PHY 474 Spring 2021
Homework 3 solutions

1. Let h be the height in the atmosphere above the photosphere. Then, the mean free path $\lambda = 1/\kappa\rho$ must satisfy the relation

$$1 \simeq \int_0^\infty \kappa\rho \, dh \tag{1}$$

Approximate the opacity κ by a power law

$$\kappa = \kappa_0^{(\mathrm{ph})} \, \rho^a T^b \tag{2}$$

where (ph) stands for the photosphere, the surface of the star that is at the base of its atmosphere. Take care that you don't confuse the use of a in the exponent for the density with the use of a for the radiation density constant (e.g., in Question 4); they are different.

Assuming an *isothermal atmosphere*, the variation of density with height is given by

$$\rho = \rho_{\mathrm{ph}} \, \exp\left(-\frac{h}{H}\right) \tag{3}$$

where the density scale height H is the same as the pressure scale height we wrote in class, that is

$$H = \frac{k_B T}{\mu m_p g} \tag{4}$$

where $g = GM/R^2$ is the *gravitational acceleration* in the atmosphere.

(a) Replacing κ in equation (1) with the expression in equation (2), then using equation (3) to substitute for ρ, show that equation (1) integrates to

$$1 \simeq \frac{H}{a+1} \kappa_{\mathrm{ph}} \, \rho_{\mathrm{ph}}$$

where $\kappa_{\mathrm{ph}} = \kappa_0^{(\mathrm{ph})} \, \rho_{\mathrm{ph}}^a \, T_{\mathrm{eff}}^b$ is the opacity of the photosphere.

Solution: First, replace κ in equation (1) with the expression in equation (2):

$$1 \simeq \int_0^\infty \left[\kappa_0^{(\mathrm{ph})} \, \rho^a T_{\mathrm{eff}}^b\right] \rho \, dh$$

where I've put $T = T_{\mathrm{eff}}$ since we're going to be assuming an isothermal atmosphere when we substitute for ρ below. Thus, we now have

$$1 \simeq \int_0^\infty \kappa_0^{(\mathrm{ph})} \, \rho^{a+1} \, T_{\mathrm{eff}}^b \, dh \tag{5}$$

Next, put ρ from equation (3) into equation (5) above to get

$$1 \simeq \int_0^\infty \kappa_0^{(\mathrm{ph})} \left[\rho_{\mathrm{ph}} \, \exp\left(-\frac{h}{H}\right)\right]^{a+1} T_{\mathrm{eff}}^b \, dh$$

so that

$$1 \simeq \int_0^\infty \kappa_0^{(\mathrm{ph})} \, \rho_{\mathrm{ph}}^{a+1} \left[\exp\left(-\left\{\frac{a+1}{H}\right\} h\right)\right] T_{\mathrm{eff}}^b \, dh \tag{6}$$

Let's gather terms from equation (6) as shown below, also taking them outside the integral:

$$1 \simeq \left(\kappa_0^{(\mathrm{ph})}\,\rho_{\mathrm{ph}}^a\,T_{\mathrm{eff}}^b\right)\rho_{\mathrm{ph}}\int_0^\infty \left[\exp\left(-\left\{\frac{a+1}{H}\right\}h\right)\right]dh$$

The terms in parentheses are just κ_{ph}, so that

$$1 \simeq \kappa_{\mathrm{ph}}\,\rho_{\mathrm{ph}}\int_0^\infty \left[\exp\left(-\left\{\frac{a+1}{H}\right\}h\right)\right]dh$$

Integrating, we get

$$1 \simeq \kappa_{\mathrm{ph}}\,\rho_{\mathrm{ph}}\left[-\left\{\frac{H}{a+1}\right\}\exp\left(-\left\{\frac{a+1}{H}\right\}h\right)\right]_0^\infty$$

so that

$$1 \simeq -\kappa_{\mathrm{ph}}\,\rho_{\mathrm{ph}}\left\{\frac{H}{a+1}\right\}\left[\exp(-\infty)-\exp(0)\right]$$

and thus

$$1 \simeq -\kappa_{\mathrm{ph}}\,\rho_{\mathrm{ph}}\left\{\frac{H}{a+1}\right\}\left[0-1\right]$$

Finally

$$1 \simeq \frac{H}{a+1}\,\kappa_{\mathrm{ph}}\,\rho_{\mathrm{ph}}$$

which is the result we were asked to obtain.

(b) Using the ideal gas law in the form $P = \rho k_B T/\mu m_p$ that we wrote in class, show that

$$P_{\mathrm{ph}} = \left[\frac{GM(a+1)}{R^2\kappa_0^{(\mathrm{ph})}}\right]^{1/(a+1)}\left[\frac{k_B}{\mu m_p}\right]^{a/(a+1)} T_{\mathrm{eff}}^{(a-b)/(a+1)}$$

Solution: Begin from the result of part (a):

$$1 \simeq \frac{H}{a+1}\,\kappa_{\mathrm{ph}}\,\rho_{\mathrm{ph}}$$

and put $\kappa_{\mathrm{ph}} = \kappa_0^{(\mathrm{ph})}\,\rho_{\mathrm{ph}}^a\,T_{\mathrm{eff}}^b$ to get

$$1 \simeq \frac{H}{a+1}\left[\kappa_0^{(\mathrm{ph})}\,\rho_{\mathrm{ph}}^a\,T_{\mathrm{eff}}^b\right]\rho_{\mathrm{ph}}$$

so that

$$1 \simeq \frac{H}{a+1}\,\kappa_0^{(\mathrm{ph})}\,\rho_{\mathrm{ph}}^{a+1}\,T_{\mathrm{eff}}^b$$

Cross multiplying, we get

$$\rho_{\mathrm{ph}}^{a+1} = \left(\frac{a+1}{H}\right)\frac{1}{\kappa_0^{(\mathrm{ph})}}\,T_{\mathrm{eff}}^{-b} \tag{7}$$

To proceed, we will use the ideal gas law.

From the ideal gas law in the form $P = \rho k_B T/\mu m_p$, we have that

$$\rho = P \left(\frac{\mu m_p}{k_B T_{\rm eff}} \right)$$

where, as before, I've put $T = T_{\rm eff}$ since we're working in an isothermal atmosphere. If we substitute this in equation (7), we will get

$$\rho_{\rm ph}^{a+1} = \left[P_{\rm ph} \left(\frac{\mu m_p}{k_B T_{\rm eff}} \right) \right]^{a+1} = \left(\frac{a+1}{H} \right) \frac{1}{\kappa_0^{\rm (ph)}} T_{\rm eff}^{-b}$$

Let's take the $1/(a+1)$th root of both sides; that is, let's raise the left and right hand sides to the power $1/(a+1)$:

$$\left\{ \left[P_{\rm ph} \left(\frac{\mu m_p}{k_B T_{\rm eff}} \right) \right]^{a+1} \right\}^{1/(a+1)} = \left\{ \left(\frac{a+1}{H} \right) \frac{1}{\kappa_0^{\rm (ph)}} T_{\rm eff}^{-b} \right\}^{1/(a+1)}$$

We can distribute this power across all three terms on the right hand side to get

$$P_{\rm ph} \left(\frac{\mu m_p}{k_B T_{\rm eff}} \right) = \left(\frac{a+1}{H} \right)^{1/(a+1)} \left\{ \frac{1}{\kappa_0^{\rm (ph)}} \right\}^{1/(a+1)} \left(T_{\rm eff} \right)^{-b/(a+1)}$$

Keeping only $P_{\rm ph}$ on the left hand side, we get

$$P_{\rm ph} = \left(\frac{k_B T_{\rm eff}}{\mu m_p} \right) \left(\frac{a+1}{H} \right)^{1/(a+1)} \left\{ \frac{1}{\kappa_0^{\rm (ph)}} \right\}^{1/(a+1)} \left(T_{\rm eff} \right)^{-b/(a+1)} \tag{8}$$

Also from equation (4), we have

$$H = \frac{k_B T}{\mu m_p g} \equiv \frac{k_B T_{\rm eff} R^2}{\mu m_p G M}$$

where, yet again, I've put $T = T_{\rm eff}$ since we're working in an isothermal atmosphere, and also substituted $g = GM/R^2$. Upon putting this expression for H into equation (8) above, we get

$$P_{\rm ph} = \left(\frac{k_B T_{\rm eff}}{\mu m_p} \right) \left[\frac{\mu\, m_p\, GM\, (a+1)}{k_B T_{\rm eff}\, R^2} \right]^{1/(a+1)} \left\{ \frac{1}{\kappa_0^{\rm (ph)}} \right\}^{1/(a+1)} \left(T_{\rm eff} \right)^{-b/(a+1)}$$

Combining terms with an eye on the answer, we get

$$P_{\rm ph} = \left(\frac{k_B T_{\rm eff}}{\mu m_p} \right)^{1-1/(a+1)} \left[\frac{GM\, (a+1)}{R^2\, \kappa_0^{\rm (ph)}} \right]^{1/(a+1)} \left(T_{\rm eff} \right)^{-b/(a+1)}$$

or

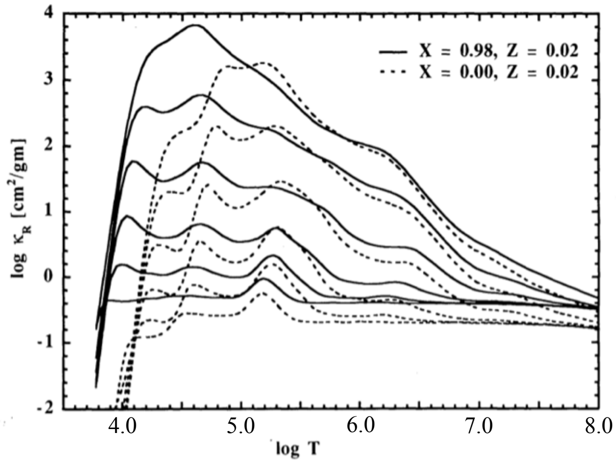
$$P_{\rm ph} = \left(\frac{k_B T_{\rm eff}}{\mu m_p} \right)^{\frac{a+1-1}{(a+1)}} \left[\frac{GM\, (a+1)}{R^2\, \kappa_0^{\rm (ph)}} \right]^{1/(a+1)} \left(T_{\rm eff} \right)^{-b/(a+1)}$$

Therefore, we get finally

$$P_{\rm ph} = \left[\frac{GM(a+1)}{R^2 \kappa_0^{\rm (ph)}} \right]^{1/(a+1)} \left[\frac{k_B}{\mu m_p} \right]^{a/(a+1)} T_{\rm eff}^{(a-b)/(a+1)}$$

which is the desired expression.

2. Consider the graph of opacity *vs.* temperature shown below.



Recall that although we wrote analytic expressions for the opacity, it is usually determined from computed tables. The plot above shows the opacity as a function of temperature for different values of the density (actually a quantity R which has a relation to the density) taken from such a table.

(a) Discuss why the opacity dips sharply at low temperatures, as seen on the left of the graph above. **Note:** I see now that some of you read the question to be about the gradient on the low T side (which is indeed due to the opacity dependence of H^-), rather than the value itself at low T; thus, "dips sharply" should be replaced by "is low" instead.

Solution: At very low temperatures, opacity comes mainly from bound-bound scattering, for which the opacity is very low, so low in fact that bound-bound processes were not considered to be relevant inside stars for a long time (although revisions to opacity tables were made starting in the 1990's based on newly derived bound-bound opacities). Thus, the opacity is low at these temperatures.

(b) Discuss why the opacity levels off at high temperatures, as seen on the right of the graph above.

Solution: At high temperatures, the opacity is mainly due to electron scattering, given by

$$\kappa_e = 0.2\,(1 + X)\,\mathrm{cm\ g}^{-1}$$

We see that this is independent of density and temperature. That explains why the opacity levels off in the high temperature region on the right of the graph.

Additional material of interest (not needed for the homework answer):

Starting from the left side of the graph, as T increases, we see a series of bumps. For the solid curve, the first bump showing a rapid rise in opacity is caused by the ionization of hydrogen. For the dotted curve corresponding to $X = 0$ (no hydrogen), the first bump is caused by the ionization of helium (which explains why it is to the right of the first bump in the solid line; ionization of helium will require a higher temperature than the ionization of hydrogen).

Subsequent bumps are due to different ionizations. For example, the bump to the right of $\log T = 5$ is more pronounced for $X = 0$ and can be ascribed, in part, to the photoionization of the electron in singly-ionized helium. Finally, at higher temperatures, notice how the dotted curves level off to a lower opacity; this is consistent with putting $X = 0$ in the expression for κ_e written above. For more details, see Rogers and Iglesias (1992), *Astrophysical Journal*, vol. 401, page 361. Note that I've changed the x-axis of the graph from their $\log T_6$ to $\log T$.

3. Consider a star that has values for relevant physical quantities near its center as given below.

r	$m(r)$	$L(r)$	$T(r)$	$\rho(r)$	κ
$0.1\ R_\odot$	$0.028\ M_\odot$	$24.4\ L_\odot$	2.2×10^7 K	3.1×10^4 kg m^{-3}	0.04 m^2 kg^{-1}

Assuming a mean molecular weight, $\mu = 0.7$, determine by appropriate calculations whether the energy transport at this location is radiative or convective. *Choosing an answer without mathematical justification may result in negative points.*

Solution: We learned in class that convective instability occurs if the gradient

$$\nabla_R > \nabla_{\rm ad}$$

where the adiabatic gradient is

$$\nabla_{\rm ad} = \frac{\Gamma_2}{\Gamma_2 - 1} = \frac{\gamma}{\gamma - 1} = \frac{5/3}{5/3 - 1} = \frac{2}{5}$$

and the radiative temperature gradient is given by

$$\nabla_R \equiv \frac{d\ln T}{d\ln P} = \frac{3k_B}{16\pi acGm_p}\,\frac{\kappa}{\mu}\,\frac{L(r)}{m(r)}\,\frac{\rho}{T^3}$$

In other words, we are checking to see how ∇_R compares to $\nabla_{\rm ad}$; if $\nabla_R < \nabla_{\rm ad}$, energy transport at this location will be radiative, and if $\nabla_R > \nabla_{\rm ad}$, energy transport will be convective.

Since the expression for ∇_R is very long, I'll put in the numbers without their units, so let's write them down here first:

- First, the standard constants have their usual values in SI units: $k_B = 1.38 \times 10^{-23}$ J/K, $c = 3 \times 10^8$ m/s, $G = 6.67 \times 10^{-11}$ N m^{-2} kg^{-2}, $m_p = 1.67 \times 10^{-27}$ kg.
- Next, $a = 7.566 \times 10^{-16}$ J m^{-3} K^{-4} is the radiation density constant that comes from the energy density of radiation, $u_R = aT^4$.
- Then, the units for κ and ρ are given in the table at the top of the page, and T is in K.
- The mean molecular weight $\mu = 0.7$.
- The luminosity $L(r)$ given in the table will need to be multiplied by $L_\odot = 3.846 \times 10^{26}$ watts, and the mass $m(r)$ will need to be multiplied by $1\ M_\odot = 1.99 \times 10^{30}$ kg.

Let's put in the numbers into the expression for ∇_R without writing their units (for brevity); the units are listed above for reference. We'll still need two lines!

$$\nabla_R = \left[\frac{3(1.38 \times 10^{-23})}{16\pi(7.566 \times 10^{-16})(3 \times 10^8)(6.67 \times 10^{-11})(1.67 \times 10^{-27})}\right]$$

$$\left[\frac{0.04}{0.7}\right]\left[\frac{24.4\,(3.846 \times 10^{26})}{0.028\,(1.99 \times 10^{30})}\right]\left[\frac{3.1 \times 10^4}{(2.2 \times 10^7)^3}\right] = 0.91$$

Since $\nabla_R > \nabla_{\rm ad} = 2/5$, $\boxed{\text{the energy transport is convective}}$ at this location.

4. In class, you showed that the luminosity $L(r)$ is given by

$$L = -\frac{4\pi r^2 ac}{3\kappa\rho}\frac{d}{dr}\left(T^4\right)$$

Replacing r by R, $-d(T^4)/dr$ by T^4/R, the opacity by a power law $\kappa \simeq \kappa_0\,\rho^\lambda T^{-\nu}$, and other appropriate steps, show that the surface luminosity of a star is given by

$$L_s \simeq \frac{ac}{\kappa_0}\left(\frac{G\mu m_p}{k_B}\right)^{4+\nu} R^{3\lambda-\nu}\,M^{3+\nu-\lambda}$$

Solution: As directed, let's begin by replacing r by R (which means that $L \equiv L_s$) and $-d(T^4)/dr$ by T^4/R in the equation for L, to get

$$L_s \simeq \frac{4\pi R^2 ac}{3\,\kappa\rho}\frac{T^4}{R}$$

Next, replace the opacity by a power law $\kappa \simeq \kappa_0\,\rho^\lambda T^{-\nu}$, so that the equation for L_s above becomes

$$L_s \simeq \frac{4\pi R^2 ac}{3\,(\kappa_0\,\rho^\lambda T^{-\nu})\,\rho}\frac{T^4}{R}$$

We see that the final expression does not have any numerical factors, so let's drop $4\pi/3$, which we can do since we are using the $\simeq$ sign to connect the left and right hand sides. Thus

$$L_s \simeq \frac{R^2 ac}{(\kappa_0\,\rho^\lambda T^{-\nu})\,\rho}\frac{T^4}{R}$$

Canceling R in the numerator and denominator, combining the ρ factors in the denominator, and writing T in one location, this becomes

$$L_s \simeq \left[\frac{Rac}{\kappa_0\,\rho^{\lambda+1}}\right] T^{4+\nu}$$

From equation (4.9) in *Jackson*, we can write $T = G\mu m_p M/k_B R$, and putting this in the equation for L_s, together with $\rho = M/R^3$, we get

$$L_s \simeq \left[\frac{Rac}{\kappa_0\,(M/R^3)^{\lambda+1}}\right]\left[\frac{G\mu m_p M}{k_B R}\right]^{4+\nu}$$

Rearrange terms with an eye on the answer:

$$L_s \simeq \frac{ac}{\kappa_0}\left(\frac{G\mu m_p}{k_B}\right)^{4+\nu}\left[\frac{R\,R^{3(\lambda+1)}}{(M)^{\lambda+1}}\right]\left[\frac{M}{R}\right]^{4+\nu}$$

or

$$L_s \simeq \frac{ac}{\kappa_0}\left(\frac{G\mu m_p}{k_B}\right)^{4+\nu}\left[R^{1+3\lambda+3-(4+\nu)}\right]\left[M^{4+\nu-(\lambda+1)}\right]$$

so that, finally we get

$$L_s \simeq \frac{ac}{\kappa_0}\left(\frac{G\mu m_p}{k_B}\right)^{4+\nu} R^{3\lambda-\nu}\,M^{3+\nu-\lambda}$$