Physics 440, Autumn 2019 **In-Class Exam 1**

Tuesday, October 8

The following equations might be useful:

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$ $\quad L = T - V$ $\quad \delta W = \sum_{\alpha=1}^{n} Q_\alpha\, \delta q_\alpha$ $\quad Q_\alpha = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_\alpha}$

$\sin^2\alpha + \cos^2\alpha = 1$ $\quad \sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$

1. Find the constant(s) of motion for the following five Lagrangians. If there are no constants of motion please explain why not.
   a. $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ (free particle)
   b. $L = \frac{1}{2}m\dot{z}^2 - mgz$ (particle in a gravitational field)
   c. $L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 - mg(l-x)\sin\alpha$
   (disk with radius R and moment of inertia I rolling down an inclined plane of length l and inclination angle α)
   d. $L = \frac{ml^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta$ (spherical pendulum)
   e. $L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}$ (planet orbiting a star)

a. All three coordinates are cyclic. The constants of motion are the linear momenta in $x, y\ and\ z$ direction: $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ $\quad \frac{\partial L}{\partial \dot{y}} = m\dot{y}$ $\quad \frac{\partial L}{\partial \dot{z}} = m\dot{z}$

b. z is the only coordinate, and it is not cyclic, so there is no constant of motion.

c. Even though it looks like θ is a cyclic it is not because of the constraint equation for rolling without slipping: $\dot{\theta} = \frac{\dot{x}}{R}$, so $L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{I}{R^2}\dot{x}^2 - mg(l-x)sin\,\alpha$ or $L = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 - mgl + mgR\theta$. There is only one generalized coordinate $(x\ or\ \theta)$ and it is not cyclic.

d. ϕ is cyclic, so $\frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2\theta$ is a constant of motion.

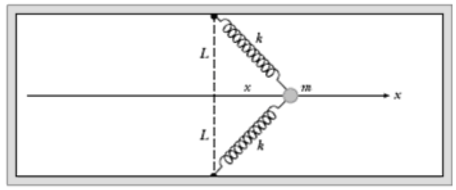
e. θ is cyclic, so $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ is a constant of motion

2. Find the Lagrangians in appropriate generalized coordinates for the following four systems

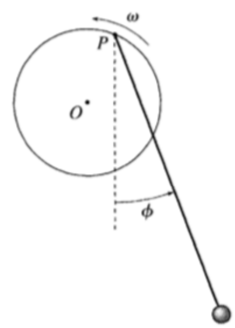
   a. A box sliding down a frictionless inclined ramp with inclination angle α.

$$L = \frac{1}{2}m(\dot{x}^2) + mgxsin\,\alpha$$

   b. A mass attached between two identical springs with spring constant k and rest length L on a horizontal frictionless tabletop. (The figure on the right shows the top view from above the table.)



$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k\left(\sqrt{(L-y)^2 + x^2} - L\right)^2$$
$$- \frac{1}{2}k\left(\sqrt{(L+y)^2 + x^2} - L\right)^2$$



   c. A pendulum of mass m and length l attached to the edge of a wheel with radius R that rotates at angular velocity ω.

In cartesian coordinates: $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

$x = R\,cos\,\omega t + l\sin\phi \quad y = R\sin\omega t - l\cos\phi$

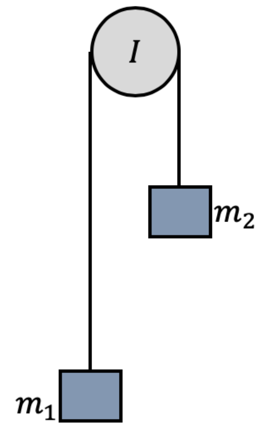
$\dot{x} = -\omega R\sin\omega t + \dot{\phi} l\cos\phi \quad \dot{y} = \omega R\cos\omega t + \dot{\phi} l\sin\phi$

$$L = \frac{1}{2}m\left((-\omega R\sin\omega t + \dot{\phi} l\cos\phi)^2 + (\omega R\cos\omega t + \dot{\phi} l\sin\phi)^2\right) - mg(R\sin\omega t - l\cos\phi)$$

$$= \frac{1}{2}m\left((-\omega R\sin\omega t + \dot{\phi} l\cos\phi)^2 + (\omega R\cos\omega t + \dot{\phi} l\sin\phi)^2\right) - mg(R\sin\omega t - l\cos\phi)$$
$$= \frac{1}{2}m(\omega^2 R^2 + \dot{\phi}^2 l^2 + 2\omega R\dot{\phi} l(-\sin\omega t\cos\phi + \cos\omega t\sin\phi) - mg(R\sin\omega t - l\cos\phi)$$

$$= \frac{1}{2}m(\omega^2 R^2 + \dot{\phi}^2 l^2 + 2\omega R\dot{\phi} l\sin(\phi - \omega t)) - mg(R\sin\omega t - l\cos\phi)$$

d. Atwood's machine with a pulley with mass M, radius R and moment of inertia $I = \frac{1}{2}MR^2$ and rope length l.



$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2 - (-m_1 g x_1 - m_2 g x_2)$$

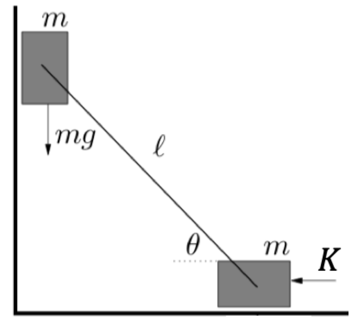
(x_1 and x_2 increase downward, $x_1 = x$, $x_2 = l - x$, $\dot{x}_2 = -\dot{x}_1$, $\dot{\theta} = \dot{x}/R$)

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{\dot{x}^2}{R^2} - (-m_1 g x - m_2 g(l - x))$$

$$= \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{x}^2 - (-m_1 g x - m_2 g l + m_2 g x)$$

$$= \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{x}^2 - (m_2 - m_1) g x$$

3. Two blocks on frictionless surfaces are constrained by a rod of length l to move together. Use the principle of virtual work to determine the force K needed to keep the system in static equilibrium.



When using the principle of virtual work we only consider applied forces, not the the forces of constraint (normal force of the wall and floor on the blocks, force of the rod on the blocks) because the constraint forces do not do any work.

The applied force on block 1 is gravity: $F_1^y = mg$

The applied force on block 2 is K: $F_2^x = K$

Principle of virtual work:

$\delta W = \sum_{\alpha=1}^{n} Q_\alpha \, \delta q_\alpha = 0$

This requires that the generalized force Q_θ is zero:

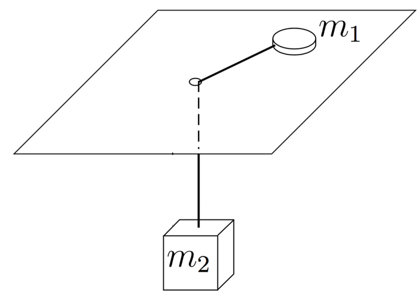
$$Q_\theta = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial \theta} = F_1^y \frac{\partial y_1}{\partial \theta} + F_2^x \frac{\partial x_2}{\partial \theta} = 0$$

$$y_1 = l \sin\theta \qquad x_2 = l\cos\theta$$

$$\frac{\partial y_1}{\partial \theta} = l\cos\theta \qquad \frac{\partial x_2}{\partial \theta} = -l\sin\theta$$

$$Q_\theta = mgl\cos\theta - Kl\sin\theta = 0 \Leftrightarrow K = mg\cot\theta$$

4. The figure on the right shows a disk of mass m_1 tethered to a hanging mass m_2 by a massless string of length l. The disk is free to move on a frictionless, horizontal table. The string in threaded through a small hole in the table. m_2 can move vertically but cannot not swing back and forth. The Lagrangian of this system is



$$L = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{r}^2 + m_2 g(l - r)$$

Derive the equation(s) of motion.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

Two generalized coordinates $r\ and\ \theta$; one equation of motion for each

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m_1\ddot{r} + m_2\ddot{r} = (m_1 + m_2)\ddot{r}$$

$$\frac{\partial L}{\partial r} = m_1 r\dot{\theta}^2 - m_2 g$$

$$\ddot{r} = (m_1 r\dot{\theta}^2 - m_2 g)/(m_1 + m_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}(m_1 r^2\dot{\theta}) = 2m_1 r\dot{r}\dot{\theta} + m_1 r^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$2m_1 r\dot{r}\dot{\theta} + m_1 r^2\ddot{\theta} = 0 \Rightarrow \ddot{\theta} = -2\frac{\dot{r}}{r}\dot{\theta}$$