PHY 411 Winter 2021

Class Summary—Week 8, Day 2—Thursday, Feb 25

Magnetostatics

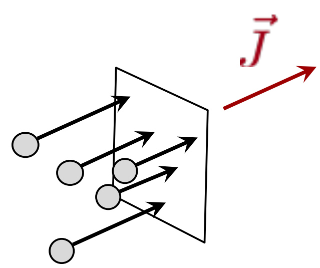
Since there are no "free magnetic charges" the basic entity in magnetic studies is the **magnetic dipole**.

In the presence of magnetic materials, the dipole tends to align itself in a certain direction. That direction is, by definition, the direction of the magnetic-flux density $\vec{B}$ (or magnetic induction), provided the dipole is sufficiently small and weak that it does not perturb the existing field. The magnitude of the flux density can be defined by the mechanical torque $\vec{N}$ exerted on the magnetic dipole:

$$\vec{N} = \vec{\mu} \times \vec{B} \tag{5.1}$$

where $\vec{\mu}$ is the **magnetic moment** of the dipole defined in some suitable set of units; we will define μ rigorously in the next class.

So, we have a more complicated situation than for the electrostatic field. A quantitative explanation of magnetic phenomena had to wait until after the connection between currents and magnetic fields was established. Recall that a current corresponds to charges in motion and is described by a **current density** $\vec{J}$, measured in units of positive charge crossing unit area per unit time (amperes per m^2), the direction of motion of the charges being the direction of $\vec{J}$ (see figure below). So $\vec{J}$ is a vector field that describes the flow of charge at any point in space.



It is worth remembering that the charge density at any point in space is related to the current density in that neighborhood by a **continuity equation**

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{5.2}$$

This is a conservation law, stemming from the physical fact that a decrease in charge inside a small volume with time must correspond to a flow of charge out through the surface of the small volume, since the total amount of charge must be conserved. In studying magnetostatics,

$$\vec{\nabla} \cdot \vec{J} = 0 \tag{5.3}$$

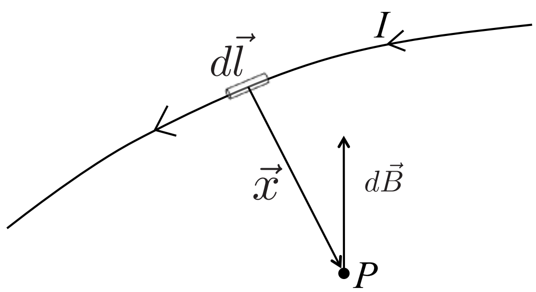
as steady-state magnetic phenomena are characterized by no change in the net charge density anywhere in space, that is, there is no growth or depletion of charge at any one point.

Biot-Savart Law

If $d\vec{l}$ is an element of length, pointing in the direction of current flow, of a filamentary wire that carries a current I and $\vec{x}$ is the coordinate vector from the element of length to an observation point P, as shown in the figure below (taken from Figure 5.1 on page 175 in Jackson), then the elemental magnetic-flux density (or magnetic induction) $d\vec{B}$ at the point P is given in magnitude and direction by

$$d\vec{B} = kI\,\frac{d\vec{l}\times\vec{x}}{|\vec{x}|^3} \tag{5.4}$$

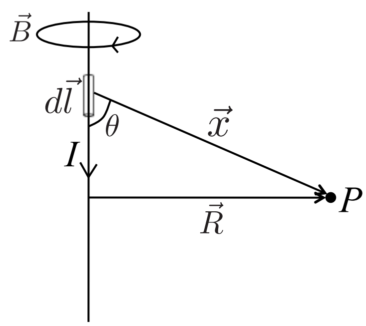
In SI units, $k = \mu_0/4\pi = 10^{-7}$ N A^{-2}.



For a straight length of wire carrying a current I, as shown in the figure below (taken from Figure 5.2 in Jackson), you may recall that the lines of $\vec{B}$ are concentric circles around the wire, with magnitude given by

$$\left|\vec{B}\right| = \frac{\mu_0}{4\pi}\,IR\int_{-\infty}^{\infty}\frac{dl}{(R^2+l^2)^{3/2}} = \frac{\mu_0 I}{2\pi R} \tag{5.6}$$

where R is the perpendicular distance from the observation point to the wire. This is known as the **Biot-Savart law**.



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Ampere's Law

With $Id\vec{l}$ in equation (5.4) written as $\vec{J}(\vec{x}\,')\,d^3x'$, the basic law for the magnetic induction can be written in general form as

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}\,') \times \frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3}\,d^3x' \tag{5.14}$$

Even though, in principle, equation (5.14) contains a description of all the phenomena of magnetostatics, it is not as convenient in some situations as the differential equations.

To obtain the differential equations, we will use the relation on page 29 in Jackson:

$$\frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)$$

to transform equation (5.14) into the form (*as you did on the worksheet*)

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\,\vec{\nabla}\times\int \frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x' \tag{5.16}$$

Since for any arbitrary vector $\vec{a}$, $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a}) = 0$ (see inside front cover of Jackson), it follows immediately from equation (5.16) that

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{5.17}$$

As you will no doubt recall, this is the first equation of magnetostatics and corresponds to $\vec{\nabla}\times\vec{E} = 0$ in electrostatics. Recall that $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$, implying that the divergence of the electric field is caused by the presence of electric charges. So, from the zero divergence of the magnetic field, as you will recall from earlier this quarter, we can infer that there are no isolated magnetic charges.

Next, we would calculate the curl of $\vec{B}$ — I won't do it here to save time, but you can see page 179 in Jackson for the procedure, and we would obtain

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{\mu_0}{4\pi}\,\vec{\nabla}\int \frac{\vec{\nabla}'\cdot\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x' \tag{5.21}$$

The reason I wrote equation (5.21) is to highlight that the path to the final result involves setting $\vec{\nabla}\cdot\vec{J} = 0$ for steady-state magnetic phenomena. Upon doing this, we get

$$\vec{\nabla}\times\vec{B} = \mu_0\,\vec{J} \tag{5.22}$$

This is the second equation of magnetostatics, corresponding to $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ in electrostatics.

Just as Gauss' law is the integral form of $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ in electrostatics, the integral equivalent of equation (5.22) is called **Ampere's law**.

To obtain Ampere's law, integrate the normal component of equation (5.22) over an open surface S bounded by a closed curve C:

$$\int_S \vec{\nabla} \times \vec{B} \cdot \hat{n}\, da = \mu_0 \int_S \vec{J} \cdot \hat{n}\, da \tag{5.23}$$

and apply Stokes theorem to get

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{J} \cdot \hat{n}\, da \tag{5.24}$$

Since the surface integral of the current density is the total current I passing through the closed curve C, Ampere's law can be written in the form

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I \tag{5.25}$$

Ampere's law can be used to calculate the magnetic field in highly symmetric situations, just like Gauss' law for the electric field, as you no doubt know from your undergrad days — you begin with the closed curve C, which is called an Amperian loop, and which plays the same role as the Gaussian surfaces in electrostatics.

As a historical note, note that Ampere's experiments did not deal directly with the determination of the relation between currents and magnetic induction, but were concerned instead with the force that one current-carrying wire experiences in the presence of another. Recall that the force exerted on a charge q moving at velocity $\vec{v}$ in a region of magnetic induction $\vec{B}$ is given by

$$\vec{F} = q\vec{v} \times \vec{B}$$

Casting this in terms of currents, replace the charge q with a wire of infinitesimal length $d\vec{l}$ carrying a current I, so that

$$d\vec{F} = I\Big(d\vec{l} \times \vec{B}\Big) \tag{5.7}$$

This gives us two things — if $\vec{B}$ is due to a closed current loop with current I_2, we can use it to write the total force experienced by a closed current loop with current I_1 as

$$\vec{F}_{12} = \frac{\mu_0}{4\pi}\, I_1 I_2 \oint \oint \frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12})}{|\vec{x}_{12}|^3} \tag{5.8}$$

The line integrals are taken around the two current loops, and $\vec{x}_{12}$ is the vector distance from line element $d\vec{l}_2$ to $d\vec{l}_1$; see, e.g., Figure 5.3 on page 177 in Jackson. We can write this in a more symmetric form (as derived by Jackson at the bottom of page 177):

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi}\, I_1 I_2 \oint \oint \frac{(d\vec{l}_1 \cdot d\vec{l}_2)\, \vec{x}_{12}}{|\vec{x}_{12}|^3} \tag{5.10}$$

The second thing we can do is use equation (5.7) to write an expression for the total force on the current distribution. Integrating equation (5.7), we get the total force:

$$\vec{F} = \int I\Big(d\vec{l} \times \vec{B}\Big)$$

Now, recall from equation (5.24) that $I = \int_S \vec{J} \cdot \hat{n}\, da$, so putting into the equation above, we get

$$\vec{F} = \int \int_S \vec{J} \cdot \hat{n}\, da \left(d\vec{l} \times \vec{B} \right)$$

The surface integral and line integral together give a volume integral, so that

$$\vec{F} = \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d^3x \tag{5.12}$$

Since the torque is $\vec{r} \times \vec{F}$, and we're using $\vec{x}$ as the symbol for $\vec{r}$, we get the total torque $\vec{N}$ to be:

$$\vec{N} = \int \vec{x} \times (\vec{J} \times \vec{B})\, d^3x \tag{5.13}$$

Vector Potential

Since $\vec{\nabla} \cdot \vec{B} = 0$ everywhere, $\vec{B}$ must be the curl of some vector field $\vec{A}(\vec{x})$, the **vector potential**; we learned this earlier in the quarter for the full electromagnetic field. So, we can write

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}) \tag{5.27}$$

Now, we've already written $\vec{B}$ in this form in equation (5.16), so from it, we get the general form of $\vec{A}$ to be

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x' + \vec{\nabla}\Psi(\vec{x}) \tag{5.28}$$

The added gradient of an arbitrary scalar function Ψ shows that for a given magnetic induction $\vec{B}$, the vector potential can be freely transformed according to

$$\vec{A} \to \vec{A} + \vec{\nabla}\Psi \tag{5.29}$$

Recall from earlier this quarter that such a transformation is called a **gauge transformation**, and such a gauge transformation gives us the freedom to make $\vec{\nabla} \cdot \vec{A}$ have any convenient functional form we wish.

Now, let's substitute equation (5.27) into $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$:

$$\begin{aligned} \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) &= \mu_o \vec{J} \\ \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} &= \mu_0 \vec{J} \end{aligned} \tag{5.30}$$

where we've used the vector formula from the inside front cover of Jackson to write the second equation above.

Let us choose $\vec{\nabla} \cdot \vec{A} = 0$; from earlier this quarter, you might remember that this is called the Coulomb gauge. Then each rectangular component of the vector potential satisfies the Poisson equation

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \tag{5.31}$$

With $\Psi =$ constant, the solution for $\vec{A}$ can be written from equation (5.28) to be

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x' \tag{5.32}$$

In the next class, we will move on to Section 5.6 on page 184 to discuss magnetic fields of a localized current distribution.