PHY 411 **Winter 2021**

Week 5—Thursday, Feb 4—Discussion Worksheet

Vector and Scalar Potentials (continued)

In the previous class, we wrote $\vec{E}$ and $\vec{B}$ in terms of the vector potential $\vec{A}$ and the scalar potential Φ:

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \text{and} \qquad \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

starting from the two homogenous Maxwell equations.

Today, we will derive two additional relations starting from the inhomogeneous Maxwell equations.

Useful Information: $\vec{\nabla} \times (\vec{\nabla} \times \vec{P}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{P}) - \nabla^2 \vec{P}$

1. **Starting from** Gauss' law:

$$\vec{\nabla} \cdot \vec{D} = \rho$$

where ρ is the charge density, **show that**

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = -\frac{\rho}{\epsilon_0} \tag{6.10}$$

$$\vec{D} = \epsilon_0 \vec{E} \rightarrow \vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho$$

$$\rightarrow \vec{\nabla} \cdot \epsilon_0 \left(-\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t} \right) = \rho$$

$$\rightarrow \epsilon_0 \left(-\vec{\nabla}^2 \Phi - \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) \right) = \rho$$

$$\rightarrow \vec{\nabla}^2 \Phi + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}$$

2. Starting from the Ampere-Maxwell law:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

where $\vec{J}$ is the current density, **show that**

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right] = -\mu_0 \vec{J} \qquad (6.11)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}(\epsilon \vec{E})$$

$$\rightarrow \vec{\nabla} \times \frac{\vec{B}}{\mu_0} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}\right)$$

$$\rightarrow \nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \mu_0 \vec{J} - \frac{1}{c^2}\vec{\nabla}\frac{\partial \Phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\rightarrow \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) + \vec{\nabla}\frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$

$$\rightarrow \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0 \vec{J}$$

Vector and Scalar Potentials: Gauge Transformations

Let's take stock. In the previous class, we wrote $\vec{E}$ and $\vec{B}$ in terms of the vector potential $\vec{A}$ and the scalar potential Φ:

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \text{and} \qquad \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

Starting from the inhomogenous Maxwell equations, you also derived on the previous two pages that

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = -\frac{\rho}{\epsilon_0} \tag{6.10}$$

and

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left[\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right] = -\mu_0 \vec{J} \tag{6.11}$$

where ρ is the charge density and $\vec{J}$ is the current density. In other words, **we have reduced the set of four Maxwell equations to two equations**. However, equation (6.10) and equation (6.11) are still coupled. Next, we will see how to uncouple them with gauge transformations.

3. Consider the **gauge transformations**

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \tag{6.12}$$

$$\Phi \to \Phi - \frac{\partial \Lambda}{\partial t} \tag{6.13}$$

Show that $\vec{B}$ and $\vec{E}$ are left unchanged by the transformations in equations (6.12) and (6.13).

$\nabla \times \nabla \psi = 0$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{B} = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\Lambda)$

$\vec{B} = (\vec{\nabla} \times \vec{A}) + \cancelto{0}{\vec{\nabla} \times (\vec{\nabla}\Lambda)}$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$

$\vec{E} = -\vec{\nabla}\left(\Phi - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \vec{A}}{\partial t}$

$\vec{E} = -\vec{\nabla}\Phi + \vec{\nabla}\frac{\partial \Lambda}{\partial t} - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\frac{\partial \Lambda}{\partial t}$

$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$

4. Consider the Lorenz condition

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{6.14}$$

Demonstrate that applying the Lorenz condition uncouples equations (6.10) and (6.11), and write down the uncoupled equations, one for Φ and the other for $\vec{A}$.

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = -\frac{\rho}{\epsilon_0} \quad (6.10)$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0 \vec{J} \quad (6.11)$$

So

$$\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$

Therefore,

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

Similarly,

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$$

Thus,

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

5. Consider again the Lorenz condition

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0 \tag{6.14}$$

Suppose potentials $\vec{A}$ and Φ **do not** satisfy the Lorenz condition. Then, make a gauge transformation to potentials $\vec{A}'$ and Φ', given by equation (6.12) and equation (6.13) respectively, that is

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$
$$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

If, now, $\vec{A}'$ and Φ' satisfy the Lorenz condition written above, equation (6.14), then derive that Λ must satisfy the condition

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) \tag{6.18}$$

Transformations $\quad \Phi = \Phi - \frac{\partial \Lambda}{\partial t}, \quad \vec{A} = \vec{A} + \vec{\nabla}\Lambda$

$$\rightarrow \vec{\nabla}\left(\vec{A} + \vec{\nabla}\Lambda\right) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\Phi - \frac{\partial \Lambda}{\partial t}\right) = 0$$

$$\rightarrow \vec{\nabla} \cdot \vec{A} + \nabla^2\Lambda + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0$$

$$\rightarrow \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\vec{\nabla}\cdot\vec{A} - \frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$