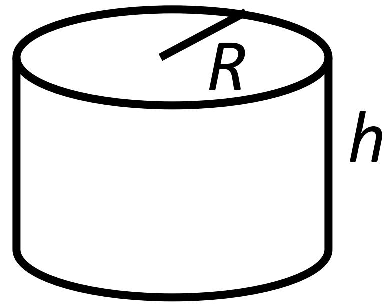
Physics 440, Autumn 2019 **Activity 9: Cylinder with minimal surface area**

Find the relationship between radius and height of a cylinder that minimizes the surface area for a given volume. Do this with and without Lagrangian multipliers.

Solution without Lagrange multipliers:



Area of the cylinder: $A = 2\pi Rh + 2\pi R^2$

Since $V = const = V_0$, radius and height of the cylinder are not independent.

$V_0 = \pi R^2 h$, so $h = \frac{V_0}{\pi R^2}$

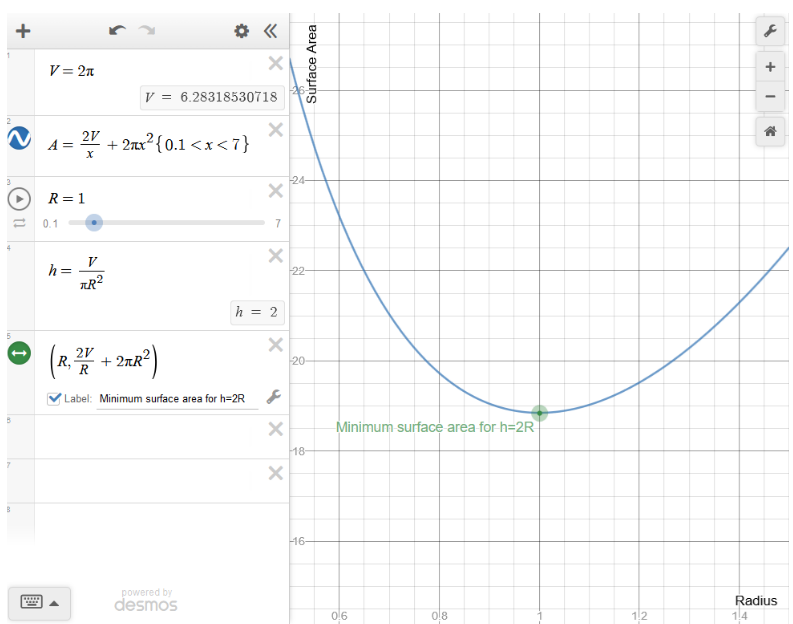
This allows us to express the surface area as a function of only R. Then we can set the derivative of A with respect to R to zero to determine the value of R for which the surface area is minimized.

$$h = \frac{V_0}{\pi R^2} \Rightarrow A = 2\pi R \frac{V_0}{\pi R^2} + 2\pi R^2 = \frac{2V_0}{R} + 2\pi R^2$$

$$\frac{\partial A}{\partial R} = -\frac{2V_0}{R^2} + 4\pi R = 0 \Rightarrow V_0 = 2\pi R^3$$

Substituting this V_0 for which A is minimized back into the definition of V_0 ($V_0 = \pi R^2 h$) we can see that the surface area is minimized when $2R = h$

Example:



https://www.desmos.com/calculator/bykqctibdd

Solution with Lagrange multipliers: (see also example 2.3)

We want to minimize

$\Phi = A = 2\pi Rh + 2\pi R^2 = 2\pi R(R + h)$

The constraint is that the volume for the cylinder is constant (V_0)

$f(R, h) = \pi R^2 h - V_0 = 0$

The condition $\delta(\Phi + \lambda f) = 0$

leads to the following three equation:

$$\frac{\partial}{\partial h}(\Phi + \lambda f) = \frac{\partial}{\partial h}[2\pi Rh + 2\pi R^2 + \lambda(\pi R^2 h - V_0)] = 2\pi R + \lambda\pi R^2 = 0$$

$$\frac{\partial}{\partial R}(\Phi + \lambda f) = \frac{\partial}{\partial R}[2\pi Rh + 2\pi R^2 + \lambda(\pi R^2 h - V_0)] = 2\pi h + 4\pi R + 2\pi\lambda Rh = 0$$

$$\frac{\partial}{\partial \lambda}(\Phi + \lambda f) = \frac{\partial}{\partial \lambda}[2\pi Rh + 2\pi R^2 + \lambda(\pi R^2 h - V_0)] = \pi R^2 h - V_0$$

The third equation is simply the constraint. The first and second equations can be solved for λ

First equation: $\lambda = -\frac{2}{R}$ Second equation: $\lambda = -\frac{h+2R}{Rh}$

Equating the two expressions for λ yields the relationship between h and R that minimizes the surface area of the cylinder:

$$\frac{2}{R} = \frac{h + 2R}{Rh} \Rightarrow R = \frac{2h - h}{2} \Rightarrow h = 2R$$