NAME: ______________________________

Final Exam

Physics 342/442, Fall 2020

There is information attached at the end of the exam that you may find useful. No books or notes allowed. Good Luck !

(1) Answer the following in a clear and concise way. Usually more words means less points.

(a) (*5 points*) **Briefly** describe the differences and similarities between convolution and correlation. The use of mathematical expressions might be helpful. Please note the emphasis on briefly.

(b) (*5 points*) **Briefly** describe the differences and similarities between the Fourier Series, Fourier Transform, and Discrete Fourier Transform. Please note the emphasis on briefly.

(c) (*5 points)* **Briefly** describe the differences between a cubic spline and fitting data to a cubic polynomial. Please note the emphasis on briefly.

2.a) *(5 points)* Convert the following 2nd order ODE to a system of first order ODEs.

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 4y = 0.$$

2.b) (*15 points*) Use Runge-Kutta to evolve the system of ODEs in part (a) by filling in the following table. The initial conditions are that $t = 0, y = 0, \dot{y} = v = 1$ and use a time step of $\Delta t = h = 0.5$. Please note that one-time step is sufficient.

t	y	v
0	0	1
1		
2		
3		

3-a (*5 points*) Consider the following data vector,

$$A = (3\ 7\ 2\ 1\ 0\ 8\ 0\ 1).$$

Lay out the terms in the manner needed to perform the *Fast Fourier Transform*(think about the odd and even sums). You do not need to do the FFT, just arrange the data and any corresponding twiddle factors. Label each twiddle factor as W^j where j is the order in which that factor appeared in your scheme. For example, the first twiddle factor would be W^1, the second, W^2, etc. Show each step of the proceess.

3-b (*10 points*) Compute the *Fast Fourier Transform* for the vector,

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \end{pmatrix}.$$

Assume that the vector contains data that was evenly sampled at a rate of 1 Hertz (= once per second). Show all your work in a way that clearly shows you are using the *fast Fourier transform* algorithm.

3-c (*5 points*) Using the results from part (3-b), compute the spectrum for this vector.

4. Consider the following partial differential equation:

$$u_t = u_x.$$

(a) (*5 points*) Write this PDE as a finite difference equation.

(b) (*25 points*) Solve the PDE using finite differences. Put your answers in the appropriate slot in the table below. Use as the boundary/initial conditions

$$u_0^j = 0; \quad u_4^j = 1 \quad u_i^0 = 0 \text{ for all } i \neq 4.$$

Use a time step of $\Delta t = 0.1$ and a spatial step of $\Delta x = 0.2$.

t	$u_{i=0}$	$u_{i=1}$	$u_{i=2}$	$u_{i=3}$	$u_{i=4}$
$t = 0.0$					
$t = 0.1$					
$t = 0.2$					

5. (*15 points*) By explicitly computing the appropriate partial derivatives, determine if the functions require linear or non-linear fits.

(a)

$$f(t) = a_1 \exp(a_2 t) + a_3 + a_4 t; \quad a_1, a_2, a_3, a_4 \text{ parameters}$$

(b)

$$F(z) = \frac{\pi \mu_o}{4} k^2 \left[\frac{1}{z^2} + \frac{1}{(z+a_1)^2} - \frac{2}{(z+a_2)^2} \right]; a_1, a_2 \text{ parameters}$$

(c)

$$f(t) = \frac{K}{1 + e^{-r(t-t_o)}}; \quad K, r, t_o \text{ parameters}$$

Useful and Useless Information

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$	$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$	$x_{N-1} = \frac{\rho_{N-j} - c_{N-j}x_{N-j+1}}{\beta_{N-j}}$
$S = \sum \left(y_i - Y(x_i, a_1, \ldots, a_m)\right)^2$	$\boldsymbol{A} = \boldsymbol{LU}$	$S_{ij} = \frac{\partial^2 S}{\partial a_i \partial a_j}$
$y_{i+1} = y_i + hf(t_i, y_i)$	$f(t_{mid}, y_{mid}) = \frac{y(t_o+h) - y(t_o)}{h}$	$y_{i+1} = y_i + hf(t_{mid}, y_{mid})$
$f_o = f(t_o, y_o)$ $f_1 = f(t_o + \frac{h}{2}, y_o + \frac{h}{2}f_o)$ $f_2 = f(t_o + \frac{h}{2}, y_o + \frac{h}{2}f_1)$ $f_3 = f(t_o + h, y_o + f_2)$ $y_{i+1} = y_i + \frac{h}{6}\left(f_o + 2f_1 + 2f_2 + f_3\right)$	$u_t \equiv u_i^j \approx \frac{u_i^{j+1} - u_i^j}{\Delta t}$ $u_x \equiv u_i^j \approx \frac{u_{i+1}^j - u_{i-1}^j}{2\Delta x}$	$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$ $y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$
$p \otimes q \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} p(\tau)q(t-\tau)d\tau$	$p \odot q \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} p^*(\tau)q(t+\tau)d\tau$	
$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{m=1}^{\infty} b_n \sin(mt)$	$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$	
$g(n\Delta\omega) = \sum_{m=0}^{N-1} f(m\Delta t)e^{-i2\pi mn/N}$	$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt dt$	
$g(n\Delta\omega) = g_{even}(n\Delta\omega) + e^{-i2\pi n/N}g_{odd}(n\Delta\omega)$	$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt dt$	