Physics 440, Autumn 2019 Activity 15: Hamiltonians and Canonical Equations

1. The Lagrangian of a one-dimensional harmonic oscillator is $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$
   a. Derive the equation of motion using the Lagrangian formalism and find the solution.
   b. Derive the Hamiltonian.
   c. Derive the equation of motion using Hamilton's canonical equations and show that it is identical to the equation of motion you derived in a.
2. For the spherical pendulum the Lagrangian is

$$L = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) - mgl\cos\theta$$

The equations of motion derived with the Lagrangian formalism are (see p. 24)

$$ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\sin\theta = 0$$

$$\frac{d}{dt}\left(ml^2\sin^2\theta\,\dot{\phi}\right) = 0$$

   a. Derive the Hamiltonian.
   b. Derive the equations of motion using the Hamiltonian formalism and show that they are identical to the ones given above.

1. a. $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \Rightarrow m\ddot{x} + kx = 0$ solution: $x(t) = A\cos\left(\sqrt{\frac{k}{m}}t - \phi_0\right) + const$

   b. $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$$H(q_i, p_i, t) = \sum_{i=1}^{n} p_i\dot{q}_i - L(q_i, \dot{q}_i t)$$

$$H(x, p_x) = p_x\dot{x} - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$H(x, p_x) = \frac{p_x^2}{2m} + \frac{1}{2}kx^2$$

   c.

$$\frac{\partial H}{\partial x} = -\dot{p}_x \qquad \frac{\partial H}{\partial p_x} = \dot{x}$$

$$\dot{p}_x = -kx \qquad \dot{x} = \frac{p_x}{m}$$

Take a second time derivative of the second equation and substitute $\dot{p}_x$ in the first equation:

$$\ddot{x} = \frac{\dot{p}_x}{m} \Rightarrow m\ddot{x} + kx = 0$$

2. a. $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \quad p_\theta^2 = m^2l^4\dot{\theta}^2 \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2\sin^2\theta\,\dot{\phi} = \alpha_\phi \quad p_\phi^2 = m^2l^4\sin^4\theta\,\dot{\phi}^2$

$$H(p_\theta, p_\phi, \theta, \phi) = ml^2\dot{\theta}^2 + ml^2\sin^2\theta\,\dot{\phi}^2 - \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgl\cos\theta$$

$$= \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2\right) + mgl\cos\theta$$

(Remember that the Hamiltonian has to be expressed as a fuction of q_i and p_i, NOT as a function of q_i and $\dot{q}_i$)

$$H\left(p_\theta, p_\phi, \theta, \phi\right) = \frac{1}{2ml^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + mgl\cos\theta$$

Four canonical equations:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}\ (i) \quad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{ml^2\sin^2\theta}\ \ (ii)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2\cos\theta}{ml^2\sin^3\theta} + mgl\cos\theta = \frac{m^2 l^4 \sin^4\theta\, \dot{\phi}^2 \cos\theta}{ml^2\sin^3\theta} + mgl\cos\theta$$
$$= ml^2\dot{\phi}^2\sin\theta\cos\theta + mgl\cos\theta \quad (iii)$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0\ (iv)$$

Take a second time derivative of (i) and plug it into (iii) to get the equation of motion for θ:

$$ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\cos\theta = 0$$

Plug the expression for the conjugate momentum p_ϕ into (iv) to the equation of motion for ϕ:

$$\frac{d}{dt}\left(ml^2\sin^2\theta\,\dot{\phi}\right) = 0$$