Learning goals today

1. Get practice on working with wave equation
2. Introduce the heat equation
3. Introduce successive over relaxation

Work on questions (1) – (3) on the worksheet.

The Heat Equation

$$\nabla^2 u(x,y,z,t) = \frac{1}{c^2} u_t, \quad c^2 = \frac{K}{\sigma\rho}$$

where

- u is the temperature at position (x,y) and time t
- K is the thermal conductivity,
- σ is the specific heat,
- ρ is the mass density

In the steady state, $u_t = 0$ so the heat equation becomes

$u_{xx} + u_{yy} = 0$ Note that there is no time

To make the system discrete, we lay out two grids. One in the *x-direction* and one in the *y-direction.* We do this as follows

$$
\begin{aligned}
x_i &= ih_x, \quad i = 0, 1, \ldots, N_x \\
y_i &= jh_y, \quad j = 0, 1, \ldots, N_y
\end{aligned}
$$

Using the notation $u_{i,j} = u(x_i, y_j)$ we get

$$
\underbrace{\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2}}_{u_{xx}} + \underbrace{\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2}}_{u_{yy}} = 0.
$$

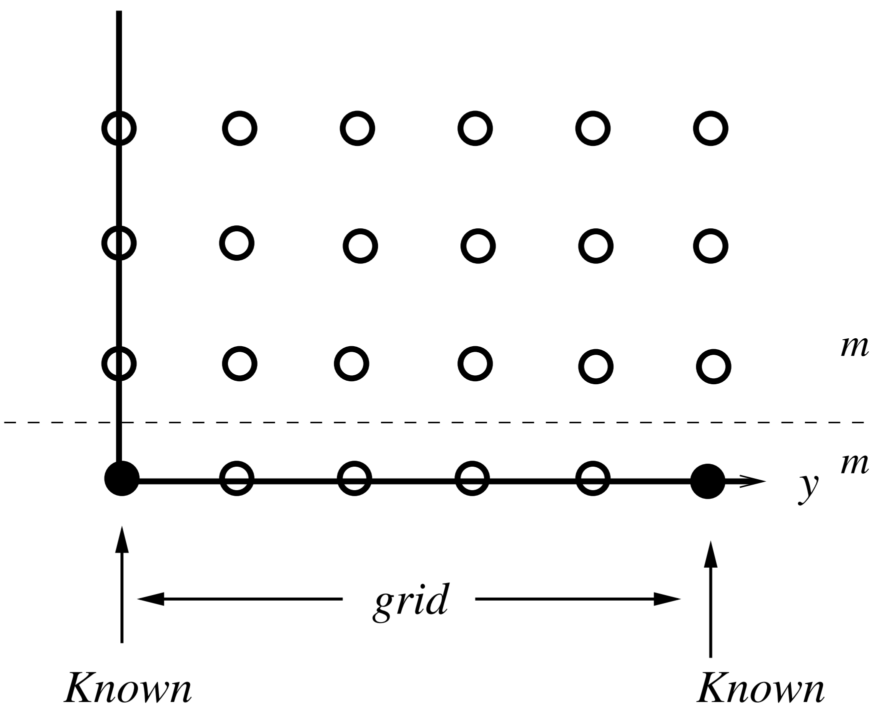
Solving for $u_{i,j}$ we get

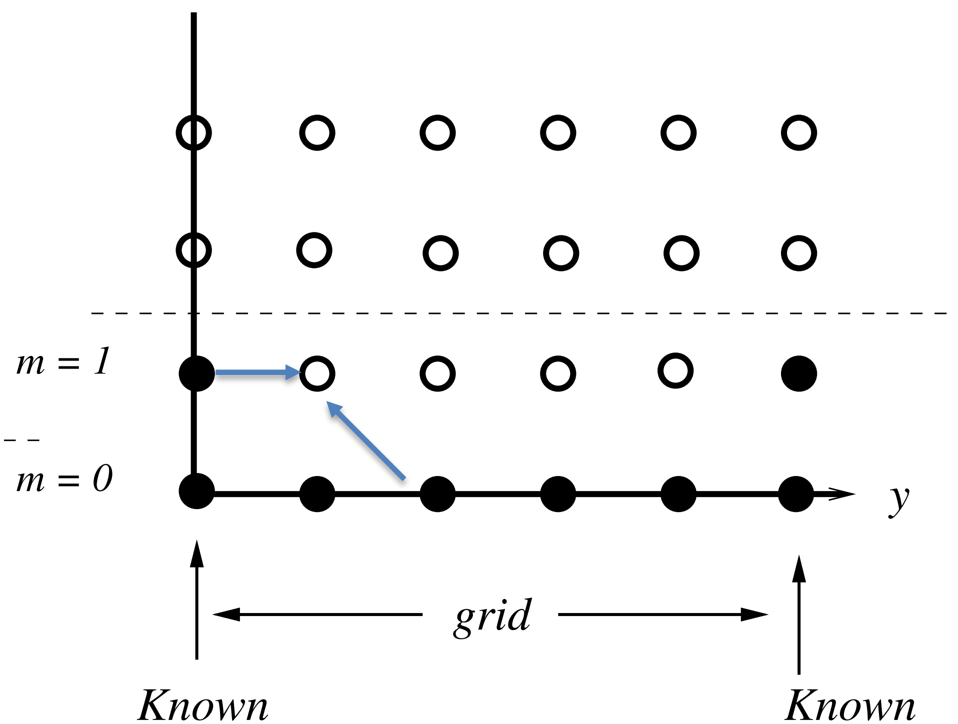
$$
u_{i,j} = \frac{h_x^2 h_y^2}{2h_x^2 + 2h_y^2} \left[\frac{u_{i+1,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} + u_{i,j-1}}{h_y^2} \right].
$$

Do question (4) on the worksheet and S T O P

(4) $$u_{i,j} = \frac{1}{4}\left[u_{i+1,\,j} + u_{i-1,\,j} + u_{i,\,j+1} + u_{i,\,j-1}\right]$$

Typically in these kinds of problems all one knows is the boundary conditions. So to solve the heat equation using finite element, we once again resort to an *iterative* process





It turns out that we can speed up the iterative process by using a technique called:

Successive Over Relaxation (SOR)

Here's the idea: We *guess* that converged result is the most recent result *plus* some factor times the *difference* between the *two* most recent results. Calling the solution, $\bar{y}$, we guess

$$\bar{y}_i^{(j)} = y_i^{(j)} + \alpha \left[y_i^{(j)} - y_i^{(j-1)} \right]$$

The idea:

1. The value of the function at each iteration is found using *normal finite differencing*
2. The value of the function is then modified by *SOR*, and it is this value that is used at the next iteration
3. Iteration ends when some *tolerance is met*.
4. The value α is less than 1, and varies from equation to equation

We are ready to solve the *heat equation*.

1. Initialize the problem
2. Apply finite differencing using $u_{i,j} = \frac{1}{4}\left[u_{i+1,\,j} + u_{i-1,\,j} + u_{i,\,j+1} + u_{i,\,j-1}\right]$
3. Adjust $u_{i,j}$ using *SOR*
4. Repeat until tolerance is met

Do questions (5) and (6) on the worksheet and S T O P

Work on the homework.