(1) A free particle has the following position-space wave function at time $t = 0$:

$$\psi(x,0) = \begin{cases} A e^{ip_0 x/\hbar}\left(b^2 - x^2\right), & |x| < b, \\ 0, & |x| \geq b. \end{cases}$$

(a) Sketch the wave function and find the normalization constant, A.

(b) Find the expectation value of x at time $t = 0$. (Maybe you don't need to integrate ...)

(c) Find the momentum-space wave function at time $t = 0$, $\phi(p,0)$.

(d) Find the expectation value of p at time $t = 0$. (Do you have to integrate ...?)

(e) Use the propagator and your answer to part (c) to find the momentum-space wave function at a later time t.

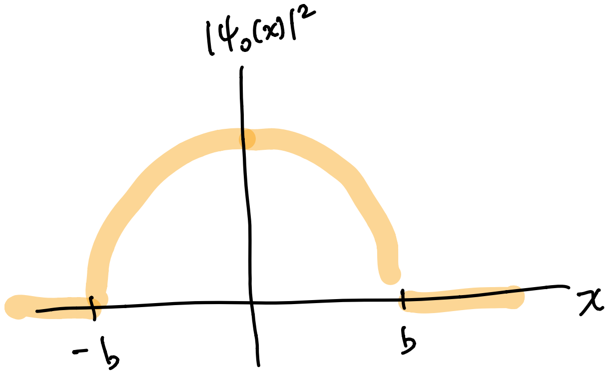
(f) Find the position-space wave function at a later time t.

(g) What is the probability that the particle will be found with $x > b$ at time t? What is the probability that the particle will be found with $x < -b$ at time t?

Note: I encourage you to use WolframAlpha or equivalent to evaluate integrals. You can also leave integrals "as is" if you like.

(a)

$$|\psi(x,0)|^2 = \begin{cases} A^2\left(b^2 - x^2\right)^2 & |x| < b \\ 0 & |x| \geq b \end{cases}$$



Normalize:

$$\int_{-b}^{b} A^2\left(b^2 - x^2\right)^2 dx = 1$$

$$1 = A^2 \int_{-b}^{b} \left[b^4 - 2b^2x^2 + x^4\right] dx$$

$$1 = A^2 \left(b^4 x - \frac{2b^2x^3}{3} + \frac{x^5}{5}\right)\Big|_{-b}^{b}$$

$$1 = A^2\left(2b^5 - \frac{4b^5}{3} + \frac{2b^5}{5}\right)$$

$$1 = A^2\left(\frac{16b^5}{15}\right) \Rightarrow A = \sqrt{\frac{15}{16b^5}}$$

(b) $|\psi(x)|^2$ is symmetric about $x = 0$

$\Rightarrow \langle x \rangle = 0$

(c)

From Eq. (5.65),

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ixp/\hbar} \psi(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-b}^{b} e^{-ipx/\hbar} \sqrt{\frac{15}{16b^5}}\, e^{ip_0x/\hbar} (b^2 - x^2)\, dx$$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \sqrt{\frac{15}{16b^5}} \int_{-b}^{b} e^{-i(p-p_0)x/\hbar} (b^2 - x^2)\, dx$$

(e) $\phi(p,t) = e^{-iHt/\hbar}\, \phi(p) = e^{-ip^2t/2m\hbar}\, \phi(p)$