

## Class Summary—Week 2, Day 1—Monday, April 5

**The Physics of Stars**

Stars are spheres of gas, but handling equations in gas can get quite complicated. An overview is given in **Section 3.1** in *Dalgaard* (page 39). The key point here is to understand what is meant by **thermodynamic equilibrium**.

On *Question 1 of today's worksheet*, you discussed three of the four expressions on which thermodynamic equilibrium is based. The existence of thermodynamic equilibrium depends on the temperatures in the following distributions being the same.

- First, we have the **Maxwellian distribution of speeds**

$$f(v) = 4\pi \left( \frac{m}{2\pi k T_{\text{kin}}} \right)^{3/2} v^2 \exp \left( -\frac{mv^2}{2k T_{\text{kin}}} \right)$$

where  $m$  is the mass of the particles, and  $f(v) dv$  gives the probability of finding particles with speeds between  $v$  and  $(v + dv)$ . The temperature that characterizes the Maxwellian distribution of speeds is known as the **kinetic temperature**,  $T_{\text{kin}}$ .

- Next we have the **Boltzmann distribution**

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp \left( -\frac{E_2 - E_1}{k T_{\text{ex}}} \right)$$

which, in the form above, gives the number of particles in the upper and lower states of a two-level system;  $g_1$  and  $g_2$  are called the statistical weights. Essentially, this temperature describes the distribution of internal energies in the particles in a system. This internal energy can be the energy of different electronic states of an atom, or the energy of rotation or vibration in a molecule. The temperature that specifies this distribution is known as the **excitation temperature**,  $T_{\text{ex}}$ .

- Then, we have the **Planck distribution**

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT_{\text{rad}}) - 1}$$

which characterizes the black body radiation spectrum of an object at the **radiation temperature**,  $T_{\text{rad}}$ .

There is also a fourth (**Saha ionization formula**), which we will get to later, and which specifies the **ionization temperature**.

## Ideal Gas

The starting point in learning about the physics of stellar interiors is the **Ideal Gas Law**. Go over all of **Section 3.2.1** of *Dalgaard* (**pages 40-43**) carefully.

In particular, *as you showed in Question 2(a) on today's worksheet*, one can start from the Ideal Gas Law,  $PV = NkT$ , where  $N$  is the number of particles, and show that

$$P = \frac{\rho kT}{\mu m_p} \quad (3.6)$$

where  $\rho$  is the (mass) density, and  $\mu$  is the (dimensionless) atomic mass. Note that I'll be writing  $k$  for the Boltzmann constant instead of  $k_B$  (in *Dalgaard*). I'll also use the proton mass  $m_p$  for  $m_u$ ; since  $1 m_u$  (or amu) =  $1/12$  mass of  $^{12}\text{C}$ , it is equal to  $m_p$  for all practical purposes.

For a monatomic ideal gas (what your text means by no internal degrees of freedom) for which the mean internal energy per particle is  $\frac{3}{2} kT$ , and there are  $n$  particles per unit volume (i.e., the particle density is  $n$ ), we get that

$$u = n \left[ \frac{3}{2} kT \right]$$

and since  $nkT = P$ , the pressure, *you showed in Question 2(b) of today's worksheet* that the **internal energy per unit volume** is

$$u = \frac{3}{2} P \quad (3.7)$$

Now, the physics of stellar interiors is governed by changes in gas properties, and is described by the **first law of thermodynamics**:

$$dQ = dU + PdV$$

which says that the added energy  $dQ$  (as heat) goes partly into changing the internal energy  $U$  of the gas, and partly into work  $PdV$  to change the volume of the gas.

If we define quantities per unit mass, so that  $U = u/\rho$ , we get from equation (3.7) that

$$U = \frac{u}{\rho} = \frac{1}{\rho} \left( \frac{3}{2} P \right)$$

Then, replacing  $P$  with equation (3.6), this becomes

$$U = \frac{1}{\rho} \left( \frac{3}{2} \frac{\rho kT}{\mu m_p} \right) = \frac{3}{2} \frac{kT}{\mu m_p} \quad (3.10)$$

Now, from the first law of thermodynamics, we get at constant volume ( $dV = 0$ ) that

$$dQ = dU + P(0) = d \left[ \frac{3}{2} \frac{kT}{\mu m_p} \right] = \left( \frac{3}{2} \frac{k}{\mu m_p} \right) dT$$

Thus, since specific heat  $c_V$  is defined from  $dQ = c_V dT$ , we get that

$$c_V = \frac{3}{2} \frac{k}{\mu m_p}$$

*as you demonstrated on Question 3(a) of today's worksheet*, where  $c_V$  is the **specific heat at constant volume**, the **amount of heat that has to be added per unit mass to raise the temperature of the gas by one degree**.

For an **adiabatic process**, one that occurs without any transfer of heat so that  $dQ = 0$ , we will now derive an important relation for  $dP/P$ .

Start from the ideal gas law  $PV = NkT$ , so that using equation (3.6) we get

$$P = \left(\frac{N}{V}\right) kT = \frac{\rho kT}{\mu m_p}$$

Thus

$$\frac{N}{V} = \frac{\rho}{\mu m_p} \quad \text{so that} \quad N = \frac{1}{\mu m_p}$$

since we have defined quantities per unit volume, so that  $V = 1/\rho$ .

Again, from the ideal gas law  $PV = NkT$ , we get

$$PdV + VdP = Nk(dT) = \left(\frac{k}{\mu m_p}\right) dT \quad (3.12)$$

Use this to replace  $PdV$  in the expression

$$dQ = dU + PdV = dU - VdP + \left(\frac{k}{\mu m_p}\right) dT$$

and then replacing  $U$  from equation (3.10), we get

$$dQ = \frac{3}{2} \left(\frac{k}{\mu m_p}\right) dT - VdP + \left(\frac{k}{\mu m_p}\right) dT = \frac{5}{2} \left(\frac{k}{\mu m_p}\right) dT - VdP$$

from which we obtain that  $c_P$ , the specific heat at constant pressure (i.e.,  $dP = 0$ ), is given by

$$c_P = \left(\frac{dQ}{dT}\right)_P = \frac{5}{2} \left(\frac{k}{\mu m_p}\right)$$

Also, for adiabatic processes,  $dQ = 0$ , so the first law of thermodynamics gives  $0 = dU + PdV$ , so that putting  $dU = c_V dT$ , we get

$$c_V dT = -PdV$$

and replacing  $dT$  from equation (3.12), we get

$$c_V \left[ \frac{PdV + VdP}{k/\mu m_p} \right] = -PdV$$

Dividing by  $PV$  on both sides, and moving  $k/\mu m_p$  to the right hand side, we get

$$c_V \left[ \frac{dV}{V} + \frac{dP}{P} \right] = - \left(\frac{k}{\mu m_p}\right) \frac{dV}{V} = \left(\frac{k}{\mu m_p}\right) \frac{d\rho}{\rho}$$

where, since  $V = 1/\rho$ , we can write

$$dV = -\frac{1}{\rho^2} d\rho = -\frac{1}{\rho} \frac{d\rho}{\rho} = -V \frac{d\rho}{\rho} \quad \text{so that} \quad -\frac{dV}{V} = \frac{d\rho}{\rho}$$

Thus,

$$c_V \frac{dP}{P} = \left(\frac{k}{\mu m_p}\right) \frac{d\rho}{\rho} - c_V \frac{dV}{V} = \left(\frac{k}{\mu m_p}\right) \frac{d\rho}{\rho} - \frac{3}{2} \left(\frac{k}{\mu m_p}\right) \left(-\frac{d\rho}{\rho}\right) = \frac{5}{2} \left(\frac{k}{\mu m_p}\right) \frac{d\rho}{\rho} = c_P \frac{d\rho}{\rho}$$

Thus, as you showed in Question 3(b) of today's worksheet, this gives

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$

where  $\gamma = c_P/c_V$  is the ratio of specific heats.

### Fully ionized gas, multiple elements

In practice, stellar matter is comprised of a mixture of different elements, the atoms of which are largely ionized. The **total pressure** in the gas is then

$$P = \sum_i P_i = \sum_i n_i kT$$

where  $n_i$  are the number densities of each type of particle.

Consider a mixture of atoms of different elements, all of which are assumed to be fully ionized. Denote atomic number of element  $j$  by  $Z_j$ , its atomic mass by  $A_j$ , and its mass fraction by  $X_j$ . When fully ionized, each atom contributes  $(Z_j + 1)$  particles ( $Z_j$  electrons and one nucleus). One can then show that  $P$  written in the expression above can be put in the form in equation (3.6), with the **mean molecular weight**  $\mu$  given by

$$\mu^{-1} = \sum_j X_j \frac{Z_j + 1}{A_j}$$

as you did in Question 4(a) of today's worksheet.

Let us denote the **mass fractions** of H and He by  $X$  and  $Y$  respectively, and the mass fraction of the remaining (so-called) heavy elements by  $Z$ . Since this is everything, we must have

$$X + Y + Z = 1$$

and we must also have  $Z \ll X, Y$  in all cases.

Now, if we take  $A_1 = 1$  for hydrogen,  $A_2 = 4$  for He, and approximate  $(Z_j + 1)/A_j$  by  $\frac{1}{2}$  for the heavy elements, then we can show that

$$\mu = \frac{4}{3 + 5X - Z}$$

as you did in Question 4(b) of today's worksheet.

See **Section 3.2.2** in *Dalgaard* (**pages 44-45**) for more details.