PHY 411 **Winter 2021**

Homework 5 solutions

1. Choice of topics for **Formal Write-up**.

Solution: Topics have been assigned and instructions posted in D2L.

2. Suppose the scalar potential Φ and vector potential $\vec{A}$ are given by

$$\Phi = 0 \qquad \text{and} \qquad \vec{A} = \hat{y}\, A_0 \sin(kx - \omega t)$$

where A_0 is a constant.

(a) Find the fields $\vec{E}$ and $\vec{B}$.

Solution: We know that $\vec{E} = -\vec{\nabla}\Phi - \dfrac{\partial \vec{A}}{\partial t}$
and so

$$\vec{E} = -\vec{\nabla}(0) - \frac{\partial}{\partial t}\left[\hat{y}\, A_0 \sin(kx - \omega t)\right]$$

so that

$$\vec{E} = 0 - \hat{y}\, A_0(-\omega)\cos(kx - \omega t) = \hat{y}\,\omega A_0 \cos(kx - \omega t)$$

Meanwhile

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & A_0 \sin(kx - \omega t) & 0 \end{vmatrix}$$

so that

$$\vec{B} = \hat{x}\,(0 - 0) + \hat{y}\,(0 - 0) + \hat{z}\left[\frac{\partial}{\partial x} A_0 \sin(kx - \omega t) - 0\right]$$

and thus

$$\vec{B} = \hat{z}\, k A_0 \cos(kx - \omega t)$$

Therefore, $\boxed{\vec{E} = \hat{y}\,\omega A_0 \cos(kx - \omega t)}$ and $\boxed{\vec{B} = \hat{z}\, k A_0 \cos(kx - \omega t)}$.

(b) Discuss whether these fields can represent an electromagnetic wave, and if so, provide a quantitative answer for the direction in which the wave would be traveling.

Solution: Since $\vec{E}$ and $\vec{B}$ are perpendicular to each other, they can in principle represent an electromagnetic wave. To be absolutely sure, you should verify they satisfy Maxwell's equations explicitly. For this homework question, though, the statement in principle will suffice.

If, indeed, $\vec{E}$ and $\vec{B}$ do represent an electromagnetic wave, it would be traveling in the $\hat{y} \times \hat{z}$, or $\hat{x}$ direction.

3. Consider the gauge transformation

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \qquad \text{and} \qquad \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

Suppose you are given the potentials:

$$\Phi(\vec{r}, t) = 0 \qquad \text{and} \qquad \vec{A}(\vec{r}, t) = -\frac{qt}{4\pi\epsilon_0 r^2}\,\hat{r}$$

and suppose the gauge function Λ is given by

$$\Lambda = -\frac{qt}{4\pi\epsilon_0 r}$$

(a) Determine explicitly the fields $\vec{E}$ and $\vec{B}$ corresponding to $\Phi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$.

Note: The cross product in spherical coordinates (r, θ, ϕ) is given on the inside back cover in Jackson, or see the Class Summary for Week 1—Day 1.

Solution: Proceeding in the same manner as in Question 2, we get

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

and thus

$$\vec{E} = -\vec{\nabla}(0) - \frac{\partial}{\partial t}\left[-\frac{qt}{4\pi\epsilon_0 r^2}\,\hat{r}\right] = \frac{q}{4\pi\epsilon_0 r^2}\,\hat{r}$$

Meanwhile,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

and the ***curl*** in spherical coordinates for a vector $\vec{A} \equiv (A_r, A_\theta, A_\phi)$ is given by

$$\vec{\nabla}\times\vec{A} = \hat{r}\,\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \hat{\phi}\,\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

and since the given $\vec{A} = (A_r, 0, 0)$, this reduces to

$$\vec{\nabla}\times\vec{A} = \hat{r}\,(0) + \hat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - 0\right] + \hat{\phi}\,\frac{1}{r}\left[0 - \frac{\partial A_r}{\partial\theta}\right]$$

Notice, however, that $A_r = -qt/4\pi\epsilon_0 r^2$ does not have a term that depends on θ, and neither does it have a term that depends on ϕ. This implies that

$$\frac{\partial A_r}{\partial\theta} = 0 \qquad \text{and} \qquad \frac{\partial A_r}{\partial\phi} = 0$$

which means that

$$\vec{\nabla}\times\vec{A} = 0$$

Therefore, $\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\,\hat{r}}$ and $\boxed{\vec{B} = 0}$.

This tells us that we are dealing with a purely electrostatic field due to a point charge q.

(b) Find the transformed potentials $\vec{A}'$ and Φ' for the gauge function Λ given above.

Solution: The gauge transformation for $\vec{A}$ is $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$, so that

$$\vec{A}' = -\frac{qt}{4\pi\epsilon_0 r^2}\,\hat{r} + \vec{\nabla}\left[-\frac{qt}{4\pi\epsilon_0 r}\right]$$

Only the $\partial/\partial r$ term of the gradient is relevant here; the θ and ϕ differentiations will give zero. Thus

$$\vec{A}' = -\frac{qt}{4\pi\epsilon_0 r^2}\,\hat{r} + \hat{r}\,\frac{\partial}{\partial r}\left[-\frac{qt}{4\pi\epsilon_0 r}\right] = -\frac{qt}{4\pi\epsilon_0 r^2}\,\hat{r} - \hat{r}\,\frac{\partial}{\partial r}\left[\frac{qt}{4\pi\epsilon_0 r}\right]$$

and

$$\vec{A}' = -\frac{qt}{4\pi\epsilon_0 r^2}\,\hat{r} - \hat{r}\left[\frac{qt}{4\pi\epsilon_0}\left(-\frac{1}{r^2}\right)\right]$$

so that we get finally

$$\vec{A}' = -\frac{qt}{4\pi\epsilon_0 r^2}\,\hat{r} + \frac{qt}{4\pi\epsilon_0 r^2}\,\hat{r} = 0$$

The gauge transformation for Φ is $\Phi' = \Phi - \partial\Lambda/\partial t$, so that

$$\Phi' = 0 - \frac{\partial}{\partial t}\left[-\frac{qt}{4\pi\epsilon_0 r}\right] = \frac{q}{4\pi\epsilon_0 r}$$

Therefore, the transformed potentials are $\boxed{\Phi' = \dfrac{q}{4\pi\epsilon_0 r}}$ and $\boxed{\vec{A}' = 0}$.

(c) Determine explicitly the fields $\vec{E}'$ and $\vec{B}'$ corresponding to Φ' and $\vec{A}'$.

Solution: The fields are

$$\vec{E}' = -\vec{\nabla}\Phi' - \frac{\partial \vec{A}'}{\partial t} \qquad \text{and} \qquad \vec{B}' = \vec{\nabla}\times\vec{A}' = 0$$

Again, only the $\hat{r}$-term is relevant in the gradient term $\vec{\nabla}\Phi'$, so that

$$\vec{E}' = -\hat{r}\frac{\partial}{\partial r}\left(\frac{q}{4\pi\epsilon_0 r}\right) = -\hat{r}\frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r^2}\right) = \frac{q}{4\pi\epsilon_0 r^2}\,\hat{r}$$

Therefore, $\boxed{\vec{E}' = \dfrac{q}{4\pi\epsilon_0 r^2}\,\hat{r}}$ and $\boxed{\vec{B}' = 0}$.

(d) Comment on your results, e.g., how is $\vec{E}'$ related to $\vec{E}$, and is this expected? Likewise for $\vec{B}'$ and $\vec{B}$. Physically, what kind of charge distribution do we have here? Could you have figured that by looking at Φ and $\vec{A}$, or did you need to do the gauge transformation to figure this out?

Solution: We see that $\vec{E}' = \vec{E}$ and $\vec{B}' = \vec{B}$. Therefore, the fields are unchanged by the gauge transformation, and thus we have explicitly proved this property of a gauge transformation. We have here a purely electrostatic field of a single charge q. This would be difficult to figure from the potentials Φ and $\vec{A}$, but the gauge transformation lets us see clearly that Φ' is the electrostatic potential due to a charge q.

4. In order to work with wave packets, we built up a superposition of solutions to the wave equation by writing

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx - i\omega(k)t}\, dk \qquad \text{where} \qquad A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx$$

where the amplitude $A(k)$ describes the properties of the linear superposition of the different waves. Consider now an approximately monochromatic plane wave packet in one dimension that has the instantaneous form

$$u(x,0) = \begin{cases} N\, e^{ik_o x} & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

(a) Calculate the wave-number spectrum $|A(k)|^2$ of this packet.

Solution: Due to the nature of $u(x,0)$, this is a straightforward integration:

$$\begin{aligned}
A(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \left[N\, e^{ik_o x}\right] e^{-ikx}\, dx \\
&= \frac{N}{\sqrt{2\pi}} \int_{-a}^{a} e^{i(k_0-k)x}\, dx \\
&= \frac{N}{\sqrt{2\pi}} \left[\frac{e^{i(k_0-k)x}}{i(k_0-k)}\right]_{-a}^{a} \\
&= \frac{N}{\sqrt{2\pi}} \left[\frac{e^{i(k_0-k)a} - e^{-i(k_0-k)a}}{i(k_0-k)}\right] \\
&= \frac{N}{\sqrt{2\pi}} \left[\frac{2i\, \sin(k_0-k)a}{i(k_0-k)}\right] \\
&= \frac{N}{\sqrt{2\pi}} \left[\frac{2\, \sin(k_0-k)a}{(k_0-k)}\right]
\end{aligned}$$

Therefore

$$|A(k)|^2 = \frac{2N^2}{\pi} \left[\frac{\sin^2(k_0-k)a}{(k_0-k)^2}\right]$$

This makes sense; $|u(x,0)|^2$ is like a single-slit, and so its Fourier transform is a sinc pattern.

(b) Draw graphs of $|u(x,0)|^2$ and $|A(k)|^2$. Attach your program and your graph to your submission. *Hand-drawn sketches will not be accepted.*

Solution: Since we are given only symbolic values (e.g., a, N), I'm going to write terms in a convenient way for plotting. First

$$|u(x,0)|^2 = (N\,e^{ik_o x})\,(N\,e^{-ik_o x}) = N^2 \qquad \text{for } |x| < a$$

so that

$$\frac{|u(x,0)|^2}{N^2} = 1 \qquad \text{for } -1 < \frac{x}{a} < 1$$

and zero outside of this range.

Meanwhile for $|A(k)|^2$, multiplying on the right hand side by a^2, we get

$$|A(k)|^2 = \frac{2N^2a^2}{\pi}\left[\frac{\sin^2(k_0-k)a}{[(k_0-k)a]^2}\right]$$

so that

$$\frac{|A(k)|^2}{N^2a^2} = \frac{2}{\pi}\left[\frac{\sin^2(k_0-k)a}{[(k_0-k)a]^2}\right]$$

The plots are shown below. Note that the plot on the right is that of the so-called sinc function, of the form $(\sin x)/x$.