PHY 411 **Winter 2021**

Week 7—Tuesday, Feb 16—Discussion Worksheet

Laplace equation in spherical coordinates (contd.)

The Laplace equation, $\nabla^2\Phi = 0$, in spherical coordinates is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{3.1}$$

In the previous class, you assumed a product form for the potential: $\Phi = \dfrac{U(r)}{r}P(\theta)Q(\phi)$ and separated variables to obtain:

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{P\,r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right)\right] + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0 \tag{3.2}$$

The last term on the left hand side is a function of ϕ only; setting it equal to $-m^2$, you should have obtained the solution

$$Q = e^{\pm im\phi} \tag{3.5}$$

In general, m need not be an integer, but we will choose m to be an integer so that we can run through the full azimuthal range and still get single-valued Q.

The next step is the separation of the r and θ terms, following which you get

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) + \left[l\,(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0 \tag{3.6}$$

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0 \tag{3.7}$$

Using a power solution method with $U = r^a$, you found the solution for the radial equation (3.7) to be

$$U = A\,r^{l+1} + B\,r^{-l} \tag{3.8}$$

where A and B are constants that will ultimately be determined from the boundary conditions, and again l is, as yet, undetermined, but will be determined from consideration of the θ-equation.

1. It is customary to write the θ-equation in terms of $x = \cos\theta$ instead of θ itself. Do this, and show that you get the **generalized Legendre equation**:

$x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right] + \left[l\,(l+1) - \frac{m^2}{1-x^2}\right]P = 0 \tag{3.9}$$

$$\frac{d}{-\sin\theta\, d\theta}\left(\sin^2\theta\,\frac{dP}{-\sin\theta\, d\theta}\right) + \left[l(l+1) - \frac{m^2}{1-\cos^2\theta}\right]P = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-\cos^2\theta)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0$$

Before we look into the solutions of equation (3.9), consider the restricted case $m = 0$ (azimuthal symmetry), for which equation (3.9) reduces to the **Legendre equation**:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right] + l\,(l+1)\,P = 0 \tag{3.10}$$

We'll skip the steps for finding a solution of equation (3.10), but knowing what follows is important enough that I've written in much detail about it in the posted class summary; ***please read it and let me know if you have trouble following any of the steps***. After going through these steps, we find that the P in equation (3.10) are a special class of solutions. Indexed by l, which turns out to be 0 or an integer, these $P_l(x)$ are called the **Legendre polynomials of order $\boldsymbol{l}$**. Let us write out the first few.

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right) \qquad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$
$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) \tag{3.11}$$

Remember that in all of these, $x = \cos\,\theta$. Notice also that each polynomial has the value 1 when its argument is 1; this is by design, and it is what Jackson means when he says that "by convention, these polynomials are normalized to have the value unity at $x = +1$" on page 97. Legendre polynomials have odd or even symmetry about the origin:

$$P_l(-x) = (-1)^l\,P_l(x)$$

A compact generating function for Legendre polynomials, known as the Rodrigues' formula, is

$$P_l(x) = \frac{1}{2^l\,l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l \tag{3.12}$$

The Legendre polynomials form a complete orthogonal set of functions on the interval $-1 \le x \le 1$, and the orthogonality condition is

$$\int_{-1}^{1} P_{l'}(x)\,P_l(x)\,dx = \frac{2}{2l+1}\,\delta_{l'l} \tag{3.21}$$

If you remember the orthogonal functions and expansions discussion from the previous chapter, then you'll know that since the Legendre polynomials form a complete set of orthogonal functions, any function $f(x)$ in the interval $-1 \le x \le 1$ can be expanded in terms of these polynomials:

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \tag{3.23}$$

where

$$A_l = \frac{2l+1}{2}\int_{-1}^{1} f(x)\,P_l(x)\,dx \tag{3.24}$$

With all of the above in place, we can now write the general solution in the case of azimuthal symmetry ($m = 0$) as

$$\Phi = \frac{U(r)}{r}\,P(\theta)\,Q(\phi) = \sum_{l=0}^{\infty}\left[\frac{A_l\,r^{l+1} + B_l\,r^{-l}}{r}\right] P_l(\cos\theta)\,e^{\pm i(0)\phi}$$

so that

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + \frac{B_l}{r^{l+1}}\right] P_l(\cos\,\theta) \tag{3.33}$$

Now let's apply this to find the potential in an electrostatic problem.

Boundary value problems with azimuthal symmetry

The general solution to the Laplace equation $\nabla^2\Phi = 0$ in spherical coordinates (r, θ, ϕ) for a problem possessing azimuthal symmetry ($m = 0$) is given by equation (3.33) on the previous page, with the coefficients A_l and B_l determined from boundary conditions. Note that azimuthal symmetry means that the problem includes the entire azimuthal range, with ϕ going from 0 to 2π. The boundary conditions *must not* depend on the azimuthal angle.

2. Consider a boundary value problem in which the potential is specified to be $V(\theta)$ on the surface of a sphere of radius a. Find the potential inside the sphere. *Leave the solution in closed integral form, since the form that V takes on the surface is not given.*

Since the problem has azimuthal symmetry, solution for potential inside the sphere is given by (3.33). To complete the problem, we must determine A_l and B_l. To do so, consider that the potential must be finite at the origin, so since the B_l's would all blow up at the origin, we must exclude them from our solution. Thus,

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l \cos\theta$$

At $r = a$, $\Phi = V(\theta)$, thus

$$\Phi(a,\theta) = \sum_{l=0}^{\infty} A_l a^l P_l \cos\theta = V(\theta)$$

This is a series in Legendre polynomials, so the coefficients A_l can be determined by invoking orthogonality of Legendre Polynomials. So, multiply both sides by $P_{l'} \cos\theta$ and apply orthogonality:

$$\sum_{l=0}^{\infty} A_l a^l \int_0^{\pi} P_l \cos\theta \, P_{l'} \cos\theta \sin\theta \, d\theta = \int_0^{\pi} V(\theta) P_{l'} \cos\theta \sin\theta \, d\theta$$

orthogonality ensures that only $l = l'$ survives, so remove summation to get

$$A_l a^l \left[\frac{2}{2l+1}\right] = \int_0^{\pi} V(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

Therefore, $$A_l = \frac{2l+1}{2a^l} \int_0^{\pi} V(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

Associated Legendre Functions and Spherical Harmonics

The ***generalized Legendre equation***, with $x = \cos\theta$, is

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P = 0 \tag{3.9}$$

Solutions to the generalized Legendre equation are known as **associated Legendre functions**.

Previously, we dealt with potential problems involving azimuthal symmetry, so we put $m = 0$ in the equation above (and obtained the ordinary Legendre equation), with solutions in terms of Legendre polynomials of order l, $P_l(x)$. The general potential problem, however, can have azimuthal variations, so that $m \neq 0$. Therefore, we need the generalization of $P_l(x)$, i.e., the solution of the generalized Legendre equation written above, with l and m both arbitrary.

The solution of the generalized Legendre equation involves the same consideration for θ as the ordinary Legendre equation we discussed previously: the whole range of cos θ is in play, including the north and south poles. Therefore, we demand that the desired solution be single valued, finite, and continuous on the interval $-1 \leq x \leq 1$, in order that it represent a physical solution. In order to have finite solutions on the interval $-1 \leq x \leq 1$, it can be shown that the parameter l must be zero or a positive integer and the integer m can only take the values $-l, -(l-1), \ldots, 0, \ldots, (l-1), l$. The solution having these properties is called an ***associated Legendre function*** $P_l^m(x)$. For positive m, it is defined by the formula:

$$P_l^m(x) = (-1)^m\,(1-x^2)^{m/2}\,\frac{d^m}{dx^m}\,P_l(x) \tag{3.49}$$

whereas $P_l^{-m}(x)$ can be obtained from $P_l^m(x)$ because they are proportional, as the generalized Legendre equation (3.9) depends only on m^2 and m is an integer:

$$P_l^{-m}(x) = (-1)^m\,\frac{(l-m)!}{(l+m)!}\,P_l^m(x) \tag{3.51}$$

The choice of the arbitrary phase factor $(-1)^m$ is by convention (see Jackson's footnote on page 108 for the original source).

If $P_l(x)$ is written explicitly using Rodrigues' formula, then the corresponding expression for $P_l^m(x)$ is valid for both positive and negative integers m:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}\,(1-x^2)^{m/2}\,\frac{d^{\,l+m}}{dx^{l+m}}\,(x^2-1)^l \tag{3.50}$$

For fixed m, the associated Legendre functions $P_l^m(x)$ form an orthogonal set in the interval $[-1, 1]$, that is, $-1 \leq x \leq 1$:

$$\int_{-1}^{1} P_{l'}^m(x)\,P_l^m(x)\,dx = \frac{2}{2l+1}\,\frac{(l-m)!}{(l+m)!}\,\delta_{l'\,l} \tag{3.52}$$

We now have the **full solution to the generalized Legendre equation (3.9)**:

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A'_{lm}\,r^l + \frac{B'_{lm}}{r^{l+1}}\right)\,P_l^m(\cos\,\theta)\,e^{im\phi}$$

3. In writing the *full solution to the generalized Legendre equation* on the previous page, why have I written only $e^{im\phi}$ in the solution, even though the ϕ-solution is actually $Q_m(\phi) = e^{\pm im\phi}$?

Moved to day 2

Now, consider the following:

- The functions $Q_m(\phi) = e^{im\phi}$ form a complete set of orthogonal functions in the index m on the interval $0 \le \phi \le 2\pi$.
- The functions $P_l^m(\cos\theta)$ form a complete orthogonal set in the index l for each m value in the interval $-1 \le \cos\theta \le 1$.

So, we can use the orthogonality relation equation (3.52), together with the factor 2π, to write normalized versions of $P_l^m(x)$ and $e^{im\phi}$ into the solution itself in the following manner:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}\,r^l + \frac{B_{lm}}{r^{l+1}}\right)\left[\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}\right]$$

This is very convenient, because we can now define the term in square brackets that is a combination of the angular factors (θ,ϕ) as a complete set of orthogonal functions $Y_{lm}(\theta,\phi)$ in the indices (l,m) that are normalized in the intervals $-1 \lesssim \cos\theta \le 1$ and $0 \le \phi \le 2\pi$ so that our solution to the generalized Legendre equation equation (3.9) now looks like:

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}\,r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\phi) \tag{3.61}$$

where the functions $Y_{lm}(\theta,\phi)$ are called the **spherical harmonics**, and are given by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi} \tag{3.53}$$

Explicit forms of $Y_{lm}(\theta,\phi)$ for $0 \le l \le 3$ are written on page 109 in Jackson. $Y_{lm}(\theta,\phi)$ form what are known as *orthonormal functions* (because they are normalized and orthogonal) over all angles (θ,ϕ) of the unit sphere. In other words

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta\,Y^*_{l'm'}(\theta,\phi)\,Y_{lm}(\theta,\phi) = \delta_{l'l}\,\delta_{m'm} \tag{3.54}$$

4. With $Y_{lm}(\theta, \phi)$ defined in equation (3.53) on the previous page, show that

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y^*_{lm}(\theta, \phi) \tag{3.55}$$

Moved to day 2

Since $Y_{lm}(\theta, \phi)$ form a complete set of functions, an arbitrary function $f(\theta, \phi)$ can be expanded in spherical harmonics:

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm}\, Y_{lm}(\theta, \phi) \tag{3.56}$$

where the coefficients C_{lm} are given by

$$C_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\,\theta\, d\theta\, f(\theta, \phi)\, Y^*_{lm}(\theta, \phi)$$