Homework 3

Timothy Holmes
PHY 411 Electrodynamics I

January 25, 2021

Problem 1

Show

$$\left[\frac{1}{\mu}(\vec{k}\times\vec{E_0}+\vec{k}''\times\vec{E_0}'')-\frac{1}{\mu'}(\vec{k}'\times\vec{E}_0')\right]\times\hat{n}=0$$

Reduces to

$$\left[\sqrt{\frac{\epsilon}{\mu}}E_0cos(i)-\sqrt{\frac{\epsilon}{\mu}}E_0''cos(i)-\sqrt{\frac{\epsilon'}{\mu'}}E_0'cos(r)\right]=0.$$

Starting from the first equation, it can be expanded to

$$\left[\frac{1}{\mu}(\vec{k}\times\vec{E_0})+\frac{1}{\mu}(\vec{k}''\times\vec{E_0}'')-\frac{1}{\mu'}(\vec{k}'\times\vec{E}_0')\right]\times\hat{n}=0.$$

the energy flow is in the direction of the , so the curl from $\vec{E}$ is $\vec{B}$. Therefore, $\vec{k}\times\vec{E_0}$ will become $\vec{B}$, $\vec{k}''\times\vec{E_0}''$ will become $\vec{B}''$, and $\vec{k}'\times\vec{E_0}'$ will become $\vec{B}'$. Therefore, the equation becomes

$$\left[\frac{1}{\mu}(\vec{B_0}\times\hat{n})+\frac{1}{\mu}(\vec{B_0}''\times\hat{n})-\frac{1}{\mu'}(\vec{B_0}'\times\hat{n})\right]=0.$$

$\vec{B_0}$ is now perpendicular to the field so this means that all the B_0 vectors are $B_0cos(i)$. Therefore, we have, for one example of $\vec{B_0}$ that

$$\frac{1}{\mu}(\vec{B_0} \times \hat{n}) = \frac{1}{\mu}\vec{B_0}cos(i)$$

which can also be expressed as

$$\frac{1}{\mu}(\vec{B_0} \times \hat{n}) = \frac{\sqrt{\mu\epsilon}}{\mu}\vec{E_0}cos(i)$$

and reduced to

$$\frac{1}{\mu}(\vec{B_0} \times \hat{n}) = \sqrt{\frac{\epsilon}{\mu}}\vec{E_0}cos(i).$$

Thus, the final equation is then

$$\left[\sqrt{\frac{\epsilon}{\mu}}E_0cos(i) - \sqrt{\frac{\epsilon}{\mu}}E_0''cos(i) - \sqrt{\frac{\epsilon'}{\mu'}}E_0'cos(r)\right] = 0.$$

Problem 2

Since

$$E_0 + E_0'' - E_0' = 0$$

and

$$\left[\sqrt{\frac{\epsilon}{\mu}}E_0 cos(i) - \sqrt{\frac{\epsilon}{\mu}}E_0'' cos(i) - \sqrt{\frac{\epsilon'}{\mu'}}E_0' cos(r)\right] = 0.$$

The equation above can be reduced to

$$\sqrt{\frac{\epsilon}{\mu}}(E_0 - E_0'')cos(i) - \sqrt{\frac{\epsilon'}{\mu'}}E_0' cos(r) = 0.$$

Now, $E_0 + E_0'' - E_0' = 0$ can also be rewrote as $E_0 + E_0'' = E_0'$ and the equation will become

$$\sqrt{\frac{\epsilon}{\mu}}(E_0 - E_0'')cos(i) = \sqrt{\frac{\epsilon'}{\mu'}}(E_0 - E_0'')cos(r).$$

From equation 7.36 we know that

$$\frac{sin(i)}{sin(r)} = \frac{k'}{k} = \sqrt{\frac{\mu'\epsilon'}{\mu\epsilon}} = \frac{n'}{n}.$$

From equation 7.44 it can be worked out that

$$i_0 = sin^{-1}\left(\frac{n'}{n}\right) \rightarrow sin(i_0) = \left(\frac{n'}{n}\right) \rightarrow sin(i_0) = \left(\frac{sin(i)}{sin(r)}\right) \rightarrow sin(r) = \left(\frac{sin(i)}{sin(i_0)}\right)$$

Finally, equation 7.45 says that

$$cos(r) = i\sqrt{\left(\frac{sin(i)}{sin(i_0}\right) - 1}$$

$$cos(r) = \sqrt{1 - \left(\frac{sin(i)}{sin(i_0}\right)}$$

$$cos(r) = \sqrt{1 - \frac{n^2}{n'^2} sin^2(i)}$$

$$cos(r) = \frac{\sqrt{n'^2 - n^2 sin^2(i)}}{n}.$$

Therefore, the equation becomes

$$\sqrt{\frac{\epsilon}{\mu}}(E_0 - E_0'')cos(i) = \sqrt{\frac{\epsilon'}{\mu'}}(E_0 - E_0'')\frac{\sqrt{n'^2 - n^2 sin^2(i)}}{n}.$$

Expanding the equation result in

$$\sqrt{\frac{\epsilon}{\mu}}E_0 cos(i) - \sqrt{\frac{\epsilon}{\mu}}E_0'' cos(i) = \sqrt{\frac{\epsilon'}{\mu'}}E_0 \frac{\sqrt{n'^2 - n^2 sin^2(i)}}{n} + \sqrt{\frac{\epsilon'}{\mu'}}E_0'' \frac{\sqrt{n'^2 - n^2 sin^2(i)}}{n}.$$

The idea here is to combine the E terms to be on one specific side such that the equation becomes

$$\sqrt{\frac{\epsilon}{\mu}}E_0 cos(i) - \sqrt{\frac{\epsilon'}{\mu'}}E_0 \frac{\sqrt{n'^2 - n^2 sin^2(i)}}{n} = \sqrt{\frac{\epsilon}{\mu}}E_0'' cos(i) + \sqrt{\frac{\epsilon'}{\mu'}}E_0'' \frac{\sqrt{n'^2 - n^2 sin^2(i)}}{n}$$

and then is factored so that

$$E_0\left(\sqrt{\frac{\epsilon}{\mu}}cos(i) - \sqrt{\frac{\epsilon'}{\mu'}}\frac{\sqrt{n'^2 - n^2 sin^2(i)}}{n}\right) = E_0''\left(\sqrt{\frac{\epsilon}{\mu}}cos(i) + \sqrt{\frac{\epsilon'}{\mu'}}\frac{\sqrt{n'^2 - n^2 sin^2(i)}}{n}\right).$$

Remembering that $\sqrt{\mu'\epsilon'/\mu\epsilon}$, we arrive at the equation

$$\frac{E_0''}{E_0} = \frac{ncos(i) - (\mu/\mu')\sqrt{n'^2 - n^2 sin^2(i)}}{ncos(i) - (\mu/\mu')\sqrt{n'^2 - n^2 sin^2(i)}}.$$

Similarly, if we let $E_0'' = (E_0' - E_0)$ then

$$\left[\sqrt{\frac{\epsilon}{\mu}}E_0cos(i) - \sqrt{\frac{\epsilon}{\mu}}(E_0' - E_0)cos(i) - \sqrt{\frac{\epsilon'}{\mu'}}E_0'cos(r)\right] = 0.$$

Expanding and collecting like terms allow us to rewrite the equation as

$$\sqrt{\frac{\epsilon}{\mu}}(2E_0 - E_0')cos(i) = \sqrt{\frac{\epsilon'}{\mu'}}E_0'cos(r).$$

Separating the E values to each side gives us

$$2\sqrt{\frac{\epsilon}{\mu}}E_0cos(i) = \sqrt{\frac{\epsilon'}{\mu'}}E_0'cos(r) + \sqrt{\frac{\epsilon}{\mu}}E_0'cos(i)$$

$$2\sqrt{\frac{\epsilon}{\mu}}E_0cos(i) = E_0\left(\sqrt{\frac{\epsilon'}{\mu'}}cos(r) + \sqrt{\frac{\epsilon}{\mu}}cos(i)\right)$$

We can rewrite the equation as

$$\frac{E_0'}{E_0} = \frac{2\sqrt{\epsilon/\mu}cos(i)}{\sqrt{\epsilon/\mu}cos(i) + \sqrt{\epsilon'/\mu'}cos(r)}$$

Remember that

$$cos(r) = \frac{\sqrt{n'^2 - n^2sin^2(i)}}{n}.$$

and

$$\sqrt{\frac{\epsilon'}{\epsilon}} = \frac{n'}{n}.$$

Thus, the final equation will becomes

$$\frac{E_0'}{E_0} = \frac{2nn'cos(i)}{ncos(i) - (\mu/\mu')\sqrt{n'^2 - n^2sin^2(i)}}.$$

Problem 3

The transmission. coefficient is gave by equation 7.13c in the class summaries as

$$T = \frac{\vec{s'} \cdot \hat{n}}{\vec{s} \cdot \hat{n}}.$$

The equation $\vec{s}$, $\vec{s'}$, and $\vec{s''}$ and are gave as

$$\vec{s} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E_0|^2\hat{k}$$

$$\vec{s'} = \frac{1}{2}\sqrt{\frac{\epsilon'}{\mu'}}|E_0'|^2\hat{k}$$

$$\vec{s''} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E_0''|^2\hat{k}$$

Now when taking the dot product of the vectors above with the unit vector $\hat{n}$, the following are the results

$$\vec{s} \cdot \hat{n} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E_0|^2 cos(i)$$

$$\vec{s'} \cdot \hat{n} = \frac{1}{2}\sqrt{\frac{\epsilon'}{\mu'}}|E_0'|^2 \cos(r)$$

$$\vec{s''} \cdot \hat{n} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E_0''|^2 \cos(r)'$$

Therefore, the transmission coefficient becomes

$$T = \sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\epsilon}{\mu}}\frac{1}{2}\frac{2}{1}\frac{|E_0'|^2}{|E_0|^2}\frac{cos(r)}{cos(i)}$$

Remember that $cos(r)$ is

$$cos(r) = \frac{\sqrt{n'^2 - n^2 sin^2(i)}}{n}$$

then T becomes

$$T = \frac{|E_0'|^2}{|E_0|^2} \frac{\sqrt{n'^2 - n^2 sin^2(i)}}{ncos(i)}.$$

Since we are going to assume that $\mu = \mu'$ and E_0'/E_0 from equation 7.41, T is now

$$T = \left| \frac{2nn'cos(i)}{n'^2 cos(i) + n\sqrt{(n'^2 sin^2(i)}} \right|^2 \frac{\sqrt{n'^2 - n^2 sin^2(i)}}{ncos(i)}$$

Similarly, the reflection coefficient is

$$R = \frac{\vec{s}'' \cdot \hat{n}}{\vec{s} \cdot \hat{n}}.$$

Following the same method R becomes

$$R = \sqrt{\frac{\mu}{\mu'}} \sqrt{\frac{\epsilon}{\epsilon'}} \frac{1}{2} \frac{2}{1} \frac{|E_0''|^2}{|E_0|^2} \frac{cos(r')}{cos(i)}.$$

Remember that $\mu = \mu'$ and $\epsilon = \epsilon'$, since we are in the same medium, the equation is reduced to

$$R = \left| \frac{n'^2 cos(i) - n\sqrt{n'^2 - n^2 sin^2(i)}}{n'^2 cos(i) + n\sqrt{n'^2 - n^2 sin^2(i)}} \right|^2 \frac{cos(r')}{cos(i)}.$$

The angle i is equal to r', therefore $cos(r')$ can be wrote as $cos(i)$. The $cos(i)$ will cancel out and the final equation becomes,

$$R = \left| \frac{n'^2 cos(i) - n\sqrt{n'^2 - n^2 sin^2(i)}}{n'^2 cos(i) + n\sqrt{n'^2 - n^2 sin^2(i)}} \right|^2.$$

Problem 4

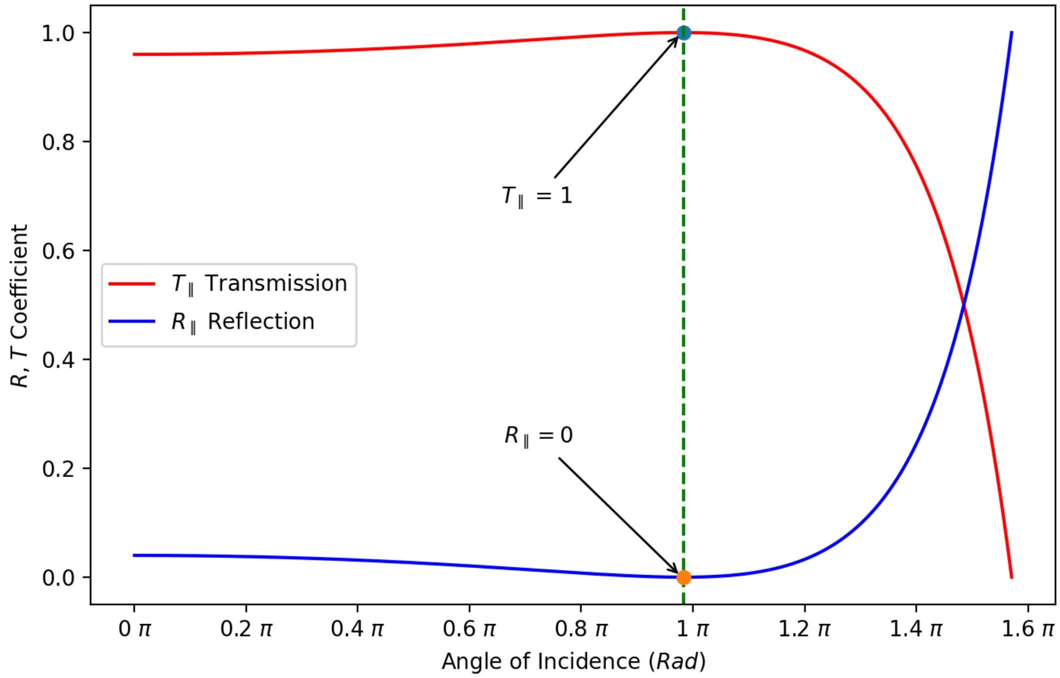

FIG 1: .

FIG 1. shows the reflective coefficient in blue and the transmission coefficient in red. Starting from the when the angle of incidence is $0°$, $T_{\|}$ is not 1 and $R_{\|}$ is not 0. Intuitively, it would be thought that $T_{\|} = 1$ and $R_{\|} = 0$ when the angle of incidence is $0°$. As the angle of incidence increases and reaches $\approx 56°$, then $T_{\|} = 1$ and $R_{\|} = 0$. Increasing the angle further, the blue and red lines intersect which means that $T_{\|} = R_{\|}$, until the angle gets to $\pi/2$ in this case $T_{\|} = 0$ and $R_{\|} = 1$. The reason why $T_{\|} = 1$ and $R_{\|} = 0$ at $\approx 56°$ is due to the Brewster angle. The specific conditions that need to be met are; $n = 1$, $n' = 1.5$, and $\mu = \mu'$. Moreover, the Brewster angle is gave by and for this example is

$$i_B = tan^{-1}\left(\frac{n'}{n}\right) = tan^{-1}\left(\frac{1.5}{1}\right) \approx 56°.$$

At this angle the 100% of the light is transmitted through the surface while 0% is reflected.

Appendix

Transmission Coefficient and Reflection Coefficient Plot

```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import matplotlib.ticker as tck

n       = 1
n_prime = 1.5
i       = np.linspace(0, np.pi/2, 10000)

T_parallel = ((2 * n * n_prime * np.cos(i)) / \
              ((n_prime**2 * np.cos(i)) + (n * np.sqrt(n_prime**2 - n**2 * np.sin(i)**2))))**2 \
              * (np.sqrt(n_prime ** 2 - n ** 2 * np.sin(i) ** 2)/(n * np.cos(i)))

R_parallel = ((n_prime**2 * np.cos(i) - n * np.sqrt(n_prime**2 - n**2 * np.sin(i)**2))/ \
               (n_prime**2 * np.cos(i) + n * np.sqrt(n_prime**2 - n**2 * np.sin(i)**2)))**2 \

T_max = T_parallel.argmax()
R_min = R_parallel.argmin()

f,ax=plt.subplots(figsize=(8,5))
ax.plot(i, T_parallel, 'r', label=r'$T_{\parallel}$ Transmission')
ax.plot(i[T_max], T_parallel[T_max],'o')
ax.plot(i, R_parallel, 'b', label=r'$R_{\parallel}$ Reflection')
ax.axvline(x=i[T_max], ymin=0, ymax=1, linestyle='--', color='g')
ax.plot(i[R_min], R_parallel[R_min],'o')
plt.xlabel(r'Angle of Incidence $(Rad)$')
plt.ylabel(r'$R$, $T$ Coefficient')
ax.xaxis.set_major_formatter(tck.FormatStrFormatter('%g $\pi$'))
ax.xaxis.set_major_locator(tck.MultipleLocator(base=0.2))
ax.annotate(r'$T_{\parallel}$ = 1', xy=(0.98, 1),  xycoords='data',
            xytext=(0.5, 0.7), textcoords='axes fraction',
            arrowprops=dict(arrowstyle="->", facecolor='black'),
            horizontalalignment='right', verticalalignment='top',
            )
ax.annotate(r'$R_{\parallel} = 0$', xy=(0.98, 0),  xycoords='data',
            xytext=(0.5, 0.3), textcoords='axes fraction',
            arrowprops=dict(arrowstyle="->", facecolor='black'),
            horizontalalignment='right', verticalalignment='top',
            )
```

```
ax.legend(loc='center left')
plt.show()
```