Homework 6

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PHY 440 Classical Mechanics

December 24, 2020

Problem 1

1a.

The Lagrangian for this system is given as

$$L = \frac{1}{2}m\dot{r}^2 + q\dot{r}\cdot A - q\phi.$$

The only generalized position in the Lagrangian is r and generalized velocity is $\dot{r}$. r can be expanded into x, y, and z components but for the sake of simplicity r will just be left as r without any loss of information. Thus, the generalized momenta is

$$\frac{\partial L}{\partial \dot{q}_i} \to p_r = m\dot{r} + q\cdot A.$$

The Hamiltonian can be found using the following equation

$$H(\dot{q}_i, p_i, t) = \sum_{i=1}^{n} \dot{q}_i p_i - L(\dot{q}_i, p_i, t).$$

Plugging the generalized momenta back into the Hamiltonian the equation becomes

$$H(r, \dot{r}, t) = m\dot{r}^2 + q\dot{r}\cdot A - \frac{1}{2}m\dot{r}^2 + q\dot{r}\cdot A - q\phi$$

The Hamiltonian can be reduced by combining like terms as such

$$H(r, \dot{r}, t) = \frac{1}{2}m\dot{r}^2 + q\phi$$

Writing it in terms of our generalized momenta we can solve for $\dot{r}$ in the generalized momentum equation. Plugging in the results for $\dot{r}$ where $\dot{r} = (p_r - q\cdot A)/m$, gives us the Hamiltonian,

$$H(r, \dot{r}, t) = \frac{1}{2m}(p_r - q\cdot A)^2 + q\phi.$$

Writing it in this form is important since it allows us to since for Hamilton's equation of motion.

1b.

The canonical equations of motion for the Hamiltonian are given by

$$\dot{p}_r = -\frac{\partial H}{\partial q_i} \to m\ddot{r} + q\dot{A}$$

and

$$\dot{r} = \frac{\partial H}{\partial p_i} \to \frac{1}{m}(p - q \cdot A).$$

We can rewrite the equation as

$$m\ddot{r} = -q\Big(\frac{\partial \phi}{\partial r} + \frac{\partial A}{\partial t}\Big) + q\Big[\dot{r}\Big(\frac{\partial A}{\partial x} - \frac{\partial A}{\partial y}\Big) + \dot{r}(\frac{\partial A}{\partial x} - \frac{\partial A}{\partial z}\Big)\Big]$$

This can be further reduced to

$$m\ddot{r} = qE + q[\dot{r}(\nabla_r \times A)].$$

The equation of motion above will finally reduce to

$$m\ddot{r} = q[E + \dot{r} \times B].$$

1c.

We have the Hamiltonian as,

$$\dot{p}_r = -\frac{\partial H}{\partial q_i} \to m\ddot{r} + q\dot{A}.$$

When the electric field and magnetic field are zero, that is $E, B(r) = 0$, the ϕ component is also zero. Therefore, the Hamiltonian will reduce down to

$$H = \frac{1}{2m}|p_r - q \cdot A|^2.$$

Since we know that $p_r = m\dot{r} + q \cdot A$. We can plug this value into the Hamiltonian such that

$$H = \frac{d}{dt}\Big[\frac{1}{2m}|m\dot{r} - q \cdot A + q \cdot A|^2\Big] \to H = \frac{d}{dt}\Big[\frac{1}{2m}|m\dot{r}|^2\Big] \to \Big[\frac{d}{dt}\frac{1}{2m}m^2\dot{r}^2\Big] = 0.$$

Finally, we can set our velocity term $\dot{r}^2 = v^2$ and one of our mass terms m will cancel out. What out time derivative tells us is that this equation of motion is constant, which leaves us with

$$\frac{d}{dt}\frac{mv^2}{2} = const.$$

1d.

The last problem had the $q\phi$ term drop off since $E = B(r) = 0$. This problem has constant non negative terms where $B = B(r)$ and $E = E_0$. Given the Hamiltonian

$$H = \frac{d}{dt}\Big[\frac{1}{2m}|m\dot{r} - q \cdot A + q \cdot A|^2 + q\phi\Big].$$

We can reduce it down like the last equation and set the differentials,

$$\frac{d}{dt}\Big[\frac{1}{2m}m^2\dot{r}^2 + q\frac{d\phi}{dt}\frac{\partial r}{\partial t} + \frac{\partial \phi}{\partial t}\Big] = 0.$$

This equation can reduce down to

$$\frac{d}{dt}\Big[\frac{1}{2}m\dot{r}^2 + q\frac{d\phi}{dt}\dot{r}\Big] = 0.$$

Like the previous equation we can set our velocity term $\dot{r}^2 = v^2$ and we get

$$\frac{d}{dt}\frac{mv^2}{2} = q(E_0 \cdot \dot{r}).$$

Problem 2

2a.

The Lagrangian that governs the system for a mass on a vertical rotating hoop is given in the form of

$$L = \frac{mR^2}{2}(\dot{\theta}^2 + \omega^2 sin^2(\theta)) + mgRcos(\theta).$$

The Hamiltonian can be found using the following equation

$$H(\dot{q}_i, p_i, t) = \sum_{i=1}^{n} \dot{q}_i p_i - L(\dot{q}_i, p_i, t).$$

Since the Lagrangian is given above, the only missing information from the Hamiltonian is the generalized momenta. The generalized momenta is found using

$$\frac{\partial L}{\partial \dot{q}_i}$$

The only generalized position in the Lagrangian is θ and generalized velocity is $\dot{\theta}$. Therefore, the only generalized momenta for the Hamiltonian is

$$p_\theta = mR^2\dot{\theta}.$$

Plugging the generalized momenta back into the Hamiltonian the equation becomes

$$H(\theta, \dot{\theta}, t) = mR^2\dot{\theta}^2 - \frac{mR^2}{2}\Big(\dot{\theta}^2 + \omega^2 sin^2(\theta)\Big) - mgRcos(\theta)$$

The Hamiltonian can be reduced by combining like terms and writing it in terms of our generalized momenta as

$$H(\theta, \dot{\theta}, t) = \frac{p_\theta^2}{2mR^2} - \frac{mR^2}{2}\Big(\omega^2 sin^2(\theta)\Big) - mgRcos(\theta).$$

The effective potential for this problem is $U(\theta) = -mR^2/2(\omega^2 sin^2(\theta)) + mgRcos(\theta)$. The final Hamiltonian equations becomes

$$H(\theta, \dot{\theta}, t) = \frac{p_\theta^2}{2mR^2} + U(\theta).$$

2b.

The canonical equations of motion for the Hamiltonian are given by

$$\dot{p_\theta} = \frac{\partial H}{\partial \theta} \rightarrow -mR^2(\omega^2 sin(\theta)cos(\theta)) - mgsin(\theta)$$

and

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \rightarrow \frac{p_\theta}{mR^2}.$$

What the canonical equations of the Hamiltonian tell us is that since $p_\theta = mR^2\dot{\theta}$ then $\ddot{p}_\theta = mR^2\ddot{\theta}$. Substituting this into the equation above we get

$$mR^2\ddot{\theta} = -mR^2(\omega^2 sin(\theta)cos(\theta)) - mgRsin(\theta)$$

This the equation of motion can be reduced into its final for of

$$\ddot{\theta} = (\omega^2 cos(\theta) - \frac{g}{R})sin(\theta)$$

Now, if we are to look at the equilibrium conditions the $cos(\theta)$ and $sin(\theta)$ terms can tell a lot. If we set the equation of motion equal to zero since in the equilibrium condition say the acceleration is zero, then $sin(\theta) = 0$ and $cos(\theta) = g/(R\omega^2)$. $sin(\theta)$ is zero whenever $\theta = 0$ or $\theta = \pi$, indicating the bead is either at the bottom of the hoop or the top of the hoop. As for $cos(\theta)$ where the equilibrium points are $\theta = \pi/2$ and $\theta = -\pi/2$.

2c.

Based on the results from above we are plotting θ vs the function $U(\theta)$. Since $U(\theta)$ components are sinusoidal, we should expect to see some sort of sinusoidal plot. Furthermore, the system is at equilibrium when θ is constant, leaving $\dot{\theta} = 0$ and $\ddot{\theta} = 0$. Thus, the equilibrium condition is

$$sin(\theta)\Big(- \omega^2 cos(\theta) + \frac{g}{R}\Big) = 0$$

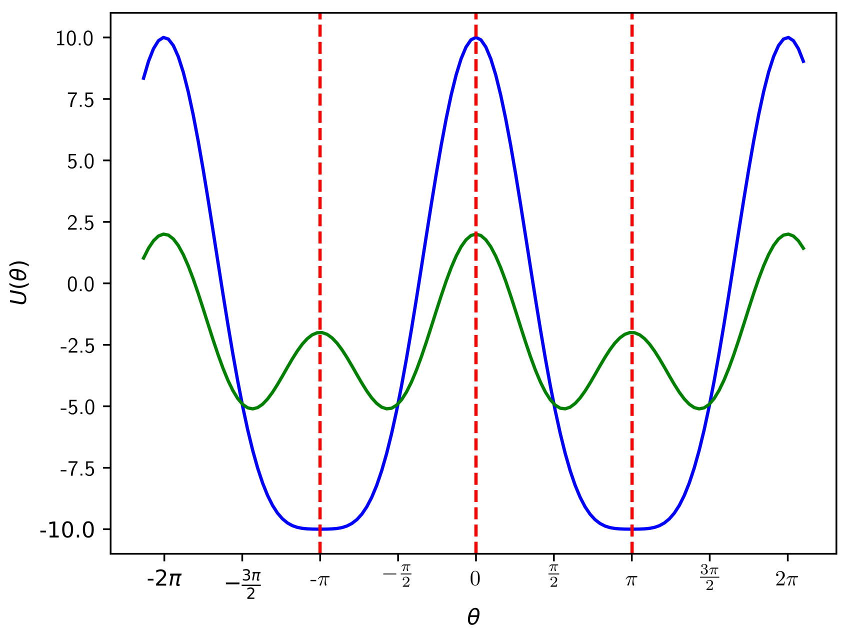
We are then left with two equilibrium points at $\theta = 0$ when the bead is at the bottom of the hoop or $\theta = \pi$ when the bead is at the top of the hoop as discussed in part b. Based on these two conditions we get the critical rotation rate is

$$\omega_c = \sqrt{\frac{g}{R}}.$$

In terms of the critical rotation the equilibrium condition becomes

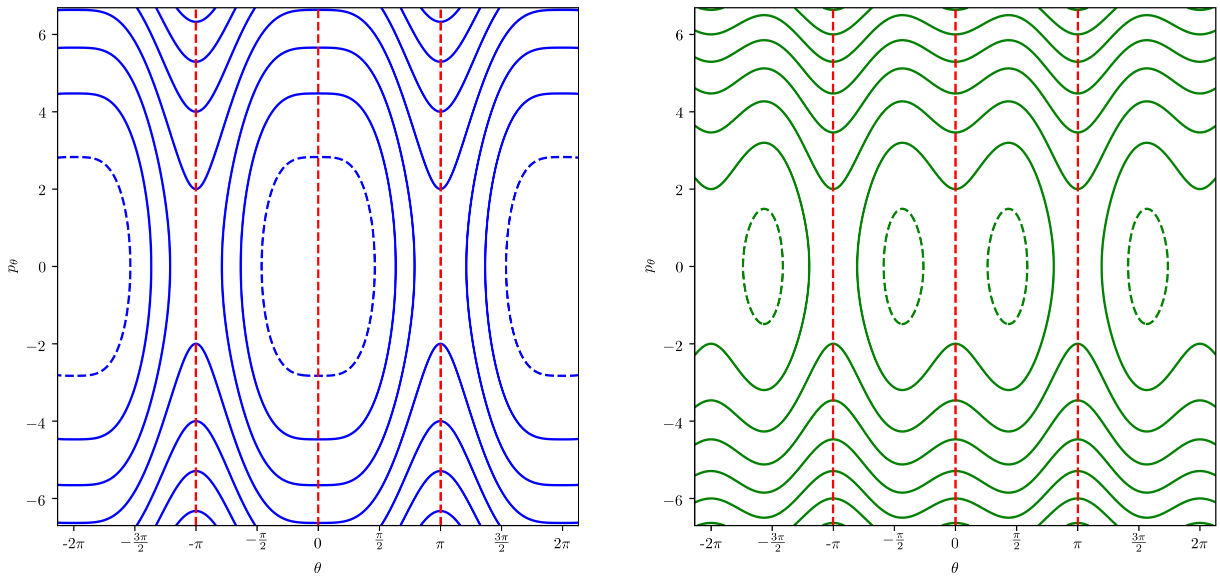
$$U(\theta) = \omega^2 sin(\theta) cos(\theta) + \omega_c^2 sin(\theta).$$

Figure 1: Plot of θ vs. $U(\theta)$.



There are two different plots in Figure 1. The blue line represents $\omega^2 > g/R$ while the green plot shows that $\omega^2 < g/R$.

2d.



(a) $\omega^2 > g/R$

(b) $\omega^2 < g/R$

Figure 2: Phase plane charts where in figure a $\omega^2 > g/R$ and in figure b $\omega^2 < g/R$.

Figure 1 compared to the two phase planes in Figure 2 show very similar trends. if we take the scenario where $\omega^2 > g/R$ i.e. the blue plots, we see in Figure 1 that at π and $-\pi$ the plot dips very low ans does not have a hump in it like the green plot. For the same reason the blue plot has a much larger amplitude to it. The blue plot in this case looks more like regular sinusoidal motion than the green plot. In the phase plane we see a dotted circle in the middle, in the green phase plane we see two dotted circles around $\pi/2$ and π. We see that the green phase plane can never complete a circle a around 0. Starting clockwise, we see p is steady, decreases, is steady, and then increases. For the green phase plane going clockwise, it increases, decreases, increases, then decreases, increases again, and finally decreases. In Figure 1 we see the same trend in the blue plot going from $-\pi$ to π. In the same range for the green plot, we see the same trend as just described. But the green plot varies more because of the hump at $-\pi$ and π.

Appendix

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl

#---------- Variables -----------
m = 1
R = 1
g = 9.81
omega = np.sqrt(g/R)
omega_c_1 = 10
omega_c_2 = 2
theta = np.arange(-6.7, 6.7, 0.1)

#---------- Figure 1 -----------
U = -(((m * (R ** 2))/2) * ((omega ** 2) * (np.sin(theta) ** 2))
    - (omega_c_1 * np.cos(theta)))

U1 = -(((m * (R ** 2))/2) * ((omega ** 2) * (np.sin(theta) ** 2))
    - (omega_c_2 * np.cos(theta)))

fig, ax = plt.subplots(figsize=(6,4.5))
labels = ax.plot(theta, U,  color='b', linestyle='-')
labels = ax.plot(theta, U1,  color='g', linestyle='-')
pi = ax.axvline(x=3.145, color='r', linestyle='--')
zero = ax.axvline(x=0, color='r', linestyle='--')
neg_pi = ax.axvline(x=-3.145, color='r', linestyle='--')
#ax.legend([labels,neg_pi,zero,pi],bbox_to_anchor=(1.05, 1), loc='upper
    left', borderaxespad=0.)

mpl.rc('text', usetex = True)
plt.xticks([-2*np.pi, -3*np.pi/2, -np.pi, -np.pi/2, 0, np.pi/2, np.pi, 3*
    np.pi/2, 2*np.pi],
           [r'-$2\pi$', r'$-\frac{3\pi}{2}$', r'-$\pi$', r'$-\frac{\pi}{2}
    $',
           '$0$', r'$\frac{\pi}{2}$', r'$\pi$', r'$\frac{3\pi}{2}$', r'$2\
    pi$'])

ax.set_xlabel(r'$\theta$')
ax.set_ylabel(r'$U(\theta)$')
plt.savefig('./plot_1.png', dpi=1200)
plt.show()

#---------- Variables -----------
m = 1
R = 1
g = 9.81
omega_c_1 = 10
omega_c_2 = 2
theta = np.arange(-6.7, 6.7, 0.01)
p = np.arange(-6.7, 6.7, 0.01)
Theta, P = np.meshgrid(theta, p)
```

```python
#---------- Figure 2 -----------
H = ((P ** 2)/(2*m*R**2) - ((m*R ** 2)/2) * (omega ** 2 * np.sin(Theta) ** 2)
    - omega_c_1 * np.cos(Theta))

fig, ax = plt.subplots(figsize=(6,6))
ax.contour(Theta, P, H, colors='blue')
pi = ax.axvline(x=3.145, color='r', linestyle='--')
zero = ax.axvline(x=0, color='r', linestyle='--')
neg_pi = ax.axvline(x=-3.145, color='r', linestyle='--')

mpl.rc('text', usetex = True)
plt.xticks([-2*np.pi, -3*np.pi/2, -np.pi, -np.pi/2, 0, np.pi/2, np.pi, 3*np.pi/2, 2*np.pi],
           [r'-$2\pi$', r'$-\frac{3\pi}{2}$', r'-$\pi$', r'$-\frac{\pi}{2}$',
           '$0$', r'$\frac{\pi}{2}$', r'$\pi$', r'$\frac{3\pi}{2}$', r'$2\pi$'])

ax.set_xlabel(r'$\theta$')
ax.set_ylabel(r'$p_{\theta}$')
plt.savefig('./phase_plane_1', dpi=1200)
plt.show()

#---------- Figure 3 -----------
H = ((P ** 2)/(2*m*R**2) - ((m*R ** 2)/2) * (omega ** 2 * np.sin(Theta) ** 2)
    - omega_c_2 * np.cos(Theta))

fig, ax = plt.subplots(figsize=(6,6))
ax.contour(Theta, P, H, colors='green')
pi = ax.axvline(x=3.145, color='r', linestyle='--')
zero = ax.axvline(x=0, color='r', linestyle='--')
neg_pi = ax.axvline(x=-3.145, color='r', linestyle='--')

mpl.rc('text', usetex = True)
plt.xticks([-2*np.pi, -3*np.pi/2, -np.pi, -np.pi/2, 0, np.pi/2, np.pi, 3*np.pi/2, 2*np.pi],
           [r'-$2\pi$', r'$-\frac{3\pi}{2}$', r'-$\pi$', r'$-\frac{\pi}{2}$',
           '$0$', r'$\frac{\pi}{2}$', r'$\pi$', r'$\frac{3\pi}{2}$', r'$2\pi$'])

ax.set_xlabel(r'$\theta$')
ax.set_ylabel(r'$p_{\theta}$')
plt.savefig('./phase_plane_2.png', dpi=1200)
plt.show()
```

Listing 1: Python Graphs