(1) On Homework Report 1, you showed that the representation of S_x in the z-state basis for spin-3/2 is

$$S_x \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}.$$

(a) Find the representation of the eigenstates of S_x in the z-state basis. Call them $|+3/2_x\rangle$, $|+1/2_x\rangle$, $|-1/2_x\rangle$, and $|-3/2_x\rangle$ (that is, just use the m value to label the states).

Hint: you already know the eigenvalues!

(b) If the state of the system is $|+3/2_x\rangle$ and you measure the spin along the z axis, what values can you obtain, and with what probabilities?

(c) Find a state of the system for which $\langle S_z\rangle = 3\hbar/4$. If you measure the spin of a system in this state along the x axis, what are the possible results of the measurement, and the probability of each result?

(a) For $|+3/2_x\rangle$, we have

$$\frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = +\frac{3\hbar}{2}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$\Rightarrow$ (dropping $\hbar/2$ everywhere)

$\sqrt{3}\,b = 3a$, $\sqrt{3}\,a + 2c = 3b$, $2b + \sqrt{3}\,d = 3c$, $\sqrt{3}\,c = 3d$

pick $b = \sqrt{3} \Rightarrow a = 1$

$\Rightarrow \sqrt{3} + 2c = 3\sqrt{3} \Rightarrow 2c = 2\sqrt{3} \Rightarrow c = \sqrt{3}$

$\Rightarrow d = 1$

$\therefore$

$$|+3/2_x\rangle \leftrightarrow A\begin{bmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{bmatrix} \qquad A = \frac{1}{\sqrt{8}} \Rightarrow |+3/2_x\rangle \leftrightarrow \frac{1}{\sqrt{8}}\begin{bmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{bmatrix}$$

For $|+\tfrac{1}{2}x\rangle$,

$\sqrt{3}\,b = a$, $\sqrt{3}\,a + 2c = b$, $2b + \sqrt{3}\,d = c$, $\sqrt{3}\,c = d$

pick $b = \sqrt{3} \Rightarrow a = 3$

$\Rightarrow 3\sqrt{3} + 2c = \sqrt{3} \Rightarrow 2c = -2\sqrt{3} \Rightarrow c = -\sqrt{3}$

$\Rightarrow d = -3$

$\therefore$

$$|+\tfrac{1}{2}x\rangle \Longleftrightarrow A\begin{bmatrix} 3 \\ \sqrt{3} \\ -\sqrt{3} \\ -3 \end{bmatrix} \qquad A = \frac{1}{\sqrt{24}} \Rightarrow |+\tfrac{1}{2}x\rangle \Longleftrightarrow \frac{1}{\sqrt{8}}\begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{bmatrix}$$

For $|-\tfrac{1}{2}x\rangle$,

$\sqrt{3}\,b = -a$, $\sqrt{3}\,a + 2c = -b$, $2b + \sqrt{3}\,d = -c$, $\sqrt{3}\,c = -d$

pick $b = 1 \Rightarrow a = -\sqrt{3}$

$\Rightarrow -3 + 2c = -1 \Rightarrow 2c = 2 \Rightarrow c = 1$

$\Rightarrow \sqrt{3} = -d \Rightarrow d = -\sqrt{3}$

$\therefore$

$$|-\tfrac{1}{2}x\rangle \Longleftrightarrow \frac{1}{\sqrt{8}}\begin{bmatrix} -\sqrt{3} \\ 1 \\ 1 \\ -\sqrt{3} \end{bmatrix}$$

Lastly, for $|-\tfrac{3}{2}x\rangle$,

$\sqrt{3}\,b = -3a$, $\sqrt{3}\,a + 2c = -3b$, $2b + \sqrt{3}\,d = -3c$, $\sqrt{3}\,c = -3d$

pick $b = \sqrt{3} \Rightarrow a = -1$

$\Rightarrow -\sqrt{3} + 2c = -3\sqrt{3} \Rightarrow 2c = -2\sqrt{3} \Rightarrow c = -\sqrt{3}$

$\Rightarrow -3 = -3d \Rightarrow d = 1$

$$|-\tfrac{3}{2}x\rangle \Longleftrightarrow \frac{1}{\sqrt{8}}\begin{bmatrix} -1 \\ \sqrt{3} \\ -\sqrt{3} \\ 1 \end{bmatrix}$$

(b)

$$|+\tfrac{3}{2}x\rangle \longleftrightarrow \frac{1}{\sqrt{8}}\begin{bmatrix}1\\ \sqrt{3}\\ \sqrt{3}\\ 1\end{bmatrix} = \begin{bmatrix}\langle +3/2 | +3/2\,x\rangle \\ \langle +1/2 | +3/2\,x\rangle \\ \langle -1/2 | +3/2\,x\rangle \\ \langle -3/2 | +3/2\,x\rangle\end{bmatrix}$$

$$P(+3/2) = \frac{1}{8}, \quad P(+1/2) = \frac{3}{8}, \quad P(-1/2) = \frac{3}{8}, \quad P(-3/2) = \frac{1}{8}$$

(c)

Pick $|\psi\rangle \longleftrightarrow \begin{bmatrix}1/\sqrt{2}\\ 1/2\\ 1/2\\ 0\end{bmatrix}$

$$\langle S_z\rangle = \frac{\hbar}{2}\begin{bmatrix}1/\sqrt{2} & 1/2 & 1/2 & 0\end{bmatrix}\begin{bmatrix}3 & & & 0\\ & 1 & & \\ & & -1 & \\ 0 & & & -3\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ 1/2\\ 1/2\\ 0\end{bmatrix}$$

$$= \frac{\hbar}{2}\begin{bmatrix}1/\sqrt{2} & 1/2 & 1/2 & 0\end{bmatrix}\begin{bmatrix}3/\sqrt{2}\\ 1/2\\ -1/2\\ 0\end{bmatrix}$$

$$= \frac{\hbar}{2}\left[\frac{3}{2} + \frac{1}{4} - \frac{1}{4} + 0\right] = \frac{3\hbar}{4} \quad \checkmark$$

$$P(+3/2\,x) = \left|\langle +3/2\,x | \psi\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{8}}\begin{bmatrix}1 & \sqrt{3} & \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ 1/2\\ 1/2\\ 0\end{bmatrix}\right|^2$$

$$= \frac{1}{8}\left(\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)^2 = \frac{1}{8}\left(\sqrt{3} + 1/\sqrt{2}\right)^2$$

$$P(+1/2\,x) = \left|\langle +1/2\,x | \psi\rangle\right|^2, \quad P(-1/2\,x) = \left|\langle -1/2\,x | \psi\rangle\right|^2$$

etc