**PHY 474/374** **Spring 2021**

## Class Summary—Week 5, Day 1—Monday, April 26

# Nuclear Energy Generation in Stars

Stars derive their energy from nuclear reactions throughout most of their lifetime; neither gravitational contraction (as you found by calculating the Kelvin-Helmholtz timescale in an earlier class) nor chemical reactions (*as you demonstrated in Question 1 on today's worksheet*) can sustain stars for very long. We will now learn some of the physics of nuclear energy generation in more detail.

You may recall that in an earlier class, we introduced the quantity $\epsilon$, the rate of energy generation per unit mass, that is, the energy generated per unit mass per unit time. The computation of this quantity can be separated into three parts:

- the cross section for reaction between a pair of nuclei, determined predominantly by the properties of the nuclei,
- the amount of energy generated per reaction, also a property of the nuclei, and
- the total reaction rate, which depends not only on the cross section, but also on the motion of the nuclei.

As nuclear processes happen, there is a gradual change in the chemical composition, and this has an impact on the evolution of the star. Thus, we must determine the rate of change of the abundances, e.g., the rate of change of the hydrogen abundance $X$ may be written as

$$\frac{dX}{dt} = r_X \tag{8.1}$$

where $r_X$ is a sum over the reactions which use up the hydrogen.

Be aware that *Dalsgaard* uses a compact notation for reactions, in which he abbreviates the reaction

$$A + a \rightarrow Y + y \qquad \text{as} \qquad A\,(a, y)\,Y \tag{8.2}$$

with the primary reactant and the primary product placed outside the parentheses; this follows standard practice in nuclear physics. To avoid any confusion, I will write out all equations (as far as possible). In equation (8.2), the left side is called the entrance channel in nuclear physics, and the right side is called the exit channel. A more detailed way of writing this reaction is

$$A + a \rightarrow Z^* \rightarrow Y + y$$

where $Z^*$ denotes an excited intermediate state called a compound nucleus, an excited composite that quickly decays into the final products of the reaction.

It is common to classify nuclear reactions according to the number of (nuclear) species in the entrance channels; thus, the reaction in equation (8.2) is a two-body reaction; one-body reactions like $A \rightarrow B + C$ and three-body reactions like $A + B + C \rightarrow D$ are also important in stellar energy production.

We will now discuss separately the three parts listed above: the cross section, the energy generated, and the reaction rate.

## The cross sections

Nuclear reactions are caused by the **strong nuclear force** acting between nucleons, the constituents of the nucleus (protons and neutrons). Of the four fundamental forces in nature, the strong nuclear force is the strongest, but its influence is limited to nuclear size scales. Thus, the interacting nuclei must be brought near each other, so that they are almost touching, in order to undergo nuclear reactions. This requires overcoming the Coulomb repulsion between like charges, and thus requires very high energy. To get an idea of the typical energies involved, consider that the height of the Coulomb barrier at the nuclear surface, $r_0 \sim 10^{-15}$ m $\equiv 10^{-13}$ cm, is

$$E_{\text{Coul}} \simeq \frac{\mathcal{Z}_1 \mathcal{Z}_2 e^2}{r_0} \simeq \mathcal{Z}_1 \mathcal{Z}_2 \text{ MeV} \tag{8.4}$$

where $\mathcal{Z}_1, \mathcal{Z}_2$ are the atomic numbers of the nuclei taking part in the reaction; note that this equation is for cgs units, ***not*** SI units. Thus, typical nuclear energies are in the MeV range.

Meanwhile, the average kinetic energy of the nucleus is

$$\langle E_{\text{kin}} \rangle = \frac{3}{2}\, k_B T \simeq (130 \text{ eV})\, T_6 \tag{8.5}$$

where $T_6 \equiv T/10^6$ K, that is, $T_6$ is the temperature expressed in units of a million K. Since $T \sim 10^7$ K in the cores of stars undergoing nuclear fusion of hydrogen, we get that

$$\langle E_{\text{kin}} \rangle \simeq (130 \text{ eV}) \left( \frac{10^7 \text{ K}}{10^6 \text{ K}} \right) = 1300 \text{ eV}$$

Thus, the average kinetic energy of nuclei is about a factor of 1000 *smaller* than the energy required to overcome the Coulomb barrier, *as you determined in Question 2 of today's worksheet.* Overcoming this would be prohibitively difficult in Classical Mechanics.

Fortunately, quantum mechanics comes to our rescue. The phenomenon of quantum tunneling implies that there is a finite probability that the nuclei may tunnel through the barrier and react. This probability is small, which explains why nuclear reactions in stellar interiors are a slow process.

Measuring the cross section for nuclei to tunnel through the Coulomb barrier under stellar conditions is difficult. Typically, lab experiments are for nuclei in MeV, whereas stellar conditions require values in keV, as calculated above. For a nice visual representation, see Figure 8.2 (*Dalsgaard*, page 105).

Instead, we consider the energy dependence of the probability that the nuclei will penetrate the Coulomb barrier. We can obtain this from $\alpha$-particle decay, since the $\alpha$-particle will have the same probability going in the opposite sense, that is, to tunnel out through the potential barrier. Gamow demonstrated that this probability is proportional to

$$\exp\left( -\frac{2\pi \mathcal{Z}_1 \mathcal{Z}_2 e^2}{\hbar v} \right) \tag{8.6}$$

where $v$ is the relative speed between the two nuclei, $e$ is the charge of an electron, $\mathcal{Z}_1, \mathcal{Z}_2$ are the atomic numbers of the nuclei, and $\hbar = h/2\pi$, where $h$ is Planck's constant.

An additional energy dependence enters into the cross sections because the nuclei have a geometrical extent given by their de Broglie wavelengths, $\lambda \propto 1/p$, where $p$ is the momentum of the nucleus. This geometrical cross section is

$$\pi \lambda^2 \propto p^{-2} \propto E^{-1} \tag{8.7}$$

where $E$ is the energy of the nucleus.

Combining the quantities considered in equation (8.6) and equation (8.7), we get that the **cross section** is given by

$$\sigma(E) \equiv \frac{S(E)}{E} \exp\left(-\frac{2\pi \mathcal{Z}_1 \mathcal{Z}_2 e^2}{\hbar v}\right) \tag{8.8}$$

where we have introduced a cross-section factor $S(E)$, which essentially describes the energy dependence of the reaction once the nuclei have penetrated the potential barrier; it is assumed to vary slowly with $E$ and to contain all energy dependence other than $E^{-1}$ from the geometrical cross section and the energy dependence in the exponential term (see Figure 8.3 on page 107 in *Dalsgaard*).

Quantities considered in these analyses are usually in the center of mass system; thus

$$E = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v^2 = \frac{1}{2} \mathcal{A} m_p v^2 \tag{8.9}$$

where $\mathcal{A}$ is the reduced atomic weight

$$\mathcal{A} = \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}_1 + \mathcal{A}_2} \tag{8.10}$$

and $\mathcal{A}_1, \mathcal{A}_2$ are the atomic weights of the nuclei.

Based on the expressions in equation (8.9) and equation (8.10), the cross section in equation (8.8) can then be written as

$$\sigma_E \equiv \frac{S(E)}{E} \exp\left(-\frac{b}{\sqrt{E}}\right) \tag{8.11}$$

where

$$b = 31.291\, \mathcal{Z}_1 \mathcal{Z}_2\, \mathcal{A}^{1/2}\ \mathrm{keV}^{1/2} \tag{8.12}$$

We see from equation (8.11) and equation (8.12) that the cross section is very sensitive to the charges of the nuclei involved, since the charges effect the extent of the Coulomb barrier. In general, therefore, reactions among nuclei of lower charges are faster, and are possible at lower temperature, than reactions involving nuclei with higher charges, *as you found in Question 3 of today's worksheet*. This may, however, be reversed by differences in the cross section factor $S(E)$, due to differences in the nuclear structure.

*Dalsgaard* points out that the measurement of nuclear cross sections is an extensive and ongoing effort in many laboratories around the world. The results are often expressed in terms of Taylor expansions of $S(E)$ around $E = 0$; see, e.g., Table 8.1 (*Dalsgaard*, page 109), where $S(E)$ extrapolated to $E = 0$ and its derivative $dS/dE$ are listed for the principal fusion reactions in the Sun.

Since the nuclei are in a Maxwellian distribution, the energy dependence of the cross section resides primarily in the factor

$$F_G \equiv \exp(-E/kT)\, \exp(-b/\sqrt{E})$$

called the **Gamow window**. The first factor $\exp(-E/kT)$ arises from the Maxwellian velocity distribution and decreases rapidly with energy, whereas the second factor $\exp(-b/\sqrt{E})$ arises from the barrier penetration, and increases rapidly with energy. Thus, the product is strongly localized in energy, and only for energies within the Gamow window are stellar nuclear reactions likely to occur.

### The release of energy

We will now compute the total energy released. To do so, we need to know the energy released by each reaction.

If we add the individual masses of all the protons and neutrons in the nucleus, then we will find that this sum is larger than the total mass of the nucleus constituted by these protons and neutrons (collectively called nucleons). Thus, when a nucleus is formed from these nucleons, the mass difference is released as energy. This difference is known as the **binding energy** of the nucleus, and is given by

$$Q(\mathcal{Z},\mathcal{N}) = c^2\Big[\mathcal{Z}\, m_p + \mathcal{N}\, m_n - m(\mathcal{Z},\mathcal{N})\Big]$$

for a nucleus with $\mathcal{Z}$ protons and $\mathcal{N}$ neutrons. *You did an example calculation on Question 4 of today's worksheet.*

Knowing about the binding energy, we can obtain the energy released by a nuclear reaction by finding the total mass of the reactants and that of the products from tables of nuclear masses, subtracting them, and converting the difference in mass to energy using the mass-energy equivalence. As an example, if we have the (generic) reaction

$$A + a \rightarrow Y + y \tag{8.13}$$

then the energy released by the reaction is

$$Q = c^2\Big[m(A) + m(a) - m(Y) - m(y)\Big] \tag{8.14}$$

where $m(A)$ is the mass of particle $A$, for example.

Since the difference in mass is very small, it is generally more convenient to work with a quantity called the **mass excess**, $\Delta m$, defined for a nucleus with $\mathcal{Z}$ protons and $\mathcal{N}$ neutrons as

$$\Delta m = m - m_p\,(\mathcal{Z} + \mathcal{N}) \tag{8.15}$$

where $m$ is the mass of the nucleus, and $m_p$ is the proton mass (or $m_u$ in *Dalsgaard*, the atomic mass unit). Then, since the combined number of protons and neutrons must be conserved in the reaction given in equation (8.13), we can write equation (8.14) in terms of mass excess as

$$Q = c^2\Big[\Delta m(A) + \Delta m(a) - \Delta m(Y) - \Delta m(y)\Big] \tag{8.16}$$

*You used this to find the energy released in a nuclear reaction in Question 5 of today's worksheet.* In reactions where electrons (or positrons) are absorbed or emitted to preserve charge neutrality, we can still use equation (8.16), but now the mass excesses refer to the atomic (rather than nuclear) masses; for details, see equation (8.17)-(8.19) in *Dalsgaard* (page 110).

The energy produced in these reactions is released as kinetic energy of the particles, and sometimes also in the form of $\gamma$-ray photons. This energy is redistributed among the gas particles via collisions, or by absorption of the photons. As long as we can assume LTE, the details of the redistribution are not relevant. Instead, what matters is the total amount of heat added to the gas. An exception to this general statement, however, is when neutrinos are emitted in the reaction. Neutrinos have tiny interaction cross sections, meaning that they rarely interact with the other gas particles; instead, they escape directly from the star. Thus, whenever neutrinos are generated, we must subtract the energy carried away by neutrinos in order to compute the energy released in the region where the nuclear reactions take place.