PHY 411 **Winter 2021**

Class Summary—Week 5, Day 2—Thursday, Feb 4

Vector and Scalar Potentials

In the previous class, we wrote $\vec{E}$ and $\vec{B}$ in terms of the vector potential $\vec{A}$ and the scalar potential Φ:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{6.7}$$

and

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \tag{6.9}$$

On the worksheet in the previous class, you showed how the definition of $\vec{B}$ and $\vec{E}$ in terms of $\vec{A}$ and Φ satisfies directly the (two homogenous of the four) Maxwell's equations:

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho & \vec{\nabla} \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \end{aligned} \tag{6.6}$$

We will now turn our attention to the inhomogenous equations in equation (6.6), which determine the dynamic behavior of $\vec{A}$ and Φ. To keep things simple, we will restrict ourselves to a vacuum, so anytime we encounter μ, we'll replace it by μ_0, and likewise replace ϵ by ϵ_0.

Starting from Gauss' law $\vec{\nabla} \cdot \vec{D} = \rho$, *you showed on the worksheet today that*

$$\nabla^2 \Phi + \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right) = -\frac{\rho}{\epsilon_0} \tag{6.10}$$

and starting from the Ampere-Maxwell law, *you showed on the worksheet today that*

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left[\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right] = -\mu_0 \vec{J} \tag{6.11}$$

where ρ is the charge density and $\vec{J}$ is the current density.

By carrying out the above procedure, we have reduced the set of four Maxwell equations to two equations. However, equation (6.10) and equation (6.11) are still coupled.

To uncouple them, we must recognize that the association of $\vec{B}$ with the vector potential through equation (6.7) is arbitrary to the extent of adding the gradient of some scalar function Λ, that is, $\vec{B}$ is left unchanged by the transformation:

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \tag{6.12}$$

as you showed on today's worksheet

If we carry out the transformation in equation (6.12), we *must also simultaneously transform* the scalar potential in order that $\vec{E}$ remains unchanged, so

$$\Phi \to \Phi - \frac{\partial \Lambda}{\partial t} \tag{6.13}$$

Again, you demonstrated this in the worksheet for today.

The freedom implied by equation (6.12) and equation (6.13) means that we can choose a set of potentials $(\vec{A}, \Phi)$ to make the quantity in square brackets in equation (6.11) equal to zero:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0 \tag{6.14}$$

This expression is known as the **Lorenz condition**[1].

Implementing the Lorenz condition written in equation (6.14) will uncouple equations (6.10) and (6.11) and leave two inhomogenous wave equations, one for Φ and one for $\vec{A}$:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{6.15}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \tag{6.16}$$

Equations (6.15) and (6.16), together with equation (6.14), **form a set of equations equivalent in all respects to the Maxwell equations in vacuum**.

Gauge Transformations

Before moving on, it is important to comprehend the theoretical underpinnings of what we just did.

The transformation in equation (6.12) and equation (6.13) is called a *gauge transformation*, and the invariance of the fields under such transformations is called *gauge invariance*.

To see that potentials can always be found to satisfy the Lorenz condition, suppose that the potentials $\vec{A}, \Phi$ that satisfy equation (6.10) and equation (6.11) do not satisfy equation (6.14).

Then, let us make a gauge transformation to potentials $\vec{A}', \Phi'$ and demand that $\vec{A}', \Phi'$ satisfy the Lorenz condition (*as you did on today's worksheet*):

$$\vec{\nabla} \cdot \vec{A}' + \frac{1}{c^2} \frac{\partial \Phi'}{\partial t} = 0 = \vec{\nabla} \cdot \left(\vec{A} + \vec{\nabla} \Lambda \right) + \frac{1}{c^2} \frac{\partial}{\partial t} \left(\Phi - \frac{\partial \Lambda}{\partial t} \right)$$

so that

$$\vec{\nabla} \cdot \vec{A}' + \frac{1}{c^2} \frac{\partial \Phi'}{\partial t} = 0 = \vec{\nabla} \cdot \vec{A} + \nabla^2 \Lambda + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} \tag{6.17}$$

Therefore, provided a gauge function Λ can be found to satisfy

$$\nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = -\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) \tag{6.18}$$

the new potentials $\vec{A}', \Phi'$ will satisfy the Lorenz condition (6.14) and the wave equations (6.15) and (6.16).

[1] Yes, the Danish physicist Ludwig V. Lorenz, not the Dutch physicist Hendrik A. Lorentz, to whom this was misattributed for many years; see Jackson's note on page 294.

Moreover, the **restricted gauge transformation**

$$\begin{aligned} \vec{A} &\to \vec{A} + \vec{\nabla}\Lambda \\ \Phi &\to \Phi - \frac{\partial \Lambda}{\partial t} \end{aligned} \tag{6.19}$$

where

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0 \tag{6.20}$$

preserves the Lorenz condition, provided $\vec{A}, \Phi$ satisfy it initially. All potentials in this restricted class are said to belong to the **Lorenz gauge**.

Why is this called restricted, since the equations are the same as for the gauge transformation in equation (6.12) and equation (6.13)? Because, unlike in the gauge transformation where Λ is any arbitrary scalar function, the value of Λ is now restricted by equation (6.20).

The Lorenz gauge is commonly used, first because it leads to the wave equations (6.15) and (6.16), which treat Φ and $\vec{A}$ on an equivalent footing, and second because it is a concept independent of the coordinate system and so fits naturally into the context of special relativity, as we will learn in the next quarter.

While we're on this topic, this is also a good time to learn about another useful gauge for the potentials: the so-called **Coulomb gauge**, also known as the **radiation gauge** or **transverse gauge** (for reasons which will become clear in Chapter 9). In this gauge, we have

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{6.21}$$

From equation (6.10), we see that the scalar potential in the Coulomb gauge satisfies the equation

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \tag{6.22}$$

Equation (6.22) is called the **Poisson equation**, and is a well known equation in electrostatics, obtained by writing the electrostatic field as the negative gradient of a scalar potential function: $\vec{E} = -\vec{\nabla}\Phi$ and using Gauss' law for an electrostatic field in vacuum: $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$, so that

$$\vec{\nabla} \cdot \Big(-\vec{\nabla}\Phi \Big) = \rho/\epsilon_0$$

which becomes equation (6.22).

In charge-free space, equation (6.22) becomes **Laplace's equation** $\nabla^2 \Phi = 0$, and we'll have occasion to work with this equation in various geometries and coordinates, when we do Chapters 2-4 later. More immediately, though, we'll be referring to the Laplacian ∇^2 when we discuss Green functions next.

Green Functions

Another mathematical technique we will need in Chapter 9 is the technique of Green functions. In essence, the Green functions provide a general way to solve inhomogenous partial (or ordinary) differential equations.

Let us write the general schematic for a Green function problem before proceeding to an example in electrostatics. Any math methods text would be good; I've compiled the following from a series of presentations obtained via a Google search (which is why I haven't provided any one specific source).

Consider the linear inhomogenous differential equation

$$\mathcal{D}\Psi(\vec{x}) = f(\vec{x}) \tag{W5.1}$$

where $\mathcal{D}$ is any linear differential operator, e.g., a well known example of an inhomogenous differential equation in electrostatics is the Poisson equation for the scalar potential $\Phi(\vec{x})$, given by

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

We know from our study of differential equations that the solution to equation (W5.1) is comprised of a solution to the associated homogenous equation Ψ_h, plus the particular solution Ψ_{part} that comes from the inhomogenous equation:

$$\Psi(\vec{x}) = \Psi_h + \Psi_{\text{part}} \tag{W5.2}$$

We do not need to ourselves with Ψ_h, since there are many techniques to solve homogenous PDE's.

Instead, we are going to look for the particular solution (Ψ_{part}) with the inhomogeneity on the right hand side, represented here by the generic function $f(\vec{x})$.

To do so, we need to consider the associated inhomogenous equation for a point source function

$$\mathcal{D}G(\vec{x},\vec{x}\,') = \delta(\vec{x}-\vec{x}\,') \tag{W5.3}$$

which defines the Green function $G(\vec{x},\vec{x}\,')$ corresponding to the operator $\mathcal{D}$.

Once we've determined $G(\vec{x},\vec{x}\,')$, then the particular solution to the inhomogenous equation can be written as

$$\Psi_{\text{part}} = \int_V G(\vec{x},\vec{x}\,')\, f(\vec{x}\,')\, d^3x' \tag{W5.4}$$

so that the solution to the differential equation (W5.1) is

$$\Psi(\vec{x}) = \Psi_h + \int_V G(\vec{x},\vec{x}\,')\, f(\vec{x}\,')\, d^3x' \tag{W5.5}$$

In summary, **the Green function may be characterized as the response of a system to a Dirac δ-function input signal, to take an electronics analogy**. Now, because any function can be expanded as a sum of Dirac signals, you can compute the output for any signal input if you know the Green function.

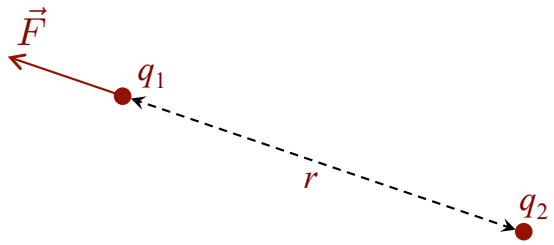
Green Function: An Example from Electrostatics

We will now introduce the formal math with an intuitive example from electrostatics.

First, let us set up the math notation (since that is usually the first hurdle to be overcome in understanding the math).

You learned in freshman physics that the magnitude of the force $\vec{F}$ exerted on a point charge q_1 due to a point charge q_2 located a distance r apart from each other is given by Coulomb's law:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} \qquad \text{(W5.6)}$$



If q_1 and q_2 have the same sign, then $\vec{F}$ is repulsive, and is directed from q_2 to q_1 along the line drawn through the two charges.

Let us express r in vector notation. Suppose the charge q_1 is located at $\vec{x}_1$ and q_2 is located at $\vec{x}_2$, as shown in the figure below.

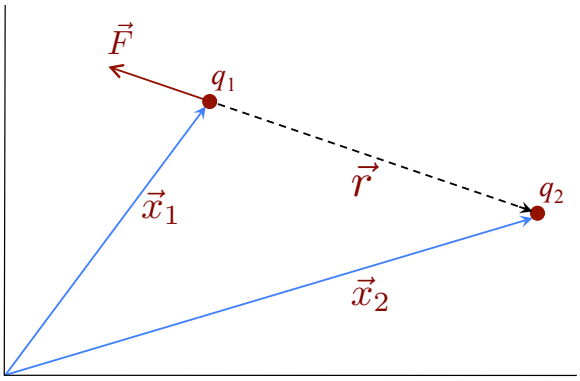
Then, it's easy to see that

$$\vec{x}_1 + \vec{r} = \vec{x}_2$$

where $\vec{r}$ is a vector along the line passing through q_1 and q_2 and directed from q_1 to q_2, so that

$$-\vec{r} = \vec{x}_1 - \vec{x}_2$$

I'm writing it in this form because $-\vec{r}$ is the direction of the force of q_2 on q_1.



So, we could write Coulomb's law in equation (W5.6) as

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|^2}$$

But, we could also use this vectorial form to insert information about the direction of the force. Again, since the force on q_1 due to q_2 is directed along the line joining the two charges, and is also the direction of the vector $-\vec{r} = \vec{x}_1 - \vec{x}_2$, we can write that

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|^2}\frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|} \qquad \text{(W5.7)}$$

By dividing $(\vec{x}_1 - \vec{x}_2)$ by its magnitude $|\vec{x}_1 - \vec{x}_2|$, I've written a unit vector in the direction of the vector $(\vec{x}_1 - \vec{x}_2)$, and since that is the direction of the force of q_2 on q_1, I can use this unit vector to specify the direction of the Coulomb force exerted by q_2 on q_1.

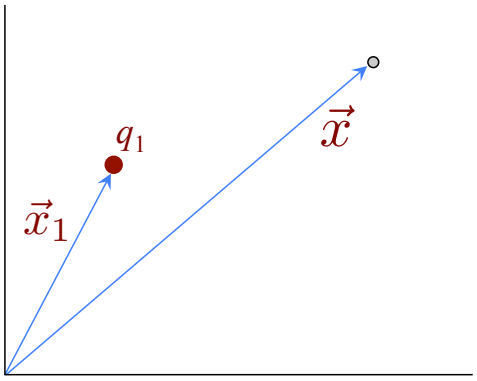
Writing equation (W5.7) in more compact form, we get the Coulomb force on charge q_1 located at $\vec{x}_1$ due to charge q_2 located at $\vec{x}_2$ to be

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} \tag{W5.8}$$

Next, let us start thinking in terms of electric fields, since that is really what we are after at this level.

We know that $\vec{F} = q\vec{E}$, where $\vec{E}$ is the electric field. This means that the electric field at a point in space is the force felt by a unit positive "test" charge placed at that point. Therefore, with the geometry set up as in the figure on the right, we get that the electric field at a point $\vec{x}$ due to a point charge q_1 at the point $\vec{x}_1$ is given by

$$\vec{E}(\vec{x}) = \frac{q_1}{4\pi\epsilon_0} \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3} \tag{W5.9}$$



Note: It is important to check that we have the vector direction correct in equation (W5.9) above. If q_1 is a positive charge, then the force on a unit positive test charge at location $\vec{x}$ will be along the direction of vector $\vec{r}$ from q_1 to the unit positive charge, so we have $\vec{x}_1 + \vec{r} = \vec{x}$, and hence $\vec{r} = \vec{x} - \vec{x}_1$.

If, now, we had a collection of n discrete point charges $q_1, q_2, q_3, \ldots, q_n$, located at $x_1, x_2, x_3, \ldots, x_n$ respectively, then we would get

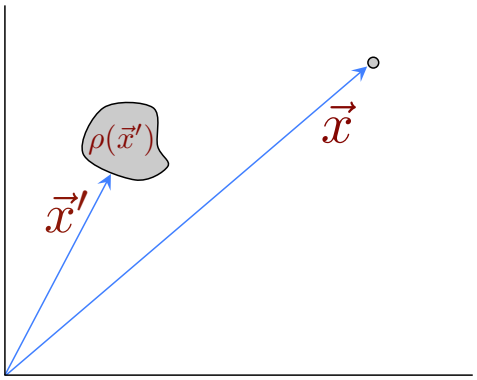
$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3} \tag{W5.10}$$

If, instead, we had a continuous charge distribution described by a charge density $\rho(\vec{x}\,')$ as shown in the figure below, then the sum would be replaced by an integral

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}\,') \frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3} \, d^3x' \tag{1.5}$$

where $d^3x' = dx'dy'dz'$ is the three-dimensional volume element at $\vec{x}\,'$.

Note how the vectors are specified in the figure, because we will be using this notation frequently. Vector $\vec{x}$ is the position in space at which we are writing the field, whereas $\vec{x}\,'$ is the location of our source charge or charge distribution that is responsible for the field.



The vector factor in the integrand of equation (1.5), viewed as a function of $\vec{x}$, is the negative gradient of the scalar $1/|\vec{x}-\vec{x}\,'|$, that is

$$\frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)$$

We can substitute this into equation (1.5):

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}\,')\left[-\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)\right] d^3x'$$

and, upon realizing that the gradient operation involves $\vec{x}$, but not $\vec{x}\,'$, we can take it outside the integral:

$$\vec{E}(\vec{x}) = \frac{-1}{4\pi\epsilon_0}\vec{\nabla}\int \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\, d^3x' \tag{1.15}$$

Now, since $\vec{E}$ is the negative gradient of the scalar potential in electrostatics, that is, since

$$\vec{E} = -\vec{\nabla}\Phi \tag{1.16}$$

we get from equation (1.15) that the scalar potential in terms of the charge density is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\, d^3x' \tag{1.17}$$

We are frequently interested in the Laplacian of the scalar potential, $\nabla^2\Phi$, and from equation (1.17), we see that we'll have to evaluate

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)$$

In other words, we need to consider the Laplacian (∇^2) of $1/r$, where $r = |\vec{x}-\vec{x}\,'|$.

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We are engaged in the process of finding the Laplacian of the scalar potential, $\nabla^2\Phi$, and from equation (1.17), we have seen that we'll have to evaluate

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)$$

i.e., we need to consider the Laplacian (∇^2) of $1/r$, where $r = |\vec{x}-\vec{x}\,'|$.

The Laplacian of $1/r$ is singular: $\nabla^2(1/r) = 0$ for $r \neq 0$, and its volume integral is -4π. Such a behavior can be represented in terms of a Dirac δ-function, that is, we can write

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right) = -4\pi\delta(\vec{x}-\vec{x}\,') \tag{1.31}$$

Now, $1/|\vec{x}-\vec{x}\,'|$ is just the potential of a unit point source charge. Following our discussion introducing the Green function, we see that equation (1.31) represents exactly what is meant by the Green function — it is the point source response. Or, as Jackson says, $1/|\vec{x}-\vec{x}\,'|$ is one of a class of functions depending on the variables $\vec{x}$ and $\vec{x}\,'$, called Green functions (or Green's functions — various authors have their own opinions on how to write it to "sound" correct).

Therefore, we can write equation (1.31) as

$$\nabla^2 G(\vec{x},\vec{x}\,') = -4\pi\delta(\vec{x}-\vec{x}\,') \tag{1.39}$$

where, in general

$$G(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|} + F(\vec{x},\vec{x}\,') \tag{1.40}$$

with the function F satisfying the Laplace equation

$$\nabla^2 F(\vec{x},\vec{x}\,') = 0 \tag{1.41}$$

Note that Jackson's equation (1.39)-(1.41) has a ∇'^2, and I'm not sure why he wrote it this way — it's the variable $\vec{x}$ specifying the locations where you want to know the field that's getting differentiated, so I believe it should be ∇^2.