PHY 420 Spring 2021

Class Summary—Week 3, Day 2—Thursday, Apr 15

Green Functions

In PHY 411, we learned about **Green functions**. We will now begin using them. Today's class is a reminder of what we learned last quarter, and additional examples on applying the mathematical technique of Green functions to electrodynamics.

The Green functions provide a general way to solve inhomogenous partial (or ordinary) differential equations. To remind ourselves of how they work, let us write down again the general schematic for a Green function problem. Any math methods text would be good; I've compiled the following from a series of presentations obtained via a Google search (which is why I haven't provided any one specific source).

Consider the linear inhomogenous differential equation

$$\mathcal{D}\Psi(\vec{x}) = f(\vec{x}) \tag{W3.1}$$

where $\mathcal{D}$ is any linear differential operator, e.g., a well known example of an inhomogenous differential equation in electrostatics is the Poisson equation for the scalar potential $\Phi(\vec{x})$, given by

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

We know from our study of differential equations that the solution to equation (W3.1) is comprised of a solution to the associated homogenous equation Ψ_h, plus the particular solution Ψ_{part} that comes from the inhomogenous equation:

$$\Psi(\vec{x}) = \Psi_h + \Psi_{\text{part}} \tag{W3.2}$$

We do not need to ourselves with Ψ_h, since there are many techniques to solve homogenous PDE's. Instead, we are going to look for the particular solution (Ψ_{part}) with the inhomogeneity on the right hand side, represented here by the generic function $f(\vec{x})$.

To do so, we need to consider the associated inhomogenous equation for a point source function

$$\mathcal{D}G(\vec{x},\vec{x}\,') = \delta(\vec{x}-\vec{x}\,') \tag{W3.3}$$

which defines the Green function $G(\vec{x},\vec{x}\,')$ corresponding to the operator $\mathcal{D}$.

Once we've determined $G(\vec{x},\vec{x}\,')$, then the particular solution to the inhomogenous equation can be written as

$$\Psi_{\text{part}} = \int_V G(\vec{x},\vec{x}\,')\, f(\vec{x}\,')\, d^3x' \tag{W3.4}$$

so that the solution to the differential equation (W3.1) is

$$\Psi(\vec{x}) = \Psi_h + \int_V G(\vec{x},\vec{x}\,')\, f(\vec{x}\,')\, d^3x' \tag{W3.5}$$

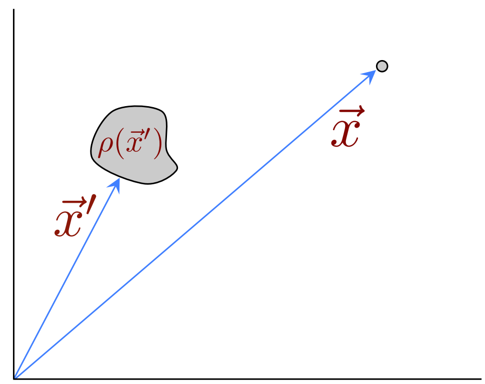
In summary, the Green function may be characterized as the response of a system to a Dirac δ-function input signal, to take an electronics analogy. Now, because any function can be expanded as a sum of Dirac signals, you can compute the output for any signal input if you know the Green function.

Green Functions in Electrostatics

We will now do an intuitive example from electrostatics. Recall that last quarter we wrote the scalar potential $\Phi(\vec{x})$ in integral form as

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\, d^3x' \tag{1.17}$$

where $d^3x' = dx'dy'dz'$ is the three-dimensional volume element at $\vec{x}\,'$. I've also included a figure showing how the vectors are specified. Vector $\vec{x}$ is the position in space at which we are writing the potential, whereas $\vec{x}\,'$ is the location of our source charge or charge distribution that is responsible for the electrostatic field.



We are frequently interested in the Laplacian of the scalar potential, $\nabla^2\Phi$, and from equation (1.17), we see that we'll have to evaluate

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)$$

In other words, we need to consider the Laplacian (∇^2) of $1/r$, where $r = |\vec{x}-\vec{x}\,'|$.

The Laplacian of $1/r$ is singular: $\nabla^2(1/r) = 0$ for $r \neq 0$, and its volume integral is -4π. Such a behavior can be represented in terms of a Dirac δ-function, that is, we can write

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right) = -4\pi\delta(\vec{x}-\vec{x}\,') \tag{1.31}$$

Now, $1/|\vec{x}-\vec{x}\,'|$ is just the potential of a unit point source charge. Following our discussion introducing the Green function, we see that equation (1.31) represents exactly what is meant by the Green function — it is the point source response. Or, as Jackson says, $1/|\vec{x}-\vec{x}\,'|$ is one of a class of functions depending on the variables $\vec{x}$ and $\vec{x}\,'$, called Green functions (or Green's functions — various authors have their own opinions on how to write it to "sound" correct).

Therefore, we can write equation (1.31) as

$$\nabla^2 G(\vec{x},\vec{x}\,') = -4\pi\delta(\vec{x}-\vec{x}\,') \tag{1.39}$$

where, in general

$$G(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|} + F(\vec{x},\vec{x}\,') \tag{1.40}$$

with the function F satisfying the Laplace equation

$$\nabla^2 F(\vec{x},\vec{x}\,') = 0 \tag{1.41}$$

Note that Jackson's equation (1.39)-(1.41) has a ∇'^2, and I'm not sure why he wrote it this way — it's the variable $\vec{x}$ specifying the locations where you want to know the field that's getting differentiated, so I believe it should be ∇^2.

We will now cast some of the equations on the preceding page in terms of the equations (W3.1)-(W3.5) on the first page of this class summary.

In electrostatics, a well known example of an inhomogenous differential equation is the Poisson equation for the scalar potential $\Phi(\vec{x})$, given by

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \tag{1.28}$$

Since the Green function is the point source response for any given differential equation, the Green function equation corresponding to the Poisson equation (1.28) would be

$$\nabla^2 G(\vec{x}, \vec{x}\,') = \delta(\vec{x} - \vec{x}\,') \tag{1.39}$$

where $\vec{x}\,'$ is the **source point** and $\vec{x}$ is the **observation point**. Note that Jackson has a -4π factor in equation (1.39); I've chosen to insert it later, but in the end it must be accounted for somewhere to make the expression for $\Phi(\vec{x})$ come out correctly.

If we can find the solution to equation (1.39) that determines $G(\vec{x}, \vec{x}\,')$, then we can write the solution to Poisson's equation as

$$\Phi(\vec{x}) = \Phi_h(\vec{x}) + \int_V G(\vec{x}, \vec{x}\,') \left[-\frac{\rho(\vec{x}\,')}{\epsilon_0} \right] d^3x' \tag{W3.6}$$

where $\Phi_h(\vec{x})$ is a solution to the homogenous (Laplace) equation $\nabla^2 \Phi = 0$.

In electrostatics, $\Phi_h(\vec{x})$ is usually viewed as a surface integral on the surface S bounding the volume V, so we'll ignore it here by assuming that there are no boundary surfaces in the problem being discussed; if you're interested in how to handle the surface term, see page 39 in Jackson. So, with no boundary surfaces present, we have

$$\Phi(\vec{x}) = \int_V G(\vec{x}, \vec{x}\,') \left[-\frac{\rho(\vec{x}\,')}{\epsilon_0} \right] d^3x' \tag{W3.7}$$

Therefore, if we can determine $G(\vec{x}, \vec{x}\,')$, we have solved the Poisson equation (1.28) and obtained $\Phi(\vec{x})$. We demonstrated on the previous page that

$$\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}\,'|} \right) = -4\pi \delta(\vec{x} - \vec{x}\,')$$

so an appropriate solution for $G(\vec{x}, \vec{x}\,')$ in equation (1.39) would be

$$G(\vec{x}, \vec{x}\,') = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}\,'|} \tag{W3.8}$$

Indeed, if we substitute this solution for $G(\vec{x}, \vec{x}\,')$ into equation (W3.7), we would get

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x' \tag{W3.9}$$

which is identical to the equation (1.17) for the scalar potential $\Phi(\vec{x})$ that we wrote on the previous page, and that is derived from the electrostatic field $\vec{E} = -\vec{\nabla}\Phi$.

Green Functions for the Wave Equation

In the previous quarter, we discovered that the four Maxwell equations could be reduced to two equations, which could be uncoupled to two equations in the Lorenz gauge: equation (6.15) for the scalar potential Φ and equation (6.16) for the vector potential $\vec{A}$, or in the Coulomb gauge: equation (6.22) for Φ (which is just the Poisson equation), and equation (6.30) for $\vec{A}$ (which we didn't specifically discuss). I won't write them again here, but just make the point that all three of these, equation (6.15), equation (6.16), and equation (6.30), have the same basic structure of the inhomogenous wave equation:

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\vec{x}, t) \tag{6.32}$$

where $f(\vec{x}, t)$ is a known source distribution. The factor c is the velocity of propagation in the medium.

We will now apply the Green function method to find a solution to the inhomogenous differential equation (6.32).

To keep things simple, we will consider a situation where there are no boundary surfaces (as we did for the electrostatic case above). Moreover, to make things even simpler, let us first start by considering the problem without the time dependence. This is easy to do, because we can remove the explicit time dependence in equation (6.32) by a Fourier transform with respect to frequency. That is, we'll do

$$\Psi(\vec{x}, t) \to \Psi(\vec{x}, \omega) \qquad \text{and} \qquad f(\vec{x}, t) \to f(\vec{x}, \omega)$$

I won't write the expressions for the Fourier transforms here since they're peripheral to what we're going to be discussing, but if you're interested, see equation (6.33) and equation (6.34) on page 243 in Jackson.

After carrying out the Fourier transforms, one finds that the Fourier transformed quantities $\Psi(\vec{x}, \omega)$ and $f(\vec{x}, \omega)$ satisfy the equation

$$\left(\nabla^2 + k^2\right)\Psi(\vec{x}, \omega) = -4\pi f(\vec{x}, \omega) \tag{6.35}$$

where $k = \omega/c$ is the wave number associated with frequency ω.

We see that equation (6.35) looks similar to the Helmholtz wave equation (7.3), but has a term on the right rendering it an inhomogenous equation. Therefore, equation (6.35) is labeled an **inhomogenous** Helmholtz wave equation.

The Green function associated with equation (6.35), therefore, has to satisfy the inhomogenous equation

$$\left(\nabla^2 + k^2\right)G_k(\vec{x}, \vec{x}\,') = -4\pi\delta(\vec{x} - \vec{x}\,') \tag{6.36}$$

In the electrostatic case, I'd moved the -4π factor to a later step, but given the cumbersome nature of the math to follow, I'll leave it in here to match Jackson; in any case, this will help make the point that the -4π factor is an accounting step.

If we can solve for $G_k(\vec{x}, \vec{x}\,')$ from equation (6.36), we'll have found a solution to the inhomogenous Helmholtz wave equation (6.35).

On the previous page, we found that if we can solve for $G_k(\vec{x}, \vec{x}\,')$ from equation (6.36), we'll have found a solution to the inhomogenous Helmholtz wave equation (6.35).

How do we find a solution for $G_k(\vec{x}, \vec{x}\,')$?

The formal way, which Jackson does on pages 243-244, is to begin by writing the Laplacian in spherical coordinates.

However, *I'm going to try a more intuitive approach here.* I find it easier to follow, and we lose nothing because we've already discussed formal solutions to the Laplace equation in different coordinate systems.

So here goes!

Notice that the δ-function in equation (6.36) localizes the source function to one point $\vec{x}\,'$ in space. Everywhere else, where there are no sources, we know that the solution will be a traveling wave — we spent a few weeks discussing exactly this in Chapter 7 last quarter. So

$$G_k \propto e^{\pm i\vec{k}\cdot\vec{x}}$$

where $\vec{k}$ is the wave vector with magnitude k being the wave number.

Since the δ-function involves $(\vec{x} - \vec{x}\,')$, shift the origin in the expression above so that

$$G_k \propto e^{\pm i\vec{k}\cdot(\vec{x}-\vec{x}\,')}$$

Now, the Green function must be spherically symmetric; there is no reason to expect otherwise given that we've localized the source function to one point via the δ-function. That means the Green function can depend only on $|\vec{x} - \vec{x}\,'|$.

Since the wave vector $\vec{k}$ is along the direction of propagation, a dependence only on $|\vec{x} - \vec{x}\,'|$ means that

$$G_k \propto e^{\pm ik|\vec{x}-\vec{x}\,'|}$$

Next, the δ-function in equation (6.36) has influence only at $|\vec{x} - \vec{x}\,'| \to 0$. In that limit, equation (6.35) reduces to the Poisson equation because $kR \ll 1$. In that case, we know from electrostatics that G_k must have a $1/|\vec{x} - \vec{x}\,'|$ dependence because it is the potential for a unit point charge.

Therefore, the Green function is

$$G_k^{(\pm)}(|\vec{x} - \vec{x}\,'|) = \frac{e^{\pm ik|\vec{x}-\vec{x}\,'|}}{|\vec{x} - \vec{x}\,'|} \tag{6.40}$$

The $(+)$ term represents an **outgoing spherical wave** and the $(-)$ terms represents an **incoming spherical wave**.

Next, we will consider the time dependence.

We will now consider the wave equation with its time dependence. We wrote previously that

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\vec{x}, t) \tag{6.32}$$

where $f(\vec{x}, t)$ is a known source distribution.

The Green function associated with equation (6.32), therefore, has to satisfy

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_k^{(\pm)}(\vec{x}, t; \vec{x}\,', t') = -4\pi\delta(\vec{x} - \vec{x}\,')\delta(t - t') \tag{6.41}$$

This time, notice that first δ-function on the right hand side localizes the source function to one point $\vec{x}\,'$ in space, and the other δ-function localizes it to one point in time. Again, everywhere else where there are no sources, we know that the solution will be a traveling wave in time, so

$$G_k \propto e^{\pm i\vec{k}\cdot\vec{x} - i\omega t}$$

where $\vec{k}$ is the wave vector and ω is the frequency of the traveling wave, with $k = \omega\sqrt{\mu\epsilon} = \omega/c$ in free space.

Again, since the δ-functions involve $(\vec{x} - \vec{x}\,')$ and $(t - t')$, shift the origins in space and time in the expression above so that

$$G_k \propto e^{\pm i\vec{k}\cdot(\vec{x} - \vec{x}\,') - i\omega(t - t')}$$

As before, localizing the source function to one point via the δ-function means that the Green function must be spherically symmetric in space, and can depend only on $|\vec{x} - \vec{x}\,'|$, so that

$$G_k \propto e^{\pm ik|\vec{x} - \vec{x}\,'| - i\omega(t - t')}$$

And, finally, we need the $1/|\vec{x} - \vec{x}\,'|$ dependence because it is the potential for a unit point charge, so that

$$G_k = \frac{e^{\pm ik|\vec{x} - \vec{x}\,'| - i\omega(t - t')}}{|\vec{x} - \vec{x}\,'|}$$

The general solution is a combination of all possible solutions. A sum over ω also brings out all possible values of k, since k and ω are connected. So, we get

$$G^{(\pm)}(R, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm ikR}}{R}\, e^{-i\omega\tau}\, d\omega \tag{6.42}$$

where, following Jackson, I've written $R = |\vec{x} - \vec{x}\,'|$ and $\tau = t - t'$ to make the expression look neat and compact.

The general solution for the Green function in infinite space is therefore a function of only the relative distance R between the source and observation point and the relative time τ between, again, source and observation point.

In free space where $k = \omega/c$, the integral in equation (6.42) is just a Dirac δ-function, so that

$$G^{(\pm)}(R,\tau) = \frac{1}{R}\,\delta\left(\tau \mp \frac{R}{c}\right) \tag{6.43}$$

or, if we wish to write it more explicitly in its full glory

$$G^{(\pm)}(\vec{x},t;\vec{x}\,',t') = \frac{\delta\left(t' - \left[t \mp \dfrac{|\vec{x}-\vec{x}\,'|}{c}\right]\right)}{|\vec{x}-\vec{x}\,'|} \tag{6.44}$$

The Green function $G^{(+)}$ is called the **retarded Green function** because the argument of the δ-function ensures that an effect observed at the point $\vec{x}$ at time t is caused by the action of a source a distance R away at an earlier or retarded time $t' = t - R/c$. The time difference R/c is just the time of propagation of the disturbance from one point to the other.

Similarly, $G^{(-)}$ is called the **advanced Green function**.

Having determined the Green function, we can now write the particular solution to the wave equation (6.32) as

$$\Psi^{(\pm)}(\vec{x},t) = \int\int G^{(\pm)}(\vec{x},t;\vec{x}\,',t')\,f(\vec{x}\,',t')\,d^3x'dt'$$

The full solution $\Psi(\vec{x},t)$ requires the solution to the homogenous part that we wrote in equation (W3.2) on the first page of this class summary.

For a concrete example, consider a source distribution $f(\vec{x}\,',t')$ that is localized in space and time. It is different from zero only for a finite interval of time around $t' = 0$.

- Consider a limiting situation where, at $t \to -\infty$, there exists a wave $\Psi_{\text{in}}(\vec{x},t)$ that satisfies the homogenous wave equation. This wave propagates in time and space; near $t' = 0$, the source $f(\vec{x}\,',t')$ turns on and generates waves of its own. The complete solution of this problem for all time is

  $$\Psi(\vec{x},t) = \Psi_{\text{in}}(\vec{x},t) + \int\int G^{(+)}(\vec{x},t;\vec{x}\,',t')\,f(\vec{x}\,',t')\,d^3x'dt' \tag{6.45}$$

  The presence of the retarded Green function $G^{(+)}$ guarantees that at remotely early times before the source has been activated, there is no contribution from the integral, and only the specified Ψ_{in} exists.

- In another situation, at remotely late times $t \to \infty$, there exists a wave $\Psi_{\text{out}}(\vec{x},t)$ that satisfies the homogenous wave equation. Then the complete solution for all times is

  $$\Psi(\vec{x},t) = \Psi_{\text{out}}(\vec{x},t) + \int\int G^{(-)}(\vec{x},t;\vec{x}\,',t')\,f(\vec{x}\,',t')\,d^3x'dt' \tag{6.46}$$

  The presence of the advanced Green function $G^{(-)}$ guarantees that no signal from the source will exist after the source shuts off.