Physics 460—Homework Report 1
Due Tuesday, Apr. 7, 1 pm **Name:** ______________________________

Complete all the problems on the accompanying assignment.

List all the problems you worked on in the space below. Circle the ones you fully completed:

Please place the problems into the following categories:

- These problems helped me understand the concepts better: ____________________
- I found these problems fairly easy: ____________________
- I found these problems very challenging: ____________________

In the space below, show your work (even if not complete) for any problems you still have questions about. Indicate where in your work the question(s) arose, and ask specific questions that I can answer.

Use the back of this sheet or attach additional paper, if necessary.

If you have no remaining questions about this homework assignment, use this space for one of the following:

- Write one or two of your solutions here so that I can give you feedback on its clarity.
- Explain how you checked that your work is correct.

(1) Using the properties of the angular momentum operators and their eigenstates, evaluate

$$J_x|j,m\rangle \quad \text{and} \quad J_y|j,m\rangle.$$

My Answer:

The easiest way to show this is to write J_x and J_y in terms of the ladder operators $J_\pm$. The ladder operators are

$$J_+ = J_x + iJ_y \quad \text{and} \quad J_- = J_x - iJ_y,$$

so we can express J_x and J_y as the sum or difference of these operators. First for J_x:

$$J_x = \frac{1}{2}\Big(J_+ + J_-\Big).$$

Applying this to the state $|j,m\rangle$, we have

$$J_x|j,m\rangle = \frac{1}{2}\Big((J_+|j,m\rangle + J_-|j,m\rangle\Big) = \frac{1}{2}\Big(C_+|j,m+1\rangle + C_-|j,m-1\rangle\Big).$$

Next, for J_y:

$$J_y = \frac{1}{2i}\Big(J_+ - J_-\Big).$$

Applying this to the state $|j,m\rangle$, we have

$$J_y|j,m\rangle = \frac{1}{2i}\Big((J_+|j,m\rangle - J_-|j,m\rangle\Big) = \frac{1}{2i}\Big(C_+|j,m+1\rangle - C_-|j,m-1\rangle\Big).$$

So each of these operators "splits" the z-basis eigenstates into two states, one moved up from m to $m+1$ and the other moved down to $m-1$>

(2) Show that

$$\langle J_x\rangle = \langle J_y\rangle = 0$$

for the states $|\Psi\rangle = |j,m\rangle$.

My Answer:

This is easy to do given the results from the previous problems. These expectation values must be zero because the angular momentum eigenstate are orthogonal, so that $\langle j,m|j,n\rangle = 0$ unless $m = n$. Therefore,

$$\langle j,m|J_x|j,m\rangle = \frac{1}{2}\Big(C_+\langle j,m|j,m+1\rangle + C_-\langle j,m|j,m-1\rangle\Big) = 0,$$

$$\langle j,m|J_y|j,m\rangle = \frac{1}{2i}\Big(C_+\langle j,m|j,m+1\rangle - C_-\langle j,m|j,m-1\rangle\Big) = 0.$$

What does this mean physically? It means that if you have an ensemble of particles in one of the angular momentum eigenstates, then if you measure the x or y component of the angular momentum, the average of all the measurements will be zero. (Of course, most individual measurements will not be zero.) In other words, these components are effectively randomized; they have no particular orientation on average for systems with a definite z component of the angular momentum. Or to put it another way, if you measure the x or y component of the angular momentum, you are equally likely to obtain a result with a positive value of m or a negative value of m.

(3) Show that

$$\langle J_x{}^2 \rangle = \langle J_y{}^2 \rangle = \hbar^2[j(j+1) - m^2]/2$$

for the states $|\Psi\rangle = |j, m\rangle$. Using a symmetry argument, explain why these values must be true, in light of the values for $\langle J_z{}^2 \rangle$ and $\langle J^2 \rangle$ for these states.

My Answer:

We'll use the same idea as before, and express $J_x{}^2$ and $J_y{}^2$ in terms of $J_\pm$. I'll start with $J_x{}^2$:

$$J_x{}^2 = \frac{1}{4}\Big(J_+ + J_-\Big)^2 = \frac{1}{4}\Big(J_+{}^2 + J_+J_- + J_-J_+ + J_-{}^2\Big).$$

Now I'll use the commutation relation between J_+ and J_- to reorder the middle terms:

$$[J_+, J_-] = 2\hbar J_z \quad \Rightarrow \quad J_+J_- = J_-J_+ + 2\hbar J_z, \quad \text{or} \quad J_-J_+ = J_+J_- - 2\hbar J_z.$$

We can use either of these in our expression for $J_x{}^2$:

$$J_x{}^2 = \frac{1}{4}\Big(J_+{}^2 + 2J_+J_- - 2\hbar J_z + +J_-{}^2\Big) = \frac{1}{4}\Big(J_+{}^2 + 2J_-J_+ + 2\hbar J_z + +J_-{}^2\Big).$$

Now we're ready to calculate $\langle J_x{}^2 \rangle$ for the states $|j, m\rangle$:

$$\langle j, m|J_x{}^2|j, m\rangle = \frac{1}{4}\langle j, m|J_+{}^2 + 2J_+J_- - 2\hbar J_z + +J_-{}^2|j, m\rangle.$$

I've just used the first expression here. The first and last terms of this expectation value will be zero because of the orthogonality of the angular momentum eigenstates. The operators $J_+{}^2$ and $J_-{}^2$ will either raise or lower $|j, m\rangle$ to $|j, m+2\rangle$ or $|j, m-2\rangle$, so we'll be left with $\langle j, m|j, m+2\rangle = 0$ and $\langle j, m|j, m-2\rangle = 0$ for these terms.

For the second term, we'll use the fact that $J_-|j, m\rangle = C_-|j, m-1\rangle$ and the fact that $J_+ = J_-^\dagger$, so that

$$\langle j, m|J_+J_-|j, m\rangle = C_-{}^2\langle j, m-1|j, m-1\rangle = C_-{}^2.$$

Of course $J_z|j, m\rangle = m\hbar|j, m\rangle$. Using these results in our expectation value, and the fact that $C_-{}^2 = j(j+1) - m(m-1)$, we have

$$\langle j, m|J_x{}^2|j, m\rangle = \frac{\hbar^2}{2}\Big(C_-{}^2 - m\Big) = \frac{\hbar^2}{2}\Big(j(j+1) - m^2\Big),$$

as desired. The only potential problem is if $m = m_{\text{min}}$, in which case $J_-|j, m_{\text{min}}\rangle = |\varnothing\rangle$, and $\langle j, m_{\text{min}}|J_+ = \langle\varnothing|$. In that case, we have to use the second expression for $J_x{}^2$, and write

$$\langle j, m|J_x{}^2|j, m\rangle = \frac{1}{4}\langle j, m|J_+{}^2 + 2J_-J_+ + 2\hbar J_z + +J_-{}^2|j, m\rangle.$$

Using the same logic as before, with $J_+|j, m\rangle = C_+|j, m+1\rangle$ and $\langle j, m|J_- = C_+\langle j, m+1|$, it's easy to show that we obtain the same result.

Of course the calculation for $\langle J_y{}^2 \rangle$ is similar. In this case we have

$$\begin{aligned} J_y{}^2 &= -\frac{1}{4}\left(J_+ - J_-\right)^2 = -\frac{1}{4}\Big(J_+{}^2 - J_+J_- - J_-J_+ + J_-{}^2\Big) \\ &= \frac{1}{4}\Big(-J_+{}^2 + J_+J_+ - J_-J_+ - J_-{}^2\Big). \end{aligned}$$

Because the only two terms that matter are the J_+J_- and J_-J_+ terms, and these are the same as before, we must get the same answer. So

$$\langle J_y{}^2 \rangle = \frac{\hbar^2}{2}\Big(j(j+1) - m^2\Big)$$

as well.

So why must the be true? We know that $J^2 = {J_x}^2 + {J_y}^2 + {J_z}^2$, so we also know that

$$\langle {J_x}^2 \rangle + \langle {J_y}^2 \rangle = \langle J^2 \rangle - \langle {J_z}^2 \rangle.$$

But we also know for these states that

$$\langle J^2 \rangle = j(j+1)\hbar^2 \quad \text{and} \quad \langle J_z \rangle^2 = m^2\hbar^2,$$

because the states $|j,m\rangle$ are eigenstates of J^2 and J_z with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$ respectively. Therefore, we know that

$$\langle {J_x}^2 \rangle + \langle {J_y}^2 \rangle = \hbar^2\Big(j(j+1) - m^2\Big).$$

There's no way to distinguish between the x direction and the y direction, so the two expectation values must be equal, and each must be equal to half this value, which is exactly what we have shown.

(4) Find the representation of the operators J_x, J_y, J_z, and J^2 using the states $|j,m\rangle$ as your basis for the case $j = 3/2$. Since there are four possible values for m, your answers should all be 4×4 matrices. Use the following representation for the basis states:

$$|3/2, 3/2\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |3/2, 1/2\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad |3/2, -1/2\rangle \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |3/2, -3/2\rangle \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

My Answer:

The matrix elements of any operator A in the z-state angular momentum basis are $\langle j, m_1|A|j, m_2\rangle$, where m_1 and m_2 can take on any of the four allowed values for $j = 3/2$. To answer this question we'll have to evaluate these sixteen matrix elements for each of the operators J_x, J_y, J_z, and J^2. Fortunately, many of these matrix elements will be zero!

I'll find the representations for J_z and J^2 first because they are the easiest, since our basis states are their eigenvalues. That means we can use

$$\langle j, m_1|J_z|j, m_2\rangle = m_2\hbar\delta_{m_1,m_2} \quad \text{and} \quad \langle j, m_1|J^2|j, m_2\rangle = j(j+1)\hbar^2\delta_{m_1,m_2}.$$

Therefore,

$$J_z \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad \text{and} \quad J^2 \leftrightarrow \frac{15\hbar}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Of course these are both diagonal matrices, and the diagonal elements of J^2 are all equal, because all the states have the same value of j, and differ only in their m values.

To find the representations of J_x and J_y, I'll use the results of the first problem:

$$\begin{aligned} J_x|j,m\rangle &= \frac{1}{2}\Big(C_+(m)|j, m+1\rangle + C_-(m)|j, m-1\rangle\Big), \\ J_y|j,m\rangle &= \frac{1}{2i}\Big(C_+(m)|j, m+1\rangle - C_-(m)|j, m-1\rangle\Big). \end{aligned}$$

I've written the constants $C_\pm$ as functions of m so that I can keep track of the m value to use in the coefficients using $C_\pm(m) = \sqrt{j(j+1) - m(m \pm 1)}\hbar$. The matrix elements for J_x and J_y are then

$$
\begin{aligned}
\langle j, m_1 | J_x | j, m_2 \rangle &= \frac{1}{2}\Big(C_+(m_2)\langle j, m_1 | j, m_2 + 1\rangle + C_-(m_2)\langle j, m_1 | j, m_2 - 1\rangle\Big), \\
&= \frac{1}{2}\Big(C_+(m_2)\delta_{m_1, m_2+1} + C_-(m_2)\delta_{m_1, m_2-1}\Big), \\
\langle j, m_1 | J_x | j, m_2 \rangle &= \frac{1}{2\mathrm{i}}\Big(C_+(m_2)\langle j, m_1 | j, m_2 + 1\rangle + C_-(m_2)\langle j, m_1 | j, m_2 - 1\rangle\Big), \\
&= \frac{1}{2\mathrm{i}}\Big(C_+(m_2)\delta_{m_1, m_2+1} - C_-(m_2)\delta_{m_1, m_2-1}\Big).
\end{aligned}
$$

These will only be nonzero if $m_1 = m_2 + 1$ or $m_1 = m_2 - 1$. The only nonzero matrix elements for J_x are

$$
\begin{aligned}
\langle 3/2, 3/2 | J_x | 3/2, 1/2 \rangle = \langle 3/2, 1/2 | J_x | 3/2, 3/2 \rangle^* &= \frac{1}{2}\sqrt{15/4 - 3/4}\hbar = \frac{\sqrt{3}}{2}\hbar, \\
\langle 3/2, 1/2 | J_x | 3/2, -1/2 \rangle = \langle 3/2, -1/2 | J_x | 3/2, 1/2 \rangle^* &= \frac{1}{2}\sqrt{15/4 + 1/4}\hbar = \hbar, \\
\langle 3/2, -1/2 | J_x | 3/2, -3/2 \rangle = \langle 3/2, -3/2 | J_x | 3/2, -1/2 \rangle^* &= \frac{1}{2}\sqrt{15/4 - 3/4}\hbar = \frac{\sqrt{3}}{2}\hbar.
\end{aligned}
$$

Here I've used the property of inner products that $\langle \Psi | \Phi \rangle = \langle \Phi | \Psi \rangle^*$. The matrix representation of J_x in the z-state basis is then

$$
J_x \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}.
$$

Similarly, the only nonzero matrix elements for J_y are

$$
\begin{aligned}
\langle 3/2, 3/2 | J_y | 3/2, 1/2 \rangle = \langle 3/2, 1/2 | J_y | 3/2, 3/2 \rangle^* &= \frac{1}{2\mathrm{i}}\sqrt{15/4 - 3/4}\hbar = -\frac{\mathrm{i}\sqrt{3}}{2}\hbar, \\
\langle 3/2, 1/2 | J_y | 3/2, -1/2 \rangle = \langle 3/2, -1/2 | J_y | 3/2, 1/2 \rangle^* &= \frac{1}{2\mathrm{i}}\sqrt{15/4 + 1/4}\hbar = -\mathrm{i}\hbar, \\
\langle 3/2, -1/2 | J_y | 3/2, -3/2 \rangle = \langle 3/2, -3/2 | J_y | 3/2, -1/2 \rangle^* &= \frac{1}{2\mathrm{i}}\sqrt{15/4 - 3/4}\hbar = -\frac{\mathrm{i}\sqrt{3}}{2}\hbar.
\end{aligned}
$$

Therefore,

$$
J_y \leftrightarrow \frac{\mathrm{i}\hbar}{2}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}.
$$

Note that both J_x and J_y are Hermitian, so they are represented by matrices that are symmetric under complex transpose, as they must be. Note also that the two diagonals that are nonzero are just above and below the main diagonal. This is the concrete representation of what we found in problem (1)—that these operators "split" the eigenstates into two parts, one with $m + 1$, and one with $m - 1$.

(5) Use your answers from Problem (4) to construct the representations of J_+ and J_- for the case $j = 3/2$. Explain why these matrices have the form they do—do they "look" like raising and lowering operators?

My Answer:

We can find the representations of J_+ and J_- using

$$J_+ = J_x + \mathrm{i}J_y \quad \text{and} \quad J_- = J_x - \mathrm{i}J_y,$$

and the representations that we found above for J_x and J_y. These representations are

$$J_+ \leftrightarrow \hbar \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad J_- \leftrightarrow \hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}.$$

Note that $J_+^\dagger = J_-$ and $J_-^\dagger = J_+$, as they must. Note also that each matrix contains just a single diagonal of nonzero elements. This is the hallmark of the representation of a raising or lowering operator. The raising operator moves each eigenstate up one, so its only nonzero elements are in the diagonal just above the main diagonal. The lowering operator moves each eigenstate down one so its only nonzero elements are in the diagonal just below the main diagonal.