1 Vector Formulas

1.1 Triple Products

$$A\cdot(B\times C)=B\cdot(C\times A)=C\cdot(A\times B)$$
$$A\times(B\times C)=B(A\cdot C)-C(A\cdot B)$$

1.2 Product Rules

$$\nabla(fg)=f(\nabla g)+g(\nabla f)$$
$$\nabla((A\cdot B)=A\times(\nabla\times B)+B\times(\nabla\times A)+(A\cdot)B+(B\cdot\nabla)A$$
$$\nabla\cdot(fA)=f(\nabla\cdot A)+A\cdot(\nabla f)$$
$$\nabla\cdot(A\times B)=B\cdot(\nabla\times A)-A\cdot(\nabla\times B)$$
$$\nabla\times(fA)=f(\nabla\times A)-A\times(\nabla f)$$
$$\nabla\times(A\times B)=(B\cdot\nabla)-(A\cdot\nabla)B+A(\nabla\cdot B)-B(\nabla\cdot A)$$

1.3 Second Derivatives

$$\nabla\cdot(\nabla\times A)=0$$
$$\nabla\times(\nabla f)=0$$
$$\nabla\times(\nabla\times A)=\nabla(\nabla\cdot A)-\nabla^2 A$$

2 Maxwell Equations

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \nabla\cdot\vec{B}=0$$
$$\vec{\nabla}\cdot\vec{D}=0 \qquad \nabla\cdot\vec{D}=\rho$$
$$\vec{\nabla}\times\vec{E}-i\omega\vec{B}=0 \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\times\vec{B}+i\omega\mu\varepsilon\vec{E}=0 \qquad \vec{\nabla}\times H=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

2.1 Constructive Relations

$$\vec{D}=\varepsilon\vec{E} \qquad \vec{B}=\mu\vec{H}$$

3 Electromagnetic Waves and Propagation

3.1 Helmholtz wave equations

$$(\nabla^2+\mu\varepsilon\omega^2)\vec{E}=0 \qquad (\nabla^2+\mu\varepsilon\omega^2)\vec{B}=0$$

3.2 Constructive Relations

1. **Wave number:** $k=\omega\sqrt{\mu\varepsilon}$
2. **Phase velocity:** $v=\frac{\omega}{k}=\frac{1}{\sqrt{\mu\varepsilon}}=\frac{c}{n}$
3. **Index of refraction of the medium:** $n=\frac{\mu\varepsilon}{\mu_0\varepsilon_0}$

3.3 Plane Electromagnetic Waves

$$\vec{E}=\vec{E}_0e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \vec{B}=\sqrt{\mu\varepsilon}\frac{\vec{k}\times\vec{E}}{k}$$

3.4 Polarization of Waves

$$\vec{E}_1=\hat{\varepsilon}_1E_1e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \vec{E}_2=\hat{\varepsilon}_2E_2e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$\vec{E}(\vec{x},t)=(\hat{\varepsilon}_1E_1+\hat{\varepsilon}_2E_2)e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

3.5 Stokes Parameters

Linear polarization basis:

$$\vec{E}(\vec{x},t)=(\hat{\varepsilon}_1E_1+\hat{\varepsilon}_2E_2)e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$E_1=a_1e^{i\delta_1} \qquad E_2=a_2e^{i\delta_2}$$

Circular polarization basis:

$$\vec{E}(\vec{x},t)=(\hat{\varepsilon_+}E_++\hat{\varepsilon_-}E_-)e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$E_+=a_+e^{i\delta_+} \qquad E_-=a_-e^{i\delta_-}$$

3.6 Reflection and Refraction: Kinematic Properties

Incident wave:

$$\vec{E}=\vec{E}_0e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \vec{B}=\sqrt{\mu\varepsilon}\frac{\vec{k}\times\vec{E}}{k}$$

Refracted wave:

$$\vec{E'}=\vec{E}'_0e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \vec{B'}=\sqrt{\mu'\varepsilon'}\frac{\vec{k}'\times\vec{E}'}{k}$$

Reflected wave:

$$\vec{E''}=\vec{E}''_0e^{i(\vec{k}''\cdot\vec{x}-\omega t)} \qquad \vec{B''}=\sqrt{\mu\varepsilon}\frac{\vec{k}''\times\vec{E}''}{k}$$

3.7 Reflection and Refraction: Boundary condition

Normal components:

$$[\varepsilon(\vec{E}_0+\vec{E}''_0)-\varepsilon'\vec{E}'_0]\cdot\hat{n}=0$$
$$[\vec{k}\times E_0+\vec{k''}\times\vec{\vec{E}''_0}-\vec{k'}\times\vec{E}'_0]\cdot\hat{n}=0$$

Tangential components:

$$[\vec{E}_0+\vec{E}''_0-\vec{E}'_0]\times\hat{n}=0$$
$$\left[\frac{1}{\mu}(\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}''_0)-\frac{1}{\mu'}(\vec{k}'\times\vec{E}'_0)\right]\times\hat{n}=0$$

3.8 Brewster's Angle

3.9 Snell's Law

3.10 Total Internal Reflection

3.11 Reflection and Transmission Coefficients

$$\vec{s}\cdot\hat{n}=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|E_0|^2cos(i) \qquad T=\frac{\vec{s}'\cdot\hat{n}}{\vec{s}\cdot\hat{n}}$$
$$\vec{s}'\cdot\hat{n}=\frac{1}{2}\sqrt{\frac{\varepsilon'}{\mu'}}|E'_0|^2\cos(r) \qquad R=\frac{\vec{s}''\cdot\hat{n}}{\vec{s}\cdot\hat{n}}$$
$$\vec{s}''\cdot\hat{n}=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|E''_0|^2\cos(r)' \qquad T+R=1$$

3.12 Dispersion Model for time-varying field

$$m[\ddot{\vec{x}}+\gamma\dot{\vec{x}}+\omega_0^2\vec{x}]=-e\vec{E}(\vec{x},t)$$

3.13 Dispersion

$$\frac{\varepsilon(\omega)}{\varepsilon_0}=1+\frac{Ne^2}{\varepsilon_0 m}\sum_j\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}$$

3.14 Attenuation of a plane wave

3.15 Propagation through Discursive Media

Fourier series:

$$u(x,t)=\frac{1}{sqrt2\pi}\int_{-\infty}^{\infty}A(k)e^{ikx-i\omega(k)t}dk$$
$$A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,0)e^{-ikx}dx$$

Group velocity:

$$v_g=\left.\frac{d\omega}{dk}\right|_{k_0}$$

3.16 Propagation through Discursive Media

$$\vec{B}=\vec{\nabla}\times\vec{A} \qquad \vec{E}=\vec{\nabla}\Phi-\frac{\partial\vec{A}}{\partial t}$$
$$\nabla^2\Phi+\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})=-\frac{\rho}{\varepsilon_0}$$
$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}-\vec{\nabla}\left[\vec{\nabla}\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right]=-\mu_0\vec{J}$$

3.17 Propagation through Discursive Media

Gauge transformation:

$$\vec{A}\to\vec{A}+\vec{\nabla}\Lambda \qquad \Phi\to\Phi-\frac{\partial\Lambda}{\partial t}$$

Restricted gauge transformation:

$$\nabla^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=0$$

3.18 Laplace's Equation in rectangular coordinates

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

Separation of variables:

$$\Phi(\vec{x})=X(x)Y(y)Z(z)$$
$$\frac{1}{X}\frac{\partial^2X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2Z}{\partial z^2}$$

3.19 Laplace's Equation in spherical coordinates

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

4 Energy conservation in the electromagnetic field

4.1 Rate of decrease of energy

$$\int_v \vec{J}\cdot\vec{E}d^3x = \int_v \left[\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right]d^3x$$

4.2 Total energy density

$$u = \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H})$$

4.3 Differential continuity equation

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot(\vec{S}) = -\vec{J}\cdot\vec{E}$$

4.4 Poynting Vector

$$\vec{S} = \vec{E}\times\vec{H}$$

5 Conservation of linear momentum

5.1 Force on a single charge q

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{b} \qquad \frac{d\vec{P}_{mech}}{dt} = \int_v(\rho\vec{E}+\vec{J}\times\vec{B})d^3x$$

5.2 Maxwell Stress Tensoer

$$T_{\alpha\beta} = \varepsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E}+c^2\vec{\beta}\cdot\vec{B})\delta_{\alpha\beta}\right]$$

6 Radiation

Harmonic time dependence:

$$\rho(\vec{x},t) = \rho(\vec{x})e^{-i\omega t} \qquad \vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$$

Wave equation for $\vec{A}$:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\vec{J}(\vec{x}')\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}d^3x'$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} \qquad \vec{E} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H}$$

6.1 Zones

1. 1. **The near (static) zone:** $d << r << \lambda$
2. **The intermediate zone** $d << r \approx \lambda$
3. **The far (radiation) zone:** $d >> \lambda << r$

6.2 Electric Dipole Radiation and Fields

$$\vec{A}(\vec{x}) = -\frac{i\omega\mu_0}{4\pi}\vec{p}\frac{e^{ikr}}{r} \quad \text{where} \quad \vec{p} = \int x'\rho(\vec{x'})d^3x'$$

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\frac{e^{ikr}}{r} + [3\hat{n}(\hat{n}\cdot\vec{p}-\vec{p}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}$$

6.3 Near Zone

$$\vec{H} = \frac{i\omega}{4\pi}(\hat{n}\times\vec{p})\frac{1}{r^2} \qquad \vec{E} = \frac{1}{4\pi\varepsilon_0}[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}]\frac{1}{r^3}$$

6.4 Far Zone $kr >> 1$

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r} \qquad \vec{E} = \frac{k^2}{4\pi\varepsilon_0}[(\hat{n}\times\vec{p})\times\hat{n}]\frac{e^{ikr}}{r}$$

6.5 Electric Dipole Radiation

Time-averaged power radiated per unit solid angle

$$\frac{dP}{d\Omega} = \frac{1}{2}Re[r^2\hat{n}\cdot\vec{E}\times\vec{H}^*]$$

7 Relativistic Electrodynamics

7.1 Lorentz Transformations

$x_0 = ct \qquad x_1 = z \qquad x_2 = x \qquad x_3 = y$

$x_0' = \gamma(x_0-\beta x_1) \qquad x_1' = \gamma(x_1-\beta x_0)$

$x_2' = x_2 \qquad x_3' = x_3 \qquad$ where

$$\beta = \frac{v}{c} \qquad \vec{\beta} = \frac{\vec{v}}{c} \qquad \gamma = \frac{1}{\sqrt{(1-\beta^2)}}$$

7.2 Relativistic Energy and Momentum

$$\vec{p} = \gamma m\vec{u} = \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\vec{E} = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$E = \sqrt{p^2c^2+m^2c^4}$$

7.3 4-vectors

4-vectors: $A^\alpha = (A^0,\vec{A}) \to (A^0,A^1,A^2,A^3)$
Relation: $dt = \gamma_u d\tau$
4-velocity: $U = (U_0,\vec{U}) = (\gamma_u c, \gamma_u\vec{u})$
4-momentum: $P = (\gamma_u mc, \gamma_u m\vec{u}) = \left(\frac{E}{c},\vec{p}\right)$

where $\gamma_u = \left(1-\frac{u^2}{c^2}\right)^{-1/2}$

7.4 Tensors

Space-time continuum: $(ct,z,x,y) = (x^0,x^1,x^2,x^3)$
Transformation: $x'^\alpha = x'^\alpha(x^0,x^1,x^2,x^3$
Lorentz transformations for general contravariant vector:

$$A'^0 = \gamma(A^0-\beta A^1) \qquad A'^2 = A^2$$

$$A'^1 = \gamma(A^1-\beta A^0) \qquad A'^3 = A^3$$

Contravariant vectors: $A'^\alpha \frac{\partial x'^\alpha}{\partial x^\beta}A^\beta$
Covariant vectors: $B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha}B_\beta$
Scalar Product: $B\cdot A = B_\alpha A^\alpha$

7.5 Metric Tensor

$(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta$
Raising indices $x_\alpha = g_{\alpha\beta}x^\beta$ **Lowering indices** $x^\alpha = g^{\alpha\beta}x_\beta$

$$g = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} \qquad x = \begin{pmatrix}x^0\\x^1\\x^2\\x^3\end{pmatrix} \qquad gx = \begin{pmatrix}x^0\\-x^1\\-x^2\\-x^3\end{pmatrix}$$

7.6 4-derivative

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0},\vec{\nabla}\right) \qquad \partial^\alpha = \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x_0},\vec{\nabla}\right)$$

7.7 Covariance of Electrodynamics

Continuity equation: $J^\alpha = (c\rho,\vec{J}$
Covariant Continuity equation: $\partial_\alpha J^\alpha = 0$

7.8 The Field Strength Tensor

Covariant form of the two inhomogenous Maxwell equation:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}J^\beta$$

Vector potential: $\mathrm{F}^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

7.9 The Dual Field Strength Tensor

Covariant form of the two homogenous Maxwell equation

$$\vec{\nabla}\cdot\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{B} = 0$$

$$\partial_\alpha F^{\alpha\beta} = 0$$

Vector potential: $\mathrm{F}^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

7.10 Covariant Equations

Wave Equations: $A^\alpha = \frac{4\pi}{c}J^\alpha$
Lorenz condition: $\partial_\alpha A^\alpha = 0$
Macroscopic Equations: $\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c}J^\beta, \quad \partial_\alpha F^{\alpha\beta}$

7.11 Covariant Equation for Force Equation

$d\tau = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta$