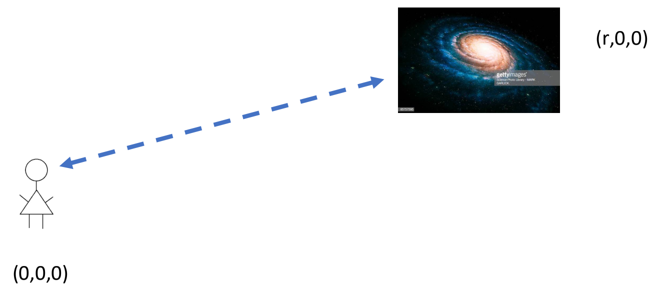
Announcements:

Class time change for Wed, Jan 22 3:30—5:00
Please make sure and choose your topic by Friday

Now that we have a metric, we ask questions about distances. We'll begin with *proper distance.*



To find distance, we must agree upon a time and fix that time, so, *dt = 0*.

$$\begin{aligned} ds^2 &= -c^2dt^2 + a(t)^2\left[dr^2 + S_k(r)^2 d\Omega^2\right] \\ &= a(t)^2\left[dr^2 + S_k(r)^2 d\Omega^2\right] \\ &= a(t)^2 dr^2 \\ ds &= a(t)dr \end{aligned}$$

The *proper distance* is found by integrating over the radial comoving component, *r*

$$d_p(t) = \int ds = a(t)\int_0^r dr' = a(t)r$$

The rate of change for the proper distance between the two points is then

$$\dot{d}_p(t) = \dot{a}(t)\, r = \frac{\dot{a}}{a}\; d_p$$

At the current time, $t = t_o$ we have a linear relationship between velocity and distance. First define

$v_p(t_o) \equiv \dot{d}_p(t_o) \quad H_o = \left(\frac{\dot{a}}{a}\right)_{t=t_o}$ Then, we arrive at $v_p(t_o) = H_o\, d_p(t_o)$

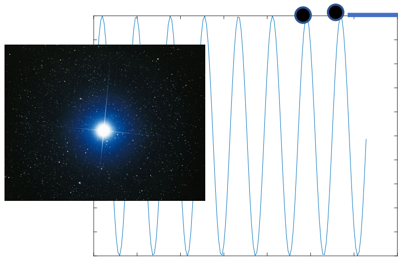
$v_p(t_o) = H_o \, d_p(t_o)$ Is well and good, but not very practical because one cannot measure proper distances

However, we can measure redshifts. While this doesn't give the proper distance, it will give us *a(t)*…progress. In the following we are using light as our tracer of distance and recall that for light *ds = 0*. Once again, the angular part plays no role in the metric since, *d* Ω = 0. From the metric we then have

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_k(r)^2 d\Omega^2 \right] = 0 \quad \text{for light}$$

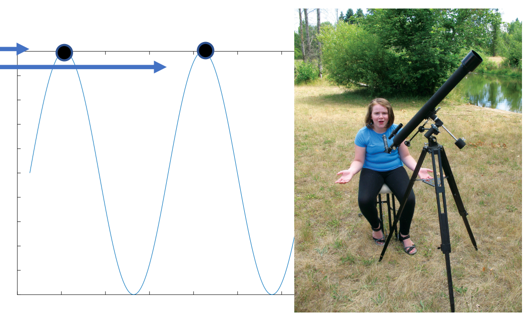
$$c^2 dt^2 = a(t)^2 dr^2 \Rightarrow c \frac{dt}{a(t)} = dr \Rightarrow \boxed{c \int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_o^r dr'}$$

$$c \int_{t_e+\lambda_e/c}^{t_o+\lambda_o/c} \frac{dt}{a(t)} = r$$



r

r



$$c \int_{t_e}^{t_o} \frac{dt}{a(t)} = r$$

$$\int_{t_o}^{t_e} \frac{dt}{a(t)} = \int_{t_e+\lambda_e/c}^{t_o+\lambda_o/c} \frac{dt}{a(t)}$$

So we are here: $\int_{t_o}^{t_e} \frac{dt}{a(t)} = \int_{t_e+\lambda_e/c}^{t_o+\lambda_o/c} \frac{dt}{a(t)}$ To proceed, we subtract $\int_{t_e+\lambda_e/c}^{t_o} \frac{dt}{a(t)}$ from both integrals and we note that over these time scales ***a(t)*** can be treated as constant so that we get

$$\frac{1}{a(t_e)} \int_{t_e}^{t_e+\lambda_e/c} dt = \frac{1}{a(t_o)} \int_{t_o}^{t_o+\lambda_o/c} dt$$

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_o}{a(t_o)}$$

Thus although we can't get distances (yet) by just measuring the redshift, we can tell how fast the universe is expanding.

Now define

$$z \equiv \frac{\lambda_o - \lambda_e}{\lambda_e}$$

so that

$$\boxed{1 + z = \frac{a(t_o)}{a(t_e)} = \frac{1}{a(t_e)}} \quad (a(t_o) = 1)$$

Last lecture we essentially introduce cosmological *kinematics*. Today we will begin our look at *dynamics.*

Dynamics is cosmology are governed by three factors

a) Einstein's General Theory of Relativity
b) The Robertson-Walker metric which arises in an isotropic and homogeneous universe
c) The Friedman equations, in which b) is applied to a) to arrive for a dynamical picture of how the universe evolves in time.
d) To do this ourselves requires an understanding of *tensors*, so we'll forgo the full derivation. We'll just present the results when needed.

The Einstein Equation(s) $\underbrace{G_{\mu\nu}}_{\substack{Spacetime \\ Geometry}} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \underbrace{8\pi T_{\mu\nu}}_{\substack{Matter \\ distribution}}$

$$\begin{aligned} G_{\mu\nu} &= \text{The Einstein tensor} \\ R_{\mu\nu} &= \text{The Ricci tensor} \\ g_{\mu\nu} &= \text{The metric tensor} \\ R &= \text{The Ricci scalar} \\ T_{\mu\nu} &= \text{The stress-energy tensor} \end{aligned}$$

16 *coupled* partial differential equations!!

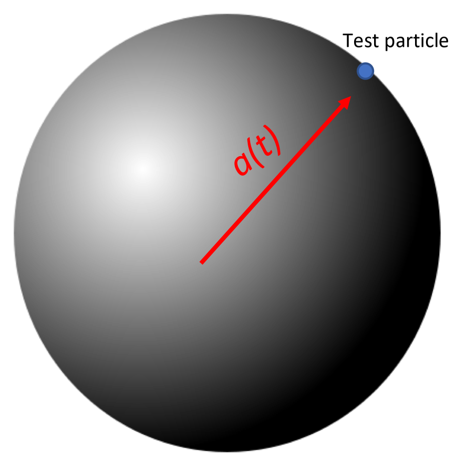
Luckily, for an *isotropic and homogeneous* universe, only the $G_{00} = T_{00}$ equation remains. Whew!

The Friedman equation--how does *a(t)* evolve

- Can derive using mostly Newtonian mechanics if we include two results from *GR*
  1. *Birkhoff's Theorem:* for a spherically symmetric system, the force due to gravity at radius *r* is determined only by the mass *interior* to that radius.
  2. *Energy* contributes to the gravitating mass density:

$$\epsilon(t) \equiv \underbrace{\rho_m}_{\text{density of matter}} + \underbrace{\frac{\mu}{c^2}}_{\text{energy density of radiation and relativistic particles}}$$

- Recall we are solving only $G_{00} = T_{00}$



Do question (1a) on the worksheet and S T O P

The Friedman equation (1a):

$$m_p \ddot{a} = \frac{-GM_s m_p}{a^2} \quad \text{but}$$

$$\rho = \frac{M_s}{(4\pi/3)a^3} \Rightarrow M_s = \tfrac{4}{3}\rho a^3 \quad \text{so}$$

$$\ddot{a} = -\frac{4\pi}{3}G\rho a$$

Now note the we can rewrite the density as $\rho = \rho_o a^{-3}$ and substituting above we get that $\ddot{a} = -\frac{4\pi}{3}\frac{G\rho_o}{a^2}$

Do question (1b) on the worksheet and S T O P.

(1b) The universe cannot be static

To proceed, we have to figure out a way to solve $\ddot{a} + \frac{4\pi}{3}\frac{G\rho_o}{a^2} = 0$. What follows is a series of ***tricks*** designed to do the integration.

Do question (1c) on the worksheet and S T O P

(1c)

$$\ddot{a} + \frac{4\pi}{3}\frac{G\rho_o}{a^2} = 0 \quad \text{multiply both sides by } \dot{a}$$

$$\dot{a}\ddot{a} + \frac{4\pi}{3}\frac{G\rho_o}{a^2}\dot{a} = 0 \quad \text{now use fact that } d(\dot{a})/dt = 2\dot{a}\ddot{a}$$

$$\frac{1}{2}\frac{d\,(\dot{a})}{dt} + \frac{4\pi}{3}\frac{G\rho_o}{a^2}\frac{da}{dt} = 0$$

Do question (1d) on the worksheet and S T O P

$$\frac{1}{2}\frac{d\,(\dot{a})}{dt} + \frac{4\pi}{3}\frac{G\rho_o}{a^2}\frac{da}{dt} = 0 \quad \text{now use } \frac{1}{a^2}\frac{da}{dt} = -\frac{d(1/a)}{dt}$$

$$\frac{1}{2}\frac{d\,(\dot{a})}{dt} - \frac{4G\rho_o\pi}{3}\frac{d(1/a)}{dt} = 0 \quad \text{factor out a } \frac{d}{dt}$$

$$\frac{d}{dt}\left[\dot{a}^2 - \frac{(8\pi G\rho_o/3)}{a}\right] = 0 \quad \text{what about the quantity in brackets?}$$

Let's finish up. $\left[\dot{a}^2 - \frac{(8\pi G\rho_o/3)}{a}\right] = -k; \quad \text{a constant}$

Recall that we had $\rho = \frac{\rho_o}{a^3} \Rightarrow \rho_o = \rho a^3$ and that *GR* tells that it is energy as well as matter that contributes to gravity . Substituting for density and dividing by ***a***^{2} gives

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2}$$

And finally replacing the density with the energy + matter gives (Ryden's form) the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{kc^2}{R_o^2}\frac{1}{a^2} \qquad (1)$$

Energy + matter

Friedmann's first equation.

Notes:

- You will see it in several different but equivalent forms
- In Ryden's form, she puts back the constants *(c)* which many authors set to 1
- In our derivation, we set $R_o = 1$. She leaves it in the more general form.

Finish question (1) on the worksheet and S T O P

The Friedmann equation: $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{kc^2}{R_o^2}\frac{1}{a^2} \qquad (1)$$

Let's play around with Eq. (1) a bit. First, the constant *k* is the curvature and can take on the values *k = +1, 0, -1* which correspond to *positive, flat, or negatively* curved universes.

Next recall that $H(t) = \frac{\dot{a}}{a}$ so that Eq (1) can also be written as

$$(H(t))^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{kc^2}{R_o^2}\frac{1}{a^2} \qquad (2)$$

H(t) is known as the *Hubble parameter*. At present time we have $H_o = H(t_o) = \left(\frac{\dot{a}}{a}\right)_{t=0} \approx 68\ \mathrm{km/(s\ Mpc)}$ and H_o is known as the *Hubble constant*

The Friedmann equation evaluated at the present moment is

$$H_o^2 = \frac{8\pi G}{3c^2}\epsilon_o - \frac{kc^2}{R_o^2}\frac{1}{(a_o^2 = 1)} \qquad (3)$$

Do question (2) on the worksheet and S T O P

(2a) $\frac{R_o^2 a(t)^2}{c^2}\left(H(t)^2 - \frac{8\pi G}{3c^2}\epsilon(t)\right) = -k$

(2b) $\frac{R_o^2 a(t)^2}{c^2}\left(H(t)^2 - \frac{8\pi G}{3c^2}\frac{3c^2}{8\pi G}H(t)^2\right) = 0 = -k$

(2c) positive, (2d) negative

(2e) $1 - \Omega(t) = -\frac{kc^2}{R_o^2 a(t)^2 H(t)^2}$

Regardless of the form used for Friedmann equation, we still do not have enough information to solve for *a(t)* because there are two unknowns

1. *a(t)*
2. $\varepsilon(t)$

Thermodynamics to the rescue. Two key facts

1. The laws of thermodynamics do not change form in *GR*
2. Except perhaps at very earliest moments, the expansion of the universe is *adiabatic.*

The first fact means we can write (1st law of thermo): $d\,Q = d\,E + P\,d\,V$

The second fact means that $d\,Q = 0$ so $0 = d\,E + P\,d\,V$.

At your table discuss why the expansion of an isotropic and homogeneous universe must be adiabatic.

For a homogeneous, isotropic, and expanding universe, the first law is (note the time evolution)

$$\dot{E} + P\dot{V} = 0$$

Let's expound on this a bit. For the *volume* of a sphere of comoving radius, r_s, that is expanding along with the universe, we have

$$\begin{aligned} V(t) &= \frac{4\pi}{3} r_s^3 a(t)^3 \quad \text{or} \\ \dot{V} &= \frac{4\pi}{3} r_s^3 \left(3a^2\dot{a}\right) = \boxed{V\left(3\frac{\dot{a}}{a}\right)} \end{aligned}$$

Having taken care of the volume, let's look now at the *E.* The internal energy of a sphere is,

$$\begin{aligned} E(t) &= \epsilon(t)V(t) \\ \dot{E} &= V\dot{\epsilon} + \epsilon\dot{V} \\ \dot{E} &= V\left(\dot{\epsilon} + 3\frac{\dot{a}}{a}\epsilon\right) \end{aligned}$$

And now substituting these into the first law we have

$$\begin{aligned} \dot{E} + P\dot{V} &= 0 \\ V\left(\dot{\epsilon} + 3\frac{\dot{a}}{a}\epsilon\right) + PV\left(3\frac{\dot{a}}{a}\right) &= 0 \end{aligned}$$

$$\boxed{\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0}$$ *Fluid equation*

The two equations we have are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{R_o^2 a^2} \tag{1}$$

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0 \tag{2}$$

These two can be combined to form a 3rd equation that is often very useful. This will be left as an exercise, but result is of combing Eqs (1) and (2) is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) \tag{3}$$

This is called the *acceleration equation*

Do question (3) on the worksheet and S T O P