PHY 411 **Winter 2021**

Week 4—Thursday, Jan 28—Discussion Worksheet

Normal and Anomalous Dispersion

In the previous class, we derived an expression for the dielectric constant and showed that it can be written with the real and imaginary parts separately

$$\chi(\omega) = \frac{\epsilon(\omega)}{\epsilon_0} - 1 = \frac{Ne^2}{\epsilon_0 m}\sum_j f_j \left\{ \frac{(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2} + i\left[\frac{\omega\gamma_j}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2}\right]\right\} \tag{7.51.a}$$

which we will be referring to as Re $\chi(\omega)$ and Im $\chi(\omega)$ respectively below. *I've rewritten the expression in terms of the susceptibility χ because that makes it easier to state the questions in the form written below.*

1. To keep matters simple, suppose we have only one type of oscillator with frequency ω_1, so that we get

$$\chi(\omega) = \frac{Ne^2 f_1}{\epsilon_0 m}\left\{ \frac{(\omega_1^2 - \omega^2)}{(\omega_1^2 - \omega^2)^2 + \omega^2\gamma_1^2} + i\left[\frac{\omega\gamma_1}{(\omega_1^2 - \omega^2)^2 + \omega^2\gamma_1^2}\right]\right\} \tag{1}$$

Now imagine scanning across in frequency ω, starting from $\omega = 0$.

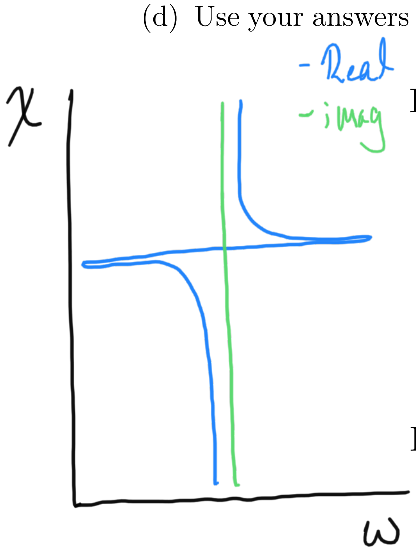
b a

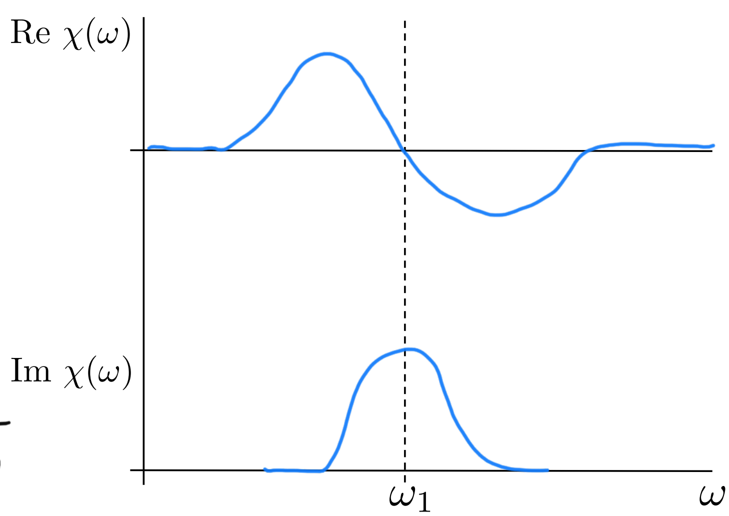
(a) For $\omega < \omega_1$, the sign of Re $\chi(\omega)$ is Positive.

(b) For $\omega > \omega_1$, the sign of Re $\chi(\omega)$ is Negative.

(c) For $\omega = \omega_1$, Re $\chi(\omega)$ is Zero.

(d) Use your answers above to predict the shape of the graph of Re $\chi(\omega)$ and Im $\chi(\omega)$ plotted *vs.* ω.





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Attenuation of a plane wave

One of the consequences of having a complex dielectric constant is that a traveling wave gets attenuated. We will now study this in greater detail.

First, though, let us ask: what is the most appropriate quantity to pick in order to describe the attenuation of a plane wave? In equation (7.51.a), we expressed the dielectric constant $\epsilon(\omega)/\epsilon_0$ as a complex number with real and imaginary parts.

But consider equation (7.5) that we wrote in a previous class: $v = \dfrac{\omega}{k} = \dfrac{c}{n}, \qquad n = \sqrt{\dfrac{\mu\epsilon}{\mu_0\epsilon_0}}$

Clearly, a complex $\epsilon(\omega)$ implies a complex index of refraction $n(\omega)$, which in turn implies a complex wave number k. Now, the wave number k features in the phase factor of a plane wave (as the magnitude of the wave vector $\vec{k}$), so it is definitely useful in describing the attenuation of a plane wave, as we shall see below.

Let us, therefore, express the attenuation of a plane wave by writing the wave number k in terms of its real and imaginary parts:

$$k = \beta + i\frac{\alpha}{2} \tag{7.53}$$

We'll find out later that we divide the imaginary part by 2 to ensure that the intensity ($|\vec{E}|^2$) falls off as $e^{-\alpha z}$, and α can be designated as the *attenuation constant* or *absorption coefficient.*

2. Starting from $k = \omega\sqrt{\mu\epsilon}$, show that for nonpermeable media ($\mu = \mu_0$)

$$\beta^2 - \frac{\alpha^2}{4} = \frac{\omega^2}{c^2}\,\mathrm{Re}\left(\frac{\epsilon}{\epsilon_0}\right) \qquad \text{and} \qquad \beta\alpha = \frac{\omega^2}{c^2}\,\mathrm{Im}\left(\frac{\epsilon}{\epsilon_0}\right) \tag{7.54}$$

$k = \dfrac{\omega}{v} = \dfrac{\omega}{c} n$, $n = \sqrt{\dfrac{\mu\epsilon}{\mu_0\epsilon_0}}$

$\rightarrow \dfrac{\omega}{c} n = \sqrt{\mu\epsilon}$

$n = \dfrac{c}{v} = \dfrac{\sqrt{\mu\epsilon}}{\sqrt{\mu_0\epsilon_0}}$

$\mu = \mu_0 \rightarrow n = \sqrt{\dfrac{\epsilon}{\epsilon_0}}$

$\rightarrow k^2 = \dfrac{\omega^2}{c^2}\dfrac{\epsilon}{\epsilon_0}$

ϵ/ϵ_0 has Real and Imag parts

Real k

$\beta^2 - \dfrac{\alpha^2}{4} = \dfrac{\omega^2}{c^2} \mathrm{Re}\left(\dfrac{\epsilon}{\epsilon_0}\right)$

Imag k

$\beta\alpha = \dfrac{\omega^2}{c^2} \mathrm{Im}\left(\dfrac{\epsilon}{\epsilon_0}\right)$

$\mathrm{Re}(k)$ $\mathrm{Im}(k)$

3. On the previous page, we wrote $k = \beta + i\alpha/2$ and obtained two useful relations.

(a) As long as $\beta \gg \alpha$, show that

$$\frac{\alpha}{\beta} \simeq \frac{\text{Im } \epsilon(\omega)}{\text{Re } \epsilon(\omega)} \qquad \text{where} \qquad \beta \simeq \frac{\omega}{c}\sqrt{\text{Re}\left(\frac{\epsilon}{\epsilon_0}\right)}$$

Hint: It is easier if you start from the right hand side.

$\beta >> \alpha$

$$\beta^2 - \cancel{\frac{\alpha^2}{4}} = \frac{\omega^2}{c^2}\frac{\epsilon}{\epsilon_0}$$

$$\beta^2 = \frac{\omega^2}{c^2}\frac{\epsilon}{\epsilon_0}$$

$$\beta = \frac{\omega}{c}\sqrt{Re\left(\frac{\epsilon}{\epsilon_0}\right)}$$

(b) Present arguments to demonstrate that the ratio Im $\epsilon(\omega)$/Re $\epsilon(\omega)$, or α/β, tells us the *fractional decrease in intensity per wavelength* (divided by 2π, of course).

Low Frequency Behavior

We will now discuss the behavior in the low frequency limit; this determines the conduction properties of the material. Next week, we will discuss the high frequency limit.

In the limit $\omega \to 0$, there is a qualitative difference in the response of the medium depending on whether or not there is a resonance at zero. If a resonance does not exist at $\omega_i = 0$ (i.e., the lowest resonant frequency is different from zero), we have a dielectric insulator whose molecular polarizability is given by equation (4.73) that we wrote in the previous class when we discussed the static case in § 4.6 to motivate the discussion for time-varying fields. If there is a resonance $\omega_0 = 0$, then we will learn below that this describes conductors.

4. Suppose some fraction f_0 of the electrons per molecule have their lowest resonance frequency at $\omega_0 = 0$. Let's call these the "free" electrons (in the sense that, since ω_i are the binding frequencies, if they have $\omega_0 = 0$, they may be considered to be "free").

(a) By separating out the contribution of the fraction f_0 of the electrons per molecule that have their lowest resonant frequency at $\omega_0 = 0$ (the so-called "free" electrons), show that equation (7.51):

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \left[\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}\right] \tag{7.51}$$

takes the form

$$\epsilon(\omega) = \epsilon_b(\omega) + i\left[\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}\right] \tag{7.56}$$

where $\epsilon_b(\omega)$ is the contribution from all the "bound" dipoles.

$$\epsilon(\omega) = 1 + \frac{Ne^2}{m} \sum_j \left[\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}\right]$$

$$= \epsilon_0\left(1 + \frac{Ne^2}{m} \sum_b \frac{f_b}{(\omega_b^2 - \omega^2 - i\omega\gamma_b)}\right) + \frac{Ne^2}{m} \sum_f \left[\frac{f_f}{(\omega_f^2 - \omega^2 - i\omega\gamma_b)}\right]$$

$$= \epsilon_b + i\left[\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}\right]$$

5. Consider the Maxwell-Ampere law: $\vec{\nabla} \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}$, where $\vec{J}$ is the current density.

(a) Assuming that the medium obeys Ohm's law $\vec{J} = \sigma \vec{E}$, where σ is the electrical conductivity, and assuming a harmonic time dependence $\sim e^{-i\omega t}$ and a "normal" dielectric constant ϵ_b, show that the Maxwell-Ampere law written above takes the form

$$\vec{\nabla} \times \vec{H} = -i\omega \left(\epsilon_b + i \frac{\sigma}{\omega} \right) \vec{E} \tag{7.57}$$

(b) On the other hand, we could set $\vec{J} = 0$ in the Maxwell-Ampere law and attribute instead all the properties of the medium, including the "currents" present, to the dielectric constant by putting $\vec{D} = \epsilon \vec{E}$. Again, assuming a harmonic time dependence $\sim e^{-i\omega t}$, show that we would get

$$\vec{\nabla} \times \vec{H} = -i\omega \left[\epsilon_b + i \left\{ \frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \right\} \right] \vec{E}$$

(c) Hence derive the Drude model for conductivity: $\sigma = \dfrac{f_0 N e^2}{m(\gamma_0 - i\omega)}$, where $f_0 N$ is the number of free electrons per unit volume in the medium.