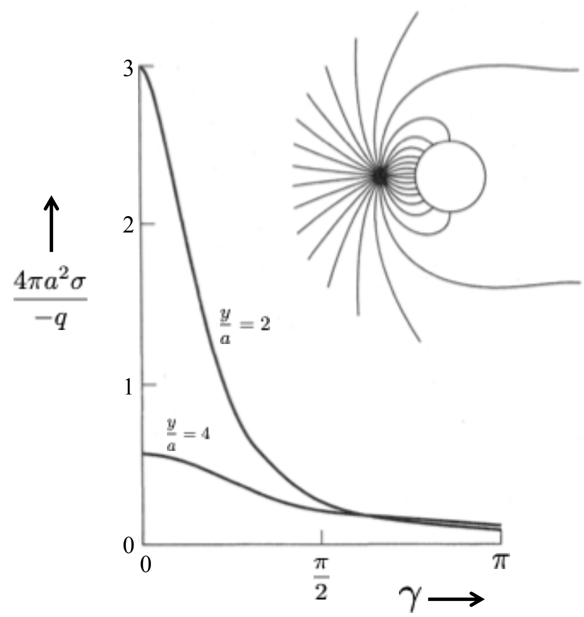
PHY 411 **Winter 2021**

Class Summary—Week 10, Day 1—Tuesday, Mar 9

Method of Images

In the previous class, we used the method of images to find the potential outside a grounded conducting sphere when a charge q is placed at a distance y from the center of the sphere. You also derived an expression for the charge density σ induced on the surface of the sphere.

Your result for σ is plotted on the right (in units of $-q/4\pi a^2$, where a is the radius of the sphere) as a function of γ for two values of y/a.



Recall that γ is the angle between the radial line from the center of the sphere to the charge q and the radial line from the center of the sphere to the observation point. Thus, γ is the angle between $\vec{x}$ and $\vec{y}$, where $\vec{x}$ is the position of the observation point and $\vec{y}$ is the position of the charge q, both with respect to an origin at the center of the sphere.

In the figure, $\gamma = 0$ corresponds to the direction of the point charge q outside the sphere, and the concentration of charge (as reflected by the highest value of the surface charge density σ) toward the direction of the point charge q is evident in this plot, especially for $y/a = 2$ (i.e., when the point charge q is at twice the distance from the radius a of the sphere). The inset on the top right of the figure above shows the lines of force for this situation, when $y = 2a$.

By direct integration, you can show that the total induced charge on the sphere is $-aq/y$, the magnitude of the image charge. This makes sense, and is what you would obtain using Gauss' Law.

To calculate the force acting on the charge q, you could proceed in one of two ways.

- You could just find the force between the charge q and the image charge q'.
- Another way would be to find the force per unit area due to the surface charge density σ, and integrate over the entire surface area. The force per unit area at the surface of a conductor is

$$\frac{d\vec{F}}{da} = \frac{\sigma^2}{2\epsilon_0}\,\hat{n}$$

  where $\hat{n}$ is a unit outward normal to the surface. If you've forgotten, see page 104 in Griffiths for a derivation. I won't go over it for this example, but you'll find the force using both methods for the charge and infinite plane conductor on the homework.

Corollaries to the problem discussed on the previous page are a point charge inside a grounded conducting sphere, a point charge in the presence of a charged conducting sphere, and a point charge near a conducting sphere at fixed potential, all of which you should be able to do based on the grounded sphere we discussed in class; see *Jackson*, pages 60-62 for details.

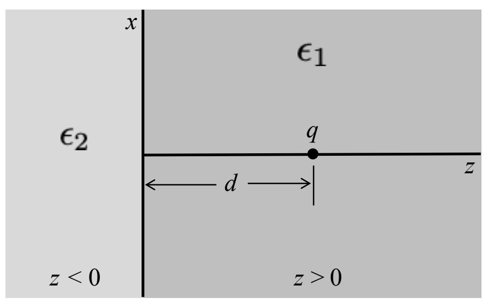
Boundary value problems with dielectrics

So far, we have discussed the image problems only for conductors. The method of images can be extended easily to handle the presence of dielectrics.

Before we begin, it is worth remembering that if problems don't involve free charges, and if $\hat{n}$ is a unit normal to the surface directed from region 1 to region 2, then

- normal $\vec{D}$ is continuous at interface, $(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = \sigma = 0$.
- tangential $\vec{E}$ is continuous at interface, $\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0$.

Consider a point charge q embedded in a semi-infinite dielectric (of permittivity) ϵ_1 at a distance d from a plane interface that separates the first medium from another semi-infinite dielectric ϵ_2, as shown in the figure below (modified from Figure 4.3, *Jackson*, page 155). The interface is taken as the plane $z = 0$.



The problem is to solve $\vec{\nabla} \cdot \vec{D} = \rho$, which takes the following forms in the two semi-infinite regions:

$$\begin{aligned} \epsilon_1 \vec{\nabla} \cdot \vec{E} &= \rho, \qquad z > 0 \\ \epsilon_2 \vec{\nabla} \cdot \vec{E} &= 0, \qquad z < 0 \end{aligned} \tag{4.41.a}$$

where we've assumed linear media to use $\vec{D} = \epsilon \vec{E}$; we'll need also Faraday's law for electrostatics:

$$\vec{\nabla} \times \vec{E} = 0, \qquad \text{everywhere} \tag{4.41.b}$$

From the boundary condition at $z = 0$ that $D_\perp$, the normal component of $\vec{D}$ is continuous (because there is no surface charge σ), we get by applying $(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = \sigma = 0$ that

$$\epsilon_1 E_z\,(z > 0) = \epsilon_2 E_z\,(z < 0) \tag{4.42.a}$$

where we've used $\vec{D} = \epsilon \vec{E}$. Meanwhile, using the boundary condition at $z = 0$ that $E_\parallel$, the tangential component of $\vec{E}$ is continuous, we get by applying $\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0$ that

$$\begin{aligned} E_x\,(z > 0) &= E_x\,(z < 0) \\ E_y\,(z > 0) &= E_y\,(z < 0) \end{aligned} \tag{4.42.b}$$

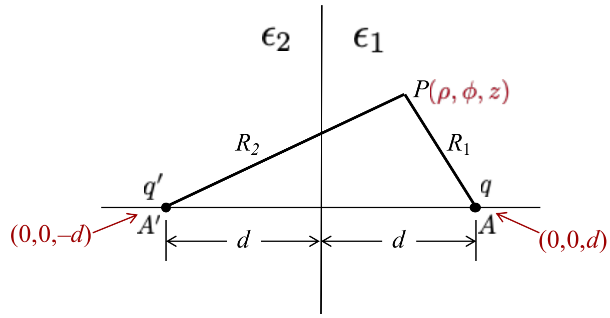
We will now apply the method of images to find the potential Φ.

To use the method of images, consider the figure on the right below.

Since q is located in $z > 0$, we'll put an image charge q' at $z = -d$, that is

- q is at $(0, 0, d)$, and
- q' is at $(0, 0, -d)$.

as shown in the figure.



Then, for $z > 0$ at a point $P(\rho, \phi, z)$ in cylindrical coordinates, the potential will be given by

$$\Phi_1 = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right), \qquad z > 0 \tag{4.43}$$

where $R_1 = \sqrt{(\rho - 0)^2 + (z - d)^2}$, we don't need ϕ since the problem is effectively 2-dimensional,

and $R_2 = \sqrt{(\rho - 0)^2 + [z - (-d)]^2} = \sqrt{\rho^2 + (z + d)^2}$.

So far the problem is completely analogous to the problem with a conducting material in place of the dielectric ϵ_2 for $z < 0$. But now we must specify the potential for $z < 0$. We can't have any charges in $z < 0$, and so the simplest assumption is that the potential in $z < 0$ is equivalent to that with a charge q'' at $z = d$, the position of the actual charge q, and thus

$$\Phi_2 = \frac{1}{4\pi\epsilon_2}\frac{q''}{R_1}, \qquad z < 0 \tag{4.44}$$

Now, evaluate the fields in order to apply boundary conditions.

First, since the normal component of $\vec{D}_1$ at $z = 0$ is given by

$$D_{1\perp}\Big|_{z=0} = \epsilon_1\, E_{1\perp}\Big|_{z=0} = -\epsilon_1\,\frac{\partial\Phi_1}{\partial z}\Big|_{z=0}$$

you showed on Question 2b of today's worksheet that

$$D_{1\perp}\Big|_{z=0} = -\frac{1}{4\pi}\left[\frac{qd}{(\rho^2 + d^2)^{3/2}} - \frac{q'd}{(\rho^2 + d^2)^{3/2}}\right]$$

Meanwhile, the normal component $D_{2\perp}$ of $\vec{D}_2$ at the boundary $z = 0$ is given by

$$D_{2\perp}\Big|_{z=0} = -\epsilon_2\,\frac{\partial\Phi_2}{\partial z}\Big|_{z=0} = -\frac{1}{4\pi}\left[\frac{q''\,d}{(\rho^2 + d^2)^{3/2}}\right]$$

as you showed on Question 3(a) of today's worksheet.

Setting $D_{1\perp}$ at $z = 0$ written above equal to $D_{2\perp}$ at $z = 0$ also written above, we get that

$$q - q' = q''$$

as you showed on Question 3(b) of today's worksheet.

Next, we will write down expressions for the tangential components of $\vec{E}$ to derive a second relation between the charge and its image charges. The tangential components of $\vec{E}$, given by

$$(\vec{E})_\parallel = (\vec{E})_\rho = -\frac{\partial \Phi}{\partial \rho}$$

First, the tangential component $(E_1)_\parallel$ at $z = 0$ is given by

$$(E_1)_\parallel\Big|_{z=0} = -\frac{\partial \Phi_1}{\partial \rho}\Big|_{z=0} = +\frac{1}{4\pi\epsilon_1}\left[\frac{q\rho}{(\rho^2+d^2)^{3/2}} + \frac{q'\rho}{(\rho^2+d^2)^{3/2}}\right]$$

as you showed on Question 4(a) of today's worksheet.

Next, the tangential component $(E_2)_\parallel$ at $z = 0$ is given by

$$(E_2)_\parallel\Big|_{z=0} = -\frac{\partial \Phi_2}{\partial \rho}\Big|_{z=0} = +\frac{1}{4\pi\epsilon_2}\left[\frac{q''\rho}{(\rho^2+d^2)^{3/2}}\right]$$

as you showed on Question 4(b) of today's worksheet.

Setting $(E_1)_\parallel$ equal to $(E_2)_\parallel$ at the boundary $z = 0$, we get that

$$\frac{1}{\epsilon_1}\Big(q + q'\Big) = \frac{1}{\epsilon_2}\, q''$$

Space left blank for student notes; class summary continues on next page.

We now have the system of equations:

$$q - q' = q'' \tag{4.45}$$

$$q + q' = \left(\frac{\epsilon_1}{\epsilon_2}\right) q'' \tag{4.46}$$

To solve for q', subtract the second equation from the first to get

$$-2q' = \left(1 - \frac{\epsilon_1}{\epsilon_2}\right) q'' = \left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2}\right) q''$$

Meanwhile, adding the two equations, we get

$$2q = \left(1 + \frac{\epsilon_1}{\epsilon_2}\right) q'' = \left(\frac{\epsilon_2 + \epsilon_1}{\epsilon_2}\right) q''$$

Thus

$$q'' = \left(\frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\right) q$$

and substituting this in the equation for q', we get

$$\begin{aligned} -2q' &= \left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2}\right) q'' \\ &= \left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2}\right) \left(\frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\right) q \\ \text{so that} \qquad -q' &= \left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\right) q \end{aligned}$$

where, in the last step, I've canceled a factor of 2 on the left hand side with a factor of 2 on the right hand side.

Therefore, we have

$$\begin{aligned} q' &= -\left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\right) q \\ q'' &= \quad \left(\frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\right) q \end{aligned} \tag{4.45}$$

If $\epsilon_1 < \epsilon_2$, charge q is attracted to the $z = 0$ boundary (of ϵ_1 and ϵ_2) because q' is negative.

If $\epsilon_1 > \epsilon_2$, charge q is repelled by the $z = 0$ boundary (of ϵ_1 and ϵ_2) because q' is positive.