# PHY 420 Spring 2021

## Week 8—Thursday, May 20—Discussion Worksheet

## Covariance of Electrodynamics

The invariance in form of the equations of electrodynamics under Lorentz transformations was shown by Lorentz and Poincare before the formulation of the Special Theory of Relativity. A more precise word for it would be covariance, meaning that the *form* of the equations does not change. Not only are Maxwell's equations invariant under Lorentz transformations, but also the Lorentz force law and the continuity equation. That is, $\rho, \vec{J}, \vec{E}$ and $\vec{B}$ transform in well defined ways under Lorentz transformations. However, we haven't shown any of that yet. In the last few classes, we've developed the mathematics that will express that invariance (covariance). Now, we will recast our basic equations in 4-tensor form, so that we can demonstrate their invariance. That is, ***since 4-tensors are invariant (covariant) under Lorentz transformations by definition, expressing equations like (11.127) below in terms of such quantities will ensure that the equations themselves are covariant under Lorentz transformations***. I won't proceed along Jackson's route, however, at least not initially, preferring instead a more direct approach via Maxwell's equations sourced from Wheeler (Reed College).

Begin with the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{11.127}$$

Now, the *left hand side* of equation (11.127) looks like the 4-divergence of a 4-vector that we wrote in equation (11.77) for a general 4-vector $A$:

$$\partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A}$$

using the notation that

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right) \tag{11.76}$$

**1.** Recalling that $ct = x^0$, write equation (11.127) with $x_0$ and other changes, if applicable, and use it to define the 4-vector $J^\alpha$.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$$\partial_\alpha = \frac{\partial}{\partial x^0} + \vec{\nabla}_0$$

$$(A^0, A)$$

$$\frac{\partial(\rho c)}{\partial(ct)} + \vec{\nabla} \cdot \vec{J} = 0$$

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$$

$J^\alpha =$ $(c\rho, J)$

With the 4-vector $J^\alpha$ in place, the continuity equation (11.127) takes the obviously covariant form

$$\partial_\alpha J^\alpha = 0 = \frac{\partial_\alpha J^\alpha}{\partial x^\alpha} \tag{11.129}$$

It is a wondrous sight to behold; the sheer economy of the expression brings to mind John Keats' "*Beauty is Truth, Truth Beauty.*"

**2.** Next, let us write Maxwell's equations. For simplicity, we consider the microscopic Maxwell equations, without $\vec{D}$ and $\vec{H}$. Consider first the two Maxwell equations with source terms:

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \text{and} \qquad \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \tag{W8.1}$$

Yes, these look different from last quarter; they are now in gaussian units (*Jackson*, page 781).

(a) Write these two equations in component form (i.e., you'll get four equations), using the notation from equation (11.76) written on the previous page. Jackson uses $x_1 = x, x_2 = y, x_3 = z$ here, and you should too, so that our expressions are comparable to his. I've done part of the first one for you as an example.

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = 4\pi\rho = \underline{\frac{4\pi}{c} J^0} \qquad \partial_x = \frac{\partial}{\partial x^1} \ldots$$

$$(\vec{\nabla} \times \vec{B})_x - \frac{1}{c}\frac{\partial E_x}{\partial t} = \frac{4\pi}{c} J^1 \Rightarrow -\partial_0 E_x + \partial_y B_z - \partial_z B_y = \frac{4\pi}{c} J^1$$

$$(\vec{\nabla} \times \vec{B})_y - \frac{1}{c}\frac{\partial E_y}{\partial t} = \frac{4\pi}{c} J^2 \Rightarrow -\partial_0 E_y - \partial_x B_z + \partial_z B_x = \frac{4\pi}{c} J^2$$

$$(\vec{\nabla} \times \vec{B})_z - \frac{1}{c}\frac{\partial E_z}{\partial t} = \frac{4\pi}{c} J^3 \Rightarrow -\partial_0 E_z + \partial_x B_y - \partial_y B_x = \frac{4\pi}{c} J^3$$

$$(\partial_0 \ \partial_x \ \partial_y \ \partial_z)\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} = \frac{4\pi}{c}\begin{pmatrix} J^0 \\ J^1 \\ J^2 \\ J^3 \end{pmatrix}$$

(b) It should become clear after you've written the four equations above that they could be put into a remarkably simple form:

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \tag{11.141}$$

where $F^{\alpha\beta}$ is called the field-strength tensor. Write down the elements of $F^{\alpha\beta}$ by inspection.

$$F^{\alpha\beta} = \begin{pmatrix} \underline{0} & \underline{-E_x} & \underline{-E_y} & \underline{-E_z} \\ \underline{E_x} & \underline{0} & \underline{-B_z} & \underline{B_y} \\ \underline{E_y} & \underline{B_z} & \underline{0} & \underline{-B_x} \\ \underline{E_z} & \underline{-B_y} & \underline{B_x} & \underline{0} \end{pmatrix}$$

**Please do not look at Jackson while answering this question.**

**3.** Consider next the two sourceless Maxwell equations:

$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \tag{W8.2}$$

(a) Using again the notation from equation (11.76) that

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)$$

write down below the four equations you get from equation (W8.2).

$$\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \partial_x B_x + \partial_y B_y + \partial_z B_z = 0$$

$$(\vec{\nabla} \times \vec{E})_x + \frac{1}{c}\frac{\partial B_x}{\partial t} = 0 \Rightarrow \partial_0 B_x + \partial_y E_z - \partial_z E_y = 0$$

$$(\vec{\nabla} \times \vec{E})_y + \frac{1}{c}\frac{\partial B_y}{\partial t} = 0 \Rightarrow \partial_0 B_y - \partial_x E_z + \partial_z E_x = 0$$

$$(\vec{\nabla} \times \vec{E})_z + \frac{1}{c}\frac{\partial B_z}{\partial t} = 0 \Rightarrow \partial_0 B_z + \partial_x E_y - \partial_y E_x = 0$$

$$(\partial_0 \ \ \partial_x \ \ \partial_y \ \ \partial_z)\begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) This time, the four equations you've written above could be put into the form:

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \tag{11.142}$$

where $\mathcal{F}^{\alpha\beta}$ is called the dual field-strength tensor. Write down the elements of $\mathcal{F}^{\alpha\beta}$.

$$\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

**Please do not look at Jackson while answering this question.**

**4.** The field-strength tensor you wrote in Question 2(b) should've looked like

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{11.137}$$

(a) Determine the field-strength tensor $F_{\alpha\beta}$ with two covariant indices, by using the raising and lowering property of the metric tensor $g_{\alpha\beta}$. In other words, do

$$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta}$$

$$F_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_y & 0 & B_z & -B_y \\ E_x & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

(b) Evidently, the elements of $F_{\alpha\beta}$ can be obtained from $F^{\alpha\beta}$ by a very simple replacement. What is it?

In $F^{\alpha\beta}$, Put $E_i \to -E_i$, $i = x, y, z$

$B_i \to B_i$, Remains unchanged

to get $F_{\alpha\beta}$

**5.** Another way to construct the dual field-strength tensor $\mathcal{F}^{\alpha\beta}$ is by using the the totally anti-symmetric $4^{th}$ rank tensor $\epsilon^{\alpha\beta\gamma\delta}$, defined by

- $\epsilon^{\alpha\beta\gamma\delta} = +1$, for $\alpha = 0, \beta = 1, \gamma = 2, \delta = 3$, or any even permutation.
- $\epsilon^{\alpha\beta\gamma\delta} = -1$, for any odd permutation of $\alpha, \beta, \gamma, \delta$.
- $\epsilon^{\alpha\beta\gamma\delta} = 0$, if any two indices are equal.

The dual field-strength tensor $\mathcal{F}^{\alpha\beta}$ can then be obtained by doing

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\,\epsilon^{\alpha\beta\gamma\delta}\,F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{11.140}$$

where summation over $\gamma$ and $\delta$ is implied. In other words, the elements of the dual tensor are obtained by putting $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ in equation (11.137).

(a) By explicitly doing $\mathcal{F}^{\alpha\beta} = \frac{1}{2}\,\epsilon^{\alpha\beta\gamma\delta}\,F_{\gamma\delta}$, verify that you get $\mathcal{F}^{01} = -B_x$.

$$\mathcal{F}^{01} = \frac{1}{2}\Big[\epsilon^{0100}F_{00} + \epsilon^{0101}F_{01} + \epsilon^{0102}F_{02} + \epsilon^{0103}F_{03}$$
$$+ \epsilon^{0110}F_{10} + \epsilon^{0111}F_{11} + \epsilon^{0112}F_{12} + \epsilon^{0113}F_{13}$$
$$+ \epsilon^{0120}F_{20} + \epsilon^{0121}F_{21} + \epsilon^{0122}F_{22} + \epsilon^{0123}F_{23}$$
$$+ \epsilon^{0130}F_{30} + \epsilon^{0131}F_{31} + \epsilon^{0132}F_{32} + \epsilon^{0133}F_{33}\Big]$$

Thus,

$$\mathcal{F}^{01} = \frac{1}{2}\Big[\epsilon^{0123}F_{23} + \epsilon^{0123}F_{32}\Big]$$

$$= \frac{1}{2}\Big[(+1)(-B_x) + (-1)(B_x)\Big] = \frac{1}{2}\Big[-2B_x\Big] = \boxed{-B_x}$$

(b) Again, by explicit evaluation, verify that you get $\mathcal{F}^{23} = E_x$.

$$\mathcal{F}^{23} = \frac{1}{2}\Big[\epsilon^{2300}F_{00} + \epsilon^{2301}F_{01} + \epsilon^{2302}F_{02} + \epsilon^{2310}F_{10}$$
$$+ 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0\Big]$$

$$= \frac{1}{2}\Big[\epsilon^{2301}F_{01} + \epsilon^{2310}F_{10}\Big]$$

$$= \frac{1}{2}\Big[(+1)E_x + (-1)(-E_x)\Big] = \frac{1}{2}\Big[2E_x\Big] \Rightarrow \mathcal{F}^{23} = E_x$$

Next, consider the wave equation for the vector potential $\vec{A}$ and the scalar potential $\Phi$ in the Lorenz gauge:

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J} \qquad \text{and} \qquad \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\,\rho \tag{11.130}$$

The differential operators on the left hand side in equation (11.130) are the 4-dimensional Laplacian operators we defined in equation (11.78):

$$\Box \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{02}} - \nabla^2$$

while the right hand sides in equation (11.130) are the component of a 4-vector.

**6.** With a 4-vector potential $A^\alpha$ formed by $\Phi$ and $\vec{A}$:

$$A^\alpha = (\Phi, \vec{A}) \tag{11.132}$$

write down the wave equations in equation (11.130) in their covariant forms.

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\vec{A} = \frac{4\pi}{c}\vec{J}$$

$$\left[\frac{\partial^2}{\partial (ct)^2} - \nabla^2\right]\vec{A} = \frac{4\pi}{c}\vec{J}$$

$$\underbrace{\left[\frac{\partial^2}{\partial x^{02}} - \nabla^2\right]}\vec{A} = \frac{4\pi}{c}\vec{J}$$

$$\Box\vec{A} = \frac{4\pi}{c}\vec{J}$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\Phi = 4\pi\rho$$

$$\Big\downarrow$$

$$\left[\quad\ldots\quad\right]\Phi = \frac{4\pi}{c}(c\rho)$$

$$\Big\downarrow$$

$$\Box A^0 = \frac{4\pi}{c}J^0$$

$$\boxed{\Box A^\alpha = \frac{4\pi}{c}J^\alpha}$$

With the definitions of $J^\alpha$ in equation (11.128) in *Jackson* (Question 1 of this worksheet), $A^\alpha$ in equation (11.132), and $F^{\alpha\beta}$ in equation (11.137), together with the wave equations in equation (11.133) in *Jackson* (Question 6 above), or the Maxwell equations in equation (11.141) and equation (11.142), the **covariance of electrodynamics is established**. To complete matters, we will put the Lorenz force equation into covariant form in the next class.