Reducing the number of generalized coordinates vs. Lagrange's λ method

n generalized coordinates $\quad k$ holonomic constraints

Reduce the number of generalized coordinates to $n-k$ using the constraint equations

Keep all n generalized coordinates

$$\delta I = \delta \int_{t_1}^{t_2} L dt = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$q's$ are independent

($n-k$ equations)

$$\ddot{q}_i = \cdots$$

$n-k$ equations of motion

No information about forces of constraint

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_i} \qquad f_j(q_1, q_2, \ldots, q_n) = 0,$$

$q's$ are not independent $\quad$ ($n+k$ equations)

$$\ddot{q}_i = \cdots$$

n equations of motion

$$Q_i^{nc} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_i}$$

k forces of constraint

Example: Beads on a hemisphere (standard approach):

$$L = \frac{1}{2}m\left(\dot{\mathrm{r}}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$

Constraint equation: $r - R = 0 \Rightarrow r = R$

$$L = \frac{1}{2}m\left(R^2\dot{\theta}^2\right) - mgR\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = mR^2\ddot{\theta} \qquad \frac{\partial L}{\partial\theta} = mgR\ sin\theta \qquad \ddot{\theta} = -\frac{g}{R}\sin\theta$$

Note that we don't get any information about the constraint force of the surface of the hemisphere on the bead that keeps the bead on the circular path. This solution assumes that the bead stays on the surface until it reaches the end. But if the bead is not held on the hemisphere by some sort of rail it might jump off.

Example: Beads on a hemisphere (Lagrange multipliers approach):

$$L = \frac{1}{2}m\left(\dot{\mathrm{r}}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$

Constraint equation: $\mathrm{f} = r - R = 0$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda\frac{\partial f}{\partial r} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial f}{\partial \theta} \qquad r - R = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda$$

$$m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) - \ mgr\sin\theta = 0$$

$$r = R, \qquad \dot{r} = 0, \qquad \ddot{r} = 0$$

Like in the standard approach, we also recover the equation of motion for θ:

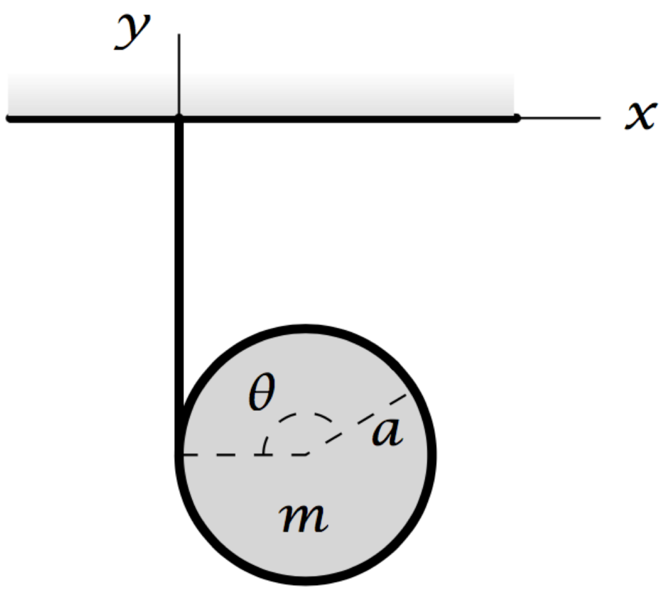
$$\ddot{\theta} = -\frac{g}{R}\sin\theta$$

But now we can also determine the constraint force that keeps the bead on the hemisphere.

$$\lambda = mg(3\cos\theta - 2) \qquad Q_r = \lambda\frac{\partial f}{\partial r} = mg(3\cos\theta - 2)$$

Setting Q_r (the generalized force in radial direction) to zero tells us that the bead will come off the hemisphere for $\theta = \pm\cos^{-1}\left(\frac{2}{3}\right) = \pm 48.2°$

Activity 13



$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \lambda \frac{\partial f}{\partial y}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial f}{\partial \theta}$$

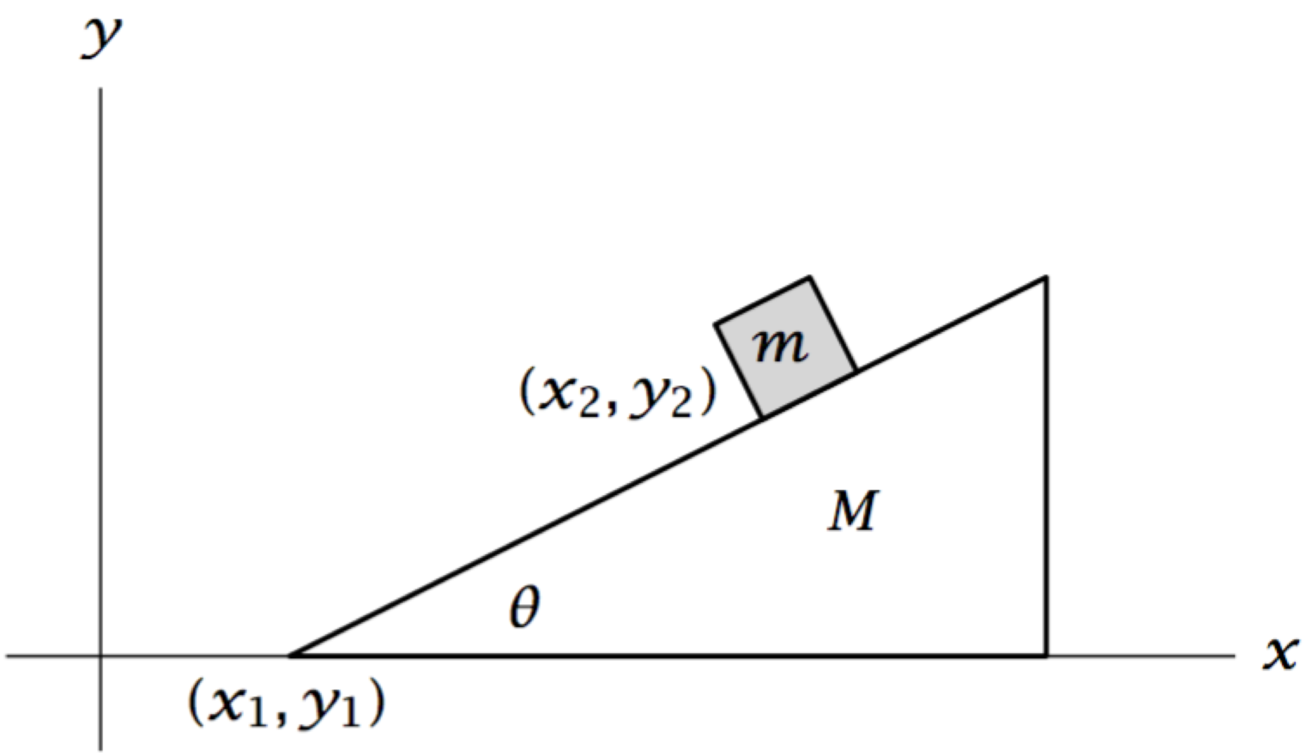
$$f = y + a\theta$$

Strategy:

1. Find the Lagrangian (remember rotational term)
2. Plug it into the two “modified” Lagrange equations
3. Solve these equations and the equation of constraint equation of constraint for λ
4. Plug λ into the equations for Q_y and Q_θ

Before you start, think about what you would expect the result to be. What do you think are the constraint forces that cause the yoyo to “roll without slipping?”

Activity 14



$$L = \frac{1}{2}M(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2) - Mgy_1 - mgy_2$$

The two holonomic constraints are that the wedge has to stay on the table:

$$f_1(x_1, y_1, x_2, y_2) = y_1 = 0,$$

and that the block has to stay on the wedge:

$$f_2(x_1, y_1, x_2, y_2) = \cdots$$

Activity 14

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = \lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1} \quad (i)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = \lambda_1 \frac{\partial f_1}{\partial x_2} + \lambda_2 \frac{\partial f_2}{\partial x_2} \quad (ii)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_1} - \frac{\partial L}{\partial y_1} = \lambda_1 \frac{\partial f_1}{\partial y_1} + \lambda_2 \frac{\partial f_2}{\partial y_1} \quad (iii)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_2} - \frac{\partial L}{\partial y_2} = \lambda_1 \frac{\partial f_1}{\partial y_2} + \lambda_2 \frac{\partial f_2}{\partial y_2} \quad (iv)$$

$$f_1 = y_1 = 0$$

$$f_2 = \cdots$$

Use these equations to solve for λ_1 and λ_2

Then derive the forces of constraint $Q_{x_1} = \lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1}$ Etc.