PHY 474/374 Spring 2021

Week 4—Wednesday, April 21—Discussion Worksheet

Mass Luminosity Relations

Today, we will learn about the luminosity of a star based on the equation of radiative transport, combined with estimates of the temperature and density of the star.

1. Let's begin with the equation of radiative transport that we encountered in Chapter 5:

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi acr^2T^3} \tag{1}$$

where κ is the opacity, and a is the radiation density constant that features in the expression for the radiation energy density, $u_R = aT^4$.

Starting from the equation (1) above, show that the luminosity L is given by

$$L = -\frac{4\pi r^2 ac}{3\kappa\rho}\frac{d}{dr}\left(T^4\right)$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi acr^2T^3}$$

$$\int -\frac{16\pi acr^2T^3}{3\kappa\rho\, dr}\, dT = L(r)$$

$$\rightarrow -\frac{16\pi acr^2T^4}{12\kappa\rho\, dr} = L(r)$$

$$\rightarrow -\frac{4\pi acr^2T^4}{3\kappa\rho}\frac{d}{dr} = L(r)$$

2. The expression for the luminosity L given on the previous page involves the **opacity**, κ. Let's look at this quantity in more detail.

Opacity and ***mean free path*** convey the same information, but opacity is the preferred quantity in the literature. Recall that the mean free path is given by

$$\lambda = \frac{1}{n\sigma}$$

where n is the number of particles (e.g., atoms or molecules) per unit volume, and σ is the cross section that describes the microscopic interaction between radiation and matter; roughly speaking, a photon will interact with an atom if it passes within an area σ of the atom.

(a) Recall that $n = \rho/(\mu m_p)$. Thus, since n is proportional to the mass density ρ, we would like to replace n with ρ in the equation for λ above. Show that the opacity κ can then be defined as the ***cross section per unit mass***, and thus

$$\kappa = \frac{\sigma n}{\rho}$$

$\lambda = \frac{1}{n\sigma}$, Since $n = \frac{\rho}{\mu m_p}$, Replace n by ρ

Thus, $\lambda = \frac{1}{n\sigma} = \frac{1}{\rho\kappa} \Rightarrow \kappa = \frac{\sigma n}{\rho}$

$$[\kappa] = \frac{cm^2\, cm^{-3}}{g\, cm^{-3}} = cm^2/g$$

(b) One of the contributions to opacity comes from photons scattering off free electrons. Such scattering, known as Thomson scattering, occurs when the free electrons have thermal motions that are much smaller than their rest mass energy, that is,

$$kT \ll m_e c^2$$

Calculate $m_e c^2/k$ to verify that temperatures in stars satisfy this condition.

$$T \ll \frac{m_e c^2}{k} = \frac{(9.11 \times 10^{-31} kg)(3 \times 10^8 m/s)^2}{1.38 \times 10^{-23} J/K}$$

$$T \ll 5.9 \times 10^9 K$$

3. As we noted on the previous page, scattering of photons off electrons is one contributor to the opacity. We will now demonstrate that if the gas is completely ionized (a reasonable assumption in a star), the opacity due to electron scattering is independent of density and temperature. For free electrons, the opacity is given by

$$\kappa_e = \frac{\sigma_e n_e}{\rho}$$

(a) Calculate σ_e, the Thomson cross-section of the electron, given **in cgs units** by

$$\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2$$

Note: The electron charge in cgs units is $e = 4.8 \times 10^{-10}$ esu, whereas $m_e = 9.11 \times 10^{-28}$ g.

$$\sigma_e = \frac{8\pi}{3}\left(\frac{(4.8\times10^{-10}\,\text{esu})^2}{9.11\times10^{-28}\,\text{g}\,(3\times10^{10}\,\text{cm/s})^2}\right)$$

$$= 6.65\times10^{-29}\ m^2$$

(b) The electron density, $n_e = \rho/\mu_e m_p$, where the mean molecular weight per electron can be obtained by modifying the expression for μ from a previous class to $\mu_e^{-1} = \sum_j X_j Z_j/A_j$. By substituting appropriate numbers to find μ_e, and then n_e, show that the opacity due to electron scattering is

$$\kappa_e = 0.2\,(1 + X)\ \text{cm}^2\,\text{g}^{-1}$$

which is independent of the density and temperature.

$$\frac{1}{\mu_e} = \sum_j \frac{X_j Z_j}{A_j} = \underbrace{X\left(\frac{1}{1}\right)}_{H:\ Z=1,\ A=1} + \underbrace{Y\left(\frac{2}{4}\right)}_{He:\ Z=2,\ A=4} + \underbrace{Z\left(\frac{1}{2}\right)}_{Z/A = 1/2}$$

So $$1/\mu_e = X + Y/2 + Z/2$$

$$= X + \frac{1-X}{2} = \frac{2X+1-X}{2} = \frac{1+X}{2}$$

$$\frac{\sigma_e n_e}{\rho} = \left(\frac{\sigma_e}{\cancel{\rho}}\right)\frac{\cancel{\rho}}{\mu_e m_p} = \left[\frac{\sigma_e}{m_p}\right]\frac{1}{\mu_e} = \left[\frac{\sigma_e}{m_p}\right]\left(\frac{1+X}{2}\right)$$

$$\kappa_e = \left[\frac{6.6\times10^{-25}\,\text{cm}^2}{1.67\times10^{-24}\,\text{g}}\right]\left(\frac{1+X}{2}\right) = 0.198(1+X) = 0.2(1+X)\ \text{cm}^2\text{g}^{-1}$$

4. In general, though, the opacity depends on the density and temperature (due to contributions from bound-free and free-free absorption), and can be approximated by a power law

$$\kappa \simeq \kappa_0\, \rho^{\lambda} T^{-\nu}$$

The luminosity that you derived in Question 1, together with the expression for κ above, then gives for the surface luminosity (as you will demonstrate on the homework) that

$$L_s \simeq \frac{ac}{\kappa_0} \left(\frac{G\mu m_p}{k_B}\right)^{4+\nu} R^{3\lambda-\nu}\, M^{3+\nu-\lambda}$$

For relatively low mass stars (including our Sun), the opacity is dominated by atomic processes, in particular bound-free transitions, and can be approximated by the so-called Kramer's law

$$\kappa = \kappa_0\, \rho\, T^{-3.5}$$

where, for bound-free transitions, the constant in the expression is given by

$$\kappa_0 \simeq 4 \times 10^{25}\, Z\,(1+X) \qquad \text{in cgs units}$$

Calculate the surface luminosity of our Sun. You will find that your estimate is higher than the actual solar luminosity of $L_\odot = 3.846 \times 10^{33}$ erg/s, not surprising in view of the approximations made, but remarkable in that the result is quite close to the actual value.

Useful Information: The radiation density constant $a = 7.565 \times 10^{-15}$ erg cm^{-3} K^{-4}. Use the hydrogen fraction $X = 0.7$, and metal fraction $Z = 0.02$ to find both κ_0 and $\mu = 4/(3+5X-Z)$. In cgs units, the mass of a proton is $m_p = 1.67 \times 10^{-24}$ g, $G = 6.67 \times 10^{-8}$ dyne cm^2 g^{-2}, and Boltzmann's constant is $k_B = 1.38 \times 10^{-16}$ erg K^{-1}. Also, $R_\odot = 6.96 \times 10^{10}$ cm, and $M_\odot = 1.99 \times 10^{33}$ g.

$$L_s = \frac{ac}{4\times10^{25} Z(1+X)} \left[\frac{G\mu m_p}{k_B}\right]^{4+3.5} R^{3(1)-3.5}\, M^{3+3.5+1}$$

$$\mu = \frac{4}{3+5(0.7)-0.02}$$

$$\mu = 0.62$$

$$L_s = 1.5\times10^{35}\ \text{erg s}^{-1}$$

5. For high mass stars, the temperature is higher and so the opacity is dominated by electron scattering. The surface luminosity of high mass stars is given by

$$L_s \simeq 3 \times 10^{35} \left(\frac{M}{M_\odot}\right)^3 \left(\frac{\mu}{0.62}\right)^4 \frac{1.7}{1+X} \text{ erg s}^{-1}$$

(a) Calculate the surface luminosity of a 20 $M_\odot$ star in erg s^{-1}. Assume that the star has the same chemical composition and hydrogen mass fraction as the Sun.

$$L_s = 3 \times 10^{35} \left[\frac{20 M_\odot}{M_\odot}\right]^3 \left[\frac{0.62}{0.62}\right]^4 \frac{1.7}{1+0.7} \text{ erg s}^{-1}$$

$$= 3 \times 10^{35} (20)^3$$

$$= 2.4 \times 10^{39} \text{ erg s}^{-1}$$

(b) Express your answer in terms of the luminosity of our Sun, where $L_\odot = 3.846 \times 10^{33}$ erg/s

$$L_s = \frac{2.4 \times 10^{39} \text{ erg/s}}{3.846 \times 10^{33} \text{ erg s}^{-1} L_\odot^{-1}}$$

$$= 6 \times 10^5 L_\odot$$