

## Class Summary—Week 4, Day 1—Tuesday, Jan 26

## Dispersion

So far, we have discussed wave propagation in non-dispersive media, in which the index of refraction was constant in frequency. In reality, this is realistic only over a limited range of frequencies, or in a vacuum. Frequently, the index of refraction is dependent on the frequency, which means that different frequency components of a wave travel at different speeds, changing the wave shape and spreading it out as it propagates through the medium; this is known as dispersion. Therefore, we need to develop a simple model of dispersion to investigate the effects that arise in dispersive media. Fortunately, all of our results from the previous classes that involved a single frequency component are still valid in the presence of dispersion. The interest, however, is in studying the new effects that arise as a result of the frequency dependence of  $\epsilon$  and  $\mu$ .

Simple model for  $\epsilon(\omega)$ : Static Case

Before jumping into the full discussion, let us go over § 4.6 in Jackson. In that section, we learn that when a static electric field is present in a medium made of bound charges (electrons and ions), it polarizes those charges.

To model this, Jackson assumed that in the presence of an electric field  $\vec{E}$  acting to displace a charge  $e$  from equilibrium, there would be a linear and harmonic restoring force on each charge acting to get the charge back into equilibrium, given by

$$\vec{F} = -m\omega_0^2 \vec{x} \quad (4.71)$$

where  $m$  is the mass of the charge, and  $\omega_0$  is the frequency of oscillation about equilibrium. Implicit in this assumption of a linear restoring force is that the polarization of the medium is a linear function of the total field in the neighborhood of the atoms.

**Note:** This is just the usual restoring force  $F = -kx$ , with the usual  $\omega_0^2 = k/m$  (see undergrad notes on simple harmonic oscillators if you've forgotten).

So the charge  $e$  feels two forces: the restoring force  $\vec{F}$  written above, and the force  $e\vec{E}$  exerted by the electric field (which you'll remember using even in your freshman class if you recall that force is charge times the electric field, or if you recall the freshman definition of the electric field).

Next, we can use Newton's law to write the net force equal to  $ma$  (with  $a = 0$  in this instance since we're discussing the static case in which the charge  $e$  is in equilibrium). We'll do this on the next page.

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On the previous page, we found that the charge  $e$  feels two forces: the restoring force  $\vec{F}$  written in equation (4.71), and the force  $e\vec{E}$  exerted by the electric field. We can then use Newton's law to write the net force felt by the charge equal to  $ma$ , except that  $a = 0$  here, since we're discussing the static case in which the charge  $e$  is in equilibrium under the action of the applied force  $e\vec{E}$  due to the electric field and the restoring force  $\vec{F}$ . So,

$$\sum F \equiv e\vec{E} - m\omega_0^2\vec{x} = 0$$

Under the action of this electric field  $\vec{E}$ , therefore, the charge is displaced from its equilibrium by an amount  $\vec{x}$  given by

$$m\omega_0^2\vec{x} = e\vec{E}$$

This allows us to define the induced dipole moment:

$$\vec{p}_{\text{mol}} = e\vec{x} = \frac{e^2}{m\omega_0^2} \vec{E} \quad (4.72)$$

The expression is easy to remember if you recall from freshman physics (but cast into the notation we're using now) that two charges  $+q$  and  $-q$  separated by a distance  $\vec{x}$  have an electric dipole  $\vec{p} = q\vec{x}$ , whose direction is defined by convention to be from the negative to the positive charge.

In reality, molecules have many bound charges, each of which has its own restoring force with a different natural frequency  $\omega_j$ .

So we have to define an average dipole moment of the molecules. This dipole moment is approximately proportional to the electric field acting on the molecule. To bring out this dependence on the electric field, we define the molecular polarizability  $\gamma_{\text{mol}}$  as the ratio of the average molecular dipole moment to  $\epsilon_0$  times the applied field at the molecule.

So, from equation (4.72), we get that

$$\gamma = \frac{p_{\text{mol}}}{\epsilon_0 E} = \frac{1}{\epsilon_0} \frac{e^2}{m\omega_0^2}$$

So, if there are a set of charges  $e_j$  with masses  $m_j$  and oscillation frequencies  $\omega_j$  in *each* molecule, then the molecular polarizability is

$$\gamma_{\text{mol}} = \frac{1}{\epsilon_0} \sum_j \frac{e_j^2}{m_j \omega_j^2} \quad (4.73)$$

Now we need a few more steps to connect this to the dielectric constant  $\epsilon/\epsilon_0$ . Before we can do that, though, we have to note that equation (4.73) is for a *single* molecule, whereas an actual medium would contain  $N$  molecules per unit volume. We will now proceed to incorporate all of this.

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Recall that at our first meeting this quarter, we discussed that the induced polarization  $\vec{P}$  is proportional to  $\vec{E}$  with a proportionality constant that is independent of direction:

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \quad (4.36)$$

where the constant  $\chi_e$  is called the electric susceptibility of the medium.

and that the electric permittivity  $\epsilon$  of the medium is given by

$$\epsilon = \epsilon_0 (1 + \chi_e) \quad (4.38)$$

where  $\epsilon/\epsilon_0$  is called the *dielectric constant*.

Now,  $\vec{E}$  in equation (4.36) is the macroscopic electric field.

- In rarefied media where the molecular separations are large, there is little difference between the macroscopic field and the field acting on any molecule or group of molecules.
- In dense media with closely packed molecules, however, the polarization of neighboring molecules leads to an internal field  $\vec{E}_i$  at any given molecule in addition to the average macroscopic field  $\vec{E}$ , so that the total field at the molecule is  $(\vec{E} + \vec{E}_i)$ .

Jackson presents arguments to demonstrate that

$$\vec{E}_i = \frac{\vec{P}}{3\epsilon_0} \quad (4.63)$$

although we will skip the details.

So, equation (4.72) for the dipole moment may now be written as an average dipole moment for all the molecules taking into account both the macroscopic field  $\vec{E}$  and the internal field  $\vec{E}_i$  as

$$\langle \vec{p}_{\text{mol}} \rangle = \sum_j \frac{e_j^2}{m_j \omega_j^2} (\vec{E} + \vec{E}_i) = \epsilon_0 \gamma_{\text{mol}} (\vec{E} + \vec{E}_i) \quad (4.67)$$

By definition, the electric polarization  $\vec{P}$  is the dipole moment per unit volume. So, we can write

$$\vec{P} = N \langle \vec{p}_{\text{mol}} \rangle$$

where  $N$  is the number of molecules (or atoms) per unit volume. Putting equation (4.67) into this expression, we get that

$$\vec{P} = N \langle \vec{p}_{\text{mol}} \rangle = N \epsilon_0 \gamma_{\text{mol}} (\vec{E} + \vec{E}_i)$$

Writing  $\vec{E}_i$  from equation (4.63), this becomes

$$\vec{P} = N \epsilon_0 \gamma_{\text{mol}} \left( \vec{E} + \frac{\vec{P}}{3\epsilon_0} \right)$$

so that finally

$$\vec{P} = N \gamma_{\text{mol}} \left( \epsilon_0 \vec{E} + \frac{1}{3} \vec{P} \right) \quad (4.68)$$

Using equation (4.36), we can show that the susceptibility is

$$\chi_e = \frac{N\gamma_{\text{mol}}}{1 - \frac{1}{3}N\gamma_{\text{mol}}} \quad (4.69)$$

Therefore, the dielectric constant is

$$\frac{\epsilon}{\epsilon_0} = \frac{1 + \frac{2}{3}N\gamma_{\text{mol}}}{1 - \frac{1}{3}N\gamma_{\text{mol}}}$$

Before moving on, it is of interest to note that another way of writing this is to cast it as

$$\gamma_{\text{mol}} = \frac{3}{N} \left[ \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} \right] \quad (4.70)$$

which is known as the Clausius-Mossotti equation.

So much for the static case. Let us now move on to the dynamic case of electromagnetic waves and see how we'll have to modify the model, and what we'll get.

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### Simple model for $\epsilon(\omega)$ : Time-varying Fields

We will now extend the static model discussed above to time-varying fields:

- For the force binding the electron, we will still use the linear and harmonic restoring force that we wrote in equation (4.71).
- Except that, now, unlike the static case, we will also have an acceleration component  $m\ddot{\vec{x}}$  in the equation of motion.
- In an undamped system feeling only the force of an electric field and the restoring force defined in equation (4.71), an electron would oscillate forever. In reality, though, the electron interacts with other electrons and loses energy in the process. Given all the possible interactions involved, we can just sweep the details under the rug by using a damping constant  $\gamma$  to describe these interactions, so that we will have a velocity-dependent damping force  $m\gamma\dot{\vec{x}}$ .

With all the considerations above, Newton's second law that the net force equals  $m\ddot{\vec{x}}$  for an electron of charge  $-e$  now gives

$$-e\vec{E}(\vec{x}, t) - m\omega_0^2\vec{x} - m\gamma\dot{\vec{x}} = m\ddot{\vec{x}}$$

so that the equation of motion for an electron of charge  $-e$  bound by the harmonic force in equation (4.71) and acted on by an electric field  $\vec{E}(\vec{x}, t)$  is

$$m\left[\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}\right] = -e\vec{E}(\vec{x}, t) \quad (7.49)$$

where  $\gamma$  is the damping constant.

Some things to realize about equation (7.49):

- To keep things simple, we've neglected the difference between the applied electric field and the local field in equation (7.49). Per our discussion earlier, therefore, the model is appropriate only for substances of relatively low density. We could remove this deficiency, if desired, by using equation (4.69), in which the expression for  $\chi_e$  is written taking into account both the macroscopic  $\vec{E}$  and the internal field  $\vec{E}_i$ .
- The effects of magnetic force are neglected in equation (7.49). This amounts to assuming that  $\epsilon(\omega)$  is both frequency-dependent and complex, but that  $\mu$  is both constant and real. This turns out to be a good approximation for most materials, especially at optical frequencies.
- We will assume that the amplitude of oscillation is small enough to permit evaluation of  $\vec{E}$  at the position of the electron.

If we assume that the field (and hence  $\vec{x}$ ) varies harmonically in time with frequency  $\omega$  — that is,  $\vec{E} \sim e^{-i\omega t}$  and  $\vec{x} = \vec{x}_0 e^{-i\omega t}$  then solving equation (7.49), we get that the dipole moment contributed by one electron is

$$\vec{p} = -e\vec{x} = \frac{e^2}{m} \left( \frac{\vec{E}}{\omega_0^2 - \omega^2 - i\omega\gamma} \right) \quad (7.50)$$

as you showed on today's Discussion Worksheet.

In reality, though, we have  $N$  molecules per unit volume with  $Z$  electrons per molecule, and instead of a single binding frequency for all, there are  $f_j$  electrons per molecule with binding frequency  $\omega_j$  and damping constants  $\gamma_j$ . Then the dielectric constant is given by

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \quad (7.51)$$

as you showed on today's Discussion Worksheet.

Since there are  $f_j$  electrons per molecule with binding frequency  $\omega_j$  and damping constants  $\gamma_j$ , and we have a total of  $Z$  electrons per molecule, we must satisfy the sum rule

$$\sum_j f_j = Z \quad (7.52)$$

So what have we found?

- $\epsilon(\omega)$  is now complex!
- The term  $1/(\omega_j^2 - \omega^2 - i\omega\gamma_j)$  can be written as the sum of real and imaginary parts:

$$\frac{1}{\omega_j^2 - \omega^2 - i\omega\gamma_j} = \frac{(\omega_j^2 - \omega^2) + i\omega\gamma_j}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2}$$

as you showed on today's Discussion Worksheet.

- Notice that it is the imaginary part that is explicitly connected to the damping constants  $\gamma_j$ . But  $\gamma_j$  are generally small compared to the compared to the binding frequencies  $\omega_j$ , e.g.,  $\gamma_i \sim 10^9 \text{ s}^{-1}$  compared to  $\omega_j \sim 10^{15} \text{ s}^{-1}$  for optical transitions in atoms.

This means that  $\epsilon(\omega)$  is approximately real for most frequencies.

- You may remember, if you've seen the term above before, that it is associated with resonances. In preparation for the discussion in the next class, let's consider the real and imaginary parts separately:

$$\frac{1}{\omega_j^2 - \omega^2 - i\omega\gamma_j} = \frac{(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2} + i \left[ \frac{\omega\gamma_j}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2} \right]$$

so that

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j f_j \left\{ \frac{(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2} + i \left[ \frac{\omega\gamma_j}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2} \right] \right\} \quad (7.51.a)$$

The factor  $(\omega_j^2 - \omega^2)$  is positive for  $\omega < \omega_j$  and negative for  $\omega > \omega_j$ .