Physics 412—Harmonic Oscillator Operators and States — Winter 2020

Starting with the definition of a, N, and H for the simple harmonic oscillator,

$$H = \frac{P^2}{2m} + \frac{m{\omega_0}^2X^2}{2}, \quad a = \frac{1}{\sqrt{2}d_0}\left[X + \frac{\mathrm{i}P}{m\omega_0}\right], \quad d_0 = \sqrt{\frac{\hbar}{m\omega_0}}, \quad N = a^\dagger a,$$

verify all the following properties, using the commutation relation for X and P, $\left[X, P\right] = \mathrm{i}\hbar$, as necessary. Note that $|E\rangle$ means an eigenstate of H with eigenvalue E and $|n\rangle$ means and eigenstate of N with eigenvalue n. You should do this without referencing the course notes!!

$$a^\dagger = \frac{1}{\sqrt{2}d_0}\left[X - \frac{iP}{m\omega_0}\right]$$

(1) $H = \hbar\omega_0\left(a^\dagger a + \frac{1}{2}\right) = \hbar\omega_0\left(N + \frac{1}{2}\right)$.

(2) $\left[a, a^\dagger\right] = I$.

(3) $\left[N, a\right] = -a$.

(4) $\left[N, a^\dagger\right] = a^\dagger$.

(5) $N\left(a|n\rangle\right) = (n-1)\left(a|n\rangle\right)$.

(6) $N\left(a^\dagger|n\rangle\right) = (n+1)\left(a^\dagger|n\rangle\right)$.

(7) $a|n\rangle = \sqrt{n}\,|n-1\rangle$.

(8) $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$.

(9) $X|n\rangle = \frac{d_0}{\sqrt{2}}\left[\sqrt{n+1}\,|n+1\rangle + \sqrt{n}\,|n-1\rangle\right]$.

(10) $P|n\rangle = \frac{\mathrm{i}\hbar}{\sqrt{2}d_0}\left[\sqrt{n+1}\,|n+1\rangle - \sqrt{n}\,|n-1\rangle\right]$.

(11) $\left[H, N\right] = 0$.

(12) $\left[H, a\right] = -\hbar\omega_0 a$.

(13) $\left[H, a^\dagger\right] = \hbar\omega_0 a^\dagger$.

(14) $H\left(a|E\rangle\right) = (E - \hbar\omega_0)\left(a|E\rangle\right)$.

(15) $H\left(a^\dagger|E\rangle\right) = (E + \hbar\omega_0)\left(a^\dagger|E\rangle\right)$.

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$$a^\dagger a = \frac{1}{2d_0^2}\left[X - \frac{iP}{m\omega_0}\right]\left[X + \frac{iP}{m\omega_0}\right]$$

$i\hbar$

$$= \frac{1}{2d_0^2}\left[X^2 + \frac{P^2}{m^2\omega_0^2} + \frac{i}{m\omega_0}\left[XP - PX\right]\right]$$

$$= \frac{1}{2}\frac{m\omega_0}{\hbar}\left[X^2 + \frac{P^2}{m^2\omega_0^2} - \frac{\hbar}{m\omega_0}\right]$$

$$\Rightarrow \frac{1}{2}\left(X^2 + \frac{P^2}{m^2\omega_0^2}\right) = \frac{\hbar}{m\omega_0}a^\dagger a + \frac{\hbar}{2m\omega_0}$$

$$\Rightarrow \frac{1}{2}\left(m\omega_0^2X^2 + \frac{P^2}{m}\right) = \hbar\omega_0\left(a^\dagger a + \frac{1}{2}\right)$$

✓

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$$aa^\dagger = \frac{1}{2d_0^2}\left[X + \frac{iP}{m\omega_0}\right]\left[X - \frac{iP}{m\omega_0}\right]$$

$-i\hbar$

$$= \frac{1}{2d_0^2}\left[X^2 + \frac{P^2}{m^2\omega_0^2} + \frac{i}{m\omega_0}(PX - XP)\right]$$

$$= \frac{1}{2}\frac{m\omega_0}{\hbar}\left[X^2 + \frac{P^2}{m^2\omega_0^2} + \frac{\hbar}{m\omega_0}\right]$$

$$\therefore \left[a, a^\dagger\right] = aa^\dagger - a^\dagger a$$

$$= \frac{1}{2}\frac{m\omega_0}{\hbar}\left(\frac{\hbar}{m\omega_0}\right) - \frac{1}{2}\frac{m\omega_0}{\hbar}\left(\frac{-\hbar}{m\omega_0}\right)$$

$$= 1$$ ✓

③ $$[N,a] = a^\dagger a a - a a^\dagger a \qquad a^\dagger a - a a^\dagger = -[a,a^\dagger] = -I$$

$$\Rightarrow a^\dagger a = \underbrace{a a^\dagger - I}$$

$$[N,a] = (a a^\dagger - I)a - a a^\dagger a$$

$$[N,a] = a a^\dagger a - a - a a^\dagger a = -a \quad ✓$$

④ $$[N,a^\dagger] = a^\dagger a a^\dagger - a^\dagger a^\dagger a = a^\dagger(a a^\dagger - a^\dagger a) = a^\dagger [a,a^\dagger] = a^\dagger \quad ✓$$

⑤ $$N(a|n\rangle) = Na|n\rangle = (Na - aN + aN)|n\rangle$$

$$= ([N,a] + aN)|n\rangle = (-a + na)|n\rangle$$

$$= (n-1)(a|n\rangle) \quad ✓$$

⑥ $$N(a^\dagger|n\rangle) = Na^\dagger|n\rangle = (Na^\dagger - a^\dagger N + a^\dagger N)|n\rangle$$

$$= ([N,a^\dagger] + a^\dagger N)|n\rangle = (a^\dagger + n a^\dagger)|n\rangle$$

$$= (n+1)(a^\dagger|n\rangle) \quad ✓$$

⑦ From ⑤, $a|n\rangle = \alpha|n-1\rangle$

$$\Rightarrow \langle n|\underbrace{a^\dagger a}_{N}|n\rangle = \langle n-1|\alpha^* \alpha|n-1\rangle$$

$$\langle n|N|n\rangle = |\alpha|^2 \langle n-1|n-1\rangle$$

$$n\langle n|n\rangle = |\alpha|^2 \langle n-1|n-1\rangle$$

$$|\alpha|^2 = n \Rightarrow \alpha = \sqrt{n} \quad ✓$$

(8) From (6), $a^\dagger|n\rangle = \beta|n+1\rangle$

$$\Rightarrow \langle n|aa^\dagger|n\rangle = \langle n+1|\beta^*\beta|n+1\rangle$$

$$\langle n|aa^\dagger - a^\dagger a + a^\dagger a|n\rangle = |\beta|^2\langle n+1|n+1\rangle$$

$$\langle n|[a,a^\dagger]+N|n\rangle = |\beta|^2$$

$$\langle n|I+n|n\rangle = |\beta|^2$$

$$\langle n|n\rangle(n+1) = |\beta|^2 \Rightarrow \beta = \sqrt{n+1}$$

(9)

$$a+a^\dagger = \frac{2}{\sqrt{2}d_0}X \Rightarrow X = \frac{d_0}{\sqrt{2}}(a+a^\dagger)$$

So, using (7) & (8)

✓

$$X|n\rangle = \frac{d_0}{\sqrt{2}}(a+a^\dagger)|n\rangle = \frac{d_0}{\sqrt{2}}\left(\sqrt{n}|n-1\rangle + \sqrt{n+1}|n+1\rangle\right)$$

(10)

$$a-a^\dagger = \frac{2iP}{\sqrt{2}d_0 m\omega_0} \Rightarrow P = \frac{m\omega_0 d_0}{\sqrt{2}i}(a-a^\dagger) = \frac{m\omega_0 d_0^2}{\sqrt{2}i d_0}(a-a^\dagger)$$

$$= \frac{\hbar}{\sqrt{2}i d_0}(a-a^\dagger)$$

$$P|n\rangle = -\frac{i\hbar}{\sqrt{2}d_0}\left(\sqrt{n}|n-1\rangle - \sqrt{n+1}|n+1\rangle\right)$$

(11) $H = \hbar\omega_0\left(N+\frac{1}{2}\right) \Rightarrow [H,N] = 0$

(12) $$[H, a] = \hbar\omega_0 [N, a] = -\hbar\omega_0 a$$ ✓

(13) $$[H, a^\dagger] = \hbar\omega_0 [N, a^\dagger] = \hbar\omega_0 a^\dagger$$

(14) $$H(a|E\rangle) = (aH + [H, a])|E\rangle$$
$$= (aE - \hbar\omega_0 a)|E\rangle$$
$$= (E - \hbar\omega_0) a|E\rangle$$ ✓

(15) $$H(a^\dagger|E\rangle) = (a^\dagger H + [H, a^\dagger])|E\rangle$$
$$= (a^\dagger E + \hbar\omega_0 a^\dagger)|E\rangle$$
$$= (E + \hbar\omega_0) a^\dagger|E\rangle$$ ✓