PHY 420 Spring 2021

Homework 1 solutions

1. The alpha particle model of the nucleus postulates that, whenever possible, the neutrons and protons in a nucleus are arranged in alpha particles. It has been around for almost a hundred years, never verified, but never disproved either! According to this model, the nucleus of ^{12}C would be comprised of three alpha particles arranged in an equilateral triangle. If each side of this triangle has length 3×10^{-15} m, calculate the potential energy W of this arrangement. You may assume that the alpha particles are point charges.

Solution: We could solve this using either equation (1.50) in Jackson:

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

or the more symmetric form in equation (1.51) in Jackson:

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i} \sum_{j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

but if you use the later, take care that you understand the summation is over all i and all j.

I'll use the former (eq. 1.50) since it will be shorter, and also makes more intuitive sense; we are multiplying charges pairwise, dividing by the distance between them, and adding. Thus

$$W = \frac{1}{4\pi\epsilon_0} \left[\frac{(2e)(2e)}{(3 \times 10^{-15}\ \mathrm{m})} + \frac{(2e)(2e)}{(3 \times 10^{-15}\ \mathrm{m})} + \frac{(2e)(2e)}{(3 \times 10^{-15}\ \mathrm{m})} \right]$$

where $2e$ is the charge of an α-particle (since it is essentially a helium nucleus). Since the three terms are identical, we get

$$W = \frac{1}{4\pi\epsilon_0} \left[\frac{3\,(2e)(2e)}{(3 \times 10^{-15}\ \mathrm{m})} \right]$$

so that

$$W = \frac{1}{4\pi\epsilon_0} \left[\frac{12\,e^2}{(3 \times 10^{-15}\ \mathrm{m})} \right]$$

or

$$W = \frac{1}{4\pi(8.85 \times 10^{-12}\ \mathrm{m}^{-3}\,\mathrm{kg}^{-1}\,\mathrm{s}^4\,\mathrm{A}^2)} \left[\frac{12\,(1.6 \times 10^{-19}\ \mathrm{C})^2}{(3 \times 10^{-15}\ \mathrm{m})} \right]$$

from which we obtain

$$W = 9.2 \times 10^{-13}\ \mathrm{J}$$

Therefore, the potential energy of this arrangement is $\boxed{9.2 \times 10^{-13}\ \mathrm{J}}$.

Note: If you use the latter equation (1.51), you will have 6 terms in the summation ($i = 1, 2, 3$, and $j = 1, 2, 3$, so there are 9 terms, but we exclude the self-energy terms $i = j$, leaving us with 6 terms). Since all 6 terms are identical, it will introduce an extra factor of 2 in the numerator, but the factor 8 in the denominator, instead of 4 as in equation (1.50), will take out this extra factor of 2, leaving you with the same answer of 9.2×10^{-13} J.

2. In class, we showed that the potential energy can be expressed as an integral of the square of the electric field over all space:

$$W = \frac{\epsilon_0}{2} \int \left|\vec{E}\right|^2 d^3x$$

Suppose we have a charge Q that is ***uniformly distributed within a sphere*** of radius R. Show that

$$W = \frac{3}{5}\left[\frac{Q^2}{4\pi\epsilon_0 R}\right]$$

Solution: Assume, without having to derive it using Gauss' Law, that the electric field is radial, and given by

$$E_r(r) = \frac{Q}{4\pi\epsilon_0}\frac{r}{R^3} \qquad \text{for} \quad r < R$$

and

$$E_r(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} \qquad \text{for} \quad r > R$$

Then

$$W = \frac{\epsilon_0}{2}\int_0^R \left|\vec{E}\right|^2 d^3x + \frac{\epsilon_0}{2}\int_R^\infty \left|\vec{E}\right|^2 d^3x$$

so that

$$W = \frac{\epsilon_0}{2}\int_0^R \left[\frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}\right]^2 d^3x + \frac{\epsilon_0}{2}\int_R^\infty \left[\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\right]^2 d^3x$$

Let's write terms common to both integrals in one place:

$$W = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \left[\int_0^R \left[\frac{r}{R^3}\right]^2 d^3x + \int_R^\infty \left[\frac{1}{r^2}\right]^2 d^3x\right]$$

and since the volume element $d^3x = r^2\,dr\,\sin\theta\,d\theta\,d\phi$, this becomes

$$W = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \left[\int_0^R \left[\frac{r}{R^3}\right]^2 r^2\,dr + \int_R^\infty \left[\frac{1}{r^2}\right]^2 r^2\,dr\right]\int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi$$

where I've written the integrals over θ and ϕ in one place, since they are the same for both. The θ-integral gives 2, and the ϕ-integral gives 2π, so we have

$$W = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \left[\frac{1}{R^6}\int_0^R r^4\,dr + \int_R^\infty \frac{dr}{r^2}\right]\left(4\pi\right)$$

and thus

$$W = \frac{Q^2}{8\pi\epsilon_0}\left\{\frac{1}{R^6}\left[\frac{r^5}{5}\right]_0^R + \left[-\frac{1}{r}\right]_R^\infty\right\} = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{R^6}\left(\frac{R^5}{5} - 0\right) - \left(0 - \frac{1}{R}\right)\right]$$

so that

$$W = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{5R} + \frac{1}{R}\right] = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1+5}{5R}\right] = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{6}{5R}\right] = \frac{Q^2}{4\pi\epsilon_0}\left[\frac{3}{5R}\right]$$

Therefore, we get

$$W = \frac{3}{5}\left[\frac{Q^2}{4\pi\epsilon_0 R}\right]$$

which is the desired result.

3. The magnetic analog of equation (4.86) for electrostatics is

$$\delta W = \int \vec{H} \cdot \delta\vec{B}\, d^3x$$

If a linear relation exists between $\vec{B}$ and $\vec{H}$, then show that the total magnetic energy will be

$$W = \frac{1}{2}\int \vec{H} \cdot \vec{B}\, d^3x$$

Solution: This problem was meant to give you practice working with magnetostatic energy, which I'd asked you to look over from the class summary, since we wouldn't have time to do worksheet problems on it.

The linearity of a medium is defined with respect to the response of the medium to an external (electric or magnetic) field. In particular, we are talking about the polarization response. Let's think in terms of electric polarization since it is easier to visualize. As Jackson notes, applying an electric field to a medium made up of a large number of atoms or molecules will cause the charges bound in each molecule to respond to the applied field and execute perturbed motions, thereby distorting the molecular charge density. Thus, the multipole moments will be different from what they were in the absence of the field. The dominant multipole moment in the presence of applied fields is the dipole, and this produces an electric polarization (dipole moment per unit volume), $\vec{P}(\vec{x})$. If we assume that the response of the medium to an applied field is linear, then we can write

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

where χ_e is called the electric susceptibility of the medium; note that it is also usual to assume the medium to be isotropic so that χ_e is independent of direction. Since $\vec{D} = \epsilon_0\vec{E} + \vec{P}$, linearity of the medium is often expressed by saying that $\vec{D} = \epsilon\vec{E}$, where ϵ is a constant quantity.

Similar considerations apply to magnetic fields, so we can say that if the response of the system to the (external) magnetic field is linear, then a linear relation exists between $\vec{B}$ and $\vec{H}$, given by $\vec{B} = \mu\vec{H}$. Thus,

$$\delta(\vec{H} \cdot \vec{B}) = \delta H \cdot \vec{B} + \vec{H} \cdot \delta B$$

so that

$$\delta(\vec{H} \cdot \vec{B}) = \delta H \cdot \left(\mu\vec{H}\right) + \vec{H} \cdot \delta B$$

or

$$\delta(\vec{H} \cdot \vec{B}) = \mu\, \delta H \cdot \vec{H} + \vec{H} \cdot \delta B = \delta\left(\mu\vec{H}\right) \cdot \vec{H} + \vec{H} \cdot \delta B$$

and thus

$$\delta(\vec{H} \cdot \vec{B}) = \delta\vec{B} \cdot \vec{H} + \vec{H} \cdot \delta B = \vec{H} \cdot \delta B + \vec{H} \cdot \delta B = 2\, \vec{H} \cdot \delta B$$

Thus, we have demonstrated that

$$\vec{H} \cdot \delta B = \frac{1}{2}\delta(\vec{H} \cdot \vec{B})$$

so that

$$\delta W = \int \vec{H} \cdot \delta\vec{B}\, d^3x = \frac{1}{2}\int \delta(\vec{H} \cdot \vec{B})\, d^3x$$

Integrating from $\vec{B} = 0$ to $\vec{B}$ takes away the δ and we get

$$W = \int dW = \frac{1}{2}\int \vec{H} \cdot \vec{B}\, d^3x$$

which is the desired result.

4. In class, we discussed how a problem of considerable interest is the change in energy when a dielectric object is placed in an electric field whose sources are fixed. Suppose initially that there exists an electric field $\vec{E}_0$ due to a distribution of charges $\rho_0(\vec{x})$ in a medium with ϵ_0, which may be a function of position. Then, with the sources fixed in position, a dielectric object of volume V_1 is introduced into the field, changing the field from $\vec{E}_0$ to $\vec{E}$. Then $\epsilon(\vec{x})$ has the value ϵ_1 inside V_1 and ϵ_0 outside V_1. The change in the energy is then given by

$$W = \frac{1}{2}\int \left(\vec{E}\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0\right) d^3x \tag{1}$$

Show that this can be written as

$$W = \frac{1}{2}\int \left(\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0\right) d^3x$$

Solution: Equation (1) and the desired result tell us that starting from terms like $\vec{E}\cdot\vec{D}$ we must end up with "cross-terms" like $\vec{E}\cdot\vec{D}_0$, so let's begin by writing

$$\left(\vec{E}+\vec{E}_0\right)\cdot\left(\vec{D}-\vec{D}_0\right) = \vec{E}\cdot\vec{D} - \vec{E}\cdot\vec{D}_0 + \vec{E}_0\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0$$

so that

$$\vec{E}\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0 = \vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0 + \left(\vec{E}+\vec{E}_0\right)\cdot\left(\vec{D}-\vec{D}_0\right)$$

Then, we can replace $(\vec{E}\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0)$ in equation (1) with the right hand side above to get

$$W = \frac{1}{2}\int \left(\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0\right) d^3x + \frac{1}{2}\int \left(\vec{E}+\vec{E}_0\right)\cdot\left(\vec{D}-\vec{D}_0\right) d^3x \tag{2}$$

That is it, provided we can show that the second term on the right goes to zero. Writing $(\vec{E}+\vec{E}_0) = -\vec{\nabla}\Phi$, the *second term on the right hand side* of equation (2) becomes

$$\frac{1}{2}\int \left(-\vec{\nabla}\Phi\right)\cdot\left(\vec{D}-\vec{D}_0\right) d^3x = -\frac{1}{2}\left[\int \vec{\nabla}\cdot\left\{\Phi\left(\vec{D}-\vec{D}_0\right)\right\} d^3x + \int \Phi\vec{\nabla}\cdot\left(\vec{D}-\vec{D}_0\right) d^3x\right]$$

where I've used a result from the inside front cover of Jackson. Using the divergence theorem, the *first term on the right above* can be written as a surface integral, so equation (2) is now

$$W = \frac{1}{2}\int \left(\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0\right) d^3x - \frac{1}{2}\left[\int_S \left\{\Phi\left(\vec{D}-\vec{D}_0\right)\right\}\hat{n}\, da + \int \Phi\vec{\nabla}\cdot\left(\vec{D}-\vec{D}_0\right) d^3x\right]$$

The first term in the square brackets is zero because Φ is a localized function — what we mean is that since the volume integral is over all space, the surface bounding it must be at infinity, and we know Φ is defined to be zero at infinity. Meanwhile, the second term in the square brackets is zero because $\vec{\nabla}\cdot(\vec{D}-\vec{D}_0) = 0$, because the source charge density $\rho_0(\vec{x})$ is assumed unaltered by the insertion of the dielectric object (i.e., if $\vec{\nabla}\cdot\vec{D}_0 = \rho_0$, $\vec{\nabla}\cdot\vec{D}$ is also equal to ρ_0). Therefore, we have our desired result that the change in energy W is given by

$$W = \frac{1}{2}\int_{V_1} \left(\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0\right) d^3x$$

In writing the expression above, I've emphasized also that we need to carry out the integration only over the volume V_1 of the object, since, outside V_1, we have $\vec{D} = \epsilon_0\vec{E}$ (meaning that the contribution to the integral is zero outside V_1).