PHY 474 Spring 2021
Homework 1 solutions

1. In class, we discussed some relevant timescales. Yet another such quantity is the timescale for photons created in the core of the Sun to reach the surface.

Note that we will work with a single photon in this problem, but in reality the same photon doesn't actually make its way to the surface; instead, photons are absorbed and re-emitted.

(a) Assuming the Sun is completely ionized, the Thomson scattering cross section is relevant in this problem, since photons will scatter off the free electrons, and it is given by

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0\, m_e c^2}\right)^2$$

in SI units, where e is the electron charge, and m_e is the electron mass.

Compute σ_T. *Show steps clearly if you want full credit.*

Solution: With $e = 1.6 \times 10^{-19}$ C, and $m_e = 9.11 \times 10^{-31}$ kg for the electron, we get

$$\sigma_T = \frac{8\pi}{3}\left[\frac{(1.6 \times 10^{-19}\ \mathrm{C})^2}{4\pi(8.85 \times 10^{-12}\ \mathrm{m^{-3}\,kg^{-1}\,s^4\,A^2})\,(9.11 \times 10^{-31}\ \mathrm{kg})(3 \times 10^8\ \mathrm{m/s})^2}\right]^2$$

Therefore

$$\sigma_T = \boxed{6.6 \times 10^{-29}\ \mathrm{m^2}}$$

(b) The photon scattering timescale is then given by $t_s = \dfrac{l}{c}$, where $l = \dfrac{1}{n_e\sigma_T}$ is the mean free path.

Calculate t_s.

Note: You'll need to find the electron density n_e (in $\mathrm{cm^{-3}}$), which you can do by using the average (mass) density of the Sun, $\rho = 1.4 \times 10^3\ \mathrm{kg\ m^{-3}}$.

Solution: Since the Sun is mostly hydrogen, the (average) electron density, n_e, can be found from the average (mass) density ρ of the Sun divided by the mass of a proton:

$$n_e = \frac{\rho}{m_p} = \frac{1.4 \times 10^3\ \mathrm{kg\,m^{-3}}}{1.67 \times 10^{-27}\ \mathrm{kg}} = 8.4 \times 10^{29}\ \mathrm{m^{-3}}$$

and so

$$t_s = \frac{1}{n_e\sigma_T c} = \frac{1}{(8.4 \times 10^{29}\ \mathrm{m^{-3}})\,(6.6 \times 10^{-29}\ \mathrm{m^2})\,(3 \times 10^8\ \mathrm{m/s})} = 6 \times 10^{-11}\ \mathrm{s}$$

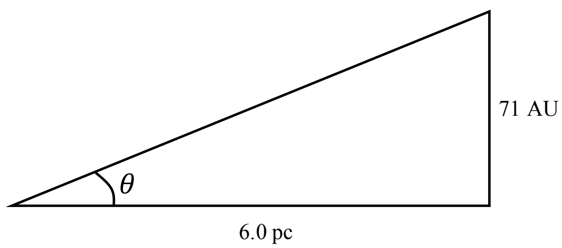
Therefore, the photon scattering timescale is 6×10^{-11} s, or of the order of $\boxed{10^{-10}\ \mathrm{s}}$.

This result is important to know, since it gives the timescale for matter and radiation to come into equilibrium; note that the timescales for proton-proton and proton-electron interactions are also short. Therefore, the timescale calculated here is for the interior to relax to thermodynamic equilibrium (more correctly, LTE). Of course, it should be kept in mind that the Sun is not globally in equilibrium; the surface radiates energy freely to space, and is therefore much colder than the core.

2. In class, we discussed the η Cas binary star system, located at a distance of 6.0 pc from us. The mean separation of the two stars, η Cas A and η Cas B, is 71 AU.

(a) When separated by 71 AU, what would be the angular separation of the two stars, η Cas A and η Cas B, as seen from Earth? Express your answer in arcseconds ($''$) if you want full credit.

Solution: This should be a straightforward matter of applying trigonometry in a right-angled triangle, as shown in the figure below.



We do require both distances in the same units. I prefer to put both in AU, since pc will give a larger number in AU. Since 1 pc = 206265 AU, we get that the angular separation between η Cas A and η Cas B is

$$\tan\theta = \frac{71 \text{ AU}}{(6.0 \text{ pc})(206265 \text{ AU/pc})} = \frac{71 \text{ AU}}{1237590 \text{ AU}}$$

so that

$$\theta = \tan^{-1}\left[\frac{71 \text{ AU}}{1237590 \text{ AU}}\right] = 0.003287^\circ$$

Since we want the answer in arcseconds ($''$), we'll need to multiply by (60×60), and upon doing so we get

$$\theta = 0.003287^\circ\,(3600''/^\circ) = 11.8''$$

Therefore, the angular separation between η Cas A and η Cas B would be $\boxed{11.8''}$ when they are separated by 71 AU.

(b) The system has very high eccentricity, and when it was discovered during the time of William Herschel, the angular separation of the two stars was only about $6.2''$. How far apart in AU were η Cas A and η Cas B at that time?

Solution: This is just like part (a), except now we know the angular separation and need to figure it out in arcseconds. So

$$\tan\left(\frac{6.2''}{3600''/^\circ}\right) = \frac{d}{1237590 \text{ AU}}$$

and thus

$$d = \left[1237590 \text{ AU}\right] \tan\left(\frac{6.2''}{3600''/^\circ}\right) = 37.2 \text{ AU}$$

Therefore, η Cas A and η Cas B were apart by $\boxed{37 \text{ AU}}$ when they were $6.2''$ apart during the time of Herschel.

3. The binary star system Albireo, or β Cyg, is sometimes called the "eye of the swan" for its location in the constellation of Cygnus at a distance of about 390 Ly from us. When observed with a V-band filter, the magnitude of the brighter component β Cyg A is 3.18, and the fainter component β Cyg B is about 5.9 times fainter than β Cyg A.

(a) Compute the *absolute* magnitude of the brighter component β Cyg A.

Solution: The apparent magnitude m and the absolute magnitude M are related by

$$m - M = 5 \log_{10} d - 5$$

where d is in pc.

Since we are given the distance to β Cyg A in Ly, we will have to convert it to pc, which is easy to do since 1 pc = 3.26 Ly. Thus

$$\frac{390 \text{ Ly}}{3.26 \text{ Ly/pc}} = 119.6 \text{ pc}$$

Then, we have

$$M = m - 5 \log_{10} d + 5$$

and so

$$M = 3.18 - 5 \log(119.6 \text{ pc}) + 5 = -2.21$$

Therefore, the absolute magnitude of the brighter component β Cyg A is $\boxed{-2.21}$.

Although a negative number might seem weird at first sight, the history of the magnitude scale allows us to understand what is going on here. Historically, the brightest stars were 0 magnitude, and the faintest were 6th magnitude (the limit of the human eye). Once telescopes started scanning the heavens and objects brighter than naked eye zeroth magnitude were discovered, the magnitude scale had to be extended. So, negative magnitude simply means brighter than whatever star has been assigned zeroth magnitude.

(b) Compute the *apparent* magnitude of the fainter component β Cyg B.

Solution: The ratio of brightnesses is linked to the magnitude scale via the relation

$$m_A - m_B = -2.5 \log_{10} \left(\frac{B_A}{B_B} \right)$$

Since β Cyg B is about 5.9 times fainter than β Cyg A, meaning that β Cyg A is about 5.9 times brighter than β Cyg B, we can write

$$3.18 - m_B = -2.5 \log_{10} \left(\frac{5.9B}{B} \right)$$

and thus

$$m_2 = 3.18 + 2.5 \log_{10} \left(5.9 \right) = 5.11$$

Therefore, the apparent magnitude of the fainter component β Cyg B is $\boxed{5.11}$.

4. In class, we learned about the Lorentzian distribution

$$\phi(\nu) = \frac{\gamma_n/4\pi^2}{(\nu-\nu_0)^2 + (\gamma_n/4\pi)^2}$$

Show that the Lorentzian distribution is normalized to unity, so that

$$\int_0^\infty \phi(\nu)\, d\nu = 1$$

Solution: We need to integrate, so

$$\int_0^\infty \phi(\nu)\, d\nu = \int_0^\infty \frac{\gamma_n/4\pi^2}{(\nu-\nu_0)^2 + (\gamma_n/4\pi)^2}\, d\nu$$

Let $x = \nu - \nu_0$, so that $dx = d\nu$. For convenience, let's set $\gamma_n/4\pi = b$, so that

$$\int_0^\infty \frac{\gamma_n/4\pi^2}{(\nu-\nu_0)^2 + (\gamma_n/4\pi)^2}\, d\nu = \int_{[0]}^{[\infty]} \frac{b/\pi}{x^2+b^2}\, dx = \frac{b}{\pi}\int_{[0]}^{[\infty]} \frac{dx}{x^2+b^2}$$

Note that I've put the limits in square brackets because I prefer to change back to the original variables before substituting them, rather than change the limits to those for the new variable x.

From integral tables, the standard integral

$$\int \frac{dx}{x^2+b^2} = \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)$$

and so we get

$$\int_0^\infty \phi(\nu)\, d\nu = \frac{b}{\pi}\left[\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right]_{[0]}^{[\infty]}$$

and thus

$$\int_0^\infty \phi(\nu)\, d\nu = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{\nu-\nu_0}{b}\right)\right]_0^\infty$$

so that

$$\int_0^\infty \phi(\nu)\, d\nu = \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1}\left(-\frac{\nu_0}{b}\right)\right]$$

or

$$\int_0^\infty \phi(\nu)\, d\nu = \frac{1}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(-\frac{\nu_0}{b}\right)\right]$$

In order to find the value of the second term, let's consider the typical values of ν_0 and b. The frequency for visible light will be of the order 10^{14} Hz, so this is also the order of magnitude for ν_0. Meanwhile, γ_n and hence b is much smaller (recall that γ_n is the sum of the Einstein A-coefficients to all the levels below n, and typically, A-coefficients are of the order 10^8 or smaller). Thus, $-\nu_0/b$ will go to $-\infty$, and we have

$$\int_0^\infty \phi(\nu)\, d\nu = \frac{1}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(-\infty\right)\right] = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

which shows that the Lorentzian distribution is normalized to unity.