Physics 460—Homework Report 3
Due Tuesday, Apr. 21, 1 pm **Name:** ______________________

Complete all the problems on the accompanying assignment.

List all the problems you worked on in the space below. Circle the ones you fully completed:

Please place the problems into the following categories:

- These problems helped me understand the concepts better: ______________________
- I found these problems fairly easy: ______________________
- I found these problems very challenging: ______________________

In the space below, show your work (even if not complete) for any problems you still have questions about. Indicate where in your work the question(s) arose, and ask specific questions that I can answer.

Use the back of this sheet or attach additional paper, if necessary.

If you have no remaining questions about this homework assignment, use this space for one of the following:

- Write one or two of your solutions here so that I can give you feedback on its clarity.
- Explain how you checked that your work is correct.

(1) Suppose that the initial wave function for a particle in a cube is

$$\psi_0(x,y,z) = -Axyz(x-L)(y-L)(z-L),$$

where A is a normalization constant.

(a) Find A.

(b) The wave function can be expressed in terms of the energy eigenstates ψ_{n_x,n_y,n_z} as

$$\psi_0 = c_{n_x,n_y,n_z}\psi_{n_x,n_y,n_z}.$$

Find the coefficients c_{n_x,n_y,n_z}.

(c) If you measure the energy of the particle, what is the probability of obtaining the result $E = 11\hbar^2\pi^2/(2mL^2)$? What is the probability of obtaining the result $E = 27\hbar^2\pi^2/(2mL^2)$? Your answers should be numbers!

My Answer:

(a) The normalization constant is given by

$$\int_0^L \mathrm{d}x \int_0^L \mathrm{d}y \int_0^L \mathrm{d}z\, A^2x^2y^2z^2(x-L)^2(y-L)^2(z-L)^2 = 1.$$

These integrals decouple, and we really have to do one integral:

$$\int_0^L \mathrm{d}x\, x^2(x-L)^2 = \int_0^L \mathrm{d}x\,(x^4 - 2Lx^3 + L^2x^2) = \frac{L^5}{30}.$$

This integral appears three times in the expression above, so

$$A^2\frac{L^{15}}{27000} = 1,$$

or

$$A = \sqrt{\frac{27000}{L^{15}}}.$$

(b) The coefficients are given by

$$c_{n_x,n_y,n_z} = \int_0^L \mathrm{d}x \int_0^L \mathrm{d}y \int_0^L \mathrm{d}z \left(-\sqrt{\frac{27000}{L^{15}}}xyz(x-L)(y-L)(z-L)\right)$$
$$\left(\sqrt{\frac{8}{L^3}}\sin\frac{n_x\pi x}{L}\sin\frac{n_y\pi y}{L}\sin\frac{n_z\pi z}{L}\right).$$

To evaluate this, we need the following integral:

$$\int_0^L x(x-L)\sin\frac{n\pi x}{L}\,\mathrm{d}x = \frac{2L^3}{n^3\pi^3}\left[\cos n\pi - 1\right] = -\frac{4L^3}{n^3\pi^3}$$

for n odd. Therefore,

$$c_{n_x,n_y,n_z} = \sqrt{\frac{27000}{L^{15}}}\sqrt{\frac{8}{L^3}}\frac{4L^3}{{n_x}^3\pi^3}\frac{4L^3}{{n_y}^3\pi^3}\frac{4L^3}{{n_z}^3\pi^3} = \frac{128\sqrt{54000}}{\pi^9{n_x}^3{n_y}^3{n_z}^3},$$

if n_x, n_y, and n_z are all odd, and zero otherwise.

(c) If $E = 11\hbar^2\pi^2/(2mL^2)$, then ${n_x}^2 + {n_y}^2 + {n_z}^2 = 11$. This will occur if two of the quantum numbers are 1 and the third is 3, which can happen three ways. Therefore, the probability of measuring this energy is

$$P = 3|c_{3,1,1}|^2 = 4.10 \times 10^{-3}.$$

If $E = 27\hbar^2\pi^2/(2mL^2)$, then ${n_x}^2 + {n_y}^2 + {n_z}^2 = 27$. This will occur if two of the quantum numbers are 1 and the third is 5, which can happen three ways, or if all three quantum numbers are 3. Therefore, the probability of measuring this energy is

$$P = 3|c_{5,1,1}|^2 + |c_{3,3,3}|^2 = 1.91 \times 10^{-4}.$$

(2) Consider a particle in a cube of sides L in initial state

$$\psi_0(x,y,z) = \begin{cases} A, & 0 \le x \le L,\ 0 \le y \le L/4,\ 0 \le z \le L/4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the normalization constant A.

(b) If you measure the energy of the particle, what are the three most likely results of this measurement, and what is the probability of each of these results?

My Answer:

(a) Similar to question (1), the normalization constant is given by

$$\int_0^L \mathrm{d}x \int_0^{L/4} \mathrm{d}y \int_0^{L/4} \mathrm{d}z\, A^2 = 1.$$

This integral is over the whole cube, but the only regions that contribute are those where the wave function is non-zero. Integrating, we find that

$$\frac{A^2L^3}{16} = 1 \quad \Rightarrow \quad A = \frac{4}{\sqrt{L^3}}.$$

(b) To answer this, we have to expand the initial state as a superposition of energy eigenstates

$$\psi_0 = c_{n_x,n_y,n_z}\psi_{n_x,n_y,n_z}.$$

The coefficients are given by

$$c_{n_x,n_y,n_z} = A\int_0^L \mathrm{d}x \int_0^{L/4} \mathrm{d}y \int_0^{L/4} \mathrm{d}z \sqrt{\frac{8}{L^3}} \sin\frac{n_x\pi x}{L} \sin\frac{n_y\pi y}{L} \sin\frac{n_z\pi z}{L}.$$

Using WolframAlpha to evaluate these integrals, we have

$$\begin{aligned} c_{n_x,n_y,n_z} &= \frac{8\sqrt{2}}{L^3}\left(\frac{2L}{n_x\pi}\sin^2\frac{n_x\pi}{2}\right)\left(\frac{2L}{n_y\pi}\sin^2\frac{n_y\pi}{8}\right)\left(\frac{2L}{n_z\pi}\sin^2\frac{n_z\pi}{8}\right) \\ &= \frac{64\sqrt{2}}{n_xn_yn_z\pi^3}\sin^2\frac{n_x\pi}{2}\sin^2\frac{n_y\pi}{8}\sin^2\frac{n_z\pi}{8}. \end{aligned}$$

The probabilities of measuring a given energy are the squares of these coefficients:

$$\mathcal{P}(n_x,n_y,n_z) = \left(\frac{64\sqrt{2}}{n_xn_yn_z\pi^3}\sin^2\frac{n_x\pi}{2}\sin^2\frac{n_y\pi}{8}\sin^2\frac{n_z\pi}{8}\right)^2.$$

We want the three most probable energies. We probably want to pick smaller integers for the different quantum numbers, but it's not completely obvious to me which values of

n_x, n_y and n_z will give the largest probability. Furthermore, since some of the states are degenerate, we have to add the appropriate probabilities for measuring a given energy.

In order to answer this part, I wrote a little MATLAB program to evaluate this expression for different combinations of the quantum numbers. This program also calculated the corresponding energy, and summed the probabilities as appropriate to account for the degeneracy. The three most probable energies are and their probabilities are

$$E = 26\frac{\hbar^2\pi^2}{2mL^2}, \quad \text{(corresponding to quantum numbers 1, 3, 4)} \quad \mathcal{P} = 0.089,$$

$$E = 14\frac{\hbar^2\pi^2}{2mL^2}, \quad \text{(corresponding to quantum numbers 1, 2, 3)} \quad \mathcal{P} = 0.089,$$

$$E = 21\frac{\hbar^2\pi^2}{2mL^2}, \quad \text{(corresponding to quantum numbers 1, 2, 4)} \quad \mathcal{P} = 0.067.$$

All of these are nominally six-fold degenerate. That is, there are six states that share each of these energies. However, for the second two, only two states contribute to the probability given here, because the expression for the probability above is zero whenever n_x is an even number. Also, as you can see, these three probabilities add up to less than 0.25, so there are plenty of other possible energies that you might obtain from such a measurement.

(3) The Hamiltonian for the isotropic harmonic oscillator in three dimensions in Cartesian coordinates is

$$H = \frac{{P_x}^2 + {P_y}^2 + {P_z}^2}{2m} + \frac{k\left(X^2 + Y^2 + Z^2\right)}{2}.$$

This is an isotropic oscillator because there is a single value of the spring constant k for the springs in all three directions.

(a) Show that the wave functions for the energy eigenstates can be written in the form

$$\psi_n(x, y, z) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z),$$

where $\psi_{n_x}(x)$ *etc.* are the energy eigenstates for the one-dimensional harmonic oscillator.

(b) Show that the energy eigenvalues for these states are

$$E_n = \left(n + \frac{3}{2}\right)\hbar\omega_0,$$

where $n = n_x + n_y + n_z$, and n_x, n_y, and n_z can take on the values 0, 1, 2, etc.

(c) Write out explicitly the wave functions for the energy eigenstates for the four states $(n_x, n_y, n_z) = (0, 0, 0)$, $(n_x, n_y, n_z) = (1, 0, 0)$, $(n_x, n_y, n_z) = (0, 1, 0)$, and $(n_x, n_y, n_z) = (0, 0, 1)$. The relevant wave functions for the one-dimensional oscillator are

$$\psi_0(x) = \frac{\mathrm{e}^{-x^2/2{d_0}^2}}{\sqrt{\sqrt{\pi}d_0}}, \quad \text{and} \quad \psi_1(x) = \sqrt{\frac{2}{\sqrt{\pi}{d_0}^3}}\,x\mathrm{e}^{-x^2/2{d_0}^2}, \quad \text{where} \quad d_0 = \sqrt{\frac{\hbar}{m\omega_0}}.$$

(d) Explain how that you know that the angular momentum eigenstates, $|l, m\rangle$, are also eigenstates of the Hamiltonian.

(e) Express your four states from part (c) in spherical coordinates. Then express the angular dependence in terms of the spherical harmonics.

(f) Using your results from part (e), if you measured the z-component of the angular momentum of each of the four states from part (c), what values could you obtain, and with what probabilities?

My Answer:

(a) If we express the eigenvalue equation for the energy in the position basis, we obtain

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi(x,y,z)+\frac{k}{2}\left(x^2+y^2+z^2\right)\psi(x,y,z)=E\psi(x,y,z).$$

If we make the guess

$$\psi(x,y,z)=\psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z),$$

this equation separates:

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_{n_x}(x)+\frac{kx^2}{2}\psi_{n_x}(x)-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_{n_y}(y)+\frac{ky^2}{2}\psi_{n_y}(y)\\ -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi_{n_z}(z)+\frac{kz^2}{2}\psi_{n_z}(z)=E.\end{aligned}$$

Each of the three pieces on the left is a one-dimensional simple harmonic oscillator:

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_{n_x}(x)+\frac{kx^2}{2}\psi_{n_x}(x)&=E_x,\\ -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_{n_y}(y)+\frac{ky^2}{2}\psi_{n_y}(y)&=E_y,\\ -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi_{n_z}(z)+\frac{kz^2}{2}\psi_{n_z}(z)&=E_z,\end{aligned}$$

where $E_x+E_y+E_z=E$. Therefore, each of $\psi_{n_x}(x)$, $\psi_{n_y}(y)$, and $\psi_{n_z}(z)$ must be an energy eigenstate for a one-dimensional simple harmonic oscillator.

(b) As an alternative to expressing the Hamiltonian in the position basis, we can rewrite it in terms of the raising and lowering operators in each dimension:

$$H=\hbar\omega_0\left(a_x^\dagger a_x+\frac{I}{2}\right)+\hbar\omega_0\left(a_y^\dagger a_y+\frac{I}{2}\right)+\hbar\omega_0\left(a_z^\dagger a_z+\frac{I}{2}\right).$$

The raising and lowering operators in different dimensions commute (because the position and momentum operators in different dimensions commute), so we can make eigenstates that are simultaneously eigenstates of the three different sets of raising and lowering operators. In other words, we can make states $|n_x,n_y,n_z\rangle$ that satisfy

$$\begin{aligned}\hbar\omega_0\left(a_x^\dagger a_x+\frac{I}{2}\right)|n_x,n_y,n_z\rangle&=\hbar\omega_0\left(n_x+\frac{1}{2}\right)|n_x,n_y,n_z\rangle,\\ \hbar\omega_0\left(a_y^\dagger a_y+\frac{I}{2}\right)|n_x,n_y,n_z\rangle&=\hbar\omega_0\left(n_y+\frac{1}{2}\right)|n_x,n_y,n_z\rangle,\\ \hbar\omega_0\left(a_z^\dagger a_z+\frac{I}{2}\right)|n_x,n_y,n_z\rangle&=\hbar\omega_0\left(n_z+\frac{1}{2}\right)|n_x,n_y,n_z\rangle.\end{aligned}$$

Therefore,

$$H|n_x,n_y,n_z\rangle=\hbar\omega_0\left(n_x+n_y+n_z+\frac{3}{2}\right)|n_x,n_y,n_z\rangle=E|n_x,n_y,n_z\rangle.$$

(c) The four desired wave functions are

$$\psi_{000} = \frac{e^{-(x^2+y^2+z^2)/2d_0^2}}{(\sqrt{\pi}d_0)^{3/2}},$$

$$\psi_{100} = \frac{\sqrt{2}x e^{-(x^2+y^2+z^2)/2d_0^2}}{(\sqrt{\pi}d_0)^{3/2}d_0},$$

$$\psi_{010} = \frac{\sqrt{2}y e^{-(x^2+y^2+z^2)/2d_0^2}}{(\sqrt{\pi}d_0)^{3/2}d_0},$$

$$\psi_{001} = \frac{\sqrt{2}z e^{-(x^2+y^2+z^2)/2d_0^2}}{(\sqrt{\pi}d_0)^{3/2}d_0}.$$

(d) If we express the Hamiltonian in spherical coordinates, the potential becomes a function of r alone, because

$$\frac{kx^2}{2} + \frac{ky^2}{2} + \frac{kz^2}{2} = \frac{kr^2}{2}.$$

Therefore, this potential is spherically symmetric, and the angular momentum eigenstates must be eigenstates of this Hamiltonian.

(e) In spherical coordinates,

$$x = r\cos\phi\sin\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\theta, \quad x^2+y^2+z^2 = r^2.$$

Therefore, in spherical coordinates these wave functions are

$$\psi_{000} = \frac{e^{-r^2/2d_0^2}}{(\sqrt{\pi}d_0)^{3/2}},$$

$$\psi_{100} = \frac{\sqrt{2}r\cos\phi\sin\theta\, e^{-r^2/2d_0^2}}{(\sqrt{\pi}d_0)^{3/2}d_0},$$

$$\psi_{010} = \frac{\sqrt{2}r\sin\phi\sin\theta\, e^{-r^2/2d_0^2}}{(\sqrt{\pi}d_0)^{3/2}d_0},$$

$$\psi_{001} = \frac{\sqrt{2}r\cos\theta\, e^{-r^2/2d_0^2}}{(\sqrt{\pi}d_0)^{3/2}d_0}.$$

If we look at the formulas for the spherical harmonics, we see that we need Y_0^0 to make ψ_{000}, Y_1^0 to make ψ_{001}, and $Y_1^{\pm 1}$ to make ψ_{100} and ψ_{010}. In terms of the spherical harmonics, these become

$$\psi_{000} = \frac{\sqrt{4\pi}}{(\sqrt{\pi}d_0)^{3/2}}\, e^{-r^2/2d_0^2}\, Y_0^0(\theta,\phi),$$

$$\psi_{100} = \frac{\sqrt{4\pi/3}}{(\sqrt{\pi}d_0)^{3/2}}\frac{r}{d_0}\, e^{-r^2/2d_0^2}\left(Y_1^1(\theta,\phi) - Y_1^{-1}(\theta,\phi)\right),$$

$$\psi_{010} = \frac{\sqrt{4\pi/3}\mathrm{i}}{(\sqrt{\pi}d_0)^{3/2}}\frac{r}{d_0}\, e^{-r^2/2d_0^2}\left(Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)\right),$$

$$\psi_{001} = \frac{\sqrt{8\pi/3}}{(\sqrt{\pi}d_0)^{3/2}}\frac{r}{d_0}\, e^{-r^2/2d_0^2}\, Y_1^0(\theta,\phi).$$

(f) Looking at the wave functions in part (e), we see that ψ_{000} has $l = 0$ and $m = 0$, so it has no angular momentum. The state ψ_{001} has $l = 1$ and $m = 0$ so it has total angular momentum $\hat{L}^2 = 2\hbar^2$ and z-component zero. The other two states have $l = 1$ and are equal mixtures of $m = \pm 1$, so you would have a 50% probability of measuring either of $\pm\hbar$ for the z-component of the angular momentum for these states.