PHY 420 Spring 2021
Week 3—Tuesday, Mar 13—Discussion Worksheet

In the previous class, we began discussing the **Lagrange equation in cylindrical coordinates** and wrote the solutions to the radial equation in terms of ***Bessel functions*** of the first kind $J_\nu(x)$, and ***Neumann functions***, or Bessel functions of the second kind, $N_\nu(x)$.

Each Bessel function has an infinite number of roots. We will work mostly with the roots of $J_\nu(x)$:

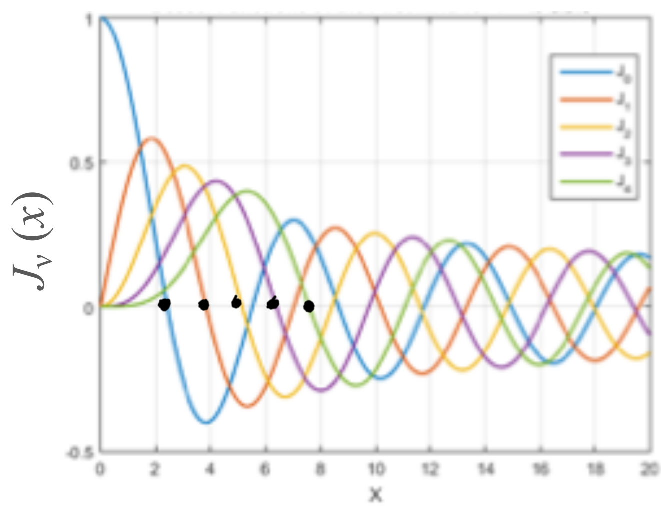
$$J_\nu(x_{\nu n}) = 0, \qquad n = 1, 2, 3, \ldots \tag{3.92}$$

where $x_{\nu n}$ is the nth root of $J_\nu(x)$.

1. Jackson provides the first three roots ($n = 1, 2, 3$) for the first three integer values of $\nu = 0, 1, 2$, and they are

$$\begin{array}{lcrrrl} \nu = 0, & x_{0n} = & 2.405, & 5.520, & 8.654, & \ldots \\ \nu = 1, & x_{1n} = & 3.832, & 7.016, & 10.173, & \ldots \\ \nu = 2, & x_{2n} = & 5.136, & 8.417, & 11.620, & \ldots \end{array}$$

Verify these by visual inspection of the plots of $J_\nu(x)$ below.



For higher roots, the asymptotic formula

$$x_{\nu n} \simeq n\pi + \left(\nu - \frac{1}{2}\right)\frac{\pi}{2}$$

is accurate to at least three figures.

Having written the solution of the radial part of the Laplace equation in terms of Bessel functions, the next step would be to investigate in what sense the Bessel functions form an orthogonal, complete set of functions. We will consider only Bessel functions of the first kind $J_\nu(x)$, which are defined in $0 \le x \le \infty$, although for our purposes, we're often interested in a physical solution over a restricted finite interval $0 \le \rho \le a$.

It can be shown, as Jackson does on pages 114 and 115 if you're interested, that $\sqrt{\rho}\, J_\nu(x_{\nu n}\rho/a)$ form an orthogonal set in the interval $0 \le \rho \le a$, for fixed $\nu \ge 0, n = 1, 2, \ldots$, and that the normalization is given by

$$\int_0^a \left[\sqrt{\rho}\, J_\nu\left(x_{\nu n'}\,\frac{\rho}{a}\right)\right]\left[\sqrt{\rho}\, J_\nu\left(x_{\nu n}\,\frac{\rho}{a}\right)\right] d\rho = \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2 \delta_{n'n}$$

or

$$\int_0^a \rho\, J_\nu\left(x_{\nu n'}\,\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\,\frac{\rho}{a}\right) d\rho = \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2 \delta_{n'n} \tag{3.95}$$

Now that we have a set of orthogonal functions, and assuming that the set is complete, we can expand an arbitrary function of ρ on the interval $0 \le \rho \le a$ in a ***Fourier-Bessel series***:

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n}\, J_\nu\left(x_{\nu n}\,\frac{\rho}{a}\right) \tag{3.96}$$

2. Show that the coefficients $A_{\nu n}$ in equation (3.96) are given by

$$A_{\nu n} = \frac{2}{a^2\, J_{\nu+1}^2(x_{\nu n})} \int_0^a \rho\, f(\rho)\, J_\nu\left(\frac{x_{\nu n}\,\rho}{a}\right) d\rho \tag{3.97}$$

Multiply Both sides by (3.96)

$$\int_0^a f(\rho)\left[\rho\, J_\nu\left(x_{\nu n'}\,\frac{\rho}{a}\right)\right] d\rho$$

$$= \sum_{n=1}^{\infty} A_{\nu n} \underbrace{\int_0^a \rho\, J_\nu\left(x_{\nu n'}\,\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\,\frac{\rho}{a}\right) d\rho}_{\frac{a^2}{a}\left[J_{\nu+1}(x_{\nu n})\right]^2 \delta_{n'n}}$$

$n' \to n$

$$\int_0^a \rho\, f(\rho)\, J_\nu\left(x_{\nu n}\,\frac{\rho}{a}\right) d\rho = A_{\nu n}\left[\frac{a^2}{2}\, J_{\nu+1}^2(x_{\nu n})\right]$$

$$A_{\nu n} = \frac{2}{a^2\, J_{\nu+1}^2(x_{\nu n})} \int_0^a \rho\, f(\rho)\, J_\nu\left(\frac{x_{\nu n}\,\rho}{a}\right) d\rho$$

Equation (3.96), together with (3.97), is the conventional Fourier-Bessel series and is particularly appropriate for functions that are zero at boundaries, e.g., Dirichlet boundary conditions on a cylinder. We will now look at an example of such a boundary value problem.

Boundary Value Problems in Cylindrical Coordinates

Consider the boundary value problem in which a cylinder of radius a and length L has its bottom surface on the xy-plane at $z = 0$, and its top surface at $z = L$. The potential on the side and the bottom of the cylinder is zero, while the top has a potential $\Phi = V(\rho, \phi)$. We wish to find the potential at any point inside the cylinder.

3. The potential $\Phi(\rho, \phi, z)$ at any point inside the cylinder can be written as the product of solutions

$$\Phi(\rho, \phi, z) = R(\rho)\, Q(\phi)\, Z(z)$$

Substituting this into the Laplace equation, we know from equation (3.76) that we get $Q(\phi) = e^{\pm i\nu\phi}$, and that ν must be an integer for the potential to be single-valued when the full azimuthal span is allowed. Setting $\nu = m$, an integer, and writing in terms of sin and cos, we get that

$$Q(\phi) = A \sin m\phi + B \cos m\phi$$

Meanwhile,

$$Z(z) = e^{\pm kz} = \cosh kz \pm \sinh kz$$

where I haven't written any constants, because they can introduced later.

(a) Apply an appropriate boundary condition to find whether $Z(z) = \cosh kz$ or $\sinh kz$ in the problem we are solving. Note that k is undetermined as of this stage.

Cosh $kz \neq 0$ at $z=0$

BC: $\varphi = 0$ at $z = 0$, so $\cosh(kz)$ can not be in the solution for $Z(z)$

Thus,

$$Z(z) = \sinh kz$$

(b) Finally, the radial solution is

$$R(\rho) = C\, J_m(k\rho) + D N_m(k\rho)$$

Examine whether $C = 0$ or $D = 0$. Explain.

$R(\rho)$ will diverge as $\rho \to 0$ if $N_m(k\rho)$ is present in the solution b/c $N(\rho) \to -\infty$, as $\rho \to 0$

Therefore $D = 0$

$$R(\rho) = C J_m(k\rho)$$

Please *do not* turn the page until after you've answered part (b).

Meanwhile, the potential must vanish at $\rho = a$ on the surface of the cylinder. Substituting this in the radial equation in Question 3(b), we get

$$0 = C\, J_m(ka)$$

4. From equation (3.92) in Jackson, we know that each Bessel function $J_\nu(x)$ of order ν has an infinite number of roots. In our current problem, $\nu = m$, an integer, the restriction coming from the $Q(\phi)$ solution. That means we must have

$$J_m(x_{mn}) = 0 \qquad n = 1, 2, 3, \ldots$$

where x_{mn} is the nth root of $J_m(x)$.

(a) Use the relations written above to show that

$$k_{mn} = \frac{x_{mn}}{a} \qquad n = 1, 2, 3, \ldots$$

where we've indexed k with the same m and n to keep track. Keep in mind that x_{mn} are the roots of $J_m(x_{mn}) = 0$.

Since $C \neq 0$, $J_m(ka) = 0 = J_m(x_{mn})$, where x_{mn} is the n^{th} root

Indexing k with same m, n gives

$$k_{mn} a = x_{mn} \Rightarrow k_{mn} = \frac{x_{mn}}{a} \quad n = 1, 2, 3$$

Combining all that we've obtained above, we get the general form of the solution for the potential inside the cylinder (for the problem we are solving in which the potential vanishes at the bottom of the cylinder $z = 0$ and the side of the cylinder $\rho = a$):

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn} z) \left[A_{mn} \sin m\phi + B_{mn} \cos m\phi\right] \tag{3.105a}$$

Applying the third boundary condition that the potential is $V(\rho, \phi)$ at $z = L$, we get

$$V(\rho, \phi) = \sum_{m,n} J_m(k_{mn}\rho) \sinh(k_{mn} L) \left[A_{mn} \sin m\phi + B_{mn} \cos m\phi\right]$$

(b) Write down in words how you will determine A_{mn}.

Multiply $V(\rho, \varphi)$ on both sides by $\sin(m' \varphi)$ and use orthogonality of sines

Then, use Fourier-Bessel orthogonality from (3.97)

Upon evaluating the Fourier-Bessel series, we get

$$\begin{aligned} A_{mn} &= \frac{2}{\pi a^2 J_{m+1}^2(k_{mn}a)\,\sinh(k_{mn}L)} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho V(\rho,\phi)\, J_m(k_{mn}\rho)\, \sin m\phi \\ B_{mn} &= \frac{2}{\pi a^2 J_{m+1}^2(k_{mn}a)\,\sinh(k_{mn}L)} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho V(\rho,\phi)\, J_m(k_{mn}\rho)\, \cos m\phi \end{aligned} \tag{3.105b}$$

Note that for $m = 0$, we will have to use $\frac{1}{2}B_{0n}$ in the series.

5. Apply the procedure in Question 4(b) to derive the relation for A_{mn} written above.

$$\int_0^{2\pi} V(\rho,\varphi)\sin(m'\varphi)\,d\varphi =$$

$$\sum_m \sum_n J_m(k_{mn}\rho)\sinh(k_{mn}L)\Big[\overbrace{\int_0^{2\pi} A_{nm}\sin(m\varphi)\sin(m'\varphi)\,d\varphi}^{A_{mn}\pi\delta_{mm'}} + \underbrace{\int_0^{2\pi} B_{nm}\cos(m\varphi)\sin(m'\varphi)\,d\varphi}_{=0}\Big]$$

$$= \int_0^{2\pi} V(\rho,\varphi)\sin(m\varphi)\,d\varphi = \sum_n J_m(k_{mn}\rho)\sinh(k_{mn}L)A_{mn}\pi$$

$$= \underbrace{\frac{1}{\pi\sinh(k_{mn}L)}\int_0^{2\pi} d\varphi\, V(\rho,\varphi)\sin(m\varphi)}_{f(\rho)} = \sum_{n=1}^{\infty} A_{mn} J_m(k_{mn}\rho)$$

Therefor, we get the top equation

The Fourier-Bessel series in equation (3.105) is appropriate for a finite interval in ρ, in this case, we had $0 \le \rho \le a$. If, however, we wish to cover all space, that is $a \to \infty$, then the series goes over into an integral just as a Fourier series goes over into a Fourier integral. For example, if the potential in charge-free space is finite for $z \ge 0$ and vanishes for $z \to \infty$, the general form of the solution for $z \ge 0$ must be

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(k\rho)\, e^{-kz}\Big[A_m(k)\sin m\phi + B_m(k)\cos m\phi\Big] \tag{3.106}$$

If the potential over the whole plane $z = 0$ is $V(\rho,\phi)$, the coefficients are determined by

$$V(\rho,\phi) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(k\rho)\Big[A_m(k)\sin m\phi + B_m(k)\cos m\phi\Big]$$

Modified Bessel Functions

Finally, it is worth noting that if we had picked the separation constant in equation (3.73) as $-k^2$ (instead of k^2), then the solutions to $Z(z)$ would have been $e^{\pm ikz}$, and the radial equation equation (3.75) for $R(\rho)$ would have instead been

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(-k^2 - \frac{\nu^2}{\rho^2}\right) R = 0 \tag{3.98}$$

6. Show that equation (3.98) can be changed into the form

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0 \tag{3.99}$$

which is called the ***modified Bessel equation***.

Divide (3.98) by k^2 to get

$$\frac{1}{k^2}\frac{d^2R}{d\rho^2} + \frac{1}{k^2\rho}\frac{dR}{d\rho} + \left(-\frac{k^2}{k^2} - \frac{\nu^2}{k^2\rho^2}\right) R = 0$$

So that

$$\frac{d^2R}{d(k\rho)^2} + \frac{1}{k\rho}\frac{dR}{d(k\rho)} + \left(-1 - \frac{\nu^2}{(k\rho)^2}\right) R = 0$$

Set $x = k\rho$

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(-1 - \frac{\nu^2}{x^2}\right) R = 0$$

and write in the form

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0$$

The solutions to equation (3.99) are known as the ***modified Bessel functions***. They are just the Bessel functions but with an imaginary argument ix instead of x. The usual choices of linearly independent solutions are denoted by $I_\nu(x)$ and $K_\nu(x)$, and defined by

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \tag{3.100}$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \tag{3.101}$$

The $i^{-\nu}$ is there to make $I_\nu(x)$ a real function for real x and ν, likewise for $K_\nu(x)$.