PHY 411 Winter 2021

Homework 8 solutions

1. A point charge q is brought to a position a distance d away from an infinite plane conductor held at zero potential. Use the method of images to find the following.

(a) Find the surface charge density induced on the plane.

Solution: Let's take the z-axis on the plane of the page going left to right as usual in such problems and put the conducting plane at $z = 0$, so that charge q is at $(0, 0, d)$ and charge $-q$ is at $(0, 0, -d)$. Then the potential Φ at an general point (x, y, z) is given by

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-0)^2+(y-0)^2+(z-d)^2}} + \frac{-q}{\sqrt{(x-0)^2+(y-0)^2+(z-\{-d\})^2}}\right]$$

or

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

The field and surface charge density σ on a conductor are given by

$$\vec{E} = \frac{\sigma}{\epsilon_0}\,\hat{n} \qquad \text{and} \qquad \sigma = -\epsilon_0\,\frac{\partial\Phi}{\partial n}$$

respectively, where $\hat{n}$ is a unit outward normal to the conductor. If you've forgotten this, see pages 89 and 103 in Griffiths.

We will need the surface charge density on the infinite conducting plane which is at $z = 0$, and $\hat{n}$ in this problem is along the direction of positive z, so

$$\sigma = -\epsilon_0\,\frac{\partial\Phi}{\partial z}\bigg|_{z=0}$$

Remembering that the derivative of $1/\sqrt{X} = -\frac{1}{2}\,X^{-\frac{1}{2}-1} = -1/2X^{3/2}$, and moving this factor of $-\frac{1}{2}$ outside of the square brackets, we get

$$\sigma = -\epsilon_0\,\frac{\partial\Phi}{\partial z}\bigg|_{z=0} = -\frac{\epsilon_0}{4\pi\epsilon_0}\left(-\frac{1}{2}\right)\left[\frac{q\,2(z-d)\,(1-0)}{\{x^2+y^2+(z-d)^2\}^{3/2}} + \frac{(-)q\,2(z+d)\,(1-0)}{\{x^2+y^2+(z+d)^2\}^{3/2}}\right]_{z=0}$$

After putting $z = 0$, we get

$$\sigma = \frac{1}{4\pi}\left[\frac{q\,(0-d)}{(x^2+y^2+d^2)^{3/2}} + \frac{(-q)\,(0+d)}{(x^2+y^2+d^2)^{3/2}}\right]$$

so that

$$\sigma = \frac{1}{4\pi}\left[\frac{-qd}{(x^2+y^2+d^2)^{3/2}} + \frac{-qd}{(x^2+y^2+d^2)^{3/2}}\right]$$

or

$$\sigma = \frac{1}{4\pi}\left[\frac{-2qd}{(x^2+y^2+d^2)^{3/2}}\right]$$

Therefore, the surface charge density induced on the plane is

$$\boxed{\sigma = \frac{-qd}{2\pi\,(x^2+y^2+d^2)^{3/2}}}$$

(b) Find the force between the plane and the charge by using Coulomb's law for the force between the charge and its image.

Solution: This is just the force between the charge and its image found using Coulomb's law. Thus, since the distance between q and its image charge $-q$ is $2d$, we get

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\,\frac{q(-q)}{(2d)^2}\,\hat{z} = \boxed{-\frac{q^2}{16\pi\epsilon_0 d^2}\,\hat{z}}$$

Notice that I've written the full vector form of the force. The charge q is positive, and it induces negative surface charge density on the plane so the force of the plane on the charge will be toward the plane, thus in the $-\hat{z}$ direction. Or, you could ask, what if we have a negative q; then, the surface charge density induced on the plane will be positive; therefore, the force will still be in the negative z-direction.

(c) Find the total force acting on the plane by integrating $\sigma^2/2\epsilon_0$ over the whole plane.

Solution: The force obtained by integrating $\sigma^2/2\epsilon_0$ over the whole plane is given by

$$\vec{F} = \hat{z}\int \frac{\sigma^2}{2\epsilon_0}\,dxdy$$

Writing σ from part (a), this becomes

$$\vec{F} = \hat{z}\int \frac{1}{2\epsilon_0}\left[\frac{-qd}{2\pi\,(x^2+y^2+d^2)^{3/2}}\right]^2 dxdy$$

so that

$$\vec{F} = \hat{z}\,\frac{q^2d^2}{4\pi^2\,(2\epsilon_0)}\int\left[\frac{1}{(x^2+y^2+d^2)^3}\right]dxdy$$

This integral is easier to do if we convert to cylindrical coordinates (r,ϕ,z), where $x^2+y^2=\rho^2$; meanwhile, the volume element in cylindrical coordinates is $\rho d\rho\, d\phi\, dz$, but we are integrating only over the xy-plane, so here we will replace $dxdy$ with $\rho d\rho\, d\phi$. Thus

$$\vec{F} = \hat{z}\,\frac{q^2d^2}{4\pi^2\,(2\epsilon_0)}\int\frac{\rho d\rho\, d\phi}{(\rho^2+d^2)^3}$$

The ϕ-integral just gives 2π, so we get

$$\vec{F} = \hat{z}\,\frac{(2\pi)\,q^2d^2}{4\pi^2\,(2\epsilon_0)}\int_0^\infty\frac{\rho d\rho}{(\rho^2+d^2)^3}$$

You can look up the integral in a table, or its easy to just put $\rho^2+d^2=R$, so that $\rho d\rho = dR/2$, and

$$\vec{F} = \hat{z}\,\frac{q^2d^2}{2\pi\,(2\epsilon_0)}\left(\frac{1}{2}\right)\int_{\rho=0}^{\infty}\frac{dR}{R^3} = \hat{z}\,\frac{q^2d^2}{4\pi\,\epsilon_0}\left(\frac{1}{2}\right)\left[-\frac{1}{2R^2}\right]_{\rho=0}^{\infty}$$

Putting back $R=\rho^2+d^2$, this becomes

$$\vec{F} = -\hat{z}\,\frac{q^2d^2}{4\pi\,\epsilon_0}\left(\frac{1}{4}\right)\left[\frac{1}{(\rho^2+d^2)^2}\right]_{\rho=0}^{\infty} = -\hat{z}\,\frac{q^2d^2}{4\pi\,\epsilon_0}\left(\frac{1}{4}\right)\left[0-\frac{1}{(0^2+d^2)^2}\right]$$

so that

$$\vec{F} = \boxed{\frac{q^2}{16\pi\epsilon_0 d^2}\,\hat{z}}$$

In magnitude, this matches the answer in part (b), as it should. It has positive sign because it is the force acting on the plane due to the charge q, and hence will be along the positive direction of the z-axis.

2. The surface of a hollow conducting sphere centered at the origin is divided into octants *numbered* as follows:

	$0 < \theta < \pi/2$	$\pi/2 < \theta < \pi$
$0 < \phi < \pi/2$	1	5
$\pi/2 < \phi < \pi$	2	6
$\pi < \phi < 3\pi/2$	3	7
$3\pi/2 < \phi < \pi$	4	8

Octants 1, 3, 6, and 8 are kept at potential V and octants 2, 4, 5, and 7 at potential $-V$.

(a) Explain why (or show that) the potential outside the sphere can be written as

$$\Phi(r,\theta,\phi) = \sum_l \sum_m \left[C_{lm}\, r^{-l-1}\, P_l^m(\cos\,\theta)\,\cos\, m\phi + D_{lm}\, r^{-l-1}\, P_l^m(\cos\,\theta)\,\sin\, m\phi\right]$$

Solution: The general solution for the potential in spherical coordinates is written in equation (3.61) in Jackson:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm}\, r^l + \frac{B_{lm}}{r^{l+1}}\right] Y_{lm}(\theta,\phi) \tag{7.1}$$

Since we want the potential outside the sphere, and we need Φ not to blow up as r goes to infinity, we must demand that all $A_{lm} = 0$. Moving the r^{l+1} term to the numerator, equation (7.1) becomes

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{lm} r^{-(l+1)}\, Y_{lm}(\theta,\phi) \tag{7.2}$$

Next, from equation (3.53) in Jackson, we can write $Y_{lm}(\theta,\phi) = B'_{lm} P_l^m(\cos\,\theta)\, e^{im\phi}$, where I've written the terms under the square root in equation (3.53) as B'_{lm}. Putting this in equation (7.2), we get

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{lm} r^{-(l+1)}\, B'_{lm} P_l^m(\cos\,\theta)\, e^{im\phi} \tag{7.3}$$

If we expand $e^{im\phi} = \cos\, m\phi + i \sin\, m\phi$, then equation (7.3) becomes

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[B_{lm}\, B'_{lm}\, r^{-(l+1)}\, P_l^m(\cos\,\theta)\,\cos\, m\phi + i\, B_{lm}\, B'_{lm}\, r^{-(l+1)}\, P_l^m(\cos\,\theta)\,\sin\, m\phi\right]$$

Therefore, if we set $B_{lm}\, B'_{lm} = C_{lm}$ and $iB_{lm}\, B'_{lm} = D_{lm}$, and write out the $r^{-(l+1)}$ term as r^{-1-1}, the above expression becomes the desired expression:

$$\Phi(r,\theta,\phi) = \sum_l \sum_m \left[C_{lm}\, r^{-l-1}\, P_l^m(\cos\,\theta)\,\cos\, m\phi + D_{lm}\, r^{-l-1}\, P_l^m(\cos\,\theta)\,\sin\, m\phi\right] \tag{7.4}$$

(b) Determine which coefficients C_{lm} and D_{lm} are zero due to the symmetry of the potential. List all (l, m) combinations for $l \leq 12$ that are non-zero based on such symmetry considerations.

Solution: The potential on the surface of the sphere, $\Phi(a, \theta, \phi)$ has the following symmetry properties.

(1) $\Phi(a, \theta, \phi)$ is an odd function in $\pm\phi$, i.e., $\Phi(a, \theta, \phi) = -\Phi(a, \theta, -\phi)$, because if you go clockwise around by the same azimuthal angle as anticlockwise, you get the opposite sign for the potential.

(2) $\Phi(a, \theta, \phi) = -\Phi(a, \theta, \frac{\pi}{2} + \phi)$, because each alternate quadrant in the azimuthal direction has the opposite sign for the potential.

(3) $\Phi(a, \theta, \phi)$ is odd in the function involving θ, because the upper and lower hemispheres in each quadrant have the opposite signs for the potential.

Since $\cos(\ldots)$ is an even function, property (1) above requires that all coefficients $C_{lm} = 0$. So the updated expression for the potential is

$$\Phi(r, \theta, \phi) = \sum_l \sum_m \left[D_{lm}\, r^{-l-1}\, P_l^m(\cos\theta)\, \sin m\phi \right] \tag{7.5}$$

Property (2) above implies that all $\sin m\phi$ in equation (7.5) have to satisfy

$$\sin m\phi = -\sin m \left(\frac{\pi}{2} + \phi\right) \tag{7.6}$$

Let's see what constraint this places on m.

- If $m = 0$, equation (7.6) is ***not*** valid, since we would not get the change in sign.
- If $m = 1$, equation (7.6) is ***not*** valid, since $\sin(\pi/2 + \phi) = +\cos\phi \neq -\sin\phi$.
- If $m = 2$, equation (7.6) ***is valid***, since $\sin 2(\pi/2 + \phi) = \sin(\pi + 2\phi) = -\sin 2\phi$.
- I'll let you work out the next few, and you'll see that next higher m value for which equation (7.6) is valid is $m = 6$, since $\sin 6(\pi/2 + \phi) = \sin(3\pi + 6\phi) = -\sin 6\phi$

So, property (2) above implies that

$$m = 2(2n + 1) \text{ where } n = 0, 1, 2, \ldots$$

Finally, property (3) implies that all functions $P_l^m(\cos\theta)$ in the expansion have to be odd in $\cos\theta$. Since $P_l^m(\cos\theta)$ equals $(\sin\theta)^m$ times a polynomial in $(\cos\theta)^{l-m}, (\cos\theta)^{l-m-2}, \ldots$ (e.g., see page 109 in Jackson), this means that $(l - m)$ needs to be odd. Since, as we found above, m needs to be even (e.g., $2, 6, \ldots$), this means that l needs to be odd (starting from $l = 3$, since $l = 1$ would need $m = 0$, which isn't allowed as noted in the first bulleted point above).

In summary, the following must vanish: all C_{lm}; all D_{lm} with even l; all D_{lm} with odd l but $m \neq 2(2n + 1)$. Therefore, the only D_{lm} that survive are for l odd **and** $m = 2(2n + 1)$.

The first few (l, m) combinations for $l \leq 12$ not excluded by symmetry are:

- $l = 3, m = 2$
- $l = 5, m = 2$
- $l = 7, m = 2, 6$
- $l = 9, m = 2, 6$
- $l = 11, m = 2, 6, 10$