PHY 420 Spring 2021

Homework 2 solutions

1. An infinitely long straight wire along the z-axis carries a uniform current I (moving upward toward the positive z-direction). A spherical shell of radius R with a total charge Q uniformly distributed over its surface is centered at the origin (through which the wire also passes, since the wire is along the z-axis).

(a) Write down an expression for $\vec{E}$ (i.e., *magnitude and direction*) produced by this configuration.

Solution: This is a straightforward use of Gauss' law. Draw a Gaussian sphere of radius r centered at the origin. The enclosed charge is zero if we're inside the sphere ($r < R$), and Q if we're outside ($r > R$), so for the latter case

$$\oint_S \vec{E} \cdot \hat{n}\, da = \frac{Q}{\epsilon_0}$$

Since $\vec{E}$ is everywhere radial by symmetry ($\vec{E} = E\hat{r}$), and $\hat{n} = \hat{r}$ for our arrangement, we get that

$$\oint_S \vec{E} \cdot \hat{n}\, da = E \oint_S da = E\Big(4\pi r^2\Big) = \frac{Q}{\epsilon_0}$$

so that the magnitude of the electric field is

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

and thus

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{r}}$$

(b) Write down an expression for $\vec{B}$ (*magnitude and direction*) produced by this configuration.

Solution: This is an Ampere's law problem, since you may assume that the magnetic field is produced only by the current in the infinitely long wire, so

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$$

With an Amperian loop of radius ρ, we get that

$$\oint_C \vec{B} \cdot d\vec{l} = B \oint_C dl = \mu_0 I$$

where the fingers of the right hand wrap around the direction of $\vec{B}$ when the thumb is pointed along the positive z-direction, the direction of flow of the current. Notice that I'm using ρ to distinguish from the r used for the electric field, since ***the two are different distances***; I'll leave them with separate symbols rather than connect them. Thus

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi\rho}\,\hat{\phi}}$$

where I've indicated the direction of $\vec{B}$ using the unit vector of the azimuthal coordinate ϕ in the spherical coordinate system. Note, we have exercised this choice to make part (c) more convenient to do, because we have the ability to make this choice. Not so for $\vec{E}$, which is radial, and so the $\hat{r}$ in $\vec{E}$ in part (a) is the unit vector in spherical coordinates.

(c) Use your results above to write down the *magnitude and direction* of the Poynting vector $\vec{S}$.

Solution: The Poynting vector $\vec{S}$ is then

$$\vec{S} = \vec{E} \times \vec{H} = \vec{E} \times \frac{1}{\mu_0}\vec{B} = \frac{1}{\mu_0}\left[\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{r}\right] \times \left[\frac{\mu_0 I}{2\pi\rho}\,\hat{\phi}\right]$$

and since $\hat{r} \times \hat{\phi} = -\hat{\theta}$, this can be written as

$$\boxed{\vec{S} = \left(-\hat{\theta}\right)\frac{1}{8\pi^2\epsilon_0}\frac{QI}{r^2\rho}}$$

Note: Some of you mistook $\hat{r}$ to be in cylindrical coordinates and thus wrote the direction of $\vec{S}$ as $\hat{z}$. However, the direction of $\vec{E}$ is radial, so $\hat{r}$ in the expression for $\vec{E}$ has to be in spherical coordinates; we don't have a choice here. We do have a choice for $\vec{B}$, because its direction is azimuthal. Since $\hat{\phi}$ in spherical coordinates is along the azimuthal direction, we chose to express the direction of $\vec{B}$ using the unit vector $\hat{\phi}$ in the spherical coordinate system to make finding the cross product for $\vec{S}$ in part (c) easier to do.

(d) Use $\vec{J} \cdot \vec{E}$ to determine the sign of the work done, and verify that this is consistent with the sign of the energy flow due to this configuration.

Solution: Since $\vec{J} = J\hat{z}$, where $\hat{z}$ is the unit vector along the z-direction in rectangular Cartesian coordinates, and $\vec{E} = E\hat{r}$, where $\hat{r}$ is the unit vector along the radial direction in spherical coordinates, we can easily find the sign of the work done from $\vec{J} \cdot \vec{E}$.

Let us do this intuitively first, and then I'll write down a formal solution based on unit vector relations.

Since $\vec{J}$ always points along the positive z-direction, its dot product with $\vec{E}$ will have either a positive or negative sign:

- In the upper half-plane $z > 0$ (i.e., above the xy-plane where z has positive values), $\vec{J} \cdot \vec{E}$ will have a positive sign. This is because the angle between $\hat{r}$ and the z-axis is an acute angle (i.e., smaller than $\pi/2$), so its cosine will be positive.
- In the lower half-plane $z < 0$ (i.e., below the xy-plane where z has negative values), $\vec{J} \cdot \vec{E}$ will have a negative sign. This is because the angle between $\hat{r}$ and the z-axis is larger than $\pi/2$ but smaller than π, so its cosine will be negative.

We can also demonstrate this formally by using the relation between $\hat{r}$ in spherical coordinates and $\hat{z}$ in rectangular coordinates:

$$\hat{r} = \sin\theta\,\cos\phi\,\hat{x} + \sin\theta\,\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

so that

$$\hat{z} \cdot \hat{r} = \cos\theta$$

which is positive for $\theta < \pi/2$ and negative for $\pi/2 < \theta < \pi$.

In summary, the work done is positive in the upper half-plane ($z > 0$), and negative in the lower half-plane ($z < 0$), the volume containing the sphere being excluded in both cases. Thus, energy is being pushed from the lower half-plane into the upper half-plane, and this is consistent with the direction of the Poynting vector on the plane $z = 0$, since $-\hat{\theta}$ points from the lower half-plane ($z < 0$) toward the upper half-plane ($z > 0$), at the interface $z = 0$; of course, unlike Cartesian coordinates, the unit vectors in spherical coordinates change directions depending on location, so $\vec{S} \sim -\hat{\theta}$ won't always point vertically upward, but $\vec{S}$ will always have a net upward component due to the direction of $-\hat{\theta}$.

2. A transverse plane wave is incident normally in vacuum on a perfectly absorbing flat screen. From the law of conservation of linear momentum that we discussed in class, we know that $\sum_\beta T_{\alpha\beta}\, n_\beta$ is the αth component of the force per unit area on the surface. Use this to show that the pressure (called radiation pressure) exerted on the screen is equal to the field energy per unit volume in the wave.

Solution: Let the screen by along the xy-plane, and let the transverse plane wave travel along the z-direction, so that the outward normal to the surface is $\hat{n} = -\hat{z}$.

Also, without any loss of generality, let the $\vec{E}$ field of the transverse plane wave be along the x-direction, so that $\vec{E} = E\hat{x}$, and let the $\vec{B}$ field be along the y-direction, so that $\vec{B} = B\hat{y}$.

Now, since

$$T_{\alpha\beta} = \epsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right)\delta_{\alpha\beta}\right]$$

per the geometry we've set up, we seek the z-component of the pressure (force per unit area) on the screen, so with $\alpha = z$, we get

$$p_z = \sum_\beta T_{\alpha\beta}\, n_\beta = \sum_\beta T_{z\beta}\, n_\beta = T_{zx}\, n_x + T_{zy}\, n_y + T_{zz}\, n_z$$

Now, since $\hat{n} = -\hat{z}$ as noted above, we must have $n_x = 0, n_y = 0$ and $n_z = -1$, so that

$$p_z = T_{zx}\,(0) + T_{zy}\,(0) + T_{zz}\,(-1) = 0 + 0 + \epsilon_0\left[E_z E_z + c^2 B_z B_z - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right)\delta_{zz}\right](-1)$$

Now, since $E_z = 0, B_z = 0$, and $\delta_{zz} = 1$, the above reduces to

$$p_z = \epsilon_0\left[-\frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right)\right](-1) = \frac{\epsilon_0}{2}\left[\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right] = \frac{\epsilon_0}{2}\left[E_x^2 + E_y^2 + E_z^2 + c^2\left(B_x^2 + B_y^2 + B_z^2\right)\right]$$

and, since $E_y = E_z = B_x = B_z = 0$, we get

$$p_z = \frac{\epsilon_0}{2}\left[E_x^2 + c^2 B_y^2\right] \tag{1}$$

We see that our choice of signs has worked out correctly, since p_z must clearly be in the positive z-direction.

All we've left to do is show that the field energy per unit volume in the wave matches equation (1).

The field energy per unit volume is given by

$$u = \frac{1}{2}\left[\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right] = \frac{1}{2}\left[\vec{E}\cdot\epsilon_0\vec{E} + \vec{B}\cdot\frac{\vec{B}}{\mu_0}\right] = \frac{\epsilon_0}{2}\left[\vec{E}\cdot\vec{E} + \frac{1}{\mu_0\epsilon_0}\vec{B}\cdot\vec{B}\right]$$

and since $\mu_0\epsilon_0 = 1/c^2$, we get after expanding the dot product above (just like we did a few lines above) that

$$u = \frac{\epsilon_0}{2}\left[E_x^2 + c^2 B_y^2\right]$$

which matches equation (1), proving that the pressure (called radiation pressure) exerted on the screen is equal to the field energy per unit volume in the wave.

3. In the neighborhood of the Earth, the flux of electromagnetic energy from the Sun is approximately 1.4 kW/m^2. If an interplanetary sailplane had a sail of mass 1 g/m^2 of area and negligible other weight, what would be its maximum acceleration in m/s^2 due to the solar radiation pressure? **Hint:** The previous problem might be of use.

Solution: In SI units, the flux of energy from the Sun is 1400 W/m^2, and the mass per unit area of the sail is 10^{-3} kg/m^2. Use the symbol E_Φ for the flux, and m/A for the mass per unit area, which means that A is the area of the sail.

To solve the problem, recall the definition of the energy flux: the energy flux of a wave is the energy per unit time which crosses unit area of a surface perpendicular to the direction of propagation of the wave. That is why its units are J/m^{2}s, or W/m^2.

In the problem we are working, consider that the wave is propagating along the z-axis with speed c, so the energy density u (the field energy per unit volume in the wave) that flows through a surface perpendicular to the direction of propagation of the wave per unit area per unit time would be uc: that would be the energy flux, E_Φ. So, since $E_\Phi = uc$, we get

$$u = \frac{E_\Phi}{c}$$

Now, in part (a), we proved that the pressure on the screen on which the wave is incident normally is equal to the field energy per unit volume in the wave. Since pressure is the force $\sum F$ per unit area A (of the screen), we have from part (a) that

$$\text{pressure, } \frac{\sum F}{A} = u = \frac{E_\Phi}{c}$$

If m is the mass of the sail, its acceleration a would be

$$a = \frac{\sum F}{m}$$

Note that this is the maximum acceleration that the problem is asking for; if the wave were incident at an angle or if all the energy didn't go into the mechanical momentum, the acceleration would be smaller.

Divide numerator and denominator by the area of the sail A, since we know the force per unit area (pressure) on the sail, and are given its mass per unit area.

So,

$$a = \frac{\sum F/A}{m/A} = \frac{E_\Phi/c}{m/A} = \frac{E_\Phi}{c(m/A)} = \frac{1400\ \text{W/m}^2}{3 \times 10^8\ \text{m/s}\ (10^{-3}\ \text{kg/m}^2)} = 4.7 \times 10^{-3}\ \text{m/s}^2$$

Check that the units work out:

$$\frac{\text{W/m}^2}{\text{m/s (kg/m}^2)} = \frac{\text{W}}{\text{m/s (kg)}} = \frac{\text{J/s}}{\text{m/s (kg)}} = \frac{\text{J}}{\text{m (kg)}} = \frac{\text{N m}}{\text{m (kg)}} = \frac{\text{kg m/s}^2}{\text{kg}} = \text{m/s}^2$$

Therefore, the maximum acceleration of the sail due to the solar radiation pressure would be $\mathbf{4.7 \times 10^{-3}\ m/s^2}$.

4. Consider a circular toroidal coil of mean radius a and N turns, with a *small* uniform cross section of area A, that is, both the height and width of the toroid are small compared to a. The toroid has a current I flowing in it. There is also a point charge Q located at the center of the toroid. Assume that the toroid is in the xy-plane, so that its axis is along the z-direction.

Calculate all the components of the electromagnetic field momentum of the system. You should find that in the plane of the toroid

$$\left(\vec{P}_{\text{field}}\right)_x = 0 \qquad \text{and} \qquad \left(\vec{P}_{\text{field}}\right)_y = 0$$

whereas the component of the field momentum along the axis of the toroid is

$$\left(\vec{P}_{\text{field}}\right)_z \approx \pm\frac{\mu_0 QINA}{4\pi a^2}$$

where the sign depends on the sense of the current flow in the coil. Assume that the electric field of the charge penetrates unimpeded into the region of nonvanishing magnetic field, as would happen for a toroid that is actually a set of N small nonconducting tubes inside which ionized gas moves to provide the energy flow.

Solution: Apply Ampere's law

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \int_S \vec{J}\cdot\hat{n}da$$

Since the height and width of the toroid are small compared to a, we can use an Amperian loop equal to a. This means there is only a ϕ-component of $\vec{B}$, obtained using Ampere's law to be

$$B_\phi\left(2\pi a\right) = \pm\mu_0 NI$$

where the $\pm$ is present to incorporate the sense of the current flow in the coil.

So, in full vector form,

$$\vec{B} = \pm\frac{\mu_0 NI}{2\pi a}\,\hat{\phi}$$

so that

$$\vec{H} = \frac{B}{\mu_0} = \pm\frac{NI}{2\pi a}\,\hat{\phi} \tag{2}$$

Meanwhile, the electric field is just that of a point charge, so

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{r} \tag{3}$$

The electromagnetic field momentum is

$$P_{\text{field}} = \mu_0\epsilon_0\int_V \vec{E}\times\vec{H}\,d^3x$$

It will be easier to do the cross product in Cartesian coordinates, using

$$\hat{r} = \hat{x}\,\sin\,\theta\,\cos\,\phi + \hat{y}\,\sin\,\theta\,\sin\,\phi + \hat{z}\,\cos\,\theta$$

$$\hat{\theta} = \hat{x}\,\cos\,\theta\,\cos\,\phi + \hat{y}\,\cos\,\theta\,\sin\,\phi - \hat{z}\,\sin\,\theta$$

$$\hat{\phi} = -\hat{x}\,\sin\,\phi + \hat{y}\,\cos\phi$$

Also, for brevity, set

$$e = \frac{Q}{4\pi\epsilon_0 r^2} \qquad \text{and} \qquad h = \pm\frac{NI}{2\pi a} \tag{4}$$

so that

$$\vec{E} = e\hat{r} \qquad \text{and} \qquad \vec{H} = h\hat{\phi}$$

Evaluate the cross product

$$\vec{E} \times \vec{H} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \\ e\sin\theta\cos\phi & e\sin\theta\sin\phi & e\cos\theta \\ \\ -h\sin\phi & h\cos\phi & 0 \end{pmatrix} \tag{5}$$

Since the toroid is in the xy-plane, we have $\theta = \pi/2$. Therefore, the cross product above becomes

$$\vec{E} \times \vec{H} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \\ e\cos\phi & e\sin\phi & 0 \\ \\ -h\sin\phi & h\cos\phi & 0 \end{pmatrix} \tag{6}$$

so that

$$\left(\vec{E} \times \vec{H}\right)_x = 0 \qquad \text{and} \qquad \left(\vec{E} \times \vec{H}\right)_y = 0 \tag{7}$$

and

$$\left(\vec{E} \times \vec{H}\right)_z = eh\cos^2\phi - (-eh\sin^2\phi) = eh\left(\cos^2\phi + \sin^2\phi\right) = eh\Big(1\Big)$$

Therefore

$$\left(\vec{E} \times \vec{H}\right)_z = \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)\left(\pm\frac{NI}{2\pi a}\right) \tag{8}$$

From equation (7), we find that

$$\left(\vec{P}_{\text{field}}\right)_x = 0 \qquad \text{and} \qquad \left(\vec{P}_{\text{field}}\right)_y = 0$$

Meanwhile, since the height and width of the toroid are small compared to a, the volume of the toroid can be written as approximately $2\pi aA$, and we can set $r = a$ in equation (8), so that

$$\begin{aligned} \left(\vec{P}_{\text{field}}\right)_z &= \mu_0\epsilon_0 \int_V \left(\vec{E} \times \vec{H}\right)_z d^3x \\ &= \mu_0\epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 a^2}\right)\left(\pm\frac{NI}{2\pi a}\right)\int_V d^3x \\ &\approx \mu_0 \left(\frac{Q}{4\pi a^2}\right)\left(\pm\frac{NI}{2\pi a}\right)\Big[2\pi aA\Big] \\ &\approx \pm\frac{\mu_0 QINA}{4\pi a^2} \end{aligned}$$