(1a) $S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $|+z\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|-z\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$S_x|+z\rangle = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\hbar}{2}|-z\rangle$$

$$S_x|-z\rangle = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\hbar}{2}|+z\rangle$$

$$S_y|+z\rangle = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 \\ i \end{bmatrix} = \frac{i\hbar}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{i\hbar}{2}|-z\rangle$$

$$S_y|-z\rangle = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} -i \\ 0 \end{bmatrix} = \frac{-i\hbar}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{-i\hbar}{2}|+z\rangle$$

(B) $|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\langle S_x\rangle = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix} b \\ a \end{bmatrix} = \frac{\hbar^2}{4}(a^*b + b^*a)$$

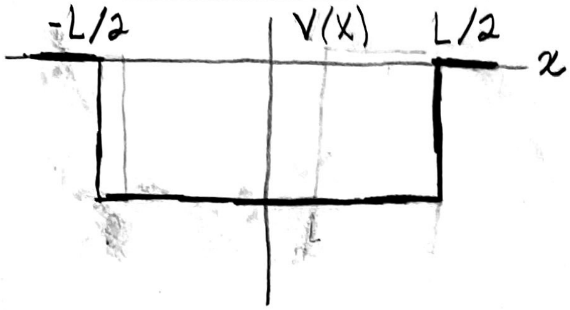
$$\langle S_y\rangle = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix} -ib \\ ia \end{bmatrix} = \frac{\hbar^2}{4}(-ia^*b + ib^*a)$$

$$\langle S_z\rangle = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ -b \end{bmatrix} = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix} a \\ -b \end{bmatrix} = \frac{\hbar^2}{4}(a^*a + b^*(-b)) = \frac{\hbar^2}{4}(|a|^2 + |b|^2)$$

$\begin{bmatrix} -1 \\ i \end{bmatrix}$ $\langle S_x\rangle = \frac{\hbar^2}{4}(-i+i) = 0$ $\langle S_y\rangle = \frac{\hbar^2}{4}(-1+(1)) = 0$ $\langle S_z\rangle = \frac{\hbar^2}{4}(1-1) = 0$

$$\boxed{|\psi\rangle = \begin{bmatrix} -1 \\ i \end{bmatrix}}$$

(2)



$$\Psi(x,0) = A\left[\sin\frac{4\pi x}{L} + i\cos\frac{\pi x}{L}\right]$$

$$\int_{-L/2}^{L/2} |\Psi(x,0)|^2\, dx = \int_{-L/2}^{L/2} A^2\, \Psi^*(x,0)\,\Psi(x,0)\, dx = 1$$

$$= A^2\int_{-L/2}^{L/2}\left[\sin\left(\frac{4\pi x}{L}\right) - i\cos\left(\frac{\pi x}{L}\right)\right]\left[\sin\left(\frac{4\pi x}{L}\right) + i\cos\left(\frac{\pi x}{L}\right)\right] = 1$$

$$\rightarrow A^2\int_{-L/2}^{L/2}\left[\sin^2\left(\frac{4\pi x}{L}\right) \underbrace{\cancel{- i\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{4\pi x}{L}\right) + i\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{4\pi x}{L}\right)}}_{0} + \cos^2\left(\frac{\pi x}{L}\right)\right] = 1$$

$$\rightarrow A^2\int_{-L/2}^{L/2}\left[\sin^2\left(\frac{4\pi x}{L}\right)\right]dx + \int_{-L/2}^{L/2}\left[\cos^2\left(\frac{\pi x}{L}\right)\right]dx = 1$$

$$\rightarrow A^2\left[\frac{L}{2}\right] + \left[\frac{L}{2}\right] = 1 \rightarrow A^2\,[L] = 1 \quad A^2 = 1/L \quad \boxed{A = 1/\sqrt{L}}$$

$$\boxed{\Psi(x,0) = \frac{1}{\sqrt{L}}\left[\sin\left(\frac{4\pi x}{L}\right) + i\cos\left(\frac{\pi x}{L}\right)\right]}$$

$$\Psi(x,t) = \sum_{n=1}^{\infty} C_n\, \psi_n(x)\, e^{-iE_n t} \rightarrow C_n = \int_{-L/2}^{L/2} \psi_n^*(x)\,\Psi(x,0)\, dx$$

$$C_1 = \int_{-L/2}^{L/2}\left(\sqrt{\frac{2}{L}}\cos\frac{\pi x}{L}\right)\frac{1}{\sqrt{L}}\left(i\cos\left(\frac{\pi x}{L}\right)\right)dx = \int_{-L/2}^{L/2}\frac{\sqrt{2}}{L}\, i\cos^2\left(\frac{\pi x}{L}\right)dx = \boxed{i/\sqrt{2}} = C_1$$

$$C_4 = \int_{-L/2}^{L/2}\left(\sqrt{\frac{2}{L}}\sin\frac{4\pi x}{L}\right)\frac{1}{\sqrt{L}}\left(\sin\left(\frac{4\pi x}{L}\right)\right)dx = \int_{-L/2}^{L/2}\frac{\sqrt{2}}{L}\sin^2\left(\frac{4\pi x}{L}\right)dx = \boxed{1/\sqrt{2}} = C_4$$

$$\boxed{\Psi(x,t) = \frac{i}{\sqrt{2L}}\cos\left(\frac{\pi x}{L}\right)e^{-it} + \frac{1}{\sqrt{2L}}\sin\left(\frac{4\pi x}{L}\right)e^{-i4t}}$$

$$P[L/4, L/2] = \int_{L/4}^{L/2} \left| -\frac{i}{\sqrt{2L}} \cos\left(\frac{\pi x}{L}\right) e^{-it} + \frac{1}{\sqrt{2L}} \sin\left(\frac{4\pi x}{L}\right) e^{-i4t} \right|^2 dx$$

$$= \int_{L/4}^{L/2} \left(+\frac{i}{\sqrt{2L}} \cos\left(\frac{\pi x}{L}\right) e^{it} + \frac{1}{\sqrt{2L}} \sin\left(\frac{4\pi x}{L}\right) e^{i4t} \right)$$

$$\cdot \left(-\frac{i}{\sqrt{2L}} \cos\left(\frac{\pi x}{L}\right) e^{-it} + \frac{1}{\sqrt{2L}} \sin\left(\frac{4\pi x}{L}\right) e^{-i4t} \right)$$

$e^{it} \cdot e^{-it} = 1$

$e^{i4t} \cdot e^{-i4t} = 1$

Cross terms will cancel

$$= \int_{L/4}^{L/2} \left[\frac{1}{\sqrt{2L}} \cos^2\left(\frac{\pi x}{L}\right) + \frac{1}{\sqrt{2L}} \sin\left(\frac{4\pi x}{L}\right) \right]$$

$$= \frac{1}{\sqrt{2L}} \left[\int_{L/4}^{L/2} \cos^2\left(\frac{x\pi}{L}\right) + \int_{L/4}^{L/2} \sin^2\left(\frac{4\pi x}{L}\right) \right]$$

$$= \frac{1}{\sqrt{2L}} \left[\frac{(\pi-2)L}{8\pi} + \frac{(2+\pi)L}{8\pi} \right] \rightarrow \frac{1}{\sqrt{2\cancel{L}}} \cdot \frac{\cancel{L}}{4} = \frac{1}{\sqrt{2}\,4} = 0.176$$

(3) $S(p_0) = e^{ip_0 X/\hbar} \rightarrow iP_0 X/\hbar + 1$

(a) $[a, S(p_0)] = a(iP_0 X/\hbar + 1) - (iP_0 X/\hbar + 1)a$

$= a\left(iP_0 \frac{d_0}{\sqrt{2}}(a+a^\dagger)/\hbar + 1\right) - \left(iP_0 \frac{d_0}{\sqrt{2}}(a+a^\dagger)/\hbar + 1\right)a$

$= aiP_0 \frac{d_0}{\sqrt{2}}(a+a^\dagger)/\hbar \cancel{+a} - iP_0 \frac{d_0}{\sqrt{2}}(a+a^\dagger)a/\hbar \cancel{+a}$

$= \frac{iP_0 d_0}{\sqrt{2}\hbar}(\cancel{aa} + aa^\dagger - \cancel{aa} + a^\dagger a) = \frac{iP_0 d_0}{\sqrt{2}\hbar}(aa^\dagger + a^\dagger a)$

$$\boxed{= \frac{iP_0 d_0}{\sqrt{2}\hbar}}$$

(B) $|\psi\rangle = S(p_0)|0\rangle$ $\quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} \quad |0\rangle = \frac{a^n}{\sqrt{n!}}|n\rangle$

$0|0\rangle + \frac{iP_0 d_0}{\sqrt{2}\hbar}|a^\dagger|0\rangle\rangle$

$|\psi\rangle = |0\rangle + \frac{iP_0 d_0}{\sqrt{2}\hbar}|1\rangle$

(c) Find $\langle X\rangle, \langle P\rangle$ $\quad X = \frac{d_0}{\sqrt{2}}(a+a^\dagger) \quad P = \frac{i\hbar}{\sqrt{2}d_0}(a^\dagger - a)$

all cross terms will go away such as $\cancel{\langle 0|0\rangle}^{0}$ $\cancel{\langle 1|1\rangle}^{0}$

$\langle X\rangle = \langle\psi|X|\psi\rangle =$

$\frac{d_0}{\sqrt{2}}\left[\langle 0|(a+a^\dagger)\frac{+iP_0 d_0}{\sqrt{2}\hbar}|1\rangle + \langle 1|\frac{-iP_0 d_0}{\sqrt{2}\hbar}(a+a^\dagger)|0\rangle\right] = 0$

$\langle X\rangle = 0$

$\langle P\rangle = \langle\psi|P|\psi\rangle$

$= \frac{i\hbar}{\sqrt{2}d_0}\left[\langle 0|(a-a^\dagger)\frac{iP_0 d_0}{\sqrt{2}\hbar}|1\rangle + \langle 1|\frac{-iP_0 d_0}{\sqrt{2}\hbar}(a-a^\dagger)|0\rangle\right]$

$= \frac{i\hbar}{\sqrt{2}d_0}\left[\left(\frac{iP_0 d_0}{\sqrt{2}\hbar}\right)\left(-\frac{iP_0 d_0}{\sqrt{2}\hbar}\right)\right] = \frac{i\hbar}{\sqrt{2}d_0}\left[-\frac{2iP_0 d_0}{\sqrt{2}\hbar}\right] = \boxed{P_0} = \langle P_0\rangle$

(3) $S(p_0) = e^{ip_0 X/\hbar}$ $\quad a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{P}{m\omega}\right)$ $\quad a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{P}{m\omega}\right)$

$\rightarrow \cos\left(\frac{p_0 X}{\hbar}\right) + i\sin\left(\frac{p_0 X}{\hbar}\right)$ $\qquad d_0 = \sqrt{m\omega/2\hbar}$

(a) $[a, S(p_0)] = a e^{ip_0 X/\hbar} - e^{ip_0 X/\hbar} a$

$d_0\left(X + i\frac{P}{m\omega}\right)e^{ip_0 X/\hbar} - e^{ip_0 X/\hbar} d_0\left(X - i\frac{P}{m\omega}\right)$

$= \cancel{d_0 X e^{ip_0 X/\hbar}} + i\frac{d_0 P}{m\omega}e^{ip_0 X/\hbar} - \left(\cancel{d_0 X e^{ip_0 X/\hbar}} - i\frac{P}{m\omega}e^{ip_0 X/\hbar} d_0\right)$

$= d_0 i\frac{P}{m\omega}e^{ip_0 X/\hbar} + d_0 i\frac{P}{m\omega}e^{ip_0 X/\hbar} = \boxed{\frac{2iP}{m\omega}e^{ip_0 X/\hbar} d_0}$

(B) $|\psi\rangle = S(p_0)|0\rangle$

$a|\psi\rangle = a e^{ip_0 X/\hbar} = 0$

(C) $|\psi\rangle = e^{ip_0 X/\hbar}|0\rangle$ $\qquad \frac{d_0}{\sqrt{2}}(a + a^+)$

$\langle X\rangle = \langle\psi|X|\psi\rangle = \langle 0|e^{-ip_0 X/\hbar}(a + a^+)e^{ip_0 X/\hbar}|0\rangle = 0$

$\langle p\rangle = \langle\psi|P|\psi\rangle = \langle 0|e^{-ip_0 X/\hbar}(a^+ - a)e^{ip_0 X/\hbar}|0\rangle = 0$

$\frac{-i\hbar}{\sqrt{2}d_0}$

$$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\langle S_x \rangle = \frac{\hbar^2}{4}\begin{bmatrix} a^* & b^* \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix} a^* & b^* \end{bmatrix}\begin{bmatrix} b \\ a \end{bmatrix} = \frac{\hbar^2}{4}(a^*b + b^*a)$$

$$\langle S_y \rangle = \frac{\hbar^2}{4}\begin{bmatrix} a^* & b^* \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix} a^* & b^* \end{bmatrix}\begin{bmatrix} -ib \\ ia \end{bmatrix} = \frac{\hbar^2}{4}(-ia^*b + ib^*a)$$

$$\langle S_z \rangle = \frac{\hbar^2}{4}\begin{bmatrix} a^* & b^* \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix} a^* & b^* \end{bmatrix}\begin{bmatrix} a \\ -b \end{bmatrix} = \frac{\hbar^2}{4}(a^*a + b^*(-b))$$

$$= \frac{\hbar^2}{4}(|a|^2 + |b|^2)$$

$\begin{bmatrix} -i \\ i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(-1-1)$ $\langle S_y \rangle = \frac{\hbar^2}{4}($

$\begin{bmatrix} 1 \\ i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(i-i) = 0$ $\langle S_y \rangle = \frac{\hbar^2}{4}(1-1)$ $\langle S_z \rangle = \frac{\hbar^2}{4}(1+1)$

$\begin{bmatrix} -1 \\ i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(-i+i) = 0$ $\langle S_y \rangle = \frac{\hbar^2}{4}(1+(-1)) = 0$ $\langle S_z \rangle = \frac{\hbar^2}{4}(+1+1) = 2\frac{\hbar^2}{4}$

$\begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(-i+i) = 0$ $\langle S_y \rangle = \frac{\hbar^2}{4}(-1-1)$

$\begin{bmatrix} -1 \\ -i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(i-i) = 0$ $\langle S_y \rangle = \frac{\hbar^2}{4}(1+1)$

$\begin{bmatrix} -1 \\ i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(-i+i) = 0$ $\langle S_y \rangle = \frac{\hbar^2}{4}(1+(-1)) = 0$ $\langle S_z \rangle = \frac{\hbar^2}{4}(1-1) = 0$

$$|\psi\rangle = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

(3) (a) $[a, S(p_0)]$

$$S(P_0) = e^{ip_0 X/\hbar} \to \cos\left(\frac{P_0 X}{\hbar}\right) + i\sin\left(\frac{P_0 X}{\hbar}\right)$$

$$= a\left[\cos\left(\frac{P_0 X}{\hbar}\right) + i\sin\left(\frac{P_0 X}{\hbar}\right)\right] + \left[\cos\left(\frac{P_0 X}{\hbar}\right) + i\sin\left(\frac{P_0 X}{\hbar}\right)\right]a$$

$$= a\cos\left(\frac{P_0 X}{\hbar}\right) + ai\sin\left(\frac{P_0 X}{\hbar}\right) + \cos\left(\frac{P_0 X}{\hbar}\right)a + i\sin\left(\frac{P_0 X}{\hbar}\right)a$$

$$e^{ip_0 d_0/\sqrt{2}\,(a+a^\dagger)}$$

$$H(X,P) = \frac{P^2}{2m} + \frac{kX^2}{2}$$

$$H = \hbar\omega(aa^\dagger - I/2)$$

$$[a, a^\dagger] = I$$