PHY 420 **Spring 2021**

Week 1—Thursday, Apr 1—Discussion Worksheet

Poynting's Theorem

Today, we will learn about the conservation of energy in the electromagnetic field, often called ***Poynting's theorem***.

1. Consider the force on a single charge q traveling at velocity v, given by $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$.

(a) If E_{kin} is the kinetic energy of the charges, then show the rate of doing work by the external electromagnetic fields $\vec{E}$ and $\vec{B}$ is

$$\frac{dE_{\text{kin}}}{dt} = q\vec{E} \cdot \vec{v}$$

$$\frac{dE_{kin}}{dt} = \vec{F} \cdot \vec{v} = (q\vec{E} + q\vec{v} \times \vec{B}) \cdot \vec{v}$$

$$= q\vec{E} \cdot \vec{v} + \cancel{q \times \vec{B}\vec{v}} \rightarrow 0$$

$$\frac{dE_{kin}}{dt} = q\vec{E} \cdot \vec{v}$$

For a continuous distribution, we replace $q\vec{v}$ by $\vec{J}\,d^3x$ and then integrate over the volume V of the charge distribution to get the total rate of doing work by the fields in a finite volume:

Harmonic Fields for Maxwell Equations

$J = (\vec{\nabla} \times \vec{H} + i\omega D)$

$$\int_V \vec{J} \cdot \vec{E}\, d^3x \tag{6.103}$$

This power represents a conversion of electromagnetic energy into mechanical or thermal energy. It must be balanced by a corresponding rate of decrease of energy in the electromagnetic field within the volume V. We will now write this conservation law explicitly.

(b) First, show that equation (6.103) becomes

$$\int_V \vec{J} \cdot \vec{E}\, d^3x = \int_V \left[\vec{E} \cdot \left(\vec{\nabla} \times \vec{H}\right) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right] d^3x \tag{6.104}$$

$$- \frac{\partial D}{\partial t}$$

$$\int_V \vec{J} \cdot \vec{E}\, d^3x = \int_V (\vec{\nabla} \times \vec{H} + i\omega D) \cdot \vec{E}\, d^3x$$

$$= \int_V \left(\vec{\nabla} \times \vec{H} - \frac{\partial D}{\partial t}\right) \cdot \vec{E}\, d^3x$$

$$= \int_V \left[\vec{E} \cdot (\vec{\nabla} \times \vec{H}) - \vec{E} \cdot \frac{\partial D}{\partial t}\right] d^3x$$

2. To write the conservation of energy explicitly, we derived equation (6.104) on the previous page. Next, use the vector identity $\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right)=\vec{H}\cdot\left(\vec{\nabla}\times\vec{E}\right)-\vec{E}\cdot\left(\vec{\nabla}\times\vec{H}\right)$, together with Faraday's law $\vec{\nabla}\times\vec{E}=-\partial B/\partial t$, to show that

$$\int\limits_V \vec{J}\cdot\vec{E}\,d^3x=-\int\limits_V\left[\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right)+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}+\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right]d^3x \tag{6.105}$$

Equation (6.105) represents the rate of decrease of energy in the electromagnetic field within the volume V; this goes into increasing the mechanical or thermal energy of the moving charges.

$$\int_V \vec{J}\cdot\vec{E}\,d^3x=\int_V\left[\vec{E}\cdot(\vec{\nabla}\times\vec{H})-\vec{E}\cdot\frac{\partial D}{\partial t}\right]d^3x$$

where $\vec{E}\cdot(\vec{\nabla}\times\vec{H})=\vec{H}\cdot(\vec{\nabla}\times\vec{E})-\vec{\nabla}\cdot(\vec{E}\times\vec{H})$

$$=\int_V\left[\vec{H}\cdot(\nabla\times\vec{E})-\vec{\nabla}\cdot(\vec{E}\times\vec{H})-\vec{E}\cdot\frac{\partial D}{\partial t}\right]d^3x$$

The harmonic fields from Maxwell equations

$\vec{\nabla}\times\vec{E}=i\omega B\rightarrow\frac{\partial B}{\partial t}$ (or Faraday's Law $\vec{\nabla}\times\vec{E}=-\frac{\partial B}{\partial t}$)

$$=\int_V\left[-\vec{H}\cdot\frac{\partial B}{\partial t}-\vec{\nabla}\cdot(\vec{E}\times\vec{H})-\vec{E}\,\frac{\partial\vec{D}}{\partial t}\right]$$

$$=-\int_V\left[\vec{\nabla}\cdot(\vec{E}\times\vec{H})+\vec{E}\,\frac{\partial\vec{D}}{\partial t}+\vec{H}\,\frac{\partial B}{\partial t}\right]$$

To proceed, we will make **two assumptions**:

- Assume that the macroscopic medium is linear in its electric properties (i.e., $\vec{D}=\epsilon\vec{E}$), and its magnetic properties ($\vec{B}=\mu\vec{H}$), with negligible dispersion or losses, so that

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E}\cdot\vec{D}\right)\qquad\text{and}\qquad\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left(\vec{B}\cdot\vec{H}\right)$$

- Assume also that the total electromagnetic energy, even for time-varying fields, is the sum of eq. (4.89) and eq. (5.148), so that the total energy density is given by

$$u=\frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right) \tag{6.106}$$

Based on these two assumptions, you will now write an expression for $\partial u/\partial t$.

3. At the bottom of the previous page, we made ***two assumptions***.

(a) Based on these two assumptions, show that you get

$$\frac{\partial u}{\partial t} = \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{B}}{\partial t}$$

$$u = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)$$

$$\frac{\partial u}{\partial t} = \underbrace{\frac{1}{2}\frac{\partial \vec{E}}{\partial t}\cdot\vec{D}}_{\text{Assumption 1}} + \underbrace{\frac{1}{2}\frac{\partial \vec{B}}{\partial t}\cdot\vec{H}}_{\text{Assumption 2}}$$

$$= \left(\vec{E}\cdot\frac{\partial \vec{D}}{\partial t}\right) + \left(\vec{H}\cdot\frac{\partial \vec{B}}{\partial t}\right)$$

$$\frac{\partial u}{\partial t} = \left(\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{B}}{\partial t}\right)$$

(b) Substitute in equation (6.105) the expression you obtained above and show that you get

$$-\int\limits_V \vec{J}\cdot\vec{E}\,d^3x = \int\limits_V \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right)\right] d^3x \tag{6.107}$$

$$= -\int_V \Big[\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) + \underbrace{\vec{E}\,\frac{\partial \vec{D}}{\partial t} + \vec{H}\,\frac{\partial B}{\partial t}}_{\frac{\partial u}{\partial t}}\Big]$$

$$-\int_V \vec{J}\cdot\vec{E}\,d^3x = \int_V \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right)\right] d^3x$$

Since the volume V is arbitrary, the integrand in equation (6.107) can be written in the form of a differential continuity equation or conservation law

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \left(\vec{S}\right) = -\vec{J} \cdot \vec{E} \tag{6.108}$$

4. The vector $\vec{S}$ represents energy flow, and is called the **Poynting vector**. By comparing equation (6.107) and equation (6.108), write down what $\vec{S}$ is equal to, and verify that it has the dimensions of energy/(area × time).

$$\vec{S} = \left(\vec{E} \times \vec{H}\right) \tag{6.109}$$

Dimensions

$$= (\text{velocity}) \left(\frac{\text{energy}}{\text{volume}}\right)$$

$$= \frac{\text{energy}}{\text{area} \times \text{time}}$$

Let us now consider the physical meaning of the integral form in equation (6.107) or the differential form in equation (6.108). Either equation tells us that the time rate of change of electromagnetic energy within a certain volume, plus the energy per unit time flowing out through the boundary surfaces of the volume, is equal to the negative of the total work done by the fields on the sources within the volume. This is a statement of the **conservation of energy**.

So far, we've emphasized the energy of the electromagnetic fields. We can also interpret Poynting's theorem for the *microscopic* fields ($\vec{E}$ and $\vec{B}$) as ***a statement of the conservation of energy of the combined system of particles and fields***. Recall that the work done per unit time per unit volume by the fields ($\vec{J} \cdot \vec{E}$) is a conversion of electromagnetic energy into mechanical or heat energy (e.g., for ohmic conductors, $\vec{J} = \sigma\vec{E}$, and $\vec{J} \cdot \vec{E}$ is converted to heat via the resistance of the material). So, since matter is ultimately composed of charged particles (electrons and atomic nuclei), we can think of this rate of conversion as a rate of increase of energy of the charged particles per unit volume.

If we denote the total energy of the particles within the volume V as E_{mech} and assume that no particles move out of the volume, we have

$$\frac{dE_{\text{mech}}}{dt} = \int_V \vec{J} \cdot \vec{E}\, d^3x \tag{6.110}$$

This allows us to write Poynting's theorem in terms of the energy in the field and of the particles.

5. We will now write Poynting's theorem in terms of the energy in the field and of the particles.

(a) Starting from the expression for u in equation (6.106), show that the total field energy within V is given by

$$U = \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H})$$

$$E_{\text{field}} = \int_V u\, d^3x = \frac{\epsilon_0}{2}\int_V \left(\vec{E}^2 + c^2\vec{B}^2\right) d^3x \tag{6.112}$$

$$= \int_V \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H})\, d^3x \rightarrow \vec{D} = \epsilon_0\vec{E},\ \vec{B} = \mu_0 H$$

$$= \int_V \frac{1}{2}(\vec{E}\cdot\epsilon_0\vec{E} + \vec{B}\cdot\vec{B}\,{}^1\!/\!\mu_0)\, d^3x$$

$$= \frac{1}{2}\int_V (\epsilon_0\vec{E}^2 + \vec{B}^2\,{}^1\!/\!\mu_0\epsilon_0) \rightarrow c^2 = ({}^1\!/\!\mu_0\epsilon_0)$$

$$= \epsilon_0/2 \int_V (\vec{E}^2 + c^2\vec{B}^2)\, d^3x$$

(b) Show that Poynting's theorem in equation (6.107) becomes

$$\frac{dE}{dt} = \frac{d}{dt}\left(E_{\text{mech}} + E_{\text{field}}\right) = -\oint_S \hat{n}\cdot\vec{S}\, da \tag{6.111}$$

$$-\int_V \vec{J}\cdot\vec{E}\, d^3x = \int_V \Big[\underbrace{\frac{\partial u}{\partial t}}_{E_{\text{field}}} + \underbrace{\vec{\nabla}\cdot(\vec{E}\times\vec{H})}_{E_{\text{mech}}}\Big] d^3x$$

$\vec{S}$

$$\rightarrow \frac{dE}{dt} = \frac{d}{dt}(E_{\text{mech}} + E_{\text{field}})$$

$$\frac{d\vec{E}}{dt} = -\int_V \vec{\nabla}\cdot\vec{S}\, d^3x \qquad \frac{d\vec{E}}{dt} = -\oint \hat{n}\cdot\vec{S}\, da$$

$\downarrow$ divergence theorem

$$= -\oint \vec{S}\cdot\hat{n}\, da$$

Thus, Poynting's theorem expresses the conservation of energy for the combined system of particles and fields as equation (6.111), where the total field energy within V is given by equation (6.112).