An object of mass m moving in two dimensions in a potential

$$V(r) = \frac{k}{n+1} r^{n+1}$$

has the following Lagrangian (in polar coordinates)

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

If $n = -2$ this potential corresponds to the gravitational or Coulomb potential, while if $n = 1$ it corresponds to the two-dimensional simple harmonic oscillator.

a. Find the conjugate momenta and the Hamiltonian.
b. The angle θ is an ignorable coordinate in the Hamiltonian, so its conjugate momentum, p_θ, is a constant of the motion, and we can replace it with a constant, $\mathcal{L}$, in the Hamiltonian. This reduces the problem to one degree of freedom. From the Hamiltonian, find the equations of motion for (r, p_r).
c. Find the equilibrium solutions for (r, p_r). What shape of trajectory do these equilibria correspond to?

a.

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L = p_r\frac{p_r}{m} + p_\theta\frac{p_\theta}{mr^2} - \frac{m}{2}\left(\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2r^4}\right) + V(r)$$

$$H = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2mr^2} + V(r)$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{k}{n+1}r^{n+1}$$

b.

$$p_\theta = \mathcal{L} = const. \Rightarrow H = \frac{p_r^2}{2m} + \frac{\mathcal{L}^2}{2mr^2} + \frac{k}{n+1}r^{n+1}$$

Equations of motion:

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{\mathcal{L}^2}{mr^3} - kr^n$$

c.

Equilibrium when

$$\dot{r} = \dot{p}_r = 0 \ \Rightarrow p_r = 0, r^{n+3} = \frac{\mathcal{L}^2}{mk} \Rightarrow r = const$$

These are circular trajectories