PHY 411 Winter 2021

Class Summary—Week 8, Day 1—Tuesday, Feb 23

Multipole Expansion of $\Phi(\vec{x})$

We will now study the potential due to a localized charge distribution and its expansion in multipoles. The larger aim is to prepare us for the expansion of the vector potential that we will need for learning about radiating systems in Chapter 9.

Consider a localized distribution of charge described by the charge density $\rho(\vec{x}\,')$ that is contained within a sphere of radius R around some origin. Note that the sphere of radius R is an arbitrary conceptual device employed merely to divide space into regions with and without charge.

Outside the sphere of radius R, the potential can be written as an expansion in spherical harmonics:

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm}\, r^l + \frac{B_{lm}}{r^{l+1}} \right] Y_{lm}(\theta,\phi) \tag{3.61}$$

where $Y_{lm}(\theta,\phi)$ are the *spherical harmonics*, given by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi} \tag{3.53}$$

Since we want to be far away from the charge distribution, but not anywhere in the vicinity of $r = 0$, we must demand that $A_l = 0$ for all l. This means that the potential becomes

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{B_{lm}}{r^{l+1}}\, Y_{lm}(\theta,\phi)$$

To match Jackson's "particular choice of constant coefficients . . . made for later convenience" we'll have to set

$$B_{lm} = \frac{1}{4\pi\epsilon_0} \frac{4\pi}{2l+1}\, q_{lm}$$

where q_{lm} is a quantity whose form we will work out below. With B_{lm} written in this manner, the expansion for the potential becomes

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1}\, q_{lm}\, \frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \tag{4.1}$$

Equation (4.1) is called a **multipole expansion**; the $l = 0$ term is called the monopole term, $l = 1$ are the dipole terms, etc. — the reason for these names should become clear below.

We must now determine the quantities q_{lm} in terms of the properties of the charge density $\rho(\vec{x}\,')$ in order to obtain to fully solve the problem. To do so, let us look at the integral form for the potential that we wrote previously in equation (1.17):

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\, d^3x'$$

and substitute in it equation (3.70) for $1/|\vec{x}-\vec{x}\,'|$ that we wrote on the previous page:

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left(\frac{r_<^l}{r_>^{l+1}} \right) Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi) \tag{3.70}$$

Since we're interested in the potential outside the charge distribution, we'll put $r_< = r'$ and $r_> = r$, so that equation (3.70) becomes

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left(\frac{r'^{\,l}}{r^{l+1}} \right) Y^*_{lm}(\theta', \phi')\, Y_{lm}(\theta, \phi)$$

and thus

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\, 4\pi \sum_{l,m} \int \frac{\rho(\vec{x}\,')}{2l+1} \left(\frac{r'^{\,l}}{r^{l+1}} \right) Y^*_{lm}(\theta', \phi')\, Y_{lm}(\theta, \phi)\, d^3x'$$

Rearranging so that all primed coordinates are written within the integral and the rest outside, we get

$$\Phi(\vec{x}) = \frac{1}{\epsilon_0} \sum_{l,m} \frac{1}{2l+1} \left[\int Y^*_{lm}(\theta', \phi')\, r'^{\,l}\, \rho(\vec{x}\,')\, d^3x' \right] \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \tag{4.2}$$

Comparing equation (4.1) and equation (4.2), we find that

$$q_{lm} = \int Y^*_{lm}(\theta', \phi')\, r'^{\,l}\, \rho(\vec{x}\,')\, d^3x' \tag{4.3}$$

These coefficients are called the **multipole moments**. To see their physical significance, let's express the first few explicitly in terms of Cartesian coordinates.

Start with q_{00}, which would be given by

$$q_{00} = \int Y^*_{00}(\theta', \phi')\, (r')^0\, \rho(\vec{x}\,')\, d^3x'$$

and since $Y_{00} = 1/\sqrt{4\pi}$, we get

$$q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(\vec{x}\,')\, d^3x'$$

The integral is just the total charge q, so

$$q_{00} = \frac{q}{\sqrt{4\pi}} \tag{4.4}$$

So the $l = 0$ term is simply proportional to the total charge q. The potential, if you just take the first term in the summation into account, is

$$\Phi(\vec{x})\Big|_{l=0} = \frac{1}{\epsilon_0} \frac{1}{2(0)+1}\, q_{00}\, \frac{Y_{00}(\theta, \phi)}{r^{0+1}} = \frac{1}{\epsilon_0} \Big[q_{00} \Big]\, Y_{00}\, \frac{1}{r}$$

Substituting the value of q_{00} from equation (4.4) and putting $Y_{00} = 1/\sqrt{4\pi}$, we get

$$\Phi(\vec{x})\Big|_{l=0} = \frac{1}{\epsilon_0} \left[\frac{q}{\sqrt{4\pi}} \right] \frac{1}{\sqrt{4\pi}}\, \frac{1}{r} = \frac{q}{4\pi\epsilon_0\, r}$$

This makes sense — if you're far away from a total charge q, it appears like a point charge, and the potential of a point charge is $q/4\pi\epsilon_0 r$. This also explains the designation of q_{00} as the **monopole moment**.

On Homework 7, you'll demonstrate that the $l = 1$ terms (q_{11}, q_{10}) are proportional to the components of the electric dipole moment $\vec{p}$, where $\vec{p}$ is given by

$$\vec{p} = \int \vec{x}\,' \rho(\vec{x}\,')\, d^3x' \tag{4.8}$$

Again, this makes sense, because the potential for the $l = 1$ term in equation (4.2) goes as $1/r^2$, and we know that the potential of a dipole goes like $1/r^2$ at large r — see Griffiths if you've forgotten.

Meanwhile, the $l = 2$ terms (q_{22}, q_{21}, q_{20}) are proportional to the quadrupole moments Q_{ij}, where the traceless quadrupole moment tensor is defined as

$$Q_{ij} = \int \left(3x_i'x_j' - r'^{\,2}\,\delta_{ij}\right) \rho(\vec{x}\,')\, d^3x' \tag{4.9}$$

Yet again, this makes sense, because the potential for the $l = 2$ term in equation (4.2) goes as $1/r^3$, and we know that the potential of a quadrupole goes as $1/r^3$.

By writing the first few terms in equation (4.1) explicitly, and expressing them in terms of the multipole moments in rectangular coordinates in equations (4.4)-(4.6), you can also show that the expansion of $\Phi(\vec{x})$ in rectangular coordinates is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\,\frac{x_i x_j}{r^5} + \ldots\right] \tag{4.10}$$

This shows us the practical application of what we've learned, by demonstrating that the potential can be expanded as a sum over monopoles, dipoles, quadrupoles, and so on.

Finally, since $\vec{E} = -\vec{\nabla}\Phi$, we can show by direct differentiation of equation (4.1) that the coordinates of the **electric field** are

$$\begin{aligned}
E_r &= \frac{(l+1)}{(2l+1)\,\epsilon_0}\, q_{lm}\, \frac{Y_{lm}(\theta,\phi)}{r^{l+2}} \\
E_\theta &= -\frac{1}{(2l+1)\,\epsilon_0}\, q_{lm}\, \frac{1}{r^{l+2}}\, \frac{\partial}{\partial\theta} Y_{lm}(\theta,\phi) \\
E_\phi &= \frac{1}{(2l+1)\,\epsilon_0}\, q_{lm}\, \frac{1}{r^{l+2}}\, \frac{im}{\sin\theta}\, Y_{lm}(\theta,\phi)
\end{aligned} \tag{4.11}$$

For a dipole $\vec{p}$ along the z-axis, the fields in equation (4.11) reduce to:

$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \qquad E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \qquad E_\phi = 0 \tag{4.12}$$

We can write this in vector form, so that the field at the observation point $\vec{x}$ due to a dipole $\vec{p}$ at the point $\vec{x}_0$ is

$$\vec{E}(\vec{x}) = \frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{4\pi\epsilon_0\,|\vec{x} - \vec{x}_0|^3} \tag{4.13}$$

where $\hat{n}$ is a unit vector directed from $\vec{x}_0$ to $\vec{x}$.