Physics 412—Quantum Mice Uncertainty Winter 2020

We will continue our studies of the size and attitude of quantum mice, moving onto observations. Quantum mice come in two sizes, small and large. The size property is represented mathematically by the operator W, with two eigenvalues, 2 and 10, and corresponding eigenstates $|s\rangle$ and $|l\rangle$. The attitude property is represented by the operator A, with eigenvalues $+1$ (happy) and -1 (unhappy), and corresponding eigenstates $|h\rangle$ and $|u\rangle$.

The relationship between size and happiness can be found from the relationship between the eigenstates of W and A:

$$|s\rangle = \frac{1}{\sqrt{5}}\Big[|h\rangle + 2|u\rangle\Big], \quad |l\rangle = \frac{1}{\sqrt{5}}\Big[-2|h\rangle + |u\rangle\Big].$$

If we use the attitude states as our basis, we can represent the operators as matrices and the states as column vectors:

$$A \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad |h\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |u\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$W \leftrightarrow \frac{1}{5}\begin{bmatrix} 42 & -16 \\ -16 & 18 \end{bmatrix}, \qquad |s\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad |l\rangle \leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Note: You can answer all of these working in the attitude basis, using the representation given above. It's probably easiest that way.

(1) Suppose that you had an ensemble of quantum mice in the state $|\Psi\rangle = \sqrt{0.9}\,|h\rangle + \sqrt{0.1}\,|u\rangle$. If you measured the attitudes of these mice, what would be the average value and uncertainty in your measurements?

(2) Suppose that you had an ensemble of quantum mice in the state $|\Psi\rangle = \sqrt{0.9}\,|h\rangle + \sqrt{0.1}\,|u\rangle$. If you measured the size of these mice, what would be the average value and uncertainty in your measurements?

(3) If you wanted to create an ensemble of quantum mice that has the maximum possible uncertainty in the measurement of the size of the mice, what state should (could?) the mice be in? What would be their average size and the uncertainty in their size?

(4) If you measured the attitude of your mice from Question 3, what average value would you obtain, and what uncertainty?

(5) Is your answer to Question 3 unique, or are there multiple states that will predict the same average and uncertainty for measurements of the mouse's size? Explain.

(6) If the answer to Question 3 is not unique, do different states give different statistics if you measure the attitude of the mouse? Explain.

① $|\psi\rangle \leftrightarrow \begin{bmatrix} \sqrt{0.9} \\ \sqrt{0.1} \end{bmatrix} = \sqrt{0.1}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \left(\sqrt{0.1}\begin{bmatrix} 3 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\sqrt{0.1}\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \frac{8}{10}$$

$$A^2 \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\langle A^2\rangle = \langle\psi|A^2|\psi\rangle = \langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle = 1$$

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} = \sqrt{1-(0.8)^2} = \sqrt{0.36} = 0.6$$

2

$$\langle W\rangle = \langle \psi | W | \psi\rangle$$

$$W|\psi\rangle = \left(\frac{1}{5}\begin{bmatrix} 42 & -16 \\ -16 & 18\end{bmatrix}\right)\left(\sqrt{0.1}\begin{bmatrix} 3 \\ 1\end{bmatrix}\right) = \frac{\sqrt{0.1}}{5}\begin{bmatrix} 42\cdot 3 - 16 \\ -16\cdot 3 + 18\end{bmatrix} = \frac{\sqrt{0.1}}{5}\begin{bmatrix} 110 \\ -30\end{bmatrix}$$

$$\langle \psi|W|\psi\rangle = \left(\sqrt{0.1}\begin{bmatrix} 3 & 1\end{bmatrix}\right)\left(\frac{\sqrt{0.1}}{5}\begin{bmatrix} 110 \\ -30\end{bmatrix}\right) = \frac{1}{50}\left(330-30\right) = 6$$

$$\langle W^2\rangle = \langle\psi|W^2|\psi\rangle = \langle \psi|WW|\psi\rangle = \left(\langle\psi|W\right)\left(W|\psi\rangle\right)$$

$$= \frac{0.1}{25}\begin{bmatrix} 110 & -30\end{bmatrix}\begin{bmatrix} 110 \\ -30\end{bmatrix} = \frac{1}{250}\left(12100+900\right)$$

$$= \frac{1300}{25} = 52$$

$$\Delta W = \sqrt{\langle W^2\rangle - \langle W\rangle^2} = \sqrt{52-36} = 4$$

3 To maximize ΔW, we want 50% small and 50% large, e.g.

$$|\psi\rangle = \frac{1}{\sqrt{2}}|s\rangle + \frac{1}{\sqrt{2}}|\ell\rangle \longleftrightarrow \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 2\end{bmatrix} + \frac{1}{\sqrt{10}}\begin{bmatrix} -2 \\ 1\end{bmatrix}$$

$$|\psi\rangle \longleftrightarrow \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3\end{bmatrix}$$

(4) $\langle A \rangle = \langle \psi | A | \psi \rangle = \frac{1}{10} \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$= \frac{1}{10} \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \frac{1}{10}(1-9)$$

$\langle A \rangle = -0.8$ not very happy!

$$\langle A^2 \rangle = \langle \psi | A^2 | \psi \rangle = \langle \psi | I | \psi \rangle = 1$$

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} = \sqrt{1-(0.8)^2} = 0.6$$

(5) No! Can change the relative phase of $|s\rangle$ & $|\ell\rangle$

$|\psi\rangle = \frac{1}{\sqrt{2}}|s\rangle + \frac{e^{i\theta}}{\sqrt{2}}|\ell\rangle$ for any θ

e.g $|\psi\rangle = \frac{1}{\sqrt{2}}|s\rangle + \frac{i}{\sqrt{2}}|\ell\rangle$

$$|\psi\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{i}{\sqrt{2}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1-2i \\ 2+i \end{bmatrix}$$

$$\langle \psi | A | \psi \rangle = \frac{1}{2} \begin{bmatrix} 1+2i & 2-i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1-2i \\ 2+i \end{bmatrix}$$

$$\langle A \rangle = \frac{1}{2}\begin{bmatrix} 1+2i & 2-i \end{bmatrix}\begin{bmatrix} 1-2i \\ -2-i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+4 & -4 \underset{?}{\pm} 1 \end{bmatrix} = 1 \;??$$

check!

$\langle A^2 \rangle$ is still 1, so

$$AA = 0 \;!!$$