PHY 420 **Spring 2021**

Week 4—Tuesday, Apr 20—Discussion Worksheet

Chapter 9: Radiation

In Chapter 7, we discussed the propagation of electromagnetic waves. In this chapter, we will study how such waves are generated. We will use a system of oscillating charges as the generator of electromagnetic radiation. To keep things simple, we'll assume that the sources are oscillating in otherwise empty space, meaning that there are no boundaries or materials present. We will also assume a harmonic time dependence for our system of charges and currents; such an assumption is justified, and loses no generality, because we can construct any time dependence using Fourier components.

1. All electromagnetic fields and potentials (e.g., $\Phi, \vec{A}, \vec{E}, \vec{B}, \vec{D}, \vec{H}$) are assumed to have the same (harmonic) time dependence.

(a) Write down expressions for $\rho(\vec{x}, t)$ and $\vec{J}(\vec{x}, t)$ with the time dependence written separately.

$$\rho(\vec{x}, t) = \rho(\vec{x}) e^{-i\omega t} \qquad J(\vec{x}, t) = J(\vec{x}) e^{-i\omega t} \tag{1}$$

(b) Now, let's begin recalling material from PHY 411; *equation numbers from the text are supplied in case you've forgotten.* Write down the two equations we obtained last quarter from the Ampere-Maxwell law in terms of the scalar potential Φ and vector potential $\vec{A}$:

$$\nabla^2 \varphi + \frac{\partial}{\partial t}(\nabla \cdot A) = -\rho/\epsilon_0 \tag{6.10}$$

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla\left(\nabla \cdot A + \frac{1}{c^2}\frac{\partial \varphi}{\partial t}\right) = -\mu_0 J \tag{6.11}$$

(c) Since the fields $\vec{E}$ and $\vec{B}$ are connected to the derivatives of the potentials Φ and $\vec{A}$, we have some freedom in choosing these potentials. Write down the gauge transformations for the potentials that leave $\vec{B}$ and $\vec{E}$ unchanged.

$$A \longrightarrow A' = A + \nabla \Lambda \tag{6.12}$$

and

$$\varphi \longrightarrow \varphi' = \varphi - \frac{\partial \Lambda}{\partial t} \tag{6.13}$$

A specific choice of potentials is called a ***gauge***. The Maxwell equations, though, are gauge invariant, meaning that they are valid regardless of what gauge we may decide to choose. One particular gauge, called the **Lorenz gauge**, is defined by the Lorenz condition.

(d) By inspection of equation (6.11), write down the ***Lorenz condition***.

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{6.14}$$

2. The Lorenz gauge can be used to uncouple equations (6.10) and (6.11).

(a) Use the Lorenz condition to uncouple equations (6.10) and (6.11), leaving two inhomogenous equations, one for Φ and one for $\vec{A}$.

$$\nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\rho/\epsilon_0 \tag{6.15}$$

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 J \tag{6.16}$$

(b) Equations (6.15) and (6.16) are wave equations that have the same basic structure. Write down the left hand side, with Ψ as a placeholder; the right hand side is already written.

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -4\pi f(\vec{x}, t) \tag{6.32}$$

Equation (6.32) is an inhomogenous partial differential equation whose solution can be broken apart into two parts

$$\Psi(\vec{x}, t) = \Psi_h + \Psi_{\text{part}}$$

where Ψ_h is the solution to the homogenous equation — we won't worry about that since we've already studied the solutions in Chapter 7: transverse plane waves propagating in free space. Instead, the ***particular solution*** is what we're interested in here, for it ***corresponds to sources creating electromagnetic radiation***. So, how do we find the particular solution? That's where the Green function comes into play.

(c) Write down the ***Green function equation*** corresponding to equation (6.32).

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_k^{(\pm)}(\vec{x}, t : \vec{x}', t') = -4\pi\, \delta(\vec{x} - \vec{x}')\,\delta(t - t') \tag{6.41}$$

(d) Write down the ***Green function*** that is the solution to equation (6.41).

$$G^{(\pm)}(x, t; x', t') = \frac{\delta\left(t' - \left[t \mp \frac{|x - x'|}{c}\right]\right)}{|x - x'|} \tag{6.44}$$

Recall that we described in the previous Class Summary how G^+ is called the **retarded Green function**; The time difference R/c is the time taken by the disturbance to propagate from the source point to the point of observation, so the effect observed at $\vec{x}$ at time t is caused at the source point $\vec{x}\,'$ at an *earlier* or ***retarded time*** $t' = t - R/c$. Meanwhile, G^- is called the **advanced Green function**.

Having written the Green function, we can write the particular solution to equation (6.32) as

$$\Psi^{(\pm)}(\vec{x}, t) = \int\int G^{(\pm)}(\vec{x}, t; \vec{x}\,', t')\, f(\vec{x}\,', t')\, d^3x'dt'$$

as we discussed last week. Recall that $\Psi^{(+)}$ corresponds to the ***outward going wave***. Jackson then discusses how we can add solutions of the homogenous equations to either of these solutions. For details, see the posted Class Summary.

Now let us apply the preceding specifically to the vector potential $\vec{A}$, for which the wave equation is given by equation (6.16): $\nabla^2 \vec{A} - (1/c^2)\,\partial^2 \vec{A}/\partial t^2 = -\mu_0 \vec{J}$. On the homework, you will write the Green function corresponding to equation (6.16), and show that the particular solution to it is

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \int dt' \,\frac{\vec{J}(\vec{x}\,',t')}{|\vec{x}-\vec{x}\,'|}\,\delta\left(t' - \left[t - \frac{|\vec{x}-\vec{x}\,'|}{c}\right]\right) \tag{9.2}$$

3. Write the current density and vector potential with the usual harmonic time dependence:

$$\vec{J}(\vec{x},t) = \vec{J}(\vec{x})\,e^{-i\omega t} \qquad \text{and} \qquad \vec{A}(\vec{x},t) = \vec{A}(\vec{x})\,e^{-i\omega t}$$

and show that equation (9.2) becomes

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}\,')\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|}\,d^3x' \tag{9.3}$$

$$\vec{A}(\vec{x})e^{-i\omega t} = \frac{\mu_0}{4\pi}\int d^3x' \int dt' \,\frac{\vec{J}(\vec{x})e^{-i\omega t}}{|\vec{x}-\vec{x}'|}\,\delta\left(t' - \left[t - \frac{|\vec{x}-\vec{x}'|}{c}\right]\right)$$

$$= \frac{\mu_0}{4\pi}\int d^3x' \,\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\underbrace{\int dt'\, e^{-i\omega t'}\,\delta\Big(t' - \underbrace{\left[t - \frac{R}{c}\right]}_{a}\Big)}$$

Property of δ-fn to integrate:
$\int f(t')\,\delta(t'-a)\,dt' = f(a)$

$$= \frac{\mu_0}{4\pi}\int d^3x' \,\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,e^{-i\omega[t-R/c]}$$

$$\vec{A}(\vec{x})\cancel{e^{-i\omega t}} = \frac{\mu_0}{4\pi}\cancel{e^{-i\omega t}}\int d^3x' \,\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,e^{+i\omega(R/c)}$$

We have that $R = |\vec{x}-\vec{x}'|$ and write $\omega/c = k$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\,\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x'$$

Equation (9.3) is a very important step, because knowing $\vec{A}$ means that we can write $\vec{B}$ and $\vec{E}$.

4. Given $\vec{A}$, we can determine $\vec{B}$ using $\vec{B} = \vec{\nabla} \times \vec{A}$, then write an expression for $\vec{E}$.

(a) Since we're in free space where $\vec{B} = \mu_0 \vec{H}$, write down an expression for $\vec{H}$ using $\vec{B}$ given above.

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} \tag{9.4}$$

(b) Starting from $\vec{\nabla} \times \vec{H} = \vec{J} + \partial \vec{D}/\partial t$, show that the electric field *outside the source* is given by

$$\vec{E} = \frac{iZ_0}{k} \vec{\nabla} \times \vec{H} \tag{9.5}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space.

$\vec{\nabla} \times \vec{H} = \partial \vec{D}/\partial t$ outside the source

$= \frac{\partial}{\partial t}(\epsilon_0 \vec{E}) = \epsilon_0 \frac{\partial}{\partial t}\left[\vec{E}(\vec{x}) e^{-i\omega t}\right]$

$\vec{\nabla} \times \vec{H} = -i\omega\epsilon_0 \vec{E}(\vec{x}) = -ik\epsilon_0 \vec{E}(\vec{x}, t)$ $\qquad k = \omega\sqrt{\mu\epsilon}$ $= \omega\sqrt{\mu_0\epsilon_0}$

$\vec{E}(\vec{x}, t) = -\frac{\sqrt{\mu_0\epsilon_0}}{ik\epsilon_0} \vec{\nabla} \times \vec{H} = -\frac{1}{ik}\sqrt{\frac{\mu_0}{\epsilon_0}} \vec{\nabla} \times \vec{H} = -\frac{Z_0}{ik} \vec{\nabla} \times \vec{H} = +\frac{iZ_0}{k} \vec{\nabla} \times \vec{H}$

Three zones of interest: Given a current distribution $\vec{J}(\vec{x}\,')$, the fields can, in principle, be determined by evaluating the integral in equation (9.3), but instead of integrating by brute force, we will study the properties of the fields in three different zones of interest, defined by their distance from the source. Assume that the source (all charge and current distributions) is confined to a region of size d (thus, a sphere of radius d about the origin) that is very small compared to a wavelength (so $d \ll \lambda$). This is known as the ***long-wavelength approximation***, and we will always work in this approximation. If, then, we let r be the distance from the source to the observation point located at $\vec{x}$, then we have $r = |\vec{x}|$. Then, there are ***three spatial regions of interest***:

- The **near (static) zone**: $d \ll r \ll \lambda$
- The **intermediate (induction) zone**: $d \ll r \sim \lambda$
- The **far (radiation) zone**: $d \ll \lambda \ll r$

We won't spend much time on the near and intermediate zones (but see the posted class summary for comments), instead focusing on the far (radiation) zone, since our detectors are always located a great number of wavelengths away from the sources, and thus it is our primary region of interest.

The far (radiation) zone: In the far zone, the observation point r is very far from the source and much larger than the wavelength of the light, so we have $d \ll \lambda \ll r$. Now, $r \gg \lambda$ means that

$$\frac{r}{\lambda} \gg 1 \qquad \text{and so} \qquad \left(\frac{2\pi}{\lambda}\right) r \gg 1$$

But $k = 2\pi/\lambda$, so in the radiation zone we have $kr \gg 1$.

5. With the observation point defined so that $r = |\vec{x}|$, and setting $r' = |\vec{x}'|$, show that in some appropriate limit, we get

$$\left|\vec{x} - \vec{x}'\right| \simeq r - \hat{n} \cdot \vec{x}' \tag{9.7}$$

where $\hat{n}$ is a unit vector in the direction of $\vec{x}$. What is that "appropriate limit" that you imposed?

$r = |\vec{x}|, \quad r = |\vec{x}'|$

$|\vec{x} - \vec{x}'| = \sqrt{|\vec{x}|^2 + |\vec{x}'| - 2\,\vec{x}\cdot\vec{x}'} = \sqrt{r^2 + (r')^2 - 2(r\hat{x})\cdot(r'x')}$

In far radiation zone, $r \gg d \Rightarrow r \gg r'$, so $r'/r \ll 1$

Expand in binomial series $(1+y)^n = 1 + ny + \frac{n(n-1)}{2!}y^2 + \ldots$

$|\vec{x} - \vec{x}| = r\Big[1 + \underbrace{\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\hat{x}\cdot\hat{x}'}_{y}\Big]^{1/2}$

$= r\left[1 + \frac{1}{2}\left\{\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\hat{x}\cdot\hat{x}'\right\} + \frac{1/2(1/2-1)}{2!}\left\{\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\hat{x}\cdot\hat{x}'\right\}^2 + \ldots\right]$

$\rightarrow |\vec{x} - \vec{x}'| \approx r\left[1 + \cancel{1/2}\left\{\cancel{\left(\frac{r'}{r}\right)^2} - \cancel{2}\left(\frac{r'}{r}\right)\hat{x}\cdot\hat{x}'\right\} + \cancel{\ldots}\right]$

$= r - \cancel{r}\left(\frac{r'}{\cancel{r}}\right)\hat{x}\cdot\hat{x}'$

$= r - \hat{n}\cdot\underbrace{(r'\hat{x}')}_{=\vec{x}'}$

$|\vec{x} - \vec{x}'| = r - \hat{n}\cdot\vec{x}'$

6. Starting from the expression for the vector potential $\vec{A}$:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}\,') \frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} \, d^3x' \tag{9.3}$$

where we have the observation point $r = |\vec{x}|$, and the sources are located at $r' = |\vec{x}\,'|$, and using equation (9.7) for the far (radiation) zone that you derived on the previous page, derive the relation

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int \vec{J}(\vec{x}\,') \, (\hat{n} \cdot \vec{x}\,')^n \, d^3x' \tag{9.9}$$

Hint: You will need

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-ik\hat{n}\cdot\vec{x}\,'} = \sum_{n=0}^{\infty} \frac{(-ik\hat{n}\cdot\vec{x}\,')^n}{n!}$$

Sub 9.7 in 9.3

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int J(\vec{x}\,') \frac{e^{ik(r-\hat{n}\cdot\vec{x})}}{r} d^3x'$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}\,') e^{-ik\hat{n}\cdot\vec{x}} d^3x'$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}\,') \left[\sum_{n=0}^{\infty} \frac{(-ik\hat{n}\cdot\vec{x}\,')^n}{n!} \right] d^3x'$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int \vec{J}(\vec{x}\,') (\hat{n}\cdot\vec{x}\,')^n d^3x'$$

In the next class, we will begin with the first term in this expression, which is responsible for the electric dipole radiation.