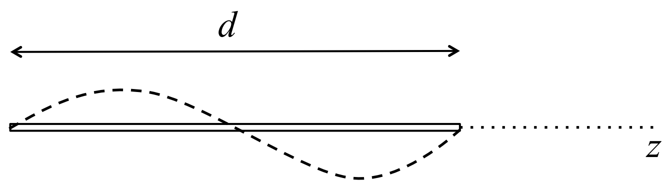
PHY 420 **Spring 2021**

Homework 5 solutions

A thin linear antenna of length d lies along the z-axis with its center at the origin, as shown in the figure below. The antenna is excited in such a way that the sinusoidal current makes a full wavelength of oscillation (as shown by the dashed line in the figure).



1. By inspection, one can write the current density as

$$\vec{J}(\vec{x})\, e^{-i\omega t} = I\, \sin(kz)\, \delta(x)\delta(y)\, e^{-i\omega t}\, \hat{z}, \qquad \text{if } -\frac{d}{2} < z < \frac{d}{2}$$

$$= 0, \qquad \text{if } |z| > \frac{d}{2}$$

Use this to show that the vector potential $\vec{A}(\vec{x})$ in the radiation zone ($kr \gg 1$) is given by

$$\vec{A}(\vec{x}) = \hat{z}\, \frac{\mu_0 I}{2\pi} \frac{e^{ikr}}{ikr} \left[\frac{\sin(\pi \cos\theta)}{\sin^2\theta}\right]$$

Solution: In the radiation zone ($kr \gg 1$), the vector potential is given by equation (9.8) in Jackson:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}\,')\, e^{-ik\hat{n}\cdot\vec{x}\,'}\, d^3x'$$

Now, recall that $\hat{n}$ is a unit vector toward the observation point, and $\vec{x}\,'$ is the position vector of the source. In this problem, the source — the linear antenna, is along the z-axis, and so

$$\hat{n}\cdot\vec{x}\,' = \hat{n}\cdot(z'\hat{z}') = z' \cos\theta$$

since in the spherical coordinate system, θ is the angle made by a vector with the z-axis.

Therefore

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}\,')\, e^{-ikz'\cos\theta}\, d^3x'$$

With $\vec{J}(\vec{x}\,')$ written from the result in the worksheet problem above, this essentially becomes an integral in z', with $\vec{A}$ along the $\hat{z}$ direction, so that

$$\vec{A}(\vec{x}) = \hat{z}\, \frac{\mu_0 I}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} \sin(kz')\, e^{-ikz'\cos\theta}\, dz'$$

To make the integral easy to do, express $\sin(kz') = (e^{ikz'} - e^{-ikz'})/2i$, so that

$$\vec{A}(\vec{x}) = \hat{z}\frac{\mu_0 I}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} \left[\frac{e^{ikz'-ikz'\cos\theta} - e^{-ikz'-ikz'\cos\theta}}{2i}\right] dz' \qquad \text{(H5.1)}$$

Integrating equation (H5.1), we get

$$\vec{A}(\vec{x}) = \hat{z}\frac{\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\frac{1}{2i}\left[\frac{e^{ikz'-ikz'\cos\theta}}{ik-ik\cos\theta} - \frac{e^{-ikz'-ikz'\cos\theta}}{-ik-ik\cos\theta}\right]_{-d/2}^{d/2} \tag{H5.2}$$

Substituting the limit values, equation (H5.2) becomes

$$\vec{A}(\vec{x}) = \hat{z}\frac{\mu_0 I}{8\pi i}\frac{e^{ikr}}{r}\left[\frac{e^{ik(\frac{d}{2})-ik(\frac{d}{2})\cos\theta} - e^{ik(-\frac{d}{2})-ik(-\frac{d}{2})\cos\theta}}{ik-ik\cos\theta} + \frac{e^{-ik(\frac{d}{2})-ik(\frac{d}{2})\cos\theta} - e^{-ik(-\frac{d}{2})-ik(-\frac{d}{2})\cos\theta}}{ik+ik\cos\theta}\right]$$

As you answered on the worksheet last week, $k = 2\pi/d$ from the geometry of the antenna (see figure on the previous page), so that $k(\frac{d}{2}) = \pi$. Therefore, the equation above gives

$$\vec{A}(\vec{x}) = \hat{z}\frac{\mu_0 I}{8\pi i}\frac{e^{ikr}}{r}\left[\frac{e^{i\pi-i\pi\cos\theta} - e^{-i\pi+i\pi\cos\theta}}{ik-ik\cos\theta} + \frac{e^{-i\pi-i\pi\cos\theta} - e^{-i\pi+i\pi\cos\theta}}{ik+ik\cos\theta}\right]$$

Now, it is easy to verify that $e^{i\pi} = -1$, as well as $e^{-i\pi} = -1$, so that

$$\vec{A}(\vec{x}) = \hat{z}\frac{\mu_0 I}{8\pi i}\frac{e^{ikr}}{r}\left[\frac{(-1)\,e^{-i\pi\cos\theta} - (-1)\,e^{i\pi\cos\theta}}{ik-ik\cos\theta} + \frac{(-1)\,e^{-i\pi\cos\theta} - (-1)\,e^{i\pi\cos\theta}}{ik+ik\cos\theta}\right]$$

Rearranging into a recognizable form

$$\vec{A}(\vec{x}) = \hat{z}\frac{\mu_0 I}{8\pi i}\frac{e^{ikr}}{r}\left[\frac{e^{i\pi\cos\theta} - e^{-i\pi\cos\theta}}{ik-ik\cos\theta} + \frac{e^{i\pi\cos\theta} - e^{-i\pi\cos\theta}}{ik+ik\cos\theta}\right]$$

so that we can put $e^{i\pi\cos\theta} - e^{-i\pi\cos\theta} = 2i\,\sin(\pi\cos\theta)$, we get

$$\vec{A}(\vec{x}) = \hat{z}\frac{\mu_0 I}{8\pi i}\frac{e^{ikr}}{r}\left[\frac{2i\,\sin(\pi\cos\theta)}{ik-ik\cos\theta} + \frac{2i\,\sin(\pi\cos\theta)}{ik+ik\cos\theta}\right]$$

Some more cleaning up:

$$\vec{A}(\vec{x}) = \hat{z}\frac{\mu_0 I}{8\pi i}\frac{e^{ikr}}{r}\left[\frac{2i\,\sin(\pi\cos\theta)}{ik(1-\cos\theta)} + \frac{2i\,\sin(\pi\cos\theta)}{ik(1+\cos\theta)}\right]$$

so that

$$\vec{A}(\vec{x}) = \hat{z}\frac{\mu_0 I}{4\pi}\frac{e^{ikr}}{ikr}\left[\frac{\sin(\pi\cos\theta)}{1-\cos\theta} + \frac{\sin(\pi\cos\theta)}{1+\cos\theta}\right]$$

and

$$\vec{A}(\vec{x}) = \hat{z}\frac{\mu_0 I}{4\pi}\frac{e^{ikr}}{ikr}\,\sin(\pi\cos\theta)\left[\frac{1+\cos\theta+1-\cos\theta}{1-\cos^2\theta}\right]$$

Therefore, we get finally

$$\vec{A}(\vec{x}) = \hat{z}\,\frac{\mu_0 I}{2\pi}\,\frac{e^{ikr}}{ikr}\left[\frac{\sin(\pi\cos\theta)}{\sin^2\theta}\right]$$

2. Use your derived expression for $\vec{A}(\vec{x})$ to find $\vec{B}$ and $\vec{E}$ in the radiation zone.

Hint: It helps to change to spherical coordinates at this stage. Also, instead of trying to differentiate $\vec{A}$ explicitly, it helps to use $\vec{B} = ik\hat{n} \times \vec{A}$, as Jackson says to do in equation (9.39).

Solution: If you try to continue working in (x, y, z) coordinates, it will become clear very quickly that you have an impossible job on your hands. But with r and θ present in $\vec{A}$, a natural next step is to gravitate toward spherical coordinates. Now, since $\vec{A}$ has only a $\hat{z}$-component, and we know that

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

meaning that only A_r and A_θ are present (and $A_\phi = 0$), looking up the cross product in spherical coordinates in the inside back cover of Jackson will show you that $\vec{B}$ will only have a $\hat{\phi}$ component. That makes perfect sense; if the current is along $\hat{z}$, we expect a magnetic field in the $\hat{\phi}$ direction. But it gets better! Of the two terms in the $\hat{\phi}$ component of $(\vec{\nabla} \times \vec{A})$ in spherical coordinates, only the $\partial(rA_\theta)/\partial r$ is of consequence in the radiation zone, and we can ignore the other, $\partial A_r/\partial\theta$. Thus, if you do the differentiation explicitly, you'll find that the net result is the same as calculated by doing

$$\vec{B} = ik\hat{n} \times \vec{A}$$

as Jackson says to do in equation (9.39). Remembering that $\hat{n}$ is in the direction of $\hat{r}$, calculating $\vec{B}$ now becomes very simple because

$$\hat{n} \times \hat{z} = \hat{r} \times \hat{z} = \hat{r} \times \hat{r}\cos\theta - \hat{r} \times \hat{\theta}\sin\theta = -(\sin\theta)\,\hat{\phi}$$

Therefore

$$\vec{B} = -ik(\sin\theta)\,\hat{\phi}\,\frac{\mu_0 I}{2\pi}\,\frac{e^{ikr}}{ikr}\left[\frac{\sin(\pi\cos\theta)}{\sin^2\theta}\right]$$

so that

$$\vec{H} = \frac{\vec{B}}{\mu_0} = -\hat{\phi}\,\frac{I}{2\pi}\,\frac{e^{ikr}}{r}\left[\frac{\sin(\pi\cos\theta)}{\sin\theta}\right]$$

and with $Z_0 = \sqrt{\mu_0/\epsilon_0}$, we get

$$\vec{E} = Z_0\,\vec{H} \times \hat{n} = -\hat{\theta}\,\frac{c\mu_0 I}{2\pi}\,\frac{e^{ikr}}{r}\left[\frac{\sin(\pi\cos\theta)}{\sin\theta}\right]$$

3. Calculate $\dfrac{dP}{d\Omega}$, the power radiated per unit solid angle.

Solution: The power radiated per unit solid angle is given by

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{r^2}{2}\,\mathrm{Re}\left[\hat{n}\cdot\vec{E}\times\vec{H}^*\right] \\
&= \frac{r^2}{2}\left\{\hat{r}\cdot\left(\hat{\theta}\times\hat{\phi}\right)\frac{c\mu_0 I^2}{4\pi^2}\,\frac{e^{ikr}e^{-ikr}}{r^2}\left[\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}\right]\right\} \\
&= \hat{r}\cdot\left(\hat{r}\right)\frac{c\mu_0 I^2}{8\pi^2}\left[\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}\right]
\end{aligned}$$

Therefore,

$$\frac{dP}{d\Omega} = \frac{c\mu_0 I^2}{8\pi^2}\left[\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}\right]$$

4. Please present three choices (only) of topic from the list below, ranked in order of preference (1, 2, 3).

I'll try and award your highest choice if there is a sufficient diversity of picks, but if not, I'll make the choice for you. Please make sure you identify your choices clearly.

Solution: This was a choice of topics for the Formal Write-up assignment, therefore no solution needs to be posted. The topics have now been assigned in D2L.