PHY 411 Winter 2021

Reading Assignment for Week 5—Thursday, Feb 4

Green's Function

Green's functions provide a general way to solve inhomogenous partial (or ordinary) differential equations. It is another mathematical technique we will need when we begin Chapter 9. Let us write the general schematic for a Green function problem before proceeding to an example in electrostatics. Any math methods text would be good; I've compiled the following from a series of presentations obtained via a Google search (thus, I haven't provided any one specific source).

Consider the linear inhomogenous differential equation

$$\mathcal{D}\Psi(\vec{x}) = f(\vec{x}) \tag{W6.1}$$

where $\mathcal{D}$ is any linear differential operator, e.g., a well known example of an inhomogenous differential equation in electrostatics is the Poisson equation for the scalar potential $\Phi(\vec{x})$, given by

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

We know from our study of differential equations that the solution to equation (W6.1) is comprised of a solution to the associated homogenous equation Ψ_h, plus the particular solution Ψ_{part} that comes from the inhomogenous equation:

$$\Psi(\vec{x}) = \Psi_h + \Psi_{\text{part}} \tag{W6.2}$$

We do not need to consider Ψ_h, since there are many techniques to solve homogenous PDE's.

Instead, we are going to look for the particular solution (Ψ_{part}) with the inhomogeneity on the right hand side, represented here by the generic function $f(\vec{x})$.

To do so, we need to consider the associated inhomogenous equation for a point source function

$$\mathcal{D}G(\vec{x}, \vec{x}\,') = \delta(\vec{x} - \vec{x}\,') \tag{W6.3}$$

which defines the Green function $G(\vec{x}, \vec{x}\,')$ corresponding to the operator $\mathcal{D}$.

Once we've determined $G(\vec{x}, \vec{x}\,')$, then the particular solution to the inhomogenous equation can be written as

$$\Psi_{\text{part}} = \int_V G(\vec{x}, \vec{x}\,')\, f(\vec{x}\,')\, d^3x' \tag{W6.4}$$

so that the solution to the differential equation (W6.1) is

$$\Psi(\vec{x}) = \Psi_h + \int_V G(\vec{x}, \vec{x}\,')\, f(\vec{x}\,')\, d^3x' \tag{W6.5}$$

In summary, the Green function may be characterized as the response of a system to a Dirac δ-function input signal, to take an electronics analogy. Now, because any function can be expanded as a sum of Dirac signals, you can compute the output for any signal input if you know the Green function.

Green Function: An Example from Electrostatics

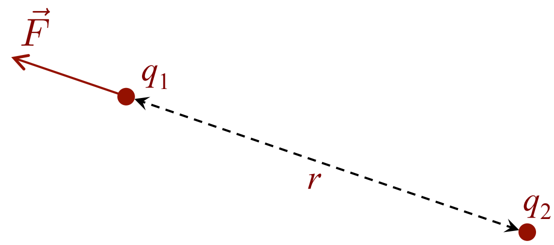
We will now introduce the formal math with an intuitive example from electrostatics.

First, let us set up the math notation (since that is usually the first hurdle to be overcome in understanding the math).

You learned in freshman physics that the magnitude of the force $\vec{F}$ exerted on a point charge q_1 due to a point charge q_2 located a distance r apart from each other is given by Coulomb's law:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} \tag{W6.6}$$

If q_1 and q_2 have the same sign, then $\vec{F}$ is repulsive, and is directed from q_2 to q_1 along the line drawn through the two charges.



Let us express r in vector notation. Suppose the charge q_1 is located at $\vec{x}_1$ and q_2 is located at $\vec{x}_2$, as shown in the figure below.

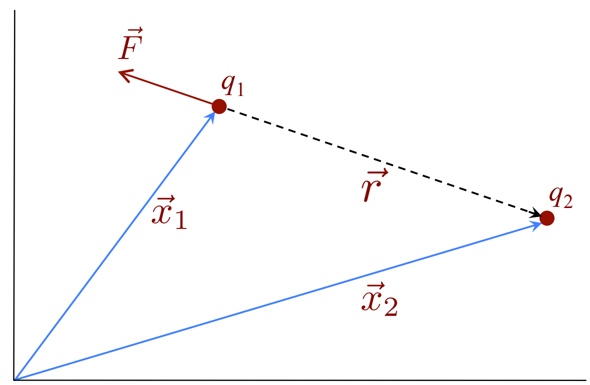
Then, it's easy to see that

$$\vec{x}_1 + \vec{r} = \vec{x}_2$$

where $\vec{r}$ is a vector along the line passing through q_1 and q_2 and directed from q_1 to q_2, so that

$$-\vec{r} = \vec{x}_1 - \vec{x}_2$$

I'm writing it in this form because $-\vec{r}$ is the direction of the force of q_2 on q_1.



So, we could write Coulomb's law in equation (W6.11) as

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|^2}$$

But, we could also use this vectorial form to insert information about the direction of the force. Again, since the force on q_1 due to q_2 is directed along the line joining the two charges, and is also the direction of the vector $-\vec{r} = \vec{x}_1 - \vec{x}_2$, we can write that

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|^2}\frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|} \tag{W6.7}$$

By dividing $(\vec{x}_1 - \vec{x}_2)$ by its magnitude $|\vec{x}_1 - \vec{x}_2|$, I've written a unit vector in the direction of the vector $(\vec{x}_1 - \vec{x}_2)$, and since that is the direction of the force of q_2 on q_1, I can use this unit vector to specify the direction of the Coulomb force exerted by q_2 on q_1.

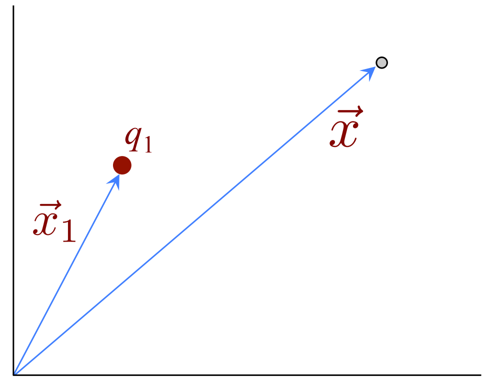
Writing equation (W6.12) in more compact form, we get the Coulomb force on charge q_1 located at $\vec{x}_1$ due to charge q_2 located at $\vec{x}_2$ to be

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} \tag{W6.8}$$

Next, let us start thinking in terms of electric fields, since that is really what we are after at this level.

We know that $\vec{F} = q\vec{E}$, where $\vec{E}$ is the electric field. This means that the electric field at a point in space is the force felt by a unit positive "test" charge placed at that point. Therefore, with the geometry set up as in the figure on the right, we get that the electric field at a point $\vec{x}$ due to a point charge q_1 at the point $\vec{x}_1$ is given by

$$\vec{E}(\vec{x}) = \frac{q_1}{4\pi\epsilon_0} \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3} \tag{W6.9}$$



Note: It is important to check that we have the vector direction correct in equation (W6.14) above. If q_1 is a positive charge, then the force on a unit positive test charge at location $\vec{x}$ will be along the direction of vector $\vec{r}$ from q_1 to the unit positive charge, so we have $\vec{x}_1 + \vec{r} = \vec{x}$, and hence $\vec{r} = \vec{x} - \vec{x}_1$.

If, now, we had a collection of n discrete point charges $q_1, q_2, q_3, \ldots, q_n$, located at $x_1, x_2, x_3, \ldots, x_n$ respectively, then we would get

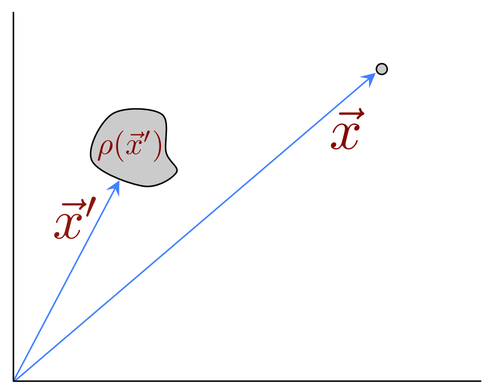
$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3} \tag{W6.10}$$

If, instead, we had a continuous charge distribution described by a charge density $\rho(\vec{x}\,')$ as shown in the figure below, then the sum would be replaced by an integral

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}\,') \frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3} \, d^3x' \tag{1.5}$$

where $d^3x' = dx'dy'dz'$ is the three-dimensional volume element at $\vec{x}\,'$.

Note how the vectors are specified in the figure, because we will be using this notation frequently. Vector $\vec{x}$ is the position in space at which we are writing the field, whereas $\vec{x}\,'$ is the location of our source charge or charge distribution that is responsible for the field.



The vector factor in the integrand of equation (1.5), viewed as a function of $\vec{x}$, is the negative gradient of the scalar $1/|\vec{x}-\vec{x}\,'|$, that is

$$\frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)$$

We can substitute this into equation (1.5):

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}\,')\left[-\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)\right] d^3x'$$

and, upon realizing that the gradient operation involves $\vec{x}$, but not $\vec{x}\,'$, we can take it outside the integral:

$$\vec{E}(\vec{x}) = \frac{-1}{4\pi\epsilon_0}\,\vec{\nabla}\int \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\, d^3x' \tag{1.15}$$

Now, since $\vec{E}$ is the negative gradient of the scalar potential in electrostatics, that is, since

$$\vec{E} = -\vec{\nabla}\Phi \tag{1.16}$$

we get from equation (1.15) that the scalar potential in terms of the charge density is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\, d^3x' \tag{1.17}$$

We are frequently interested in the Laplacian of the scalar potential, $\nabla^2\Phi$, and from equation (1.17), we see that we'll have to evaluate

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)$$

In other words, we need to consider the Laplacian (∇^2) of $1/r$, where $r = |\vec{x}-\vec{x}\,'|$.

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We are engaged in the process of finding the Laplacian of the scalar potential, $\nabla^2\Phi$, and from equation (1.17), we have seen that we'll have to evaluate

$$\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}\,'|} \right)$$

i.e., we need to consider the Laplacian (∇^2) of $1/r$, where $r = |\vec{x} - \vec{x}\,'|$.

The Laplacian of $1/r$ is singular: $\nabla^2(1/r) = 0$ for $r \neq 0$, and its volume integral is -4π. Such a behavior can be represented in terms of a Dirac δ-function, that is, we can write

$$\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}\,'|} \right) = -4\pi\delta(\vec{x} - \vec{x}\,') \tag{1.31}$$

Now, $1/|\vec{x} - \vec{x}\,'|$ is just the potential of a unit point source charge. Following our discussion introducing the Green function, we see that equation (1.31) represents exactly what is meant by the Green function — it is the point source response. Or, as Jackson says, $1/|\vec{x} - \vec{x}\,'|$ is one of a class of functions depending on the variables $\vec{x}$ and $\vec{x}\,'$, called Green functions (or Green's functions — various authors have their own opinions on how to write it to "sound" correct).

Therefore, we can write equation (1.31) as

$$\nabla^2 G(\vec{x}, \vec{x}\,') = -4\pi\delta(\vec{x} - \vec{x}\,') \tag{1.39}$$

where, in general

$$G(\vec{x}, \vec{x}\,') = \frac{1}{|\vec{x} - \vec{x}\,'|} + F(\vec{x}, \vec{x}\,') \tag{1.40}$$

with the function F satisfying the Laplace equation

$$\nabla^2 F(\vec{x}, \vec{x}\,') = 0 \tag{1.41}$$

Note that Jackson's equation (1.39)-(1.41) has a ∇'^2, and I'm not sure why he wrote it this way — it's the variable $\vec{x}$ specifying the locations where you want to know the field that's getting differentiated, so I believe it should be ∇^2.

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Green Functions in Electrostatics

In electrostatics, a well known example of an inhomogenous differential equation is the Poisson equation for the scalar potential $\Phi(\vec{x})$, given by

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \tag{1.28}$$

The Green function equation corresponding to the Poisson equation would be

$$\nabla^2 G(\vec{x}, \vec{x}\,') = -4\pi\delta(\vec{x} - \vec{x}\,') \tag{1.39}$$

because, as we have learned already, the Green function is the point source response for any given differential equation. Recall also that $\vec{x}\,'$ is the source point and $\vec{x}$ is the observation point.

If we can find the solution to equation (1.39) that determines $G(\vec{x}, \vec{x}\,')$, then we can write the solution to Poisson's equation as

$$\Phi(\vec{x}) = \Phi_h(\vec{x}) + \int_V G(\vec{x}, \vec{x}\,') \left[\frac{\rho(\vec{x}\,')}{\epsilon_0}\right] d^3x' \tag{W6.11}$$

where $\Phi_h(\vec{x})$ is a solution to the homogenous (Laplace) equation $\nabla^2\Phi = 0$.

In electrostatics, $\Phi_h(\vec{x})$ is usually viewed as a surface integral on the surface S bounding the volume V, so we'll ignore it here by assuming that there are no boundary surfaces in the problem being discussed; if you're interested in how to handle the surface term, see page 39 in Jackson.

So, with no boundary surfaces present, we have

$$\Phi(\vec{x}) = \int_V G(\vec{x}, \vec{x}\,') \left[\frac{\rho(\vec{x}\,')}{\epsilon_0}\right] d^3x' \tag{W6.12}$$

Therefore, if we can determine $G(\vec{x}, \vec{x}\,')$, we have solved the Poisson equation (1.28) and obtained $\Phi(\vec{x})$. We demonstrated on one of the previous pages that

$$\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}\,'|}\right) = -4\pi\delta(\vec{x} - \vec{x}\,')$$

so an appropriate solution for $G(\vec{x}, \vec{x}\,')$ in equation (1.39) would be

$$G(\vec{x}, \vec{x}\,') = \frac{1}{|\vec{x} - \vec{x}\,'|} \tag{W6.13}$$

Indeed, if we substitute this solution for $G(\vec{x}, \vec{x}\,')$ into equation (W6.12), we would get

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x' \tag{W6.14}$$

which is identical to the equation (1.17) for the scalar potential $\Phi(\vec{x})$ that we derived on one of the previous pages from $\vec{E} = -\vec{\nabla}\Phi$ for the electrostatic case.