Physics 412—Quantum Mice Uncertainty Principle Winter 2020

We will continue our studies of the properties of quantum mice, moving onto the uncertainty principle. The four properties of our quantum mice are attitude (operator A, eigenstates $|h\rangle$ and $|u\rangle$), behavior (operator B, eigenstates $|p\rangle$ and $|a\rangle$), energy (operator H, eigenstates $|4\rangle$ and $|2\rangle$), and size (operator W, eigenstates $|s\rangle$ and $|l\rangle$). If we use the attitude states as our basis, we can represent the operators as matrices and the states as column vectors as follows:

$$
\begin{aligned}
A &\leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, & |h\rangle &\leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & |u\rangle &\leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
B &\leftrightarrow \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, & |p\rangle &\leftrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}, & |a\rangle &\leftrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix}, \\
H &\leftrightarrow \frac{2}{5}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}, & |4\rangle &\leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, & |2\rangle &\leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \\
W &\leftrightarrow \frac{2}{5}\begin{bmatrix} 21 & -8 \\ -8 & 9 \end{bmatrix}, & |s\rangle &\leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, & |l\rangle &\leftrightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.
\end{aligned}
$$

Note that H and W share eigenstates because they commute. None of the other operators commute.

You can answer all of these working in the attitude basis, using the representation given above. It's probably easiest that way.

If two Hermitian operators A and B do not commute, then their commutator can be written as $[A, B] = \mathrm{i}C$, where C is Hermitian. The uncertainty principle states that

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{\langle C\rangle^2}{4}.$$

It may or may not be most illuminating (or possible!) to pick an eigenstate of one of the operators when answering these questions.

(1) The commutation relation between the attitude and behavior operators is given by

$$C \leftrightarrow \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}.$$

Find a state for which $\langle C\rangle = 0$, and calculate $(\Delta A)^2$ and $(\Delta B)^2$ for your state. Are either one equal to zero?

(2) Can you find a state for which $\langle C\rangle = 0$ and for which neither $(\Delta A)^2$ nor $(\Delta B)^2$ is equal to zero? Warning! I don't know if this is possible!!

(3) Find a state for which $\langle C\rangle \neq 0$ and calculate $(\Delta A)^2$ and $(\Delta B)^2$ for this state. You should find that the uncertainty inequality holds!

(4) The commutation relation between the attitude and energy operators is given by

$$C \leftrightarrow \frac{8}{5}\begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}.$$

Repeat questions (1) – (3) for the pair A, H as you see fit.

1. We want $|\psi\rangle \leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix}$ such that

$$\langle C\rangle = \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$= \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} -2b \\ -2a \end{bmatrix} = -2a^*b - 2b^*a = 0$$

$$\Rightarrow \boxed{a^*b = -b^*a}$$

Could pick $a = \frac{1}{2}$, $b = \frac{\sqrt{3}}{2}i$ $\Rightarrow \frac{1}{2}\left(\frac{\sqrt{3}}{2}i\right) = -\left(-\frac{\sqrt{3}}{2}i\right)\frac{1}{2}$

also $|a|^2 + |b|^2 = 1$, of course

$$|\psi_1\rangle \leftrightarrow \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3}i \end{bmatrix} \quad \text{or} \quad |\psi_2\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

or ...

$$A|\psi_1\rangle \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3}i \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{3}i \end{bmatrix}$$

$$\langle A\rangle = \langle\psi_1|A|\psi_1\rangle = \frac{1}{4}\begin{bmatrix} 1 & -\sqrt{3}i \end{bmatrix}\begin{bmatrix} 1 \\ -\sqrt{3}i \end{bmatrix} = \frac{1}{4}(1-3) = -\frac{1}{2}$$

$$\langle A^2\rangle = \langle\psi_1|AA|\psi_1\rangle = \frac{1}{4}\begin{bmatrix} 1 & \sqrt{3}i \end{bmatrix}\begin{bmatrix} 1 \\ -\sqrt{3}i \end{bmatrix} = \frac{1}{4}(1+3) = 1$$

$$\Delta A^2 = \langle A^2\rangle - \langle A\rangle^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$B|\psi_1\rangle \leftrightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3}i \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ i \end{bmatrix}$$

$$\langle B\rangle = \langle\psi_1|B|\psi_1\rangle = \frac{1}{4}\begin{bmatrix} 1 & -\sqrt{3}i \end{bmatrix}\begin{bmatrix} \sqrt{3} \\ i \end{bmatrix} = \frac{1}{4}\left(\sqrt{3}+\sqrt{3}\right) = \frac{\sqrt{3}}{2}$$

$$\langle B^2\rangle = \langle\psi_1|BB|\psi_1\rangle = \frac{1}{4}\begin{bmatrix} \sqrt{3} & -i \end{bmatrix}\begin{bmatrix} \sqrt{3} \\ i \end{bmatrix} = \frac{1}{4}\left(3+1\right) = 1$$

$$\Delta B^2 = \langle B^2\rangle - \langle B\rangle^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Delta A^2\,\Delta B^2 = \frac{3}{4}\cdot\frac{1}{4} = \frac{3}{16} > 0$$

(2) Yes! see above.

(3) Try $|\psi_3\rangle \leftrightarrow \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$

$$\langle\psi_3|C|\psi_3\rangle = \frac{1}{4}\begin{bmatrix} 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} -2\sqrt{3} \\ -2 \end{bmatrix}$$

$$= -\frac{1}{2}\left(\sqrt{3}+\sqrt{3}\right) = -\sqrt{3} \neq 0 \checkmark$$

$$A|\psi_3\rangle \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$$

$$B|\psi_3\rangle \leftrightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} -\sqrt{3}i \\ i \end{bmatrix}$$

$$\langle A \rangle = \frac{1}{4}\begin{bmatrix} 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} = \frac{1}{4}\left(1 - 3\right) = -\frac{1}{2}$$

$$\langle A^2 \rangle = \frac{1}{4}\begin{bmatrix} 1 & -\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} = \frac{1}{4}\left(1 + 3\right) = 1$$

$$\Delta A^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\langle B \rangle = \frac{1}{4}\begin{bmatrix} 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} -\sqrt{3}i \\ i \end{bmatrix} = \frac{1}{4}\left(-\sqrt{3}i + \sqrt{3}i\right) = 0$$

$$\langle B^2 \rangle = \frac{1}{4}\begin{bmatrix} +\sqrt{3}i & -i \end{bmatrix}\begin{bmatrix} -\sqrt{3}i \\ i \end{bmatrix} = \frac{1}{4}\left(3 + 1\right) = 1$$

$$\Delta B^2 = 1$$

$$\Delta A^2 \Delta B^2 = \frac{3}{4} \qquad \frac{\langle C \rangle^2}{4} = \frac{(-\sqrt{3})^2}{4} = \frac{3}{4}$$ phew!