PHY 411 **Winter 2021**

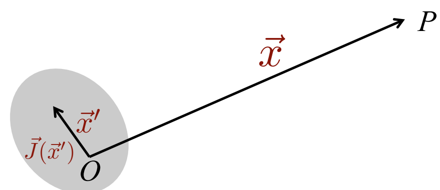
Class Summary—Week 9, Day 1—Tuesday, Mar 2

Magnetic Fields of a Localized Current Distribution

In the previous class, we used $\vec{B} = \vec{\nabla} \times \vec{A}$, and the form of $\vec{B}$ in equation (5.16), to write the vector potential $\vec{A}(\vec{x})$ as

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x' \tag{7.32}$$

Consider a **localized current distribution** as shown in the figure on the right (taken from Figure 5.6 on page 185 in Jackson). Assuming that the distance to the observation point $|\vec{x}| \gg |\vec{x}\,'|$, the dimensions of the source, we will expand the denominator of equation (7.32) in powers of $\vec{x}\,'$ measured relative to a suitable origin in the localized current distribution:



$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{(|\vec{x}|^2 - 2\vec{x}\cdot\vec{x}\,' + |\vec{x}\,'|^2)^{1/2}} = \frac{1}{|\vec{x}|}\left[1 - \frac{2\vec{x}\cdot\vec{x}\,'}{|\vec{x}|^2} + \frac{|\vec{x}\,'|^2}{|\vec{x}|^2} + \ldots\right]^{-1/2} \approx \frac{1}{|\vec{x}|}\left[1 + \frac{\vec{x}\cdot\vec{x}\,'}{|\vec{x}|^2} + \ldots\right]$$

so that

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}\,'}{|\vec{x}|^3} + \ldots \tag{5.50}$$

Then, the ith component of the vector potential will have the expansion

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{1}{|\vec{x}|}\int J_i(\vec{x}\,')\, d^3x' + \frac{\vec{x}}{|\vec{x}|^3}\cdot\int J_i(\vec{x}\,')\,\vec{x}\,'\, d^3x' + \ldots\right] \tag{5.51}$$

The first term in equation (5.51) corresponds to the magnetic field generated by a monopole. However, there are no magnetic monopoles, so this term should be zero. You proved it mathematically *on Question 1 of today's Discussion Worksheet* by showing that

$$\int J_i(\vec{x}\,')\, d^3x' = 0$$

Meanwhile the integral in the second term of equation (5.51) can be written as

$$\vec{x}\cdot\int \vec{x}\,'\, J_i(\vec{x}\,')\, d^3x' = -\frac{1}{2}\left[\vec{x}\times\int\left(\vec{x}\,'\times\vec{J}\right) d^3x'\right]_i$$

as you showed in Question 2 on the Discussion Worksheet for today.

So, if we keep the second term in the expansion in equation (5.51) and drop all higher terms, and put back all three components $i = x, y, z$ to write $\vec{A}(\vec{x})$, the vector potential is

$$\vec{A}(\vec{x}) = -\frac{1}{2}\,\frac{\mu_0}{4\pi}\,\frac{1}{|\vec{x}|^3}\left\{\vec{x}\times\int\left(\vec{x}\,'\times\vec{J}\right) d^3x'\right\}$$

We can get rid of the minus sign by reversing the order of the cross product.

After reversing the order of the cross product to remove the minus sign in the expression written on the previous page, we get the **vector potential**

$$\vec{A}(\vec{x}) = \frac{1}{2}\,\frac{\mu_0}{4\pi}\,\frac{1}{|\vec{x}|^3}\left[\int \left(\vec{x}\,' \times \vec{J}\right) d^3x'\right] \times \vec{x}$$

where we've kept only the second term in the expansion for $\vec{A}(\vec{x})$ and dropped all higher terms.

Now, recall equation (4.10) in Chapter 4, where we wrote the dipole term in the multipole expansion for the electrostatic field; it was

$$\Big[\Phi(\vec{x})\Big]_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\,\frac{\vec{p}\cdot\vec{x}}{|\vec{x}|^3}$$

The expression for $\vec{A}(\vec{x})$ written above looks very similar in form to the electrostatic expression if we define the term in square brackets as a magnetic dipole moment $\vec{m}$. It is customary to define first the *magnetic moment density* or *magnetization* as

$$\vec{\mathcal{M}}(\vec{x}) = \frac{1}{2}\left[\vec{x} \times \vec{J}(\vec{x})\right] \tag{5.53}$$

and its integral as the **magnetic dipole moment** $\vec{m}$:

$$\vec{m} = \frac{1}{2}\int \vec{x}\,' \times \vec{J}(\vec{x}\,')\, d^3x' \tag{5.54}$$

so that the expression for $\vec{A}(\vec{x})$ is now

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3} \tag{5.55}$$

This is the lowest nonvanishing term in the expansion of $\vec{A}(\vec{x})$ for a localized steady-state current distribution. The magnetic induction $\vec{B}$ outside the localized source can be calculated directly by evaluating the curl of equation (5.55), and it is given by

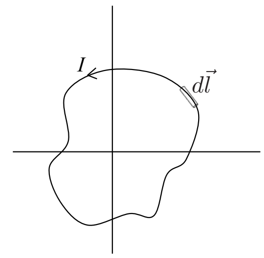
$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}\right] \tag{5.56}$$

where $\hat{n}$ is a unit vector in the direction of $\vec{x}$. The magnetic induction $\vec{B}$ has exactly the form of the field of a dipole. Far away from any localized current distribution, $\vec{B}$ is that of a magnetic dipole with dipole moment given by equation (5.54).

On the next page, we will interpret the expression for the magnetic moment.

Space left blank for student notes; class summary continues on next page.

If the current is confined to a plane, but otherwise arbitrary loop (see figure on the right), the magnetic moment can be expressed in a simple form. If the current I flows in a closed circuit whose line element is $d\vec{l}$, equation (5.54) becomes



$$\vec{m} = \frac{I}{2} \oint \vec{x} \times d\vec{l}$$

This is easy to derive, *as you did in Question 3(b) on today's worksheet*): we've just replaced $\vec{J}(\vec{x}\,')\,d^3x'$ by $Id\vec{l}$ in equation (5.54). Note that Jackson has replaced primed variables with unprimed ones, which is confusing; remember, however, that they are just variables of integration, so you can make such replacements. The variables still refer to the primed ones, however, i.e., the integrand is still $\vec{x}\,' \times d\vec{l'}$. I wrote this with primed variables on the worksheet to avoid confusion.

For a plane loop such as the one shown above, the magnetic moment is perpendicular to the plane of the loop.

Moreover, since $\frac{1}{2}|\vec{x} \times d\vec{l}| = da$, where da is the triangular element of the area defined by the two ends of $d\vec{l}$ and the origin, the loop integral gives the total area of the loop, and so the magnetic moment has the magnitude

$$\left|\vec{m}\right| = I \times (\text{area of the loop}) \tag{5.57}$$

regardless of the shape of the circuit. *This is the solution to Question 3(c) on today's worksheet.*

If the current distribution is provided by a number of charged particles with charges q_i and masses M_i in motion with velocities $\vec{v}_i$, the magnetic moment can be expressed in terms of the orbital angular momentum of the particles. The **current density** is

$$\vec{J} = \sum_i q_i \vec{v}_i \delta(\vec{x} - \vec{x}_i)$$

where $\vec{x}_i$ is the position of the ith particle. Then the magnetic moment in equation (5.54) becomes

$$\vec{m} = \frac{1}{2} \sum_i q_i \Big(\vec{x}_i \times \vec{v}_i\Big)$$

The vector product $(\vec{x}_i \times \vec{v}_i)$ is proportional to the ith particle's orbital angular momentum, $\vec{L}_i = M_i\,(\vec{x}_i \times \vec{v}_i)$. Then, we get

$$\vec{m} = \sum_i \frac{q_i}{2M_i}\,\vec{L}_i \tag{5.58}$$

If all the particles in motion have the same charge-to-mass ratio ($q_i/M_i = e/M$), the magnetic moment can be written in terms of the total orbital angular momentum $\vec{L}$:

$$\vec{m} = \frac{e}{2M} \sum_i \vec{L}_i = \frac{e}{2M}\,\vec{L} \tag{5.59}$$

This is the well-known classical connection between the angular momentum and the magnetic moment, which holds for orbital motion even on the atomic scale. Of course, the classical connection fails for the intrinsic moment of electrons and other elementary particles. For electrons, the intrinsic moment is slightly more than twice as large as implied by equation (5.59), with the spin angular momentum $\vec{S}$ replacing $\vec{L}$, so we say the electron has a g-factor of $2\,(1.00116)$. The departure of the magnetic moment from its classical value has its origins in relativistic and quantum-mechanical effects which we won't discuss here.

Macroscopic Equations

So far, we have assumed that the current density $\vec{J}$ is a known function of position, but in macroscopic problems this is often not true. The electrons in atoms in matter often cause effective atomic currents, and their current density is a rapidly fluctuating quantity. Only its average over a macroscopic volume is known or pertinent. Furthermore, atomic electrons contribute intrinsic magnetic moments in addition to those from their orbital motion; these moments can give rise to dipole fields that vary appreciably on atomic scales.

Just as in electrostatics, the averaging of the microscopic equation $\vec{\nabla}\cdot\vec{B}_{\text{micro}} = 0$ leads to the same equation

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{5.75}$$

for the macroscopic magnetic induction. This is good, because it allows us to use the concept of a vector potential $\vec{A}(\vec{x})$ whose curl gives $\vec{B}$.

The large number of atoms or molecules per unit volume, each with its molecular magnetic moment $\vec{m}_i$ gives rise to an average macroscopic magnetization or magnetic moment density

$$\vec{M}(\vec{x}) = \sum_i N_i\langle\vec{m}_i\rangle \tag{5.76}$$

where N_i is the average number per unit volume of molecules of type i and $\langle\vec{m}_i\rangle$ is the average molecular moment in a small volume at the point $\vec{x}$. Such a magnetization may be induced, or it may already exist if we're dealing with permanent magnets. The freshman-level picture is to think of numerous small regions of magnetic moments that are called domains. If we're not dealing with permanent magnets, then the magnetic moments of these domains point in random directions and cancel each other out on average. However, when an external magnetic field is applied, or a current density creates an applied magnetic field, then the domains feel a torque and align, creating an average magnetization.

In addition to the bulk magnetization, we assume that there is a macroscopic current density $\vec{J}(\vec{x})$ from the flow of free charges in the medium.

Then the vector potential from a small volume ΔV at the point $\vec{x}\,'$ will be

$$\Delta\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{\vec{J}(\vec{x}\,')\,\Delta V}{|\vec{x}-\vec{x}\,'|} + \frac{\vec{M}(\vec{x}\,')\times(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}\,\Delta V\right]$$

The first term is due to the macroscopic current density resulting from the flow of free charges in the medium and the second term is the dipole vector potential that arises due to the existence of an average macroscopic magnetization written in equation (5.76) above. *This is the answer to Question 4 on today's worksheet.*

If we let ΔV become the macroscopically infinitesimal d^3x', then the total vector potential at $\vec{x}$ can be written as the integral over all space:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} + \frac{\vec{M}(\vec{x}\,')\times(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}\right]d^3x' \tag{5.77}$$

Equation (5.77) can be put into the form

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,') + \vec{\nabla}' \times \vec{M}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \, d^3x' \tag{5.78}$$

You derived this expression in Question 5 on today's worksheet.

From equation (5.78), we see that the magnetization is contributing an effective current density

$$\vec{J}_M = \vec{\nabla} \times \vec{M} \tag{5.79}$$

so that the macroscopic equivalent of $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ is

$$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \vec{\nabla} \times \vec{M} \right] \tag{5.80}$$

implying that $(\vec{J} + \vec{J}_M)$ plays the role of the current density in the macroscopic equivalent.

The $\vec{\nabla} \times \vec{M}$ term can be combined with $\vec{B}$ to define a new macroscopic field $\vec{H}$, called the **magnetic field**

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} \tag{5.81}$$

Then, the macroscopic equations are

$$\begin{aligned} \vec{\nabla} \times \vec{H} &= \vec{J} \\ \vec{\nabla} \cdot \vec{B} &= 0 \end{aligned} \tag{5.82}$$

The first is the answer to Question 6(a) on today's worksheet.

To complete the description, there must be a constitutive relation between $\vec{B}$ and $\vec{H}$, analogous to electrostatics. For isotropic diamagnetic and paramagnetic substances, there is a simple linear relation

$$\vec{B} = \mu \vec{H} \tag{5.84}$$

where μ is called the **magnetic permeability**, and is a characteristic parameter of the medium.

Substituting and rearranging equation (5.81), we get

$$\vec{M} = \frac{\vec{B}}{\mu_0} - \vec{H} = \frac{\mu \vec{H}}{\mu_0} - \vec{H} = \left[\frac{\mu}{\mu_0} - 1 \right] \vec{H}$$

which is the solution to Question 6(b) on today's worksheet.

In paramagnetic materials, $\mu > \mu_0$, so

$$\vec{M} = + \left| \frac{\mu}{\mu_0} - 1 \right| \vec{H}$$

Therefore, in a paramagnetic material, the magnetization $\vec{M}$ points in the same direction as $\vec{H}$ and amplifies the applied field. *This is the solution to Question 6(c) on today's worksheet.*

In diamagnetic materials, $\mu < \mu_0$, so

$$\vec{M} = -\left|\frac{\mu}{\mu_0} - 1\right| \vec{H}$$

Therefore, in a diamagnetic material, the magnetization $\vec{M}$ points in the opposite direction as $\vec{H}$ and weakens the applied field. *This is the solution to Question 6(d) on today's worksheet.*

For ferromagnetic substances, equation (5.84) must be replaced by a nonlinear functional relationship

$$\vec{B} = \vec{F}(\vec{H}) \tag{5.85}$$

If you remember the phenomenon of hysteresis that you studied in freshman physics (or, e.g., see Figure 5.8 on page 193 in Jackson if you've forgotten), then you'll agree that $\vec{B}$ for ferromagnetic materials is not a single-valued function of $\vec{H}$.

Boundary Conditions

We've already gone over the boundary conditions while discussing reflection and transmission of waves, so I'll just write them down here for reference.

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0 \tag{5.86}$$

$$\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K} \tag{5.87}$$

where $\hat{n}$ is a unit normal pointing from region 1 to region 2, and $\vec{K}$ is the surface current density.

For linear media with $\vec{B} = \mu\vec{H}$, and when $\vec{K} = 0$, the boundary conditions can also be written as

$$\vec{B}_2 \cdot \hat{n} = \vec{B}_1 \cdot \hat{n} \qquad \text{and} \qquad \frac{1}{\mu_2}\,\vec{B}_2 \times \hat{n} = \frac{1}{\mu_1}\,\vec{B}_1 \times \hat{n} \tag{5.88}$$

or

$$\mu_2\,\vec{H}_2 \cdot \hat{n} = \mu_1\,\vec{H}_1 \cdot \hat{n} \qquad \text{and} \qquad \vec{H}_2 \times \hat{n} = \vec{H}_1 \times \hat{n} \tag{5.89}$$

Methods of solving boundary value problems in magnetostatics

I will go over this section briefly, especially since a couple of the writeup assignments in PHY 420 will be based on the material presented here.

The basic equations of magnetostatics are

$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{H} = \vec{J} \tag{5.90}$$

along with the constitutive relations between $\vec{B}$ and $\vec{H}$. We will now survey different techniques for solving boundary value problems in magnetostatics.

A. Generally applicable method of the vector potential

We can always define a vector potential $\vec{A}(\vec{x})$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

For linear media with $\vec{B} = \mu \vec{H}$, the second equation in (5.90): $\vec{\nabla} \times \vec{H} = \vec{J}$, becomes

$$\vec{\nabla} \times \left(\frac{1}{\mu} \vec{\nabla} \times \vec{A} \right) = \vec{J} \tag{5.91}$$

which can be written as

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} \tag{5.92}$$

and if we choose the Coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$), this becomes

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

which is like equation (5.31), but with a modified current density $(\mu/\mu_0)\vec{J}$.

Solutions of equation (5.92) must be matched across boundary surfaces using appropriate boundary conditions on the magnetic field written in equation (5.88) and equation (5.89).

B. Magnetic Scalar Potential (for $\vec{J} = 0$)

If the current density vanishes in some finite region of space, the second equation in equation (5.90) becomes $\vec{\nabla} \times \vec{H} = 0$. This means that we can introduce a *magnetic scalar potential* Φ_M such that

$$\vec{H} = -\vec{\nabla} \Phi_M \tag{5.93}$$

analogous to $\vec{E} = -\vec{\nabla}\Phi$ in electrostatics.

If we have a linear medium in which $\vec{B} = \mu \vec{H}$, then $\vec{\nabla} \cdot \vec{B} = 0$ can be written as

$$\vec{\nabla} \cdot (\mu \vec{\nabla} \Phi_M) = 0 \tag{5.94}$$

If μ is piecewise constant in each chosen region, then the magnetic scalar potential satisfies the Laplace equation

$$\nabla^2 \Phi_M = 0$$

We can then solve the Laplace equation in the different regions, and connect the solutions via the boundary conditions in equation (5.89), or with the linear media we can write $\vec{B} = -\vec{\nabla}\Phi_M$ with $\nabla^2 \Phi_M = 0$ and use the boundary conditions in equation (5.88).

C. Hard Ferromagnets ($\vec{M}$ given, and $\vec{J} = 0$)

A common practical situation concerns so-called "hard" ferromagnets, in which the magnetization is essentially independent of applied fields for moderate field strengths. Such materials can be treated as if they had a fixed, specified magnetization $\vec{M}(\vec{x})$. See pages 196 and 197 if you're interested.