PHY 411 Winter 2021

Class Summary—Week 2, Day 1—Tuesday, Jan 12

Chapter 7: Electromagnetic Waves and Propagation

The genius of Maxwell becomes evident as soon as we begin playing around with his equations and realize that they lead naturally to traveling wave solutions that transport energy from one point to another. We will now study the simplest and most fundamental electromagnetic waves: transverse plane waves.

To keep things simple, let us study the properties of fields in free space where there are no sources[1] (i.e., with $\rho = 0$ and $\vec{J} = 0$). In such regions, Maxwell's equations are

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \qquad & \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{D} &= 0 \qquad & \vec{\nabla} \times \vec{H} &= \frac{\partial \vec{D}}{\partial t} \end{aligned} \tag{7.1}$$

Let us assume solutions with harmonic time dependence

$$\vec{E}(\vec{x},t) = \vec{E}(\vec{x})\, e^{-i\omega t}, \qquad \vec{B}(\vec{x},t) = \vec{B}(\vec{x})\, e^{-i\omega t}, \qquad \vec{D}(\vec{x},t) = \vec{D}(\vec{x})\, e^{-i\omega t}, \qquad \vec{H}(\vec{x},t) = \vec{H}(\vec{x})\, e^{-i\omega t}$$

To avoid crowding, we will just write $\vec{E}$ instead of $\vec{E}(\vec{x})$. Clearly, this could be confused with $\vec{E}(\vec{x},t)$, but I've retained it to be consistent with Jackson's notation, especially because he is going to move to a completely different notation before introducing plane waves! So the current notation is only for deriving Helmholtz's equation.

You've already shown in the worksheet for the previous class that with the choice of time dependence $\sim e^{-i\omega t}$ written above, we get

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \qquad & \vec{\nabla} \times \vec{E} - i\omega \vec{B} &= 0 \\ \vec{\nabla} \cdot \vec{D} &= 0 \qquad & \vec{\nabla} \times \vec{H} + i\omega \vec{D} &= 0 \end{aligned}$$

Previously, we had also discussed that for uniform, isotropic, linear media, we have

$$\vec{D} = \epsilon \vec{E} \qquad \text{and} \qquad \vec{B} = \mu \vec{H}$$

In general, the permittivity ϵ and the permeability μ may be complex functions of ω. For now, though, we'll assume they are real and positive (no losses).

Again, *you've already shown in the worksheet for the previous class* that with $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$, we can modify the curl equations for $\vec{E}$ and $\vec{H}$ written above to

$$\vec{\nabla} \times \vec{E} - i\omega \vec{B} = 0 \qquad \text{and} \qquad \vec{\nabla} \times \vec{B} + i\omega\mu\epsilon \vec{E} = 0 \tag{7.2}$$

[1] Why can we do this? Is it realistic? The answer is yes! Remember, we are learning about propagation of waves; thus, it is reasonable to assume that the waves are now sufficiently far away from whatever sources produced them. Later in Chapter 9, we will learn about the generation of these waves where we will have to relax such assumptions.

You showed in the worksheet for the previous class that the equations above lead to the Helmholtz wave equation. Although I don't generally repeat the procedures from previous worksheets, I'll do so for this one because it is an important series of steps. Begin, as we discussed on the worksheet, by taking the curl of the $\vec{E}$-equation in equation (7.2):

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) - i\omega\left(\vec{\nabla} \times \vec{B}\right) = 0$$

then use the result from the inside front cover of Jackson for $\vec{\nabla} \times (\vec{\nabla} \times \vec{E})$ in the equation above:

$$\left[\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 E\right] - i\omega\left(\vec{\nabla} \times \vec{B}\right) = 0$$

Since from equation (7.1), we have $\vec{\nabla} \cdot \vec{D} = 0$, and so $\vec{\nabla} \cdot \vec{E} = 0$, the above equation becomes

$$-\nabla^2 E - i\omega\left(\vec{\nabla} \times \vec{B}\right) = 0$$

and replacing $\vec{\nabla} \times \vec{B}$ from equation (7.2), we get

$$-\nabla^2 E - i\omega\left(- i\omega\mu\epsilon\vec{E}\right) = 0$$

or

$$-\nabla^2 E + i^2\omega^2\mu\epsilon\vec{E} = 0$$

Since $i^2 = -1$, this becomes

$$-\nabla^2 E - \omega^2\mu\epsilon\vec{E} = 0$$

which leads us finally to

$$\begin{aligned} \left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{E} &= 0 \\ \text{and } \left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{B} &= 0 \end{aligned} \tag{7.3}$$

where the latter may be obtained by taking the curl of the $\vec{B}$-equation in equation (7.2).

The relations in equation (7.3) are known as the **Helmholtz wave equations**.

Space left blank for student notes; class summary continues on next page.

Let us write equation (7.3) on the previous page as

$$\left(\nabla^2 + k^2\right)\vec{u} = 0 \tag{7.3.a}$$

where $\vec{u}$ stands for $\vec{E}$ or $\vec{B}$ and k is called the **wave number** — we'll see why below.

By direct comparison of equation (7.3) and equation (7.3.a), we get that $k^2 = \mu\epsilon\omega^2$, so that

$$k = \omega\sqrt{\mu\epsilon} \tag{7.4}$$

and so we obtain

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \tag{7.4.a}$$

The right hand side will now give us an interesting number in a vacuum.

On the Discussion Worksheet in class today, you calculated the value of $1/\sqrt{\mu_0\epsilon_0}$ and found that it is equal to c, the speed of light in vacuum. This was one of the original spectacular results of Maxwell's work.

Thus, the quantity ω/k has the dimension of speed, and is called the **phase velocity** of the wave. Therefore, equation (7.4.a) can be written as

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \tag{7.5.a}$$

Next, let us multiply the numerator and denominator by $1/\sqrt{\mu_0\epsilon_0}$ in equation (7.5.a) above:

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1/\sqrt{\mu_0\epsilon_0}}{\sqrt{\mu\epsilon}/\sqrt{\mu_0\epsilon_0}} = \frac{c}{n}$$

where n is called the index of refraction of the medium, and is defined as

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \tag{7.5.b}$$

where ϵ is the permittivity of the medium and μ is its permeability. Yes, this is yet another use of the symbol n (!!!) — get used to it, it happens a lot. We'll do a lot more with the index of refraction n when we do § 7.3 in Jackson.

The complete equation (7.5) in Jackson is then

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}, \qquad \text{where } n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \tag{7.5}$$

as you derived in *Question 2 of today's Discussion Worksheet.*

Plane Wave Solutions

We will now look at plane wave solutions to the Helmholtz wave equation.

Plane Wave Solutions: One-dimensional Case

Before looking at the general plane wave solution of equation (7.3.a) in three dimensions, let us follow Jackson and consider a solution for a plane wave propagating along the x-direction.

Presumably, you remember that whenever you worked with plane waves propagating along one dimension, you used $\sin(kx - \omega t)$ or $\cos(kx - \omega t)$ to represent such waves. To be more general, and also because it makes the math easier to manipulate, we'll use $e^{i(kx-\omega t)}$ as a possible solution, recognizing of course that we've put back the t-dependence in the wave equation in equation (7.3) or equation (7.3.a). Such a plane wave, as you must know, is traveling in the positive x-direction. Moreover, a plane wave propagating in the negative x-direction, given by $e^{-i(kx-\omega t)}$ is also a solution to equation (7.3) or equation (7.3.a). On *Question 3 of the Discussion Worksheet for today*, you verified by explicit substitution that both are indeed solutions to the wave equation.

Since $e^{i(kx-\omega t)}$ and $e^{-i(kx-\omega t)}$ both satisfy the wave equation, the most general solution to equation (7.3.a) is

$$u(x,t) = a\, e^{i(kx-\omega t)} + b\, e^{-i(kx+\omega t)} \tag{7.6}$$

where u is standing in for $\vec{E}$ or $\vec{B}$, and a and b are arbitrary constants.

By pulling out k, we can write equation (7.6) as

$$u_k(x,t) = a\, e^{ik(x-\{\frac{\omega}{k}\}t)} + b\, e^{-ik(x+\{\frac{\omega}{k}\}t)}$$

We see that if $(\omega/k)\,t$ is to have a dimension of length (since it is subtracted from x), then (ω/k) must have a dimension of speed. Indeed, by Fourier superposition of all the different $u_k(x,t)$, we see that in a non-dispersive medium, where μ, ϵ are independent of frequency, the most general solution of the wave equation in one-dimension is

$$u(x,t) = f(x - vt) + g(x + vt) \tag{7.7}$$

where $f(\ldots)$ and $g(\ldots)$ are arbitrary functions, and v is the speed with which the wave travels along the x-axis, known as the ***phase velocity*** of the wave (to distinguish it from another important quantity called the group velocity, which we will discuss later). As is clear from the equation above, the phase velocity is $v = \omega/k$. Also, we see now why k is called the wave number, because $kv = \omega$.

In summary, any wave that preserves its shape and travels along the x-axis with speed v is a solution to the Helmholtz wave equation (as we can verify by direct substitution), and the general solution is just a superposition of the sets of two possible solutions — waves traveling along the positive and negative x-directions via Fourier superposition.

If there is dispersion in the medium (i.e., $\mu\epsilon$, and hence velocity, is a function of frequency), then equation (7.7) no longer holds. Each Fourier component is still an exponential as in equation (7.6), but their velocities are different, so the wave packet spreads out and changes shape as it propagates — we will study this later in the chapter.

Plane Wave Solutions: The General Case

Consider now the general case of a three-dimensional wave by considering a general electromagnetic plane wave of frequency ω and wave vector $\vec{k} = k\vec{n}$, where k is the wave number and $\vec{n}$ is the direction of propagation of the wave. We will demonstrate below that $\vec{n}$ must be a unit vector $\hat{n}$. You might be wondering at this stage why we don't just apportion magnitudes so that $\vec{k} = k\hat{n}$, but we need to be sure that the wave vector $\vec{k}$ can indeed be split up like this, with the wave number k being its magnitude.

We must make this three-dimensional plane wave satisfy equation (7.3), but note that $\vec{E}$ and $\vec{B}$ are not independent, they are linked via Maxwell's equations, so even if the wave is traveling in the direction indicated by the direction of $\vec{n}$, the amplitudes of $\vec{E}$ and $\vec{B}$ are linked; we cannot choose them separately. Let us write the (3-dimensional) plane waves as

$$\begin{aligned} \vec{E}(\vec{x},t) &= \vec{\mathcal{E}}\, e^{i(k\vec{n}\cdot\vec{x}-\omega t)} \\ \vec{B}(\vec{x},t) &= \vec{\mathcal{B}}\, e^{i(k\vec{n}\cdot\vec{x}-\omega t)} \end{aligned} \tag{7.8}$$

Remember that this isn't the final expression — we will demonstrate shortly that $\vec{n}$ has to be a unit vector. Notice also that $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ in equation (7.8) are defined as constant vectors; that is, their spatial and time dependencies are in the exponential term.

Each component of $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ from equation (7.8), e.g., $E_x(\vec{x},t), B_x(\vec{x},t), E_y(\vec{x},t)$, etc., satisfies the Helmholtz equation (7.3) provided

$$k^2\, \vec{n}\cdot\vec{n} = \mu\epsilon\omega^2 \tag{7.9}$$

You demonstrated this explicitly for $E_x(\vec{x},t)$ in *today's Discussion Worksheet.* Recall that it is important in carrying out this proof to be aware of how $E_x(\vec{x},t)$ would look; on the Discussion Worksheet for today, we wrote that

$$E_x(\vec{x},t) = \mathcal{E}_x\, e^{i(k\vec{n}\cdot\vec{x}-\omega t)} = \mathcal{E}_x\, e^{i(k\{n_x x+n_y y+n_z z\}-\omega t)}$$

where $\mathcal{E}_x$ is the x-component of the constant vector $\vec{\mathcal{E}}$.

In order to recover equation (7.4) that $k = \omega\sqrt{\mu\epsilon}$, equation (7.9) above imposes the constraint that $\vec{n}\cdot\vec{n} = 1$. This can only be the case if $\vec{n} = \hat{n}$, a unit vector, such that

$$\hat{n}\cdot\hat{n} = 1$$

Henceforth, therefore, we will always write $\hat{n}$ for the unit vector instead of $\vec{n}$.

With all of the above considerations, let us write equation (7.8) again, but now with the unit vector $\hat{n}$:

$$\begin{aligned} \vec{E}(\vec{x},t) &= \vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \\ \vec{B}(\vec{x},t) &= \vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \end{aligned} \tag{7.8.a}$$

Remember that $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are *constant* vectors; in other words, the spatial and time dependencies of $\vec{E}(\vec{x},t)$ are in the exponential term.

The above has been mostly mathematics following from the wave equation. In the next class, we will apply physics from Maxwell's equations (7.1) and see what that tells us about electromagnetic waves in particular.