Physics 412—Quantum Mice Basics Winter 2020

Consider a quantum lab mouse and some of its properties. One property might be the size of the mouse, which we could measure by putting the mouse on a quantum scale. Since the mouse is quantized, its weight can take on only one of two values, either $w = 2$ (small mouse), or $w = 10$ (large mouse). If a quantum mouse is small, it's in the state $|s\rangle$ (labeled s for small), while if it's large, it's in the state $|l\rangle$ (labeled l for large). These are eigenstates of the weight operator W, with eigenvalue equations

$$W|s\rangle = 2|s\rangle \quad \text{and} \quad W|l\rangle = 10|l\rangle.$$

Being either large or small is normal for a quantum mouse, so the states $|s\rangle$ and $|l\rangle$ are assumed to be normalized. Because the operator W corresponds to a measurable quantity, it is Hermitian.

A second property of the quantum mice is their attitude, which can be measured by looking at a mouse's expression, yielding either a smile (attitude $= +1$), or a frown (attitude $= -1$). We'll call the corresponding quantum states $|h\rangle$ (for happy) and $|u\rangle$ (for unhappy). These are eigenstates of the attitude operator A, with eigenvalue equations

$$A|h\rangle = +|h\rangle \quad \text{and} \quad A|u\rangle = -|u\rangle.$$

As with the size states, the attitude states are normalized, and the operator A is Hermitian.

(1) What can you say about the inner products $\langle s|s\rangle$, $\langle l|l\rangle$, $\langle s|l\rangle$, $\langle h|h\rangle$, $\langle u|u\rangle$, and $\langle h|u\rangle$? If you think some of these inner products are zero, prove it! (See page 16 of the course notes, starting with Eq. (2.17).)

$\langle s|s\rangle = 1 \quad \langle l|l\rangle = 1 \quad \langle h|h\rangle = 1 \quad \langle u|u\rangle = 1$

Following (2.17) from the course notes,

$$\langle l|W|s\rangle = \langle s|W|l\rangle^*$$

$$\langle l|2|s\rangle = \langle s|10|l\rangle^*$$

$$2\langle l|s\rangle = 10\langle s|l\rangle^* = 10\langle l|s\rangle$$

$$\Rightarrow \langle l|s\rangle = 0$$

(2) There is a relationship between size and attitude for the quantum mice. Suppose that

$$|s\rangle = \frac{1}{\sqrt{5}}|h\rangle + \frac{2}{\sqrt{5}}|u\rangle.$$

I guess that means that small quantum mice are more than a little bit stressed! (Do you see that from the equation?)

(a) Expand the "large" size state in the "attitude basis": $|l\rangle = a|h\rangle + b|u\rangle$. Find the constants a and b. (Use your results from Question 1.)

(b) Represent all four states $|s\rangle$, $|l\rangle$, $|h\rangle$, and $|u\rangle$ as column vectors in the attitude basis.

(c) Find the representations of the operators W and A in the attitude basis $\{|h\rangle, |u\rangle\}$. Express your answers as 2×2 matrices. Verify that the representations of W and A satisfy the condition for a Hermitian operator.

(d) Invert the equations relating the size states to the attitude states to represent $|h\rangle$ and $|u\rangle$ as column vectors in the size basis.

(e) Find the representations of the operators W and A in the size basis $\{|s\rangle, |l\rangle\}$. Express your answers as 2×2 matrices. Verify that the representations of W and A satisfy the condition for a Hermitian operator.

(f) Are the size states $|s\rangle$ and $|l\rangle$ eigenstates of the attitude operator A? Explain.

(g) Are the attitude states $|h\rangle$ and $|u\rangle$ eigenstates of the size operator W? Explain.

a) $|l\rangle = a|h\rangle + b|u\rangle \qquad s = \frac{1}{\sqrt{5}}|h\rangle + \frac{2}{\sqrt{5}}|u\rangle$

$$\langle s|l\rangle = \left[\frac{1}{\sqrt{5}}\langle h| + \frac{2}{\sqrt{5}}\langle u|\right]\left[a|h\rangle + b|u\rangle\right]$$

$$= \frac{a}{\sqrt{5}} + \frac{2b}{\sqrt{5}} = 0 \Rightarrow a = -2b$$

$$|l\rangle = -2b|h\rangle + b|u\rangle$$

$$\langle l|l\rangle = \left[a^*\langle h| + b^*\langle u|\right]\left[a|h\rangle + b|u\rangle\right]$$

$$= |a|^2 + |b|^2 = 1$$

$$= |2b|^2 + |b|^2 = 1 = 5|b|^2$$

$$\Rightarrow b = \frac{1}{\sqrt{5}}, \quad a = -\frac{2}{\sqrt{5}}$$

$$|l\rangle = -\frac{2}{\sqrt{5}}|h\rangle + \frac{1}{\sqrt{5}}|u\rangle$$

(b)

$$\left.\begin{array}{ll} |h\rangle \longleftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} & |u\rangle \longleftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ |s\rangle \longleftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} & |l\rangle \longleftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} \end{array}\right\} \text{Attitude basis}$$

(d)

$$2|s\rangle + |l\rangle = 2\left[\frac{1}{\sqrt{5}}|h\rangle + \frac{2}{\sqrt{5}}|u\rangle\right] + \frac{-2}{\sqrt{5}}|h\rangle + \frac{1}{\sqrt{5}}|u\rangle$$

$$= \frac{5}{\sqrt{5}}|u\rangle \Rightarrow \boxed{|u\rangle = \frac{2}{\sqrt{5}}|s\rangle + \frac{1}{\sqrt{5}}|l\rangle}$$

$$|s\rangle - 2|l\rangle = \left[\frac{1}{\sqrt{5}}|h\rangle + \frac{2}{\sqrt{5}}|u\rangle\right] - 2\left[\frac{-2}{\sqrt{5}}|h\rangle + \frac{1}{\sqrt{5}}|u\rangle\right]$$

$$= \frac{5}{\sqrt{5}}|h\rangle \Rightarrow \boxed{|h\rangle = \frac{1}{\sqrt{5}}|s\rangle - \frac{2}{\sqrt{5}}|l\rangle}$$

$$\left.\begin{array}{ll} |s\rangle \longleftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} & |l\rangle \longleftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ |h\rangle \longleftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix} & |u\rangle \longleftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{array}\right\} \text{size basis}$$

(c) Attitude basis:

$$A \longleftrightarrow \begin{bmatrix} \langle h|A|h\rangle & \langle h|A|u\rangle \\ \langle u|A|h\rangle & \langle u|A|u\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle h|1|h\rangle & \langle h|-1|u\rangle \\ \langle u|1|h\rangle & \langle u|-1|u\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$W \longleftrightarrow \begin{bmatrix} \langle h|W|h\rangle & \langle h|W|u\rangle \\ \langle u|W|h\rangle & \langle u|W|u\rangle \end{bmatrix}$$

$$W|h\rangle = W\left[\frac{1}{\sqrt5}|s\rangle - \frac{2}{\sqrt5}|e\rangle\right] = \frac{2}{\sqrt5}|s\rangle - \frac{20}{\sqrt5}|e\rangle$$

$$\langle h|W|h\rangle = \underbrace{\left[\frac{1}{\sqrt5}\langle s| - \frac{2}{\sqrt5}\langle e|\right]\left[\frac{2}{\sqrt5}|s\rangle - \frac{20}{\sqrt5}|e\rangle\right]}_{\begin{bmatrix} \frac{1}{\sqrt5} & -\frac{2}{\sqrt5}\end{bmatrix}\cdot\begin{bmatrix} 2/\sqrt5 \\ -\frac{20}{\sqrt5}\end{bmatrix} = \frac{2}{5} + \frac{40}{5} = \frac{42}{5}}$$

$$= \frac{2}{5} + \frac{40}{5} = \frac{42}{5}$$

$$\langle u|W|h\rangle = \begin{bmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5}\end{bmatrix}\begin{bmatrix} 2/\sqrt5 \\ -20/\sqrt5\end{bmatrix}$$

$$= \frac{4}{5} - \frac{20}{5} = -\frac{16}{5}$$

$$\langle h|W|u\rangle = \langle u|W|h\rangle^* = -\frac{16}{5}$$

$$W|u\rangle = W\left[\frac{2}{\sqrt5}|s\rangle + \frac{1}{\sqrt5}|e\rangle\right] = \frac{4}{\sqrt5}|s\rangle + \frac{10}{\sqrt5}|e\rangle$$

$$\langle u|W|u\rangle = \begin{bmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5}\end{bmatrix}\begin{bmatrix} \frac{4}{\sqrt5} \\ 10/\sqrt5\end{bmatrix} = \frac{8}{5} + \frac{10}{5} = \frac{18}{5}$$

$$W \longleftrightarrow \frac{1}{5}\begin{bmatrix} 42 & -16 \\ -16 & 18 \end{bmatrix}$$

(e) Size basis:

$$W \longleftrightarrow \begin{bmatrix} \langle s|W|s\rangle & \langle s|W|\ell\rangle \\ \langle \ell|W|s\rangle & \langle \ell|W|\ell\rangle \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix}$$

$$A \longleftrightarrow \begin{bmatrix} \langle s|A|s\rangle & \langle s|A|\ell\rangle \\ \langle \ell|A|s\rangle & \langle \ell|A|\ell\rangle \end{bmatrix}$$

$$A|s\rangle = A\left[\frac{1}{\sqrt5}|h\rangle + \frac{2}{\sqrt5}|u\rangle\right] = \frac{1}{\sqrt5}|h\rangle - \frac{2}{\sqrt5}|u\rangle$$

$$\langle s|A|s\rangle = \begin{bmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{bmatrix}\begin{bmatrix} 1/\sqrt5 \\ -2/\sqrt5 \end{bmatrix} = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}$$

$$\langle \ell|A|s\rangle = \begin{bmatrix} -\frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{bmatrix}\begin{bmatrix} 1/\sqrt5 \\ -2/\sqrt5 \end{bmatrix} = -\frac{2}{5} - \frac{2}{5} = -\frac{4}{5}$$

$$\langle s|A|\ell\rangle = \langle \ell|A|s\rangle^* = -4/5$$

$$A|\ell\rangle = A\left[-\frac{2}{\sqrt5}|h\rangle + \frac{1}{\sqrt5}|u\rangle\right] = -\frac{2}{\sqrt5}|h\rangle - \frac{1}{\sqrt5}|u\rangle$$

$$\langle \ell|A|\ell\rangle = \begin{bmatrix} -\frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{bmatrix}\begin{bmatrix} -2/\sqrt5 \\ -1/\sqrt5 \end{bmatrix} = \frac{4}{5} - \frac{1}{5} = \frac{3}{5}$$

$$A \longleftrightarrow \begin{bmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{bmatrix}$$

(f) $|e\rangle = a|h\rangle + b|u\rangle$

For $|e\rangle$ to be an eigenvector of A

$$A|e\rangle = \lambda|e\rangle$$

$$= aA|h\rangle + bA|u\rangle$$

$$= +a|h\rangle - b|u\rangle \neq \lambda\left[a|h\rangle + b|u\rangle\right]$$

Similar argument for $|s\rangle$

(g) $|h\rangle = \alpha|s\rangle + \beta|e\rangle$

for $|h\rangle$ to be an eigenvector of W

$$W|h\rangle = \lambda|h\rangle$$

$$= \alpha W|s\rangle + \beta W|e\rangle$$

$$= 2\alpha|s\rangle + 10\beta|e\rangle$$

$$\neq \lambda\left[\alpha|s\rangle + \beta|e\rangle\right]$$