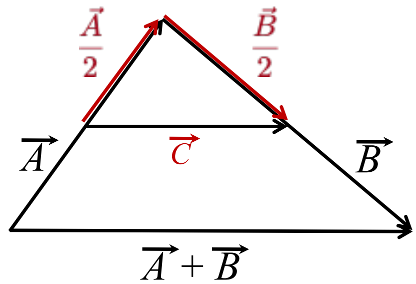
PHY 411 **Winter 2021**

Homework 1 solutions

1. By constructing an appropriate set of vectors, prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

Solution: Consider the figure below.



In order to take advantage of the triangle law of vector addition, I've chosen vectors $\vec{A}$ and $\vec{B}$ as shown in the figure above, so that the third side of the triangle is given by $\vec{A}+\vec{B}$.

Then, choose the midpoint of the sides represented by $\vec{A}$ and $\vec{B}$. The vector $\vec{A}/2$ will then be given by the dark red vector marked in the figure; $\vec{A}/2$ must have the same direction as $\vec{A}$, and be half its magnitude, so choosing it to stretch from the midpoint to the apex of the triangle is a good choice.

Meanwhile, the vector $\vec{B}/2$ must be in the same direction as $\vec{B}$, and equal to half of $\vec{B}$in magnitude, so choosing it to start at the apex of the triangle with its arrowhead at the midpoint of the side designated by the vector $\vec{B}$is also a good choice.

Then, from the triangle law of vector addition in the smaller triangle formed by the vectors $\vec{A}/2$, $\vec{B}/2$, and $\vec{C}$, we get

$$\frac{\vec{A}}{2}+\frac{\vec{B}}{2}=\vec{C}$$

which means that vector $\vec{C}$ is given by

$$\vec{C}=\frac{1}{2}\,(\vec{A}+\vec{B})$$

This allows us to conclude that:

- The vector $\vec{C}$ must be parallel to $(\vec{A}+\vec{B})$, since $\vec{C}$ is a scalar multiple of $(\vec{A}+\vec{B})$.
- The length of $\vec{C}$ is half the length of $(\vec{A}+\vec{B})$.

Therefore, we have demonstrated that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

2. The ability to handle pages of vector math is a critical and necessary component of electrodynamics problems. With that in mind, show by *explicit computation* that

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

for any arbitrary vector $\vec{A}$. **Do not** use Levi-Civita notation; I'm trying to get you to see all the terms and the cancellations.

Solution: The easiest (and quickest) way to do this is to start independently on both sides and show that they meet somewhere in the middle. On the other hand, the agility that comes from experience allows one to start on the left and end up on the right hand side. I'll adopt the latter approach to demonstrate how it can be done:

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \;& \hat{x}\left[\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] \\
& + \hat{y}\left[\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] \\
& + \hat{z}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]
\end{aligned}$$

With an eye on the result, I'm going to add and subtract appropriate terms. I've put these in purple font for better visibility. Note that I'm writing the same term as, e.g., $\partial^2 A_x/\partial x^2$ in one place but $\partial^2 A_x/\partial x \partial x$ in the other place to put them into the format I'll need for the first and second terms respectively on the right hand side.

$$\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \;& \hat{x}\left[\frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial x \partial x}\right] \\
& + \hat{y}\left[\frac{\partial^2 A_z}{\partial z \partial y} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial y \partial y}\right] \\
& + \hat{z}\left[\frac{\partial^2 A_x}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial z}\right]
\end{aligned}$$

Collect terms into two groups that will eventually become the right hand side and rearrange:

$$\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \;& \hat{x}\left[\frac{\partial^2 A_x}{\partial x \partial x} + \frac{\partial^2 A_y}{\partial y \partial x} + \frac{\partial^2 A_z}{\partial z \partial x}\right] - \hat{x}\left[\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right] \\
& + \hat{y}\left[\frac{\partial^2 A_x}{\partial x \partial y} + \frac{\partial^2 A_y}{\partial y \partial y} + \frac{\partial^2 A_z}{\partial z \partial y}\right] - \hat{y}\left[\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right] \\
& + \hat{z}\left[\frac{\partial^2 A_x}{\partial x \partial z} + \frac{\partial^2 A_y}{\partial y \partial z} + \frac{\partial^2 A_z}{\partial z \partial z}\right] - \hat{z}\left[\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right]
\end{aligned}$$

I will write the expression from the previous page again here, so you can follow the thread more easily. So far, we have

$$\begin{aligned}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}\,) = \ & \hat{x}\left[\frac{\partial^2 A_x}{\partial x\partial x}+\frac{\partial^2 A_y}{\partial y\partial x}+\frac{\partial^2 A_z}{\partial z\partial x}\right]-\hat{x}\left[\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}\right]\\ &+\hat{y}\left[\frac{\partial^2 A_x}{\partial x\partial y}+\frac{\partial^2 A_y}{\partial y\partial y}+\frac{\partial^2 A_z}{\partial z\partial y}\right]-\hat{y}\left[\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2}\right]\\ &+\hat{z}\left[\frac{\partial^2 A_x}{\partial x\partial z}+\frac{\partial^2 A_y}{\partial y\partial z}+\frac{\partial^2 A_z}{\partial z\partial z}\right]-\hat{z}\left[\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2}\right]\end{aligned}$$

Keep rearranging with an eye on the right hand side:

$$\begin{aligned}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}\,) = \ & \hat{x}\frac{\partial}{\partial x}\left[\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right]-\hat{x}\Big[\nabla^2 A_x\Big]\\ &+\hat{y}\frac{\partial}{\partial y}\left[\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right]-\hat{y}\Big[\nabla^2 A_y\Big]\\ &+\hat{z}\frac{\partial}{\partial z}\left[\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right]-\hat{z}\Big[\nabla^2 A_z\Big]\end{aligned}$$

Each group of three terms in square brackets is just $\vec{\nabla}\cdot\vec{A}$, so we get

$$\begin{aligned}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}\,) = \ & \hat{x}\frac{\partial}{\partial x}\Big[\vec{\nabla}\cdot\vec{A}\Big]-\hat{x}\Big[\nabla^2 A_x\Big]\\ &+\hat{y}\frac{\partial}{\partial y}\Big[\vec{\nabla}\cdot\vec{A}\Big]-\hat{y}\Big[\nabla^2 A_y\Big]\\ &+\hat{z}\frac{\partial}{\partial z}\Big[\vec{\nabla}\cdot\vec{A}\Big]-\hat{z}\Big[\nabla^2 A_z\Big]\end{aligned}$$

which works out to

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}\,) = \ \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\Big[\vec{\nabla}\cdot\vec{A}\Big]-\nabla^2\Big(\hat{x}A_x+\hat{y}A_y+\hat{z}A_z\Big)$$

and so finally

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}\,) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

which is the relation we set out to prove.

3. Prove the vector identity

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

You may do so by explicit computation or by using the Levi-Civita notation (*your choice*).

Solution: Explicit computation would require a rearrangement of terms, and as long as there are no errors in signs and writing components, it should work out fine. Here, I'll present a proof using Levi-Civita notation.

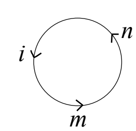
From the Class Summary for Week 1—Day 1, we know that

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = A_i\, \delta_{ij}\, (\epsilon_{mnj}\, B_m C_n)$$

Clearly, we seek a rearrangement of vector components to get the solution, but we have to be careful. Remember, repeated indices imply summation, and so there are summations present in the above over i, m, and n; I'm not mentioning j because the Kronecker δ-function, δ_{ij}, forces terms with $i \neq j$ to go to zero. Knowing this, let us rearrange the order of writing:

$$\begin{aligned}
\vec{A} \cdot (\vec{B} \times \vec{C}) &= A_i\, \delta_{ij}\, (\epsilon_{mnj}\, B_m C_n) \\
&= A_i\, (\epsilon_{mni}\, B_m C_n) \\
&= \epsilon_{mni}\, B_m C_n A_i \\
&= B_m\, (\epsilon_{mni} C_n A_i) \\
&= B_m\, (\epsilon_{nim} C_n A_i) \qquad \text{(H1.1)} \\
&= B_m\, \delta_{mj}\, (\epsilon_{nij} C_n A_i) \qquad \text{(H1.2)} \\
&= \vec{B} \cdot (\vec{C} \times \vec{A})
\end{aligned}$$

There are different ways to do the rearrangement, but I've written it so it is, hopefully, easy to follow. On the right, I've included a figure that will help us keep track of the permutations of the indices of the Levi-Civita tensor.



Notice that the Levi-Civita tensor initially has indices (mnj), and hence (mni) because of the δ_{ij}, so that the index of the first vector component, A_i, is the last of the indices in ϵ_{mni}. Therefore, when we rearrange the order of the vector components with B_m coming first to get $\vec{B} \cdot (\vec{C} \times \vec{A})$, we will also need to rearrange the order of the indices in the Levi-Civita tensor and verify that the sign doesn't change. The original order of indices is (mni) and I've drawn this order counterclockwise in the figure above. Thus, if we want B_m first, then C_n, followed by A_i, the indices of the Levi-Civita tensor will need to be (nim), as in equation (H1.1) above. But (nim) is still in counterclockwise order (see figure above), meaning that (nim) is a cyclic permutation, leaving ϵ_{nim} positive, as written in equation (H1.1). In equation (H1.2), I've restored the Kronecker δ-function with the appropriate indices to be consistent with the way the scalar triple product is written.

To make sure you've learned this procedure, go ahead and show that $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B})$ using similar steps as above, and let me know if you have any issues.

4. In spherical coordinates (r, θ, ϕ), a charge Q is uniformly distributed over a spherical shell of radius R. Express this charge distribution as a three-dimensional charge density $\rho(\vec{x})$ using Dirac δ-functions.

$$\rho(\vec{x}) = C\,\delta(r - R)$$

and then determine the constant C by integrating $\rho(\vec{x})$ over all space.

Solution: This is easy enough that we could work it out without any math! The charge is spread over the surface of the sphere, so $Q/4\pi R^2$ is the surface charge density, and the Dirac δ-function $\delta(r - R)$ localizes the charge to the surface of the sphere. Therefore, we know right away that we should get as our answer

$$\rho(\vec{x}) = \frac{Q}{4\pi R^2}\,\delta(r - R)$$

In the spirit of learning about the Dirac δ-function, however, let us work it out explicitly.

Since the charge Q is uniformly distributed over a spherical shell of radius R, we can write the charge density as

$$\rho(\vec{x}) = C\,\delta(r - R)$$

where C is a constant, and we've used the Dirac δ function to localize the charge to the surface of the sphere. We can find C by realizing that integrating $\rho(\vec{x})$ over all space should give back the total charge Q. So

$$\int_V \rho(\vec{x})\,d^3x = Q$$

and since the volume element in spherical coordinates is $d^3x = r^2 dr\,\sin\theta\,d\theta\,d\phi$, we get

$$C \int_0^\infty r^2\,\delta(r - R)\,dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi = Q$$

The limits are the usual ones for spherical coordinates; θ goes from 0 to π, ϕ goes from 0 to 2π, and r can go from 0 to ∞ since the Dirac δ-function has already localized the charge to $r = R$ (i.e., you don't accumulate anything else in the integral).

Remembering that the δ-function picks out the value of the function at a given point a, i.e.,

$$\int f(r)\delta(r - a)\,dr = f(a)$$

since $f(r) = r^2$ in the r-integral, and the given point a is just R, the result of doing the r-integral is just R^2. The θ and ϕ integrals are straightforward; the ϕ-integral gives 2π and the θ-integral gives 2. Thus, after doing the integrals, we get

$$C\Big(R^2\Big)\Big(2\Big)\Big(2\pi\Big) = Q$$

Therefore

$$C = \frac{Q}{4\pi R^2}$$

and so the charge density is

$$\rho(\vec{x}) = \frac{Q}{4\pi R^2}\,\delta(r - R)$$

which is exactly what we predicted!