PHY 411 **Winter 2021**

Week 7—Thursday, Feb 18—Discussion Worksheet

Associated Legendre Functions and Spherical Harmonics

The ***generalized Legendre equation***, with $x = \cos\theta$, is

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P = 0 \tag{3.9}$$

Solutions to the generalized Legendre equation are known as **associated Legendre functions**.

Previously, we dealt with potential problems involving azimuthal symmetry, so we put $m = 0$ in the equation above (and obtained the ordinary Legendre equation), with solutions in terms of Legendre polynomials of order l, $P_l(x)$. The general potential problem, however, can have azimuthal variations, so that $m \neq 0$. Therefore, we need the generalization of $P_l(x)$, i.e., the solution of the generalized Legendre equation written above, with l and m both arbitrary.

The solution of the generalized Legendre equation involves the same consideration for θ as the ordinary Legendre equation we discussed previously: the whole range of cos θ is in play, including the north and south poles. Therefore, we demand that the desired solution be single valued, finite, and continuous on the interval $-1 \leq x \leq 1$, in order that it represent a physical solution. In order to have finite solutions on the interval $-1 \leq x \leq 1$, it can be shown that the parameter l must be zero or a positive integer and the integer m can only take the values $-l, -(l-1), \ldots, 0, \ldots, (l-1), l$. The solution having these properties is called an ***associated Legendre function*** $P_l^m(x)$. For positive m, it is defined by the formula:

$$P_l^m(x) = (-1)^m\,(1-x^2)^{m/2}\,\frac{d^m}{dx^m}\,P_l(x) \tag{3.49}$$

whereas $P_l^{-m}(x)$ can be obtained from $P_l^m(x)$ because they are proportional, as the generalized Legendre equation (3.9) depends only on m^2 and m is an integer:

$$P_l^{-m}(x) = (-1)^m\,\frac{(l-m)!}{(l+m)!}\,P_l^m(x) \tag{3.51}$$

The choice of the arbitrary phase factor $(-1)^m$ is by convention (see Jackson's footnote on page 108 for the original source).

If $P_l(x)$ is written explicitly using Rodrigues' formula, then the corresponding expression for $P_l^m(x)$ is valid for both positive and negative integers m:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}\,(1-x^2)^{m/2}\,\frac{d^{\,l+m}}{dx^{l+m}}\,(x^2-1)^l \tag{3.50}$$

For fixed m, the associated Legendre functions $P_l^m(x)$ form an orthogonal set in the interval $-1 \leq x \leq 1$:

$$\int_{-1}^{1} P_{l'}^m(x)\,P_l^m(x)\,dx = \frac{2}{2l+1}\,\frac{(l-m)!}{(l+m)!}\,\delta_{l'\,l} \tag{3.52}$$

We now have the **full solution to the generalized Legendre equation (3.9)**:

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A'_{lm}\, r^l + \frac{B'_{lm}}{r^{l+1}} \right) P_l^m(\cos\,\theta)\, e^{im\phi}$$

1. Why have I written only $e^{im\phi}$ above, even though the ϕ-solution is actually $Q_m(\phi) = e^{\pm im\phi}$?

The $\pm$ m is still in the equation, we can tell from

$$\sum_{m=-l}^{l}$$

Now, consider the following:

- The functions $Q_m(\phi) = e^{im\phi}$ form a complete set of orthogonal functions in the index m on the interval $0 \le \phi \le 2\pi$.
- The functions $P_l^m(\cos\,\theta)$ form a complete orthogonal set in the index l for each m value in the interval $-1 \le \cos\,\theta \le 1$.

So, we can use the orthogonality relation equation (3.52), together with the factor 2π, to write normalized versions of $P_l^m(x)$ and $e^{im\phi}$ into the solution itself in the following manner:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm}\, r^l + \frac{B_{lm}}{r^{l+1}} \right) \left[\sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\,\theta)\, e^{im\phi} \right]$$

This is very convenient, because we can now define the term in square brackets that is a combination of the angular factors (θ,ϕ) as a complete set of orthogonal functions $Y_{lm}(\theta,\phi)$ in the indices (l,m) that are normalized in the intervals $-1 \le \cos\,\theta \le 1$ and $0 \le \phi \le 2\pi$ so that our solution to the generalized Legendre equation equation (3.9) now looks like:

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm}\, r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_{lm}(\theta,\phi) \tag{3.61}$$

where the functions $Y_{lm}(\theta,\phi)$ are called the **spherical harmonics**, and are given by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\,\theta)\, e^{im\phi} \tag{3.53}$$

Explicit forms of $Y_{lm}(\theta,\phi)$ for $0 \le l \le 3$ are written on page 109 in Jackson. $Y_{lm}(\theta,\phi)$ form what are known as orthonormal functions (because they are normalized and orthogonal) over all angles (θ,ϕ) of the unit sphere. In other words

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\,\theta\, d\theta\, Y^*_{l'm'}(\theta,\phi)\, Y_{lm}(\theta,\phi) = \delta_{l'l}\, \delta_{m'm} \tag{3.54}$$

2. With $Y_{lm}(\theta, \phi)$ defined in equation (3.53), show that

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi) \tag{3.55}$$

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta) e^{im\varphi}$$

$$Y_{l-m}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l+m)!}{(l-m)!}}\, P_l^{-m}(\cos\theta) e^{-im\varphi}$$

$$P_l^{-m} = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m$$

$$Y_{l-m}(\theta, \varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m \cos\theta\, e^{-im\varphi}$$

$$\Rightarrow \quad Y_{l,-m} = (-1)^m Y_{lm}^*(\theta, \varphi)$$

Since $Y_{lm}(\theta, \phi)$ form a complete set of functions, an arbitrary function $f(\theta, \phi)$ can be expanded in spherical harmonics:

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} Y_{lm}(\theta, \phi) \tag{3.56}$$

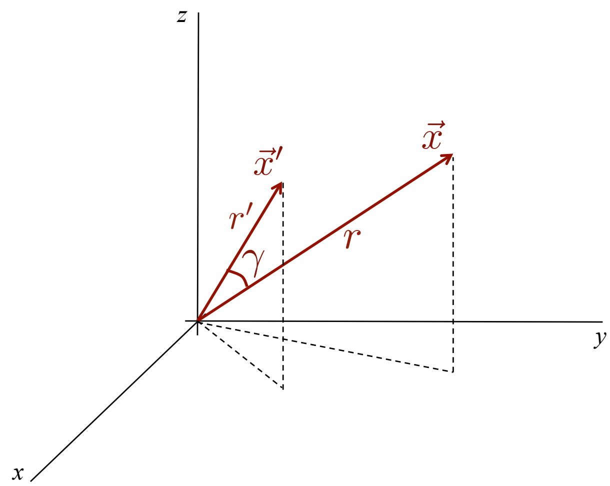
where the coefficients C_{lm} are given by

$$C_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, f(\theta, \phi)\, Y_{lm}^*(\theta, \phi)$$

A useful expansion

Now that we have the machinery of the expansion in Legendre polynomials, we can take advantage of it to render the expansion of $1/|\vec{x} - \vec{x}\,'|$, the potential of a unit point charge at $\vec{x}\,'$.

Consider the figure on the right (taken from Figure 3.3 on page 102 in Jackson), where the potential is sought at the observation point $\vec{x}$ due to a unit point charge at the source point $\vec{x}\,'$; notice that the angle between $\vec{x}$ and $\vec{x}\,'$ is γ.



We know that the potential at the observation point $\vec{x}$ due to the unit point charge at $\vec{x}\,'$ is

$$\Phi(\vec{x}) = \frac{1}{|\vec{x} - \vec{x}\,'|}$$

where we've ignored factors of $4\pi\epsilon_0$ for now.

With $r = |\vec{x}|$, and $r' = |\vec{x}\,'|$, the expansion can be written as

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}}$$

Without any loss of generality, we can rotate axes so that $\vec{x}\,'$ lies along the z-axis. Then, the angle γ is just the angle θ of the observation point $\vec{x}$. More important, though, letting $\vec{x}\,'$ lie along the z-axis means that the potential is azimuthally symmetric, so we can express it in terms of Legendre polynomials (with γ playing the role of θ):

$$\Phi(\vec{x}) = \sum_{l=0}^{\infty} \left[A_l\, r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\gamma)$$

Setting the two expressions for $\Phi(x)$ above equal to each other, and substituting the expression for $1/|\vec{x} - \vec{x}\,'|$ above, we now have

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}} = \sum_{l=0}^{\infty} \left[A_l\, r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\gamma)$$

To find the coefficients A_l and B_l, we note that this relation must be valid for all θ (being called γ here), so we can simplify our task by choosing a particular θ; let's choose $\gamma = 0$ (which is what Jackson means when he says "if the point $\vec{x}$ is on the z-axis"):

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'}} = \frac{1}{\sqrt{(r - r')^2}} = \frac{1}{|r - r'|}$$

We have a choice of sign for the square root and pick the positive, since we're interested in the magnitude. So, now we have

$$\frac{1}{|r - r'|} = \sum_{l=0}^{\infty} \left[A_l\, r^l + \frac{B_l}{r^{l+1}} \right]$$

3. Consider the expansion of $1/|\vec{x} - \vec{x}\,'|$ that we wrote on the previous page.

For $r < r'$, that is, when the *observation point is closer to the origin than the source charge*, show that we get

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{r'} \sum_{l=0}^{\infty} \left(\frac{r}{r'}\right)^l$$

Since $(r - r') < 0$, we can write $|r - r'| = (r' - r)$

$$\frac{1}{r' - r} = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right]$$

Take r' out so that $1 - r/r'$

$$\frac{1}{1 - r/r'} = \sum_{l=0}^{\infty} \left[A_l r' r^l + r' \frac{B_l}{r^{l+1}} \right]$$

$$\rightarrow \sum_{l=0}^{\infty} \left(\frac{r}{r'}\right)^l = \sum_{l=0}^{\infty} \left[A_l r' r^l + \frac{r' B_l}{r^{l+1}} \right]$$

$$\frac{1}{r' - r} = \sum_{l=0}^{\infty} \left[A_l r^l + 0 \right] = \sum_{l=0}^{\infty} \frac{1}{(r')^{l+1}} r^l = \frac{1}{r'} \sum_{l=0}^{\infty} \left(\frac{r}{r'}\right)^l$$

thus,

$$\left.\frac{1}{|\vec{x} - \vec{x}\,'|}\right|_{\gamma = 0} = \left.\frac{1}{|r - r'|}\right|_{r < r'} = \frac{1}{r' - r} = \frac{1}{r'} \sum_{l=0}^{\infty} \left(\frac{r}{r'}\right)^l$$

For additional practice, you should demonstrate that for $r > r'$, that is, when the *observation point is farther from the origin than the source charge*, we get

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l$$

4. Consider the expressions written on the previous page.

(a) We can write the results for the two cases ($r < r'$ and $r > r'$) in one equation by writing

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l$$

Explain what we mean by $r_<$ and $r_>$ for each case ($r < r'$ and $r > r'$).

In the case $r' > r$ or in the case $r > r'$

$r_<$ means that $r_<$ is lesser of the two quantities r or r'. while, it is similarly true to say that $r_>$ is greater of the two quantities r and r'

$r_< = \min(r, r')$ while $r_> = \max(r, r')$

(b) The above expression was derived by choosing $\gamma \equiv \theta = 0$, i.e., only for points $\vec{x}$ lying on the z-axis. For points off the axis, all we need to do is multiply by $P_l(\cos\gamma)$. Show that you would then get

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left(\frac{r_<^l}{r_>^{l+1}}\right) Y_{lm}^*(\theta', \phi')\, Y_{lm}(\theta, \phi) \tag{3.70}$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l$$

$$\rightarrow \frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l \cdot P_l(\cos\gamma)$$

$$\rightarrow P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^m(\theta, \varphi)\, Y_l^m(\theta', \varphi')^*$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left(\frac{r^l}{r^{l+1}}\right) Y_{lm}^*(\theta', \varphi')\, Y_{lm}(\theta, \varphi)$$