1 Partial Differential Equations

As more advanced topics in physics are encountered, partial differential equations enter the discussion. The equations of classical mechanics in more direction, the equations governing the behavior of electric and magnetic fields, quantum mechanics, fluids, etc., all involve partial differential equations.

In this course we will only consider 2nd order and linear, PDEs. The most general form in two variables, $u(x,t)$ where x,t are the independent variables and $u(x,t)$ the quantity of interest is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + D\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0 \tag{1}$$

where in general, A, B, C are functions of x, t and D is a function of u, x, t and the partials of u.

These kinds of PDEs have solutions and solution methods which are fundamentally different depending on the classification of the PDE. Second order PDEs can be classified as follows

1. If
$$B^2 - 4AC > 0$$
the PDE is said to be hyperbolic. An example of this kind of PDE is the *wave equation.*

2. If
$$B^2 - 4AC = 0$$
the PDE is said to be parabolic. An example of this kind of PDE is the *diffusion equation.*

3. If
$$B^2 - 4AC < 0$$
the PDE is said to be elliptic. An example of this kind of PDE is the *Laplace equation.*

It is common in PDEs to adopt the following notation:

$$\frac{\partial u}{\partial x} \equiv u_x; \quad \frac{\partial^2 u}{\partial x^2} \equiv u_{xx}; \quad \frac{\partial^2 u}{\partial x \partial t} \equiv u_{x,t}.$$

(1) Consider the partial differential equation,

$$u_{tt} - c^2 u_{xx} = 0,$$

where $c > 0$ and a constant. Is the equation a hyperbolic, parabolic, or elliptic PDE?

PDEs are inherently boundary value problems. They may also involve one or more initial conditions. Boundary value problems come in two forms, both of which can appear in the same problem.

Dirichelet Boundary conditions have the value of the field (the quantity we are solving for) specified at the boundary.

Neumann Boundary conditions have the value of the derivative of the field at the boundary.

(2) In each of the following, $u = u(x,t)$. Identify the following as either initial conditions or boundary conditions (assume the variable t is time). If boundary conditions, identify whether the boundary condition is Dirichelet or Neumann.

(a) $u(x, 0) = t_o$, t_o a constant.

(b) $u(L, t) = u_o$, L, u_o constants.

(c) $u_t(x, 0) = t_o$ t_o a constant.

(d) $u_x(L, t) = u_o$, L, u_o constants.

(3) In finite differences, derivatives are replaced by difference equations at a discrete set of points. The central differences are given as

$$\begin{aligned} y'(x) &= \frac{y(x+h) - y(x-h)}{2h} \\ y''(x) &= \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} \end{aligned}$$

Using the notation,

$$y_i \equiv y(x_i); \quad y_{i+1} \equiv y(x_i + h); \quad y_{i-1} \equiv y(x_i - h)$$

rewrite the following ODE as a difference equation and solve for the the y_i term.

$$y''(x) - 5y'(x) + 10y = 10x$$

(4) Write a `MatLab` function that uses the Jacobi finite differencing scheme to solve the ODE given in problem (3).

(5) Write a `MatLab` function that uses the Gauss-Seidel finite differencing algorithm to solve the ODE given in problem (3).

(6) Download the code `MyHeatEq.m` from the course *D2L* page. Identify by line number, where the code is doing what the algorithm you developed needs done. For a heat equation on a Cartesian grid, it can be shown that the optimal α for use in the SOR process is

$$\alpha = \frac{4}{2 + \sqrt{4 - \left[\cos\frac{\pi}{N_x} + \cos\frac{\pi}{N_y}\right]^2}} - 1.$$

Make sure you identify exactly where the *SOR* process is occuring. Also try to understand what lines 25 – 28 do.