PHY 420 **Spring 2021**

Class Summary—Week 2, Day 1—Tuesday, Apr 6

Conservation of Linear Momentum

In the previous class, we discussed the conservation of energy (Poynting's Theorem). Today we will discuss another important conservation law — the **conservation of linear momentum**.

Begin by writing the total electromagnetic force on a charged particle:

$$\vec{F} = q\Big(\vec{E} + \vec{v} \times \vec{B}\Big) \tag{6.113}$$

From Newton's second law, force is the rate of change of momentum, so we can write $\vec{F} = d\vec{P}/dt$. Now, if the sum of all the momenta of all the particles in the volume V is denoted by $\vec{P}_{\text{mech}}$, then we can write equation (6.113) in the form

$$\frac{d\vec{P}_{\text{mech}}}{dt} = \int\limits_V \Big(\rho\vec{E} + \vec{J} \times \vec{B}\Big)\, d^3x \tag{6.114}$$

where we've converted the sum over particles to an integral by replacing $q\vec{v}$ by $\vec{J}\, d^3x$ (as we did before in deriving Poynting's theorem), and this time, also replacing the q (that multiplies $\vec{E}$) by $\rho\, d^3x$.

Again, as we did before for Poynting's theorem, use the Maxwell equations to eliminate ρ and $\vec{J}$ from equation (6.114). So $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ and $\vec{\nabla} \times \vec{H} = \vec{J} + \partial\vec{D}/\partial t$ gives

$$\begin{aligned} \rho &= \epsilon_0\, \vec{\nabla} \cdot \vec{E} \\ \vec{J} &= \frac{1}{\mu_0}\, \vec{\nabla} \times \vec{B} - \epsilon_0\, \frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{6.115}$$

With equation (6.115) substituted into equation (6.114), the integrand becomes

$$\rho\vec{E} + \vec{J} \times \vec{B} = \Big(\epsilon_0\, \vec{\nabla} \cdot \vec{E}\Big)\vec{E} + \left(\frac{1}{\mu_0}\, \vec{\nabla} \times \vec{B} - \epsilon_0\, \frac{\partial\vec{E}}{\partial t}\right) \times \vec{B}$$

or

$$\rho\vec{E} + \vec{J} \times \vec{B} = \epsilon_0\, \vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) - \left[\vec{B} \times \left(\frac{1}{\mu_0}\, \vec{\nabla} \times \vec{B} - \epsilon_0\, \frac{\partial\vec{E}}{\partial t}\right)\right]$$

or

$$\rho\vec{E} + \vec{J} \times \vec{B} = \epsilon_0\, \vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) - \epsilon_0\left[\vec{B} \times \left(\frac{1}{\mu_0\epsilon_0}\, \vec{\nabla} \times \vec{B} - \frac{\partial\vec{E}}{\partial t}\right)\right]$$

and, since $c = 1/\sqrt{\mu_0\epsilon_0}$, we get

$$\rho\vec{E} + \vec{J} \times \vec{B} = \epsilon_0\, \vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) - \epsilon_0\left[\vec{B} \times \left(c^2\, \vec{\nabla} \times \vec{B} - \frac{\partial\vec{E}}{\partial t}\right)\right]$$

so that

$$\rho\vec{E} + \vec{J} \times \vec{B} = \epsilon_0\left[\vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) - c^2\vec{B} \times \left(\vec{\nabla} \times \vec{B}\right) + \vec{B} \times \frac{\partial\vec{E}}{\partial t}\right]$$

as you showed in Question 1 of today's worksheet in class.

Now

$$\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t} \qquad \text{so that} \qquad \vec{B}\times\frac{\partial\vec{E}}{\partial t} = -\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

Replacing $\vec{B}\times\partial\vec{E}/\partial t$ in the expression for $\rho\vec{E}+\vec{J}\times\vec{B}$ written on the previous page, we get

$$\rho\vec{E}+\vec{J}\times\vec{B} = \epsilon_0\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - c^2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right) - \frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) + \vec{E}\times\frac{\partial\vec{B}}{\partial t}\right]$$

Now add $c^2\,\vec{B}(\vec{\nabla}\cdot\vec{B})$ to the square bracket — we can do this without subtracting a term because $\vec{\nabla}\cdot\vec{B}=0$. Thus

$$\rho\vec{E}+\vec{J}\times\vec{B} = \epsilon_0\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) + c^2\,\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right) + \vec{E}\times\frac{\partial\vec{B}}{\partial t}\right] - \epsilon_0\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)$$

where I've written the time derivative of the $(\vec{E}\times\vec{B})$ term separately. *This was Question 2(a) of today's worksheet.*

Use Faraday's law, $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$, to replace $\partial\vec{B}/\partial t$ in the last term inside the square brackets:

$$\rho\vec{E}+\vec{J}\times\vec{B} = \epsilon_0\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) + c^2\,\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right] - \epsilon_0\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)$$

Rearrange terms inside the square brackets:

$$\rho\vec{E}+\vec{J}\times\vec{B} = \epsilon_0\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right) + c^2\,\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)\right] - \epsilon_0\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)$$

and put all this back into the integrand in equation (6.114): $\dfrac{d\vec{P}_{\text{mech}}}{dt} = \displaystyle\int_V \left(\rho\vec{E}+\vec{J}\times\vec{B}\right)d^3x$,

so that

$$\frac{d\vec{P}_{\text{mech}}}{dt} = \epsilon_0\int_V \left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right) + c^2\,\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)\right]d^3x - \epsilon_0\int_V \frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)d^3x$$

Move the last term on the right, the volume integral, to the left hand side and interchange the derivative and integral in it, i*as you did in Question 2(b) of today's worksheet.*:

$$\begin{aligned}\frac{d\vec{P}_{\text{mech}}}{dt} &+ \frac{d}{dt}\int_V \epsilon_0\left(\vec{E}\times\vec{B}\right)d^3x \\ &= \epsilon_0\int_V \left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right) + c^2\,\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)\right]d^3x\end{aligned} \tag{6.116}$$

We will now identify the nature of the volume integral on the left — the second term on the left hand side. For now, because the quantity in the integral on the left has units of momentum density (i.e., momentum per unit volume), let us tentatively call the second term on the left hand side the total **electromagnetic momentum** $\vec{P}_{\text{field}}$ in the volume V:

$$\vec{P}_{\text{field}} = \epsilon_0\int_V \vec{E}\times\vec{B}\,d^3x = \mu_0\epsilon_0\int_V \vec{E}\times\vec{H}\,d^3x \tag{6.117}$$

Putting $\mu_0\epsilon_0 = 1/c^2$, the integrand in equation (6.117) may be interpreted as the **electromagnetic momentum density** (the momentum per unit volume):

$$\vec{g} = \frac{1}{c^2}\left(\vec{E}\times\vec{H}\right) \tag{6.118}$$

Note that this (field) momentum density is proportional to the energy flux density $\vec{S}$ with proportionality constant c^{-2}.

In order to put our interpretation of $\vec{P}_{\text{field}}$ (or momentum density $\vec{g}$) on a firm footing, we will need to convert the volume integral on the right hand side of equation (6.116) into a surface integral of some quantity that can be identified as momentum flow into or out of a volume. To do so, let us begin by letting the Cartesian coordinates by denoted by x_α, with $\alpha = 1, 2, 3$. The $\alpha = 1$ component of the electric part of the integrand in equation (6.116) is given explicitly by

$$\begin{aligned}
&\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]_1 \\
&\qquad = E_1\left(\vec{\nabla}\cdot\vec{E}\right) - \left[\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]_1 \\
&\qquad = E_1\left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right) - \left[E_2\left(\vec{\nabla}\times\vec{E}\right)_3 - E_3\left(\vec{\nabla}\times\vec{E}\right)_2\right] \\
&\qquad = E_1\left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right) - \left[E_2\left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right) - E_3\left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}\right)\right] \\
&\qquad = E_1\frac{\partial E_1}{\partial x_1} + E_1\frac{\partial E_2}{\partial x_2} + E_1\frac{\partial E_3}{\partial x_3} - E_2\frac{\partial E_2}{\partial x_1} + E_2\frac{\partial E_1}{\partial x_2} + E_3\frac{\partial E_1}{\partial x_3} - E_3\frac{\partial E_3}{\partial x_1}
\end{aligned}$$

as you showed in Question 3 of today's worksheet.

Rearrange and manipulate these terms: Write the first term on the right hand side above in the form shown in the first set of square brackets below on the right hand side, gather the second and fifth terms on the right hand side above into the second set of square brackets on the right hand side below, gather the third and sixth terms on the right hand side above into the third set of square brackets on the right hand side below, and write the fourth and seventh terms above at the end below, so that we now have

$$\begin{aligned}
&\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]_1 \\
&\quad = \left[\frac{\partial}{\partial x_1}\left(E_1^2\right) - E_1\frac{\partial E_1}{\partial x_1}\right] + \left[E_1\frac{\partial E_2}{\partial x_2} + E_2\frac{\partial E_1}{\partial x_2}\right] + \left[E_1\frac{\partial E_3}{\partial x_3} + E_3\frac{\partial E_1}{\partial x_3}\right] - E_2\frac{\partial E_2}{\partial x_1} - E_3\frac{\partial E_3}{\partial x_1}
\end{aligned}$$

Then, group the second term in the first set of square brackets on the right hand side above with the last two terms on the right hand side, so that we get

$$\begin{aligned}
&\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]_1 \\
&\quad = \left[\frac{\partial}{\partial x_1}\left(E_1^2\right)\right] + \left[E_1\frac{\partial E_2}{\partial x_2} + E_2\frac{\partial E_1}{\partial x_2}\right] + \left[E_1\frac{\partial E_3}{\partial x_3} + E_3\frac{\partial E_1}{\partial x_3}\right] - E_1\frac{\partial E_1}{\partial x_1} - E_2\frac{\partial E_2}{\partial x_1} - E_3\frac{\partial E_3}{\partial x_1}
\end{aligned}$$

The term in the second set of square brackets on the right hand side above is just $\dfrac{\partial}{\partial x_2}\left(E_1E_2\right)$.

The term in the third set of square brackets on the right hand side above is just $\dfrac{\partial}{\partial x_3}\left(E_1E_3\right)$.

With the groupings identified on the previous page, we now have

$$\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right)-\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]_1$$
$$=\frac{\partial}{\partial x_1}\left(E_1^2\right)+\frac{\partial}{\partial x_2}\left(E_1E_2\right)+\frac{\partial}{\partial x_3}\left(E_1E_3\right)-\left\{E_1\frac{\partial E_1}{\partial x_1}+E_2\frac{\partial E_2}{\partial x_1}+E_3\frac{\partial E_3}{\partial x_1}\right\}$$

You should be able to verify by inspection that the term in curly brackets on the right hand side above can be written in the form below

$$\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right)-\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]_1$$
$$=\frac{\partial}{\partial x_1}\left(E_1^2\right)+\frac{\partial}{\partial x_2}\left(E_1E_2\right)+\frac{\partial}{\partial x_3}\left(E_1E_3\right)-\frac{1}{2}\frac{\partial}{\partial x_1}\left(E_1^2+E_2^2+E_3^2\right)$$

as you demonstrated in Question 4(a) of today's worksheet.

This means that we can write the αth component of the electric part of the integrand in equation (6.116) as

$$\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right)-\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]_\alpha=\sum_\beta\frac{\partial}{\partial x_\beta}\left(E_\alpha E_\beta-\frac{1}{2}\,\vec{E}\cdot\vec{E}\,\delta_{\alpha\beta}\right) \tag{6.119}$$

which you can verify by explicitly writing out the right hand side for $\alpha = 1$ and matching it to the second expression from the top of this page, *as you did in Question 4(b) of today's worksheet.*

Inspection of equation (6.119) shows that we have the right hand side in the form of a divergence, which is what we need in order to write this as a conservation law. You will see when we discuss tensors that the right hand side is in the form of a divergence of a second rank tensor. Meanwhile, since we made the form for $\vec{B}$ similar in equation (6.116), except for a multiplication by c^2, we should expect a similar form to equation (6.119) for $\vec{B}$.

With the above considerations in mind, define the **Maxwell stress tensor** $T_{\alpha\beta}$ as

$$T_{\alpha\beta}=\epsilon_0\left[E_\alpha E_\beta+c^2B_\alpha B_\beta-\frac{1}{2}\left(\vec{E}\cdot\vec{E}+c^2\vec{B}\cdot\vec{B}\right)\delta_{\alpha\beta}\right] \tag{6.120}$$

so that, following equation (6.119), the αth component of the integrand on the right hand side of equation (6.116) becomes

$$\epsilon_0\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right)-\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)+c^2\,\vec{B}(\vec{\nabla}\cdot\vec{B})-c^2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)\right]_\alpha$$
$$=\sum_\beta\frac{\partial}{\partial x_\beta}\left\{\epsilon_0\left[E_\alpha E_\beta+c^2B_\alpha B_\beta-\frac{1}{2}\left(\vec{E}\cdot\vec{E}+c^2\vec{B}\cdot\vec{B}\right)\delta_{\alpha\beta}\right]\right\}$$
$$=\sum_\beta\frac{\partial}{\partial x_\beta}T_{\alpha\beta}$$

Then the αth component of equation (6.116) is

$$\frac{d}{dt}\left[\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_\alpha\right] = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\, d^3x \tag{6.121}$$

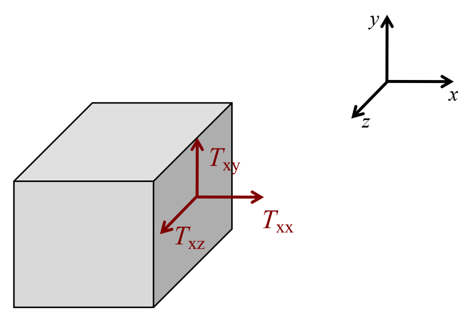
Applying the divergence theorem to the volume integral on the right hand side, we get

$$\frac{d}{dt}\left[\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_\alpha\right] = \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta\, da \tag{6.122}$$

where $\hat{n}$ is the outward normal to the closed surface S.

Let us unpack the meaning of equation (6.122). First, consider $T_{\alpha\beta}$:

The subscripts α and β refer to coordinates x, y, z, and so $T_{\alpha\beta}$ has 9 components. Since the left side of equation (6.122) above is a force, we see that $T_{\alpha\beta}$ tells us the force per unit area acting on the surface. More precisely, $T_{\alpha\beta}$ is the force per unit area in the α^{th} direction acting on an element of the surface oriented in the β^{th} direction. Thus the diagonal elements of $T_{\alpha\beta}$ are *pressures*, and the off-diagonal elements are *shears*, as we can see from the figure on the right.



If equation (6.122) represents a statement of the conservation of momentum, then $\sum_\beta T_{\alpha\beta}\, n_\beta$ is the αth component of the flow per unit area of momentum across the surface S into the volume V.

In other words, it is the force per unit area transmitted across the surface S and acting on the combined system of particles and fields inside V.

The equation (6.122) can therefore be used to calculate the forces acting on material objects in electromagnetic fields by enclosing the objects with a boundary surface S and adding up the total electromagnetic force according to the right hand side of equation (6.122).

The conservation of angular momentum of the combined system of particles and fields can be treated in the same way as energy and linear momentum treated above. Jackson leaves this as Problem 6.10 at the end of the chapter.

The extension of the above treatment to fluids and solids is more difficult because it requires separation of mechanical, thermodynamic, and electromagnetic contributions to momentum. Jackson references a couple of books on this subject if you're interested. Despite this difficulty, the **momentum associated with the electromagnetic field**, $\vec{P}_{\text{field}}$, is still given by equation (6.117):

$$\vec{P}_{\text{field}} = \frac{1}{c^2}\int_V \vec{E}\times\vec{H}\, d^3x$$

and the generally accepted expression for the **momentum density** (i.e., momentum per unit volume) associated with the electromagnetic field, even in a medium, is

$$\vec{g} = \frac{1}{c^2}\,\vec{E}\times\vec{H} = \mu_0\epsilon_0\,\vec{E}\times\vec{H} = \frac{1}{c^2}\,\vec{S} \tag{6.123}$$