PHY 411 **Winter 2021**

Class Summary—Week 7, Day 2—Thursday, Feb 18

Associated Legendre Functions and Spherical Harmonics

Recall that the **generalized Legendre equation**, with $x = \cos\theta$, is

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P = 0 \tag{3.9}$$

Previously, we dealt with potential problems involving azimuthal symmetry, so we put $m = 0$ in the equation above (and obtained the ordinary Legendre equation), with solutions in terms of Legendre polynomials of order l, $P_l(\cos\theta)$. The general potential problem, however, can have azimuthal variations, so that $m \neq 0$. Therefore, we need the generalization of $P_l(\cos\theta)$, i.e., the solution of the generalized Legendre equation written above, with l and m both arbitrary.

For the generalized Legendre equation (3.9) to have finite solutions on the interval $-1 \leq x \leq 1$, the parameter l must be zero or a positive integer and the integer m can only take the values $-l, -(l-1), \ldots, 0, \ldots, (l-1), l$. The solution having these properties is called an **associated Legendre function** $P_l^m(x)$. For positive m, it is defined by the formula:

$$P_l^m(x) = (-1)^m\,(1-x^2)^{m/2}\,\frac{d^m}{dx^m}\,P_l(x) \tag{3.49}$$

whereas $P_l^{-m}(x)$ can be obtained from $P_l^m(x)$ because they are proportional, as the generalized Legendre equation (3.9) depends only on m^2 and m is an integer:

$$P_l^{-m}(x) = (-1)^m\,\frac{(l-m)!}{(l+m)!}\,P_l^m(x) \tag{3.51}$$

The choice of the arbitrary phase factor $(-1)^m$ is by convention (see Jackson's footnote on page 108 for the original source).

If $P_l(x)$ is written explicitly using Rodrigues' formula, then the corresponding expression for $P_l^m(x)$ is valid for both positive and negative integers m:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}\,(1-x^2)^{m/2}\,\frac{d^{\,l+m}}{dx^{l+m}}\,(x^2-1)^l \tag{3.50}$$

For fixed m, the associated Legendre functions $P_l^m(x)$ form an orthogonal set in the interval $-1 \leq x \leq 1$:

$$\int_{-1}^{1} P_{l'}^m(x)\,P_l^m(x)\,dx = \frac{2}{2l+1}\,\frac{(l-m)!}{(l+m)!}\,\delta_{l'\,l} \tag{3.52}$$

We now have the full solution to the generalized Legendre equation (3.9):

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A'_{lm}\,r^l + \frac{B'_{lm}}{r^{l+1}}\right) P_l^m(\cos\theta)\,e^{im\phi}$$

Note that I've written the ϕ-solution: $Q_m(\phi) = e^{\pm im\phi}$ as just $e^{im\phi}$ because I've covered the case for $-m$ in the summation from $-l$ to l. In this solution, A'_{lm} and B'_{lm} are constants that must be determined from the boundary conditions.

Now, consider the following:

- The functions $Q_m(\phi) = e^{im\phi}$ form a complete set of orthogonal functions in the index m on the interval $0 \le \phi \le 2\pi$.
- The functions $P_l^m(\cos\,\theta)$ form a complete orthogonal set in the index l for each m value in the interval $-1 \le \cos\,\theta \le 1$.

So, we can use the orthogonality relation equation (3.52), together with the factor 2π, to write normalized versions of $P_l^m(x)$ and $e^{im\phi}$ into the solution itself in the following manner:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}\,r^l + \frac{B_{lm}}{r^{l+1}}\right)\left[\sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\,\theta)\,e^{im\phi}\right]$$

This is very convenient, because we can now define the term in square brackets that is a combination of the angular factors (θ,ϕ) as a complete set of orthogonal functions $Y_{lm}(\theta,\phi)$ in the indices (l,m) that are normalized in the intervals $-1 \le \cos\,\theta \le 1$ and $0 \le \phi \le 2\pi$ so that our solution to the generalized Legendre equation equation (3.9) now looks like:

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}\,r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\phi) \tag{3.61}$$

where the functions $Y_{lm}(\theta,\phi)$, called the **spherical harmonics**, and given by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\,\theta)\,e^{im\phi} \tag{3.53}$$

form what are known as orthonormal functions (because they are normalized and orthogonal) over all angles (θ,ϕ) of the unit sphere.

We also need $Y_{l,-m}(\theta,\phi)$, so from equation (3.53)

$$\begin{aligned} Y_{l,-m}(\theta,\phi) &= \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-(-m))!}{(l+(-m))!}}\,P_l^{-m}(\cos\,\theta)\,e^{i(-m)\phi} \\ &= \sqrt{\frac{2l+1}{4\pi}\,\frac{(l+m)!}{(l-m)!}}\left[(-1)^m\,\frac{(l-m)!}{(l+m)!}\,P_l^m(\cos\,\theta)\right]e^{-im\phi} \end{aligned}$$

where I've written $P_l^{-m}(\cos\,\theta)$ from equation (3.51). Simplifying the above expression by canceling common terms, we get

$$Y_{l,-m}(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\,\theta)\,e^{-im\phi}$$

The right hand side is just $(-1)^m$ times $Y_{lm}^*(\theta,\phi)$, the complex conjugate of $Y_{lm}(\theta,\phi)$ written in equation (3.53). Therefore

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y_{lm}^*(\theta,\phi) \tag{3.54}$$

Since $Y_{lm}(\theta,\phi)$ are orthonormalized on the unit sphere, the orthogonality condition looks like

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\,\theta\,d\theta\,Y_{l'm'}^*(\theta,\phi)\,Y_{lm}(\theta,\phi) = \delta_{l'l}\,\delta_{m'm} \tag{3.55}$$

The explicit forms of $Y_{lm}(\theta,\phi)$ for $0 \le l \le 3$ are written on page 109 in Jackson.

For $m = 0$, the spherical harmonics don't depend on the azimuthal angle, so we just get a Legendre polynomial:

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\,\theta) \tag{3.57}$$

Since $Y_{lm}(\theta, \phi)$ form a complete set of functions, an arbitrary function $f(\theta, \phi)$ can be expanded in spherical harmonics:

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm}\, Y_{lm}(\theta, \phi) \tag{3.58}$$

where the coefficients C_{lm} are given by

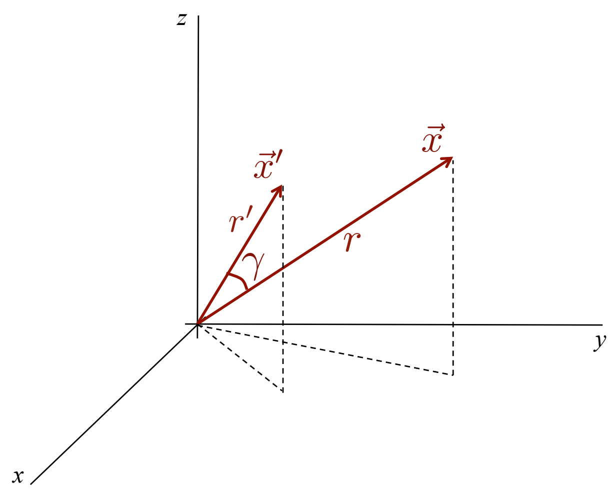
$$C_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\,\theta\, d\theta\, f(\theta, \phi)\, Y^*_{lm}(\theta, \phi)$$

Addition Theorem for Spherical Harmonics

There is a very useful rule called the **addition theorem** for spherical harmonics.

The addition theorem tells us that a Legendre polynomial $P_l(\cos\,\gamma)$ having as its argument the angle γ between two vectors $\vec{x}$ and $\vec{x}\,'$ is expanded in terms of the spherical harmonics of $\vec{x}$ and $\vec{x}\,'$.

Consider two coordinate vectors $\vec{x}$ and $\vec{x}\,'$ with spherical coordinates (r, θ, ϕ) and (r', θ', ϕ') respectively. Suppose the angle between the two vectors is γ, as shown in the figure on the right (modified from Figure 3.7 on page 111 in Jackson).



Then the ***addition theorem*** says that

$$P_l(\cos\,\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi')\, Y_{lm}(\theta, \phi) \tag{3.62}$$

where

$$\cos\,\gamma = \cos\,\theta\,\cos\,\theta' + \sin\,\theta\,\sin\,\theta'\,\cos\,(\phi - \phi')$$

We won't prove this theorem, but if you're interested in the proof, it is given on pages 110-111 in Jackson.

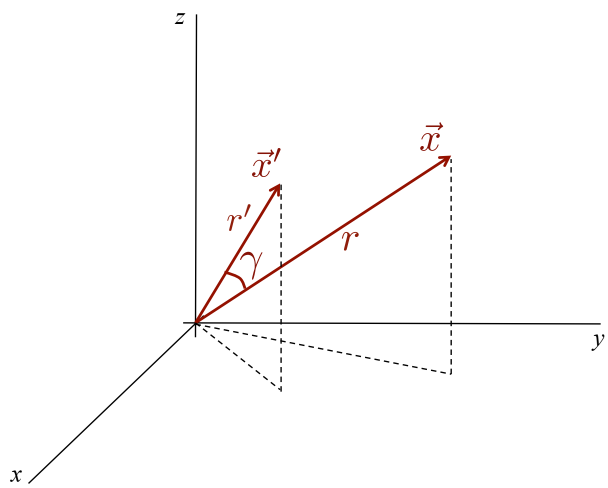
If the angle γ goes to zero, we get a sum rule for the squares of Y_{lm}'s, given by

$$\sum_{m=-l}^{l} \Big|Y_{lm}(\theta, \phi)\Big|^2 = \frac{2l+1}{4\pi} \tag{3.69}$$

A useful expansion

Now that we have the machinery of the expansion in Legendre polynomials, we can take advantage of it to render the expansion of $1/|\vec{x} - \vec{x}\,'|$, the potential at $\vec{x}$ due to a unit point charge at $\vec{x}\,'$.

Consider the figure below (taken from Figure 3.3 on page 102 in Jackson), where the potential is sought at the observation point $\vec{x}$ due to a unit point charge at the source point $\vec{x}\,'$; notice that the angle between $\vec{x}$ and $\vec{x}\,'$ is γ.



We know that the potential at the observation point $\vec{x}$ due to the unit point charge at $\vec{x}\,'$ is

$$\Phi(\vec{x}) = \frac{1}{|\vec{x} - \vec{x}\,'|}$$

where we've ignored factors of $4\pi\epsilon_0$ for now.

The expansion can be written as

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}}$$

where $r = |\vec{x}|$, and $r' = |\vec{x}\,'|$, as shown in the figure above.

Without any loss of generality, we can rotate axes so that $\vec{x}\,'$ lies along the z-axis. Then, the angle γ is just the angle θ of the observation point $\vec{x}$.

More important, though, letting $\vec{x}\,'$ lie along the z-axis means that the potential is azimuthally symmetric, so we can express it in terms of Legendre polynomials (with γ playing the role of θ):

$$\Phi(\vec{x}) = \sum_{l=0}^{\infty} \left[A_l\, r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\gamma)$$

With $\phi(\vec{x}) = 1/|\vec{x} - \vec{x}\,'|$ written in terms of r and r' above, we then get

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}} = \sum_{l=0}^{\infty} \left[A_l\, r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\gamma)$$

On the previous page, we obtained that

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}} = \sum_{l=0}^{\infty} \left[A_l\, r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\gamma)$$

To find the coefficients A_l and B_l, we note that this relation must be valid for all θ (being called γ here), so we can simplify our task by choosing a particular θ; let's choose $\gamma = 0$ (which is what Jackson means when he says "if the point $\vec{x}$ is on the z-axis"):

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'}} = \frac{1}{\sqrt{(r - r')^2}} = \frac{1}{|r - r'|}$$

We have a choice of sign for the square root and pick the positive, since we're interested in the magnitude.

So, now we have

$$\frac{1}{|r - r'|} = \sum_{l=0}^{\infty} \left[A_l\, r^l + \frac{B_l}{r^{l+1}} \right]$$

- For $r < r'$, that is, when the observation point is closer to the origin than the source charge, *you showed on the worksheet* that we get

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{r'} \sum_{l=0}^{\infty} \left(\frac{r}{r'} \right)^l$$

- For $r > r'$, that is, when the observation point is farther from the origin than the source charge, *you showed on the worksheet* that we get

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r} \right)^l$$

We can write the results from the two cases ($r < r'$ and $r > r'$) in one equation by writing:

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>} \right)^l$$

where we will write $r_< \equiv r$ and $r_> \equiv r'$, if $r < r'$, whereas the converse applies if $r > r'$, i.e., we will then write $r_< \equiv r'$ and $r_> \equiv r$, if $r > r'$.

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On the previous page, we wrote that

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = \frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l$$

where $r_< \equiv r$ and $r_> \equiv r'$, if $r < r'$, whereas the converse applies if $r > r'$, i.e., $r_< \equiv r'$ and $r_> \equiv r$, if $r > r'$.

Remember, though, that the above expression was derived by choosing $\gamma \equiv \theta = 0$, i.e., only for points $\vec{x}$ lying on the z-axis.

For points off the axis, all we need to do is multiply by $P_l(\cos\gamma)$.

Therefore, the expansion for the potential at the observation point $\vec{x}$ due to a unit point charge at the source point $\vec{x}\,'$ (ignoring factors of $4\pi\epsilon_0$ for now), is

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = \sum_{l=0}^{\infty} \left(\frac{r_<^l}{r_>^{l+1}}\right) P_l(\cos\gamma) \tag{3.38}$$

where $r = |\vec{x}|, r' = |\vec{x}\,'|$ and, as stated above, $r_<$ is the smaller of r and r' (or, equivalently, $r_>$ is the larger of r and r'), and γ is the angle between $\vec{x}$ and $\vec{x}\,'$.

This can be put into an even more explicit form by using the addition theorem that we wrote above. Substituting for $P_l(\cos\gamma)$ from equation (3.62) for the addition theorem, we get

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left(\frac{r_<^l}{r_>^{l+1}}\right) Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi) \tag{3.70}$$

This will be a very useful expansion, as we will see shortly. Note that, since the left hand side is the potential for a unit point charge, equation (3.70) is also the expansion of the Green function in spherical coordinates.