

## Class Summary—Week 5, Day 1—Tuesday, Feb 2

## High Frequency Limit

As a reminder, we are discussing the propagation of plane waves in a dispersive medium in which the dielectric constant  $\epsilon(\omega)/\epsilon_0$  is a complex quantity.

We have modeled the problem by considering the electrons in a substance as harmonic oscillators of mass  $m$  and various binding frequencies  $\omega_j$  subjected to a driving force due to the electric field  $\vec{E}(\vec{x}, t)$ , with the interactions between electrons and ions that prevent a perpetual oscillatory motion modeled by a damping constant  $\gamma$ . Stipulating  $f_j$  oscillators with binding frequencies  $\omega_j$ , so that  $\sum_j f_j = Z$ , the number of electrons per molecule, we get the dielectric constant:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \left[ \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \right] \quad (7.51)$$

In the previous class, we discussed what happens at the lower limits  $\omega \ll \omega_j$ . Now, we will discuss what happens in the high frequency limit.

At frequencies far above the highest resonant frequency ( $\omega \gg \omega_j$ ), the dielectric constant takes on a simple form, *as you showed on today's worksheet*

$$\frac{\epsilon(\omega)}{\epsilon_0} \simeq 1 - \frac{\omega_P^2}{\omega^2} \quad (7.59)$$

where

$$\omega_P^2 = \frac{NZe^2}{\epsilon_0 m} \quad (7.60)$$

where  $N$  is the number of molecules per unit volume, and  $Z$  is the number of electrons per molecule; thus  $NZ$  is the total number of electrons per unit volume. This  $\omega_P$  is called the **plasma frequency** of the medium; notice that it depends only on  $NZ$ , the total number of electrons per unit volume.

The wave number in this high frequency limit is given, *as you showed in the worksheet for today*, by

$$ck = \sqrt{\omega^2 - \omega_P^2} \quad (7.61)$$

Sometimes, equation (7.61) is written as

$$\omega^2 = \omega_P^2 + c^2 k^2$$

This is called a **dispersion relation** for  $\omega = \omega(k)$ .

In dielectric media, equation (7.59) applies only for  $\omega^2 \gg \omega_P^2$ . The dielectric constant is then close to unity, although slightly less, meaning that it is essentially real. It also increases with frequency somewhat (e.g., see the highest frequency part of the upper curve for  $\text{Re } \epsilon(\omega)$  in Figure 7.8 on page 311 in Jackson).

In certain situations, such as the ionosphere or a tenuous plasma in the lab, all the electrons are essentially “free” which means that the damping is negligible (recall that we wrote a damping constant  $\gamma$  to account for such damping when we wrote the equation of motion for the electron). In that case, equation (7.59) can hold for a wide range of frequencies, including  $\omega < \omega_P$ .

Equation (7.61) tells us that for frequencies lower than the plasma frequency ( $\omega < \omega_p$ ), the wave number is purely imaginary. Therefore, waves incident on a plasma are reflected and the fields inside fall off exponentially with distance from the surface. Such an expulsion of electric flux from within a plasma is a well known effect in controlled thermonuclear processes.

At  $\omega = 0$ , the attenuation constant is

$$\alpha_{\text{plasma}} \simeq \frac{2\omega_p}{c} \quad (7.62)$$

This is easy to show (*as you did on today's worksheet*) using the expression we wrote for  $k$  in equation (7.53):

$$k = \beta + i \frac{\alpha}{2} \quad (7.53)$$

For purely imaginary wave numbers such as we get in a plasma when  $\omega < \omega_p$ , we would have

$$k_{\text{plasma}} \simeq i \left( \frac{\alpha_{\text{plasma}}}{2} \right)$$

so that

$$k_{\text{plasma}}^2 \simeq i^2 \frac{\alpha_{\text{plasma}}^2}{4} = -\frac{\alpha_{\text{plasma}}^2}{4}$$

Substitute this into equation (7.61):  $ck = \sqrt{\omega^2 - \omega_p^2}$ , to get

$$c^2 \left( -\frac{\alpha_{\text{plasma}}^2}{4} \right) \simeq \omega^2 - \omega_p^2$$

At  $\omega = 0$ , this would be

$$c^2 \left( -\frac{\alpha_{\text{plasma}}^2}{4} \right) \simeq -\omega_p^2$$

which gives

$$\alpha_{\text{plasma}}^2 \simeq \frac{4\omega_p^2}{c^2}$$

The square root of this expression is equation (7.62).

**Space left blank for student notes; class summary continues on next page.**

## Reflectivity in Metals

The reflectivity of metals at optical and higher frequencies is caused by essentially the same behavior as for the tenuous plasma.

Recall that we showed in equation (7.56) that the permittivity of a metal can be written as the sum of the permittivity of bound electrons  $\epsilon_b(\omega)$  and that due to free electrons:

$$\epsilon(\omega) = \epsilon_b(\omega) + i \left[ \frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \right] \quad (7.56)$$

where  $f_0$  is the number of free electrons per molecule,  $N$  is the number of molecules per unit volume, and  $\gamma_0$  is the damping constant.

At frequencies for which  $\omega \gg \gamma_0$  (which isn't saying much since, as we have already discussed,  $\gamma_0$  is usually very small compared to optical frequencies), the dielectric constant of a metal given by equation (7.56) above takes the form, *as you showed on today's worksheet*

$$\epsilon(\omega) \simeq \epsilon_b(\omega) - \frac{\omega_p^2}{\omega^2} \epsilon_0 \quad (7.59.a)$$

where  $\omega_p$  is the plasma frequency of the conduction electrons, given by

$$\omega_p = \sqrt{\frac{Ne^2 f_0}{m\epsilon_0}} \quad (7.60.a)$$

As long as  $\omega \ll \omega_p$ , the dielectric constant is negative, implying that  $k$  is purely imaginary. We can see this easily from equation (7.59.a), where the second term on the right dominates for  $\omega \ll \omega_p$ , so that

$$\epsilon(\omega) \simeq \epsilon_b(\omega) - \frac{\omega_p^2}{\omega^2} \epsilon_0 \approx -\frac{\omega_p^2}{\omega^2} \epsilon_0$$

so that the wave number  $k = \omega\sqrt{\mu\epsilon}$  is imaginary. For  $\omega \ll \omega_p$ , therefore, the behavior of light incident on the metal is approximately the same as for the plasma discussed previously — the light penetrates only a short distance into the metal and is almost entirely reflected.

- If the plane electromagnetic wave is traveling in the  $z$ -direction into the conductor, for example, the electric field inside the conductor will decay as  $e^{-\frac{\alpha}{2}z}$  (as derived in more detail previously for an attenuated plane wave). Notice from this expression that  $\alpha/2$  has dimensions of 1/length.
- Therefore, the penetration into the metal is characterized by a parameter known as the **skin depth**  $\delta$ , given by

$$\delta = \frac{2}{\alpha}$$

- To find an expression for  $\delta$ , consider the following: In equation (7.57), we wrote  $\epsilon = \epsilon_b + i\sigma/\omega$ , and so  $k^2 = \omega^2\mu\epsilon = \omega^2\mu(\epsilon_b + i\sigma/\omega)$ . Therefore, the imaginary part of  $k^2$  is  $\omega^2\mu\sigma/\omega = \omega\mu\sigma$ . Therefore

$$\delta = \frac{2}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Finally, when the frequency is increased into the domain where the dielectric constant becomes positive,  $\epsilon(\omega) > 0$ , which is typically in the ultraviolet, metals are suddenly able to transmit light, and become transparent. That's why the Chandra X-ray telescope can't use a mirror to reflect and focus the X-rays at the telescope.

## Wave Superposition and Group Velocity

We have now learned that there are no truly monochromatic waves, and even in the most monochromatic light source, one has to deal with a finite (although perhaps small) spread of frequencies or wavelengths.

In principle, since the basic equations of monochromatic waves are linear, we can just make an appropriate linear superposition of solutions with different frequencies. However, several new features arise:

- If the medium is dispersive, meaning that the dielectric constant is a function of the frequency of the fields, then the phase velocity is no longer the same for each frequency component of the wave. As a result, different components of the wave travel with different speeds and tend to change phase with respect to one another.
- If the medium is dispersive, the velocity of energy flow may differ greatly from the phase velocity, or may even lack precise meaning.
- If the medium is dissipative, a pulse of radiation will be attenuated as it travels with (or without) distortion, depending on whether the dissipative effects are (or are not) functions of frequency.

A general solution taking dispersive and dissipative effects into account can be built up via a Fourier series.

For simplicity, consider plane waves in one-dimension only. The scalar amplitude  $u(x, t)$  will stand in for one of the components of the electromagnetic field (e.g.,  $E_x, E_y$ , etc.).

We wrote the basic solution to the wave equation in equation (7.6):

$$u(x, t) = ae^{ikx - i\omega t} + be^{-ikx - i\omega t} \quad (7.6)$$

whereas the relation between the frequency  $\omega$  of the electromagnetic field and wave number  $k$  is given by equation (7.4):

$$k = \omega \sqrt{\mu\epsilon} \quad (7.4)$$

In making a linear superposition, either  $\omega$  or  $k$  can be thought of as the independent variable — we'll find it convenient in the treatment below to use  $k$  as the independent variable, and consider  $\omega$  as a general function of  $k$  in order to allow for dispersion:

$$\omega = \omega(k) \quad (7.79)$$

Now, the dispersive properties of the medium shouldn't depend on whether the wave travels to the left or to the right, so  $\omega$  must be an even function of  $k$ :  $\omega(-k) = \omega(k)$ .

For most wavelengths,  $\omega$  is a smoothly varying function of  $k$  (as we saw when we discussed normal dispersion), but at certain frequencies there are regions of “anomalous dispersion” where  $\omega$  varies rapidly over a narrow interval of wavelengths.

For the discussion below, we will exclude dissipative effects by assuming that  $k$  and  $\omega(k)$  are real.

Let's use the solution equation (7.6) written above to build up a general solution of the form

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx - i\omega(k)t} dk \quad (7.80)$$

where the factor  $1/\sqrt{2\pi}$  has been inserted to keep the forms of the direct and inverse Fourier transform formulas the same. The amplitude  $A(k)$  describes the properties of the linear superposition of the different waves, and is given by

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, 0) e^{-ikx} dx \quad (7.81)$$

The way this works is that if  $u(x, 0)$  represents a harmonic wave  $e^{ik_0x}$  for all  $x$ ,  $A(k)$  is such that we get a monochromatic traveling wave. If, on the other hand,  $u(x, 0)$  at  $t = 0$  represents a finite wave train with a length of order  $x$  (see Figure 7.13 on page 324 in Jackson), then the amplitude  $A(k)$  is a peaked function with a breadth of order  $\Delta k$  and centered around a wave number  $k_0$ , which is the dominant wave number in the modulated wave  $u(x, 0)$ .

Now, how does the pulse or wave train at  $t = 0$  behave as it begins to move? The different frequency or wave number components in it move at different phase velocities, leading to the original coherence being lost and distorting the pulse shape. Extreme cases of highly dispersive media or sharp pulses with a great spread of wave numbers are difficult to treat. More typical cases where the frequency depends weakly on wave number can be handled by expanding  $A(k)$  in a Taylor series at some value  $k_0$  around which  $A(k)$  is sharply peaked (see pages 324-325 in Jackson if you're interested in the formal mathematical development). Upon doing this, we'll find that, apart from an overall phase factor, the pulse travels along undistorted in shape with a velocity

$$v_g = \left. \frac{d\omega}{dk} \right|_{k_0} \quad (7.86)$$

called the **group velocity**.

Physically, the amplitudes of all the waves have been superposed to produce a wave packet, and the group velocity tells us the velocity at which this wave packet is moving. Each of the individual waves that make up this packet has its own phase velocity, and the group velocity is different from all these phase velocities. Moreover, the amplitude of the wave packet is connected to the energy transported by the wave, so the **energy in the wave is propagated at the group velocity**. This makes sense if you think about it conceptually; a monochromatic wave traveling at phase velocity  $v_p$  keeps the same shape in space and time, so it cannot carry along information, whereas group velocity is related to how a disturbance in the wave propagates and useful information may be conveyed as disturbances relative to a steady-state condition (e.g., amplitude modulation or frequency modulation).

Note that the considerations above usually break down in regions of anomalous dispersion, where the assumptions made to derive equation (7.86) are not valid (because you have a rapid variation of  $\omega$  with  $k$ ) and group velocity is not a useful concept. In fact, if you force the issue and find the group velocity in regions of anomalous dispersion from equation (7.86) above, you will find it to be greater than  $c$  or even negative! This is not a problem with special relativity, however, it is just a problem with you trying to carry through by force a calculation that shouldn't be performed (because the initial assumptions made to generate that calculation are not fulfilled).

## Vector and Scalar Potentials

So far, we have learned about waves propagating through free space and ponderable media. In Chapter 9, we will be discussing the emission of radiation by localized systems of oscillating charge and current densities.

Before starting that discussion, though, we need some information from previous chapters: vector (and scalar) potentials, and Green functions.

From undergrad E&M, you should be familiar with the scalar potential  $\Phi$  from electrostatics. You might also have studied the vector potential  $\vec{A}$  in magnetostatics, but let's introduce both from first principles here.

First, recall Maxwell's equations:

$$\begin{aligned}\vec{\nabla} \cdot \vec{D} &= \rho & \vec{\nabla} \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0\end{aligned}\tag{6.6}$$

Since Maxwell's equation  $\vec{\nabla} \cdot \vec{B} = 0$ , we can define a quantity  $\vec{A}$  called the **vector potential** such that

$$\vec{B} = \vec{\nabla} \times \vec{A}\tag{6.7}$$

based on the identity  $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{P}) = 0$  for any arbitrary vector  $\vec{P}$  (as written on the inside front cover of Jackson).

Then Faraday's law  $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$  can be written as

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = 0$$

so that we get

$$\vec{\nabla} \times \left( \vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0\tag{6.8}$$

This means that the quantity in parentheses in equation (6.8) which has a vanishing curl can be written as the gradient of some scalar function, a scalar potential  $\Phi$ :

$$\begin{aligned}\vec{E} + \frac{\partial \vec{A}}{\partial t} &= -\vec{\nabla} \Phi \\ \text{or} \quad \vec{E} &= -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t}\end{aligned}\tag{6.9}$$

The definition of  $\vec{B}$  and  $\vec{E}$  in terms of the vector potential  $\vec{A}$  and the scalar potential  $\Phi$  in equation (6.7) and equation (6.9) respectively satisfies directly the two homogenous Maxwell equations.

Next, we turn our attention to the inhomogenous equations in equation (6.6), which will determine the dynamic behavior of  $\vec{A}$  and  $\Phi$ . We will do so in the next class.