PHY 420 **Spring 2021**

Week 6—Tuesday, May 4—Discussion Worksheet

The Special Theory of Relativity

Einstein became such a popular figure in the public imagination that the word relativity has become synonymous with him. People forget that relativity has been around since the time of Galileo. Specifically, it has been known since the time of Galileo that the laws of mechanics are the same in different reference frames moving uniformly relative to one another. Einstein's contribution was to extend this principle to all physical laws.

To understand this in context, let us write down first the equations of Galilean relativity. Consider two reference frames K and K' moving with velocity $\vec{v}$ relative to each other. Then

$$\begin{aligned} \vec{x}\,' &= \vec{x} - \vec{v}t \\ t' &= t \end{aligned} \tag{11.1}$$

Mathematically, all we mean by relativity is the question: If we write a physical law in frame K, does it take the same form in K'? For example,

$$F'_x = m\ddot{x}' = m\,\frac{d^2}{dt^2}\Big(x - v_x t\Big) = m\,\frac{d}{dt}\Big(\dot{x} - v_x\Big) = m\ddot{x} = F_x$$

meaning that Newton's second law has the *same* form in frame K' as in frame K. In fact, it can be shown that Newton's law has the same form in every inertial frame. This is an example of Galilean relativity.

So, the next question to ask is: are Maxwell's equations invariant under Galilean relativity?

1. First, show by explicit differentiation that under Galilean transformations

$$\frac{\partial \psi}{\partial t'} = \Big(\vec{v} \cdot \vec{\nabla}\Big)\,\psi + \frac{\partial \psi}{\partial t}$$

where $\psi \equiv \psi(x, y, z, t)$.

$$\frac{\partial \psi}{\partial t'} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial t'} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial t'} + \frac{\partial \psi}{\partial t}\underbrace{\frac{\partial t}{\partial t'}}_{=1} \quad \text{b/c } t = t'$$

$\downarrow$ $V_x = \partial x / \partial t = \partial x / \partial t'$ $t = t'$ in Galilean relativity

$$\frac{\partial \psi}{\partial t'} = V_x \frac{\partial \psi}{\partial x} + V_y \frac{\partial \psi}{\partial y} + V_z \frac{\partial \psi}{\partial z}$$

$$= \underbrace{\left[V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} + V_z \frac{\partial}{\partial z}\right]}\psi + \frac{\partial \psi}{\partial t}$$

$$= \left(\vec{V} \cdot \vec{\nabla}\right)\psi + \frac{\partial \psi}{\partial t}$$

2. We are examining whether Maxwell's equations are invariant under Galilean relativity

(a) Show that under Galilean transformations: $\dfrac{\partial\psi}{\partial x'} = \dfrac{\partial\psi}{\partial x}$, implying that $\nabla'^2\psi = \nabla^2\psi$.

$$\frac{\partial \psi}{\partial x'} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial \psi}{\partial t}\frac{\partial t}{\partial x'}$$

$$= \frac{\partial \psi}{\partial x}\frac{\partial}{\partial x'}\left[x' + V_x t\right] = \frac{\partial \psi}{\partial x}\left[\frac{\partial x'}{\partial x'} + V_x \frac{\partial t}{\partial x'}\right] = \frac{\partial \psi}{\partial x}\left[1 + 0\right]$$

$$\frac{\partial \psi}{\partial x'} = \frac{\partial \psi}{\partial x} \longrightarrow \nabla'^2 = \nabla^2$$

(b) Now, begin by assuming that the Maxwell equations are valid in the frame K', so that the Helmholtz wave equation in K' is

$$\left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right]\psi = 0$$

Transform this wave equation into the frame K, and write down what that allows you to conclude about the behavior of Maxwell's equations under Galilean transformations.

$$0 = \left[(\nabla')^2 - \frac{1}{c^2}\frac{\partial}{(\partial t')^2}\right]\psi$$

$$= \nabla^2\psi - \frac{1}{c^2}\left[\vec{V}\cdot\vec{\nabla} + \frac{\partial}{\partial t}\right]\left[\vec{V}\cdot\vec{\nabla} + \frac{\partial}{\partial t}\right]\psi$$

$$= \nabla^2\psi - \frac{1}{c^2}\left(\vec{V}\cdot\vec{\nabla}\right)^2\psi - \underbrace{\frac{2}{c^2}\left(\vec{V}\cdot\vec{\nabla}\right)\frac{\partial\psi}{\partial t}} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\frac{\partial}{\partial t}\left[\left(\vec{V}\cdot\vec{\nabla}\right)\psi\right] = \left(\vec{V}\cdot\vec{\nabla}\right)\frac{\partial\psi}{\partial t}$$

$$\frac{\partial\vec{V}}{\partial t} = 0 \rightarrow \frac{\partial}{\partial t}\left(\vec{\nabla}\right) = 0$$

Thus, $\left[\nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\right]\psi = \dfrac{2}{c^2}\left(\vec{V}\cdot\vec{\nabla}\right)\dfrac{\partial\psi}{\partial t} + \left(\dfrac{\vec{V}\cdot\vec{\nabla}}{c}\right)^2\psi \neq 0$

Helmholtz equation, and extension, Maxwell's equation are not invariant under Galilean transformation

Lorentz Transformations

Consider two inertial reference frames K and K' moving relative to each other. For convenience, we'll consider the two frames to be oriented such that the coordinate axes in the two frames are parallel and K' is moving in the positive z direction with speed v, as viewed from K. Also, for simplicity, we will consider the origins of the coordinates in K and K' to be coincident at $t = t' = 0$.

Introduce the notation

$$x_0 = ct, \qquad x_1 = z, \qquad x_2 = x, \qquad x_3 = y$$

Then the time and space coordinates in the frames K and K' are related by the Lorentz transformation

$$\left.\begin{aligned} x_0' &= \gamma\left(x_0 - \beta x_1\right) \\ x_1' &= \gamma\left(x_1 - \beta x_0\right) \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}\right\} \tag{11.16}$$

where

$$\begin{aligned} \beta &= |\vec{\beta}|, \qquad \text{and: } \vec{\beta} = \frac{\vec{v}}{c} \\ \gamma &= \frac{1}{\sqrt{(1-\beta^2)}} \end{aligned} \tag{11.17}$$

As you are no doubt aware already, equation (11.16) reflects that the coordinates perpendicular to the direction of relative motion are unchanged, while the parallel coordinate and the time are transformed.

The Lorentz transformations can be generalized (as you'll do on the homework) to the case where the velocity $\vec{v}$ of the frame K' in frame K is in an arbitrary direction:

$$\left.\begin{aligned} x_0' &= \gamma\left(x_0 - \vec{\beta}\cdot\vec{x}\right) \\ \vec{x}\,' &= \vec{x} + \frac{\gamma-1}{\beta^2}\left(\vec{\beta}\cdot\vec{x}\right)\vec{\beta} - \gamma\vec{\beta}\,x_0 \end{aligned}\right\} \tag{11.18}$$

3. Write down the inverse Lorentz transformations **by inspection** of equation (11.16).

$$\left.\begin{aligned} X_0 &= \gamma(X_0' + \beta X_1') \\ X_1 &= \gamma(X_1' + \beta X_0') \\ X_2 &= X_2' \\ X_3 &= X_3' \end{aligned}\right\}$$

$\beta = |\vec{\beta}|$ and $\vec{\beta} = \dfrac{\vec{V}}{C}$

$$\gamma = \frac{1}{\sqrt{1/(1-\beta^2)}}$$

Jackson problem 11.3: Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1 + v_2}{1 + (v_1 v_2/c^2)}$$

4. Begin by defining coordinate systems. Let frame K' be moving at v_1 with respect to K, and frame K'' be moving at v_2 with respect to K'.

(a) Write down the Lorentz transformation equations **from** (x_0, x_1, x_2, x_3) **to** (x'_0, x'_1, x'_2, x'_3), and **from** (x'_0, x'_1, x'_2, x'_3) **to** $(x''_0, x''_1, x''_2, x''_3)$. Use $\gamma_1 = \left(1 - v_1^2/c^2\right)^{-1/2}$ and $\gamma_2 = \left(1 - v_2^2/c^2\right)^{-1/2}$

$K_0 \to K_1$

$$x'_0 = \gamma_1 \left(x_0 - \frac{v_1}{c} x_1\right)$$

$$x'_1 = \gamma_1 \left(x_1 - \frac{v_1}{c} x_0\right)$$

$$x'_2 = x_2$$

$$x'_3 = x_3$$

$K_1 \to K_2$

$$x''_0 = \gamma_2 \left(x'_0 - \frac{v_2}{c} x'_1\right)$$

$$x''_1 = \gamma_2 \left(x'_1 - \frac{v_2}{c} x'_0\right)$$

$$x''_2 = x'_2$$

$$x''_3 = x'_3$$

(b) Use the equations you wrote in part (a) to show that

$$x''_0 = \gamma_1 \gamma_2 \left(1 + \frac{v_1 v_2}{c^2}\right) \left[x_0 - \frac{v_1 + v_2}{1 + (v_1 v_2/c^2)} \frac{x_1}{c}\right]$$

and **discuss what needs to be demonstrated** to complete the problem.

$$x''_0 = \gamma_2 \Big(\underbrace{x'_0}_{\gamma_1 (x_0 - \frac{v_1}{c} x_1)} - \frac{v_2}{c} \underbrace{x'_1}_{\gamma_1 (x_1 - \frac{v_1}{c} x_0)} \Big)$$

$$x''_0 = \gamma_2 \left(\left[\gamma_1 \left(x_0 - \frac{v_1}{c} x_1\right)\right] - \frac{v_2}{c} \left[\gamma_1 \left(x_1 - \frac{v_1}{c} x_0\right)\right] \right)$$

$$= \underbrace{\gamma_1 \gamma_2 \left(1 + \frac{v_1 v_2}{c^2}\right)}_{\text{Need to show } \gamma = 1/\sqrt{1-\beta^2}} \underbrace{\left[x_0 - \frac{v_1 + v_2}{1 + (v_1 v_2/c^2)} \frac{x_1}{c}\right]}_{\left[x_0 - \frac{v}{c} x_1\right]}$$

5. Based on your discussion on the previous page of what needs to be done in order to complete **Jackson problem 11.3**, go ahead and finish the problem.

$$\gamma_1\gamma_2\left(1-\frac{V^2}{c^2}\right)^{-1/2} = \left[1-\frac{1}{c^2}\left\{\frac{V_1+V_2}{1+(V_1V_2/c^2)}\right\}^2\right]^{-1/2}$$

$$= \left[1-\frac{1}{\cancel{c^2}}\,\frac{(V_1+V_2)^2}{(c^2+V_1V_2)^2/\cancel{c^4}}\right]^{-1/2}$$

$$= \left[\frac{(c^2+V_1V_2)^2-(V_1+V_2)^2c^2}{(c^2+V_1V_2)^2}\right]^{-1/2}$$

$$\gamma = \left[\frac{(c^2+V_1V_2)^2}{(c^2+V_1V_2)^2-(V_1+V_2)^2c^2}\right]^{+1/2} = \frac{c^2+V_1V_2}{\sqrt{c^4+V_1^2V_2^2+2c^2V_1V_2-(V_1^2+V_2^2+2V_1V_2)c^2}}$$

$$= \frac{c^2+V_1V_2}{\sqrt{c^4+V_1^2V_2^2+\cancel{2c^2V_1V_2}-V_1^2c^2-V_2^2c^2-\cancel{2c^2V_1V_2}}}$$

Thus, $$\gamma = \frac{c^2+V_1V_2}{\sqrt{c^4-(V_1^2+V_2^2)c^2+V_1^2V_2^2}}$$

Therefore,

$$\gamma = \left(1-\frac{V^2}{c^2}\right)^{-1/2} = \gamma_1\gamma_2\left(1+\frac{V_1V_2}{c^2}\right),$$ if we set $$V = \frac{V_1+V_2}{1+(V_1V_2/c^2)}$$

So,

$$x_0'' = \gamma\left[x_0-\frac{V}{c}\,x_1\right],$$ with $$V = \frac{V_1+V_2}{1+(V_1V_2/c^2)}$$