Final: Nov 18, 2:30 –4:45. Note the time!

Mid-term Stuff

1. There will be ***no*** *MatLab* coding
2. Material from this part of course is covered in *the text.*
3. The exam will be ***closed book*** and ***no notes*** will be allowed
   i. I will provide a *sheet with formulae*
4. You will be asked to solve numerical problems by *hand.*
   i. Please use a calculator, not your phone.
5. There may be interpretation or conceptual problems
   i. For example, why does a particular numerical method *fail and why does it fail*
6. You may be asked to apply a particular numerical technique *graphically.*
7. There may be some derivations, but they will not be too involved.
8. You may be asked to explain the advantage/disadvantage of one numerical technique over another.
9. Make sure you can apply numerical methods *by hand.*

Since Mid Term

1. *PDEs*

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + D\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$$

The functions *A,B,C*, can be used to characterize some types of PDEs

- If $B^2(x_o,t_o) - 4A(x_o,t_o)C(x_o,t_o) > 0$ for all ***x,t*** the equation is called *hyperbolic*
- If $B^2(x_o,t_o) - 4A(x_o,t_o)C(x_o,t_o) = 0$ for all ***x,t*** the equation is called *parabolic*
- If $B^2(x_o,t_o) - 4A(x_o,t_o)C(x_o,t_o) < 0$ for all ***x,t*** the equation is called *elliptic*.

Notation: $\dfrac{\partial u}{\partial x} = u_x; \quad \dfrac{\partial^2 u}{\partial x^2} = u_{xx}; \quad \dfrac{\partial^2 u}{\partial x \partial t} = u_{xt}.$

2. *Finite Differences*

$$u_t \equiv u_i^j \approx \frac{u_i^{j+1} - u_i^j}{\Delta t}$$

$$u_{xx} \approx \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{h_x^2}$$

Example: Set up a finite difference scheme for the 1-D heat equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}; \qquad u(0,t) = 0 \qquad u(L,t) = 100$$
$$u_t(x,0) = g(x)$$

You have to apply both *boundary and initial conditions* correctly.

Make sure you fully understand the notation so you can implement algorithm

3. *Implicit Schemes*. These solve for *all* the points simultaneously. Eventually you must solve *tri-diagonal* system of equations.

Stuff since mid-term

4. *Fourier Series.*

$$f(x) = a_o(1) + \sum_n a_n \cos(nx) + \sum_m b_m \sin(mx)$$

$$\int_{-\pi}^{\pi} \cos(nt)\cos(mt) = \begin{cases} \pi\delta_{m,n}, & m \neq n \\ 2\pi, & m = n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nt)\sin(mt) = 0$$

$$\int_{-\pi}^{\pi} \sin(nt)\sin(mt) = \begin{cases} 1 & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt$$

Function must be periodic on *$-\pi$ to π.* How would you change expressions for a function that is periodic on *$-T$* to *T*

5. Fourier Transform.

When dealing with *non-periodic*, or functions over infinite extant, we move to *Fourier Transform*

Inverse Fourier Transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega \quad (1)$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad (2)$$

Fourier Transform

6. *Discrete Fourier Transform*
   1. In physical applications, the underlying function is only *sampled*
      i. No instrument has *infinite resolution*
      ii. No experiment has *infinite time to run*

Discrete Fourier Transform

$$g(n\Delta\omega) = \sum_{0}^{N-1} f(m\Delta t)e^{-in\Delta\omega m\Delta t} = \sum_{0}^{N-1} f(m\Delta t)e^{-i2\pi mn/N}$$

Inverse Discrete Fourier Transform

$$f(m\Delta t) = \frac{1}{N}\sum_{n=0}^{N-1} g(n\Delta\omega)\, e^{i2\pi mn/N}$$

Make sure you fully understand the notation so you can apply it.

Example. Find the discrete Fourier transform for the data vector *A = (1 2 3 4)* Assume the data is evenly sampled once every second.

Answer: *g(0) = 10; g(1) = -2 + 2i; g(2) = -2; g(3) = -2 -2i*

7. *Fast Fourier Transform.*

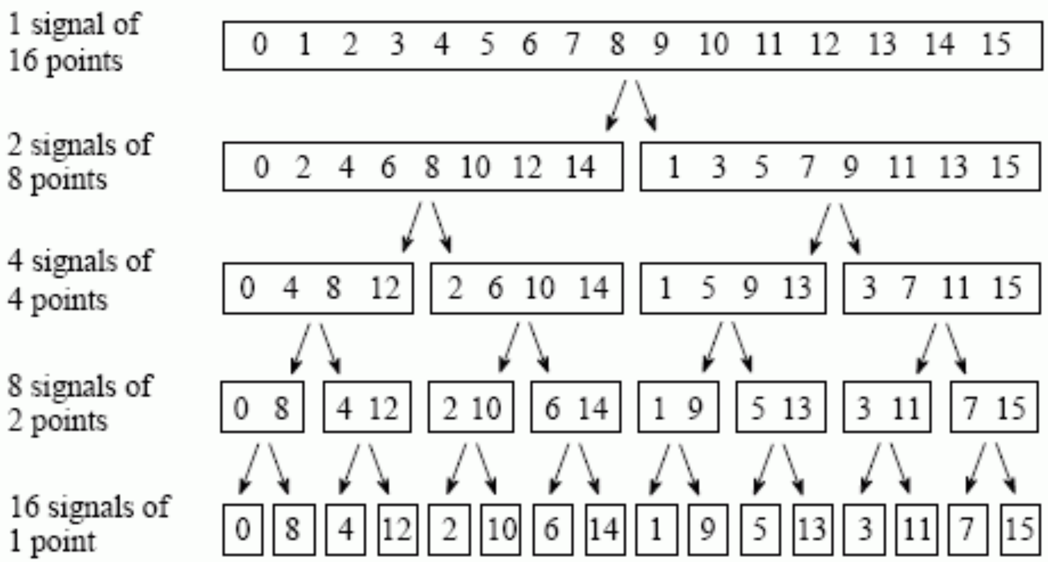

FIGURE 12-2
The FFT decomposition. An N point signal is decomposed into N signals each containing a single point. Each stage uses an *interlace decomposition*, separating the even and odd numbered samples.

$$g(n\Delta\omega) = \sum_{0}^{N-1} f(m\Delta t) e^{-in\Delta\omega m\Delta t} = \sum_{0}^{N-1} f(m\Delta t) e^{-i2\pi mn/N}$$

$$g(n\Delta\omega) = g_{even}(n\Delta\omega) + e^{-i2\pi n/N} g_{odd}(n\Delta\omega).$$

Understand how/why only first *N/2* terms correspond to *positive frequencies*, the latter half of the terms correspond to *negative frequencies*.

Detailed example, N = 4

$$g(n\Delta\omega) = \underbrace{\sum_{j=0}^{N/2-1} f(2j\Delta t)e^{-i2\pi jn/(N/2)}}_{E} + \underbrace{W_N^n \sum_{j=0}^{N/2-1} f((2j+1)\Delta t)e^{-i2\pi jn/(N/2)}}_{O}$$

$$E = EE + E0 \qquad O = OE + OO.$$

$$\begin{aligned} EE &= \sum_{j=0}^{N/(2\cdot 2)-1} f(2(2j\Delta t))e^{-i2\pi(2j)n/(N/2)} \\ &= \sum_{j=0}^{N/4-1} f(4j\Delta t)e^{-i2\pi jn/(N/4)} \\ EO &= \sum_{j=0}^{N/(2\cdot 2)-1} f(2(2j+1)\Delta t)e^{\frac{-i2\pi(2j+1)n}{(N/2)}} \\ &= \sum_{j=0}^{N/4-1} f((4j+2)\Delta t)e^{\frac{-i2\pi jn}{(N/4)}} e^{\frac{-i2\pi n}{(N/2)}} \\ &= W_{\frac{N}{2}}^n \sum_{j=0}^{N/4-1} f((4j+2)\Delta t)e^{\frac{-i2\pi jn}{(N/4)}} \end{aligned}$$

$$\begin{aligned} OE &= W_N^n \sum_{j=0}^{N/(2\cdot 2)-1} f((2(2j)+1)\Delta t)e^{\frac{-i2\pi(2j)n}{(N/2)}} \\ &= W_N^n \sum_{j=0}^{N/4-1} f((4j+1)\Delta t)e^{\frac{-i2\pi jn}{(N/4)}} \\ OO &= W_N^n \sum_{j=0}^{N/(2\cdot 2)-1} f((2(2j+2)+1)\Delta t)e^{\frac{-i2\pi(2j+1)n}{(N/2)}} \\ &= W_N^n \sum_{j=0}^{N/4-1} f((4j+3)\Delta t)e^{\frac{-i2\pi jn}{(N/4)}} e^{\frac{-i2\pi n}{(N/2)}} \\ &= W_N^n W_{\frac{N}{2}}^n \sum_{j=0}^{N/4-1} f((4j+3)\Delta t)e^{\frac{-i2\pi jn}{(N/4)}} \end{aligned}$$

$$g(n\Delta\omega) = f(0\Delta t) + W_2^n f(2\Delta t) + W_4^n f(1\Delta t) + W_4^n W_2^n f(3\Delta t). \quad W_N^n = e^{-i2\pi n/N}$$

Find the Fast Fourier transform for *f(0) = 0; f(1) = 1; f(2) = 4; f(3) = 9*

Answer: *14; -4.0 + 8.0 i; -6.0; -4.0 – 8.0 i*

8. *Convolution and Correlation*

Convolution:

$$\mathcal{F}(p) = \frac{\mathcal{F}(p \otimes q)}{\mathcal{F}(q)}$$

or that

$$p(t) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(p \otimes q)}{\mathcal{F}(q)}\right)$$

Correlation

$$p \odot p = s \odot s + n \odot n = s \odot s + |\langle n(t) \rangle|^2.$$

9. *Spectrum Analysis*

Energy of each frequency = *Fourier coefficient squared*

Example. Find the spectrum of data *f(0) = 0; f(1) = 1; f(2) = 4; f(3) = 9*

Write the second order ODE as 2 coupled first order and use Runge-Kutta to solve.

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 6 = 0$$

$$\begin{aligned}
f_o &= f(t_o, y_o) \\
f_1 &= f(t_o + \tfrac{h}{2}, y_o + \tfrac{h}{2} f_o) \\
f_2 &= f(t_o + \tfrac{h}{2}, y_o + \tfrac{h}{2} f_1) \\
f_3 &= f(t_o + h, y_o + f_2) \\
y_{i+1} &= y_i + \tfrac{h}{6}\left(f_o + 2f_1 + 2f_2 + f_3\right)
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dt} &= v \\
\frac{dv}{dt} &= 3v + 6
\end{aligned}$$

$$S = \begin{pmatrix} y \\ v \end{pmatrix}$$

$$F = \begin{pmatrix} v \\ 3v + 6 \end{pmatrix}$$

$$\frac{dS}{dt} = F$$