PHY 420 Spring 2021

Homework 6 solutions

1. The inverse Lorentz transformation equations for a frame K' traveling at velocity v along the positive x-direction of a frame K are given by

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \qquad x = \gamma\left(x' + vt'\right) \qquad y = y' \qquad z = z'$$

By explicit differentiation, derive the Lorentz transformation law for velocities:

$$u_x = \frac{u'_x + v}{1 + \dfrac{vu'_x}{c^2}} \qquad \text{and} \qquad u_y = \frac{u'_y}{\gamma\left(1 + \dfrac{vu'_x}{c^2}\right)}$$

Solution: By direct differentiation, we get

$$u_x = \frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt} = \gamma\frac{d}{dt'}\left[x' + vt'\right]\left(\frac{dt}{dt'}\right)^{-1} \tag{1}$$

Let's find dt/dt' first:

$$\frac{dt}{dt'} = \frac{d}{dt'}\left[\gamma\left(t' + \frac{vx'}{c^2}\right)\right] = \gamma\left[1 + \frac{v}{c^2}\frac{dx'}{dt'}\right]$$

But $dx'/dt' = u'_x$, so the equation above becomes

$$\frac{dt}{dt'} = \gamma\left[1 + \frac{vu'_x}{c^2}\right] \tag{2}$$

Substituting this in equation (1), we get

$$u_x = \gamma\frac{d}{dt'}\left[x' + vt'\right]\left(\gamma\left[1 + \frac{vu'_x}{c^2}\right]\right)^{-1} = \frac{\gamma\left[\dfrac{dx'}{dt'} + v\dfrac{dt'}{dt'}\right]}{\gamma\left[1 + \dfrac{vu'_x}{c^2}\right]}$$

Canceling γ from the numerator and denominator, and putting $dx'/dt' = u'_x$, we obtain finally the first of the two desired relations

$$u_x = \frac{u'_x + v}{1 + \dfrac{vu'_x}{c^2}}$$

In similar manner, we get that

$$u_y = \frac{dy}{dt} = \frac{dy}{dt'}\frac{dt'}{dt} = \frac{dy'}{dt'}\frac{dt'}{dt} = \frac{dy'}{dt'}\left(\frac{dt}{dt'}\right)^{-1} \tag{3}$$

where the key point of difference from u_x is that, now, $y = y'$. Putting $dy'/dt' = u'_y$ and writing dt/dt' from equation (2), we get from equation (3) that

$$u_y = \frac{u'_y}{\gamma\left(1 + \dfrac{vu'_x}{c^2}\right)}$$

which is the second of the two desired relations.

2. Now consider the more general case of the frame K' moving with velocity $\vec{v} = c\vec{\beta}$ with respect to the frame K. Then, the components of velocity transform according to

$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + \dfrac{\vec{v}\cdot\vec{u}'}{c^2}} \qquad\qquad \vec{u}_{\perp} = \frac{\vec{u}'_{\perp}}{\gamma_v\left(1 + \dfrac{\vec{v}\cdot\vec{u}'}{c^2}\right)} \tag{4}$$

where $u_{\parallel}$ and $\vec{u}_{\perp}$ refer to components of velocity parallel and perpendicular, respectively, to $\vec{v}$, and the subscript on γ_v explicitly identifies the relationship to be $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) For the simple case where u' is parallel to the direction of v, use equation (4) to find an expression for u.

Solution: If u' is parallel to v, then $\vec{v}\cdot\vec{u}' = (v)(u')\cos 0 = vu'$, and $u_{\perp} = 0$. Substituting into the equation on the left above, we get

$$u = \frac{u' + v}{1 + vu'/c^2} \tag{5}$$

where I've written $u_{\parallel} = u$, and $u'_{\parallel} = u'$.

(b) If $u' = c$, then use the expression you derived in part (a) to find u. Comment on how this result reflects a key aspect of Special Relativity.

Solution: If $u' = c$, then substituting in equation (5), we get

$$u = \frac{c + v}{1 + vc/c^2} = \frac{c + v}{1 + v/c}$$

This can be written as

$$u = \frac{c + v}{(c + v)/c}$$

and upon canceling $(c + v)$ in the numerator and denominator, we end up with

$$u = c$$

This upholds the second postulate of Special Relativity that the speed of light (in vacuum) is the same in all inertial frames.

(c) For speeds u' and v both small compared to c, show that the velocity addition law reduces to the Galilean result: $u = u' + v$.

Solution: Starting from the velocity addition law in equation (5) above, if u' and v are both small compared to c, then we get that

$$\frac{vu'}{c^2} \approx 0$$

and thus equation (5) becomes

$$u = \frac{u' + v}{1 + vu'/c^2} \approx \frac{u' + v}{1 + 0}$$

and thus we get the Galilean result

$$u = u' + v$$

3. In class, we derived the 4-velocity

$$U = \Big(\gamma_u c, \gamma_u \vec{u}\Big)$$

where $\gamma_u = (1 - u^2/c^2)^{-1/2}$, and $\vec{u} = d\vec{x}/dt$ is the usual 3-dimensional velocity.

(a) Find the norm or invariant length U^2 of this 4-velocity. Reduce to the simplest possible form.

Solution: U^2 is just the scalar product, similar to the way it is defined for a 3-vector, except we have to be careful with the signs. So

$$U^2 = U \cdot U = \gamma_u^2 c^2 - \gamma_u^2 \Big(u_x^2 + u_y^2 + u_z^2\Big) = \gamma_u^2 \Big(c^2 - u^2\Big)$$

Putting in the form for γ_u given above, we get

$$U^2 = \frac{c^2 - u^2}{1 - u^2/c^2} = \frac{c^2 - u^2}{(c^2 - u^2)/c^2} = c^2$$

Therefore, the norm or invariant length of the 4-velocity is just $\boxed{c^2}$.

(b) Starting from U, write down an expression for the 4-acceleration A.

Solution: The 4-acceleration can be found from the derivative of the 4-velocity with respect to proper time, so that

$$A = \frac{dU}{d\tau} = \frac{dU}{dt}\frac{dt}{d\tau} \tag{6}$$

From equation (2), we know that if $dt' = d\tau$, the proper time, then $dt/d\tau = \gamma_u$, where the expression for $\gamma_u = (1 - u^2/c^2)^{-1/2}$ is written in the statement of the problem above.

Then, equation (6) gives

$$A = \frac{dU}{dt}\gamma_u = \gamma_u \frac{d}{dt}\Big[\gamma_u c, \gamma_u \vec{u}\Big]$$

Therefore, the 4-acceleration A is given by

$$A = \gamma_u \left[c\frac{d\gamma_u}{dt}, \left(\vec{u}\,\frac{d\gamma_u}{dt} + \gamma_u \vec{a}\right)\right] \tag{7}$$

(c) Find the scalar product $U \cdot A$ of the 4-velocity and the 4-acceleration.

Solution: This is straightforward, if we use the result in part (a), where we showed that $U^2 = c^2$:

$$U \cdot A = U \cdot \frac{dU}{d\tau} = \frac{1}{2}\frac{d}{d\tau}\Big[U \cdot U\Big]$$

But in part (a), we showed that $U \cdot U = c^2$, so that

$$U \cdot A = \frac{1}{2}\frac{d}{d\tau}\Big[U \cdot U\Big] = \frac{1}{2}\frac{d}{d\tau}\Big[c^2\Big] = 0$$

Thus, we have proved that $U \cdot A = 0$.

4. The Lorentz transformation equations are given by

$$x'_0 = \gamma\,(x_0 - \beta x_1). \qquad\qquad x'_2 = x_2$$

$$x'_1 = \gamma\,(x_1 - \beta x_0) \qquad\qquad x'_3 = x_3$$

where

$$\beta = \frac{v}{c} \qquad \text{and} \qquad \gamma = (1-\beta^2)^{-1/2}$$

(a) If we introduce the parametrization $\beta = \tanh\zeta$, then **show that** the relations above imply that

$$\gamma = \cosh\,\zeta \qquad \text{and} \qquad \gamma\beta = \sinh\zeta$$

where ζ is known as the *boost parameter* or *rapidity*.

Solution: Putting $\beta = \tanh\zeta$ in the expression for γ, we get

$$\gamma = (1-\beta^2)^{-1/2} = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\tanh^2\zeta}} = \frac{1}{\sqrt{\operatorname{sech}^2\zeta}} = \frac{1}{\operatorname{sech}\zeta} = \cosh\,\zeta$$

which **proves the first of the two relations**.

Meanwhile

$$\gamma\beta = \cosh\zeta\,\tanh\zeta = \cosh\zeta\left(\frac{\sinh\zeta}{\cosh\zeta}\right) = \sinh\zeta$$

which **proves the second of the two relations**.

(b) Using the parametrization in part (a), show that the Lorentz transformation equations written above can be put in the form

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{8}$$

Solution: The first of the Lorentz transformation equations, with γ now parametrized as $\cosh\zeta$, and $\gamma\beta = \sinh\zeta$, becomes

$$x'_0 = \gamma\,(x_0 - \beta x_1) = \gamma\,x_0 - \gamma\beta x_1 = (\cosh\zeta)\,x_0 - (\sinh\zeta)\,x_1$$

and to put it in the form of the matrix equation written above, we can write

$$x'_0 = (\cosh\zeta)\,x_0 - (\sinh\zeta)\,x_1 + (0)\,x_2 + (0)\,x_3$$

Likewise, the second the Lorentz transformation equations written above becomes

$$x'_1 = \gamma\,(x_1 - \beta x_0) = \gamma\,x_1 - \gamma\beta x_0 = (\cosh\zeta)\,x_1 - (\sinh\zeta)\,x_0$$

or swapping the terms and adding the last two with coefficients zero, we get

$$x'_1 = -(\sinh\zeta)\,x_0 + (\cosh\zeta)\,x_1 + (0)\,x_2 + (0)\,x_3$$

It is trivial to show x'_3 and x'_4, and thus we have demonstrated that the Lorentz transformation equations can be put in the form written in equation (8) above.