Phase Space Trajectories. Example 1: Simple pendulum

Step 1: Derive the Hamiltonian a Legendre transformation.

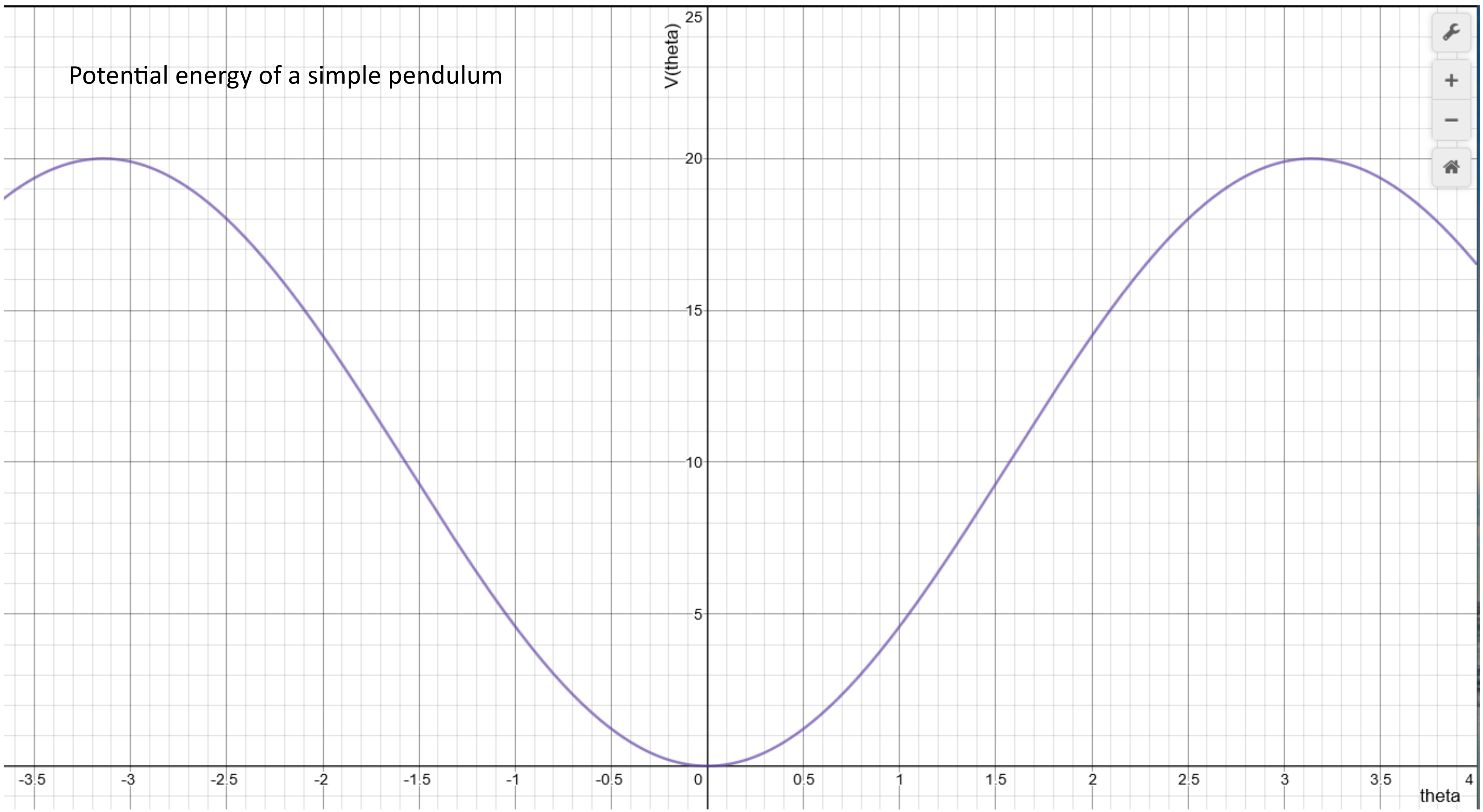
$$L = \frac{ml^2\dot{\theta}^2}{2} - mgl\,(1-\cos\theta)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$H = p_\theta\dot{\theta} - \frac{ml^2\dot{\theta}^2}{2} + mgl\,(1-\cos\theta) = ml^2\dot{\theta}\dot{\theta} - \frac{ml^2\dot{\theta}^2}{2} + mgl\,(1-\cos\theta)$$

$$= \frac{ml^2\dot{\theta}^2}{2} + mgl\,(1-\cos\theta)$$

$$H = \frac{p_\theta^2}{2ml^2} + mgl\,(1-\cos\theta)$$


Potential energy of a simple pendulum
V(theta)
25
20
15
10
5
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
theta

Step 2: Find the equilibrium solutions using Hamilton's equations

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$

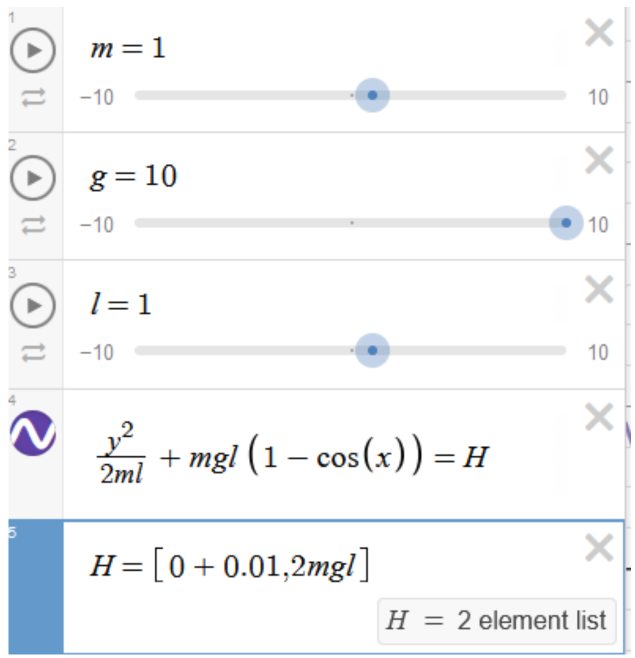
$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta$$

In equilibrium: $\dot{\theta} = 0, \dot{p}_\theta = 0$

$p_\theta = 0$ and $\theta = 0, \pm\pi, \pm 2\pi$ ...

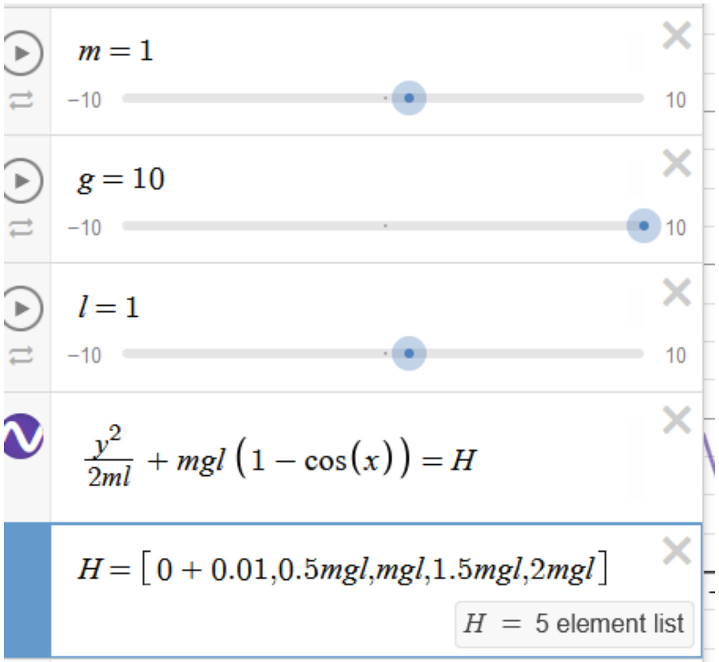
$$H = \frac{p_\theta^2}{2ml^2} + mgl\,(1 - \cos\theta) = 0 + mgl(1 \mp 1) = 0 \text{ and } 2mgl$$

Step 3: Make a contour plot

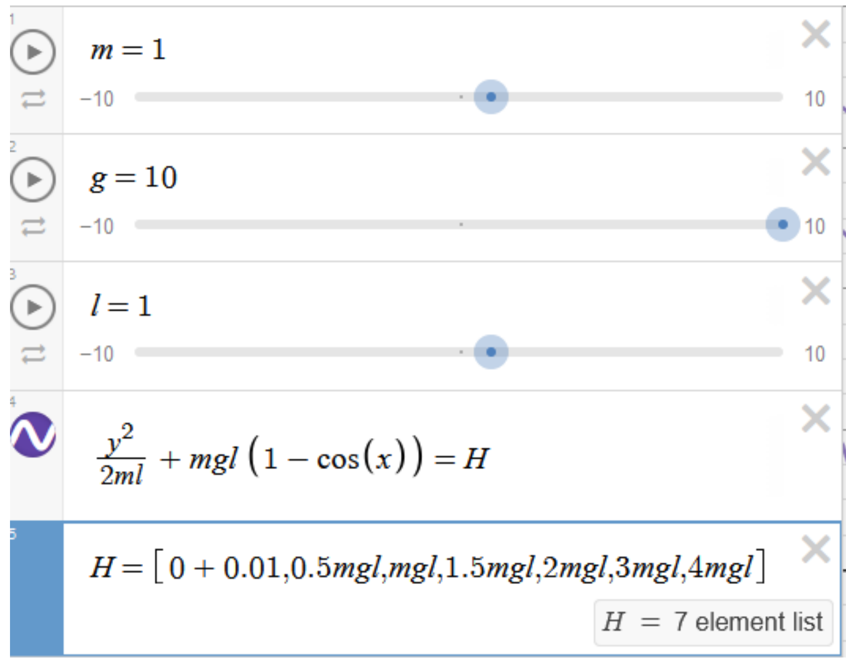


Equilibrium solutions

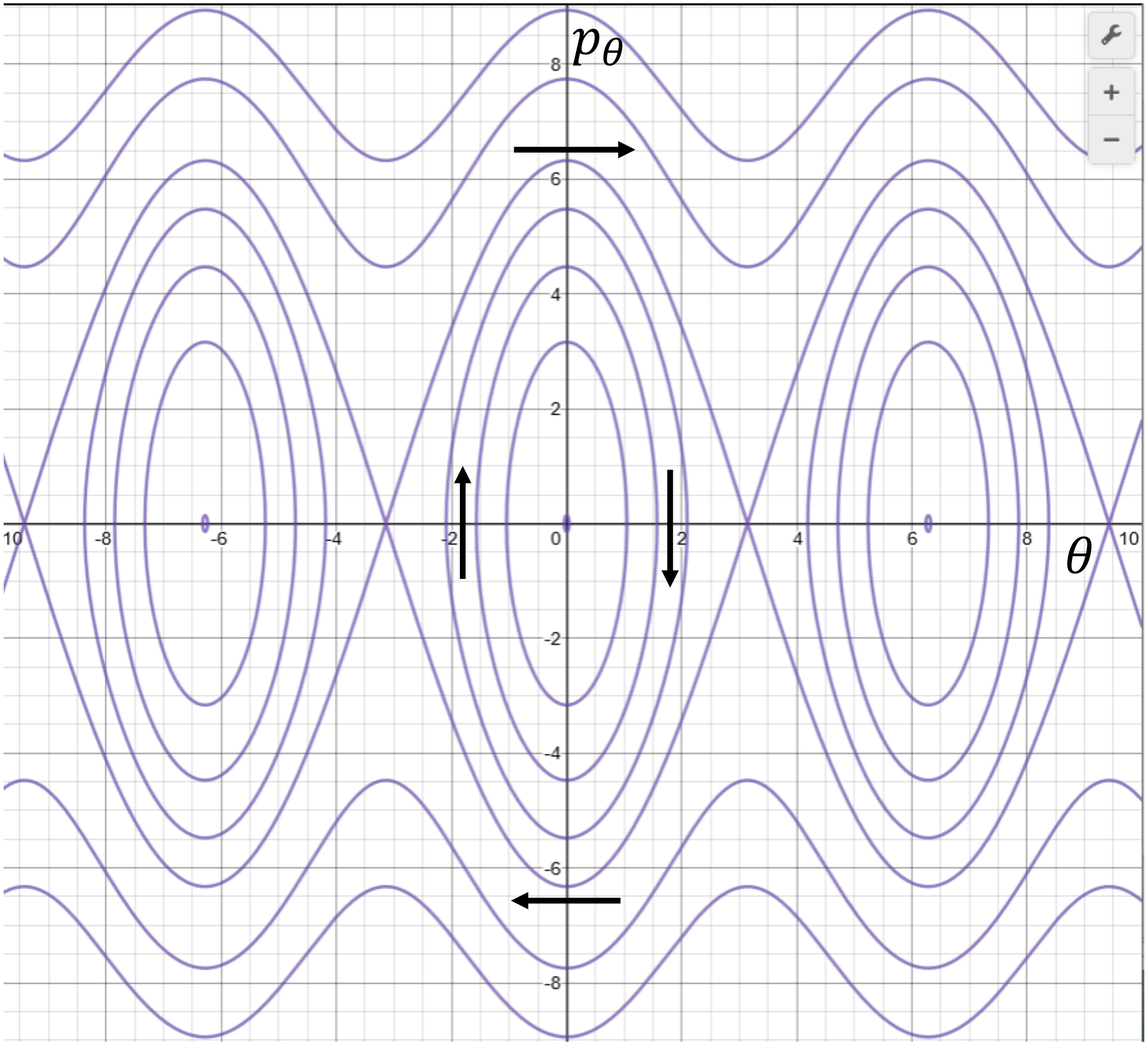
Step 4: Add additional contours for different values of H



$H < 2mgl$



$H > 2mgl$



Each contour represents a different value of H. In what direction is the particle moving along the countour lines?

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta$$

$$p_\theta > 0 \;\Rightarrow \dot{\theta} > 0$$

$$p_\theta < 0 \;\Rightarrow \dot{\theta} < 0$$

$$0 < \theta < \pi \;\Rightarrow\; \dot{p}_\theta < 0$$

$$-\pi < \theta < 0 \;\Rightarrow\; \dot{p}_\theta > 0$$

Hamilton's equations give us the slope and direction of the motion in the phase diagram

Phase Space Trajectories. Example 2: Harmonic Oscillator

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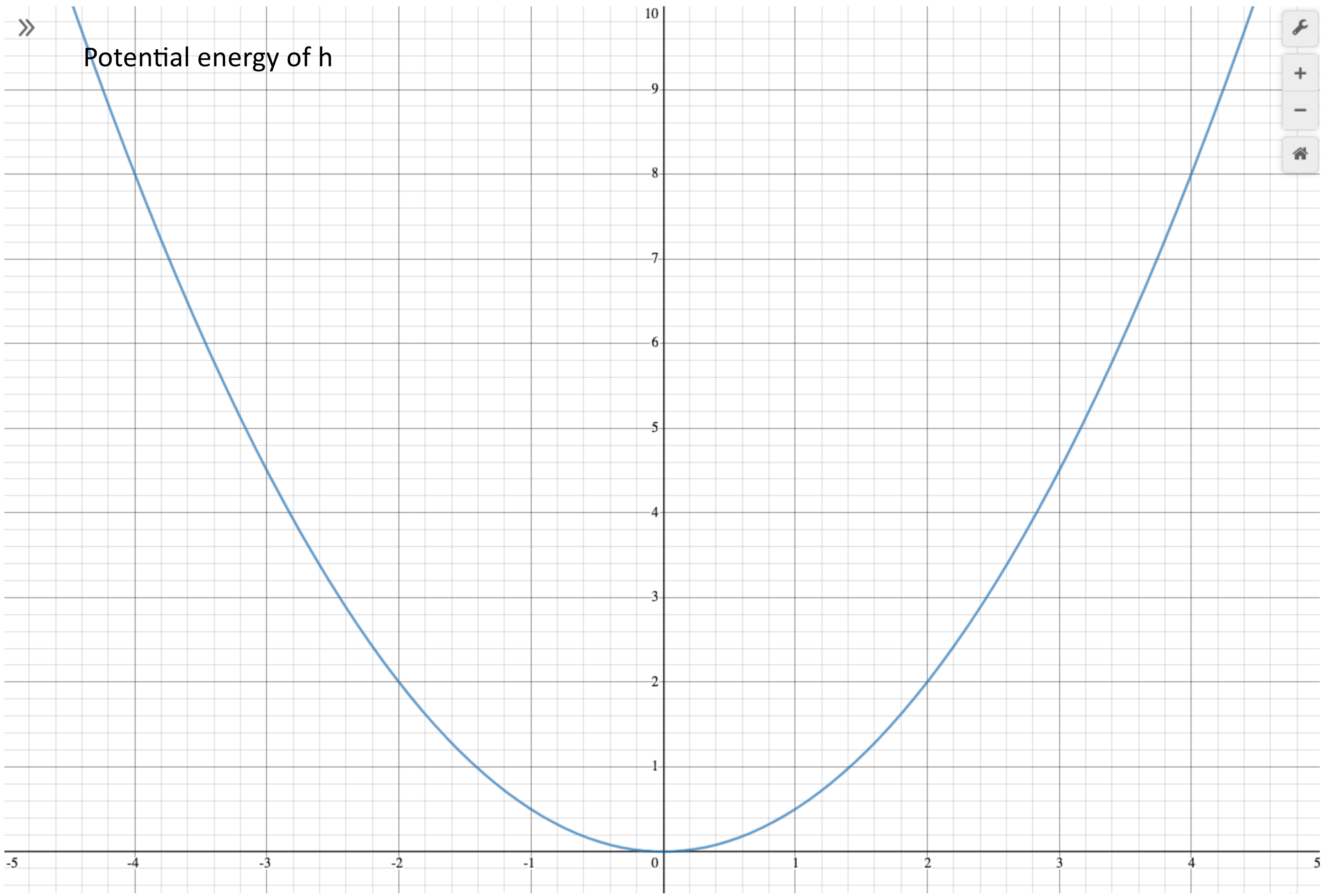
$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H = p_x\dot{x} - \frac{m\dot{x}^2}{2} + \frac{kx^2}{2} = m\dot{x}^2 - \frac{m\dot{x}^2}{2} + \frac{kx^2}{2}$$

$$= \frac{m\dot{x}^2}{2} + \frac{kx^2}{2}$$

$$H = \frac{p_x^2}{2m} + \frac{kx^2}{2}$$



Potential energy of h
10
9
8
7
6
5
4
3
2
1
0
-5
-4
-3
-2
-1
1
2
3
4
5

Step 2: Find the equilibrium solutions using Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

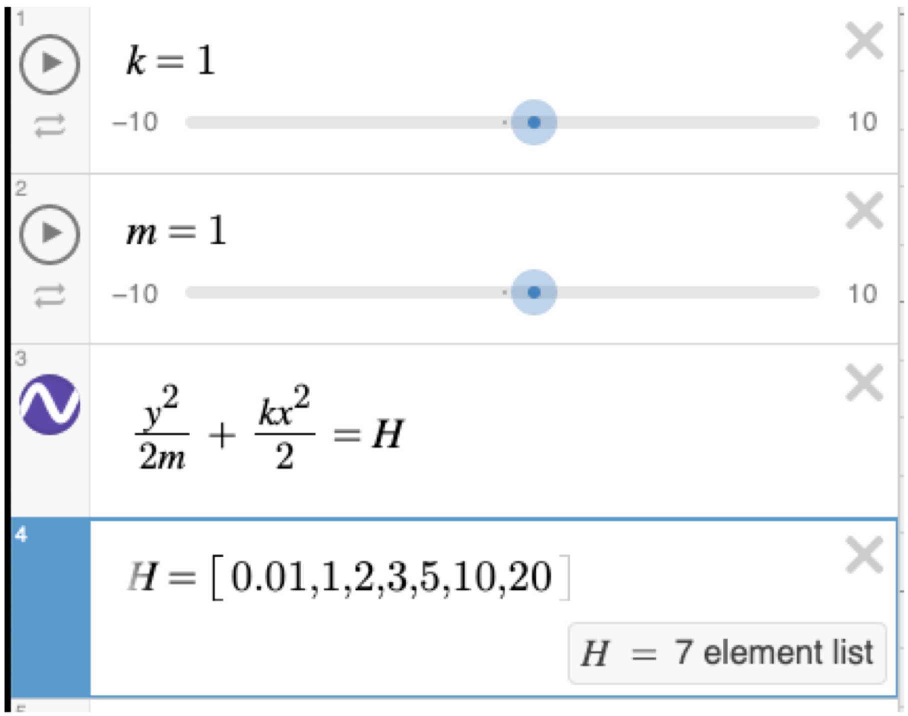
$$\dot{p}_x = -\frac{\partial H}{\partial x} = -kx$$

In equilibrium: $\dot{x} = 0, \dot{p}_x = 0$

$p_x = 0, x = 0$

The only equilibrium solution is when the oscillator is not oscillating

Plot contours for different values of H



Phase Space Trajectories. Example 3: Disk rolling without slipping down an inclined plane

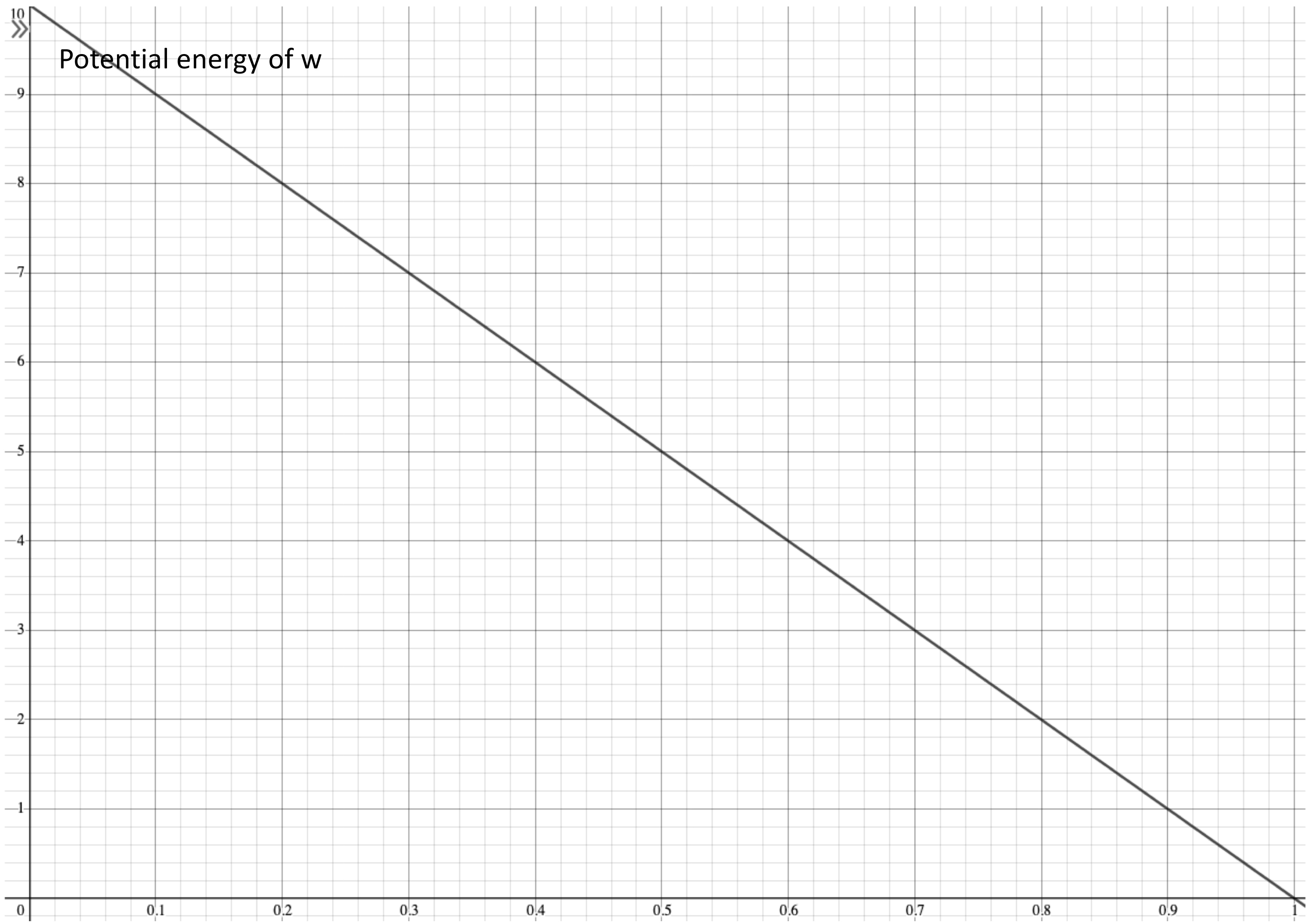
$$L = \frac{1}{2}m\dot{y}^2 - mg(l-y)\sin\alpha$$

$$p_y = m\dot{y}$$

$$H = p_y\dot{y} - \frac{1}{2}m\dot{y}^2 - mg(l-y)\sin\alpha$$

$$= m\dot{y}^2 - \frac{1}{2}m\dot{y}^2 + mg(l-y)\sin\alpha = \frac{1}{2}m\dot{y}^2 + mg(l-y)\sin\alpha$$

$$H = \frac{p_y^2}{2m} + mg(l-y)\sin\alpha$$


Potential energy of w
10
9
8
7
6
5
4
3
2
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Step 2: Find the equilibrium solutions using Hamilton's equations

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{2p_y}{3m}$$

$$\dot{p}_y = -\frac{\partial H}{\partial x} = mg\sin\alpha$$

In equilibrium: $\dot{p}_y = 0$ not possible, so no equilibrium solutions.

Plot contours for different values of H