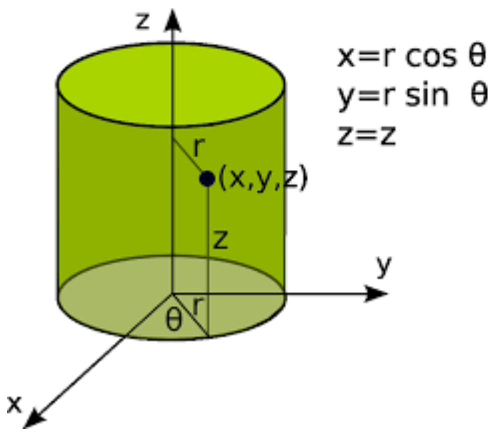
 Physics 440, Autumn 2019

Show that the shortest path between two points on the surface of a cylinder of radius r is

$$\theta(z) = \alpha z + \beta$$



The path length is $L = \int ds$ integrated from point 1 to point 2. $ds = \sqrt{dx^2 + dy^2 + dz^2}$

$$dx = -r\sin\theta\, d\theta \quad dy = r\cos\theta\, d\theta \quad dz = dz$$

$$dx^2 = r^2\sin^2\theta\, d\theta^2 \quad dy^2 = r^2\cos^2\theta\, d\theta^2 \quad dz^2 = dz^2$$

$$ds = \sqrt{r^2\sin^2\theta\, d\theta^2 + r^2\cos^2\theta\, d\theta^2 + dz^2} = \sqrt{r^2 d\theta^2 + dz^2} = \sqrt{r^2\frac{d\theta^2}{dz^2} + 1}\, dz$$

$$= \sqrt{r^2\theta'^2 + 1}\, dz$$

We want to minimize the path length $L = \int ds = \int \sqrt{r^2\theta'^2 + 1}\, dz$

Where $\Phi(\theta, \theta', z) = \sqrt{r^2\theta'^2 + 1}$ is the functional

$\int \sqrt{r^2\theta'^2 + 1}\, dz = \int \Phi(\theta, \theta', z)\, dz$ is minimized if the functional satisfies the Euler Lagrange equation:

$$\frac{d}{dz}\frac{\partial\Phi}{\partial\theta'} = \frac{\partial\Phi}{\partial\theta}$$

$\Phi(\theta, \theta', z)$ does not depend on θ, so $\frac{\partial\Phi}{\partial\theta} = 0$

$$\frac{\partial\Phi}{\partial\theta'} = \frac{\partial\left(\sqrt{r^2\theta'^2 + 1}\right)}{\partial\theta'} = \frac{1}{2}\frac{1}{\sqrt{r^2\theta'^2 + 1}}(2r^2\theta') = \frac{r^2\theta'}{\sqrt{r^2\theta'^2 + 1}}$$

$$\frac{d}{dz}\frac{\partial\Phi}{\partial\theta'} = \frac{d}{dz}\left(\frac{r^2\theta'}{\sqrt{r^2\theta'^2 + 1}}\right) = 0 \Rightarrow \frac{r^2\theta'}{\sqrt{r^2\theta'^2 + 1}} = c \Rightarrow \frac{r^4\theta'^2}{r^2\theta'^2 + 1} = c^2$$

$$\theta'^2 = \frac{c^2(r^2\theta'^2 + 1)}{r^4} = \frac{c^2r^2\theta'^2}{r^4} + \frac{c^2}{r^4} \Rightarrow \left(1 - \frac{c^2}{r^2}\right)\theta'^2 = \frac{c^2}{r^4} \Rightarrow \theta'^2 = \frac{c^2}{r^4 - c^2r^2}$$

$$\Rightarrow \theta' = \sqrt{\frac{c^2}{r^4 - cr^2}} = \text{constant} = \alpha$$

$$\therefore \theta(z) = \alpha z + \beta$$