Homework 8

Timothy Holmes
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Problem 1

To prove that a pure magnetic field in one frame cannot be a pur electric field in another frame we can do the following: For a general Lorentz transformation from K to a frame K' moving with velocity $\vec{v}$ relative to K, the transformation of the fields is

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{E})$$
$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{B})$$

From here we will $\vec{E} = 0$ so the transformation is

$$\vec{E}' = \gamma(0 + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot 0) = \gamma(\vec{\beta} \times \vec{B}).$$

Then for the magnetic field, when $\vec{E} = 0$, we have

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times 0) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{B}) = \gamma\vec{B} - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{B}).$$

Now, taking the cross product of both side with $\vec{\beta}$ will give

$$\vec{\beta} \times \vec{B}' = \gamma(\vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}\beta \times \vec{\beta}(\vec{\beta} \cdot \vec{B}).$$

Now, $\beta \times \beta$ is zero, so the final result is

$$\vec{\beta} \times \vec{B}' = \gamma(\vec{\beta} \times \vec{B}) = \vec{E}'.$$

If we substitute this into the equation we have for the magnetic field we get

$$\vec{E}' = \gamma(\vec{\beta} \times \vec{B}).$$

Therefore, we get

$$\vec{E}' = \vec{\beta} \times \vec{B}'$$

Now if we set $\vec{E}' = 0$ we have

$$0 = \vec{\beta} \times \vec{B}' = 0.$$

This satisfies a purely magnetic field in frame K to be a purely electric field in field K'.

Problem 2

We can show that the electric field in terms of the present position of the charge is given by

$$\vec{E} = \frac{q\vec{r}}{r^3\gamma^2(1-\beta^2 sin^2\phi)^{3/2}}$$

where r is the radial distance from the present position of the charge to the observer, and the angle $\phi = cos^{-1}(\hat{n}\cdot\hat{v})$ is between the direction of $\hat{n}$ and $\hat{v}$, where $\hat{n}$ is a unit radial vector from the present position of the charge to the observation point, and $\vec{v}$ is along the positive x_1-axis.

The point P in frame K' has coordinates

$$x_1 = -vt' \qquad x_2 = b \qquad x_3 = 0.$$

Then from Coulomb's law we get

$$E_1' = -\frac{qvt'}{r'^3} \qquad E_2' = \frac{qb}{r'^3} \qquad E_3' = 0$$
$$B_1' = 0 \qquad B_2' = 0 \qquad B_3' = 0.$$

To write this in the K frame we have $r'^2 = b^2 + v^2t'^2$ and $ct' = \gamma ct$. Therefore, $r'^2 = b^2 + v^2\gamma^2t^2$. and we now have that

$$E_1' = -\frac{q\gamma vt'}{(b^2+v^2\gamma^2t^2)^{3/2}} \qquad E_2' = -\frac{q}{(b^2+v^2\gamma^2t^2)^{3/2}} \qquad E_3' = 0.$$

Inverting these fields gives us

$$E_1 = -\frac{q\gamma vt'}{(b^2+v^2\gamma^2t^2)^{3/2}} \qquad E_2 = -\frac{q\gamma b}{(b^2+v^2\gamma^2t^2)^{3/2}} \qquad E_3 = 0.$$

Using Biot-Savart's law we find that $vb = vrsin\psi$. The electric field is directed along $\hat{n}$, this can be found by

$$\frac{E_1}{E_2} = -\frac{vt}{b}.$$

We can solve the distance between point P and charge q. We find that the distance between these points is $r^2 - \beta^2r^2sin^2\psi$ where $b^2 = \beta^2r^2sin^2\psi$ and $r^2 = b^2 + v^2t^2$. Or equivalently this distance is $\gamma^{-2}(b^2+\gamma^2v^2t^2)^{1/2}$ which happens to be the denominator to or electric field. Furthermore, the magnitude of our field is

$$\begin{aligned} E &= \frac{q\gamma}{(b^2+v^2\gamma^2t^2)^{3/2}}(b^2+v^2t^2)^{1/2} \\ &= \frac{q\vec{r}}{\gamma^2(r^2-\beta^2r^2sin^2\psi)^{3/2}} \end{aligned}$$

rearranging some terms we eventually arrive at

$$\vec{E} = \frac{q\vec{r}}{r^3\gamma^2(1-\beta^2sin^2\psi)^{3/2}}.$$