PHY 411 **Winter 2021**

Week 1—Thursday, Jan 7—Discussion Worksheet

The Maxwell Equations

Let us begin with the Maxwell equations, which provide the basis for describing the behavior of electromagnetic fields. First, let us look at the situation before Maxwell. You know that if you enclose charge q inside a Gaussian surface, you'll get

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

where ρ is the charge density. In writing this equation, we made no distinction between macroscopic and microscopic fields. When an electric field is applied to a medium comprised of a large number of molecules, however, the charges in each molecule will respond to the applied field and distort the molecular charge density. Without going into the details, let's just say this produces an electric polarization $\vec{P}$ — think of dipoles lined up along the field if you want a physical picture. This distortion of the charge density means that if you consider any small volume, there can be a net increase or decrease of charge in that volume. This change in the charge density is accounted for by the introduction of the divergence of $\vec{P}$, so that the equation for the divergence of $\vec{E}$ above now becomes

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho - \vec{\nabla} \cdot \vec{P}}{\epsilon_0} \tag{4.33}$$

1. Defining $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$, where $\vec{D}$ is known as the electric displacement, derive that

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{4.35}$$

$\vec{\nabla} = \hat{x}\left(\frac{\partial}{\partial x}\right) + \hat{y}\left(\frac{\partial}{\partial y}\right) + \hat{z}\left(\frac{\partial}{\partial z}\right)$

$\vec{\nabla} \cdot \vec{D} = \rho \rightarrow \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho \rightarrow \epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho - \vec{\nabla} \cdot \vec{P}$

$\Rightarrow \rightarrow \underline{\epsilon_0 \vec{\nabla} \cdot \vec{E}} + \vec{\nabla} \cdot \vec{P} = \rho$

$\hookrightarrow \vec{\nabla} \cdot (f\vec{A}) = f \vec{\nabla} \cdot \vec{A}$

distributive property

$\vec{\nabla} \cdot (\vec{A} + \vec{B}) = \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{B}$

$\Rightarrow \vec{\nabla} \cdot (\epsilon_0 \vec{E}) + \vec{\nabla} \cdot \vec{P} = \rho$

$\Rightarrow \vec{\nabla} \cdot \underbrace{(\epsilon_0 \vec{E} + \vec{P})} = \rho$

$\Rightarrow \vec{\nabla} \cdot \vec{D} = \rho$

Equation (4.35) is the macroscopic counterpart of $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$. It is the *first of the four* pre-Maxwell equations written by Jackson in equation (6.1) on page 237. Next, for the magnetic case, we get

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{5.75}$$

the *fourth of the four* pre-Maxwell equations written by Jackson in equation (6.1).

We will now work out a relation between $\vec{D}$ and $\vec{E}$, assuming that the response of the system to an applied field is linear, and that the medium is isotropic (i.e., it has the same properties in all directions). With these two assumptions, the polarization $\vec{P}$ is parallel to $\vec{E}$ with a coefficient of proportionality that is independent of direction:

$$\vec{P} = \epsilon_0 \, \chi_e \, \vec{E} \tag{4.36}$$

where ϵ_0 is the (electric) permittivity of free space; χ_e is the electric susceptibility of the medium.

2. Use $\vec{P} = \epsilon_0 \, \chi_e \, \vec{E}$ in the relation $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ to derive that

$$\vec{D} = \epsilon \vec{E} \tag{4.37}$$

where $\epsilon = \epsilon_0 \left(1 + \chi_e\right)$ is called the electric permittivity, and ϵ/ϵ_0 is called the dielectric constant of the medium.

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\Rightarrow \vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$

$$\Rightarrow \vec{D} = \epsilon_0 (\vec{E} + \chi_e \vec{E})$$

$$\Rightarrow \vec{D} = \epsilon_0 \vec{E} (1 + \chi_e) \Rightarrow \vec{D} = \vec{E} \underbrace{\epsilon_0 (1 + \chi_e)}_{\epsilon}$$

$$\Rightarrow \vec{D} = \epsilon \vec{E}$$

Meanwhile, Faraday's law that changing magnetic fields lead to electric fields:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5.143}$$

gives us the *third of the four* pre-Maxwell equations written by Jackson in equation (6.1).

So far, we have written down the first, third, and fourth of the four pre-Maxwell equations written by Jackson in equation (6.1). We turn now to the second equation in this set of four equations. Begin with the curl equation for the magnetic field:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{5.22}$$

where μ_0 is the permeability of free space and $\vec{J}$ is known as the current density, whose magnitude is measured in units of positive charge crossing unit area per unit time (A/m^2); $\vec{J}$ is in the direction of motion of the charges. Extending equation (5.22) to obtain a macroscopic description involves taking into account that the large number of molecules will each have their own magnetic moments $\vec{m}_i$, which results in an average macroscopic magnetization or magnetic moment density

$$\vec{M}(\vec{x}) = \sum_i N_i \Big\langle \vec{m}_i \Big\rangle \tag{5.76}$$

where N_i is the average number per unit volume of molecules of type i and $\langle \vec{m}_i \rangle$ is the average molecular moment in a small volume at the point $\vec{x}$. The magnetization $\vec{M}$ contributes an effective current density

$$\vec{J}_M = \vec{\nabla} \times \vec{M} \tag{5.79}$$

Additionally, we suppose that there is also a macroscopic current density $\vec{J}(\vec{x})$ due to the flow of charge in the medium. Note that Griffiths calls $\vec{J}_M$ the bound current density (because it is there due to magnetization — "it results from the conspiracy of many aligned atomic dipoles") and $\vec{J}$ the free current (because it involves actual transport of charge).

Extending equation (5.22) to the macroscopic case, we get that

$$\vec{\nabla} \times \vec{B} = \mu_0 \Big[\vec{J} + \vec{J}_M \Big]$$

3. Show that if we define $\vec{H} = \dfrac{1}{\mu_0}\vec{B} - \vec{M}$, the equation above can be used to derive Ampere's Law:

$$\vec{\nabla} \times \vec{H} = \vec{J} \tag{5.82}$$

$$\underbrace{\frac{1}{\mu_0}\vec{\nabla} \times \vec{B}} = \vec{J} + \vec{J}_m$$

$$= \vec{J} + \vec{\nabla} \times \vec{M}, \text{ using } (5.79)$$

Thus, $\vec{\nabla} \times \frac{1}{\mu_0}\vec{B} - \vec{\nabla} \times \vec{M} = \vec{J}$

$$\Rightarrow \vec{\nabla} \times \underbrace{\left[\frac{1}{\mu_0}\vec{B} - \vec{M}\right]} = \vec{J}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

Equation (5.82) is the *second of the four* pre-Maxwell equations written by Jackson in equation (6.1) on page 237. As in the case of electrostatics, we need a relation between $\vec{B}$ and $\vec{H}$, and for linear media, we have $\vec{B} = \mu\vec{H}$ (equation 5.84 in Jackson), where μ is the magnetic permeability of the medium.

We have now written all four of the pre-Maxwell equations written in equation (6.1) in Jackson. However, they are not a consistent set of equations, at least not when time-varying fields are taken into account. The main culprit is Ampere's law — equation (5.82) on the previous page.

4. Take the divergence of both sides of Ampere's law, $\vec{\nabla} \times \vec{H} = \vec{J}$, and show that this leads to

$$\vec{\nabla} \cdot \vec{J} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

Take the Divergence of Both sides

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}$$

Divergence of a Curl is Zero

i.e., $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ for any vector $\vec{A}$

Applying this result, the left hand side

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}$$

Thus,

$$\vec{\nabla} \cdot \vec{J} = \emptyset$$

Remember, of course, that it was actually the other way around; Ampere's law was derived based on the assumption of magnetostatics that $\vec{\nabla} \cdot \vec{J} = 0$.

So, how should one proceed to rectify matters? Recall that a current corresponds to charges in motion and is described by $\vec{J}$ — the current density measured as the amount of charge crossing unit area per unit time. This suggests that there is a relation between the charge density ρ and the current density $\vec{J}$, and indeed there is one! Conservation of charge requires the charge density at any point in space to be related to the current density in the neighborhood of that point by a continuity equation

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{6.3}$$

Equation (6.3) tells us that a decrease in charge inside a small volume as time passes must correspond to a flow of charge out through the surface of that small volume, thereby conserving the total amount of charge. Equation (6.3) also tells us why $\vec{\nabla} \cdot \vec{J} = 0$ in magnetostatics — because in magnetostatics (what Jackson calls "steady-state phenomena"), there is no change in the net charge density anywhere in space.

5. Starting from the continuity equation: $\vec{\nabla}\cdot\vec{J}+\dfrac{\partial\rho}{\partial t}=0$, show that you get a vanishing divergence relation

$$\vec{\nabla}\cdot\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)=0 \qquad \vec{\nabla}\cdot\vec{J}=0 \tag{6.4}$$

We know $\rho = \vec{\nabla}\cdot\vec{D}$

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J}+\left(\vec{\nabla}\cdot\frac{\partial\vec{D}}{\partial t}\right)=0$$

$$\Rightarrow \vec{\nabla}\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)=0$$

With this vanishing divergence available, Maxwell replaced $\vec{J}$ in Ampere's law by its generalization

$$\vec{J}\rightarrow\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

for time-dependent fields. With this replacement, Ampere's law, the second of the four relations in equation (6.1), became

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \tag{6.5}$$

Equation (6.5) is still the same experimentally verified law for magnetostatic phenomena, but is now mathematically consistent with the continuity equation (6.3) for time-dependent fields. Maxwell gave the additional term in equation (6.5) the name *displacement current*. The presence of this term implies that *a changing electric field causes a magnetic field*, even if a current is not present — it is the converse of Faraday's law. This additional term is of crucial importance for rapidly fluctuating fields, for without it there would be no electromagnetic radiation!

We now have the full set of four equations that are known today as the ***Maxwell equations***.

$$\begin{aligned} \vec{\nabla}\cdot\vec{D}&=\rho & \qquad \vec{\nabla}\times\vec{H}&=\vec{J}+\frac{\partial\vec{D}}{\partial t} \\ \vec{\nabla}\cdot\vec{B}&=0 & \qquad \vec{\nabla}\times\vec{E}&=-\frac{\partial\vec{B}}{\partial t} \end{aligned} \tag{6.6}$$

6. In the absence of sources (ρ and $\vec{J}$), the Maxwell equations are

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

(a) Assuming solutions with harmonic time dependence: $\vec{E} = \vec{E}\,e^{-i\omega t}$, etc., show that the two inhomogenous equations above take the form

$$\vec{\nabla}\times\vec{H} = -i\omega\vec{D} \qquad \text{and} \qquad \vec{\nabla}\times\vec{E} = i\omega\vec{B}$$

$$\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}e^{-i\omega t}}{\partial t} \qquad \vec{\nabla}\times\vec{E} = \frac{-\partial\vec{B}e^{-i\omega t}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\times\vec{H} = \vec{D}(-i\omega) \qquad \Rightarrow \vec{\nabla}\times\vec{E} = \vec{B}(i\omega)$$

(b) With $\vec{D} = \epsilon\vec{E}, \vec{B} = \mu\vec{H}$, show that the above equations become

$$\vec{\nabla}\times\vec{B} + i\omega\mu\epsilon\vec{E} = 0 \qquad \text{and} \qquad \vec{\nabla}\times\vec{E} - i\omega\vec{B} = 0$$

$\vec{\nabla}\times\vec{H} = -i\omega\vec{D}$

$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu}\right) = -i\omega(\epsilon\vec{E})$

Thus, $\vec{\nabla}\times\vec{B} = -i\omega\mu\epsilon\vec{E}$

$\vec{\nabla}\times\vec{E} = i\omega\vec{B}$

$\Rightarrow \vec{\nabla}\times\vec{E} - i\omega\vec{B} = 0$

(c) Starting from the above equations, derive the Helmholtz equation for $\vec{E}$:

$$\left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{E} = 0$$

Take equation $\vec{\nabla}\times\vec{E} - i\omega\vec{B} = 0$

So, $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) - i\omega(\vec{\nabla}\times\vec{B}) = 0$

$\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}) - \nabla^2\vec{E} - i\omega(\vec{\nabla}\times\vec{B}) = 0$

No Sources present $\to \rho = 0 \to$ $\vec{\nabla}\cdot\vec{D} = \rho = 0$, $\vec{\nabla}\cdot\vec{E} = 0$

$0 - \nabla^2\vec{E} - i\omega(\vec{\nabla}\times\vec{B}) = 0$

or $-\nabla^2\vec{E} - i\omega\left[-i\omega\mu\epsilon\vec{E}\right] = 0$

or $-\nabla^2 E + \underbrace{i^2}_{-1}\omega^2\mu\epsilon\vec{E} = 0$

$\Rightarrow (\nabla^2 + \mu\epsilon\omega^2)\vec{E} = 0$