Single and multi-component universes

(1) In the lecture we saw that

$$\frac{\Omega_{\Lambda,o}}{\Omega_{m,o}} = 2.3$$

where the subscript, o indicates current time. However, this situation was not always true as you will now show. You'll make use of the fact that $\epsilon_m \to 1/a^3$

(a) Step one, write the term

$$\frac{\epsilon_\Lambda(a)}{\epsilon_m(a)}$$

in terms of the current time values, $\epsilon_{\Lambda,o}, \epsilon_{m,o}$, using the fact that $\epsilon_m(a) \approx 1/a^3$ and that $\epsilon_\Lambda(a)$ is a constant.

(b) Set $\epsilon_\Lambda(a) = \epsilon_m(a)$ and find the scale factor a at which the energy densities of these two components was equal. Does the result surprise you? Interpret the meaning of this result.

(c) Repeat the same calculation for matter and radiation. For these two components we have currently that

$$\frac{\epsilon_{m,o}}{\epsilon_{r,o}} \approx 3600.$$

Recall that $\epsilon_r \to 1/a^4$.

(d) Interpret this result. In particular, discuss the implications of a single component being the dominant component contributing to the Friedmann and fluid equations.

(2) The Friedmann equation (without Λ) is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{R_o^2 a^2}$$

(a) In an empty universe, $\epsilon = 0$. Solve the Friedmann equation for the case when the curvature, $k = 0$. Describe this universe.

(b) Again for an empty universe, can one have $k = 1$. Explain in detail.

(c) Solve for $a(t)$ for an empty universe with $k = -1$.

(d) Define $t_o = R_o/c$ and recall that the Hubble constant is $H_o = \dot{a}_{t_o}/a_{t_o}$. Find the relationship between the Hubble constant and t_o.(*Hint:* Recall that $1 + z = 1/a(t_e)$)

(e) While an empty universe seems unrealistic, remember that the density of matter and radiation, for example, are near zero. So this kind of universe can serve as an approximation to the real universe in some cases. In this universe, we can very clearly find the time of emission of signal. Show how t_e is found in this universe by using the relation,

$$1 + z = \frac{1}{a_{t_e}}.$$

(f) In any universe governed by a Robertson-Walker metric, the proper distance observed by an observer to a light source (for example, a distant galaxy) is

$$d_p(t_o) = c\int_{t_e}^{t_o} \frac{dt}{a(t)}.$$

Find the expression for proper distance in an empty universe.

(g) The proper distance grows directly proportional to the scale factor. Find the proper time $d_p(t_e)$ at the time of emission.

(h) Recap what you have learned about the evolution of empty universes.

(3) In the lecture we've shown that for a spatially flat, single-component universe, we have

$$a(t) = \left(\frac{t}{t_o}\right)^{2/(3+3w)} \quad w \neq -1;\ t_o = \frac{1}{1+w}\left(\frac{c^2}{6\pi G\epsilon_o}\right)$$

(a) Find the Hubble constant,

$$H_o = \left(\frac{\dot{a}}{a}\right)_{t=t_o}.$$

(b) Use

$$1+z = \frac{a(t_o)}{a(t_e)}$$

to find t_e in terms of z

(c) Use

$$d_p(t_o) = c \int_{t_e}^{t_o} \frac{dt}{a(t)}$$

to find the expression for the proper distance.

(d) Set $t_e = 0$ in the previous answer to find the distance to the horizon.

(4) In the lecture we have developed the general expressions for a spatially flat, single component universe.

(a) Set $w = 0,$ (matter), $w = 1/3$ (radiation), and $w = -1$ (Λ) and find t_o as a function of the Hubble constant, the proper distance, and the horizon distance for all these universes. For Λ find only the proper distances.

(b) Interpret your results

(5) Recap the important topics covered today. Compare and contrast your results with others at your table.

Homework 02–Due Friday Jan 31

1. Problem 4.4
2. Problem 4.5
3. Problem 5.1
4. Problem 5.3

Additional Grad Student Problem(s)

5. Consider a flat universe universe with a single component characterized by the equation of state parameter, $w = -1$.

   (a) Show that in such a universe the Friedmann equation takes the form
   $$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon_\Lambda a^2 \tag{1}$$

   (b) Show that the Hubble constant in a Λ-dominated universe is given by
   $$H_o \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_o} = \left(\frac{8\pi G\epsilon_\Lambda}{3c^2}\right)^{1/2} \tag{2}$$

   (c) Solve Eq. (1) and show that the dependence of the scale factor on time is given by
   $$a(t) = e^{H_o(t-t_o)}$$