Homework 7

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Problem 1

Starting from

$$A = e^{-\vec{\omega}\cdot\vec{S}-\vec{\zeta}\cdot\vec{K}}$$

and in the case of rotation about the x^3 axis without any boost, we have $\vec{\omega} = \omega\hat{\epsilon_3}$ and $\vec{\zeta} = 0$. Thus, A will become

$$A = e^{-\omega S_3}$$

and Taylor expanding this equation gives

$$A = I + \omega S_3 + \frac{1}{2!}(-\omega S_3)^2 - \frac{1}{3!}(-\omega S_3)^3 + ...$$

Which can be rearranged and wrote as

$$A = (I + S_3^2) - S_3 sinh(\omega) - S_3^2 cosh(\omega)$$

In matrix for this is

$$A = \left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}\right) - \begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix} sinh(\omega) - \begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix} cosh(\omega)$$

$$= \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0&0\\0&0&-sinh(\omega)&0\\0&sinh(\omega)&0&0\\0&0&0&0\end{pmatrix} - \begin{pmatrix}0&0&0&0\\0&-cosh(\omega)&0&0\\0&0&-cosh(\omega)&0\\0&0&0&0\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0&0\\0&cosh(\omega)&sinh(\omega)&0\\0&-sinh(\omega)&cosh(\omega)&0\\0&0&0&1\end{pmatrix}$$

Problem 2

Express the Lorentz. scalar $F^{\alpha\beta}F_{\alpha\beta}$ in terms of $\vec{E}$ and $\vec{B}$, where $F^{\alpha\beta}$ is given by

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and $F^{\alpha\beta}$ can be obtained from $F^{\alpha\beta}$ by the procedure you worked out on the class worksheet (i.e., by putting $E_i \to -E_i$, and leaving B_i unchanged).

The tensor with two covariant indices can be found by setting all the the E_i components to $-E_i$ and leaving all the B_i components unchanged. This gives us the following

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Therefore,

$$\begin{aligned} F^{\alpha\beta}F_{\alpha\beta} &= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} E_x^2 + E_y^2 + E_z^2 & -E_yB_z + E_zB_y & E_xB_z - E_zB_x & -E_xB_y + E_yB_x \\ B_zE_y - B_yE_z & E_x^2 - B_z^2 - B_y^2 & E_xE_y + B_yB_x & E_xE_z + B_zB_x \\ -B_zE_x + B_xE_z & E_yE_x + B_xB_y & E_y^2 - B_z^2 - B_x^2 & E_yE_z + B_zB_y \\ B_yE_x - B_xE_y & E_zE_x + B_xB_z & E_zE_y + B_yB_z & E_z^2 - B_y^2 - B_x^2 \end{pmatrix} \end{aligned}$$

Problem 3

Consider the fundamental matrices $S_1, S_2, S_3, K_1, K_2, K_3$ written in equation (11.19) in Jackson. By explicit matrix multiplication, find the commutators

$$[S_2, S_3], \qquad [S_2, K_3], \qquad \text{and} \qquad [K_2, K_3]$$

$$[S_2, S_3] = S_2 S_3 - S_3 S_2$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = S_1$$

$$[S_2, K_3] = S_2 K_3 - K_3 S_2$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = K_1$$

$$[K_2, K_3] = K_2K_3 - K_3K_2$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -S_1$$

Problem 4

(a)

Recall that the fields $\vec{E}$ and $\vec{B}$ can be expressed in terms of the potentials as

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi \qquad \text{and} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

The components of $\vec{E}$ and $\vec{B}$ using the ∂^α notation for the x component is

$$\vec{E_x} = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0) \quad \text{and} \quad \vec{B_x} = \frac{\partial A_z}{\partial y} - \frac{\partial A^y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$$

where

$$\partial^\alpha = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right).$$

For the y component

$$\vec{E_y} = -\frac{1}{c}\frac{\partial A_y}{\partial t} - \frac{\partial \Phi}{\partial y} = -(\partial^0 A^2 - \partial^2 A^0) \quad \text{and} \quad \vec{B_y} = \frac{\partial A_z}{\partial y} - \frac{\partial A^y}{\partial z} = -(\partial^1 A^3 - \partial^3 A^1)$$

where

$$\partial^\alpha = \left(\frac{\partial}{\partial y^0}, -\vec{\nabla}\right).$$

For the z component

$$\vec{E_z} = -\frac{1}{c}\frac{\partial A_z}{\partial t} - \frac{\partial \Phi}{\partial z} = -(\partial^0 A^3 - \partial^3 A^0) \quad \text{and} \quad \vec{B_z} = \frac{\partial A_z}{\partial y} - \frac{\partial A^y}{\partial z} = -(\partial^1 A^2 - \partial^2 A^1)$$

where

$$\partial^\alpha = \left(\frac{\partial}{\partial z^0}, -\vec{\nabla}\right).$$

(b)

The element obtained above are the elements of the field tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

The following matrix can be used to match terms:

$$F^{\alpha\beta} = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Thus, we have $F^{\alpha 0}$ which gives

$$F^{00} = (\partial^0 A^0 - \partial^0 A^0) = 0 \qquad F^{10} = (\partial^1 A^0 - \partial^0 A^1) = E_x$$
$$F^{20} = (\partial^2 A^0 - \partial^0 A^2) = E_y \qquad F^{30} = (\partial^3 A^0 - \partial^0 A^3) = E_z$$

$F^{\alpha 1}$ which gives

$$F^{01} = (\partial^0 A^1 - \partial^1 A^0) = -E_x \qquad F^{11} = (\partial^1 A^1 - \partial^1 A^1) = 0$$
$$F^{21} = (\partial^2 A^1 - \partial^1 A^2) = B_z \qquad F^{31} = (\partial^3 A^1 - \partial^1 A^3) = -B_y$$

$F^{\alpha 2}$ which gives

$$F^{02} = (\partial^0 A^2 - \partial^2 A^0) = -E_y \qquad F^{12} = (\partial^1 A^2 - \partial^2 A^1) = -B_z$$
$$F^{22} = (\partial^2 A^2 - \partial^2 A^2) = 0 \qquad F^{32} = (\partial^3 A^2 - \partial^2 A^3) = B_x$$

$F^{\alpha 3}$ which gives

$$F^{03} = (\partial^0 A^3 - \partial^3 A^0) = -E_z \qquad F^{13} = (\partial^1 A^3 - \partial^3 A^1) = -B_y$$
$$F^{23} = (\partial^2 A^3 - \partial^3 A^2) = -B_x \qquad F^{33} = (\partial^3 A^3 - \partial^3 A^3) = 0$$