A certain quantum mechanical operator A has eigenvalues a_1, a_2, and a_3, with corresponding eigenstates

$$|a_1\rangle \leftrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \mathrm{i}\\ 0\end{bmatrix}, \quad |a_2\rangle \leftrightarrow \frac{1}{\sqrt{3}}\begin{bmatrix}\mathrm{i}\\ 1\\ 1\end{bmatrix}, \quad \text{and} \quad |a_3\rangle \leftrightarrow \frac{1}{\sqrt{6}}\begin{bmatrix}\mathrm{i}\\ 1\\ -2\end{bmatrix}.$$

(1) Find the representations of the projection operators that correspond to measurements of a_1, a_2, and a_3.

$$P_{a_1} = |a_1\rangle\langle a_1| \longleftrightarrow \frac{1}{2}\begin{bmatrix}1\\ i\\ 0\end{bmatrix}\begin{bmatrix}1 & -i & 0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -i & 0\\ i & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$P_{a_2} = |a_2\rangle\langle a_2| \longleftrightarrow \frac{1}{3}\begin{bmatrix}i\\ 1\\ 1\end{bmatrix}\begin{bmatrix}-i & 1 & 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & i & i\\ -i & 1 & 1\\ -i & 1 & 1\end{bmatrix}$$

$$P_{a_3} = |a_3\rangle\langle a_3| \longleftrightarrow \frac{1}{6}\begin{bmatrix}i\\ 1\\ -2\end{bmatrix}\begin{bmatrix}-i & 1 & -2\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1 & i & -2i\\ -i & 1 & -2\\ 2i & -2 & 4\end{bmatrix}$$

$$P_{a_1} + P_{a_2} + P_{a_3} \longleftrightarrow \frac{1}{6}\begin{bmatrix}3 & -3i & 0\\ 3i & 3 & 0\\ 0 & 0 & 0\end{bmatrix} + \frac{1}{6}\begin{bmatrix}2 & 2i & 2i\\ -2i & 2 & 2\\ -2i & 2 & 2\end{bmatrix} + \frac{1}{6}\begin{bmatrix}1 & i & -2i\\ -i & 1 & -2\\ 2i & -2 & 4\end{bmatrix}$$

(2) Verify that your projection operators sum to the identity matrix.

$$= \frac{1}{6}\begin{bmatrix}3+2+1 & -3i+2i+1 & 2i-2i\\ 3i-2i+1 & 3+2+1 & 2-2\\ -2i+2i & 2-2 & 2+4\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

(3) If the state of the system is

$$|\Psi\rangle \leftrightarrow \frac{1}{2}\begin{bmatrix} \mathrm{i} \\ 1 \\ 1-\mathrm{i} \end{bmatrix},$$

use the appropriate projection operator to find

(a) the probability of obtaining each of the three possible values a_1, a_2, or a_3 if you measure A.

(b) the state of the system after the measurement.

$$\mathcal{P}(a_1) = \langle \psi | P_{a_1} | \psi \rangle$$

$$= \left(\frac{1}{2}\begin{bmatrix} -i & 1 & 1+i \end{bmatrix}\right)\left(\frac{1}{2}\begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)\left(\frac{1}{2}\begin{bmatrix} i \\ 1 \\ 1-i \end{bmatrix}\right)$$

$$= \left(\frac{1}{2}\begin{bmatrix} -i & 1 & 1+i \end{bmatrix}\right)\left(\frac{1}{4}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = 0$$

$$\mathcal{P}(a_2) = \langle \psi | P_{a_2} | \psi \rangle$$

$$= \left(\frac{1}{2}\begin{bmatrix} -i & 1 & 1+i \end{bmatrix}\right)\left(\frac{1}{3}\begin{bmatrix} 1 & i & i \\ -i & 1 & 1 \\ -i & 1 & 1 \end{bmatrix}\right)\left(\frac{1}{2}\begin{bmatrix} i \\ 1 \\ 1-i \end{bmatrix}\right)$$

$$= \left(\frac{1}{2}\begin{bmatrix} -i & 1 & 1+i \end{bmatrix}\right)\left(\frac{1}{6}\begin{bmatrix} 3i+1 \\ 3-i \\ 3-i \end{bmatrix}\right) = \frac{1}{12}\left(3-i+3-i+4+2i\right)$$

$$= \frac{5}{6}$$

$$P(a_3) = \langle \psi | P_{a_3} | \psi \rangle$$

$$= \left(\frac{1}{2} \begin{bmatrix} -i & 1 & 1+i \end{bmatrix} \right) \left(\frac{1}{6} \begin{bmatrix} 1 & i & -2i \\ -i & 1 & -2 \\ 2i & -2 & 4 \end{bmatrix} \right) \left(\frac{1}{2} \begin{bmatrix} i \\ 1 \\ 1-i \end{bmatrix} \right)$$

$$= \left(\frac{1}{2} \begin{bmatrix} -i & 1 & 1+i \end{bmatrix} \right) \left(\frac{1}{12} \begin{bmatrix} -2 \\ 2i \\ -4i \end{bmatrix} \right) = \frac{1}{24} \left(+2i + 2i - 4i + 4 \right)$$

$$= \frac{1}{6}$$