PHY 420 Spring 2021

Class Summary—Week 3, Day 1—Tuesday, Mar 13

In the previous class, we wrote down the solution to the Laplace equation in cylindrical coordinates (for the case when we have only one of the end-faces of the cylinder at zero potential). In particular, for this case, the solutions to the $R(\rho)$ equation can be written in terms of Bessel functions of the first kind, $J_{\pm\nu}(x)$, where $x = k\rho$; k is the separation constant from the $Z(z)$ equation and, as of now, is still unconstrained but will be below by its connection to the roots of $J_\nu(x)$. Since ν must be an integer to obtain single-valued functions for the potential $\Phi(\vec{x})$, and $J_{\pm\nu}(x)$ are not linearly independent when ν is an integer, we also wrote the solutions to the $R(\rho)$ equation as a linear combination of $J_\nu(x)$ and $N_\nu(x)$, where the latter are known as Neumann functions, or *Bessel functions of the second kind*.

The linear independence of $J_\nu(x)$ and $N_\nu(x)$ allows us to define an alternative pair of solutions to the Bessel differential equation known as *Bessel functions of the third kind*, or Hankel functions, defined by

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + i\,N_\nu(x) \\ H_\nu^{(2)}(x) &= J_\nu(x) - i\,N_\nu(x) \end{aligned} \tag{3.86}$$

The Hankel functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ form a fundamental set of solutions to the Bessel equation, just like $J_\nu(x)$ and $N_\nu(x)$.

The Bessel functions all satisfy the recursion formulas

$$\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) = \frac{2\nu}{x}\,\Omega_\nu(x) \tag{3.87}$$

$$\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) = 2\,\frac{d\Omega_\nu(x)}{dx} \tag{3.88}$$

where $\Omega_\nu(x) = J_\nu(x)$ or $N_\nu(x)$ or $H_\nu^{(1)}(x)$ or $H_\nu^{(2)}(x)$.

Each Bessel function has an infinite number of roots. We will work mostly with the roots of $J_\nu(x)$:

$$J_\nu(x_{\nu n}) = 0, \qquad n = 1, 2, 3, \ldots \tag{3.92}$$

where $x_{\nu n}$ is the nth root of $J_\nu(x)$.

Jackson provides the first three roots ($n = 1, 2, 3$) for the first three integer values of $\nu = 0, 1$, and 2, and they are

$$\begin{array}{llllll} \nu = 0, & x_{0n} = & 2.405, & 5.520, & 8.654, & \ldots \\ \nu = 1, & x_{1n} = & 3.832, & 7.016, & 10.173, & \ldots \\ \nu = 2, & x_{2n} = & 5.136, & 8.417, & 11.620, & \ldots \end{array}$$

You should be able to verify these (roughly) by looking at the plots of $J_\nu(x)$, *like you did in Question 1 of today's worksheet.* For higher roots, the asymptotic formula

$$x_{\nu n} \simeq n\pi + \left(\nu - \frac{1}{2}\right)\frac{\pi}{2}$$

is accurate to at least three figures.

Let us take a moment to figure out **what we've achieved so far**. We separated variables in the Laplace equation, and we've found a solution of the radial part of the Laplace equation in terms of Bessel functions.

The **next step** would be to investigate in what sense the Bessel functions form an orthogonal, complete set of functions. We will consider only Bessel functions of the first kind $J_\nu(x)$, which are defined in $0 \le x \le \infty$, although for our purposes, we're often interested in a physical solution over a restricted finite interval $0 \le \rho \le a$.

It can be shown, as Jackson does on pages 114 and 115 if you're interested, that $\sqrt{\rho}\, J_\nu(x_{\nu n}\rho/a)$ form an orthogonal set in the interval $0 \le \rho \le a$, for fixed $\nu \ge 0, n = 1, 2, \ldots$, and that the normalization is given by

$$\int_0^a \left[\sqrt{\rho}\, J_\nu \left(x_{\nu n'}\, \frac{\rho}{a}\right)\right] \left[\sqrt{\rho}\, J_\nu \left(x_{\nu n}\, \frac{\rho}{a}\right)\right] d\rho = \frac{a^2}{2} \left[J_{\nu+1}(x_{\nu n})\right]^2 \delta_{n'n}$$

or

$$\int_0^a \rho\, J_\nu \left(x_{\nu n'}\, \frac{\rho}{a}\right)\, J_\nu \left(x_{\nu n}\, \frac{\rho}{a}\right)\, d\rho = \frac{a^2}{2}\, J^2_{\nu+1}(x_{\nu n})\, \delta_{n'n} \tag{3.95}$$

Now that we have a set of orthogonal functions, and assuming that the set is complete, we can expand an arbitrary function of ρ on the interval $0 \le \rho \le a$ in a Fourier-Bessel series:

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n}\, J_\nu \left(x_{\nu n}\, \frac{\rho}{a}\right) \tag{3.96}$$

where the coefficients $A_{\nu n}$ are given by

$$A_{\nu n} = \frac{2}{a^2\, J^2_{\nu+1}(x_{\nu n})} \int_0^a \rho\, f(\rho)\, J_\nu \left(\frac{x_{\nu n}\, \rho}{a}\right)\, d\rho \tag{3.97}$$

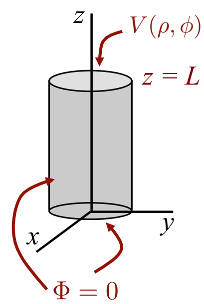
Equation (3.96), together with (3.97) is the conventional Fourier-Bessel series and is particularly appropriate for functions that are zero at boundaries, e.g., Dirichlet boundary conditions on a cylinder. We will now look at such an example of a boundary value problem.

But before we begin the boundary value problem, it is worth noting that an alternative expansion is also possible in a series of functions $\sqrt{\rho}\, J_\nu(y_{\nu n}\rho/a)$, where $y_{\nu n}$ is the nth root of the equation $dJ_\nu(x)/dx = 0$.

Such an expansion in terms of functions $\sqrt{\rho}\, J_\nu(y_{\nu n}\rho/a)$ is especially useful for expanding functions with vanishing slope at boundaries $\rho = a$, that is, for Neumann boundary conditions.

Boundary Value Problems in Cylindrical Coordinates

Consider the boundary value problem in which a cylinder of radius a and length L has its bottom surface on the xy-plane at $z = 0$, and its top surface at $z = L$ (as shown in the figure below). The potential on the side and the bottom of the cylinder is zero, while the top has a potential $\Phi = V(\rho, \phi)$. We wish to find the potential at any point *inside* the cylinder.



We know from our discussion so far that the potential $\Phi(\rho, \phi, z)$ at any point inside the cylinder can be written as the product of solutions

$$\Phi(\rho, \phi, z) = R(\rho)\, Q(\phi)\, Z(z)$$

Substituting this into the Laplace equation, we know from above that we get

$$Q(\phi) = e^{\pm i\nu\phi}$$

and that ν must be an integer for the potential to be single-valued when the full azimuthal span is allowed. Setting $\nu = m$, an integer, and writing in terms of sine and cosine, we get that

$$Q(\phi) = A \sin m\phi + B \cos m\phi$$

Meanwhile, we know that, in general

$$Z(z) = e^{\pm kz} = \cosh kz \pm \sinh kz$$

where I haven't written any constants, because they can introduced later when we multiply together $R(r)Q(\phi)Z(z)$.

Since $\cosh kz \neq 0$ at $z = 0$, and our boundary condition is that $\Phi = 0$ at $z = 0$, we cannot have $\cosh kz$ in our solution for $Z(z)$. Therefore, the boundary conditions constrain our z-solution to be

$$Z(z) = \sinh kz$$

Note that k is, as yet, undetermined.

Finally, the radial solution is

$$R(\rho) = C\, J_m(k\rho) + DN_m(k\rho)$$

Since we're interested in the potential inside the cylinder, and $N_m(k\rho)$ diverges at $\rho = 0$, we must exclude it from our solution. So, we set $D = 0$ and

$$R(\rho) = C\, J_m(k\rho)$$

Meanwhile, the potential must vanish at $\rho = a$ on the surface of the cylinder. Substituting this in the radial equation above, we get

$$0 = C\, J_m(ka)$$

From equation (3.92), we know that each Bessel function $J_\nu(x)$ of order ν has an infinite number of roots, so that we can write

$$J_\nu(x_{\nu n}) = 0 \qquad n = 1, 2, 3, \ldots$$

where $x_{\nu n}$ is the nth root of $J_\nu(x)$.

Now $\nu = m$, an integer, in our current problem, the restriction coming from the $Q(\phi)$ solution. That means we must have

$$J_m(x_{mn}) = 0$$

Putting the boundary condition and the condition above next to each other, we get that

$$J_m(x_{mn}) = J_m(ka) = 0$$

which leads us to conclude that

$$k_{mn} = \frac{x_{mn}}{a} \qquad n = 1, 2, 3, \ldots$$

where we've indexed k with the same m and n to keep track. Keep in mind that x_{mn} are the roots of $J_m(x_{mn}) = 0$.

Combining all that we've obtained above, we get the general form of the solution for the potential inside the cylinder:

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho)\, \sinh(k_{mn}\, z) \Big[A_{mn}\, \sin m\phi + B_{mn}\, \cos m\phi\Big] \tag{3.105a}$$

Note, of course, that the particular form of equation (3.105a) is dictated by the boundary conditions that the potential vanish at the bottom of the cylinder $z = 0$ and the side of the cylinder $\rho = a$. For different boundary conditions, the solution would take on a different form. You will do an example on this in Homework 3, where you have zero potential on the end faces of the cylinder.

We've already applied two of the three boundary conditions so far; the third boundary condition is that the potential is $V(\rho, \phi)$ at $z = L$. Therefore

$$V(\rho, \phi) = \sum_{m,n} J_m(k_{mn}\rho)\, \sinh(k_{mn}\, L) \Big[A_{mn}\, \sin m\phi + B_{mn}\, \cos m\phi\Big]$$

This is a Fourier series in ϕ and a Fourier-Bessel series in ρ. So, to find A_{mn}, we'll need to multiply on both sides by sin $m'\phi$ and use orthogonality of sines, and then use the Fourier-Bessel relation from equation (3.97). Likewise, to find B_{mn}, we'll need to multiply on both sides by cos $m'\phi$ and use orthogonality of cosines, and then use the Fourier-Bessel relation from equation (3.97). *As you showed in Question 5 of today's worksheet* for A_{mn}, this gives finally

$$\begin{aligned} A_{mn} &= \frac{2}{\pi a^2 J_{m+1}^2(k_{mn}a)\, \sinh(k_{mn}L)} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho V(\rho, \phi)\, J_m(k_{mn}\rho)\, \sin m\phi \\ B_{mn} &= \frac{2}{\pi a^2 J_{m+1}^2(k_{mn}a)\, \sinh(k_{mn}L)} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho V(\rho, \phi)\, J_m(k_{mn}\rho)\, \cos m\phi \end{aligned} \tag{3.105b}$$

with the caveat that for $m = 0$, we will have to use $\frac{1}{2}B_{0n}$ in the series.

The Fourier-Bessel series in equation (3.105) above is appropriate for a finite interval in ρ; in the case discussed here, we had $0 \le \rho \le a$. If, however, we wish to cover all space, that is $a \to \infty$, then the series goes over into an integral just as a Fourier series goes over into a Fourier integral.

For example, if the potential in charge-free space is finite for $z \ge 0$ and vanishes for $z \to \infty$, the general form of the solution for $z \ge 0$ must be

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(k\rho)\, e^{-kz} \Big[A_m(k) \sin m\phi + B_m(k) \cos m\phi \Big] \tag{3.106}$$

If the potential is specified over the whole plane $z = 0$ to be $V(\rho, \phi)$, the coefficients are determined by

$$V(\rho, \phi) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(k\rho) \Big[A_m(k) \sin m\phi + B_m(k) \cos m\phi \Big]$$

Since the ϕ-variation is just a Fourier series

$$\begin{aligned} \frac{1}{\pi} \int_0^{2\pi} V(\rho, \phi) \sin m\phi \, d\phi &= \int_0^{\infty} J_m(k'\rho)\, A_m(k')\, dk' \\ \frac{1}{\pi} \int_0^{2\pi} V(\rho, \phi) \cos m\phi \, d\phi &= \int_0^{\infty} J_m(k'\rho)\, B_m(k')\, dk' \end{aligned} \tag{3.107}$$

These radial integral equations are called *Hankel transforms*. To solve them, we can invert equation (3.107) by using the integral relation

$$\int_0^{\infty} x\, J_m(kx)\, J_m(k'x)\, dx = \frac{1}{k}\, \delta(k' - k) \tag{3.108}$$

Multiplying both sides by $\rho\, J_m(k\rho)$ and integrating over ρ, we find that

$$\begin{aligned} A_m(k) &= \frac{k}{\pi} \int_0^{\infty} d\rho\, \rho \int_0^{2\pi} d\phi\, V(\rho, \phi)\, J_m(k\rho) \sin m\phi \\ B_m(k) &= \frac{k}{\pi} \int_0^{\infty} d\rho\, \rho \int_0^{2\pi} d\phi\, V(\rho, \phi)\, J_m(k\rho) \cos m\phi \end{aligned} \tag{3.109}$$

Again, as usual, we must use $\frac{1}{2} B_0(k)$ for $m = 0$.

Note that for fixed $\nu, \mathrm{Re}(\nu) > -1$, the functions $J_\nu(kx)$ form a complete orthogonal set (in k) of functions in the interval $0 < x < \infty$, so that for each m value (and fixed ϕ and z), the expansion in k in equation (3.106) is a special case of the expansion

$$\begin{aligned} A(x) &= \int_0^{\infty} A_t(k)\, J_\nu(kx)\, dk \\ \text{where } A_t(k) &= k \int_0^{\infty} x A(x)\, J_\nu(kx)\, dx \end{aligned} \tag{3.110}$$

All of the above discussion was for non-periodic boundary conditions in the z-direction. Another type of problem that we are likely to encounter is when there are periodic boundary conditions along the z-direction. We will now briefly discuss such a possibility, and prepare you to solve a problem of this kind on Homework 3.

Modified Bessel Functions

Before ending our discussion of the properties of Bessel functions, it is worth noting that if we had picked the separation constant in equation (3.73) as $-k^2$ (instead of k^2), then the solutions to $Z(z)$ would have been $e^{\pm ikz}$, and the radial equation equation (3.75) for $R(\rho)$ would have instead been

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(-k^2 - \frac{\nu^2}{\rho^2}\right) R = 0 \tag{3.111}$$

As before, multiplying by k^2 gives

$$\frac{1}{k^2}\frac{d^2R}{d\rho^2} + \frac{1}{k^2}\frac{1}{\rho}\frac{dR}{d\rho} + \frac{1}{k^2}\left(-k^2 - \frac{\nu^2}{\rho^2}\right) R = 0$$

so that

$$\frac{d^2R}{d(k\rho)^2} + \frac{1}{k\rho}\frac{dR}{d(k\rho)} + \left(-1 - \frac{\nu^2}{[k\rho]^2}\right) R = 0$$

and with $x = k\rho$, this becomes

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(-1 - \frac{\nu^2}{x^2}\right) R = 0$$

or

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0 \tag{3.112}$$

The solutions of this equation are called the **modified Bessel functions**.

In other words, to get the solutions, we just take the Bessel functions but with an imaginary argument ix instead of x.

The usual choices of linearly independent solutions are denoted by $I_\nu(x)$ and $K_\nu(x)$, and defined by

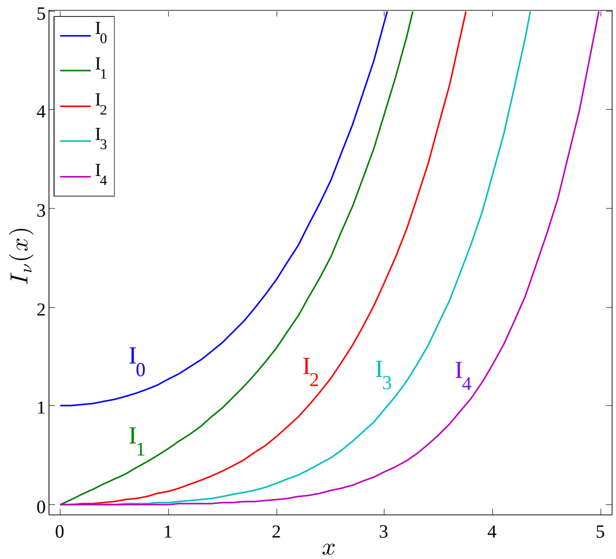
$$I_\nu(x) = i^{-\nu}\, J_\nu(ix) \tag{3.100}$$

$$K_\nu(x) = \frac{\pi}{2}\, i^{\nu+1}\, H_\nu^{(1)}(ix) \tag{3.101}$$

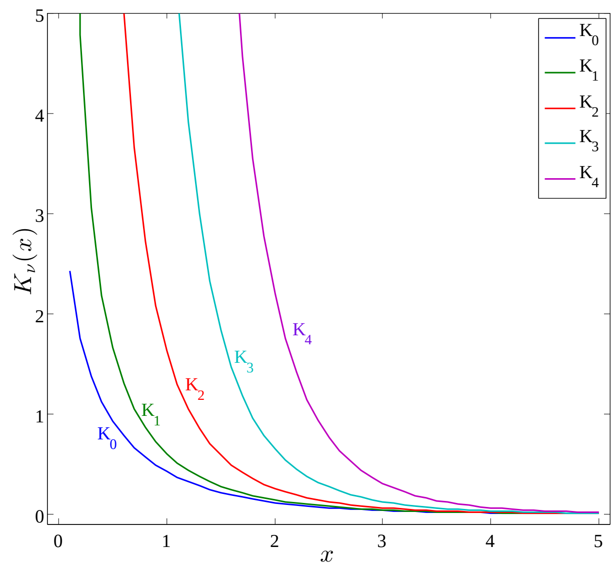
The $i^{-\nu}$ is there to make $I_\nu(x)$ a real function for real x and ν, likewise $K_\nu(x)$ is real for real x and ν.

Plots of $I_\nu(x)$ and $K_\nu(x)$ are shown on the next page.

The plot below shows the modified Bessel function of the first kind, $I_\nu(x)$.



The plot below shows the modified Bessel function of the second kind, $K_\nu(x)$.



Note that since $K_\nu(x)$ diverges at $x = 0$, any solution that includes $x = 0$ cannot have $K_\nu(x)$ in it.