PHY 420 **Spring 2021**

Class Summary—Week 5, Day 2—Thursday, Apr 29

So far, we have expanded the vector potential $\vec{A}(\vec{x})$ in powers of $1/r$, and learned about the first term in the series (the electric dipole term). Today, we will discuss the next term in the expansion, the magnetic dipole and electric quadrupole. That is it for this approach; it turns out to be too difficult to go beyond the quadrupole term in this method (in fact, we're only going to be looking at the quadrupole term in the far field). A better way is to work with the so-called spherical vector multipoles.

Magnetic Dipole and Electric Quadrupole Radiation

In equation (9.3), we wrote the **vector potential** as

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}\,') \frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|}\, d^3x' \tag{9.3}$$

We will begin today by expanding the $|\vec{x}-\vec{x}\,'|$ terms in the integrand in powers of $1/r$. We did this in the previous class, but I'm going to do it more carefully this time.

With $r = |\vec{x}|, r' = |\vec{x}\,'|$, let us write

$$|\vec{x}-\vec{x}\,'| = \left[r^2 - 2\vec{x}\cdot\vec{x}\,' + (r')^2\right]^{1/2} = r\,\left[1 - 2\,\frac{\vec{x}\cdot\vec{x}\,'}{r^2} + \left(\frac{r'}{r}\right)^2\right]^{1/2}$$

Given our interest in the region $r \gg r'$, let us expand this so that we get

$$|\vec{x}-\vec{x}\,'| \approx r\,\left[1 - \frac{1}{2}\left\{2\,\frac{\vec{x}\cdot\vec{x}\,'}{r^2} + \left(\frac{r'}{r}\right)^2\right\} + \ldots\right] = r\,\left[1 - \frac{\vec{x}\cdot\vec{x}\,'}{r^2} - \frac{1}{2}\left(\frac{r'}{r}\right)^2 + \ldots\right]$$

Applying similar considerations, we obtain that

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = \left[r^2 - 2\vec{x}\cdot\vec{x}\,' + (r')^2\right]^{-1/2} \approx \frac{1}{r}\,\left[1 + \frac{\vec{x}\cdot\vec{x}\,'}{r^2} + \frac{1}{2}\left(\frac{r'}{r}\right)^2 + \ldots\right]$$

Dropping terms of order $(r'/r)^2$ or higher, we get from these two expansions that

$$\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} \approx \frac{1}{r}\,\left[1 + \frac{\vec{x}\cdot\vec{x}\,'}{r^2}\right]\, e^{ik\,r\left[1-\frac{\vec{x}\cdot\vec{x}\,'}{r^2}\right]} = \frac{1}{r}\,\left[1 + \frac{\vec{x}\cdot\vec{x}\,'}{r^2}\right]\, e^{ik\,r}\, e^{-ik\left(\frac{\vec{x}\cdot\vec{x}\,'}{r}\right)}$$

Now expand the second exponential term using $e^x = 1 + x + x^2/2! + \ldots$ to get

$$\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} \approx \frac{1}{r}\,\left[1 + \frac{\vec{x}\cdot\vec{x}\,'}{r^2}\right]\, e^{ik\,r}\,\left[1 - ik\frac{\vec{x}\cdot\vec{x}\,'}{r} + \ldots\right]$$

Rearranging terms in the expression at the bottom of the previous page, we get

$$\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} = \frac{e^{ik\,r}}{r}\left[1+\frac{\vec{x}\cdot\vec{x}\,'}{r^2}+\ldots\right]\left[1-ik\frac{\vec{x}\cdot\vec{x}\,'}{r}+\ldots\right]$$

I've replaced $\approx$ back to an equals sign because we've written the terms again in a series (i.e., we could restore the higher terms, in principle). Multiplying the terms in square brackets, we get

$$\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} = \frac{e^{ik\,r}}{r}\left[1+\frac{\vec{x}\cdot\vec{x}\,'}{r^2}-ik\frac{\vec{x}\cdot\vec{x}\,'}{r}+\ldots\right]$$

so that finally

$$\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} = \frac{e^{ikr}}{r}\left[1+\left\{\frac{1}{r}-ik\right\}\frac{\vec{x}\cdot\vec{x}\,'}{r}+\ldots\right]$$

When we kept only the first term in the expansion, we obtained the electric dipole term.

The **next term** in the expansion above, substituted in equation (9.3), gives

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}\,')\,\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|}\,d^3x' = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}\,')\left\{\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\frac{\vec{x}\cdot\vec{x}\,'}{r}\right\}d^3x'$$

so that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int \vec{J}(\vec{x}\,')\left(\frac{\vec{x}\cdot\vec{x}\,'}{r}\right)d^3x'$$

and since $\vec{x}=r\hat{n}$, this becomes

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int \vec{J}(\vec{x}\,')\,(\hat{n}\cdot\vec{x}\,')\,d^3x' \tag{9.30}$$

We can write the integrand in equation (9.30) as the sum of a part symmetric in $\vec{J}$ and $\vec{x}\,'$ and an antisymmetric part:

$$(\hat{n}\cdot\vec{x}\,')\,\vec{J} = \frac{1}{2}\left[(\hat{n}\cdot\vec{x}\,')\,\vec{J}+(\hat{n}\cdot\vec{J})\vec{x}\,'\right]+\frac{1}{2}\,(\vec{x}\,'\times\vec{J})\times\hat{n} \tag{9.31}$$

The first term in square brackets, the symmetric part, is related to the **electric quadrupole**. The second term is related to the **magnetic dipole**, as we can tell from the expression for the magnetic moment density:

$$\vec{m} = \frac{1}{2}\int \vec{x}\,'\times\vec{J}(\vec{x}\,')\,d^3x' \tag{5.54}$$

So, the **vector potential for the magnetic dipole term** is

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int \frac{1}{2}\left[\vec{x}\,'\times\vec{J}(\vec{x}\,')\right]\times\hat{n}\,d^3x'$$

Inserting $\vec{m}$ from equation (5.54), this becomes

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)(\vec{m}\times\hat{n})$$

We can now write $\vec{A}(\vec{x})$ in the form found in Jackson (by drawing out ik and reversing the order of $\vec{m}$ and $\hat{n}$:

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\left(\hat{n}\times\vec{m}\right)\frac{e^{ikr}}{r}\left[1-\frac{1}{ikr}\right] \tag{9.33}$$

as you did in Question 1 on today's worksheet.

To determine the fields, we can proceed in one of two ways: either calculate them directly using

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} \qquad \text{and} \qquad \vec{E} = \frac{ic\mu_0}{k}\vec{\nabla}\times\vec{H}$$

or, we can realize that $\vec{A}(\vec{x})$ from equation (9.33) looks very similar to $\vec{H}$ for an electric dipole written in equation (9.18)!

Let's put them one above the other so we can see this. The vector potential for the magnetic dipole is

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\left(\hat{n}\times\vec{m}\right)\frac{e^{ikr}}{r}\left[1-\frac{1}{ikr}\right]$$

and the magnetic field for the electric dipole is

$$\vec{H}_{\text{elec dipole}} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$

and I've highlighted the terms that are not the same; everything else is identical between $\vec{A}$ for the magnetic dipole and $\vec{H}$ for the electric dipole. Remember, **this is just pattern matching; they are not similar physically**; one is a vector potential, the other is a field.

So, with this pattern matching, $\vec{H}_{\text{elec dipole}}$ becomes $\vec{A}$ for a magnetic dipole if we replace

$$ck^2\,\vec{p} \qquad \to \qquad ik\mu_0\,\vec{m}$$

or, equivalently, if we replace $\vec{p}$ by $\vec{m}$ in the equation for $\vec{H}_{\text{elec dipole}}$, then

$$\vec{H}_{\text{elec dipole}} \to \vec{A}\left(\frac{ck}{i\mu_0}\right)$$

So, how do we find $\vec{H}$ for a magnetic dipole using this pattern matching? The answer comes from the field $\vec{E}$ for the electric dipole that we also wrote in equation (9.18). Recall that to find $\vec{E}_{\text{elec dipole}}$ from $\vec{H}_{\text{elec dipole}}$, you would have done

$$\vec{E}_{\text{elec dipole}} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H}_{\text{elec dipole}}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$. Replacing $\vec{H}_{\text{elec dipole}}$ as discussed above, we get

$$\vec{E}_{\text{elec dipole}} = \frac{iZ_0}{k}\left(\vec{\nabla}\times\vec{A}\right)\left(\frac{ck}{i\mu_0}\right)$$

so that

$$\vec{E}_{\text{elec dipole}} = \frac{cZ_0}{\mu_0}\left(\vec{\nabla}\times\vec{A}\right)$$

where $\vec{A}$ is for a magnetic dipole; since that's what we are discussing I won't subscript it with the words magnetic dipole.

Now, to find $\vec{H}$ for a magnetic dipole, we would do

$$\vec{H} = \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{A}\right)$$

Dividing the expression at the bottom of the previous page by cZ_0 on both sides, we have

$$\frac{1}{c\,Z_0}\,\vec{E}_{\text{elec dipole}} = \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{A}\right)$$

and comparing the quantities on the left hand side, it is clear what we have to do! To find $\vec{H}$ for a magnetic dipole, just take $\vec{E}_{\text{elec dipole}}$, and divide it by cZ_0 (also replace $\vec{p}$ by $\vec{m}$).

So, begin with $\vec{E}$ for an electric dipole from equation (9.18):

$$\vec{E}_{\text{elec dipole}} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}$$

multiply it by $1/cZ_0$ and replace $\vec{p}\to\vec{m}$ to get $\vec{H}$ for a magnetic dipole:

$$\vec{H} = \frac{1}{cZ_0}\,\frac{1}{4\pi\epsilon_0}\left\{k^2\,(\hat{n}\times\vec{m})\times\hat{n}\,\frac{e^{ikr}}{r} + \left[3\hat{n}\,(\hat{n}\cdot\vec{m})-\vec{m}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}$$

Since $Z_0 = \sqrt{\mu_0/\epsilon_0}$ and $c = 1/\sqrt{\mu_0\epsilon_0}$, we get

$$\vec{H} = \sqrt{\mu_0\epsilon_0}\sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{1}{4\pi\epsilon_0}\left\{k^2\,(\hat{n}\times\vec{m})\times\hat{n}\,\frac{e^{ikr}}{r} + \left[3\hat{n}\,(\hat{n}\cdot\vec{m})-\vec{m}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}$$

All factors outside the curly brackets cancel, except for 4π in the denominator, and we get

$$\vec{H} = \frac{1}{4\pi}\left\{k^2(\hat{n}\times\vec{m})\times\hat{n}\frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n}\cdot\vec{m})-\vec{m}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \tag{9.35}$$

Likewise, since $\vec{E} = -\vec{\nabla}\Phi - \partial\vec{A}/\partial t = -\partial\vec{A}/\partial t$ for a magnetic dipole (for which Φ is absent), we get that $\vec{E} = -\partial[\vec{A}(\vec{x})\,e^{-i\omega t}]/\partial t = i\omega\,\vec{A}(\vec{x})$, and since $\vec{H}_{\text{elec dipole}} \to \vec{A}\,(ck/i\mu_0)$ for a magnetic dipole, we have that for a magnetic dipole, the electric field is given by

$$\vec{E} = i\omega\vec{A} = i\omega\left[\left(\frac{i\mu_0}{ck}\right)\vec{H}_{\text{elec dipole}}\right] = -\mu_0\,\vec{H}_{\text{elec dipole}}$$

since $\omega = ck$. So, we get, after remembering to replace $\vec{p}$ by $\vec{m}$ that

$$\vec{E} = -\mu_0\left[\frac{ck^2}{4\pi}\left(\hat{n}\times\vec{m}\right)\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\right]$$

and since $\mu_0\,c = \mu_0/\sqrt{\mu_0\epsilon_0} = \sqrt{\mu_0/\epsilon_0} = Z_0$, we get finally that $\vec{E}$ for a magnetic dipole is

$$\vec{E} = -\frac{Z_0}{4\pi}\,k^2(\hat{n}\times\vec{m})\,\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) \tag{9.36}$$

- Equations (9.35) and (9.36) show that for magnetic dipoles, all arguments concerning the behavior of fields in the near and far zone are the same as for the electric dipole, with replacements $\vec{E}_{\text{elec dipole}} \to cZ_0\vec{H}$, $\vec{H}_{\text{elec dipole}} \to -c\vec{E}/Z_0$, and $\vec{p}\to\vec{m}$.
- The radiation pattern and total power radiated are the same for the two kinds of dipoles.
- For electric dipoles, recall that $\vec{E}$ lies in the plane defined by $\hat{n}$ and $\vec{p}$, whereas for magnetic dipoles, $\vec{E}$ is perpendicular to the plane defined by $\hat{n}$ and $\vec{m}$.

Electric Quadrupole Radiation

After "integration by parts and some rearrangement," hefty math that we'll avoid at this stage, Jackson finds that the term in square brackets in equation (9.31) gives

$$\frac{1}{2}\int\left[(\hat{n}\cdot\vec{x}\,')\,\vec{J}+(\hat{n}\cdot\vec{J})\vec{x}\,'\right]d^3x' = -\frac{i\omega}{2}\int\vec{x}\,'\,(\hat{n}\cdot\vec{x}\,')\,\rho(\vec{x}\,')\,d^3x' \tag{9.37}$$

The **vector potential for the electric quadrupole term** is then

$$\vec{A}(\vec{x}) = -\frac{\mu_0 ck^2}{8\pi}\,\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\int\vec{x}\,'(\hat{n}\cdot\vec{x}\,')\,\rho(\vec{x}\,')\,d^3x' \tag{9.38}$$

It is difficult to write the general solution — even for Jackson!!!

So, we will consider only the fields in the far-field radiation zone, where we keep only the part varying as e^{ikr}/r, and use $\vec{\nabla}\to\hat{n}\frac{\partial}{\partial r}$, so that

$$\vec{H} = \frac{1}{\mu_0}\,\vec{\nabla}\times\vec{A} = \frac{1}{\mu_0}\,ik\,(\hat{n}\times\vec{A}) = -\frac{ick^3}{8\pi}\,\frac{e^{ikr}}{r}\left[\hat{n}\times\int\vec{x}\,'\,(\hat{n}\cdot\vec{x}\,')\rho(\vec{x}\,')\,d^3x'\right] \tag{9.40}$$

This can be written in terms of the electric quadrupole moment tensor. Recall that, in PHY 411, we expanded the (scalar) potential in spherical harmonics:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\,q_{lm}\,\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \tag{4.1}$$

where q_{lm} are called the *multipole moments*, given by

$$q_{lm} = \int Y^*_{lm}(\theta',\phi')\,r'^{\,l}\,\rho(\vec{x}\,')\,d^3x' \tag{4.2}$$

The multipoles q_{20}, q_{21}, and q_{22} can be expressed in terms of the so-called **quadrupole moment tensor** Q_{ij}, given by

$$Q_{ij} = \int\left(3x'_i x'_j - r'^{\,2}\,\delta_{ij}\right)\rho(\vec{x}\,')\,d^3x' \tag{9.41}$$

Jackson uses weird notation here, so let us make sure we understand what he is doing. He writes the integral in equation (9.40) in terms of elements of the quadrupole moment tensor:

$$\hat{n}\times\int\vec{x}\,'\,(\hat{n}\cdot\vec{x}\,')\rho(\vec{x}\,')\,d^3x' = \frac{1}{3}\,\hat{n}\times\vec{Q}(\hat{n}) \tag{9.42}$$

which you verified in Question 3 on today's worksheet.

But where did the vector quantity $\vec{Q}$ come from, since Q_{ij} are elements of a tensor? The answer is that Jackson has constructed a vector $\vec{Q}(\hat{n})$ out of the tensor elements by writing

$$Q_\alpha = \sum_j Q_{ij}\,n_j \tag{9.43}$$

or, perhaps, better put as (which he doesn't do):

$$\vec{Q}(\hat{n}) = \sum_j Q_{ij}\,n_j\,\hat{e}_j$$

If you've experience with tensors, the short hand notation for this is $\vec{Q}(\hat{n}) = \overleftrightarrow{Q}.\hat{n}$, where $\overleftrightarrow{Q}$ identifies the quantity (Q) as a tensor.

Using equation (9.42), we get from equation (9.40) that

$$\vec{H} = -\frac{ick^3}{24\pi}\,\frac{e^{ikr}}{r}\,\hat{n}\times\vec{Q}(\hat{n}) \tag{9.44}$$

as you demonstrated in Question 4(a) on today's worksheet.

Meanwhile, since $\vec{E} = Z_0\vec{H}\times\hat{n} = c\mu_0\vec{H}\times\hat{n}$ from equation (9.19), we get

$$\vec{E} = -\frac{ic^2k^3\mu_0}{24\pi}\,\frac{e^{ikr}}{r}\left[\hat{n}\times\vec{Q}(\hat{n})\right]\times\hat{n}$$

as you demonstrated in Question 4(b) on today's worksheet.

The time-averaged power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = \frac{r^2}{2}\,\mathrm{Re}\left\{\hat{n}\cdot\vec{E}\times\vec{H}^*\right\}$$

so that, *as you showed in Question 5 on today's worksheet*

$$\frac{dP}{d\Omega} = \frac{c^2Z_0}{1152\pi^2}\,k^6\left|\left\{\hat{n}\times\vec{Q}(\hat{n})\right\}\times\hat{n}\right|^2 \tag{9.45}$$

where the vector within the absolute value symbol gives the polarization. The general angular distribution is complicated, but we can see that the radiated power varies as the sixth power of the frequency for fixed quadrupole moments, a much stronger dependence compared to the fourth power for dipole radiation.

If the angular distribution is of interest to you, Jackson provides a very simple example of a spheroidal charge distribution along the z-axis (a charge distribution that would look like an elongated cigar along z). The radiated power pattern would be a four-lobed pattern, an example of which is shown in Figure 9.2 on page 416 in Jackson, and reproduced below (source: utexas.edu).