Physics 460—Homework Report 4

Due Tuesday, Apr. 28, 1 pm **Name:** ______________________________

Complete all the problems on the accompanying assignment.

List all the problems you worked on in the space below. Circle the ones you fully completed:

Please place the problems into the following categories:

- These problems helped me understand the concepts better: ____________________
- I found these problems fairly easy: ____________________
- I found these problems very challenging: ____________________

In the space below, show your work (even if not complete) for any problems you still have questions about. Indicate where in your work the question(s) arose, and ask specific questions that I can answer.

Use the back of this sheet or attach additional paper, if necessary.

If you have no remaining questions about this homework assignment, use this space for one of the following:

- Write one or two of your solutions here so that I can give you feedback on its clarity.
- Explain how you checked that your work is correct.

(1) The wave functions for the energy eigenstates of the particle in a ball are

$$\psi_{nlm}(r,\theta,\phi) = A j_l(k_{nl} r) Y_l^m(\theta,\phi).$$

Here A is a normalization constant and the wavenumber k_{nl} is given by

$$k_{nl} = \frac{z_{ln}}{a},$$

where z_{ln} are the zeros of the spherical Bessel functions. The wave functions for the states with zero angular momentum ($l = 0$) have a particularly simple form:

$$\psi_{n00}(r,\theta,\phi) = A\,\frac{\sin(k_{n0} r)}{k_{n0} r}\,Y_0^0(\theta,\phi),$$

where $k_{n0} = n\pi/a$.

(a) Find the normalization constant A for these states.

(b) If the particle is in one of these energy eigenstates, what is the probability that the measurement of the position of the particle will have result inside $r = a/2$ from the origin? Does this probability depend on n?

(c) Calculate $\langle r \rangle$ for these states. Does this value depend on n?

(d) Figure 2.6 of the course notes shows the shape of the radial wave function, $j_0(k_{n0} r)$, for these states. This wave function clearly has a peak at $r = 0$ and gets smaller as r gets larger. Interpret your answers for parts (b) and (c) in light of this picture. Do they make sense? What's going on here?

(2) An electron in a hydrogen atom is in a state with the (unnormalized) wave function

$$\psi(x,y,z) = A\,(x + y + z)\,\mathrm{e}^{-r/2a_0}$$

(a) Express this state as a superposition of energy eigenstates for the electron in the hydrogen atom.

(b) Find the normalization constant A.

(c) If you measure the energy of the electron, what values could you measure, and with what probabilities?

(d) If you measure the magnitude of the angular momentum of the electron, what values could you measure, and with what probabilities?

(e) If you measure the z-component of the angular momentum of the electron, what values could you measure, and with what probabilities?

(3) Tritium is an isotope of hydrogen that has a nucleus with one proton and two neutrons. Tritium is unstable, with one of the neutrons decaying into a proton, an electron and an anti-neutrino. The electron and anti-neutrino fly away, and the proton remains in the nucleus. The result is an ionized helium atom. This problem is concerned with the tritium atom's original electron.

The Hamiltonians that describe the orbiting electron before and after the neutron decays are

$$H_1 = \frac{p^2}{2m} - \frac{e^2}{r} \quad \text{and} \quad H_2 = \frac{p^2}{2m} - \frac{2e^2}{r}.$$

Thus, both before and after the decay, the atom is "hydrogenic," and the results from the course notes for the wave functions of hydrogenic atoms apply.

Before the decay, the electron is in its ground state, $|\Psi_0\rangle = |1,0,0\rangle$. For the purposes of this problem, assume that the decay happens instantaneously, and that the wave function of the orbiting electron is unchanged by the decay, so that the state of the electron is still $|\Psi_0\rangle$ immediately after the nucleus decays.

(a) Calculate $\langle r \rangle$ and $\langle 1/r \rangle$ both before and after the decay. Express your answers in terms of the Bohr radius for hydrogen, $a_1 = \hbar^2/me^2$.

(b) Find the expected value of the electron's energy after the decay. Hint: make use of the facts that the initial state is an energy eigenstate of H_1 and that $H_2 = H_1 - e^2/r$. Give the expected value of the energy in electron volts.

(c) What bound states are possible for the electron after the nucleus decays? In other words, what are the possible values of n, l, and m for the electron after the decay? (You should be able to answer this based on the initial state $|\Psi_0\rangle$ without doing any detailed calculations.)

(d) Calculate $\mathcal{P}_1$, the probability that the electron is in the ground state after the nucleus decays. What is the energy of this state, in eV?

(e) A detailed calculation gives the following results regarding the probabilities of the electron ending in a state with $n = 2, 3, 4,$ *etc*:

$$\mathcal{P}_2 = 0.25, \qquad \sum_{n=3}^{\infty} \mathcal{P}_n = 0.02137, \qquad \sum_{n=3}^{\infty} \frac{\mathcal{P}_n}{n^2} = 0.00177.$$

Calculate the probability that the orbiting electron remains bound to the helium atom after the decay. Also find the expected energy of the electron, if it remains bound. Discuss you result, and answer the following: is it possible for the orbiting electron to be ejected (along with the created electron and the anti-neutrino)? If so, in what percentage of decays will this occur, and what will be the average kinetic energy of the ejected electron?

Homework 4

(1) The wave functions for energy eigenstates of

$$\psi_{n\ell m}(r,\theta,\varphi) = A j_0(k_{n0} r) Y_0^0(\theta,\varphi)$$

(a) $$k_{n0} = \frac{z_{0n}}{a}$$

$$A^2\int_0^a r^2 |j_0(k_{n0} r)|^2 dr = 1 \text{ where } k_{n\ell} = \frac{z_{n0}}{a}$$

let $x = \frac{r}{a} \Rightarrow dr = a\,dx \quad r^2 = a^2x^2 \quad k_{n0} r = z_{n0} x$

$$A^2 a^3 \int_0^1 x^2 |j_0(\tfrac{n\pi}{a} x)|^2 dx = 1$$

$$1 = \frac{1}{2\pi^2 n^2 a^2} a^3 A^2 \Rightarrow \boxed{A = \frac{\sqrt{2}\pi n}{\sqrt{a}}}$$

$$\psi_{n00}(r,\theta,\varphi) = \frac{\sqrt{2}\pi n}{\sqrt{a}} \frac{\sin(n\pi/a\, r)}{n\pi/a\, r} Y_0^0(\theta,\varphi)$$

(B) $$A^2\int_0^{r/a} r^2 |j_0(k_{n0} r)|^2 dr \quad r_{00} = x_{00} a =$$

$$A^2\int_0^{r/a} r^2 |j_0(\tfrac{n\pi}{a} r)|^2 dr$$

$$\Rightarrow A^2 a^3 \int_{z_{0n}}^{z_{0n}} x^2 |j_0(z_{0n} x)|^2 dx \boxed{= 1/2}$$

Not dependent on

(C) $$\langle r \rangle = \int_0^\infty r^2 dr \left(r |\psi(r)|^2 \right)$$

$$\int_0^\infty \frac{\sqrt{2}\,\pi n}{\sqrt{a}} \left(\frac{\sin(n\pi/a\, r)}{n\pi/a\, \cancel{r}} \sqrt{\frac{1}{4\pi}} \right)^2 r^{\cancel{2}}\, dr$$

$$\frac{2\pi^{\cancel{2}} n^2}{a} \frac{1}{4\cancel{\pi}} \int_0^\infty \frac{\sin(n\pi/a\, r)}{n\pi/a} r^2\, dr$$

$$= \boxed{1/2a}$$

(D) Spherical Bessel Function is not $\cancel{0}$ at $r=0$