

## Practice Assessment 4

The normalized energy eigenstates for the hydrogen atom are

$$\Psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_l^m(\theta, \phi), \text{ with energies } E_n = -\frac{e^2}{2a_0 n^2}$$

The first few normalized radial and angular wave functions are

$$R_{10}(r) = \frac{2}{\sqrt{a_0^3}} e^{-r/a_0}, \quad R_{20}(r) = \frac{2}{\sqrt{(2a_0)^3}} \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0}$$

$$R_{21}(r) = \frac{1}{\sqrt{3(2a_0)^3}} \frac{r}{a_0} e^{-r/2a_0}$$

and

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta,$$

$$Y_1^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

(1) A hydrogen atom is prepared in the state

$$\Psi(r, \theta, \phi) = A \left[ 3\Psi_{210}(r, \theta, \phi) - \Psi_{211}(r, \theta, \phi) \right]$$

(a) Find  $A$  and explain why you don't have to evaluate any integrals to do so.

We can write the state  $\Psi$  in Bra-ket notation such that

$$|\Psi_{r,\theta,\phi}\rangle = A \left[ 3|2, 1, 0\rangle - |2, 1, 1\rangle \right]$$

To Normalize the state we can do the following,  
 $A = 1/\sqrt{\langle \Psi | \Psi \rangle}$ . So the state becomes

$$1 = \langle \Psi | \Psi \rangle = A^2 \left[ (3\langle 2,1,0| - \langle 2,1,1|)(3|2,1,0\rangle - |2,1,1\rangle) \right]$$

$$1 = A^2 \left[ 9\langle 2,1,0|2,1,0\rangle + 1\langle 2,1,1|2,1,1\rangle \right] \Rightarrow 1 = A^2 [10]$$

$$A^2 10 = 1 \Rightarrow A^2 = 1/10 \Rightarrow \boxed{A = 1/\sqrt{10}}$$

$$\boxed{|\Psi_{r,\theta,\varphi}\rangle = \frac{1}{\sqrt{10}} [3|2,1,0\rangle - |2,1,1\rangle]}$$

(B)

$$E_n = -\frac{13.6 \text{ eV}}{n^2} \rightarrow E_2 = -\frac{13.6 \text{ eV}}{2^2} = -3.4 \text{ eV}$$

$$P_{E_2} = \sum_{l=0}^1 \sum_{m=-1}^1 |\langle 2,l,m | \Psi \rangle|^2 = |\langle 2,1,0 | \Psi \rangle|^2 + |\langle 2,1,1 | \Psi \rangle|^2$$

$$= \left| \frac{3}{\sqrt{10}} \right|^2 + \left| \frac{1}{\sqrt{10}} \right|^2 = 1$$

(C)  $|L| = \sqrt{1(1+1)} \hbar = \sqrt{2} \hbar$

$$P_{\sqrt{2}\hbar} = \sum |\langle n,l,m | \Psi \rangle|^2 = |\langle 2,1,0 | \Psi \rangle|^2 + |\langle 2,1,1 | \Psi \rangle|^2$$

$$= \left| \frac{3}{\sqrt{10}} \right|^2 + \left| \frac{1}{\sqrt{10}} \right|^2 = 1$$

$$L_z = 0 \quad P_0 = |\langle 2,1,0 | \Psi \rangle|^2 = \left| \frac{3}{\sqrt{10}} \right|^2 = 9/10$$

$$L_z = \hbar \quad P_{\hbar} = |\langle 2,1,1 | \Psi \rangle|^2 = \left| \frac{1}{\sqrt{10}} \right|^2 = 1/10$$

(D)  $P(r,\theta,\varphi) = |\Psi(r,\theta,\varphi)|^2$

$$= \frac{1}{10} \left[ 3\psi_{210}^*(r,\theta,\varphi) - \psi_{211}^*(r,\theta,\varphi) \right] \frac{1}{10} \left[ 3\psi_{210}(r,\theta,\varphi) \right]$$

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$$- \psi_{21}(r, \theta, \varphi) \Big] \\ = \frac{1}{10} R_{21}(r)^* R_{21}(r) \left[ 3Y_1^0(\theta, \varphi) - Y_1^0(\theta, \varphi) \right] \left[ 3Y_1^0(\theta, \varphi) - Y_1^0(\theta, \varphi) \right]$$

$$= \frac{1}{10} \left[ |R_{21}(r)|^2 \right] \left[ 9 |Y_1^0(\theta, \varphi)|^2 - Y_1^0(\theta, \varphi) Y_1^0(\theta, \varphi) - 3Y_1^0(\theta, \varphi) Y_1^0(\theta, \varphi) + |Y_1^0(\theta, \varphi)|^2 \right]$$

$$\langle r \rangle = \int_0^\infty \frac{1}{10} r^3 |R_{21}|^2 dr \int_0^{2\pi} \int_0^\pi \left[ 9 |Y_1^0|^2 - \cancel{3Y_1^0 Y_1^0} - \cancel{3Y_1^0 Y_1^0} + |Y_1^0|^2 \right] \sin \theta d\theta d\varphi$$

$$\langle r \rangle = \int_0^\infty r^3 |R_{21}(r)|^2 dr$$

$$= \int_0^\infty r^3 \left| \frac{1}{\sqrt{3(2a_0)^3}} \frac{r}{a_0} e^{-r/2a_0} \right|^2 dr = \int_0^\infty \frac{r^5}{24a_0^5} e^{-r/2a_0} dr$$

Wolfram

$$\rightarrow 5a_0$$