

Week 4—Tuesday, Jan 26—Discussion Worksheet

Dispersion

So far, we have discussed wave propagation in non-dispersive media, in which the index of refraction was constant in frequency. In reality, the index of refraction is dependent on the frequency, which means that different frequency components of a wave travel at different speeds, changing the wave shape and spreading it out as it propagates through the medium; this is known as dispersion. We will now develop a simple model of dispersion to investigate the frequency dependence of ϵ .

Simple model for $\epsilon(\omega)$: Static Case

When a static electric field is present in a medium made of bound charges (electrons and ions), it polarizes those charges. Jackson models this as

$$\vec{F} = -m\omega_0^2 \vec{x} \quad (4.71)$$

where m is the mass of the charge, and ω_0 is the frequency of oscillation about equilibrium. See page 4 of this worksheet for more details, including the definition of the induced dipole moment and electric polarization, especially in dense media.

Simple model for $\epsilon(\omega)$: Time-varying Fields

For the dynamic case of electromagnetic waves, the equation of motion for an electron of charge $-e$ bound by a harmonic force and acted on by an electric field $\vec{E}(\vec{x}, t)$ is

$$m[\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}] = -e\vec{E}(\vec{x}, t) \quad (7.49)$$

where γ is the damping constant.

1. If we assume that the field (and hence \vec{x}) varies harmonically in time with frequency ω . that is, $\vec{x} = \vec{x}_0 e^{-i\omega t}$ then solve equation (7.49) and show that the dipole moment for one electron is

\swarrow take derivative $i\omega$
 \downarrow of $\vec{x} = \vec{x}_0 e^{-i\omega t}$

$$\vec{p} = -e\vec{x} = \frac{e^2}{m} \left(\frac{\vec{E}}{\omega_0^2 - \omega^2 - i\omega\gamma} \right) \quad (7.50)$$

$$m[\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}] = -e\vec{E}(\vec{x}, t)$$

$$\rightarrow m(-\omega^2 + i\omega\gamma + \omega_0^2)\vec{x}(\omega) = -e\vec{E}(\vec{x}, t)$$

$$\rightarrow \vec{x}(\omega) = \frac{-e}{m} \left(\frac{\vec{E}}{\omega_0^2 - \omega^2 - i\omega\gamma} \right)$$

$$\rightarrow -e\vec{x}(\omega) = \frac{e^2}{m} \left(\frac{\vec{E}}{\omega_0^2 - \omega^2 - i\omega\gamma} \right)$$

2. If we have N molecules per unit volume with Z electrons per molecule, and instead of a single binding frequency for all, there are f_j electrons per molecule with binding frequency ω_j and damping constants γ_j , then show that the dielectric constant is given by

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \quad (7.51)$$

$$\vec{p} = \frac{e^2}{m} \left(\frac{\vec{E}}{\omega_0^2 - \omega^2 - i\omega\gamma} \right) \cdot Z$$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \chi_e \quad \chi \vec{E} \cdot \epsilon_0 = \frac{e^2}{m} \left(\frac{\vec{E}}{\omega_0^2 - \omega^2 - i\omega\gamma} \right) \sum f_j$$


$$\rightarrow \chi = \frac{Ne^2}{m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}$$

$$\rightarrow \chi + 1 = 1 + \frac{Ne^2}{m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}$$

$$\rightarrow \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}$$

3. Show that equation (7.51) can also be written as

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j f_j \left\{ \frac{(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \omega^2 \gamma_j^2} + i \left[\frac{\omega \gamma_j}{(\omega_j^2 - \omega^2)^2 + \omega^2 \gamma_j^2} \right] \right\}$$



$$\frac{1}{\omega_j^2 - \omega^2 - i\omega\delta} = \frac{(\omega_j^2 - \omega^2) + i\omega\delta}{(\omega_j^2 - \omega^2)^2 + \omega^2 \delta^2}$$

$$= \frac{(\omega_j^2 - \omega^2)^2}{(\omega_j^2 - \omega^2)^2 + \omega^2 \delta^2} + \frac{i\omega\delta}{(\omega_j^2 - \omega^2)^2 + \omega^2 \delta^2}$$

$$= \frac{(\omega_j^2 - \omega^2)^2}{(\omega_j^2 - \omega^2)^2 + \omega^2 \delta^2} + i \left[\frac{\omega\delta}{(\omega_j^2 - \omega^2)^2 + \omega^2 \delta^2} \right]$$

Additional Notes for class today on $\epsilon(\omega)$: Static Case

Recall that Jackson assumes that in the presence of an electric field \vec{E} acting to displace a charge e from equilibrium, there will be a linear and harmonic restoring force on each charge acting to get the charge back into equilibrium, given by equation (4.71): $\vec{F} = -m\omega_0^2 \vec{x}$, where m is the mass of the charge, and ω_0 is the frequency of oscillation about equilibrium. So the charge e feels two forces: the restoring force \vec{F} written above, and the force $e\vec{E}$ exerted by the electric field. If the charge e is in equilibrium under the action of $e\vec{E}$ and \vec{F} , then we must have

$$\sum F \equiv e\vec{E} - m\omega_0^2 \vec{x} = 0$$

This allows us to define the induced dipole moment

$$\vec{p}_{\text{mol}} = e\vec{x} = \frac{e^2}{m\omega_0^2} \vec{E} \quad (4.72)$$

In reality, molecules have many bound charges, each of which has its own restoring force with a different natural frequency ω . If we define the molecular polarizability γ_{mol} as the ratio of the average molecular dipole moment to ϵ_0 times the applied field at the molecule, and consider a set of charges e_j with masses m_j and oscillation frequencies ω_j in *each* molecule, then we get

$$\gamma_{\text{mol}} = \frac{p_{\text{mol}}}{\epsilon_0 E} = \frac{1}{\epsilon_0} \sum_j \frac{e_j^2}{m_j \omega_j^2} \quad (4.73)$$

By definition, the electric polarization \vec{P} is the dipole moment per unit volume. So, we can write $\vec{P} = N \langle \vec{p}_{\text{mol}} \rangle$, where N is the number of molecules (or atoms) per unit volume. Now, recall that in the first week of this quarter, we discussed that the induced polarization \vec{P} is proportional to \vec{E} with a proportionality constant that is independent of direction:

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \quad (4.36)$$

where the constant χ_e is called the electric susceptibility of the medium, and that

$$\epsilon = \epsilon_0 (1 + \chi_e) \quad (4.38)$$

where ϵ/ϵ_0 is called the *dielectric constant*.

In equation (4.36), \vec{E} is the macroscopic electric field. In rarefied media where the molecular separations are large, there is little difference between the macroscopic field and the field acting on any molecule or group of molecules. In dense media with closely packed molecules, however, the polarization of neighboring molecules leads to an internal field \vec{E}_i at any given molecule in addition to the average macroscopic field \vec{E} , so that the total field at the molecule is $(\vec{E} + \vec{E}_i)$. Jackson presents arguments to demonstrate that $\vec{E}_i = \frac{\vec{P}}{3\epsilon_0}$ in equation (4.63). So, equation (4.72) for the dipole moment may now be written as an average dipole moment for all the molecules taking into account both the macroscopic field \vec{E} and the internal field \vec{E}_i as

$$\langle \vec{p}_{\text{mol}} \rangle = \sum_j \frac{e_j^2}{m_j \omega_j^2} (\vec{E} + \vec{E}_i) = \epsilon_0 \gamma_{\text{mol}} (\vec{E} + \vec{E}_i) \quad (4.67)$$

With $\vec{P} = N \langle \vec{p}_{\text{mol}} \rangle$ we can show that dielectric constant is then

$$\frac{\epsilon}{\epsilon_0} = \frac{1 + \frac{2}{3} N \gamma_{\text{mol}}}{1 - \frac{1}{3} N \gamma_{\text{mol}}}$$