PHY 474/374 Spring 2021

Class Summary—Week 3, Day 2—Wednesday, April 14

Stellar Atmospheres

Today we will learn about **stellar atmospheres**. It is important to know about the atmosphere of a star, since radiation passes through it on its way to us.

What, though, is the atmosphere of a star? For a solid body like the Earth, an atmosphere is easy to define, but how do we define an atmosphere for a ball of gas?

By definition, the atmosphere of a star is where the opaque interior becomes semi-transparent. This can be quantified by the ratio of mean free path λ to the characteristic length to escape.

- In the interior, the mean free path λ is very small, of the order of a few cm, and much smaller than the stellar radius R.
- In the star's atmosphere, on the other hand, λ is comparable to the scale height H (which itself is comparable to size of atmosphere). We can find the scale height from

$$H = \frac{kT}{\mu m_p g} \qquad \text{where} \qquad g = \frac{Gm}{r^2} \tag{4.33}$$

So let's begin by exploring some of the differences between the stellar interior and the stellar atmosphere.

First, there is a question of **scale**. The stellar interior is much larger than the atmosphere. Using numbers for our Sun (whose atmosphere is about 2000 km in extent), we can show that the atmosphere is just a narrow surface layer, $\sim 10^{-3}\ R$, *as you did in Question 1(a) on today's worksheet*. Note that by atmosphere, we are including only the chromosphere, and not the solar corona which extends out to distances $\gg R_\odot$. Meanwhile, the scale height $H = 280$ km in our Sun's atmosphere, *as you determined in Question 1(b) on today's worksheet.*

Another difference between the stellar interior and the atmosphere is that the **radiation field** in the interior is nearly isotropic, with only a tiny fraction of photons (of order $\lambda/R \ll 1$) going up (i.e., outward, away from the center of the star) than down (i.e., inward, toward the center of the star). On the other hand, the radiation field is highly anisotropic in the atmosphere, with most photons going up (outward) and little to none coming down (from nearly empty space above).

Moreover, in the interior, radiation and matter are strongly coupled, so the same T describes the radiation field and the kinetic motion of atoms; in other words, we can assume that the interior is in **LTE** (Local Thermodynamic Equilibrium). In the atmosphere, however, we cannot assume LTE. Thus, radiation transport in the atmosphere of a star is complicated, and requires detailed radiative transfer which involves spatial integrals over the emitting volume.

Given all of the above, computing models of stellar atmospheres is a complicated problem. Still, we can develop some useful relations using approximations. For example, we can make the reasonable approximation that the atmosphere is isothermal, and has a constant temperature T that is a little below T_{eff}, the temperature at the photosphere. The bottom of the atmosphere is called the **photosphere** and corresponds roughly to the visible surface of the star. We will now learn about the stellar atmosphere in more detail.

On the previous page, we introduced the term **scale height** of a stellar atmosphere. Let's learn about this in more detail.

Recall that we discussed in a previous class how there exist simple solutions to the equation of hydrostatic equilibrium. One such example is for the case of an isothermal gas, which is a valid assumption in a stellar atmosphere. We may assume also that the atmosphere is small in extent compared to the radius of the star, so that the gravitational acceleration $g = GM/R^2$ can be assumed to be constant in the atmosphere. Starting from the equation of stellar structure (*Dalsgaard*, eq. 4.4), given by

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$$

we can show, *as you did in Question 2(a) of today's worksheet*, that

$$\frac{dP}{dr} = -\frac{P}{H} \tag{4.32}$$

where $P = \rho kT/\mu m_p$ from the ideal gas law, and the scale height H is given in equation (4.33) written on the previous page.

Now, let's integrate equation (4.32) by first separating variables *as you did in Question 2(b) of today's worksheet*, so that

$$\frac{dP}{P} = -\frac{1}{H}\,dr$$

from which we obtain, upon integrating, that

$$\ln\, P = -\frac{r}{H} + C$$

where C is a constant of integration. To determine C, consider that at $r = r_0, P = P_0$, where r_0 defines some arbitrary reference level in the atmosphere. Then, we get

$$\ln\, P_0 = -\frac{r_0}{H} + C \qquad \text{so that} \qquad C = \ln\, P_0 + \frac{r_0}{H}$$

Putting this value of C into the equation for P above, we have

$$\ln\, P = -\frac{r}{H} + \left[\ln\, P_0 + \frac{r_0}{H}\right]$$

or

$$\ln\, P - \ln\, P_0 = -\frac{r}{H} + \frac{r_0}{H}$$

and so

$$\ln\, \frac{P}{P_0} = -\left[\frac{r - r_0}{H}\right]$$

Therefore, we get

$$P = P_0\, \exp\left(-\frac{h}{H}\right) \tag{4.34}$$

where $h = r - r_0$, and as noted above r_0 defines some arbitrary reference level in the atmosphere where P has the value P_0; in other words, P_0 is the value of P at $h = 0$.

We can use equation (4.34) to interpret the scale height H: for each increase in height by H, the pressure drops by a factor of $1/e = 0.468$.

Next, consider the **pressure on the photosphere**, the bottom of the atmosphere that corresponds roughly to the visible surface of the star. Even though we have been using zero pressure at the stellar surface in our equations, there is some pressure at the bottom of the star's atmosphere (on the photosphere), $P_{\rm ph}$; let's figure out what it is.

Let h be the height of the atmosphere over the photosphere. The probability that a photon escapes from a height h is

$$\exp\left(-\int_h^\infty \frac{dh}{\lambda}\right)$$

Dalsgaard considers a reasonable measure for the location of the bottom of the atmosphere, where $h = 0$, to be the location where this probability is e^{-1}, so that

$$1 \simeq \int_0^\infty \frac{dh}{\lambda} = \int_0^\infty \kappa\rho\, dh \tag{5.16}$$

since the mean free path, $\lambda = 1/\kappa\rho$, where κ is the opacity, and ρ is the mass density.

Assuming an isothermal atmosphere, the variation of density with height is given by

$$\rho = \rho_{\rm ph}\, \exp\left(-\frac{h}{H}\right) \tag{5.17}$$

where the density scale height H is the same as the pressure scale height we wrote in equation (4.33) on page 1, that is

$$H = \frac{k_B T}{\mu m_p g} \tag{5.18}$$

where $g = GM/R^2$ is the gravitational acceleration in the atmosphere, and I've now subscripted the Boltzmann constant as k_B to avoid confusion with the opacity κ. Approximate the opacity κ by a power law

$$\kappa = \kappa_0^{\rm (ph)}\, \rho^a T^b \tag{5.19}$$

where "ph" stands for photosphere. Since opacity should increase with increasing temperature, we expect $b > 0$. Take care that you don't confuse the use of a in the exponent for the density with the use of a for the radiation density constant.

Putting equation (5.17) and equation (5.19) in equation (5.16), integrating, and then using the ideal gas law in the form written in equation (3.6), it can be shown that

$$P_{\rm ph} = \left[\frac{GM(a+1)}{R^2\kappa_0^{\rm (ph)}}\right]^{1/(a+1)} \left[\frac{k_B}{\mu m_p}\right]^{a/(a+1)} T_{\rm eff}^{(a-b)/(a+1)} \tag{5.22}$$

as you will demonstrate on the homework. The proof assumes that the temperature in the atmosphere, $T \simeq T_{\rm eff}$, the temperature of the photosphere. *In Question 3(a) on today's worksheet,* you used equation (5.22) to calculate the pressure on the Sun's photosphere, and found it to be 338 N/m^2, so it makes sense that we take the surface pressure to be zero, since the pressure at the center of the star is $P_c \sim 10^{16}$ N/m^2. Assuming again an ideal gas, the density in the Sun's photosphere is given by

$$\rho_{\rm ph} = \left[\frac{GM(a+1)}{R^2\kappa_0}\right]^{1/(a+1)} \left[\frac{k_B}{\mu m_p}\right]^{-1/(a+1)} \left[T_{\rm eff}\right]^{-(b+1)/(a+1)} \tag{5.23}$$

as you demonstrated in Question 3(b) on today's worksheet. Again, take care that you don't confuse the a in this equation with its use for the radiation density constant in the previous class (i.e., a here is different, the exponent to the density in the expression for the opacity).

The Energy Equation

In the previous class, we derived an equation for the temperature gradient, dT/dr, in a star. We also need an equation for the luminosity as a function of r.

To find such an expression, consider that during most of a star's lifetime, the energy is produced by nuclear reactions. Let ϵ be the **rate of energy production per unit mass** by nuclear reactions, that is, it is the energy production per unit mass per unit time.

Consider a spherical shell of radius r and thickness dr. If we can find the energy per unit time produced in it, it should lead to an increase dL in the luminosity of the shell.

Since ϵ is the energy produced per unit mass per unit time, then the energy produced per unit time in a shell of radius r and thickness dr is given by $\epsilon\, dm$, where the element of mass dm is given by

$$dm = \rho\, dV = \rho\Big(4\pi r^2 dr\Big)$$

But the energy produced per unit time in this shell of mass dm is just dL, the luminosity of the shell. Thus, we can write

$$dL = \epsilon\, dm = \epsilon\left[\rho\Big(4\pi r^2 dr\Big)\right] \tag{5.24}$$

from which we obtain, *as you did in Question 4(a) of today's worksheet*, that

$$\frac{dL}{dr} = 4\pi r^2 \rho\epsilon$$

This expression gives us the luminosity as a function of stellar radius during most of the lifetime of the star, when it is producing energy via nuclear reactions.

The expression we just derived for dL/dr, however, does not describe the liberation of energy during gravitational contraction. Thus, the expression we just derived cannot be used in the early phases of stellar evolution. A more complete description can be obtained by starting from the first law of thermodynamics

$$dQ = dU + PdV \tag{5.25}$$

where V is the volume of the system, U its internal energy, and dQ is the heat added to or subtracted from the system.

Let's consider the system to have unit mass, so that

$$V = \frac{1}{\rho} \qquad \text{and} \qquad U = \frac{u}{\rho}$$

where u is the internal energy per unit volume.

The heat input during the time dt has two contributions. One is the heat ϵ liberated by nuclear reactions. The other is the heat deposited or extracted from the energy flowing through the layer. Since the mass of the shell between r and $(r + dr)$ is $4\pi r^2 dr$, this contribution in the time dt, per unit mass, is just

$$\frac{[L(r) - L(r + dr)]\, dt}{\rho\,(4\pi r^2 dr)}$$

We will now put all this into equation (5.25).

On the previous page, we determined that the heat per unit mass added or subtracted to the system has two parts. Adding them, and setting them equal to the right hand side of equation (5.25) in quantities per unit mass, we get

$$\epsilon\, dt + \frac{[L(r) - L(r+dr)]\, dt}{\rho\,(4\pi r^2 dr)} = d\left(\frac{u}{\rho}\right) + P\, d\left(\frac{1}{\rho}\right) \tag{5.26}$$

from which we obtain

$$\frac{dL}{dr} = 4\pi r^2 \left[\rho\epsilon - \rho \frac{d}{dt}\left(\frac{u}{\rho}\right) + \frac{P}{\rho}\frac{d\rho}{dt}\right] \tag{5.27}$$

You interpreted this equation *in Question 4(b) on today's worksheet*: the energy radiated comes from the energy produced by nuclear reactions ($\rho\epsilon$), **minus** the energy needed to raise the internal energy of the gas ($-\rho\, d(u/\rho)/dt$ comes from dU), and **minus** the energy needed to do work on the gas. Note that the third term ends up having a positive sign because

$$P\, dV(\text{per unit mass}) = P\, d\left(\frac{1}{\rho}\right) = -\frac{P}{\rho}\, d\rho$$

has switched sign in moving to the other side.

In the equation for dL/dr in equation (5.27), the second and third term are essentially negligible during the normal nuclear burning phases. To see this, consider the Kelvin-Helmholtz timescale that we discussed earlier, which can be written in the present context as

$$t_{\mathrm{KH}} = \frac{U_{\mathrm{tot}}}{L_s} \tag{5.28}$$

where $U_{\mathrm{tot}} \simeq R^3 u$ is the total internal energy of the star. You may recall from previous calculations that $t_{\mathrm{KH}} \sim 10^7$ yr, where the nuclear burning timescale is $t_{\mathrm{nuc}} \sim 10^{10}$ yr for our Sun. By replacing dL/dr by L_s/R, and du/dt by u/t_{nuc}, *you showed in Question 5 on today's worksheet* that the magnitude of the second term is

$$\frac{\left|4\pi r^2 \rho\, \dfrac{d}{dt}\left(\dfrac{u}{\rho}\right)\right|}{\dfrac{dL}{dr}} \simeq \frac{t_{\mathrm{KH}}}{t_{\mathrm{nuc}}} \ll 1 \tag{5.29}$$