PHY 474/374 Spring 2021

Week 1—Wednesday, March 31—Discussion Worksheet

Stellar Spectra

Measuring magnitudes really only tells us about the surface temperature of a star, and that too if only it isn't subject to large interstellar absorption. To get useful information about the star, we need spectral line information. Since energy levels are discrete, we would expect that spectral lines would occur at exact frequencies corresponding to the energy difference between the two levels, $\Delta E = E_2 - E_1$. In reality however, spectral lines have finite width due to line broadening effects. On this worksheet, we will discuss ***natural broadening***, ***collisional or pressure broadening***, and ***thermal Doppler broadening***.

1. **Natural broadening** arises due to Heisenberg's Uncertainty Principle, $\Delta E\, \Delta t \geq \hbar$. If we consider a two-state system, then the less time an atom spends in the upper state, the more uncertain is its energy. Stated another way, the Einstein A-coefficient, A_{21}, gives the probability that the system will transition out of its upper state in a certain period of time. Thus, $1/A_{21}$ is the time the system will spend in the upper state, and so it is also the uncertainty in the transition time, Δt, since the system will transition at some point in this time.

(a) Show that the spread in wavelength due to natural broadening, $\Delta\lambda$, is given by

$$\Delta\lambda = \frac{\lambda^2}{2\pi c}\frac{1}{\Delta t}$$

Hint: Start with $E = hc/\lambda$, and find $dE/d\lambda$ (but drop the minus sign).

$$\frac{dE}{d\lambda} = -\frac{hc}{\lambda^2} \rightarrow \Delta\lambda \frac{\cancel{h}c}{\lambda^2} = \frac{\cancel{h}}{2\pi}$$

$$\Delta E = \Delta\lambda \frac{hc}{\lambda^2} \rightarrow \boxed{\Delta\lambda = \frac{\lambda^2}{2\pi c}}$$

(b) The Lyman-α line of hydrogen has a wavelength of 121.5 nm and $A_{21} = 6.24\times10^{8}\ \mathrm{s}^{-1}$. Calculate $\Delta\lambda$ due to natural broadening, and also the relative line broadening $\Delta\lambda/\lambda$.

$$\Delta\lambda = \frac{(121.5\times10^{-9}\,m)^2}{2(3.14)(3\times10^{8}\,m/s)}\frac{1}{6.24\times10^{8}\,s^{-1}} = 4.89\times10^{-15}\,m$$

$$\Delta\lambda/\lambda = 4.02\times10^{-8}$$

2. In general, an atom that is in an excited state n can drop down to any lower state n'. If the spontaneous decay of an atomic state n (to all lower energy levels n') proceeds at the rate

$$\gamma_n = \sum_{n'} A_{nn'}$$

then the probability of an emitted photon having a frequency ν is given by the distribution function

$$\phi(\nu)\, d\nu = \frac{\gamma_n/4\pi}{(\nu-\nu_0)^2 + (\gamma_n/4\pi)^2} \frac{d\nu}{\pi}$$

where ν_0 is the center (peak) frequency of the line. This is known as a **Lorentzian profile**.

(a) You saw an example of a Lorentzian profile in the PowerPoint slides today. Calculate the maximum value of $\phi(\nu)$, and express your answer in terms of γ_n. **Hint:** This occurs at $\nu = \nu_0$.

If $\nu = \nu_0$ $\rightarrow \varphi(\nu_0)\, d\nu = \frac{\cancel{\gamma_n/4\pi}}{(\gamma_n/4\pi)^{\cancel{2}}} \frac{d\nu}{\pi}$

$\rightarrow \varphi(\nu_0)\, \cancel{d}\nu = \frac{\cancel{d}\nu}{(\gamma_n/4\cancel{\pi})\cancel{\pi}}$

$\rightarrow \varphi(\nu_0) = \frac{4}{\gamma_n}$

(b) Describe *in words* how γ_n affects the peak value you found in part (a) above. For reasons that will become obvious from this answer, γ_n is also known as the damping constant.

as γ_n increases our $\varphi(\nu_0)$ will decrease

3. We will now find the width of a Lorentzian distribution.

(a) Find an expression for $\Delta\lambda/\lambda$ at the point where $\phi(\nu)$ drops to half its peak value that you found in Question 2(a).

Peak value $\quad \varphi(\nu) = 4/\gamma_n$

half peak $\quad \varphi(\nu) = 2/\gamma_n$

$$\varphi(\nu)\Big|_{1/2\ \text{peak}} = 2/\gamma_n = \frac{\gamma_n/4\pi}{(\nu-\nu_0)^2 + (\gamma_n/4\pi)^2} = 1/\pi$$

$$\frac{\gamma}{2} = \left[\frac{(\nu-\nu_0)^2 + (\gamma_n/4\pi)^2}{\gamma_n/4}\right]\pi$$

$$(\nu-\nu_0)^2 = \gamma_n/4\pi\left[\frac{\gamma_n}{2\pi} - \frac{\gamma_n}{4\pi}\right] = \frac{\gamma_n}{4\pi}\left[\frac{2\gamma_n - \gamma_n}{4\pi}\right] = \frac{\gamma_n^2}{16\pi^2}$$

$$\nu - \nu_0 = \frac{\gamma_n}{4\pi} \rightarrow \frac{\Delta\nu}{\nu} = \frac{\gamma_n}{4\pi\nu}$$

$$\nu = \frac{c}{\lambda} = |\Delta\nu| = \frac{c}{\lambda^2}\Delta\lambda$$

$$\rightarrow \frac{\Delta\nu}{\nu} = \frac{(c/\lambda^2)\Delta\lambda}{c/\lambda} \rightarrow \frac{\Delta\nu}{\nu} = \frac{\Delta\lambda}{\lambda}$$

$$\boxed{\frac{\Delta\lambda}{\lambda} = \frac{\gamma_n}{4\pi\nu}}$$

(b) Describe in words how the damping constant γ_n affects the width of a Lorentzian distribution.

The ↑ the damping constant,

the broader the Lorentz distribution

4. **Collisional or pressure broadening** results from the fact that atoms are not isolated (as we have assumed in order to figure the natural broadening). Instead, they interact with their neighbors. They will collide directly with some (neutral) neighbors, and they will also feel the electric fields of nearby (charged) particles (e.g., electrons). Both of these effects also produce a Lorentzian profile. We can use the same expression for the broadening that we developed in Question 1(a), except now Δt is the average time between collisions, given by

$$\Delta t = \frac{l}{v}$$

where l is the mean free path, given by

$$l = \frac{1}{n\sigma}$$

where n is the particle density, and σ is the collisional cross section. Assuming that the gas particles are described by a Maxwell-Boltzmann distribution, we can use for v the most probable velocity given by $\sqrt{2kT/m}$.

(a) Find $\Delta\lambda/\lambda$ for a star with a surface temperature of 6000 K and particle density 10^{23} m^{-3}. Find σ using H-H collisions, where the radius of a hydrogen atom is 5.3×10^{-11} m.

$$\sigma = \pi(2r)^2 = \pi[2(5.3\times10^{-11}\,m)]^2 = 3.53\times10^{-20}\,m^2$$

$$l = \frac{1}{n\sigma}, \quad \Delta t = \frac{l}{V} = \frac{1}{n\sigma V} = \frac{1}{(10^{23}m^{-3})(3.53\times10^{-20}m^2)(v)}$$

Where $V = \sqrt{\frac{2KT}{m}}$,

$$\Delta t = \frac{1}{n\sigma}\sqrt{\frac{m}{2KT}} = \frac{1}{(10^{23}m^{-3})(3.53\times10^{-20}m^2)}\left[\frac{1.67\times10^{-27}kg}{2(1.38\times10^{-23}J/k)(6000k)}\right]^{1/2}$$

$$\Delta t = 2.8\times10^{-8}s$$

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda}{2\pi c}\frac{1}{\Delta t} = \frac{121.5\times10^{-9}m}{2\pi(3\times10^{8}m/s)}\frac{1}{2.8\times10^{-8}s} = 2.3\times10^{-9}$$

(b) In which kinds of stars will pressure broadening be more significant, giant stars or dwarf stars?

n will be bigger, broadening will be more significant

$$\frac{\Delta\lambda}{\lambda} \propto n, \text{ number Density}$$

5. **Doppler broadening** arises because the atoms in a gas move around randomly with a distribution of speeds that is described by the Maxwell-Boltzmann distribution. Along the line of sight, the distribution of velocities follows a Gaussian distribution with dispersion $\sigma = \sqrt{kT/m}$, so in the nonrelativistic case the wavelengths of light emitted by atoms in the gas are Doppler-shifted by $\Delta\lambda/\lambda = \sigma/c$.

(a) Calculate $\Delta\lambda/\lambda$ due to Doppler broadening for hydrogen atoms in the Sun's photosphere, where $T = 5800$ K. **Note:** The mass of a proton, and hence a hydrogen atom, is 1.67×10^{-27} kg.

$$\frac{\Delta\lambda}{\lambda} = \frac{\sigma}{c} = \frac{1}{c}\sqrt{\frac{kT}{m}}$$

$$= \frac{1}{(3\times10^{8}\,m/s)}\left[\frac{(1.38\times10^{-23}\,J/K)(5800\,K)}{(1.67\times10^{-27}\,kg)}\right]^{1/2}$$

$$\frac{\Delta\lambda}{\lambda} = 2.3\times10^{-5}$$

(b) Calculate the Doppler broadening of the Hα line ($\lambda = 656.3$ nm) in the Sun's photosphere. The Hα (or Balmer-α) line is produced by a jump from the $n = 3$ (second excited state) to the $n = 2$ (first excited state) of hydrogen.

$$\Delta\lambda = \lambda(2.3\times10^{-5}) = (656.3\,nm)(2.3\times10^{-5})$$

$$= 0.015\,nm$$

(c) How does your answer compare to the natural broadening of the Hα line? Use $\Delta t = 10^{-8}$ s for both the first and second excited states of hydrogen. Unlike for Lyman-α in Question 1(b), you will need to adapt the equation from Question 1(a). Why? How should you modify it?

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda}{2\pi c}\left[\frac{1}{\Delta t_u} + \frac{1}{\Delta t_l}\right] = \frac{656.3\times10^{-9}\,m}{2\pi(3\times10^{8}\,m/s)}\left[2\times10^{8}\,s\right]$$

$$= 6.96\times10^{-8}$$

$$\Delta\lambda = (656.3\,nm)(6.96\times10^{-8}) = 0.000046\,nm$$

$$\Delta\lambda_{\text{Doppler}} \gg \Delta\lambda_{\text{natural Broadening}}$$