Physics 460—A Different Spin Operator Spring 2020

(1) Consider the unit vector $\hat{n} = \sin\theta\,\hat{\imath} + \cos\theta\,\hat{k}$ and its associated spin operator

$$S_{\hat{n}} = \vec{S}\cdot\hat{n} = \sin\theta\, S_x + \cos\theta\, S_z.$$

We'll work with spin-1/2 for the time being.

(a) What physical quantity does $S_{\hat{n}}$ represent?

(b) Write down the representation of $S_{\hat{n}}$ in the z-state basis, $\{|+_z\rangle,\ |-_z\rangle\}$.

(c) Find the eigenvalues and eigenvectors of $S_{\hat{n}}$. (Call them $|+_{\hat{n}}\rangle$ and $|-_{\hat{n}}\rangle$.) Again, express your answers in the z-state basis.

(d) If the state of the system is $|+_x\rangle$ and you measured the physical quantity associated with $S_{\hat{n}}$, what values could you measure and with what probabilities?

(a) It represents the spin of the particle along an axis in the $\hat{n}$ direction: in the x-z plane at an angle θ with respect to the z axis.

(b) For spin-½ we know that

$$S_x \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S_z \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore\ S_{\hat{n}} \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

(c) We know the eigenvalues must be $\pm\hbar/2$

$$\therefore\ |+_{\hat{n}}\rangle \Rightarrow \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = +\frac{\hbar}{2}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\left.\begin{aligned} a\cos\theta + b\sin\theta &= a \\ a\sin\theta - b\cos\theta &= b \end{aligned}\right) \text{re}$$

Pick $a = \sin\theta \Rightarrow \cos\theta + b = 1$

$b = 1 - \cos\theta$

$$|+\hat{n}\rangle \leftrightarrow A\begin{bmatrix}\sin\theta \\ 1-\cos\theta\end{bmatrix}$$

normalize: $A^2\left(\sin^2\theta + (1-\cos\theta)^2\right) = 1$

$$A^2\left(\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta\right) = 1$$

$$A^2\left(2-2\cos\theta\right) = 1$$

$$A = \frac{1}{\sqrt{2-2\cos\theta}}$$

$$|+\hat{n}\rangle \leftrightarrow \frac{1}{\sqrt{2-2\cos\theta}}\begin{bmatrix}\sin\theta \\ 1-\cos\theta\end{bmatrix}$$ when $\theta = \pi/2$ $|+\hat{n}\rangle = |+_x\rangle$ ✓

Similarly, $|-\hat{n}\rangle \leftrightarrow A\begin{bmatrix}\sin\theta \\ -1-\cos\theta\end{bmatrix}$ $\quad A = \frac{1}{\sqrt{2+2\cos\theta}}$

$$|-\hat{n}\rangle \leftrightarrow \frac{1}{\sqrt{2+2\cos\theta}}\begin{bmatrix}\sin\theta \\ -1-\cos\theta\end{bmatrix}$$ when $\theta = \frac{\pi}{2}$ $|-\hat{n}\rangle = |-_x\rangle$ ✓

(d) $|+_x\rangle \leftrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ 1\end{bmatrix}$

$$\langle +\hat{n}|+_x\rangle = \frac{1}{\sqrt{4+4\cos\theta}}\begin{bmatrix}\sin\theta & 1-\cos\theta\end{bmatrix}\begin{bmatrix}1 \\ 1\end{bmatrix}$$

$$= \frac{1}{\sqrt{4+4\cos\theta}}\left(\sin\theta + 1 - \cos\theta\right)$$

$$P(+\hbar/2) = \frac{1}{4+4\cos\theta}\left(\sin\theta + 1 - \cos\theta\right)^2$$

$$\langle -\hat{n} | +_x \rangle = \frac{1}{\sqrt{4+4\cos\theta}} \begin{bmatrix} \sin\theta & -(1+\cos\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{4+4\cos\theta}} \left(\sin\theta - 1 - \cos\theta \right)$$

$$P(-\hbar/2) = \frac{1}{4+4\cos\theta} \left(\sin\theta - 1 - \cos\theta \right)^2$$