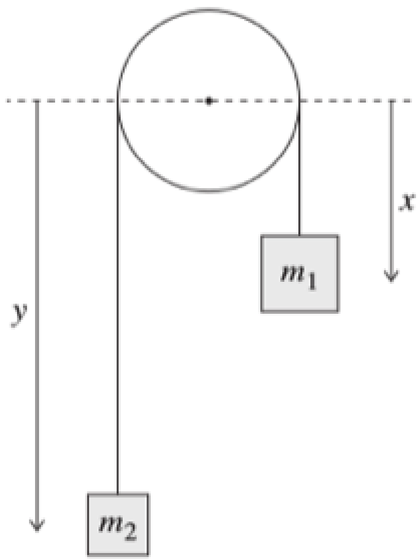
Example 1



$$L = T - V = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_1 g x + m_2 g y$$

$$f = x + y - l = 0$$

If eliminated y using the constraint equation we would get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

But we keep both x and y, so instead we get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \lambda\frac{\partial f}{\partial x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \lambda\frac{\partial f}{\partial y}$$

$$x + y - l = 0$$

$$m_1\ddot{x} - m_1 g = \lambda$$
$$m_2\ddot{y} - m_2 g = \lambda$$
$$x + y - l = 0$$

(1) $m_1\ddot{x} - m_1 g = \lambda$
(2) $m_2\ddot{y} - m_2 g = \lambda$
(3) $x + y - l = 0$

(1) and (2): $m_1\ddot{x} - m_1 g = m_2\ddot{y} - m_2 g$
(3): $m_1\ddot{x} - m_1 g = -m_2\ddot{x} - m_2 g \;\Rightarrow \ddot{x} = \frac{(m_1 - m_2)}{(m_1 + m_2)} g \quad \ddot{y} = -\frac{(m_1 - m_2)}{(m_1 + m_2)} g$
These are the equations of motion of the two blocks

(1): $\lambda = m_1 \frac{(m_1 - m_2)}{(m_1 + m_2)} g - m_1 g = m_1 g \left(\frac{m_1 - m_2}{m_1 + m_2} - 1\right) = m_1 g \left(\frac{(m_1 - m_2) - (m_1 + m_2)}{m_1 + m_2}\right) = -\frac{2 m_1 m_2 g}{m_1 + m_2}$

$$Q_i^{nc} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_i}$$

$$Q_x = \lambda \frac{\partial(x + y - l)}{\partial x} = \lambda = -\frac{2 m_1 m_2 g}{m_1 + m_2}$$

$$Q_y = Q_x$$

This is the tension in the rope, which is the constraint force that is needed to keep the length of the rope constant so that $x + y - l = 0$

Physical interpretation of the Lagrange multipliers:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \lambda \frac{\partial f}{\partial x}$$

We can write it like this:

$$\underbrace{\frac{\partial L}{\partial x}}_{-\frac{\partial V}{\partial x}\ \text{Impressed force}} + \underbrace{\frac{\partial(\lambda f)}{\partial x}}_{\text{Constraint force}} = \underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}}_{m_1 \ddot{x}}$$

λf looks like the potential energy of the constraint force

The constraint force is $Q_x = \lambda \frac{\partial f}{\partial x}$. The partial derivative probes the rate at which the constraint changes when one of the generalized coordinates is changed (in this case x). The more it changes with the coordinates the stronger the constraint force. The Lagrange multiplier λ then scales this partial derivative to obtain the correct constraint force in the direction of the coordinate.

Example 2

(Hamill example 3.1) Consider a disk of radius R rolling down an inclined plane of length l and angle α. Find the equations of motion, the angular acceleration, and the force of constraint. See Figure 3.2.

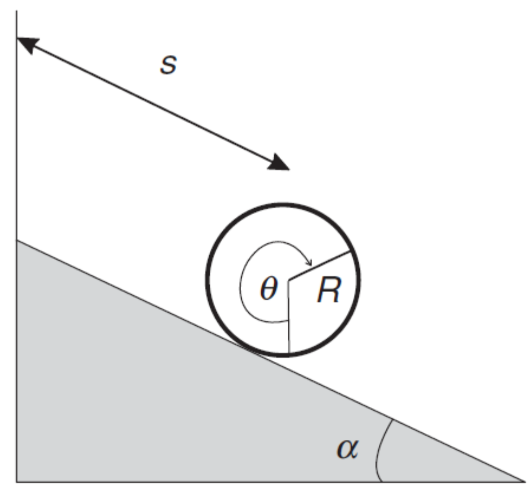


Figure 3.2 A disk rolling without slipping on an inclined plane.

$$L = \frac{1}{2}M\dot{s}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mg(s - l)\sin\alpha .$$

We are not eliminating one of the two coordinates like we normally do (or like you should have done on the exam ;)

Three equations for three unknows:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = \lambda \frac{\partial f}{\partial s} \qquad M\ddot{s} - Mg\sin\alpha = \lambda \qquad (1)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial f}{\partial \theta} \qquad \frac{1}{2}MR^2\ddot{\theta} = -\lambda R \qquad (2)$$

Notice the typo in the textbook (extra exponent 2)

$$f = s - R\theta = 0 \qquad (3)$$

(1) and (2): $M\ddot{s} - Mg\sin\alpha + \frac{1}{2}MR\ddot{\theta} = 0 \quad \ddot{s} - g\sin\alpha + \frac{1}{2}\ddot{s} = 0 \Rightarrow \ddot{s} = \frac{2}{3}g\sin\alpha$

(3): $\ddot{\theta} = \frac{2}{3}\frac{g\sin\alpha}{R}$

(1): $\lambda = -\frac{1}{3}Mg\sin\alpha$

There are two forces of Constraint that are necessary for rolling without slipping: :

$$Q_i^{nc} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_i}$$

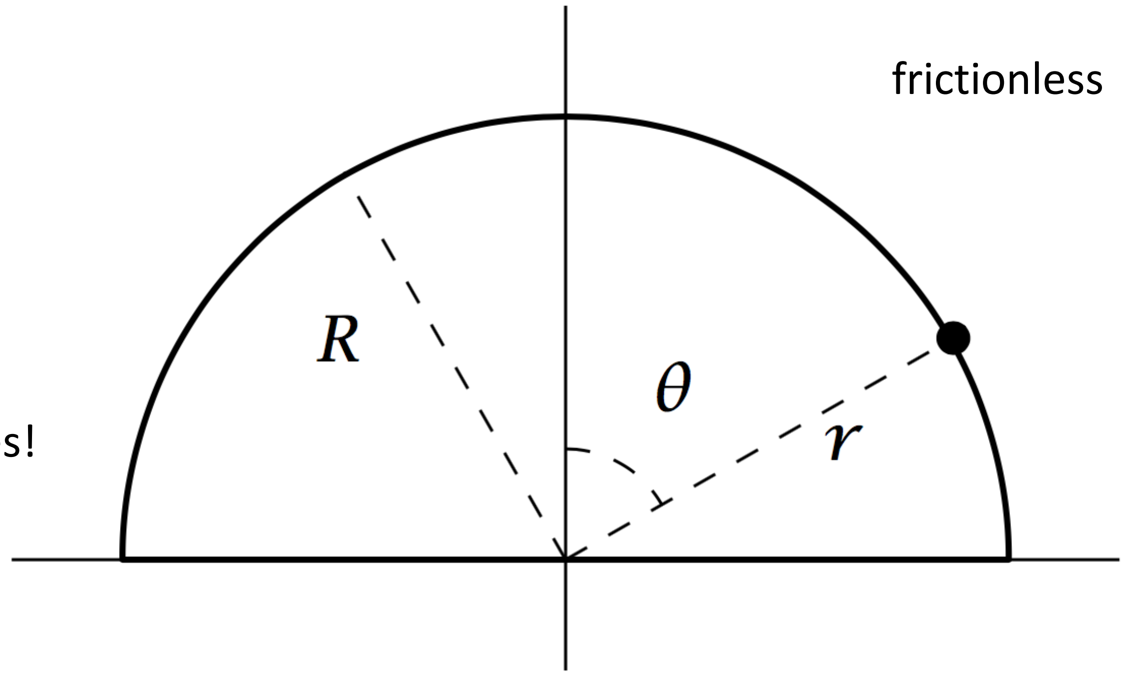
$$Q_s = \lambda \frac{\partial (s - R\theta)}{\partial s} = \lambda = -\frac{1}{3} Mg \sin \alpha$$

This is the force of friction between the ramp and the disk, which is pointing up the ramp. (Note that the book incorrectly calls this the normal force.)

$$Q_\theta = \lambda \frac{\partial (s - R\theta)}{\partial \theta} = -\lambda R = \frac{1}{3} MgR \sin\alpha$$

This is the torque of the friction force on the disk.

Activity 12



Only one degree of freedom, but two generalized coordinates!

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda \frac{\partial f}{\partial r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial f}{\partial \theta}$$

$$f = r - R = 0$$

Strategy:

1. Find the Lagrangian
2. Plug it into the two modified Lagrange equations
3. Solve the to modified Lagrange equations and equation of constraint for λ
4. Plug λ into the equation for the force of constraint in radial direction:

   $Q_r = \lambda \frac{\partial f}{\partial r}$ Before doing any math, what do you guess will happen to Q_r as the particle slips down the hill?
5. What's Q_θ? Why?

How would you solve these two equations so that you get the equation for $\lambda = \lambda(\theta)$ you need for the constraint force $Q_r = \lambda \frac{\partial f}{\partial r}$?

$$-mR\dot{\theta}^2 + mg\cos\theta = \lambda \quad (i)$$

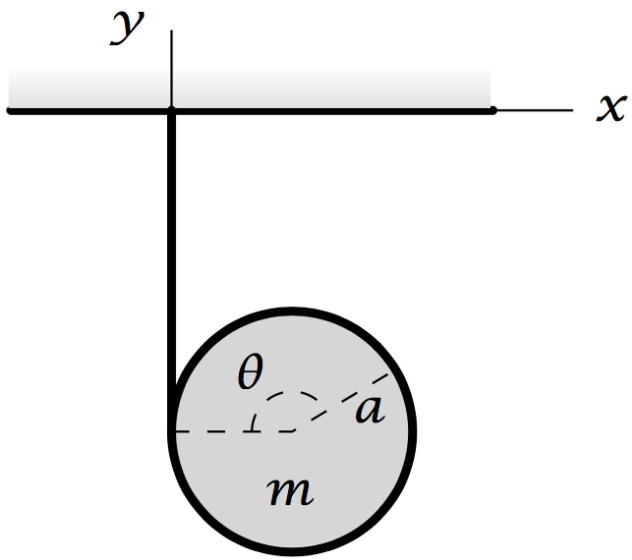
$$\ddot{\theta} - \frac{g}{R}\sin\theta = 0 \quad (ii)$$

Could you just integrate (ii) like this

$\dot{\theta} = \frac{g}{R}\int \sin\theta \, dt$ (and then plug the result into (i)? Why not?

What if you multiplied (ii) by $\dot{\theta}$, and then integrate it?

Activity 13



$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \lambda \frac{\partial f}{\partial y}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial f}{\partial \theta}$$

This is very similar to the disk rolling down the inclined plane!!

Strategy:

1. Find the Lagrangian (remember rotational term)
2. Plug it into the two modified Lagrange equations
3. Solve the to modified Lagrange equations and equation of constraint for λ
4. Plug λ into the equations for F_y and F_θ