PHY 420 Spring 2021

Class Summary—Week 7, Day 2—Thursday, May 13

Tensors (continued)

In the previous class, we wrote the **dot product** as the product of the components of a covariant and a contravariant vector:

$$B \cdot A \equiv B_\alpha A^\alpha \tag{11.66}$$

where we've used the Einstein summation convention over repeated indices, that is

$$B \cdot A \equiv B_\alpha A^\alpha = B_0 A^0 + B_1 A^1 + B_2 A^2 + B_3 A^3$$

Note that this is a general relation for a 4-dimensional vector, and we will need to write it differently for it to conform to the way the invariant is structured in Special Relativity. First though, it is worth reminding ourselves that in equation (11.66), A is a contravariant vector defined by the transformation

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta \tag{11.61}$$

and B is a covariant vector defined by

$$B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta \tag{11.62}$$

where, again, the Einstein summation over repeated indices is implied in both (11.61) and (11.62).

We can show explicitly that the scalar product is an invariant or scalar under the transformation

$$x'^\alpha = x'^\alpha \left(x^0, x^1, x^2, x^3\right) \tag{11.60}$$

where $\alpha = 0, 1, 2, 3$, *as you did in Question 1 of today's worksheet.* To do so, let us find the scalar product $B' \cdot A'$:

$$B' \cdot A' = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^\gamma} B_\beta A^\gamma$$

where we've used the relation from equation (11.62) for the covariant vector B and equation (11.61) for the contravariant vector A.

Proceeding, we get

$$B' \cdot A' = \frac{\partial x^\beta}{\partial x^\gamma} B_\beta A^\gamma = \delta_{\beta\gamma} B_\beta A^\gamma$$

Note that the δ-function comes from the fact that $\partial x^\beta / \partial x^\gamma = 0$ if $\beta \neq \gamma$.

Finally, therefore

$$B' \cdot A' = \delta_{\beta\gamma} B_\beta A^\gamma = B_\gamma A^\gamma = B \cdot A$$

which proves the property of invariance of the scalar product under the transformation in equation (11.60).

The Metric Tensor

As noted on the previous page, the results or definitions that we have learned so far are general, and could apply to any tensor. We will now transition to considering the specific geometry of the space-time of special relativity, which is defined by the invariant interval s^2 given by

$$s^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{11.59}$$

where I've introduced a modification of Jackson's equation to use superscripts on the coordinates because of the conventions defined above.

In differential form, the **infinitesimal interval** ds is

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{11.67}$$

Notice how this looks when the contravariant components are written out explicitly:

$$(ds)^2 = dx^0 dx^0 - dx^1 dx^1 - dx^2 dx^2 - dx^3 dx^3$$

That is, if we are to write this relation in terms of contravariant components like dx^α, then we'll need something to give us the correct signs. Notice that we will need a plus sign when we're contracting out the components dx^0 involving time, and minus signs when we're contracting out the components dx^1, dx^2, dx^3 of the spatial part. Since we are dealing with 4-dimensional quantities, there are in principle 4×4 possible combinations, so we need a 16-element tensor. This special tensor is given a name; it is called the **metric tensor** $g_{\alpha\beta}$ and is defined so that

$$(ds)^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{11.68}$$

Comparison with equation (11.67) tells us that in the flat space-time of special relativity (as opposed to the curved space-time of general relativity), the metric tensor takes a very simple form *as you wrote out in Question 2 on today's worksheet*, with elements

$$g_{00} = 1, \qquad g_{11} = g_{22} = g_{33} = -1 \tag{11.69}$$

Thus, as is clear from equation (11.68), the metric tensor is symmetric:

$$g_{\alpha\beta} = g_{\beta\alpha}$$

and based on equation (11.69), the metric tensor is also diagonal.

Also, for flat space-time, the contravariant and covariant tensors are the same:

$$g^{\alpha\beta} = g_{\alpha\beta} \tag{11.70}$$

Finally, the contraction of the contravariant and covariant metric tensors *that you worked out in Question 3(a) on today's worksheet* gives the **Kronecker δ in four dimensions**:

$$g_{\alpha\gamma}\, g^{\gamma\beta} = \delta_\alpha^\beta \tag{11.71}$$

where $\delta_\alpha^\beta = 0$ for $\alpha \neq \beta$, and $\delta_\alpha^\alpha = 1$ for $\alpha = 0, 1, 2, 3$. Thus, in matrix form

$$\delta_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Next, compare the invariant length element $(ds)^2$ in equation (11.68) with the invariant scalar product in equation (11.66). Let's write equation (11.66) first:

$$B \cdot A \equiv \boxed{B_\alpha}\, A^\alpha$$

and then write equation (11.68) below:

$$(ds)^2 = \boxed{g_{\alpha\beta}\, dx^\alpha}\, dx^\beta$$

I've put boxes over the comparable quantities in both equations. Notice that we have a covariant vector in the top equation and a contravariant vector multiplied by the metric tensor in the bottom equation. This suggests that the covariant coordinate 4-vector x_α can be obtained from the contravariant x^β by contraction with the metric tensor $g_{\alpha\beta}$:

$$x_\alpha = g_{\alpha\beta}\, x^\beta \tag{11.72}$$

and its inverse

$$x^\alpha = g^{\alpha\beta}\, x_\beta \tag{11.73}$$

In fact, contraction with the metric tensor $g_{\alpha\beta}$ or $g^{\alpha\beta}$ is the procedure for raising or lowering indices for tensors in general, and so converting covariant to contravariant, and vice versa. Thus

$$\begin{aligned} F^{\cdots\alpha\cdots}_{\cdots} &= g^{\alpha\beta}\, F^{\cdots}_{\cdots\beta} \\ G^{\cdots}_{\cdots\alpha\cdots} &= g_{\alpha\beta}\, G^{\cdots\beta}_{\cdots} \end{aligned} \tag{11.74}$$

We are now able to see another way how covariant and contravariant vectors differ. From our discussion of the metric tensor in equation (11.69), it follows that if a contravariant 4-vector A^α has components A^0, A^1, A^2, A^3, its covariant partner has components

$$A_0 = A^0, \qquad A_1 = -A^1, \qquad A_2 = -A^2, \qquad A_3 = -A^3 \tag{11.75.a}$$

as you showed in Question 3(b) on today's worksheet.

We can use equation (11.75.a) to write the contravariant and covariant vectors in concise form as

$$A^\alpha = (A^0, \vec{A}), \qquad A_\alpha = (A_0, -\vec{A}) \tag{11.75}$$

where the 3-vector $\vec{A}$ has components A^1, A^2, A^3. Thus, **covariant vectors are just spatially inverted contravariant vectors**.

Finally, let's apply equation (11.66) to write the scalar product:

$$B \cdot A \equiv B_\alpha A^\alpha = B_0 A^0 + B_1 A^1 + B_2 A^2 + B_3 A^3$$

Applying equation (11.75.a) to this, we get

$$B \cdot A \equiv B_\alpha A^\alpha = B^0 A^0 - B^1 A^1 - B^2 A^2 - B^3 A^3$$

Therefore, application of equation (11.75) allows us to write the scalar product written in the form we would like to have

$$B \cdot A \equiv B_\alpha A^\alpha = B^0 A^0 - \vec{B} \cdot \vec{A}$$

because this form of the scalar product makes it look like the invariant 4-vector in equation (11.24).

Now, while the discussion on the preceding pages tells us how ordinary quantities transform, we are also interested in how the partial derivative operators with respect to x^α and x_α transform, as they are involved in the construction of dynamical systems in Special Relativity. The transformation properties of these operators can be established directly by using the rules of implicit differentiation. For example, since

$$\frac{\partial \psi}{\partial x'^\alpha} = \frac{\partial \psi}{\partial x^\beta} \frac{\partial x^\beta}{\partial x'^\alpha}$$

we get

$$\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta}$$

Compare with the definition of a covariant vector in equation (11.62): $B'_\alpha = \dfrac{\partial x^\beta}{\partial x'^\alpha} B_\beta$, and you will see that

Differentiation with respect to contravariant coordinates form a covariant 4-vector.

Conversely,

Differentiation with respect to covariant coordinates forms a contravariant 4-vector.

Given these two results, we introduce the following simplifying notation:

$$\begin{aligned} \partial_\alpha &\equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right) \\ \partial^\alpha &\equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla} \right) \end{aligned} \tag{11.76}$$

to avoid having to write out all of the partial derivatives with respect to the various components,

Now let's write down the **4-divergence** of a 4-vector A:

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A} \tag{11.77}$$

This is similar in form to the continuity equation of charge and current density, the Lorenz condition on the scalar and vector potentials, etc. As Jackson notes, these examples give a first inkling of how the covariance of a physical law emerges provided suitable Lorentz transformation properties are attributed to the quantities entering the equation.

Meanwhile, the **4-dimensional Laplacian operator** is defined to be the invariant contraction

$$\square \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{02}} - \nabla^2 \tag{11.78}$$

which is just (the negative of) the operator of the wave equation in vacuum! We see that many important quantities in electrodynamics are Lorentz scalars, as we anticipated they would be so.

Matrix Representation of Lorentz Transformations

We will now introduce a matrix representation. In this representation, the components of a contravariant 4-vector will form the elements of a column vector. The coordinates x^0, x^1, x^2, x^3 thus define a coordinate vector whose representative is

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{11.79}$$

Keep an eye on the notation — we don't use an arrow (or boldface font in textbooks) to indicate a 4-vector, those are reserved for the spatial part of the 4-vector only (i.e., for the 3-vector).

Matrix scalar products of 4-vectors (a, b) are defined in the usual way by summing over the products of the elements of a and b, or equivalently, by the matrix multiplication:

$$(a, b) \equiv \tilde{a}b \tag{11.80}$$

where $\tilde{a}$ is the transpose of a (and hence a row vector).

Now, recall that in the flat space-time of special relativity, the metric tensor $g_{\alpha\beta}$ is diagonal with elements given by equation (11.69):

$$g_{00} = 1, \qquad g_{11} = g_{22} = g_{33} = -1$$

Therefore, the **metric tensor** $g_{\alpha\beta}$ has as its representative the square 4×4 matrix

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{11.81}$$

with

$$g^2 = I$$

where I is the 4×4 unit matrix.

Next, knowing that the covariant 4-vector x_α can be obtained from the contravariant x^β by contraction with the metric tensor $g_{\alpha\beta}$, as given in equation (11.72):

$$x_\alpha = g_{\alpha\beta}\, x^\beta$$

then, by matrix multiplication of g in equation (11.81) on x from equation (11.79), we find the covariant coordinate vector is

$$gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{11.82}$$

as you showed in Question 5 on today's worksheet.

In the compact notation we have introduced, the scalar product of two 4-vectors defined in equation (11.66) can be written as

$$a \cdot b = (a, gb) = (ga, b) = \tilde{a} g b \tag{11.83}$$

or, if we write out the elements explicitly

$$a \cdot b = \tilde{a} g b = a^\alpha g_{\alpha\beta} b^\beta = a^\alpha b_\alpha$$

Now, let us get to the purpose of all that we've introduced so far: we seek a group of linear transformations that leaves $(x, gx) = x \cdot x$ invariant. Since (x, gx) or $x \cdot x$ is the norm of a 4-vector, we are effectively seeking a group of transformations that preserves the "length" in the 4-dimensional metric. In other words, we seek all square 4×4 matrices A on the coordinates

$$x' = Ax \tag{11.84}$$

that leave the norm (x, gx) invariant:

$$x' \cdot x' = \tilde{x}' g x' = \tilde{x} g x = x \cdot x \tag{11.85}$$

that is, they leave $x \cdot x = x' \cdot x'$ invariant, which after all is a mathematical statement of the postulate of Special Relativity.

From equation (11.84), we get for the transpose that

$$\tilde{x}' = \tilde{x}\tilde{A}$$

and substituting this, and $x' = Ax$ from equation (11.84) into equation (11.85), we get

$$\tilde{x}\tilde{A} g A x = \tilde{x} g x$$

Since this must hold for all coordinate vectors x, A must satisfy the matrix equation

$$\tilde{A} g A = g \tag{11.86}$$

Certain properties of the transformation matrix A can be deduced immediately from equation (11.86). Taking the determinant of both sides of equation (11.86) gives

$$\det(\tilde{A} g A) = \det g \, (\det A)^2 = \det g$$

But $\det g = -1 \neq 0$, so we get

$$\det A = \pm 1$$

which is a constraint on the allowed matrices A, and hence the allowed transformations. *You demonstrated this in Question 6 on today's worksheet.*

In the next class, we will discuss more details about the transformations with $\det A = +1$.