left(\frac{n'}{n}\right) \tag{7.44}$$

(a) For $i > i_0$, show that $\sin r > 1$.

$$\frac{\sin i}{\sin r} = \frac{k'}{k} = \sqrt{\frac{\mu'\epsilon'}{\mu\epsilon}} = \frac{n'}{n} \quad \text{So } \sin i_0 = \left(\frac{\sin i}{\sin r}\right)$$

$$\Rightarrow \sin r = \frac{\sin i}{\sin i_0} \Rightarrow r = \sin^{-1}\left(\frac{\sin i}{\sin i_0}\right)$$

(b) For $i > i_0$, show that r is a complex angle with a purely imaginary cosine given by

$$\cos r = i_m \sqrt{\left(\frac{\sin i}{\sin i_0}\right)^2 - 1} \tag{7.45}$$

where i_m is the usual imaginary number i (i.e. $i_m^2 = -1$); I've written it as i_m here to avoid confusion with the angle of incidence in the same expression. (In the posted class summary, I'll write it as i, but in purple font.)

Since $\frac{\sin i}{\sin i_0} = \sin r$, then

$$\cos r = \sqrt{1 - \sin^2 r}$$

$$= \sqrt{1 - \left(\frac{\sin i}{\sin i_0}\right)^2}$$

will always be > 1 so will be i_m

$$\sqrt{m} = \sqrt{i_m^2 m} = i_m\sqrt{m}$$

$$\cos r = i_m \sqrt{\left(\frac{\sin i}{\sin i_0}\right)^2 - 1}$$

4. We will now look at the implications of complex quantities like equation (7.45).

(a) Show that the spatial part of the phase factor for the refracted wave $e^{i_m \vec{k}' \cdot \vec{x}}$ is given by

$$e^{i_m \vec{k}' \cdot \vec{x}} = e^{i_m k' (x \sin r + z \cos r)} = e^{-k' \left(\sqrt{(\sin i / \sin i_0)^2 - 1} \right) z} e^{i_m k' (\sin i / \sin i_0)\, x} \tag{7.46}$$

where, again, i_m is the usual imaginary number i (i.e. $i_m^2 = -1$) which I've written as i_m to avoid confusion with the angle of incidence in the same expression.

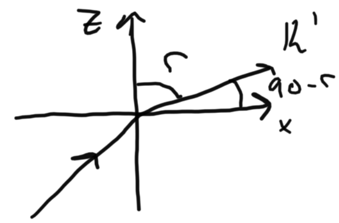
$$e^{i_m (\vec{k}' \cdot \vec{x})}$$

$$\downarrow$$

$$\vec{k}' \cdot x = k' \hat{k}' \left[x\hat{x} + y\hat{y} + z\hat{z} \right]$$

$$= (k'x)\, \hat{k}' \cdot \hat{x} + \dots$$

$$= (k'x) \cos(90 - r)$$



(b) What does equation (7.46) tell us about the behavior of the fields in the medium of refractive index n' in the space $z > 0$?

The general expression for the fields in the $z > 0$ medium with (μ', ϵ') is

$$\vec{E}' = \vec{E}'_0 e^{i(\vec{k}'\cdot\vec{x}-\omega t)}, \qquad \vec{B}' = \sqrt{\mu'\epsilon'}\,\frac{\vec{k}'\times\vec{E}'}{k'}$$

5. Since $\hat{n}$ points perpendicular to the interface from the medium (μ, ϵ) in $z < 0$ to the medium (μ', ϵ') in $z > 0$ (see figure on the first page of this worksheet), the time-averaged normal component of the Poynting vector $\vec{S}'$ just inside the surface ($z > 0$) is

$$\vec{S}'\cdot\hat{n} = \frac{1}{2}\mathrm{Re}\left[\hat{n}\cdot(\vec{E}'\times\vec{H}'^{*})\right] \tag{7.47}$$

(a) Starting from equation (7.47), show that

$$\vec{S}'\cdot\hat{n} = \frac{1}{2\mu'\omega}\mathrm{Re}\left[k'\cos r\left|E'_0\right|^2\right]$$

(b) The transmission coefficient T is defined as $T = \dfrac{\vec{S}'\cdot\hat{n}}{\vec{S}\cdot\hat{n}}$.

What is the value of T for $i > i_0$? Explain why, and what it tells us about the behavior of incident radiation for $i > i_0$.