PHY 420 Spring 2021

Homework 8 solutions

1. In class, we learned that a pure electric field in one frame cannot be a pure magnetic field in another frame. Prove this quantitatively.

Solution: From equation (11.149) in Jackson, we have

$$\begin{aligned} \vec{E}\,' &= \gamma\left(\vec{E} + \vec{\beta}\times\vec{B}\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right) \\ \vec{B}' &= \gamma\left(\vec{B} - \vec{\beta}\times\vec{E}\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right) \end{aligned} \tag{11.149}$$

In frame K, we have a *purely electric field*, so $\vec{B} = 0$. Therefore, equation (11.149) above changes to

$$\begin{aligned} \vec{E}\,' &= \gamma\,\vec{E} - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right) \\ \vec{B}' &= -\gamma\left(\vec{\beta}\times\vec{E}\right) \end{aligned} \tag{1}$$

In frame K', we need a purely magnetic field, so ***let us set*** $\vec{E}\,' = 0$. The first relation in equation (1) then becomes

$$0 = \gamma\,\vec{E} - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)$$

Canceling a γ and moving $\vec{E}$ to the left gives

$$\vec{E} = \frac{\gamma}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right) \tag{2}$$

We can then substitute equation (2) into the second relation in equation (1) to find an expression for $\vec{B}\,'$ in frame K':

$$\vec{B}' = -\gamma\left(\vec{\beta}\times\vec{E}\right) = -\gamma\left(\vec{\beta}\times\left\{\frac{\gamma}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)\right\}\right)$$

so that

$$\vec{B}' = -\frac{\gamma^2}{\gamma+1}\left(\vec{\beta}\cdot\vec{E}\right)\left(\vec{\beta}\times\vec{\beta}\right)$$

But the cross product of a vector with itself is zero, so $\vec{\beta}\times\vec{\beta} = 0$. Therefore

$$\vec{B}' = 0$$

Thus, we have demonstrated that if we set $\vec{E}' = 0$ in frame K', we will get $\vec{B}' = 0$ also. But this cannot be, since it would imply that even though there is a purely electric field in frame K, there are no fields in frame K'. We were led to this conclusion by assuming that $\vec{E}' = 0$, thus this assumption cannot be correct, and we must have a nonzero $\vec{E}'$ in frame K'.

Therefore, we have proved that if we have a purely electric field in one reference frame, when we transform to another inertial frame, we cannot have a purely magnetic field in that frame.

2. In class, we looked at the time dependence of the fields E_1, E_2, and B_3 (at a fixed observation point). An alternative is to look at the spatial distribution of the fields at a fixed instant in time (or, as Jackson puts it, "relative to the instantaneous present position of the charge in the laboratory"). Show that the electric field in terms of the present position of the charge is then given by

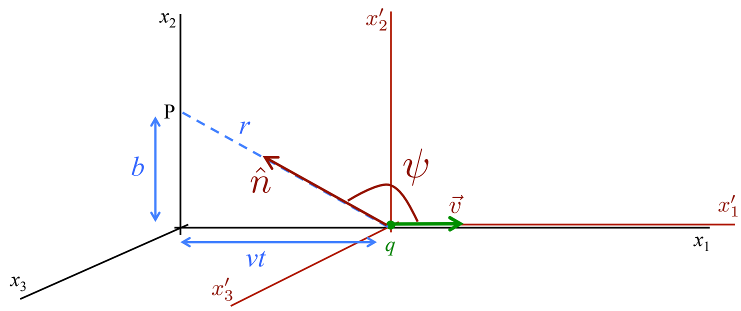
$$\vec{E} = \frac{q\vec{r}}{r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}}$$

where r is the radial distance from the present position of the charge to the observer (as shown in Figure 11.8 on page 559 in Jackson), and the angle $\psi = \cos^{-1}(\hat{n}\cdot\hat{v})$ is between the direction of $\hat{n}$ and $\vec{v}$, where $\hat{n}$ is a unit radial vector from the present position of the charge to the observation point (i.e., a unit vector along the direction of $\vec{r}$), and $\vec{v}$ is along the positive x_1-axis (see Figure 11.8 in Jackson).

Solution: From equation (11.152) in Jackson, we have

$$E_1 = -\frac{\gamma q v t}{(b^2+\gamma^2v^2t^2)^{3/2}} \qquad \text{and} \qquad E_2 = \frac{\gamma q b}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

We'll need Figure 11.8 from Jackson, so I'm including it below.



From the figure, we see that

$$b = r\sin(180^\circ - \psi) = r(\sin 180^\circ\,\cos\psi - \cos 180^\circ\sin\psi) = r\Big(0 - [-1]\sin\psi\Big) = r\sin\psi$$

where I've used the trigonometric formula sin (A−B) = sin A cos B − cos A sin B, and

$$vt = r\cos(180^\circ - \psi) = r(\cos 180^\circ\,\cos\psi + \sin 180^\circ\sin\psi) = r\Big([-1]\cos\psi + 0\Big) = -r\cos\psi$$

where I've used cos (A−B) = cos A cos B + sin A sin B. Then, we get

$$\begin{aligned}
b^2 + \gamma^2v^2t^2 &= \Big(r\sin\psi\Big)^2 + \gamma^2\Big(-r\cos\psi\Big)^2 \\
&= r^2\gamma^2\left[\frac{\sin^2\psi}{\gamma^2} + \cos^2\psi\right] \\
&= r^2\gamma^2\left[\frac{\sin^2\psi}{\gamma^2} + (1-\sin^2\psi)\right] \\
&= r^2\gamma^2\left[1 - \sin^2\psi\left(1-\frac{1}{\gamma^2}\right)\right]
\end{aligned}$$

On the next page, we will demonstrate that we can replace the term in parenthesis with β^2.

On the previous page, we derived that

$$b^2 + \gamma^2 v^2 t^2 = r^2\gamma^2 \left[1 - \sin^2\psi\,\left(1 - \frac{1}{\gamma^2}\right)\right] \tag{3}$$

Now, since

$$\gamma^2 = \left[\frac{1}{\sqrt{1-\beta^2}}\right]^2 = \frac{1}{(1-\beta^2)}$$

we have

$$1 - \beta^2 = \frac{1}{\gamma^2}$$

so that

$$\beta^2 = 1 - \frac{1}{\gamma^2}$$

Substituting this in equation (3), we get

$$b^2 + \gamma^2 v^2 t^2 = r^2\gamma^2 \Big[1 - \beta^2 \sin^2\psi\,\Big]$$

Substituting the equation above into equation (11.152) written on the previous page, we get

$$E_1 = -\frac{\gamma q(-r\cos\psi)}{\Big(r^2\gamma^2\Big[1-\beta^2\sin^2\psi\,\Big]\Big)^{3/2}} \qquad \text{and} \qquad E_2 = \frac{\gamma q(r\sin\,\psi)}{\Big(r^2\gamma^2\Big[1-\beta^2\sin^2\psi\,\Big]\Big)^{3/2}}$$

so that

$$E_1 = \frac{\gamma q r\cos\psi}{r^3\gamma^3\Big[1-\beta^2\sin^2\psi\,\Big]^{3/2}} \qquad \text{and} \qquad E_2 = \frac{\gamma q r\;\sin\,\psi}{r^3\gamma^3\Big[1-\beta^2\sin^2\psi\,\Big]^{3/2}}$$

and finally

$$E_1 = \frac{q r\cos\psi}{r^3\gamma^2\Big[1-\beta^2\sin^2\psi\,\Big]^{3/2}} \qquad \text{and} \qquad E_2 = \frac{q r\;\sin\,\psi}{r^3\gamma^2\Big[1-\beta^2\sin^2\psi\,\Big]^{3/2}} \tag{4}$$

Put directions on these so that we get $\vec{E}$:

$$\vec{E} = \hat{x_1}E_1 + \hat{x_2}E_2 + \hat{x}_3E_3$$

With $E_3 = 0$, we get from equation (4) that

$$\vec{E} = \frac{q\Big[\hat{x_1}(r\cos\psi) + \hat{x_2}(r\sin\psi)\Big]}{r^3\gamma^2\Big[1-\beta^2\sin^2\psi\,\Big]^{3/2}} \tag{5}$$

Now, notice that

$$\hat{x_1}(r\cos\psi) + \hat{x_2}(r\sin\psi) = \hat{x_1}\Big(-r\cos[180^\circ - \psi]\Big) + \hat{x_2}\Big(r\sin[180 - \psi]\Big) = \vec{r}$$

Therefore, equation (5) becomes

$$\vec{E} = \frac{q\vec{r}}{r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}}$$

which is the relation we are asked to derive.