PHY 411 **Winter 2021**

Class Summary—Week 1, Day 2—Thursday, January 7

Maxwell's Equations

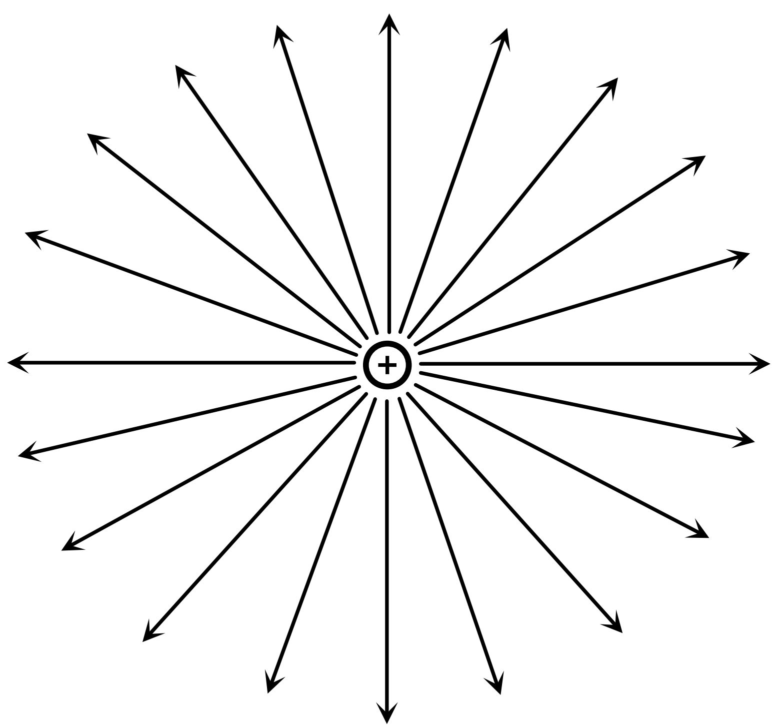
Magnetism has been known since antiquity. Likewise, the ancient Greeks (and likely other cultures) discovered that rubbing fur on amber (fossilized tree resin) caused an attraction between the two, so they knew of static electricity. Throughout most of human history, however, the phenomena of electricity and magnetism were believed to be different and unconnected.

Michael Faraday's experiments on the behavior of currents in circuits placed in time-varying magnetic fields in 1831 provided the first quantitative observations linking electric and magnetic fields. As time-varying magnetic fields give rise to electric fields and vice versa, our vocabulary changes to the study of electromagnetic fields, rather than electric or magnetic fields only. Later, in the context of Special Relativity in Chapter 11, we will see how much closer is the connection between the two, in that a purely electric or magnetic field in one coordinate system will appear as a mixture of electric and magnetic fields in another coordinate frame. For now, though, we will work on understanding the basic phenomena of electromagnetism.

We will begin by studying Maxwell's equations, which provide the basis for describing the behavior of electromagnetic fields.

First, though, let us look at the situation pre-Maxwell.

You know from introductory physics that the electric field around a point charge looks like this.



You will also recall that if you draw a Gaussian surface (in this case a sphere, or circle if you're visualizing in 2-D), you'll get the following results:

- If you put it around a charge-free region of space, you'll get

$$\vec{\nabla} \cdot \vec{E} = 0$$

  This is easy to comprehend if you understand what you mean by the divergence, $\vec{\nabla} \cdot (\ldots)$, of a quantity. Imagine drawing the Gaussian surface as a circle in the figure on the previous page in a region of space that is charge-free, i.e., away from the charge at the center of the figure. If you look at this circle, you will see that the electric field lines are diverging on one side of the Gaussian surface but converging on the other side, so in the net, the divergence of the field over the entire surface is zero.

- If you enclose charge q inside the Gaussian surface, you'll get

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

  where ρ is the charge density. For a discrete set of n point charges q_i located at the points $\vec{x}_i$, the charge density can be defined using the Dirac δ-function:

$$\rho(\vec{x}) = \sum_{i=1}^{n} q_i \, \delta(\vec{x} - \vec{x}_i) \tag{1.6}$$

  The Dirac δ-function is discussed on page 26 in Jackson. Additional comments are on the last page of the Week1—Day 1 (Tue, Jan 8) class summary posted in D2L.

Note: *Whenever equations are taken from Jackson's text, equation numbers in all reading assignments and class summaries will be identical to Jackson (3rd edition) to anchor you in the text, especially as we will be jumping around quite a bit. For those equations not from Jackson's text (e.g., Week 1—Day 1 class summary), the format will be the week number followed by the equation number, e.g., (W1.5).*

Space left blank for student notes; class summary continues on next page.

In writing the equation for $\vec{\nabla} \cdot \vec{E}$ above, we made no distinction between macroscopic and microscopic fields. While the equation we wrote above is strictly valid only in a vacuum, the density of air is low enough that it is also applicable in air. But once we start talking about ponderable media (as Jackson calls it; a very old word that means "considerable enough to be weighed"), the electric response of the medium must also be taken into account. We won't concern ourselves with the details for now (reserving it for a detailed discussion later), except to note that in ponderable media, we need to work with the electric displacement $\vec{D}$.

In order to understand $\vec{D}$, let us discuss briefly what happens when an electric field is applied to a medium made up of a large number of atoms or molecules — the charges in each molecule will respond to the applied field and this distorts the molecular charge density. Without going into the details, let's just say this produces an electric polarization $\vec{P}$ — think of dipoles lined up along the field if you want a physical picture (and that is quite close to reality, since the dipole is the dominant multipole).

This distortion of the charge density means that if you consider any small volume, there can be a net increase or decrease of charge in that volume — an example of this is shown schematically by Jackson in Figure 4.2 on page 153. This change in the charge density is accounted for by the introduction of the divergence of $\vec{P}$ (the derivation is given on pages 152-153 if you're interested, but it isn't required reading), so that the equation for the divergence of $\vec{E}$ that we wrote earlier ($\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$) now becomes

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho - \vec{\nabla} \cdot \vec{P}}{\epsilon_0} \tag{4.33}$$

Multiply on both sides by ϵ_0:

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho - \vec{\nabla} \cdot \vec{P}$$

Since ϵ_0 is just a constant, we can write the left hand side as

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho - \vec{\nabla} \cdot \vec{P}$$

Move the polarization term to the left hand side

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} + \vec{\nabla} \cdot \vec{P} = \rho$$

and then write as

$$\vec{\nabla} \cdot \left[\epsilon_0 \vec{E} + \vec{P} \right] = \rho$$

As you demonstrated on today's class worksheet (Problem 1), if we define the electric displacement $\vec{D}$ as

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \tag{4.34}$$

then we will get

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{4.35}$$

Equation (4.35) is the macroscopic counterpart of $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$. ***It is the first of the four pre-Maxwell equations written by Jackson in equation (6.1) on page 237***.

At this stage, it would also be a good idea to work out a relation between $\vec{D}$ and $\vec{E}$.

- To get such a relation, we *assume that the response of the system to an applied field is linear*. As long as the field strengths do not become extremely large, this is a reasonable assumption that only excludes ferroelectric materials.
- Furthermore, we *assume that the medium is isotropic* (i.e., it has the same properties in all directions).

Under the two assumptions above, the polarization $\vec{P}$ is parallel to $\vec{E}$ with a coefficient of proportionality that is independent of direction:

$$\vec{P} = \epsilon_0 \, \chi_e \, \vec{E} \tag{4.36}$$

where ϵ_0 is the (electric) permittivity of free space, and χ_e is the electric susceptibility of the medium.

As you demonstrated on today's class worksheet (Problem 2), substitution of equation (4.36) in equation (4.34) should lead you to the relation that

$$\vec{D} = \epsilon \vec{E} \tag{4.37}$$

where ϵ is called the electric permittivity, and

$$\epsilon = \epsilon_0 \Big(1 + \chi_e \Big) \tag{4.38}$$

Meanwhile, ϵ/ϵ_0 is called the dielectric constant of the medium (we'll need to remember all these terms for the duration of the course, at least).

Derivation: (*solution of Problem 2 on the worksheet for today*)

Start with equation (4.34):

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

and substitute equation (4.36) in it:

$$\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \, \chi_e \, \vec{E}$$

from which we obtain

$$\vec{D} = \Big(\epsilon_0 + \epsilon_0 \, \chi_e \Big) \, \vec{E}$$

and, furthermore,

$$\vec{D} = \epsilon_0 \, \Big(1 + \chi_e \Big) \, \vec{E}$$

Therefore, we have derived that

$$\vec{D} = \epsilon \, \vec{E}$$

where $\epsilon = \epsilon_0 \Big(1 + \chi_e \Big)$.

Next, let us write $\vec{\nabla} \cdot (\ldots)$ for magnetism.

Magnetic phenomena are significantly different from electric phenomena because there are *no free magnetic monopoles*, that is, there is nothing like a magnetic charge, unlike the electric charges that we deal with in electrostatics. The basic entity in magnetism is then the *magnetic dipole*. In the presence of magnetic materials, the magnetic dipole tends to align itself in a certain direction. That direction is then defined to be the direction of the *magnetic flux density* $\vec{B}$ (also called the *magnetic induction*), provided the dipole is sufficiently small and weak that it does not perturb the existing field.

If you understood the discussion of $\vec{\nabla} \cdot \vec{E}$, you should be able to figure immediately that the absence of magnetic monopoles (and hence the situation that there are no magnetic charges from which magnetic fields can diverge) leads to the relation

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{5.17}$$

Equation (5.17) is the fourth of the four pre-Maxwell equations written by Jackson in equation (6.1) on page 237.

Just as in the electric case, you might ask — presumably, equation (5.17) above is for the microscopic case, so what happens in ponderable media? We will certainly need to do that, but first we have to understand that we have a more complicated situation than in the electric case. It was only after the connection between currents and magnetic fields was established that a quantitative treatment of magnetic phenomena became possible.

A current corresponds to charges in motion and is described by a current density $\vec{J}$.

- The magnitude of $\vec{J}$ is measured in units of positive charge crossing unit area per unit time.
- The direction of $\vec{J}$ is the direction of motion of the charges.

In SI units, $\vec{J}$ is measured in coulombs per square meter-second (C/m^{2}s) or ampere per square meter (A/m^2).

In the microscopic case, we can assume that the current density $\vec{J}$ is a completely known function of position. In macroscopic problems, this is often not the case. The electrons in atoms in the medium create effective atomic currents, for which the current density is a rapidly fluctuating quantity. Therefore, only the average over a macroscopic volume is known or pertinent. The atomic electrons also contribute intrinsic magnetic moments in addition to those from their orbital motion. All of these moments can lead to dipole fields that vary appreciably on atomic scales.

Once again, we won't worry about the process of averaging microscopic equations to obtain a macroscopic description of magnetic fields in ponderable media. All we need to know for now is that the averaging of the microscopic equation (5.17), namely $\vec{\nabla} \cdot \vec{B}_{\text{micro}} = 0$, leads to the same equation for the macroscopic case, namely

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{5.75}$$

Let us return now to consideration of the electric field.

Previously, we wrote an equation for $\vec{\nabla} \cdot \vec{E}$, but that is not enough to specify completely the three components of $\vec{E}(\vec{x})$. You may be aware that a vector field can be specified almost completely (up to the gradient of a scalar function) if its divergence and curl are given everywhere in space. So, we look for an equation specifying the curl of $\vec{E}$.

To write this equation, let us go back to the picture of the electrostatic field of a single charge q that we drew earlier. We see from this figure that the electrostatic field only diverges, but does not curl. So we can write

$$\vec{\nabla} \times \vec{E} = 0 \tag{1.14}$$

If you're interested in a more formal derivation, you should start from Coulomb's law — see pages 29-30 in Jackson for hints if you're interested, but you'll have to work out the steps by yourself, and it is not required reading.

Equation (1.14) is only for the microscopic case. If you carry out the averaging for the macroscopic electrostatic field $\vec{E}$, though, you will find that the microscopic equation (1.14) above also holds for the *macroscopic electrostatic field* $\vec{E}$, namely,

$$\vec{\nabla} \times \vec{E} = 0 \tag{4.27}$$

and so we don't need to write this equation in terms of $\vec{D}$.

But what about time-dependent electric and magnetic fields?

As we mentioned earlier, it was Faraday's work that provided the connection. Jackson summarizes Faraday's observations very succinctly on page 208, so let's repeat them here. Faraday observed that a transient current is induced in a circuit if

- the steady current flowing in an adjacent circuit is turned on or off.
- the adjacent circuit with a steady current is moved relative to the first circuit.
- a permanent magnet is thrust into or out of the circuit.

Faraday attributed the transient current flow to a change in the magnetic flux linked to the circuit. This changing flux induces an electric field around the circuit, and we know from undergrad physics that the line integral of this is called the electromotive force, and according to Ohm's law, the electromotive force causes a current to flow in the circuit. Therefore, equation (4.27) above must be modified appropriately. We will skip the derivation (and you're welcome to take a look on pages 208-211 if you're interested, but it is not required reading).

Instead, based on the understanding that changing magnetic fields lead to electric fields, we will just write down the differential form of Faraday's law, which is

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5.143}$$

Equation (5.143) is the third of the four pre-Maxwell equations written by Jackson in equation (6.1) on page 237.

Finally, let us get to the curl equation for the magnetic case.

Again, we will skip the details for now. If you're interested, the curl of $\vec{B}$ is calculated by Jackson on page 179, but it is not required reading. Furthermore, Griffiths works it out from basic principles for the restricted case of a straight current-carrying wire, and that might be easier to follow. For magnetostatics, we get that

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{5.22}$$

where μ_0 is the permeability of free space and $\vec{J}$ is the current density that we discussed earlier.

Once again, equation (5.22) is a microscopic equation, and we need a macroscopic counterpart for ponderable media. As we discussed earlier, averaging to obtain a macroscopic description involves taking into account that the large number of atoms or molecules per unit volume will each have their own magnetic moments $\vec{m}_i$. For a physical example of $\vec{m}$, see page 186 in Jackson, where he shows that $\vec{m}$ for a plane current loop carrying current I has magnitude I times the area of the loop, and direction perpendicular to the plane of the loop.

These $\vec{m}_i$ give rise to an average macroscopic magnetization or magnetic moment density

$$\vec{M}(\vec{x}) = \sum_i N_i \Big\langle \vec{m}_i \Big\rangle \tag{5.76}$$

where N_i is the average number per unit volume of molecules of type i and $\langle \vec{m}_i \rangle$ is the average molecular moment in a small volume at the point $\vec{x}$.

Some fiddling around with a quantity called the vector potential, which we'll get to in great detail later so don't worry about it now (or see page 192 in Jackson if the suspense of waiting for us to get to it in class is too much for you) then gives us the result that the magnetization $\vec{M}$ contributes an effective current density

$$\vec{J}_M = \vec{\nabla} \times \vec{M} \tag{5.79}$$

Additionally, we suppose that there is also a macroscopic current density $\vec{J}(\vec{x})$ due to the flow of charge in the medium.

Note that Griffiths calls $\vec{J}_M$ the bound current density (because it is there due to magnetization — "it results from the conspiracy of many aligned atomic dipoles") and $\vec{J}$ the free current (because it involves actual transport of charge, e.g., because "somebody hooked up a wire to a battery").

Now, recall that in equation (5.22) above, we wrote that $\vec{\nabla} \times \vec{B}_{\text{micro}} = \mu_0 \vec{J}_{\text{micro}}$ (subscripting now with "micro" to explicitly identify that this equation is for the microscopic case).

Based on the discussion above, though, we are led to the conclusion that in the macroscopic equivalent, $(\vec{J} + \vec{J}_M)$ takes the place of $\vec{J}_{\text{micro}}$. So we get for the macroscopic case that

$$\vec{\nabla} \times \vec{B} = \mu_0 \Big[\vec{J} + \vec{J}_M \Big]$$

or, using equation (5.79):

$$\vec{\nabla} \times \vec{B} = \mu_0 \Big[\vec{J} + \vec{\nabla} \times \vec{M} \Big] \tag{5.80}$$

Now, on to something you demonstrated on today's worksheet.

You showed on the worksheet for today (Problem 3) that if we define

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \tag{5.81}$$

then the macroscopic equivalent of equation (5.22) is

$$\vec{\nabla} \times \vec{H} = \vec{J} \tag{5.82}$$

Equation (5.82) is the second of the four pre-Maxwell equations written by Jackson in equation (6.1) on page 237.

As in the case of electrostatics, we need a relation between $\vec{B}$ and $\vec{H}$.

For linear media, we would get

$$\vec{B} = \mu\vec{H} \tag{5.84}$$

where μ is the magnetic permeability of the medium. Note that equation (5.84) is not valid for ferromagnetic substances (which we can ignore for now; see page 193 in Jackson if you're interested).

Pre-Maxwell's Equations

Finally, therefore, we are at the starting point of Chapter 6 — the basic laws of electricity and magnetism until the time before Maxwell's work:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{D} &= \rho \\
\vec{\nabla} \times \vec{H} &= \vec{J} \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \cdot \vec{B} &= 0
\end{aligned} \tag{6.1}$$

It is important to emphasize that $\vec{E}$ and $\vec{B}$ are the fundamental fields. The derived fields $\vec{D}$ and $\vec{H}$ are introduced as a matter of convenience to take into account in an average way the contributions to ρ and $\vec{J}$ of the atomic charges and currents.

Of the four equations, three are derived from statics; Faraday's law is the only one that takes time-varying fields into account.

It was Maxwell's insight that allowed for the four equations to be made into a consistent set. As we will see shortly, Maxwell's breakthrough was revolutionary!

Maxwell's Brilliant Insight

Equation (6.1) written on the previous page constituted the basic laws of electricity and magnetism until the time before Maxwell's brilliant insight. As mentioned already, however, equation (6.1) is not a consistent set of equations, at least not when time-varying fields are taken into account. The main culprit is Ampere's law, the second of the four equations in equation (6.1). If you looked at the derivation, you already know that it was derived under the assumption of magnetostatics that $\vec{\nabla} \cdot \vec{J} = 0$. In fact, you can show from the equation itself that this has to be the case, as you showed on Problem 4 of today's worksheet by taking the divergence of both sides of Ampere's law.

So, how should one proceed? Recall that a current corresponds to charges in motion and is described by $\vec{J}$ — the current density measured as the amount of charge crossing unit area per unit time. This suggests that there is a relation between the charge density ρ and the current density $\vec{J}$, and indeed there is one! Conservation of charge requires the charge density at any point in space to be related to the current density in the neighborhood of that point by a continuity equation

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{6.3}$$

Equation (6.3) tells us that a decrease in charge inside a small volume as time passes must correspond to a flow of charge out through the surface of that small volume, thereby conserving the total amount of charge. Equation (6.3) also tells us why $\vec{\nabla} \cdot \vec{J} = 0$ in magnetostatics — because in magnetostatics (what Jackson calls "steady-state phenomena"), there is no change in the net charge density anywhere in space.

Maxwell substituted the first of the four equations in equation (6.1) into the continuity equation (6.3) and realized that it gave a vanishing divergence, that is, in

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

he wrote $\vec{\nabla} \cdot \vec{D}$ in place of ρ:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial (\vec{\nabla} \cdot \vec{D})}{\partial t} = 0$$

Since $\vec{\nabla} \cdot (\ldots)$ is a spatial operation and $\partial/\partial t$ is in time, we can interchange the order of the two operations:

$$\vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \frac{\partial \vec{D}}{\partial t} = 0$$

so that we get (as you showed on Problem 5 of today's worksheet)

$$\vec{\nabla} \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0 \tag{6.4}$$

With this vanishing divergence available, Maxwell replaced $\vec{J}$ in Ampere's law by its generalization

$$\vec{J} \to \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

for time-dependent fields.

With the modification of replacing $\vec{J}$ by its generalization

$$\vec{J} \rightarrow \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

for time-dependent fields (see previous page), Ampere's law, the second of the four relations in equation (6.1), became

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{6.5}$$

Equation (6.5) is still the same experimentally verified law for magnetostatic phenomena, but is now mathematically consistent with the continuity equation (6.3) for time-dependent fields.

Maxwell gave the additional term in equation (6.5) the name *displacement current*. The presence of this term implies that *a changing electric field causes a magnetic field*, even if a current is not present — it is the converse of Faraday's law. This additional term is of crucial importance for rapidly fluctuating fields, for without it there would be no electromagnetic radiation!

The Maxwell Equations

We now have, therefore, the full set of four equations that are known today as the Maxwell equations.

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho & \qquad \vec{\nabla} \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \qquad \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned} \tag{6.6}$$

These four equations form the basis of classical electrodynamics. Together with the Lorentz force equation and Newton's second law of motion, they provide a complete description of the classical dynamics of interacting charged particles and electromagnetic fields.

This ends § 6.1 (pages 237-239) in Jackson (including references to necessary materials in Chapters 1-5). We will now leave Chapter 6 (returning to it later to discuss vector potentials and gauges that we will need for Chapter 9). In the next class, we will move on to the study of electromagnetic waves in Chapter 7. You already got a preview of the beginning of Chapter 7 in Problem 4 of today's worksheet.