PHY 411 **Winter 2021**

Week 2—Thursday, Jan 14—Discussion Worksheet

Plane Wave Solutions to Maxwell's Equations

In the previous class, we wrote down the ***Helmholtz wave equation***:

$$\left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{E} = 0$$
$$\left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{B} = 0 \tag{7.3}$$

and learned that plane wave solutions to the Helmholtz wave equation can be written in the form

$$\vec{E}(\vec{x},t) = \vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$
$$\vec{B}(\vec{x},t) = \vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \tag{7.8.a}$$

where $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are ***constant vectors***, and $\hat{n}$ is a ***unit vector*** along the direction of propagation.

The above has been mostly mathematics following from the wave equation. But now, let us apply some physics from the Maxwell equations that we wrote in equation (7.1) of the previous lecture, and see what it tells us about electromagnetic waves in particular.

1. First, consider Gauss' Law $\vec{\nabla}\cdot\vec{E} = 0$ in a source-free region of space (i.e., the right hand side of Gauss' Law is zero because $\rho = 0$). In that case, we get from equation (7.8.a) that

$$\vec{\nabla}\cdot\vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = 0$$

Show by writing the components explicitly that if $\vec{\mathcal{E}}$ is a constant vector, then the expression written above is equivalent to

$$\vec{\mathcal{E}}\cdot\vec{\nabla}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = 0$$

$$\frac{\partial}{\partial x}\left[\mathcal{E}_x \underbrace{e^{i(k\hat{n}\cdot\vec{x}-\omega t)}}\right] + \frac{\partial}{\partial y}\left[\mathcal{E}_y e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right] + \frac{\partial}{\partial z}\left[\mathcal{E}_z e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right]$$

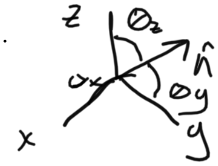
$$\Rightarrow \mathcal{E}_x \frac{\partial}{\partial x}\left[e^{()}\right] + \mathcal{E}_y \frac{\partial}{\partial y}\left[e^{()}\right] + \mathcal{E}_z \frac{\partial}{\partial z}\left[e^{()}\right]$$

$$\Rightarrow \left(\mathcal{E}_x \frac{\partial}{\partial x} + \mathcal{E}_y \frac{\partial}{\partial y} + \mathcal{E}_z \frac{\partial}{\partial z}\right)\left[e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right] = 0$$

$$\Rightarrow \left(\hat{x}\,\mathcal{E}_x + \hat{y}\,\mathcal{E}_y + \hat{z}\,\mathcal{E}_z\right)\cdot\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left[e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right] = 0$$

$$\Rightarrow \vec{\mathcal{E}}\cdot\vec{\nabla}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = 0 \rightarrow \text{proved}$$

2. We will now derive a useful expression and use it to prove an important result.



(a) Show by writing the components explicitly that

$$\vec{\nabla}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = ik\hat{n}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$

This is a very important step, and you should remember to apply it whenever you see such an expression.

$$= \hat{x}\,\frac{\partial}{\partial x}\left[e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right] + \hat{y}\,\frac{\partial}{\partial y}\left[e^{\cdots}\right] + \hat{z}\,\frac{\partial}{\partial z}\left[e^{\cdots}\right]$$

$$= \hat{x}\,\frac{\partial}{\partial x}\left[e^{ik|\hat{n}|x\cos\Theta_x}\, e^{ik|\hat{n}|y\cos\Theta_y}\, e^{ik|\hat{n}|z\cos\Theta_z}\, e^{-i\omega t}\right] + \hat{y}\,\frac{\partial}{\partial y}\left[e^{\cdots}\right] + \hat{z}\,\frac{\partial}{\partial y}\left[e^{\cdots}\right]$$

$$= \hat{x}\left[ik|\hat{n}|\cos\Theta_x\, e^{ik|\hat{n}|x\cos\Theta_x}\, e^{ik|\hat{n}|y\cos\Theta_y}\, e^{ik|\hat{n}|z\cos\Theta_z}\, e^{-i\omega t}\right] + \ldots$$

$$\vec{\nabla} e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = ik\left[\hat{x}\,|n|\cos\Theta_x + \hat{y}\,|\hat{n}|\cos\Theta_y + \hat{z}\,|\hat{n}|\cos\Theta_z\right] e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$

$$= ik\hat{n}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$

(b) Combine your results in Question 1 and part (a) above to show that

$$\hat{n}\cdot\vec{\mathcal{E}} = 0$$

$$\vec{\nabla}\cdot\vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = 0$$

$$\vec{\mathcal{E}}\cdot\vec{\nabla} e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = 0$$

$$\vec{\mathcal{E}}\cdot ik\hat{n}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = 0$$

$$ik\,\hat{n}\cdot\vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = 0$$

$$\Rightarrow \hat{n}\cdot\vec{\mathcal{E}} = 0$$

(c) What does your result in part (b) tell you about the direction of $\vec{E}$ with respect to the direction of propagation $\hat{n}$?

As long as both vectors have non-zero magnitude, which is certainly true in this case, they are perpendicular to eachother. thus, $\vec{E}$ is perpendicular, to the direction of propagation $\hat{n}$.

Repeating the above procedure for $\vec{B}$, we get that $\hat{n}\cdot\vec{\mathcal{B}} = 0$, which tells us that $\vec{B}$ is also perpendicular to the direction of propagation $\hat{n}$.

So, what have we found? Writing both the above equations together:

$$\hat{n}\cdot\vec{\mathcal{E}} = 0 \qquad \text{and} \qquad \hat{n}\cdot\vec{\mathcal{B}} = 0 \tag{7.10}$$

we conclude that *both* $\vec{E}$ and $\vec{B}$ are each perpendicular to the direction of propagation $\hat{n}$.

What else can we learn about $\vec{E}$ and $\vec{B}$?

Putting equation (7.8.a) into the curl equations, say, Faraday's law $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$, and borrowing the result to switch $\vec{\nabla}$ and $\vec{\mathcal{E}}$ from the dot product procedure in Question 1(a), but now switching the sign since we're working with the cross product, we get

$$-\vec{\mathcal{E}} \times \vec{\nabla}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = -\frac{\partial}{\partial t}\left[\vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right]$$

On Homework 2, you will show that this leads to

$$\vec{\mathcal{B}} = \sqrt{\mu\epsilon}\,\left(\hat{n}\times\vec{\mathcal{E}}\right) \tag{7.11}$$

except that in Homework 2, I've written $\hat{n} = \vec{k}/k$ (and the full $\vec{E}$ and $\vec{B}$, instead of just $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$).

Equation (7.11) tells us that $\vec{B}$ is perpendicular to $\vec{E}$. Moreover, we've already shown in equation (7.10) that $\vec{E}$ and $\vec{B}$ are each *separately* perpendicular to the direction of propagation $\hat{n}$. Therefore, we have demonstrated that ***plane electromagnetic waves are transverse waves***, in which the electric and magnetic fields are perpendicular to each other, and each is separately perpendicular to the direction of propagation.

3. Show that equation (7.11) can be written in the following forms.

(a) Show that equation (7.11) can be written as

$$\vec{\mathcal{B}} = \frac{n}{c}\left(\hat{n}\times\vec{\mathcal{E}}\right)$$

Careful! There are two different uses of the symbol in the same formula: n is the index of refraction and $\hat{n}$ is a unit vector specifying the direction of $\vec{k}$.

Since $v = \omega/k = 1/\sqrt{\mu\epsilon} = c/n$ from eq 7.5

$\sqrt{\mu\epsilon} = n/c \qquad \vec{\mathcal{B}} = \sqrt{\mu\epsilon}\,(\hat{n}\times\vec{\mathcal{E}})$

$= \vec{\mathcal{B}} = \frac{n}{c}(\hat{n}\times\vec{\mathcal{E}})$

(b) Show that equation (7.11) can also be written as

$$\vec{\mathcal{H}} = \frac{\hat{n}\times\vec{\mathcal{E}}}{Z}$$

where $Z = \sqrt{\mu/\epsilon}$ is called the impedance.

$\vec{\mathcal{B}} = \mu\vec{\mathcal{H}}$

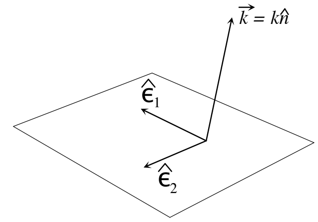
$\vec{\mathcal{B}} = \mu\vec{\mathcal{H}} = \sqrt{\mu\epsilon}\,(\hat{n}\times\vec{\mathcal{E}})$

Thus, $\vec{\mathcal{H}} = \frac{\sqrt{\mu\epsilon}}{\mu}(\hat{n}\times\vec{\mathcal{E}}) = \sqrt{\frac{\epsilon}{\mu}}(\hat{n}\times\vec{\mathcal{E}}) = \frac{\hat{n}\times\vec{\mathcal{E}}}{\sqrt{\mu/\epsilon}}$

$\therefore \vec{\mathcal{H}} = \frac{n\times\vec{\mathcal{E}}}{Z}$

Polarization of Plane Waves

In our discussion so far, we have implicitly assumed that $\hat{n}$ is real, although it doesn't have to be. We'll defer discussion of complex $\hat{n}$ to the next page.



If $\hat{n}$ is real, equation (7.11) implies that we can introduce a set of real mutually orthogonal unit vectors $(\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{n})$, as shown in Figure 7.1 of Jackson (page 297), and reproduced on the right.

We can then use this set of unit vectors to express the field strengths. One possibility is to write

$$\vec{\mathcal{E}} = \hat{\epsilon}_1 E_0 \qquad \text{and} \qquad \vec{\mathcal{B}} = \hat{\epsilon}_2 \sqrt{\mu\epsilon}\, E_0 \tag{7.12}$$

where E_0 is a constant that may be complex.

Another option is

$$\vec{\mathcal{E}} = \hat{\epsilon}_2 E_0' \qquad \text{and} \qquad \vec{\mathcal{B}} = -\hat{\epsilon}_1 \sqrt{\mu\epsilon}\, E_0' \tag{7.12$'$}$$

where, again, E_0' is a constant that may be complex, and the minus sign for $\vec{\mathcal{B}}$ arises from the right-handed nature of the coordinate system (i.e., fingers of the right hand curled from $\vec{E}$ to $\vec{B}$ must advance the thumb in the direction of $\hat{n}$).

4. In Electrodynamics, the so-called Poynting vector points in the direction of the energy flow. Consider the choice of $\vec{\mathcal{E}} = \hat{\epsilon}_1 E_0$ in equation (7.12), where E_0 is a constant that may be complex.

(a) Show that the time-averaged Poynting vector $\vec{S} = \dfrac{1}{2}\vec{E} \times \vec{H}^*$ becomes

$$\vec{S} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left|E_0\right|^2 \hat{n}$$

$$\vec{S} = \frac{1}{2}\left(\hat{\mathcal{E}}_2 E_0' \times [\hat{n} \times \vec{\mathcal{E}}_2 E_0'/Z]\right)$$

$$= \frac{1}{2}\left(\hat{\mathcal{E}}_2 E_0' \times [-\hat{\mathcal{E}}_1 E_0'/Z]\right)$$

$$= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,|E_0|^2\,\hat{n}$$

(b) What does your derived expression for $\vec{S}$ tell you about the ***direction*** of the energy flow?

Direction of energy flow is along $\hat{n}$

Wave propagation is also along $\hat{n}$

In equation (7.12), we wrote the expression for a wave with its *electric field vector always pointing in the direction* $\hat{\epsilon}_1$ (for reference, see the figure on the previous page). Such a wave is said to be ***linearly polarized*** with polarization vector $\hat{\epsilon}_1$. Likewise, the wave described in equation (7.12′) is linearly polarized with polarization vector $\hat{\epsilon}_2$. These ($\hat{\epsilon}_1$ and $\hat{\epsilon}_2$) are evidently two (linearly) independent polarizations of a transverse plane wave. Let us use this to write the most general homogenous wave propagating in the direction $\vec{k} = k\hat{n}$ by writing

$$\vec{E}_1 = \hat{\epsilon}_1\, E_1\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \text{and} \qquad \vec{E}_2 = \hat{\epsilon}_2\, E_2\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{7.18}$$

with the magnetic field written from equation (7.11) as

$$\vec{B}_1 = \sqrt{\mu\epsilon}\left[\hat{n}\times\vec{E}_1\right] \qquad \text{and} \qquad \vec{B}_2 = \sqrt{\mu\epsilon}\left[\hat{n}\times\vec{E}_2\right]$$

Now combine $\vec{E}_1$ and $\vec{E}_2$ to get the most general homogenous plane wave propagating in the direction $\vec{k} = k\hat{n}$:

$$\vec{E}(\vec{x},t) = \left(\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2\right) e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{7.19}$$

where E_1 and E_2 are, as usual, complex amplitudes to allow for the possibility that fields that are linearly polarized in different directions have different phases. The polarization of the plane wave is then specified by the relative *direction, magnitude, and phase of the electric field component* of the wave. Some well-known cases are described in the Class Summary for today. Also described there are the Stokes parameters, which you'll need for Homework 2.

Complex unit vector $\hat{n}$

In the discussion so far, we've assumed that $\hat{n}$ is a real unit vector. But, in general, $\hat{n}$ could be complex, that is, $\vec{k} = k\hat{n}$ is complex, while the wave number k remains real. We will now examine the implications.

5. If $\hat{n}$ is complex, and written as $\hat{n} = \hat{n}_R + i\hat{n}_I$, then show that the requirement that $\hat{n}\cdot\hat{n} = 1$ leads to the following relations for the real and imaginary parts:

$$n_R^2 - n_I^2 = 1 \qquad \text{and} \qquad \hat{n}_R\cdot\hat{n}_I = 0$$

$\hat{n}$ is a unit vector so $\hat{n}\cdot\hat{n} = 1$

$(\hat{n}_R + i\hat{n}_I)\cdot(\hat{n}_R + i\hat{n}_I) = 1$

$n_R^2 + i^2 n_I^2 = 1 \Rightarrow n_R^2 - n_I^2 = 1$

$\hat{n}_R\cdot\hat{n}_I + \hat{n}_I\cdot\hat{n}_R = 0 \Rightarrow \hat{n}_R\cdot\hat{n}_I = 0$

The second condition shows that $\hat{n}_R$ and $\hat{n}_I$ are perpendicular to each other. Thus, if $\hat{n}$ is complex, then the exponential part of the fields in equation (7.8) becomes

$$e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = e^{-k\hat{n}_I\cdot\vec{x}}\, e^{i(k\hat{n}_R\cdot\vec{x}-\omega t)} \tag{7.15.a}$$

Such a wave is called an *inhomogenous plane wave*. It exhibits exponential growth or decay (due to the $e^{-k\hat{n}_I\cdot\vec{x}}$ term) in some directions, while remaining a "plane wave" in directions perpendicular to them (due to the $e^{i(k\hat{n}_R\cdot\vec{x}-\omega t)}$ term); the "perpendicular" part comes from the second relation above, $\hat{n}_R\cdot\hat{n}_I = 0$.