(1a) $S_x = \frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $S_y = \frac{\hbar}{2}\begin{bmatrix}0&-i\\i&0\end{bmatrix}$ $S_z = \frac{\hbar}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $|+z\rangle = \begin{bmatrix}1\\0\end{bmatrix}$

$|-z\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

$$S_x|+z\rangle = \frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}0\\1\end{bmatrix} = \frac{\hbar}{2}|-z\rangle$$

$$S_x|-z\rangle = \frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}1\\0\end{bmatrix} = \frac{\hbar}{2}|+z\rangle$$

$$S_y|+z\rangle = \frac{\hbar}{2}\begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}0\\i\end{bmatrix} = \frac{i\hbar}{2}\begin{bmatrix}0\\1\end{bmatrix} = \frac{i\hbar}{2}|-z\rangle$$

$$S_y|-z\rangle = \frac{\hbar}{2}\begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}-i\\0\end{bmatrix} = -\frac{i\hbar}{2}\begin{bmatrix}1\\0\end{bmatrix} = -\frac{i\hbar}{2}|+z\rangle$$

(B) $|\psi\rangle = \begin{bmatrix}a\\b\end{bmatrix}$

$$\langle S_x\rangle = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix}b\\a\end{bmatrix} = \frac{\hbar^2}{4}(a^*b + b^*a)$$

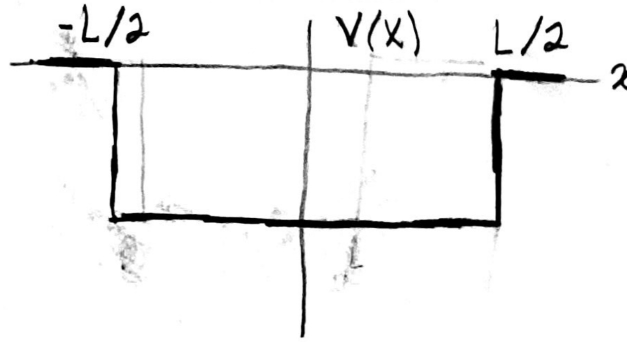
$$\langle S_y\rangle = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix}-ib\\ia\end{bmatrix} = \frac{\hbar^2}{4}(-ia^*b + ib^*a)$$

$$\langle S_z\rangle = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}a\\-b\end{bmatrix} = \frac{\hbar^2}{4}[a^* \ b^*]\begin{bmatrix}a\\-b\end{bmatrix} = \frac{\hbar^2}{4}(a^*a + b^*(-b)) = \frac{\hbar^2}{4}(|a|^2 + |b|^2)$$

$\begin{bmatrix}-1\\i\end{bmatrix}$ $\langle S_x\rangle = \frac{\hbar^2}{4}(-i+i) = 0$ $\langle S_y\rangle = \frac{\hbar^2}{4}(1+(-1)) = 0$ $\langle S_z\rangle = \frac{\hbar^2}{4}(1-1) = 0$

$$\boxed{|\psi\rangle = \begin{bmatrix}-1\\i\end{bmatrix}}$$

(2)



$$\psi(x,0) = A\left[\sin\frac{4\pi x}{L} + i\cos\frac{\pi x}{L}\right]$$

$$\int_{-L/2}^{L/2} |\psi(x,0)|^2 dx = \int_{-L/2}^{L/2} A^2 \left[\psi^*(x,0)\,\psi(x,0)\right] \overset{dx}{} = 1$$

$$= A^2 \int_{-L/2}^{L/2} \left[\sin\left(\frac{4\pi x}{L}\right) - i\cos\left(\frac{\pi x}{L}\right)\right]\left[\sin\left(\frac{4\pi x}{L}\right) + i\cos\left(\frac{\pi x}{L}\right)\right] = 1$$

$$\to A^2 \int_{-L/2}^{L/2} \left[\sin^2\left(\frac{4\pi x}{L}\right) \underbrace{\cancel{- i\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{4\pi x}{L}\right) + i\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{4\pi x}{L}\right)}}_{\to 0} + \cos^2\left(\frac{\pi x}{L}\right)\right] =$$

$$\to A^2 \int_{-L/2}^{L/2} \left[\sin^2\left(\frac{4\pi x}{L}\right)\right] dx + \int_{-L/2}^{L/2} \left[\cos^2\left(\frac{\pi x}{L}\right)\right] dx = 1$$

$$\to A^2 \left[\frac{L}{2}\right] + \left[\frac{L}{2}\right] = 1 \to A^2\,[L] = 1 \quad A^2 = 1/L \quad \boxed{A = 1/\sqrt{L}}$$

$$\boxed{\psi(x,0) = \frac{1}{\sqrt{L}}\left[\sin\left(\frac{4\pi x}{L}\right) + i\cos\left(\frac{\pi x}{L}\right)\right]}$$

$$\psi(x,t) = \sum_{n=1}^{\infty} C_n \psi_n(x) e^{-iE_n t} \to C_n = \int_{-L/2}^{L/2} \psi_n^*(x)\,\psi(x,0)\,dx$$

$$C_1 = \int_{-L/2}^{L/2} \left(\sqrt{\frac{2}{L}}\cos\frac{\pi x}{L}\right)\frac{1}{\sqrt{L}}\left(i\cos\left(\frac{\pi x}{L}\right)\right) dx = \int_{-L/2}^{L/2} \frac{\sqrt{2}}{L}\, i\cos^2\left(\frac{\pi x}{L}\right) dx = \boxed{i/L} = C_1$$

$$C_4 = \int_{-L/2}^{L/2} \left(\sqrt{\frac{2}{L}}\sin\frac{4\pi x}{L}\right)\frac{1}{\sqrt{L}}\left(\sin\left(\frac{4\pi x}{L}\right)\right) dx = \int_{-L/2}^{L/2} \frac{\sqrt{2}}{L}\sin^2\left(\frac{4\pi x}{L}\right) dx = \boxed{1/\sqrt{2}} = C_4$$

$$\boxed{\psi(x,t) = \frac{i}{L}\,\frac{1}{\sqrt{L}}\, i\cos\left(\frac{\pi x}{L}\right) e^{-it} + \frac{1}{\sqrt{2L}}\sin\left(\frac{4\pi x}{L}\right) e^{-i4t}}$$

$$P'[L/4, L/2] = \int_{L/4}^{L/2} -\frac{1}{L\sqrt{L}} \cos\left(\frac{\pi x}{L}\right) e^{-it} + \frac{1}{\sqrt{2L}} \sin\left(\frac{4\pi x}{L}\right) e^{-i4t} \, dx$$

$$= -\frac{1}{L\sqrt{L}} e^{-it} \int_{L/4}^{L/2} \cos\left(\frac{\pi x}{L}\right) dx + \frac{1}{\sqrt{2L}} e^{-i4t} \int_{L/4}^{L/2} \sin\left(\frac{4\pi x}{L}\right) dx$$

$$= -\frac{e^{-it}}{L\sqrt{L}} \left(-\frac{(\sqrt{2}-2)L}{2\pi}\right) + \frac{e^{-i4t}}{\sqrt{2L}} \left(-\frac{L}{2\pi}\right)$$

$$= \boxed{\frac{e^{-it}(\sqrt{2}-2)}{\sqrt{L}\, 2\pi} - \frac{e^{-i4t} L}{2\pi\sqrt{2L}}}$$

(3) $S(p_0) = e^{ip_0X/\hbar}$ $\quad a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + i\frac{P}{m\omega}\right)$ $\quad a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(X - i\frac{P}{m\omega}\right)$

$\to \cos\left(\frac{p_0X}{\hbar}\right) + i\sin\left(\frac{p_0X}{\hbar}\right)$ $\qquad d_0 = \sqrt{m\omega/2\hbar}$

(a) $[a, S(p_0)] = ae^{ip_0X/\hbar} - e^{ip_0X/\hbar}a$

$d_0\left(X + i\frac{P}{m\omega}\right)e^{ip_0X/\hbar} - e^{ip_0X/\hbar}d_0\left(X - i\frac{P}{m\omega}\right)$

$= \cancel{d_0Xe^{ip_0X/\hbar}} + i\frac{d_0P}{m\omega}e^{ip_0X/\hbar} - \left(\cancel{d_0Xe^{ip_0X/\hbar}} - i\frac{P}{m\omega}e^{ip_0X/\hbar}d_0\right)$

$= d_0 i\frac{P}{m\omega}e^{ip_0X/\hbar} + d_0 i\frac{P}{m\omega}e^{ip_0X/\hbar} = \boxed{\frac{2iP}{m\omega}e^{ip_0X/\hbar}d_0}$

(B) $|\psi\rangle = S(p_0)|0\rangle$

$a|\psi\rangle = ae^{ip_0X/\hbar} = 0$

(C) $|\psi\rangle = e^{ip_0X/\hbar}|0\rangle$ $\qquad \frac{d_0}{\sqrt{2}}(a+a^+)$

$\langle X\rangle = \langle\psi|X|\psi\rangle = \langle 0|e^{-ip_0X/\hbar}(a+a^+)e^{ip_0X/\hbar}|0\rangle = 0$

$\langle p\rangle = \langle\psi|P|\psi\rangle = \langle 0|e^{-ip_0X/\hbar}(a^+-a)e^{ip_0X/\hbar}|0\rangle = 0$

$\frac{i\hbar}{\sqrt{2}d_0}$

$\cancel{|\psi\rangle = \begin{bmatrix}1\\0\end{bmatrix}}$

$$\frac{\hbar^2}{4}[10]\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \cdot \quad [1\ 0]\begin{bmatrix}0\\1\end{bmatrix} = \frac{\hbar^2}{4}\cdot 0 = 0$$

$$\frac{\hbar^2}{4}[1\ 0]\begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{\hbar^2}{4}[1\ 0]\begin{bmatrix}0\\i\end{bmatrix} = \frac{\hbar^2}{4}0 = 0$$

$$\cancel{\frac{\hbar^2}{4}[1\ 0]\begin{bmatrix}0&-i\\i&0\end{bmatrix}}$$

$$\frac{\hbar^2}{4}[1\ 0]\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{\hbar^2}{4}[1\ 0]\cancel{\begin{bmatrix}1\\0\end{bmatrix} = \frac{\hbar^2}{4}}$$

$|\psi\rangle = \begin{bmatrix}-1\\1\end{bmatrix}$

$$\frac{\hbar^2}{4}[-1\ 1]\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = [-1\ 1]\begin{bmatrix}1\\-1\end{bmatrix} = \frac{\hbar^2}{4}(-1-1)$$

$$\frac{\hbar^2}{4}[-1\ 1]\begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = [-1\ 1]\cancel{\begin{bmatrix}-i\\-i\end{bmatrix} = \frac{\hbar^2}{4}(i-i)} = 0$$

$|\psi\rangle$

$$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\langle S_x \rangle = \frac{\hbar^2}{4}[a^* \; b^*]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar^2}{4}[a^* \; b^*]\begin{bmatrix} b \\ a \end{bmatrix} = \frac{\hbar^2}{4}(a^* b + b^* a)$$

$$\langle S_y \rangle = \frac{\hbar^2}{4}[a^* \; b^*]\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar^2}{4}[a^* \; b^*]\begin{bmatrix} -ib \\ ia \end{bmatrix} = \frac{\hbar^2}{4}(-ia^* b + ib^* a)$$

$$\langle S_z \rangle = \frac{\hbar^2}{4}[a^* \; b^*]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar^2}{4}[a^* \; b^*]\begin{bmatrix} a \\ -b \end{bmatrix} = \frac{\hbar^2}{4}(a^* a + b^*(-b))$$

$$= \frac{\hbar^2}{4}(|a|^2 + |b|^2)$$

$\begin{bmatrix} -i \\ i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(-1 - 1)$ $\langle S_y \rangle = \frac{\hbar^2}{4}($

$\begin{bmatrix} 1 \\ i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(i - i) = 0$ $\langle S_y \rangle = \frac{\hbar^2}{4}(1 - 1)$ $\langle S_z \rangle = \frac{\hbar^2}{4}(1 + 1)$

$\begin{bmatrix} -1 \\ i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(-i + i) = 0$ $\langle S_y \rangle = \frac{\hbar^2}{4}(1 + (-1)) = 0$ $\langle S_z \rangle = \frac{\hbar^2}{4}(+1 + 1) = 2\frac{\hbar^2}{4}$

$\begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(-i + i) = 0$ $\langle S_y \rangle = \frac{\hbar^2}{4}(-1 - 1)$

$\begin{bmatrix} -1 \\ -i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(i - i) = 0$ $\langle S_y \rangle = \frac{\hbar^2}{4}(1 + 1)$

$\begin{bmatrix} -1 \\ i \end{bmatrix}$ $\langle S_x \rangle = \frac{\hbar^2}{4}(-i + i) = 0$ $\langle S_y \rangle = \frac{\hbar^2}{4}(1 + (-1)) = 0$ $\langle S_z \rangle = \frac{\hbar^2}{4}(1 - 1) = 0$

$$|\psi\rangle = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

(3) (a) $[a, S(p_0)]$

$$S(P_0) = e^{ip_0 x/\hbar} = \cos\left(\frac{p_0 x}{\hbar}\right) + i\sin\left(\frac{p_0 x}{\hbar}\right)$$

$$= a\left[\cos\left(\frac{p_0 x}{\hbar}\right) + i\sin\left(\frac{p_0 x}{\hbar}\right)\right] - \left[\cos\left(\frac{p_0 x}{\hbar}\right) + i\sin\left(\frac{p_0 x}{\hbar}\right)\right]a$$

$$= a\cos\left(\frac{p_0 x}{\hbar}\right) + ai\sin\left(\frac{p_0 x}{\hbar}\right) - \cos\left(\frac{p_0 x}{\hbar}\right)a - i\sin\left(\frac{p_0 x}{\hbar}\right)a$$

$$e^{ip_0 d/\sqrt{2}\,(a+a^\dagger)}$$

$$H(x,p) = \frac{p^2}{2m} + \frac{kx^2}{2}$$

$$H = \hbar\omega(aa^\dagger - 1/2)$$

$$[a, a^\dagger] = I$$