PHY 474/374 **Spring 2021**

Week 5—Monday, April 26—Discussion Worksheet

Nuclear Energy Generation in Stars

Stars derive their energy from nuclear reactions throughout most of their lifetime. In this chapter, we will learn some of the physics of nuclear energy generation in more detail.

1. First, let's be sure about this statement. The luminosity of the Sun is 3.8×10^{26} watts, and its mass is 1.99×10^{30} kg.

(a) Begin by figuring out the rate of mass conversion to energy needed to sustain the Sun's luminosity. Use $E = mc^2$ to find $\Delta m/\Delta t$, the rate of mass conversion to energy.

$$\Delta E = \Delta m c^2$$

$$\rightarrow \Delta m = \Delta E / c^2$$

$$\rightarrow \Delta m = 3.8 \times 10^{26} \text{ watts} / (3.0 \times 10^8 \text{ ms}^{-1})^2$$

$$\rightarrow \Delta m = 4.2 \times 10^9 \text{ kg/s}$$

(b) Could the Sun's energy be produced by chemical means, as opposed to nuclear? A gallon of gasoline (about 2.85 kg) produces 1.3×10^8 J of energy. If the Sun were a ball of gasoline, how long would it last (assuming it produced energy at its current luminosity)?

$$1.99 \times 10^{30} \text{ kg} / 2.85 \text{ kg} \cdot (1 \text{ gallon}) = 6.98 \times 10^{29} \text{ gal}$$

$$\rightarrow 6.98 \times 10^{29} \text{ gal} \cdot 1.3 \times 10^8 \text{ J} = 9.07 \times 10^{37} \text{ J}$$

$$\rightarrow 9.07 \times 10^{37} \text{ J} / 3.8 \times 10^{26} \text{ watts}$$

7,600 years

2. Nuclear reactions are caused by the ***strong nuclear force***. Although the strongest of the four fundamental forces in nature, its influence is limited to nuclear size scales. Thus, the interacting nuclei must be brought very close each other for nuclear reactions. This requires overcoming the Coulomb repulsion between like charges, thus requires very high energy.

(a) To get an idea of the typical energies involved, calculate the height of the Coulomb barrier at the nuclear surface, $r_0 \sim 10^{-15}$ m $\equiv 10^{-13}$ cm, given (in ***cgs units***) by

$$E_{\rm Coul} \simeq \frac{\mathcal{Z}_1 \mathcal{Z}_2 e^2}{r_0}$$

for two interacting protons; thus $\mathcal{Z}_1 = \mathcal{Z}_2 = 1$. Express your answer in eV, keV, or MeV (as appropriate). You will need $e = 4.8 \times 10^{-10}$ esu, and 1 erg $= 6.242 \times 10^{11}$ eV.

$$E_{coul} \simeq \frac{Z_1 Z_2 e^2}{r_0}$$

$$E_{coul} \simeq \frac{(1)(1)(4.8\times10^{-10}\,esu)^2}{10^{-13}\,cm}$$

$$E_{coul} \simeq 2.304\times10^{-6}\,erg \cdot \frac{6.242\times10^{11}\,eV}{1\,erg}$$

$$E_{coul} \simeq 1.4\times10^{6}\,eV = 1.4\,MeV$$

(b) Meanwhile, the average kinetic energy of the nucleus is

$$\langle E_{\rm kin} \rangle = \frac{3}{2}\, k_B T$$

Calculate the average kinetic energy in the cores of stars undergoing nuclear fusion of hydrogen, where $T \sim 10^7$ K. Express your answer in eV, keV, or MeV (as appropriate), and comment on how this compares to the energy required to overcome the Coulomb barrier.

$$\langle E_{kin} \rangle = \frac{3}{2} k_B T$$

$$\langle E_{kin} \rangle = \frac{3}{2}(8.6\times10^{-5}\,eV\cdot K^{-1})(10^7\,K)$$

$$\langle E_{kin} \rangle = 1290\,eV$$

3. The cross sections for nuclear reactions inside stars can be written in terms of the following: the energy dependence of the probability that the nuclei will penetrate the Coulomb barrier (which we can obtain this from α-particle decay), the geometrical extent of the nuclei given by their de Broglie wavelengths, and a factor $S(E)$ that varies slowly with the energy and describes the energy dependence of the reaction once the nuclei have penetrated the potential barrier. The cross section is then given by

$$\sigma(E) \equiv \frac{S(E)}{E} \exp\left(-\frac{2\pi \mathcal{Z}_1 \mathcal{Z}_2 e^2}{\hbar v}\right)$$

where v is the relative speed between the two nuclei, e is the charge of an electron, $\mathcal{Z}_1, \mathcal{Z}_2$ are the atomic numbers of the nuclei, E is the energy of the nucleus and $\hbar = h/2\pi$, where h is Planck's constant.

Based on this expression, which reaction would you expect at lower temperatures in general, reactions among nuclei of lower charges ($\mathcal{Z}$) or higher charges? Discuss.

Small Z means that the temperature is put in the denominator

larger cross section

$$\sigma(E) = \frac{1}{e^{(Z_1 Z_2)}}$$

4. In order to compute the total energy released in a nuclear reaction, we need to be aware of the following. If we add the individual masses of all the protons and neutrons in the nucleus, then we will find that this sum is larger than the total mass of the nucleus constituted by these protons and neutrons (collectively called nucleons). Thus, when a nucleus is formed from these nucleons, the mass difference is released as energy. This difference is known as the ***binding energy*** of the nucleus, and is given by

$$Q(\mathcal{Z},\mathcal{N}) = c^2\Big[\mathcal{Z}\,m_p + \mathcal{N}\,m_n - m(\mathcal{Z},\mathcal{N})\Big]$$

for a nucleus with $\mathcal{Z}$ protons and $\mathcal{N}$ neutrons.

(a) Calculate the binding energy of ^{4}He, an alpha particle.

Note: The mass of a proton is 1.007825 u, that of a neutron is 1.008665 u, and the mass of ^{4}He is 4.002602 u, where u stands for atomic mass unit, and 1 u = 931.5 MeV/c^2.

$^4He = 4.002602\ U$

$\rightarrow \dfrac{4.002602\,U}{1\,U} \cdot 931.5\ MeV/c^2$

$\rightarrow 3.73 \times 10^3\ MeV/c^2$

28.3 MeV

(b) It is also customary to find the binding energy per nucleon. Compute it for ^{4}He.

2p + 2n = 4 nucleons, so

$\dfrac{Q}{\text{Nucleon}} = \dfrac{28.3\ MeV}{4\ \text{nucleons}} = 7.075 = 7.1\ MeV/\text{nucleon}$

5. We can obtain the energy released by a nuclear reaction by finding the total mass of the reactants and that of the products, subtracting them, and converting the difference in mass to energy using the mass-energy equivalence. Since the difference in mass is very small, it is generally more convenient to work with the ***mass excess***, Δm, defined for a nucleus with $\mathcal{Z}$ protons and $\mathcal{N}$ neutrons as

$$\Delta m = m - m_u\left(\mathcal{Z} + \mathcal{N}\right)$$

where m is the mass of the nucleus, and m_u is the atomic mass unit. Then, since the combined number of protons and neutrons must be conserved, we can write

$$Q = c^2\Big[\Delta m(A) + \Delta m(a) - \Delta m(Y) - \Delta m(y)\Big]$$

for the nuclear reaction $A + a \to Y + y$.

Find the energy released in the nuclear reaction

$${}^2H + {}^{12}C \to {}^1H + {}^{13}C$$

The tabulated excess masses are

$$\Delta m({}^2H) = 13.136 \text{ MeV}, \qquad \Delta m({}^1H) = 7.289 \text{ MeV}, \qquad \Delta m({}^{13}C) = 3.1246 \text{ MeV},$$

and $\Delta m({}^{12}C) = 0$ (by definition).

$$Q = c^2\Delta m({}^2H) + c^2\Delta m({}^{12}C) - c^2\Delta m({}^1H) - c^2\Delta m({}^{13}C)$$

$$= 13.136 \text{ MeV} + 0 - 7.289 \text{ MeV} - 3.1246 \text{ MeV}$$

$$= 2.7224 \text{ MeV}$$

$$= 2.72 \text{ MeV}$$

(+) value (exothermic) energy is liberated in this reaction, appears to be kinetic energy of the products, so their internal excitation.