PHY 420 **Spring 2021**

Week 3—Thursday, Apr 15—Discussion Worksheet

Green Functions

We will now begin using the **Green functions** that were introduced in PHY 411. After reminders from last quarter, we will look at additional examples applying the mathematical technique of Green functions to electrodynamics. Recall that the Green functions provide a general way to solve inhomogenous partial (or ordinary) differential equations. The following equations from the PowerPoint slides may be of use for this worksheet.

$$\mathcal{D}\Psi(\vec{x}) = f(\vec{x}) \tag{W3.1}$$

$$\Psi(\vec{x}) = \Psi_h + \Psi_{\text{part}} \tag{W3.2}$$

$$\mathcal{D}G(\vec{x}, \vec{x}\,') = \delta(\vec{x} - \vec{x}\,') \tag{W3.3}$$

$$\Psi_{\text{part}} = \int_V G(\vec{x}, \vec{x}\,')\, f(\vec{x}\,')\, d^3x' \tag{W3.4}$$

$$\Psi(\vec{x}) = \Psi_h + \int_V G(\vec{x}, \vec{x}\,')\, f(\vec{x}\,')\, d^3x' \tag{W3.5}$$

Green Functions in Electrostatics

Recall that last quarter we wrote the ***scalar potential*** $\Phi(\vec{x})$ in integral form as

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x' \tag{1.17}$$

where $d^3x' = dx'dy'dz'$ is the three-dimensional volume element at $\vec{x}\,'$, $\vec{x}$ is the position in space at which we are writing the potential, whereas $\vec{x}\,'$ is the location of our source charge or charge distribution that is responsible for the electrostatic field.

We are frequently interested in the Laplacian of the scalar potential, $\nabla^2\Phi$, and from equation (1.17), we see that we'll have to evaluate

$$\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}\,'|} \right)$$

In other words, we need to consider the Laplacian (∇^2) of $1/r$, where $r = |\vec{x} - \vec{x}\,'|$.

1. The Laplacian of $1/r$ is such that $\nabla^2(1/r) = 0$ for $r \neq 0$, and its volume integral is -4π. How would you represent such a behavior in terms of a Dirac δ-function? That is, write down the right hand side of the equation below.

Note

$$\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}\,'|} \right) = -4\pi\,\delta(\vec{x} - \vec{x}) \tag{1.31}$$

$$\int \delta(x-a)\,dx = 1$$

$$\int f(x)\,\delta(x-a) = f(a)$$

$$\int_{\Delta V} \delta(x - \vec{x})\,d^3x = \begin{cases} 1, \text{ if } \Delta V \text{ contains } \vec{x} \\ 0, \text{ if } \Delta V \text{ does not contain } \vec{x} \end{cases}$$

Now, $1/|\vec{x}-\vec{x}\,'|$ is just the potential of a unit point source charge. Following our discussion introducing the Green function, we see that equation (1.31) represents exactly what is meant by the Green function — ***it is the point source response***. Therefore, we can write equation (1.31) as

$$\nabla^2 G(\vec{x},\vec{x}\,') = -4\pi\delta(\vec{x}-\vec{x}\,') \tag{1.39}$$

where, in general

$$G(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|} + F(\vec{x},\vec{x}\,') \tag{1.40}$$

with the function F satisfying the Laplace equation

$$\nabla^2 F(\vec{x},\vec{x}\,') = 0 \tag{1.41}$$

We will now cast some of the equations above in terms of equations (W3.1)-(W3.5) on page 1.

2. In electrostatics, a well known example of an inhomogenous differential equation is the ***Poisson equation*** for the scalar potential $\Phi(\vec{x})$, given by

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{1.28}$$

(a) Since the Green function is the point source response for any given differential equation, write down the Green function equation corresponding to the Poisson equation (1.28).

$$\nabla^2 G(\vec{X},\vec{x}\,') = \delta(\vec{X}-\vec{X}\,') \tag{1.39}$$

If we can find the solution to equation (1.39) that determines $G(\vec{x},\vec{x}\,')$, then we can write the solution to Poisson's equation as

$$\Phi(\vec{x}) = \int_V G(\vec{x},\vec{x}\,')\left[-\frac{\rho(\vec{x}\,')}{\epsilon_0}\right] d^3x' \tag{W3.7}$$

where I've ignored $\Phi_h(\vec{x})$, the solution to the homogenous (Laplace) equation $\nabla^2\Phi = 0$. Therefore, if we can determine $G(\vec{x},\vec{x}\,')$, we have solved the Poisson equation (1.28) and obtained $\Phi(\vec{x})$.

(b) Write down an appropriate solution for $G(\vec{x},\vec{x}\,')$ in equation (1.39), and then verify by substituting this solution in equation (W3.7) above that you get an expression for the scalar potential $\Phi(\vec{x})$ that matches equation (1.17).

From Question 1 $\nabla^2\left(\frac{1}{|\vec{X}-\vec{X}'|}\right) = -4\pi\delta(\vec{X}-\vec{X}')$

an appropriate Solution for $G(\vec{X},\vec{X}')$ in (9.39) : $G(\vec{X},\vec{X}') = -\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|}$

Substituting into (W3.7)

$$\Phi(\vec{X}) = \int_V \underbrace{G(\vec{X},\vec{X}')}\left[-\frac{\rho(\vec{X}')}{\epsilon_0}\right] d^3X'$$

$$= \int_V \left[-\frac{1}{4\pi}\frac{1}{|\vec{X}-\vec{X}'|}\right]\left[-\frac{\rho(\vec{X}')}{\epsilon_0}\right] d^3X = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{X}')}{|\vec{X}-\vec{X}'|} d^3X'$$

Green Functions for the Wave Equation

In the previous quarter, we discovered that the four Maxwell equations could be reduced to two equations, which could be uncoupled to two equations both of which have the same basic structure of the inhomogenous wave equation:

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\vec{x}, t) \tag{6.32}$$

where $f(\vec{x}, t)$ is a known source distribution and c is the velocity of propagation in the medium.

We will now apply the Green function method to find a solution to the inhomogenous differential equation (6.32). To keep things simple, we will consider a situation where there are no boundary surfaces. Moreover, to make things even simpler, let us first start by considering the problem without the time dependence. This is easy to do, because we can remove the explicit time dependence in equation (6.32) by a Fourier transform with respect to frequency. That is, we'll do

$$\Psi(\vec{x}, t) \to \Psi(\vec{x}, \omega) \qquad \text{and} \qquad f(\vec{x}, t) \to f(\vec{x}, \omega)$$

After carrying out the Fourier transforms, one finds that the Fourier transformed quantities $\Psi(\vec{x}, \omega)$ and $f(\vec{x}, \omega)$ satisfy the equation

$$\left(\nabla^2 + k^2\right)\Psi(\vec{x}, \omega) = -4\pi f(\vec{x}, \omega) \tag{6.35}$$

where $k = \omega/c$ is the wave number associated with frequency ω. We see that equation (6.35) looks similar to the Helmholtz wave equation (7.3), but has a term on the right rendering it an inhomogenous equation. Therefore, equation (6.35) is called an *inhomogenous Helmholtz wave equation.*

3. Write down the Green function equation associated with equation (6.35).

$$\left(\nabla^2 + k^2\right) G_k(\vec{x}, \vec{x}\,') = -4\pi\,\delta(\vec{x} - \vec{x}\,') \tag{6.36}$$

$$\mathcal{D}\Psi(\vec{x}) = f(\vec{x})$$

$$\mathcal{D}G(\vec{x}, \vec{x}\,') = \delta(\vec{x} - \vec{x}\,')$$

If we can solve for $G_k(\vec{x}, \vec{x}\,')$ from equation (6.36), we'll have found a solution to the inhomogenous Helmholtz wave equation (6.35).

How do we find a solution for $G_k(\vec{x}, \vec{x}\,')$? The formal way, which Jackson does on pages 243-244, is to begin by writing the Laplacian in spherical coordinates. However, *I'm going to try a more intuitive approach here.* I find it easier to follow, and we lose nothing because we've already discussed formal solutions to the Laplace equation in different coordinate systems.

4. Notice that the δ-function in equation (6.36) localizes the source function to one point $\vec{x}\,'$ in space. Everywhere else, where there are no sources, we know that the solution will be a traveling wave, so

$$G_k \propto e^{\pm i\vec{k}\cdot\vec{x}}$$

where $\vec{k}$ is the wave vector with magnitude k being the wave number.

(a) Since the δ-function involves $\vec{x} - \vec{x}\,'$, shift the origin in the expression above. What do you get?

$$G_k \propto e^{\pm i\vec{k}\cdot(\vec{x}-\vec{x}\,')}$$

(b) Now, the Green function must be spherically symmetric; there is no reason to expect otherwise given that we've localized the source function to one point via the δ-function. That means the Green function can depend only on $|\vec{x} - \vec{x}\,'|$. How does this change the expression you wrote in part (a) above? **Hint:** The wave vector $\vec{k}$ is along the direction of propagation.

$$G_k \propto e^{\pm ik|\vec{x}-\vec{x}\,'|}$$

(c) Next, the δ-function in equation (6.36) has influence only at $|\vec{x} - \vec{x}\,'| \to 0$. In that limit, equation (6.35) reduces to the Poisson equation because $kR \ll 1$. In that case, we know from electrostatics that G_k must have a $1/|\vec{x} - \vec{x}\,'|$ dependence because it is the potential for a unit point charge. Therefore, the Green function is

$$G_k^{(\pm)}(|\vec{x} - \vec{x}\,'|) = \frac{e^{\pm ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} \tag{6.40}$$

(d) Notice that the Green function in equation (6.40) is written with a $\pm$, i.e., a G_k^+ term and a G_k^- term. What do these signs represent.

The (+) term represents an Outgoing Spherical wave.

The (−) term represents an Ingoing Spherical wave.

Next, we will consider the time dependence.

(e) Write down the Green function equation associated with equation (6.32).

$$\mathcal{D}\psi(\vec{x}) = f(\vec{x}) \quad (\text{w}3.1)$$

$$\mathcal{D}G(\vec{x},\vec{x}\,') = \delta(\vec{x}-\vec{x}\,') \quad (\text{w}3.3) \qquad \left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi = -4\pi \tag{6.41}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_k^{\pm}(\vec{x},t;\vec{x}\,',t') = -4\pi\,\delta(\vec{x}-\vec{x}\,')\,\delta(t-t')$$

Notice that the first δ-function on the right hand side localizes the source function to one point $\vec{x}\,'$ in space, and the other δ-function localizes it to one point in time.

5. Parallel the development of the time-independent case and write down an expression for $G_k^{\pm}$.

$G_k \propto e^{\pm i \vec{k}\cdot\vec{x} - i\omega t}$

$\vec{k}$: wave vector

$k : \omega\sqrt{\mu\epsilon} = \omega/c$

Shift origin in time and space

$G_k \propto e^{\pm i \vec{k}\cdot(\vec{x}-\vec{x}\,') - i\omega(t-t')}$

Green function must be spherically symmetric in space, and depend only on $|\vec{x}-\vec{x}\,'|$

$G_k \propto e^{\pm i k|\vec{x}-\vec{x}\,'| - i\omega(t-t')}$

$1/|\vec{x}-\vec{x}\,'|$ dependence for a unit point charge

$$G_k = \frac{e^{\pm i k|\vec{x}-\vec{x}\,'| - i\omega(t-t')}}{|\vec{x}-\vec{x}\,'|} = \frac{e^{\pm ikR - i\omega\tau}}{R}$$

The general solution is a combination of all possible solutions. A sum over ω also brings out all possible values of k, since k and ω are connected. So, we get

$$G^{(\pm)}(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm ikR}}{R}\, e^{-i\omega\tau}\, d\omega \tag{6.42}$$

where, following Jackson, I've written $R = |\vec{x}-\vec{x}\,'|$ and $\tau = t - t'$ to make the expression look neat and compact. The general solution for the Green function in infinite space is therefore a function of only the relative distance R between the source and observation point and the relative time τ between, again, source and observation point.

In free space where $k = \omega/c$, the integral in equation (6.42) is just a Dirac δ-function, so that

$$G^{(\pm)}(R,\tau) = \frac{1}{R}\,\delta\left(\tau \mp \frac{R}{c}\right) \tag{6.43}$$

or, if we wish to write it more explicitly in its full glory

$$G^{(\pm)}(\vec{x},t;\vec{x}\,',t') = \frac{\delta\left(t' - \left[t \mp \frac{|\vec{x}-\vec{x}\,'|}{c}\right]\right)}{|\vec{x}-\vec{x}\,'|} \tag{6.44}$$

The Green function $G^{(+)}$ is called the ***retarded Green function*** because the argument of the δ-function ensures that an effect observed at the point $\vec{x}$ at time t is caused by the action of a source a distance R away at an earlier or retarded time $t' = t - R/c$. The time difference R/c is just the time of propagation of the disturbance from one point to the other. Similarly, $G^{(-)}$ is called the ***advanced Green function***.