PHY 474 — Spring 2021

Homework 4 solutions

1. Consider the PP-I chain that we learned in class. Calculate the energy released in ***each of the three steps*** of the PP-I chain. Express your answer in MeV if you want full credit.

Note: The mass of a proton (p) is 1.0073 u, a deuteron (^{2}H or ^{2}D) is 2.0141 u, the light helium nucleus (^{3}He) is 3.0160 u, and that of the ^{4}He nucleus is 4.0015 u, where the unified atomic mass unit is 1 u = $1.660538782 \times 10^{-27}$ kg (nist.gov).

Solution: The first step in the PP-I chain is

$$^1\mathrm{H} + {}^1\mathrm{H} \to {}^2\mathrm{H} + e^+ + \nu_e$$

and since the mass of ^{1}H is just the proton mass, m_p, we get that

$$E_{\mathrm{p\,+\,p}} = \Big[2m_p - m(^2\mathrm{H})\Big]\, c^2 = \Big[2(1.0073)\, u - 2.0141\, u\Big]\, c^2$$

Now, since $u = 1.660538782 \times 10^{-27}$ kg is common to both terms inside the square brackets, we can do

$$E_{\mathrm{p\,+\,p}} = \Big[2(1.0073) - 2.0141\Big]\, u\, c^2 = \Big[0.0005\Big]\, (1.660538782 \times 10^{-27}\ \mathrm{kg})\, (3 \times 10^8\ \mathrm{m/s})^2$$

so that

$$E_{\mathrm{p\,+\,p}} = 7.47 \times 10^{-14}\ \mathrm{J} \left(\frac{1\ \mathrm{eV}}{1.602 \times 10^{-19}\ \mathrm{J}}\right) = 4.66 \times 10^5\ \mathrm{eV} = \boxed{0.466\ \mathrm{MeV}}$$

I haven't counted the additional 0.511 MeV that goes into the positron, and the energy that is carried away by the neutrino. Remember also that two of these constitute the first step, so the energy released is $E_{\mathrm{step}\ 1} = 2\, E_{\mathrm{p\,+\,p}}$ if you're counting the full first step as $2(2) = 4$ protons.

Next, the second step is

$$^2\mathrm{H} + {}^1\mathrm{H} \to {}^3\mathrm{He} + \gamma$$

so the energy generated is

$$E_{\mathrm{D\,+\,p}} = \Big[m(^2\mathrm{H}) + m_p - m(^3\mathrm{He})\Big]\, c^2 = \Big[2.0141 + 1.0073 - 3.0160\Big]\, u\, c^2 = 0.0054\ u\, c^2$$

so that

$$E_{\mathrm{D\,+\,p}} = 0.0054\ (1.660538782 \times 10^{-27}\ \mathrm{kg})\, (3 \times 10^8\ \mathrm{m/s})^2 = 8.07 \times 10^{-13}\ \mathrm{J} \equiv \boxed{5.04\ \mathrm{Mev}}$$

Again, keep in mind that there are two of these, so the energy released is $E_{\mathrm{step}\ 2} = 2\, E_{\mathrm{D\,+\,p}}$ if you're counting the full second step.

Finally, the third step is

$$^3\mathrm{He} + {}^3\mathrm{He} \to {}^4\mathrm{He} + {}^1\mathrm{H} + {}^1\mathrm{H}$$

so that

$$E_{\mathrm{step}\ 3} = \Big[2\, m(^3\mathrm{He}) - m(^4\mathrm{He}) - 2\, m_p\Big]\, c^2 = \Big[2(3.0160) - 4.0015 - 2(1.0073)\Big]\, u\, c^2 = 0.0159\, u\, c^2$$

and thus

$$E_{\mathrm{step}\ 3} = 0.0159\ (1.660538782 \times 10^{-27}\ \mathrm{kg})\, (3 \times 10^8\ \mathrm{m/s})^2 = 2.376 \times 10^{-12}\ \mathrm{J} \equiv \boxed{14.8\ \mathrm{Mev}}$$

2. Nuclear fusion converts H to He, so the mass fractions X and Y are going to change from the beginning of the Main Sequence (ZAMS) to the end of the Main Sequence (also known as TAMS). In this problem, you will calculate X and Y at TAMS.

(a) Begin by calculating the mass reduction rate dM/dt associated with the Sun's current luminosity, $L_\odot = 3.828 \times 10^{26}$ watts. State your answer in both kg/s and $M_\odot$/yr (you'll see why!).

Solution: : Start from $E = Mc^2$, which gives $M = E/c^2$, so that

$$\frac{dM}{dt} = \frac{1}{c^2}\frac{dE}{dt}$$

But $dE/dt \sim E/t$, the luminosity L, so this gives

$$\frac{dM}{dt} = \frac{L}{c^2} = \frac{3.828 \times 10^{26} \text{ watts}}{(3 \times 10^8 \text{ m/s})^2} = \boxed{4.25 \times 10^9 \text{ kg/s}}$$

and

$$\frac{dM}{dt} = 4.25 \times 10^9 \text{ kg/s} \left(\frac{1\ M_\odot}{1.99 \times 10^{30} \text{ kg}}\right) \left[\frac{(3600)(24)(365) \text{ s}}{\text{yr}}\right] = \boxed{6.7 \times 10^{-14}\ M_\odot/\text{yr}}$$

(b) For H fusion to He, the efficiency of energy released is only $\epsilon \approx 0.007$. Use this to calculate the rate of reduction in the mass of hydrogen in the Sun, dM_H/dt. Again, state your answer in both kg/s and $M_\odot$/yr.

Solution: Since the efficiency is only $\epsilon = 0.007$, this means that $\epsilon\, dM_\text{H}/dt = dM/dt$, because only 0.007 times the change in hydrogen mass was able to produce the energy output of the Sun (its luminosity). Thus

$$\frac{dM_\text{H}}{dt} = \frac{1}{\epsilon}\left(\frac{dM}{dt}\right) = \frac{4.25 \times 10^9 \text{ kg/s}}{0.007} = \boxed{6.1 \times 10^{11} \text{ kg/s}} \equiv \boxed{9.6 \times 10^{-12}\ M_\odot/\text{yr}}$$

(c) Next, calculate the (total) decrease in the mass of H over the Main Sequence lifetime of the Sun. Express your answer in $M_\odot$.

Solution: Using 10 billion yr as the Sun's Main Sequence lifetime, we get

$$\Delta M_\text{H} = 9.6 \times 10^{-12}\ M_\odot/\text{yr}\left(10 \times 10^9 \text{ yr}\right) = 0.096\ M_\odot = \boxed{0.1\ M_\odot}$$

(d) At ZAMS, the Sun had a mass fraction of H given by $X = 0.72$, and the mass fraction of He was $Y \approx 0.26$. Use your calculations above to compute the average mass fractions X and Y at TAMS, the end of the Sun's Main Sequence lifetime.

Solution: The mass fractions at the ends of the Sun's Main Sequence lifetime are X_TAMS and Y_TAMS, given respectively by

$$X_\text{TAMS} = X - \left(\frac{\Delta M_\text{H}}{M_\odot}\right) = 0.72 - \frac{0.1\ M_\odot}{1\ M_\odot} = 0.72 - 0.1 \quad \text{so that} \quad \boxed{X_\text{TAMS} = 0.62}$$

$$Y_\text{TAMS} = Y + \left(\frac{\Delta M_\text{H}}{M_\odot}\right) = 0.26 + \frac{0.1\ M_\odot}{1\ M_\odot} = 0.26 + 0.1 \quad \text{so that} \quad \boxed{Y_\text{TAMS} = 0.36}$$

3. The energy generation rate in the PP chain and CNO cycle are given by, respectively

$$\epsilon_{\rm pp} = (1.08\times10^{-12})\,\rho X^2 T_6^4 \qquad \text{and} \qquad \epsilon_{\rm pp} = (8.24\times10^{-31})\,\rho\, X\, X_{\rm CNO}\, T_6^{20}$$

where T_6 is the temperature in units of 10^6 K, and the constants within parentheses are both in units of W $\mathrm{m}^3\,\mathrm{kg}^{-2}$. Use $\rho = 10^3$ kg m^{-3}, $X = 0.7$, and $X_{\rm CNO} = 0.02$.

(a) Compute the temperature T at which the PP chain and CNO cycle will generate the same energy per unit time.

Solution: This can be found easily by setting the two expressions equal to each other, so

$$(1.08\times10^{-12})\,\rho X^2 T_6^4 = (8.24\times10^{-31})\,\rho\, X\, X_{\rm CNO}\, T_6^{20}$$

so that

$$\frac{T_6^{20}}{T_6^4} = \frac{(1.08\times10^{-12})\,\rho X^2}{(8.24\times10^{-31})\,\rho\, X\, X_{\rm CNO}} = \frac{(1.08\times10^{-12})}{(8.24\times10^{-31})}\,\frac{X}{X_{\rm CNO}}$$

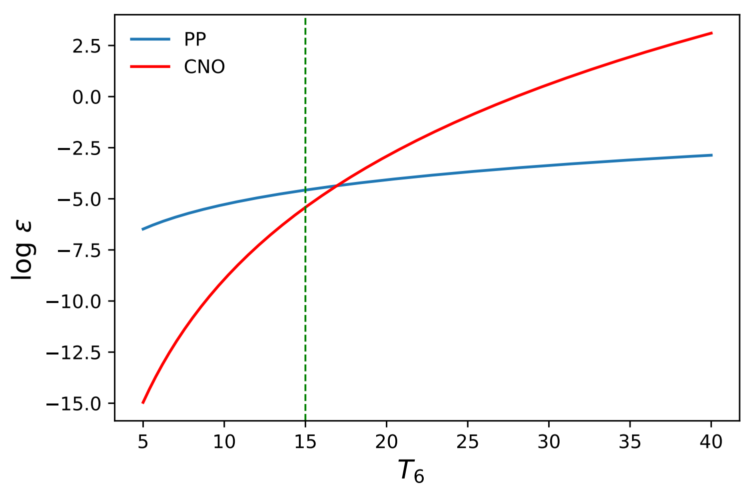
Putting in the values $X = 0.7$, and $X_{\rm CNO} = 0.02$, this gives

$$\frac{T_6}{T_6} = \left[\frac{(1.08\times10^{-12})}{(8.24\times10^{-31})}\,\frac{0.7}{0.02}\right]^{1/16} = 16.9 \approx 17$$

Therefore, at $T_6 = 17$, or $\boxed{17\times10^6\ \mathrm{K}}$, the PP chain and CNO cycle will generate the same energy per unit time.

(b) Draw a graph of $\log_{10}\epsilon$ *vs.* $\log_{10} T_6$ for the PP chain and the CNO cycle, putting both on the same plot.

Solution: The graph is shown below. I decided to plot $\log_{10}\epsilon$ *vs.* T_6 rather than a log-log plot, since I realized after I'd assigned the homework that putting T_6 directly on the axis would bring out more clearly the value of the point where the two curves intersect. If you did the log-log plot, you should've gotten two straight lines instead.



We see that the two plots intersect at about $17\,T_6$, thus at the temperature $\sim 17\times10^6$ K obtained in part (a). If you did the log-log plot, you should've gotten them to intersect at ~ 1.2. I've also marked with a green line where the Sun is at the moment (15 million K); we see that the PP-chain dominates the energy production at this temperature.

4. Recall the *Hayashi track* for pre-main sequence stars.

(a) Discuss what is meant by the *Hayashi forbidden region*.

Solution: We know that the Hayashi track occurs in fully convective pre-main sequence stars, as the star of a given mass barrels down the HR diagram at almost constant surface temperature, T_e (also known as effective temperature). This line represents a border between an allowed region (to its left) and a forbidden region (to its right in the HR diagram). Numerical models demonstrate that objects in the Hayashi forbidden region cannot achieve hydrostatic equilibrium, and thus no stable protostars can exist in this region. Recalling that surface temperature is lower to the right in the HR diagram, the implication of this important result is that a star cannot reach hydrostatic equilibrium if its surface is too cool.

(b) The mass-luminosity relation for convective stars is

$$\frac{L_s}{L_\odot} \simeq 0.03 \left(\frac{M}{M_\odot}\right)^{-7} \left(\frac{T_s}{2400\ \mathrm{K}}\right)^{-40}$$

where L_s and T_s are the luminosity and temperature on the surface of the star respectively, and M is the mass of the star. Plot $\log\ L_s$ *vs.* $\log\ T_s$ for $T = 2400$ to 2800 K.

Solution: The plot is shown below. The Hayashi track is the expected straight line that goes down the HR diagram at almost constant T_s.