Homework 7

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Problem 1

The potential of a localized distribution of charge described by the charge density $\rho(\vec{x})$ is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

where the *multipole moments* q_{lm} are given by

$$q_{lm} = \int Y_{lm}^*(\theta',\phi')r'^l\rho(\vec{x})d^3x'.$$

The evaluation for q_{11} means that $l = 1$ and $m = 1$. Therefore, starting with the equation gave above we have

$$q_{11} = \int Y_{11}^*(\theta',\phi')r'^1\rho(\vec{x})d^3x'$$

where Y_{11}^* is

$$Y_{lm}^* = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(cos\theta')e^{-im\phi'}$$

$$Y_{11}^* = -\sqrt{\frac{3}{4\pi}\frac{1}{2}}(cos\theta')e^{-i\phi'}$$

$$Y_{11}^* = -\sqrt{\frac{3}{8\pi}}(cos\theta' cos(\phi') - icos\theta' sin(\phi'))$$

Entering this back into the equation q_{11} will produce the equation

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int \rho(\vec{x})((r'cos\theta')cos(\phi') - i(r'cos\theta')sin(\phi'))d^3x'$$

From Spherical coordinates it is know that to Cartesian coordinates it is known that

$$x = rsin\theta cos\phi \qquad y = rsin\theta sin\phi \qquad z = rcos\theta$$

Therefore, rewriting the problem in Cartesian coordinates the equation becomes

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int \rho(\vec{x})((x - iy)d^3x'.$$

Now, p is given by

$$\vec{p} = \int \vec{x}' d^3x'$$

Thus, the final equation is

$$q_{11} = -\sqrt{\frac{3}{8\pi}}(p_x - ip_y).$$

The evaluation for q_{10} means that $l = 1$ and $m = 0$. Therefore, starting with the equation gave above we have

$$q_{10} = \int Y^*_{10}(\theta', \phi')r'^1\rho(\vec{x})d^3x'$$

where Y^*_{10} is

$$Y^*_{lm} = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P^m_l(cos\theta')e^{-im\phi'}$$
$$Y^*_{10} = \sqrt{\frac{3}{4\pi}\frac{1}{1}}(cos\theta')$$
$$Y^*_{10} = \sqrt{\frac{3}{4\pi}}(cos\theta')$$

Entering this all back in gives

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int \rho(\vec{x}) r' cos\theta' d^3x'.$$

Therefore, rewriting the problem in Cartesian coordinates the equation becomes

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int \rho(\vec{x}) z' d^3x'.$$

With our relation to what p is the final equation becomes

$$q_{10} = \sqrt{\frac{3}{4\pi}} p_z.$$

Problem 2

Show that

$$q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}(Q_{13} - iQ_{23})$$

where Q_{ij} is the quadrupole moment tensor given by

$$Q_{ij} = \int (3x'_i x'_j - r'^2\delta_{ij})\rho(\vec{x'})d^3x'.$$

The evaluation for q_{21} means that $l = 2$ and $m = 1$. Therefore, starting with the equation gave above we have

$$q_{21} = \int Y^*_{21}(\theta', \phi')r'^2\rho(\vec{x})d^3x'$$

where Y^*_{21} is

$$\begin{aligned}
Y^*_{lm} &= (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P^m_l(cos\theta')e^{-im\phi'} \\
Y^*_{21} &= \sqrt{\frac{15}{8\pi}}sin(\theta')\cos(\theta')e^{-i\phi'} \\
&= \sqrt{\frac{15}{8\pi}}sin(\theta')(cos(\theta')cos(\phi') - icos(\theta')sin(\phi'))
\end{aligned}$$

Entering this back into the main equation gives

$$q_{21} = \sqrt{\frac{15}{8\pi}}sin(\theta')(cos(\theta')cos(\phi') - icos(\theta')sin(\phi'))r'^2\rho(\vec{x})d^3x'$$

Converting from spherical coordinates gives

$$q_{21} = \sqrt{\frac{15}{8\pi}}(x'z' - iy'z')\rho(\vec{x})d^3x'$$

Where

$$Q_{13} = \int (3x'z')\rho(\vec{x})d^3x' \qquad\qquad Q_{23} = \int (3y'z')\rho(\vec{x})d^3x'$$

Divide a 3 out for the equations above and the final equation is

$$q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}(Q_{13} - iQ_{23}).$$

Problem 3

$$E_r = \frac{(l+1)}{(2l+1)\epsilon_0} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+2}} \quad E_\theta = -\frac{1}{(2l+1)\epsilon_0} q_{lm} \frac{1}{r^{l+2}} \frac{\partial}{\partial\theta} Y_{lm}(\theta,\phi) \quad E_\phi = \frac{1}{(2l+1)\epsilon_0} q_{lm} \frac{1}{r^{l+2}} \frac{im}{sin\theta} Y_{lm}(\theta,\phi)$$

For a dipole $\vec{p}$ along the z-axis, show that the fields above reduce to:

$$E_r = \frac{2pcos\theta}{4\pi\epsilon_0 r^3} \qquad E_\theta = -\frac{psin\theta}{4\pi\epsilon_0 r^3} \qquad E_\phi = 0$$

The dipole $\vec{p}$ is given along the z-axis. From problem 1 we found that

$$q_{11} = -\sqrt{\frac{3}{8\pi}}(p_x - ip_y).$$

But since this dipole is along the z-axis that means that $p_x = 0$ and $p_y = 0$. With both $p_x = 0$ and $p_y = 0$ this means that $q_{11} = 0$. On the other hand, from problem 1 we also found that

$$q_{10} = \sqrt{\frac{3}{4\pi}} p_z.$$

Therefore, this is the only q value that will survive. More importantly this means that $l = 1$ and $m = 0$. From Jackson

$$Y_{10} = \sqrt{\frac{3}{4\pi}} cos(\theta)$$

Substituting all of this into the equation E_r gives us

$$\begin{aligned} E_r &= \frac{(1+1)}{(2(1)+1)\epsilon_0} q_{10} \frac{Y_{10}(\theta,\phi)}{r^{1+2}} \\ &= \frac{2}{3\epsilon_0} \sqrt{\frac{3}{4\pi}} p_z \frac{1}{r^3} \sqrt{\frac{3}{4\pi}} cos(\theta) \end{aligned}$$

This reduces to

$$E_r = \frac{2pcos\theta}{4\pi\epsilon_0 r^3}$$

Applying the same process for E_θ will give

$$E_\theta = -\frac{1}{(2(1)+1)\epsilon_0} q_{10} \frac{1}{r^{1+2}} \frac{\partial}{\partial\theta} Y_{10}(\theta,\phi)$$
$$= -\frac{1}{3\epsilon_0}\sqrt{\frac{3}{4\pi}} p_z \frac{1}{r^3} \frac{\partial}{\partial\theta}\sqrt{\frac{3}{4\pi}} cos(\theta)$$

where

$$\frac{\partial}{\partial\theta} Y_{10}(\theta,\phi) = \frac{\partial}{\partial\theta}\sqrt{\frac{3}{4\pi}} cos(\theta) = -\sqrt{\frac{3}{4\pi}} sin(\theta).$$

Therefore, the equation will reduce to

$$E_\theta = -\frac{psin\theta}{4\pi\epsilon_0 r^3}.$$

Finally, applying the same process for E_ϕ will give

$$E_\phi = \frac{1}{(2(1)+1)\epsilon_0} q_{10} \frac{1}{r^{1+2}} \frac{i(0)}{sin\theta} Y_{10}(\theta,\phi)$$

Since $m = 0$ and there is a fraction in this equation that is $(im)/sin\theta$ the entire equation will be zero. Thus,

$$E_\phi = 0.$$

Problem 4

Suppose that we have a uniform magnetic field $\vec{B_0} = B_0\hat{z}$, where B_0 is a constant.

(a)

Examine whether

$$\vec{A} = \frac{\vec{B_0}}{2} \times \vec{x}$$

is an appropriate vector potential for this given field.

The uniform magnetic field can be entered into the vector potential as

$$\begin{aligned}
\vec{A} &= \frac{B_0\hat{z}}{2} \times \vec{x} \\
&= \frac{\vec{B_0}\hat{z}}{2} \times (x\vec{x} + y\vec{y} + z\vec{z}) \\
&= \frac{\vec{B_0}}{2}(x\hat{y} - y\hat{x})
\end{aligned}$$

The curl of $\vec{A}$ has to be zero for this to be an appropriate vector. Therefore,

$$\begin{aligned}
\vec{\nabla} \times \vec{A} &= \left(\hat{x}\left(\frac{\partial}{\partial x}\right) + \hat{y}\left(\frac{\partial}{\partial y}\right) + \hat{z}\left(\frac{\partial}{\partial z}\right)\right) \times \frac{\vec{B_0}}{2}(x\hat{y} - y\hat{x}) \\
&= \frac{\partial}{\partial x}x\hat{z} - \frac{\partial}{\partial z}x\hat{x} + \frac{\partial}{\partial y}y\hat{x} + \frac{\partial}{\partial z}y\hat{y} \\
&= \hat{z} + \hat{x}
\end{aligned}$$

This will obviously not be zero so it is not an appropriate vector.

(b)

Does this vector potential satisfy the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$?

Given

$$\begin{aligned}\vec{\nabla}\cdot\vec{A} &= \left(\hat{x}\Big(\frac{\partial}{\partial x}\Big)+\hat{y}\Big(\frac{\partial}{\partial y}\Big)+\hat{z}\Big(\frac{\partial}{\partial z}\Big)\right)\cdot\frac{\vec{B_0}}{2}(x\hat{y}-y\hat{x})\\ &= \frac{\vec{B_0}}{2}(\frac{\partial}{\partial y}x-\frac{\partial}{\partial x}y)\end{aligned}$$

$$\begin{aligned}\vec{\nabla}\cdot\vec{A} &= \left(\hat{x}\Big(\frac{\partial}{\partial x}\Big)+\hat{y}\Big(\frac{\partial}{\partial y}\Big)+\hat{z}\Big(\frac{\partial}{\partial z}\Big)\right)\cdot\frac{\vec{B_0}}{2}(x\hat{y}-y\hat{x})\\ &= \frac{\vec{B_0}}{2}(\frac{\partial}{\partial y}x-\frac{\partial}{\partial x}y)\end{aligned}$$

where

$$\frac{\partial}{\partial y}x=0 \qquad\qquad \frac{\partial}{\partial x}y=0$$

so

$$\vec{\nabla}\cdot\vec{A}=0$$