PHY 420 Spring 2021

Class Summary—Week 9, Day 1—Tuesday, May 25

In the previous class, we wrote several important equations of electrodynamics in **covariant** form. By covariance, we mean invariance in form, meaning that the form of the equations does not change when we transform from one (inertial) frame to another.

The key to writing electrodynamics equations in covariant form is to write all the quantities as 4-vectors. For example, by putting together the charge density ρ and the current density $\vec{J}$ into a 4-vector J^α, given by

$$J^\alpha = (c\rho, \vec{J}) \tag{11.128}$$

we wrote the **continuity equation** $\dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$ in the covariant form

$$\partial_\alpha J^\alpha = 0 \tag{11.129}$$

where we are using the notation from equation (11.76) that $\quad \partial_\alpha \equiv \dfrac{\partial}{\partial x^\alpha} = \left(\dfrac{\partial}{\partial x^0}, \vec{\nabla} \right)$

We will now put the **Lorentz force equation** into covariant form. For a particle of charge q, the Lorentz force is given by

$$\frac{d\vec{p}}{dt} = q \left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right) \tag{11.124}$$

Recall that we wrote in an earlier class the 4-momentum (or energy-momentum 4-vector):

$$p^\alpha = (p_0, \vec{p}) = (E/c, \vec{p}) = m\,(U_0, \vec{U}) = m\,(\gamma_u c, \gamma_u \vec{u})$$

where $\gamma_u = (1 - u^2/c^2)^{-1/2}$. Also recall that $dt = \gamma_u\, d\tau$, where $d\tau$ is the proper time.

The space component $\vec{p}$ of the 4-momentum p^α then takes the form

$$\frac{d\vec{p}}{d\tau} = \frac{q}{c} \left(U_0\, E + \vec{U} \times \vec{B} \right) \tag{11.125}$$

as you showed in Question 1 on today's worksheet. It is worth repeating here; first, use the chain rule to write

$$\frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt} \frac{dt}{d\tau}$$

Now, $dt = \gamma_u\, d\tau$, where $\gamma_u = (1 - u^2/c^2)^{-1/2}$, as written above. Thus, we get

$$\frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt} \Big(\gamma_u \Big) = q \left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right) \gamma_u$$

where I've substituted equation (11.124) for $d\vec{p}/dt$. Then, after pulling $1/c$ outside and pushing γ_u inside the parentheses, we get

$$\frac{d\vec{p}}{d\tau} = \frac{q}{c} \left(\gamma_u c\, \vec{E} + \gamma_u \vec{u} \times \vec{B} \right)$$

But $\gamma_u\, c = U_0$, and $\gamma_u \vec{u} = \vec{U}$, so that above expression gives finally equation (11.125):

$$\frac{d\vec{p}}{d\tau} = \frac{q}{c} \left(U_0\, E + \vec{U} \times \vec{B} \right)$$

You also showed in Question 2(a) on today's worksheet that the corresponding time component of the 4-momentum is just the time rate of change of energy of the particle, which can be written as

$$\frac{dp_0}{d\tau} = \frac{q}{c}\,\vec{U}\cdot\vec{E} \tag{11.126}$$

The **Lorentz force equation** then takes the **covariant form**

$$\frac{dp^\alpha}{d\tau} = m\,\frac{dU^\alpha}{d\tau} = \frac{q}{c}\,F^{\alpha\beta}\,U_\beta \tag{11.144}$$

as you showed in Question 2(b) on today's worksheet explicitly for $\alpha = 0$. Let's write this out explicitly below, given its importance.

For $\alpha = 0$, starting from the right hand side of equation (11.144), we get

$$\frac{q}{c}\,F^{\alpha\beta}\,U_\beta = \frac{q}{c}\left[F^{00}\,U_0 + F^{01}\,U_1 + F^{02}\,U_2 + F^{03}\,U_3\right]$$

where I've written out explicitly the four terms remembering that the Einstein summation convention applies to the summation over the index β here.

Now, if you examine the field-strength tensor, you'll find that

$$F^{00} = 0, F^{01} = -E_1, F^{02} = -E_2, F^{03} = -E_3$$

Substituting these in the expression above, we get

$$\frac{q}{c}\,F^{\alpha\beta}\,U_\beta = \frac{q}{c}\left[(0)\,U_0 + (-E_1)\,U_1 + (-E_2)\,U_2 + (-E_3)\,U_3\right]$$

This is almost the dot product in equation (11.126), except that we need to get rid of a bunch of minus signs. We can do so by writing the covariant components as contravariant, since

$$E^1 = -E_1, \qquad E^2 = -E_2, \qquad E^3 = -E_3$$

and thus

$$\frac{q}{c}\,F^{\alpha\beta}\,U_\beta = \frac{q}{c}\left[E^1\,U_1 + E^2\,U_2 + E^3\,U_3\right] = \frac{q}{c}\left[U_1\,E^1 + U_2\,E^2 + U_3\,E^3\right] = \frac{q}{c}\,\vec{U}\cdot\vec{E}$$

where I've swapped the order of $E^i U_i$ to make it visually discernible as the dot product. Then, using equation (11.126), we get

$$\frac{q}{c}\,F^{\alpha\beta}\,U_\beta = \frac{q}{c}\,\vec{U}\cdot\vec{E} = \frac{dp_0}{d\tau} = \frac{dp^0}{d\tau}$$

where I've replaced p_0 with p^0, since the time-component of the covariant form does not change sign when changed into the contravariant form.

You should try $\alpha = 1, 2, 3$ on your own using the result in equation (11.125); they work very similarly to what you did for $\alpha = 0$ in Question 2(b) on today's worksheet, except that you'll need to be careful with the minus signs when changing contravariant to covariant forms, or vice versa.

Transformation of Electromagnetic Fields

We will discuss how electromagnetic fields transform between frames.

Since $\vec{E}$ and $\vec{B}$ are the elements of a second-rank tensor $F^{\alpha\beta}$, their values in one inertial frame K' can be expressed in terms of the values in another inertial frame K according to

$$F'^{\,\alpha\beta} = \frac{\partial x'^{\,\alpha}}{\partial x^\gamma}\,\frac{\partial x'^{\,\beta}}{\partial x^\delta}\,F^{\gamma\delta} \tag{11.146}$$

In the matrix notation that we have adopted (taken from Section 11.7 of Jackson), this can be written as

$$F' = AF\tilde{A} \tag{11.147}$$

where F and F' are 4×4 matrices as written in equation (11.137), with F given by

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

and F' looks the same but with all elements replaced by primed elements. Meanwhile, A is the Lorentz transformation matrix of equation (11.93):

$$A = e^{-\vec{\omega}\cdot\vec{S} - \vec{\zeta}\cdot\vec{K}}$$

where $\vec{\omega}$ and $\vec{\zeta}$ are constant 3-vectors, their three components (each) corresponding to the six parameters of the transformation (three rotations and three Lorentz boosts). Recall that we wrote the six fundamental matrices S_1, S_2, S_3, and K_1, K_2, K_3 in equation (11.91), although I won't write them again here because we won't need those specific forms here.

Jackson notes that the subscripts $1, 2, 3$ in $F^{\alpha\beta}$ above, and in the discussion to follow, represent ordinary Cartesian spatial components and not covariant indices. Remember, however, that according to his treatment in Section 11.3, he set x_1 equal to z, but in writing the elements of $F^{\alpha\beta}$ above, I've written E_x as E_1. None of this should matter, as long as you keep using x_0, x_1, x_2, x_3 and pay attention to the fact that the boost is along, e.g., the x_1 axis, which is the case we will begin discussing on the next page.

We will now focus on a particular, and simple, situation in which there is a **Lorentz boost along the x_1 axis (and no rotation)**, so that $\vec{\omega} = 0$, and $\vec{\zeta} = \zeta\,\hat{e}_1$. We have already shown in a previous class that the matrix A then takes the form given in equation (11.95):

$$A = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To proceed, it is convenient to replace these with γ, β and/or $\gamma\beta$, *as you did in Question 3 on today's worksheet*, where from equation (11.20), we have

$$\beta = \tanh\zeta, \qquad \gamma = \cosh\zeta, \qquad \gamma\beta = \sinh\zeta$$

For the specific Lorentz transformation that results in the matrix A written at the bottom of the previous page, taken from equation (11.95), corresponding to a boost along the x^1 axis with speed $c\beta$ from the unprimed frame to the primed frame, the explicit **equations of transformation** are

$$\begin{aligned} E_1' &= E_1 & B_1' &= B_1 \\ E_2' &= \gamma\left(E_2 - \beta B_3\right) & B_2' &= \gamma\left(B_2 + \beta E_3\right) \\ E_3' &= \gamma\left(E_3 + \beta B_2\right) & B_3' &= \gamma\left(B_3 - \beta E_2\right) \end{aligned} \tag{11.148}$$

To derive these relations, you obtained first *in Question 4 on today's worksheet* the product $AF\tilde{A}$ by explicit matrix multiplication. You then wrote $F' = AF\tilde{A}$ *in Question 5 on today's worksheet*, and used it to demonstrate the transformation equations for E_1' and E_2'. We will verify transformations for $E_3', B_1', B_2', and B_3'$ in the next class.