PHY 420 Spring 2021

Class Summary—Week 4, Day 1—Tuesday, Apr 20

Chapter 9: Radiation

In Chapter 7, we discussed the propagation of electromagnetic waves. In this chapter, we will study how such waves are generated.

We know that radiation is produced by accelerated charges, so we will use a system of oscillating charges as our generator of radiation. To keep things simple, we'll assume that the sources are oscillating in otherwise empty space, meaning that there are no boundaries or materials present.

We already know that, due to our ability to handle the Fourier components of a signal, we lose no generality by assuming a harmonic time dependence for a system of charges and currents that vary in time. So we can write

$$\begin{aligned} \rho(\vec{x},t) &= \rho(\vec{x})\, e^{-i\omega t} \\ \vec{J}(\vec{x},t) &= \vec{J}(\vec{x})\, e^{-i\omega t} \end{aligned} \tag{9.1}$$

All electromagnetic fields and potentials (e.g., $\Phi, \vec{A}, \vec{E}, \vec{B}, \vec{D}, \vec{H}$) are assumed to have the same time dependence.

Now, recall what we learned in PHY 411:

- Starting from the Ampere-Maxwell law in equation (6.6), we obtained **two equations** in terms of the scalar potential Φ and vector potential $\vec{A}$:

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0} \tag{6.10}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right] = -\mu_0\vec{J} \tag{6.11}$$

- Since the fields $\vec{E}$ and $\vec{B}$ are connected to the derivatives of the potentials Φ and $\vec{A}$, we have some freedom in choosing these potentials. In particular, $\vec{B}$ and $\vec{E}$ are left unchanged by the transformation

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \tag{6.12}$$

  and

$$\Phi \to \Phi - \frac{\partial\Lambda}{\partial t} \tag{6.13}$$

- A specific choice of potentials is called a gauge. The Maxwell equations, though, are gauge invariant, meaning that they are valid regardless of what gauge we may decide to choose.

- One particular gauge, called the **Lorenz gauge**, is defined by the Lorenz condition

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0 \tag{6.14}$$

  Use of the Lorenz gauge uncouples the pair of equations (6.10) and (6.11) written above, and leaves two inhomogenous equations, one for Φ and one for $\vec{A}$.

- So, using the Lorenz gauge to uncouple equations (6.10) and (6.11), we get:

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{6.15}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \tag{6.16}$$

- Equations (6.15) and (6.16) are wave equations that have the same basic structure

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(\vec{x}, t) \tag{6.32}$$

- Equation (6.32) is an inhomogenous partial differential equation whose solution can be broken apart into two parts

$$\Psi(\vec{x}, t) = \Psi_h + \Psi_{\text{part}}$$

where Ψ_h is the solution to the homogenous equation — we won't worry about that here, we've already studied the solutions in Chapter 7: transverse plane waves propagating in free space.

- The particular solution Ψ_{part} is what we're interested in here, for it corresponds to sources creating electromagnetic radiation.

- So, how do we find the particular solution, Ψ_{part}? That's where the **Green function** comes in — recall that in the previous class, we wrote that the Green function corresponding to equation (6.32) must satisfy

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_k^{(\pm)}(\vec{x}, t; \vec{x}\,', t') = -4\pi\delta(\vec{x} - \vec{x}\,')\delta(t - t') \tag{6.41}$$

- We then showed that the solution to equation (6.41) gives the Green function

$$G^{(\pm)}(\vec{x}, t; \vec{x}\,', t') = \frac{\delta\left(t' - \left[t \mp \dfrac{|\vec{x} - \vec{x}\,'|}{c}\right]\right)}{|\vec{x} - \vec{x}\,'|} \tag{6.44}$$

Recall that we described in the previous Class Summary how G^{+} is called the **retarded Green function**; it is essentially the electrostatic Green function equation (1.40) from last week's class multiplied by a Dirac δ-function which ensures causality — that is, the effect produced by a source located at position $\vec{x}\,'$ and time t' takes time to propagate and can only be experienced at the observation point $\vec{x}$ at time $t = t' + R/c$, where $R = |\vec{x} - \vec{x}\,'|$. That is, the time difference R/c is the time taken by the disturbance to propagate from the source point to the point of observation, so the effect observed at $\vec{x}$ at time t is caused at the source point $\vec{x}\,'$ at an *earlier* or retarded time $t' = t - R/c$. Meanwhile, G^{-} is called the **advanced Green function**.

- Having written the Green function, we can write the particular solution to equation (6.32) as

$$\Psi^{(\pm)}(\vec{x}, t) = \int\int G^{(\pm)}(\vec{x}, t; \vec{x}\,', t')\, f(\vec{x}\,', t')\, d^3x' dt'$$

as we discussed last week. Recall that $\Psi^{(+)}$ corresponds to the outward going wave.

- Jackson then discusses how we can add solutions of the homogenous equations to either of these solutions.

- Having written the particular solution $\Psi^{(\pm)}(\vec{x},t)$ to the wave equation equation (6.32), Jackson discusses how to add in the homogenous solution for a particular physical problem, e.g., if we have a source distribution $f(\vec{x}\,',t')$ that is localized in time and space, and is different from zero only for a finite interval of time around $t'=0$. In one of two limiting situations, Jackson assumes that at $t\to-\infty$, a wave labeled $\Psi_{\rm in}(\vec{x},t)$ satisfies the homogenous wave equation. As this wave propagates in time and space, the source turns on and generates waves of its own. Thus, the complete solution is

$$\Psi(\vec{x},t) = \Psi_{\rm in} + \int\int G^{(+)}(\vec{x},t;\vec{x}\,',t')\,f(\vec{x}\,',t')\,d^3x'dt' \tag{6.45}$$

  where the second term on the right is taken from $\Psi^{\pm}(\vec{x},t)$ written on the previous page. The presence of G^+ effectively guarantees that at early times before the source has been activated, there is no contribution from the term with the integral on the right hand side, and only $\Psi_{\rm in}$ exists. It is usual to take $\Psi_{\rm in}(\vec{x},t)=0$ in practice, so that we are dealing only with the particular solution.

Now let us apply all this specifically to the **vector potential** $\vec{A}$, for which the wave equation is

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \tag{6.16}$$

On the homework, you will write the Green function corresponding to equation (6.16), and show that the particular solution to equation (6.16) is

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\int dt'\,\frac{\vec{J}(\vec{x}\,',t')}{|\vec{x}-\vec{x}\,'|}\,\delta\left(t' - \left[t - \frac{|\vec{x}-\vec{x}\,'|}{c}\right]\right) \tag{9.2}$$

Notice that we have kept only the outward-going solution (which makes sense, since we're interested in radiation).

In equation (9.1), we wrote the current density with a harmonic time dependence, and remarked that the potentials would have the same time dependence, i.e., we can substitute in equation (9.2) that

$$\vec{J}(\vec{x},t) = \vec{J}(\vec{x})\,e^{-i\omega t} \qquad \text{and} \qquad \vec{A}(\vec{x},t) = \vec{A}(\vec{x})\,e^{-i\omega t}$$

With these substitutions, equation (9.2) becomes

$$\vec{A}(\vec{x})e^{-i\omega t} = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\int dt' e^{-i\omega t'}\,\delta\left(t' - \left[t - \frac{|\vec{x}-\vec{x}\,'|}{c}\right]\right)$$

remembering that we are substituting for $\vec{A}(\vec{x},t)$, i.e., unprimed quantities, and $\vec{J}(\vec{x}\,',t')$, i.e., primed quantities inside the integral.

Now, to carry out the t'-integration, use the property of the δ-function that $\int f(t)\delta(t-a)dt = f(a)$, which gives

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,e^{i\omega|\vec{x}-\vec{x}\,'|/c}$$

as you did on Question 3 of today's worksheet. Notice that after doing the integration, we get a factor $e^{-i\omega t}$ on the right hand side, which can be pulled outside the integral and canceled with the $e^{-i\omega t}$ that was present on the left hand side.

Since $\omega/c = k$, the wave number, the expression for $\vec{A}$ becomes

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}\,') \frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|}\, d^3x' \tag{9.3}$$

In summary, we have solved the wave equation (6.16) using the Green function method to obtain an expression for the vector potential $\vec{A}(\vec{x})$ of a localized system of charges and currents that vary sinusoidally in time.

In fact, we have completed a very important step, because knowing $\vec{A}$ means that we can find $\vec{B}$ from

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

and since we're in free space where $\vec{B} = \mu_0\, \vec{H}$, the magnetic field is given by

$$\vec{H} = \frac{1}{\mu_0}\, \vec{\nabla} \times \vec{A} \tag{9.4}$$

as you wrote in Question 4(a) of today's workshee.

Meanwhile, outside the source, the electric field is given by

$$\vec{E} = \frac{iZ_0}{k}\, \vec{\nabla} \times \vec{H} \tag{9.5}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space, *as you showed in Question 4(b) of today's worksheet.*

Three zones of interest

Given a current distribution $\vec{J}(\vec{x}\,')$, the fields can, in principle, be determined by evaluating the integral in equation (9.3).

Instead of integrating equation (9.3) by brute force, however, we choose to study the properties of the fields in three different zones of interest, defined by their distance from the source.

To do so, we will assume that the source is confined to a region of size d that is very small compared to a wavelength. In other words, all the charge and current distributions are assumed to be contained in a sphere of radius d about the origin. So, if the wavelength is $\lambda = 2\pi c/\omega$, then we'll assume that $d \ll \lambda$. This is known as the long-wavelength approximation. We will always work in this long-wavelength approximation.

Let r be the distance from the source to the observation point. If we also consider the observation point to be located at $\vec{x}$, then we have $r = |\vec{x}|$. Then, there are three spatial regions of interest:

- The near (static) zone: $d \ll r \ll \lambda$
- The intermediate (induction) zone: $d \ll r \sim \lambda$
- The far (radiation) zone: $d \ll \lambda \ll r$

The near (static) zone

The near field (or static) zone is the region of space that is very far from the source, $d \ll r$, but the distance to the observation point $\vec{x}$, written as $r = |\vec{x}|$, is much smaller than the wavelength of the light, so $r \ll \lambda$, so that $kr \ll 1$.

In this near field limit, the exponential in equation (9.3) becomes 1, and the vector potential is given by

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x'$$

But this is exactly the same solution as in magnetostatics; e.g., see equation (5.28) on page 180 in Jackson.

In the near zone, therefore, even though the fields oscillate harmonically in time (due to the $e^{-i\omega t}$ term), they have the **character of static fields**. This is evident from the fact that the spatial parts of the fields are identical to the fields of magnetostatics (which is why we also call the near zone the *static* zone). The near zone is of interest to atomic and molecular physicists, but we will not consider it any further here since we've already discussed it while discussing Chapter 5.

The intermediate (induction) zone

In the intermediate zone, we cannot make any approximations. The exponential $e^{ik|\vec{x}-\vec{x}\,'|}$ and $1/|\vec{x} - \vec{x}\,'|$ factor in equation (9.3) must be expanded in vector spherical harmonics and integrated term by term, so we will not pursue this regime.

The far (radiation) zone

In the far (or radiation) zone, the observation point r is very far from the source and much larger than the wavelength of the light, so we have $d \ll \lambda \ll r$. Now, $r \gg \lambda$ means that

$$\frac{r}{\lambda} \gg 1$$

and so

$$\left(\frac{2\pi}{\lambda}\right) r \gg 1$$

But $k = 2\pi/\lambda$, so in the radiation zone we have $kr \gg 1$.

This is the zone that we'll spend the most time discussing, since it is where we are most of the time for the major sources of electromagnetic radiation — our detectors are always located a great number of wavelengths away from the sources.

In the far zone, where $kr \gg 1$, we can take advantage of the relation

$$\left|\vec{x} - \vec{x}\,'\right| \simeq r - \hat{n} \cdot \vec{x}\,' \tag{9.7}$$

where $\hat{n}$ is a unit vector in the direction of $\vec{x}$. *You proved this relation on Question 5 of today's worksheet.* You should realize from the derivation that the approximation in equation (9.7) is valid for $r \gg d$ (independent of kr), so it is applicable even in the near zone.

Substitute equation (9.7) for $|\vec{x} - \vec{x}\,'|$ in the exponential inside the integral in equation (9.3). Moreover, if only the leading term in kr is desired, then the $|\vec{x} - \vec{x}\,'|$ factor in the denominator of equation (9.3) can be replaced by just r. Equation (9.3) for the vector potential then becomes

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}\,') \frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|}\, d^3x' \simeq \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}\,') \frac{e^{ik(r-\hat{n}\cdot\vec{x}\,')}}{r}\, d^3x'$$

Therefore, the vector potential when $kr \gg 1$ is given by

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}\,')\, e^{-ik\hat{n}\cdot\vec{x}\,'}\, d^3x' \tag{9.8}$$

Recall from our discussion in the previous class that e^{ikr}/r is just an outgoing spherical wave, so equation (9.8) tells us that in the far zone the vector potential behaves as an outgoing spherical wave times a coefficient that depends on an integral over the source.

In the long-wavelength approximation, the source dimensions are small compared to the wavelength ($d \ll \lambda$). We can then take advantage of the expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{to write} \qquad e^{-ik\hat{n}\cdot\vec{x}\,'} = \sum_{n=0}^{\infty} \frac{(-ik\hat{n}\cdot\vec{x}\,')^n}{n!}$$

for the exponential inside the integral in equation (9.8). Equation (9.8) then becomes

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}\,') \left[\sum_{n=0}^{\infty} \frac{(-ik\hat{n}\cdot\vec{x}\,')^n}{n!} \right] d^3x'$$

We can take the summation and some terms outside the integral, so that

$$\lim_{kr\to\infty} \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int \vec{J}(\vec{x}\,')\, (\hat{n}\cdot\vec{x}\,')^n\, d^3x' \tag{9.9}$$

as you showed in Question 6 of today's worksheet.

The magnitude of the n^{th} term in equation (9.9) above is given by

$$\frac{1}{n!} \int \vec{J}(\vec{x}\,')\, (k\hat{n}\cdot\vec{x}\,')^n\, d^3x' \tag{9.10}$$

where I've moved the k^n term inside the integral. Since the order of magnitude of $\vec{x}\,'$ is d and $kd \ll 1$, the successive terms in the expansion of $\vec{A}(\vec{x})$ written in equation (9.9) fall off rapidly with n. Consequently, the radiation emitted from the source will come mainly from the **first non-vanishing term** in the expansion of equation (9.9).