Physics 460—Homework Report 6
Due Tuesday, May 12, 1 pm **Name:** Timothy Holmes

Complete all the problems on the accompanying assignment.

List all the problems you worked on in the space below. Circle the ones you fully completed:

1 2

Please place the problems into the following categories:

- These problems helped me understand the concepts better: ______________
- I found these problems fairly easy: 1, 2
- I found these problems very challenging: ______________

In the space below, show your work (even if not complete) for any problems you still have questions about. Indicate where in your work the question(s) arose, and ask specific questions that I can answer.

Use the back of this sheet or attach additional paper, if necessary.

If you have no remaining questions about this homework assignment, use this space for one of the following:

- Write one or two of your solutions here so that I can give you feedback on its clarity.
- Explain how you checked that your work is correct.

Used course worksheets
- other Books
- Course Notes

(1) An electron in a hydrogen atom is in an excited state with $n = 3$ and $l = 2$.

(a) Suppose that the z-component of the orbital angular momentum is $\hbar$ and the z-component of the electron spin is $\hbar/2$. If you measure the total angular momentum of the electron, what values can you obtain, and with what probabilities? (Remember that the total angular momentum is $\sqrt{j(j+1)}\hbar$, where j is the total angular momentum quantum number.)

(b) Suppose that the total angular momentum of the electron is $\sqrt{35}\hbar/2$ (so that $j = 5/2$), and the z-component of the total angular momentum of the electron is $\hbar/2$. Give the probability of finding the electron with a given set of values (m_l, m_s) for the z-components of the orbital angular momentum and spin.

(c) Suppose that you measure the z-component of the spin for the electron in part (b) and obtain the result $-\hbar/2$. If you then measure the total angular momentum of the electron, what values can you obtain, and with what probabilities?

(d) The electron from part (b) emits a photon, which carries off one unit of angular momentum, (reducing j from $5/2$ to $3/2$). The photon also carries off a z-component of angular momentum of $-\hbar$. Give the probability of finding the electron with a given set of values (m_l, m_s) for the z-components of the orbital angular momentum and spin after the photon has been emitted.

(2) An electron in a hydrogen atom is the state

$$|\Psi\rangle = \frac{1}{\sqrt{3}}\left\{|311\rangle \otimes |-\rangle + \Big[|210\rangle - |211\rangle\Big] \otimes |+\rangle\right\}.$$

(a) If you measured the magnitude of the orbital angular momentum of the electron, what values could you obtain, and with what probabilities? If you measured the z-component of the orbital angular momentum of the electron, what values could you obtain, and with what probabilities?

(b) If you measured the magnitude of the spin of the electron, what values could you obtain, and with what probabilities? If you measured the z-component of the spin of the electron, what values could you obtain, and with what probabilities?

(c) If you measured the magnitude of the total angular momentum of the electron, what values could you obtain, and with what probabilities? If you measured the z-component of the total angular momentum of the electron, what values could you obtain, and with what probabilities?

(3) Positronium is the bound state of an electron and a positron, which is the anti-particle of an electron. The positron is is a spin-1/2 particle like an electron, and the bound states of positronium are very similar to the bound states of the hydrogen atom. The energy levels of positronium are given by

$$E_n = -\frac{me^4}{4\hbar^2 n^2} = \frac{E_H(n)}{2};$$

the bound state energies are half of those of hydrogen.

When we include the spin of the positron and electron, the orbital ground state becomes four-fold degenerate, because there are four possible spin states, which we can write as

$$\{|+;+\rangle, |+;-\rangle, |-;+\rangle, |-;-\rangle\} \quad \text{or} \quad \{|1,+1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle\}.$$

The first basis uses the individual spins of the particles, while the second is the triplet-singlet basis that uses the total angular momentum of the system. Each of these can be attached to an orbital state, so the full state of the system is then something like

$$|\Psi\rangle = \sum_{n,l,m_l,m_1,m_2} c_{nlm_lm_1m_2}|E_{nlm_l}\rangle \otimes |m_1; m_2\rangle,$$

where E_{nlm_l} are the energies of the orbital state of the electron and m_1 and m_2 are the spins of the electron and positron respectively. If the orbital state is the ground state, then the four possible states of the system are

$$|\Psi\rangle = |E_{100}\rangle|+;+\rangle, \quad |\Psi\rangle = |E_{100}\rangle|+;-\rangle, \quad |\Psi\rangle = |E_{100}\rangle|-;+\rangle, \quad |\Psi\rangle = |E_{100}\rangle|-;-\rangle.$$

In the rest of this problem we'll ignore the orbital part of the state, and concentrate on the spin part of the state.

We will investigate the breaking of the degeneracy of the ground state that results from the spin-spin interaction of the positron and electron and from the application of an external magnetic field. These are called the hyper-fine splitting and the Zeeman effect, and both are due to the existence of spin for the positron and the electron.

(a) The hyper-fine splitting can be described by the Hamiltonian

$$H_{\mathrm{hf}} = \frac{A}{\hbar^2}\left(\vec{S}_1 \cdot \vec{S}_2\right) + H_A.$$

Here $\vec{S}_1$ is the spin operator for the electron and $\vec{S}_2$ is the spin operator for the positron, $A = mc^2\alpha^4$ (where m is the mass of the electron and $\alpha = e^2/\hbar c = 1/137.036$ is the fine structure constant), and H_A is defined by the following:

$$H_A|1,m\rangle = 0, \quad H_A|0,0\rangle = -\frac{3A}{4}|0,0\rangle.$$

Working in the triplet-singlet basis, find the eigenstates of H_{hf} and their associated energies. [Hint: $\vec{S}_1 \cdot \vec{S}_2 = (S^2 - {S_1}^2 - {S_2}^2)/2$.] What is the degeneracy of the positronium ground state once the hyper-fine splitting is accounted for? Find the numerical value for the splitting energy and express it as a percentage of the ground state energy.

(b) If positronium is put in a magnetic field $\vec{B} = B_0\hat{\mathrm{k}}$, the Hamiltonian becomes

$$H = H_{\mathrm{hf}} + H_{\mathrm{Zeeman}},$$

where

$$H_{\mathrm{Zeeman}} = -\frac{eB_0}{m}(S_{z1} - S_{z2}).$$

Letting $\omega_0 = eB_0/m$, write this Hamiltonian as a matrix in the triplet-singlet basis.

(c) Find the eigenstates of H and their associated energies. What is the degeneracy of the positronium ground state once when it is in this magnetic field? If the strength of the magnetic field is $B_0 = 1\,\mathrm{T}$, find the numerical value of the energy level splitting.

Homework 6

(a) $l = 2$ $s = 1/2$

From Clebsh-Gordan table $2 \times 1/2$

$j = 5/2$ and $m = 5/2$

$\rightarrow \sqrt{j(j+1)}\,\hbar = \sqrt{35}\,\hbar/6$ $P = 1$

(B) angular momentum of electron is $\sqrt{35}\,\hbar/2$, $j = 5/2$
$m = [-2, 2]$

J_1	J_2	m_1	m_2	
$5/2$	$+5/2$	$+2$	$+1/2$	$P = 1$
$5/2$	$+3/2$	$+2$	$-1/2$	$P = 4/5$
$5/2$	$+1/2$	$+1$	$-1/2$	$P = 3/5$
$5/2$	$-1/2$	0	$-1/2$	$P = 2/5$
$5/2$	$-3/2$	-1	$-1/2$	$P = 1/5$
$5/2$	$-5/2$	-2	$-1/2$	$P = 1$

(C)

J_1	J_2	m_1	m_2		
$5/2$	$5/2$	2	$1/2$	$J = \sqrt{55}\,\hbar/6$	$P = 1$
$5/2$	$3/2$	1	$1/2$	$J = \sqrt{55}\,\hbar/6$	$P = 4/5$
$3/2$	$3/2$	1	$1/2$	$J = \sqrt{15}\,\hbar/2$	$P = 4/5$
$5/2$	$1/2$	0	$1/2$	$J = \sqrt{55}\,\hbar/6$	$P = 3/5$
$3/2$	$1/2$	0	$1/2$	$J = \sqrt{15}\,\hbar/2$	$P = 4/5$
$5/2$	$-1/2$	-1	$1/2$	$J = \sqrt{55}\,\hbar/6$	$P = 2/5$
$3/2$	$-1/2$	-1	$1/2$	$J = \sqrt{15}\,\hbar/2$	$P = 4/5$
$5/2$	$-3/2$	-2	$1/2$	$J = \sqrt{55}\,\hbar/6$	$P = 1/5$
$3/2$	$-3/2$	-2	$1/2$	$J = \sqrt{15}\,\hbar/2$	$P = 4/5$

(D)

(2) $|\psi\rangle = \frac{1}{\sqrt{3}}\left\{|311\rangle \otimes |-\rangle + \left[|210\rangle - |211\rangle\right] \otimes |+\rangle\right\}$

$j = 3/2 \; ; \; m = +1/2$

$|3/2 ; +1/2\rangle = \sqrt{\frac{1}{3}}\,|+1 ; -1/2\rangle + \sqrt{\frac{2}{3}}\,|-1 ; +1/2\rangle$

(B)

$|\langle +1 ; +1/2 | +1 ; -1/2\rangle|^2 = |\sqrt{1/3}\,|^2 = 1/3$

$|\langle -1 ; -1/2 | -1 ; -1/2\rangle|^2 = |\sqrt{2/3}\,|^2 = 2/3$

$|\langle \pm 1 ; \mp 1/2 | \mp 1 ; \pm 1/2\rangle|^2 = |\sqrt{1/3}\cdot 0 + \sqrt{2/3}\cdot 0|^2 = 0$

(c)

J_1	J_2	m_1	m_2	
3/2	3/2	1	1/2	P = 1
3/2	1/2	1	-1/2	P = 1/3
3/2	-1/2	0	-1/2	P = 2/3
3/2	-3/2	-1	-1/2	P = 1

↑

(3a) $$H_{hf} = \frac{A}{\hbar^2}\left(\vec{S}_1 \cdot \vec{S}_2\right) + H_A$$