PHY 420 — Spring 2021

Week 2—Tuesday, Apr 6—Discussion Worksheet

Conservation of Linear Momentum

In the previous class, we learned about energy conservation in the electromagnetic field (Poynting's theorem). Today, we will look at the ***conservation of linear momentum***.

Begin by writing the total electromagnetic force on a charged particle:

$$\vec{F} = q\Big(\vec{E} + \vec{v} \times \vec{B}\Big) \tag{6.113}$$

From Newton's second law, force is the rate of change of momentum, so we can write $\vec{F} = d\vec{P}/dt$. Now, if the sum of all the momenta of all the particles in the volume V is denoted by $\vec{P}_{\text{mech}}$, then we can write equation (6.113) in the form

$$\frac{d\vec{P}_{\text{mech}}}{dt} = \int\limits_V \Big(\rho\vec{E} + \vec{J} \times \vec{B}\Big)\, d^3x \tag{6.114}$$

where we've converted the sum over particles to an integral by replacing $q\vec{v}$ by $\vec{J}\, d^3x$ (as we did before in deriving Poynting's theorem), and this time, also replacing the q (that multiplies $\vec{E}$) by $\rho\, d^3x$. Again, as we did before for Poynting's theorem, use the Maxwell equations to eliminate ρ and $\vec{J}$ from equation (6.114). So $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ and $\vec{\nabla} \times \vec{H} = \vec{J} + \partial\vec{D}/\partial t$ gives

$$\rho = \epsilon_0\, \vec{\nabla} \cdot \vec{E} \qquad \text{and} \qquad \vec{J} = \frac{1}{\mu_0}\vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{6.115}$$

1. Using equation (6.115), show that the integrand of equation (6.114) becomes

$$\rho\vec{E} + \vec{J} \times \vec{B} = \epsilon_0 \left[\vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) - c^2 \vec{B} \times \left(\vec{\nabla} \times \vec{B}\right) + \vec{B} \times \frac{\partial \vec{E}}{\partial t}\right]$$

$$\rho\vec{E} + \vec{J} \times \vec{B} = (\epsilon_0 \vec{\nabla} \cdot \vec{E})\vec{E} + \left[\frac{1}{\mu_0}\vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right] \times \vec{B}$$

$$= \epsilon_0 \vec{E}(\vec{\nabla} \cdot \vec{E}) - \vec{B} \times \left[\frac{1}{\mu_0}\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t}\right]$$

$$= \epsilon_0 \vec{E}(\vec{\nabla} \cdot \vec{E}) - \epsilon_0\left[\epsilon_0 \times \left(\frac{1}{\mu_0 \epsilon_0}\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t}\right)\right]$$

$$= \epsilon_0 \vec{E}(\vec{\nabla} \cdot \vec{E}) - \epsilon_0\left[\vec{B} \times \left(c^2 \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t}\right)\right]$$

$$= \epsilon_0\left[\vec{E}(\vec{\nabla} \cdot \vec{E}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times \frac{\partial \vec{E}}{\partial t}\right]$$

2. We are in the process of examining how the field exchanges momentum with particles.

(a) Use

$$\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) = \frac{\partial \vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial \vec{B}}{\partial t}$$

to replace $\vec{B}\times\partial\vec{E}/\partial t$, and add $c^2\,\vec{B}(\vec{\nabla}\cdot\vec{B})$ to the square bracket (we can do this without subtracting a term because $\vec{\nabla}\cdot\vec{B}=0$), and show that you get

$$\rho\vec{E}+\vec{J}\times\vec{B} = \epsilon_0\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) + c^2\,\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right) + \vec{E}\times\frac{\partial\vec{B}}{\partial t}\right] - \epsilon_0\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)$$

$$\rightarrow \vec{B}\times\frac{\partial\vec{E}}{\partial t} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

$$\rho\vec{E}+\vec{J}\times\vec{B} = \epsilon_0\Big[\vec{E}(\vec{\nabla}\cdot\vec{E}) - c^2\vec{B}\times(\vec{\nabla}\times\vec{B}) + \underbrace{\vec{B}\times\frac{\partial\vec{E}}{\partial t}}_{-\frac{\partial}{\partial t}(\vec{E}\times\vec{B})+\vec{E}\times\frac{\partial\vec{B}}{\partial t}}\Big]$$

$$= \epsilon_0\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - c^2\vec{B}\times(\vec{\nabla}\times\vec{B}) + \vec{E}\times\frac{\partial\vec{B}}{\partial t}\right] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

Since $\vec{\nabla}\times\vec{B}=0$, we can add $c^2\vec{B}(\vec{\nabla}\cdot\vec{B})$ so we can write the equation above

(b) Use Faraday's law, $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$, to replace $\partial\vec{B}/\partial t$ in the last term inside the square brackets, put it all back into the integrand in equation (6.114), rearrange terms, and show that

$$\frac{d\vec{P}_{\text{mech}}}{dt} + \frac{d}{dt}\int_V \epsilon_0\left(\vec{E}\times\vec{B}\right)d^3x$$
$$= \epsilon_0\int_V\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right) + c^2\,\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)\right]d^3x \qquad (6.116)$$

$$\rightarrow \rho\vec{E}+\vec{J}\times\vec{B} = \epsilon_0\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + c^2\vec{B}(\vec{\nabla}\cdot\vec{B}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) - c^2\vec{B}\times(\vec{\nabla}\times\vec{B})\right] - \epsilon_0\,\partial/\partial t\,(\vec{E}\times\vec{B})$$

$$\rightarrow \frac{d\vec{P}_{mech}}{dt} + \frac{d}{dt}\epsilon_0\int_V(\vec{E}\times\vec{B})\,dv$$

$$= \int_V\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + c^2\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times(\vec{\nabla}\times\vec{B})\right]d^3x$$

We will now identify the nature of the second term on the left hand side of equation (6.116). For now, let's tentatively call it the total electromagnetic momentum $\vec{P}_{\text{field}}$ in the volume V:

$$\vec{P}_{\text{field}} = \epsilon_0 \int_V \vec{E} \times \vec{B}\, d^3x = \mu_0 \epsilon_0 \int_V \vec{E} \times \vec{H}\, d^3x \tag{6.117}$$

and putting $\mu_0\epsilon_0 = 1/c^2$, the integrand may be interpreted as a density of the electromagnetic momentum

$$\vec{g} = \frac{1}{c^2}\left(\vec{E} \times \vec{H}\right) \tag{6.118}$$

Since $\vec{S} = \vec{E} \times \vec{H}$, we should remember that the momentum density $\vec{g}$ is proportional to the energy flux density $\vec{S}$ with proportionality constant c^{-2}.

3. The association of the second term on the left hand side of equation (6.116) with the electromagnetic momentum will be established if we can convert the volume integral on the right hand side of equation (6.116) into a surface integral of the normal component of something that can be identified as momentum flow. To do so, begin by letting the Cartesian coordinates be denoted by x_α, with $\alpha = 1, 2, 3$. Show that the $\alpha = 1$ component of the electric part of the integrand in equation (6.116) is given explicitly by

$$\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E} \times \left(\vec{\nabla} \times \vec{E}\right)\right]_1$$

$$= E_1 \frac{\partial E_1}{\partial x_1} + E_1 \frac{\partial E_2}{\partial x_2} + E_1 \frac{\partial E_3}{\partial x_3} - E_2 \frac{\partial E_2}{\partial x_1} + E_2 \frac{\partial E_1}{\partial x_2} + E_3 \frac{\partial E_1}{\partial x_3} - E_3 \frac{\partial E_3}{\partial x_1}$$

(L.H.S.)

$$\begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ E_1 & E_2 & E_3 \\ (\vec{\nabla}\times\vec{E})_1 & (\vec{\nabla}\times\vec{E})_2 & (\vec{\nabla}\times\vec{E})_3 \end{vmatrix} \qquad \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ E_1 & E_2 & E_3 \end{vmatrix}$$

$$= E_1(\vec{\nabla}\cdot\vec{E}) - \left[\vec{E} \times (\vec{\nabla}\times\vec{E})\right]_1$$

$$= E_1\left[\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right] - \left[E_2\,\underbrace{(\vec{\nabla}\times\vec{E})_3} - E_3\,\underbrace{(\vec{\nabla}\times\vec{E})_2}\right]$$

$$= E_1\left[\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right] - \left[E_2\left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right) - E_3\left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}\right)\right]$$

Thus, we get the equation above

4. We are working with the electric part of the integrand ($\alpha = 1$ component) in equation (6.116).

(a) By careful rearrangement and manipulation of terms in the expression you obtained on the previous page, show that it becomes

$$\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]_1 = \frac{\partial}{\partial x_1}\left(E_1^2\right) + \frac{\partial}{\partial x_2}\left(E_1 E_2\right) + \frac{\partial}{\partial x_3}\left(E_1 E_3\right) - \frac{1}{2}\frac{\partial}{\partial x_1}\left(E_1^2 + E_2^2 + E_3^2\right)$$

Q3

$$= E_1\frac{\partial E_1}{\partial x_1} + E_1\frac{\partial E_2}{\partial x_2} + E_1\frac{\partial E_3}{\partial x_3} - E_2\frac{\partial E_2}{\partial x_1} + E_2\frac{\partial E_2}{\partial x_2} + E_3\frac{\partial E_3}{\partial x_1} + E_3\frac{\partial E_3}{\partial x_3}$$

$$= \left[\frac{\partial}{\partial x_1}\left(E_1^2\right) - E_1\frac{\partial E_1}{\partial x_1}\right] + \left[E_1\frac{\partial E_2}{\partial x_2} + E_2\frac{\partial E_1}{\partial x_2}\right] + \left[E_1\frac{\partial E_3}{\partial x_3} + E_3\frac{\partial E_1}{\partial x_3}\right] - E_2\frac{\partial E_2}{\partial x_1} - E_3\frac{\partial E_3}{\partial x_1}$$

$$= \frac{\partial}{\partial x_1}\left(E_1^2\right) + \left[E_1\frac{\partial E_2}{\partial x_2} + E_2\frac{\partial E_1}{\partial x_2}\right] + \left[E_1\frac{\partial E_3}{\partial x_3} + E_3\frac{\partial E_1}{\partial x_3}\right] - E_1\frac{\partial E_1}{\partial x_1} - E_2\frac{\partial E_2}{\partial x_1} - E_3\frac{\partial E_3}{\partial x_1}$$

$$= \frac{\partial}{\partial x_1}\left(E_1^2\right) + \frac{\partial}{\partial x_2}\left(E_1 E_2\right) + \frac{\partial}{\partial x_3}\left(E_1 E_3\right) - \left[E_1\frac{\partial E_1}{\partial x_1} + E_2\frac{\partial E_2}{\partial x_1} + E_3\frac{\partial E_3}{\partial x_1}\right]$$

$$= \frac{\partial}{\partial x_1}\left(E_1^2\right) + \frac{\partial}{\partial x_2}\left(E_1 E_2\right) + \frac{\partial}{\partial x_3}\left(E_1 E_3\right) - \frac{1}{2}\frac{\partial}{\partial x_1}\left(E_1^2 + E_2^2 + E_3^2\right)$$

(b) Show that your result above means that we can write the αth component of the electric part of the integrand in equation (6.116) as

$$\left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta}\left(E_\alpha E_\beta - \frac{1}{2}\vec{E}\cdot\vec{E}\,\delta_{\alpha\beta}\right) \tag{6.119}$$

Expand RHS for $\alpha = 1$

$$\sum_\beta \frac{\partial}{\partial x_\beta}\left(E_1 E_\beta - \tfrac{1}{2}\vec{E}\cdot\vec{E}_{1\beta}\,\delta_{1\beta}\right) = \frac{\partial}{\partial x_1}\left(E_1 E_1 - \tfrac{1}{2}\vec{E}\cdot\vec{E}_{11}\cancelto{1}{\delta_{11}}\right) + \frac{\partial}{\partial x_2}\cdots\cancelto{0}{\delta_{12}} + \frac{\partial}{\partial x_3}\cdots\cancelto{0}{\delta_{13}}$$

$$= \frac{\partial}{\partial x_1}\left(E_1^2\right) - \frac{1}{2}\frac{\partial}{\partial x_1}\left(\vec{E}\cdot\vec{E}\right) + \frac{\partial}{\partial x_2}\left(E_1 E_2\right) + \frac{\partial}{\partial x_3}\left(E_1 E_3\right)$$

$$= \frac{\partial}{\partial x_1}\left(E_1^2\right) + \frac{\partial}{\partial x_2}\left(E_1 E_2\right) + \frac{\partial}{\partial x_3}\left(E_1 E_3\right) - \frac{1}{2}\frac{\partial}{\partial x_1}\left(E_1^2 + E_2^2 + E_3^2\right)$$

$$= \left[\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]_1$$

By Symmetry this will work for $\alpha = 2$ and $\alpha = 3$

Inspection of equation (6.119) shows that we have the right hand side in the form of a divergence, which is what we need in order to write this as a conservation law. Since we made the form for $\vec{B}$ similar in equation (6.116), except for a multiplication by c^2, we should expect a similar form to equation (6.119) for $\vec{B}$. As we will see when we discuss tensors, the right hand side is in the form of the divergence of a second rank tensor.

5. With the above considerations in mind, define the **Maxwell stress tensor** $T_{\alpha\beta}$ as

$$T_{\alpha\beta} = \epsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right)\delta_{\alpha\beta}\right] \tag{6.120}$$

(a) Show that, following equation (6.119), the αth component of the integrand on the right hand side of equation (6.116) becomes

$$\epsilon_0 \left[\underbrace{\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)}_{6.119} + c^2\,\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)\right]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta} T_{\alpha\beta}$$

$$= \epsilon_0 \left[\sum_B \frac{\partial}{\partial X_B}\left(E_\alpha E_B - \tfrac{1}{2}\vec{E}\cdot\vec{E}\,\delta_{\alpha B}\right) + c^2 \sum_B \frac{\partial}{\partial X_B}\left(B_\alpha B_B - \tfrac{1}{2}\vec{B}\cdot\vec{B}\,\delta_{\alpha B}\right)\right]$$

$$= \epsilon_0 \sum_B \frac{\partial}{\partial X_B}\left[E_\alpha E_B + c^2 B_\alpha B_B - \tfrac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right)\delta_{\alpha B}\right]$$

$$= \sum_B \frac{\partial}{\partial X_B} T_{\alpha B}$$

(b) Show that the αth component of equation (6.116) is then

$$\frac{d}{dt}\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_\alpha = \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta\, da$$

where $\hat{n}$ is the outward normal to the closed surface S.

$$\frac{d}{dt}\int_V \epsilon_0\left(\vec{E}\times\vec{B}\right)d^3x = \frac{d}{dt}\vec{P}_{\text{field}}$$

$$\frac{d}{dt}\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_\alpha = \epsilon_0 \int_V \left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + c^2 B^2(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times(\vec{\nabla}\times\vec{B})\right]_\alpha d^3x$$

$$= \int_V \sum_B \frac{\partial}{\partial X_B} T_{\alpha B} \Big] \rightarrow \text{Divergence theorem}$$

$$= \oint \sum_{\alpha B} n_B\, da, \text{ where } \hat{n} \text{ is outward normal to } S.$$

If the equation in part (b) above represents a statement of the conservation of momentum, then $\sum_\beta T_{\alpha\beta}\, n_\beta$ is the αth component of the flow per unit area of momentum across the surface S into the volume V. In other words, it is the force per unit area transmitted across the surface S and acting on the combined system of particles and fields inside V.