PHY 420 Spring 2021

Week 5—Tuesday, Apr 27—Discussion Worksheet

Electric Dipole Fields

In the previous class, we learned that for a localized system of charges and currents that vary in time as $e^{-i\omega t}$, the vector potential $\vec{A}(\vec{x})$ is given by

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}\,') \frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} \, d^3x' \tag{9.3}$$

In the far (radiation) zone, the observation point r is very far from the source and much larger than the wavelength of the light, so that $(r/\lambda) \gg 1$, and thus $kr \gg 1$, since $k = 2\pi/\lambda$. Thus, we can write

$$\left|\vec{x} - \vec{x}\,'\right| \simeq r - \hat{n}\cdot\vec{x}\,' \tag{9.7}$$

where $\hat{n}$ is a unit vector in the direction of $\vec{x}$. In fact, *from your derivation of this approximation on the worksheet from last week*, you know that equation (9.7) is valid for $r \gg d$ (independent of kr), so it is a reasonable approximation even in the near zone. Using the approximation in equation (9.7), and writing the exponential term as a series expansion, you showed on the worksheet last week that $\vec{A}(\vec{x})$ can be written as a series, for which the successive terms fall off rapidly with n, implying that the radiation emitted from the source will come mainly from the first non vanishing term in this series. Thus, if we keep only the first term (corresponding to n=0), then we get

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} \int \vec{J}(\vec{x}\,')\, d^3x' \tag{9.13}$$

We will now write $\vec{A}(\vec{x})$ in terms of the electric dipole moment $\vec{p}$. Before proceeding, it is worth noting that equation (9.13) is a reasonable approximation everywhere outside the source, not just in the far zone (since equation (9.7) is valid for $r \gg d$, independent of the value of kr).

1. Begin by proving the useful relation

$$\vec{\nabla}\cdot(x\vec{J}) = J_x + x\,\vec{\nabla}\cdot\vec{J}$$

Hint: The following vector identity from the inside front cover of Jackson might prove useful:

$$\vec{\nabla}\cdot(\psi\vec{a}) = \vec{a}\cdot\vec{\nabla}\psi + \psi\vec{\nabla}\cdot\vec{a}$$

$$\vec{\nabla}\cdot(x\vec{J}) = \underbrace{\vec{J}\cdot\vec{\nabla}x}\, + x\,\vec{\nabla}\cdot\vec{J}$$

$$= (\hat{x}J_x + \hat{y}J_y + \hat{z}J_z)\cdot\left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right]x$$

$$= (\hat{x}J_x + \hat{y}J_y + \hat{z}J_z)\cdot\left[\hat{x}\frac{\partial}{\partial x}x\right]$$

$$= J_x \;\ldots$$

$$\boxed{= J_x + x\,\vec{\nabla}\cdot\vec{J}}$$

2. We will now use the result of Question 1 to derive an expression that we will need in order to express $\vec{A}(\vec{x})$ in terms of $\vec{p}$.

(a) Starting from the expression that you proved in Question 1, and writing analogous results for the y and z components, derive the relation

$$\vec{\nabla}\cdot(\vec{x}\vec{J}) = \vec{J} + \vec{x}\,(\vec{\nabla}\cdot\vec{J})$$

$$\vec{\nabla}\cdot(\vec{x}\,\vec{J}) = \vec{\nabla}\cdot\left[(\hat{x}x+\hat{y}y+\hat{z}z)\vec{J}\right] = \vec{\nabla}\cdot(\hat{x}x\vec{J}) + \vec{\nabla}\cdot(\hat{y}y\vec{J}) + \vec{\nabla}\cdot(\hat{z}z\vec{J})$$

$$= \hat{x}\,\vec{\nabla}\cdot(x\vec{J}) + \hat{y}\,\vec{\nabla}\cdot(y\vec{J}) + \hat{z}\,\vec{\nabla}\cdot(z\vec{J})$$

$$= \hat{x}\left[J_x + x(\vec{\nabla}\cdot\vec{J})\right] + \hat{y}\left[J_y + y(\vec{\nabla}\cdot\vec{J})\right] + \hat{z}\left[J_z + z(\vec{\nabla}\cdot\vec{J})\right]$$

$$= (\hat{x}J_x + \hat{y}J_y + \hat{z}J_z) + (\hat{x}x + \hat{y}y + \hat{z}z)\,\vec{\nabla}\cdot\vec{J}$$

$$\vec{\nabla}\cdot(\vec{x}\,\vec{J}) = \vec{J} + \vec{x}\,(\vec{\nabla}\cdot\vec{J})$$

(b) Use the vector identity you derived in part (a) to show that

$$\int \vec{J}(\vec{x}')\,d^3x' = -\int \vec{x}'\,(\vec{\nabla}'\cdot\vec{J})\,d^3x'$$

$$\underbrace{\int_V \vec{\nabla}'\cdot(\vec{x}'J)\,d^3x'}_{\text{Divergence Theorem}} = \int_V \vec{J}\,d^3x' + \int_V \vec{x}'(\vec{\nabla}'\cdot\vec{J})\,d^3x'$$

$$\downarrow$$

$$\underbrace{\int_S (\vec{x}'\vec{J})\cdot\hat{n}\,da}_{0} = \underbrace{\int_V \vec{J}\,d^3x' + \int_V \vec{x}'(\nabla'\cdot\vec{J})\,d^3x'}$$

No Current is Crossing the Surface

$$\int \vec{J}(\vec{x}')\,d^3x' = -\int \vec{x}'(\vec{\nabla}'\cdot J)\,d^3x$$

3. We now have everything in place to write $\vec{A}(\vec{x})$ in terms of the electric dipole moment.

(a) Use the expression you derived in Question 2(b), together with the continuity equation (which I've written in primed coordinates $\vec{x}'$ and t' to make it easier to substitute inside the integral):

$$\vec{\nabla}' \cdot \vec{J}(\vec{x}', t') = -\frac{\partial \rho(\vec{x}', t')}{\partial t'}$$

$$\rho(\vec{x}, t) = \rho(\vec{x}) e^{-i\omega t}$$
$$J(\vec{x}, t) = J(\vec{x}) e^{-i\omega t}$$

to show that equation (9.13) can be written as

$$\vec{A}(\vec{x}) = -\frac{i\omega\mu_0}{4\pi}\, \vec{p}\, \frac{e^{ikr}}{r} \tag{9.16}$$

where $\vec{p}$ is the electric dipole moment.

Continuity equation

$$\vec{\nabla}' \cdot \vec{J}(\vec{x}') e^{-i\omega t'} = -\rho(\vec{x}') \frac{\partial}{\partial t}\left[e^{-i\omega t}\right]$$

$$\vec{\nabla}' \cdot \vec{J}(\vec{x}') \cancel{e^{-i\omega t'}} = -\rho(\vec{x}') \frac{\partial}{\partial t}\left[\cancel{e^{-i\omega t}}\right]$$

Sub into Q2(b)

$$\int J(\vec{x}') d^3x' = -\int \vec{x}' (\vec{\nabla}' \cdot \vec{J}) d^3x'$$
$$= -i\omega \int \vec{x}' \rho(\vec{x}') d^3x'$$

Sub into eq 9.13

$$A(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}') d^3x' = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left[-i\omega \int \vec{x}' \rho(\vec{x}') d^3x'\right]$$

$$A(\vec{x}) = -\frac{i\omega\mu_0}{4\pi} \vec{p} \frac{e^{ikr}}{r}$$

(b) Write down the form of $\vec{p}$ from your derivation in part (a) above.

$$\vec{p} = \int \vec{x}' \rho(\vec{x}') d^3x'$$

4. Having determined the vector potential, we can write the fields (as you'll do on the homework):

Same direction as $\vec{k}$

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \tag{9.18}$$

Note that $\hat{n}$ in both the expressions above is a unit vector in the direction of $\vec{x}$, the observation point. Strictly speaking, therefore, $\hat{n}$ is the radial vector. However, since we have a tiny source at the origin and the waves are propagating radially outward, the radial vector along $\hat{n}$ and the wave vector $\vec{k}$ along which the wave is propagating point in the same direction.

(a) What is the direction of the magnetic field $\vec{H}$ in relation to the wave vector $\vec{k}$ along which the wave is propagating?

Has components perpendicular to the direction of the wave

(b) What is interesting about the components of the electric field $\vec{E}$ in relation to the wave vector $\vec{k}$ along which the wave is propagating? *Don't get stressed by this answer; things will work out as expected in the far (radiation) zone!*

Has components both parallel and perpendicular to the direction of the wave

(c) In the far (radiation) zone, show that the magnetic field takes on the limiting form

$$\vec{H} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}$$

let $r/\lambda >> 1 \rightarrow 2\pi r/\lambda >> 1$

where $k = 2\pi/\lambda$

if $kr >> 1$

$\rightarrow \frac{1}{ikr} \rightarrow 0$

so $\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}(1-0)$

$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}$

5. We will now verify that fields in the far (radiation) zone behave as expected.

(a) In the far zone, show that the electric field takes on the limiting form

$$\vec{E} = \frac{k^2}{4\pi\epsilon_0}\left[(\hat{n}\times\vec{p})\times\hat{n}\right]\frac{e^{ikr}}{r}$$

Equation 9.18

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{ k^2(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r} + [3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\left\{ k^2(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r} + \cancel{[3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}]\underbrace{\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)}_{kr \gg 1}e^{ikr}}\right\}$$

0

$$\vec{E} = \frac{k^2}{4\pi\epsilon_0}\left[(\hat{n}\times\vec{p})\times\hat{n}\right]\frac{e^{ikr}}{r}$$

(b) Discuss whether we are back in familiar territory! Are the directions for $\vec{H}$ and $\vec{E}$ consistent with what we learned about propagating transverse waves?

It now looks like a transverse EM wave. It now looks a plane wave

So, what have we written in equation (9.18) and the equations in Question 4(c) and 5(a)?

If we take an electric dipole with dipole moment $\vec{p}$ and let it oscillate harmonically, it will emit electromagnetic waves. In the long wavelength limit ($r \gg d$), the electric and magnetic fields of these electromagnetic waves are given by equation (9.18), whereas the electric and magnetic fields of the electromagnetic waves in the far zone are given by the equations written in Question 4(c) and 5(a); presumably, you found in part (b) above that we have transverse waves carrying energy away from the oscillating source. Notice that both fields fall off as $1/r$.

Power Radiated

To obtain an expression for the power radiated, begin with the (time-averaged) Poynting vector

$$\vec{S} = \frac{1}{2}\,\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right]$$

where the factor 1/2 comes from the time average.

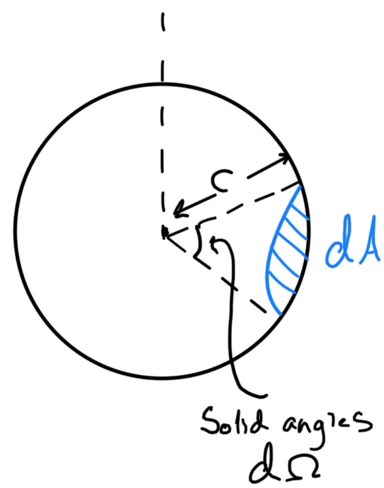
6. The time-averaged power radiated per unit surface area by the oscillating electric dipole is given by $\hat{n}\cdot\vec{S}$. Use this to show that the time-averaged power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[r^2\,\hat{n}\cdot\vec{E}\times\vec{H}^*\right]$$

Note: At any given point in space, the (time-averaged) Poynting vector $\vec{S}$ gives the *energy per unit area per unit time* flowing past that point. In other words, $\vec{S}$ is the ***time-averaged power radiated per unit surface area***.

In this question, though, we are more interested in the ***time-averaged power radiated per unit solid angle***. Thus, you will need to **begin by** writing a relation between the solid angle $d\Omega$ and the surface area dA; ***ask if you've forgotten how to do so***.

$$dA = r^2\,d\Omega$$



Since $\hat{n}\cdot\vec{S}$ is the time-averaged Power per unit Surface area (perpendicular to dA), we can write

$$dP = (\hat{n}\cdot\vec{S})\,dA$$
$$= (\hat{n}\cdot\vec{S})\left[r^2\,d\Omega\right]$$
$$= \hat{n}\cdot \tfrac{1}{2}\,\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right]r^2\,d\Omega$$

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[\hat{n}\cdot\vec{E}\times\vec{H}^*\right]r^2$$

$$= \frac{1}{2}\,\mathrm{Re}\left[r^2\,\hat{n}\cdot\vec{E}\times\vec{H}^*\right]$$