PHY 420 Spring 2021

Week 7—Thursday, May 13—Discussion Worksheet

Tensors (continued)

In the previous class, we wrote the **dot product** as the product of the components of a covariant and a contravariant vector:

$$B \cdot A \equiv B_\alpha A^\alpha \tag{11.66}$$

where we've used the **Einstein summation convention** over repeated indices, that is

$$B \cdot A \equiv B_\alpha A^\alpha = B_0 A^0 + B_1 A^1 + B_2 A^2 + B_3 A^3$$

Note that this is a general relation for a 4-dimensional vector, and we will need to write it differently for it to conform to the way the invariant is structured in Special Relativity. First though, it is worth reminding ourselves that in equation (11.66), A is a **contravariant vector** defined by the transformation

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta \tag{11.61}$$

and B is a **covariant vector** defined by

$$B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta \tag{11.62}$$

where, again, the Einstein summation over repeated indices is implied in both (11.61) and (11.62).

Now, consider the transformation

$$x'^\alpha = x'^\alpha \left(x^0, x^1, x^2, x^3\right) \tag{11.60}$$

where $\alpha = 0, 1, 2, 3$.

1. Show explicitly that the scalar product is an invariant under the transformation defined in equation (11.60). In other words, show explicitly that $B' \cdot A' = B \cdot A$.

$B \cdot A = B_\alpha A^\alpha$

$B' \cdot A' = \left[\frac{\partial x^\beta}{\partial x'^\alpha} B_\beta\right]\left[\frac{\partial x'^\alpha}{\partial x^\gamma} A^\gamma\right] = \left(\frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^\gamma}\right) B_\beta A^\gamma$

Thus, $B' \cdot A' = \left(\frac{\partial x^\beta}{\partial x^\gamma}\right) B_\beta A^\gamma$, and $\left.\begin{array}{l}\frac{\partial x^\beta}{\partial x^\gamma} = 1, \text{ if } \beta = \gamma \\ \quad = 0, \text{ if } \beta = 0\end{array}\right\} \delta_{\beta\gamma}$

$\downarrow$

$= \partial_{\beta\gamma} B_\beta A^\gamma$

$= B_\gamma A^\gamma = B \cdot A \Rightarrow B' \cdot A' = B \cdot A$

The Metric Tensor

The results or definitions that we have learned so far are general, and could apply to any tensor. We will now transition to considering the specific geometry of the space-time of special relativity, for which the **invariant** (or **norm**), in differential form, is

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{11.67}$$

Take care to differentiate between superscripted indices and powers in equation (11.67).

2. If we write ds in terms of contravariant components like dx^α, then we'll need something to give us the correct signs, so that we have a plus sign when we're contracting out the components involving time in the differential (dx^0), and minus signs when we're contracting out the components of the spatial part of the differential (dx^1, dx^2, dx^3). This is done by way of a special tensor called the **metric tensor** $g_{\alpha\beta}$ so that

$$(ds)^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{11.68}$$

(a) By inspection of equation (11.67) and (11.68), write down the **elements of the metric tensor**.

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} = \underline{1} & g_{01} = \underline{0} & g_{02} = \underline{0} & g_{03} = \underline{0} \\ g_{10} = \underline{0} & g_{11} = \underline{-1} & g_{12} = \underline{0} & g_{13} = \underline{0} \\ g_{20} = \underline{0} & g_{21} = \underline{0} & g_{22} = \underline{-1} & g_{23} = \underline{0} \\ g_{30} = \underline{0} & g_{31} = \underline{0} & g_{32} = \underline{0} & g_{33} = \underline{-1} \end{pmatrix}$$

(b) Is the metric tensor **symmetric**, i.e., is $g_{\alpha\beta} = g_{\beta\alpha}$?

yes, Because everything is on the diagonal

Next, consider the definition of the invariant scalar product in equation (11.66). We have that

$$B \cdot A = B_\alpha A^\alpha$$

where, of course, Einsteinian summation over the repeated index α is assumed. We should be able to write ds^2 like this, as a scalar product; that is, we should be able to write $ds^2 = dx_\alpha dx^\alpha$, except that we have to get the correct signs. But the metric tensor does just that, so we can write

$$(ds)^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta = dx_\alpha dx^\alpha$$

By inspection (and dropping the differentials for brevity), it is easy to see that the covariant coordinate 4-vector x_α can be obtained from the contravariant x^β by contraction with the metric tensor $g_{\alpha\beta}$:

$$x_\alpha = g_{\alpha\beta}\, x^\beta \tag{11.72}$$

We see that its inverse is

$$x^\alpha = g^{\alpha\beta}\, x_\beta \tag{11.73}$$

where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$.

3. Since $g_{\alpha\beta}$ is symmetric and diagonal, it is equal to its inverse, and so the ***contravariant and covariant tensors are the same for flat space-time***:

$$g^{\alpha\beta} = g_{\alpha\beta} \tag{11.70}$$

(a) Show that by doing the contraction of the contravariant and covariant metric tensors, we get the **Kronecker** δ in four dimensions:

$$g_{\alpha\gamma}\, g^{\gamma\beta} = \delta_\alpha^\beta \tag{11.71}$$

where $\delta_\alpha^\beta = 0$ for $\alpha \neq \beta$, and $\delta_\alpha^\alpha = 1$ for $\alpha = 0, 1, 2, 3$.

$$\left.\begin{array}{l} X_\alpha = g_{\alpha\gamma} X^\gamma \\ X^\gamma = g^{\gamma\beta} X_\beta \end{array}\right\} X_\alpha X^\gamma = \underbrace{g_{\alpha\gamma} g^{\gamma\beta}}_{= \delta_\alpha^\beta} X^\gamma X_\beta$$

$= \delta_\alpha^\beta$, in order for left hand side to be equal to right hand side.

Equations (11.72) and (11.73) on the previous page demonstrate that contraction with the metric tensor $g_{\alpha\beta}$ or $g^{\alpha\beta}$ is the procedure for raising or lowering indices to convert covariant to contravariant, and vice versa. Thus

$$F^{\dots\alpha\dots}_{\dots} = g^{\alpha\beta}\, F^{\dots}_{\dots\beta} \qquad \text{and} \qquad G^{\dots}_{\dots\alpha\dots} = g_{\alpha\beta}\, G^{\dots\beta}_{\dots} \tag{11.74}$$

(b) We are now able to see how covariant and contravariant vectors are related. If a contravariant 4-vector has components A^0, A^1, A^2, A^3, then show that its covariant partner has components

$$A_0 = A^0, \qquad A_1 = -A^1, \qquad A_2 = -A^2, \qquad A_3 = -A^3$$

Use

$$X_\alpha = g_{\alpha\beta} X^\beta \Rightarrow A_0 = g_{00} A^0 + g_{01} A^1 + g_{02} A^2 + g_{03} A^3$$

$$\Downarrow \qquad \Rightarrow A_{00} = g_{00} A^0 = (1) A^0 \Rightarrow A_0 = A^0$$

$$A_1 = \cancel{g_{10}}^{\,0} A^0 + g_{11} A^1 + \cancel{g_{12}}^{\,0} A^2 + \cancel{g_{13}}^{\,0} A^3 \Rightarrow A_1\ g_{11} A^1 = (-1) A^1$$

$$A_1 = -A^1, \quad A_2 = -A^2, \quad A_3 = -A^3$$

We can write this result concisely as

$$A^\alpha = (A^0, \vec{A}), \qquad A_\alpha = (A_0, -\vec{A}) \tag{11.75}$$

where the 3-vector $\vec{A}$ has components A^1, A^2, A^3. Therefore, ***covariant vectors are just spatially inverted contravariant vectors***.

4. Use the definition in equation (11.75), and additional appropriate information, to show that

$$B \cdot A \equiv B_\alpha A^\alpha = B^0 A^0 - \vec{B} \cdot \vec{A}$$

$$B \cdot A = B_\alpha A^\alpha = B_0 A^0 + B_1 A^1 + B_2 A^2 + B_3 A^3$$

$$\downarrow B_0 = B^0 \quad \downarrow B_1 = -B^1$$

$$= B_0 A^0 + (-B^1) A^1 + (-B^2) A^2 + (-B^3) A^3$$

$$B \cdot A = B^0 A^0 - \vec{B} \cdot \vec{A}$$

Now, while the above discussion tells us how ordinary quantities transform, we are also interested in how the partial derivative operators with respect to x^α and x_α transform, as they are involved in the construction of dynamical systems in Special Relativity. The transformation properties of these operators can be established directly by using the rules of implicit differentiation. For example, since

$$\frac{\partial \psi}{\partial x'^\alpha} = \frac{\partial \psi}{\partial x^\beta} \frac{\partial x^\beta}{\partial x'^\alpha} \qquad \text{we get} \qquad \frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta}$$

Compare with the definition of a covariant vector in equation (11.62): $B'_\alpha = \dfrac{\partial x^\beta}{\partial x'^\alpha} B_\beta$. Thus:

differentiation with respect to contravariant coordinates forms a covariant 4-vector, and conversely:

differentiation with respect to covariant coordinates forms a contravariant 4-vector.

Given these two results, we introduce the following simplifying notation:

$$\begin{aligned} \partial_\alpha &\equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right) \\ \partial^\alpha &\equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla} \right) \end{aligned} \tag{11.76}$$

Now let's write down the **4-divergence** of a 4-vector A:

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A} \tag{11.77}$$

This is similar in form to the continuity equation of charge and current density, the Lorenz condition on the scalar and vector potentials, etc. Meanwhile, the **4-dimensional Laplacian operator** is defined to be the invariant contraction

$$\Box \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{0 2}} - \nabla^2 \tag{11.78}$$

which is just (the negative of) the operator of the wave equation in vacuum! We see that many important quantities in electrodynamics are Lorentz scalars.

Matrix Representation of Lorentz Transformations

We will now introduce a ***matrix representation***. In this representation, the components of a contravariant 4-vector will form the elements of a column vector.

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{11.79}$$

Keep an eye on the notation — we don't use an arrow (or boldface font in textbooks) to indicate a 4-vector, those are reserved for the spatial part of the 4-vector only (i.e., for the 3-vector).

Matrix scalar products of 4-vectors (a, b) are defined in the usual way by summing over the products of the elements of a and b, or equivalently, by the matrix multiplication:

$$(a, b) \equiv \tilde{a}b \tag{11.80}$$

where $\tilde{a}$ is the **transpose** of a (and hence a row vector).

Now, you found in Question 2 (a) that in the flat space-time of special relativity, the metric tensor $g_{\alpha\beta}$ is diagonal, and can be represented by the square 4×4 matrix

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{11.81}$$

with $g^2 = I$, where I is the 4×4 unit matrix.

5. Knowing that the covariant 4-vector x_α can be obtained from the contravariant x^β by contraction with the metric tensor $g_{\alpha\beta}$, as given in equation (11.72): $x_\alpha = g_{\alpha\beta}\, x^\beta$, show using equation (11.79) that the covariant vector is

$$X = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{11.82}$$

$$gX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} = \begin{pmatrix} X^0 \\ -X^1 \\ -X^2 \\ -X^3 \end{pmatrix} \rightarrow \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$X_0 = +X^0$

$X_1 = -X^1$

$X_2 = -X^2$

$X_3 = -X^3$

In the compact notation we have introduced, the scalar product in equation (11.66) of two 4-vectors is defined to be

$$a \cdot b = (a, gb) = (ga, b) = \tilde{a}gb \tag{11.83}$$

or, if we write out the elements explicitly

$$a \cdot b = \tilde{a}gb = a^\alpha g_{\alpha\beta} b^\beta = a^\alpha b_\alpha$$

Now, let us get to the purpose of introducing everything that we have dealt with so far: We seek a group of linear transformations that leaves $(x, gx) = x \cdot x$ invariant. Since (x, gx) or $x \cdot x$ is the norm of a 4-vector, we are effectively seeking a group of transformations that preserves the "length" in the 4-dimensional metric. In other words, we seek all square 4×4 matrices A on the coordinates

$$x' = Ax \tag{11.84}$$

that leave the norm (x, gx) invariant:

$$x' \cdot x' = \tilde{x}'gx' = \tilde{x}gx = x \cdot x \tag{11.85}$$

that is, they leave $x \cdot x = x' \cdot x'$ invariant, which after all is a mathematical statement of the postulate of Special Relativity.

6. From equation (11.84), we also get for the transpose that $\tilde{x}' = \tilde{x}\tilde{A}$.

(a) Show that A must satisfy the matrix equation

$$\tilde{A}gA = g \tag{11.86}$$

$$\tilde{A}xgAx = \tilde{x}gx$$

$$\rightarrow \tilde{x}\vec{A}gAx = \tilde{x}gx$$

$$\rightarrow \tilde{A}gA = g$$

(b) Show that

$$\det A = \pm 1$$

$$\det(\tilde{A}gA)$$

$$= \det(\tilde{A})\det(g)\det(A)$$

$$= \det(g) = -1$$

$$(\det(A))^2 = \pm 1$$

$$\det(A) = \pm 1$$

In the next class, we will investigate what kinds of transformations are allowed under the condition $\det A = \pm 1$.