PHY 411 **Winter 2021**

Week 5—Tuesday, Feb 2—Discussion Worksheet

The High Frequency Limit

Recall again that we obtained the following expression for the dielectric constant:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \left[\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \right] \tag{7.51}$$

At frequencies far above the highest resonant frequency, the dielectric constant takes on a simple form.

1. Show that in the high frequency limit $\omega \gg \omega_j$, we get

$$\frac{\epsilon(\omega)}{\epsilon_0} \simeq 1 - \frac{\omega_P^2}{\omega^2} \tag{7.59}$$

with

$$\omega_P^2 = \frac{NZe^2}{\epsilon_0 m} \tag{7.60}$$

where ω_P is called the plasma frequency of the medium, and NZ is the total number of electrons per unit volume.

Starting from equation 7.51

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \left[\frac{f_i}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \right]$$

$$\epsilon(\omega) = \epsilon_0 \left(1 + \frac{Ne^2}{m} \sum_j \left[\frac{f_j}{(\omega_j^2 - \omega^2 - j\omega\gamma_j)} \right] \right)$$

$$\epsilon(\omega) = \epsilon_0 \left(1 - \frac{Ne^2}{\omega^2 m} \sum_j \left[\frac{f_j}{(1 + i\frac{\gamma_j}{\omega} - \frac{\omega_j^2}{\omega})} \right] \right)$$

$$\epsilon(\omega) = \left(1 - \frac{NZe^2}{\omega^2 m} \right)$$

$$\epsilon(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2} \right)$$

$$\omega_p^2 = \frac{NZe^2}{\epsilon_0 m}$$

2. Consider again the high frequency limit $\omega \gg \omega_j$.

(a) Show that the wave number in the high frequency limit is given by

$$k = \frac{\omega}{v} = \frac{\omega}{c} n, \quad n = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$$

$$ck = \sqrt{\omega^2 - \omega_p^2} \tag{7.61}$$

also $\omega^2 = \omega_p^2 + c^2 k^2$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}$$

$$c^2 k^2 = \frac{\omega^2 \mu \epsilon}{\mu_0 \epsilon_0}$$

$$\mu = \mu_0$$

$$c^2 k^2 = \omega^2 \frac{\epsilon}{\epsilon_0}$$

$$\omega^2 \frac{\epsilon(\omega)}{\epsilon_0} = \omega^2 - \omega_p^2$$

$$c^2 k^2 = \omega^2 - \omega_p^2$$

$$ck = \sqrt{\omega^2 - \omega_p^2}$$

(b) In certain situations, such as the ionosphere or a tenuous plasma in the lab, equation (7.59) can hold for a wide range of frequencies, including $\omega < \omega_p$. In that case, equation (7.61) tells us that wave number is purely imaginary, so that waves incident on a plasma are reflected and the fields inside fall off exponentially with distance from the surface. Show that at $\omega = 0$, the attenuation constant is

$$\alpha_{\text{plasma}} \simeq \frac{2\omega_p}{c} \tag{7.62}$$

$$ck = \sqrt{\omega^2 - \omega_p^2}$$

$$\rightarrow ck = \sqrt{-\omega_p^2}$$

$$\rightarrow ck = i\omega_p$$

R.H.S is imaginary so $\rightarrow c\,\mathrm{Im}(k) = i\omega_p$

$k = \beta + i\frac{\alpha}{2}$ so $\mathrm{Im}(k) = \frac{i\alpha}{2}$

Thus, $c\left(\frac{i\alpha}{2}\right) = i\omega_p \Rightarrow \alpha = \frac{2\omega_p}{c}$

Propagation through Dispersive Media: Group Velocity

The speed of a wave may be found by multiplying its wavelength (λ) by its frequency (f). We will call this the phase velocity, v_p; in words, it is the speed at which a point of constant phase (in a wave) travels as the wave propagates. If a medium is dispersive (i.e., dielectric constant is a function of frequency), then the phase velocity will no longer be the same for each frequency component of the wave. Putting together components traveling at different speeds implies that the phase velocity of a particular wave may no longer provide a meaningful value for the speed of propagation. For such cases, we define the group velocity, $v_g = d\omega/dk$, which may be interpreted as the speed at which a disturbance in the wave propagates.

3. Although you must have learned about the distinction between phase and group velocities before, the exercise below will serve as a useful reminder.

(a) Starting from $v_p = \lambda f$, show that $v_p = \omega/k$.

Hint: Recall that λ is connected to the magnitude of the wave vector $\vec{k}$ via $k = 2\pi/\lambda$.

$\lambda = \frac{2\pi}{k}$

$$V_p = \lambda f = \left(\frac{\cancel{2\pi}}{k}\right)\left(\frac{\omega}{\cancel{2\pi}}\right)$$

$$V_p = \frac{\omega}{k}$$

(b) In some nonpermeable media ($\mu = \mu_0$), the dielectric constant, $\epsilon(\omega)/\epsilon_0$, varies as the square of frequency over a narrow range of frequencies centered at ω_0, so that we can write

$$\frac{\epsilon}{\epsilon_0} = B\left(\frac{\omega}{\omega_0}\right)^2 \qquad C = 1/\sqrt{\mu_0\epsilon_0}$$

where B is a positive real constant. Find v_p and v_g in such media, and show that $v_g = v_p/2$.

$$V_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\mu_0}\frac{1}{\sqrt{\epsilon_0 B(\omega/\omega_0)}} = \frac{C}{\sqrt{B}}\left(\frac{\omega_0}{\omega}\right)$$

$$V_g = \frac{d\omega}{dk} = \left[\frac{dk}{d\omega}\right]^{-1} = \left[\frac{d}{d\omega}\left(\omega\sqrt{\mu\epsilon}\right)\right]^{-1} = \left[\frac{d}{d\omega}\left(\omega\sqrt{\mu_0 B}\,\frac{\omega}{\omega_0}\right)\right]^{-1}$$

$$V_g = \left[\frac{\sqrt{\mu_0\epsilon_0 B}}{\omega_0}\frac{d}{d\omega}(\omega^2)\right]^{-1} = \left[\frac{B}{C\omega_0}(2\omega)\right]^{-1}$$

Thus

$$V_p = \frac{C}{\sqrt{B}}\left(\frac{\omega_0}{\omega}\right) \qquad V_g = \frac{C}{2\sqrt{B}}\left(\frac{\omega_0}{\omega}\right) \Rightarrow V_g = \frac{V_p}{2}$$

Vector and Scalar Potentials

First, recall Maxwell's equations:

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{6.6}$$

Since Maxwell's equation $\vec{\nabla} \cdot \vec{B} = 0$, we can define a quantity $\vec{A}$ called the *vector potential* such that

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{6.7}$$

based on the identity $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{P}) = 0$ for any arbitrary vector $\vec{P}$ (inside front cover of Jackson).

Useful Information: $\vec{\nabla} \times (\vec{\nabla} \times \vec{P}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{P}) - \nabla^2 \vec{P}$

4. Starting from Faraday's law $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$, show that

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \tag{6.9}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = 0$$

$$\Rightarrow \vec{\nabla} \times \vec{E} + \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla} \times \left[\vec{E} + \frac{\partial \vec{A}}{\partial t}\right] = 0$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\Phi \Rightarrow \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$