PHY 474/374 Spring 2021

Class Summary—Week 4, Day 2—Wednesday, April 21

Mass-Luminosity Relations

In the last couple of classes, we have discussed how energy produced in the core of a star is transferred from the interior to its surface. We have discussed both radiative transport and the convective transfer of energy. Using the equation of radiative transport, combined with estimates of the temperature and density, we will now estimate the luminosity of the star.

Let's begin with the equation of radiative transport that we encountered in Chapter 4:

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi acr^2T^3} \tag{5.8}$$

where κ is the opacity, and a is the radiation density constant that features in the expression for the radiation energy density, $u_R = aT^4$.

Starting from equation (5.8) above, we can show that the luminosity L is given by

$$L = -\frac{4\pi r^2 ac}{3\kappa\rho}\frac{d}{dr}\left(T^4\right) \tag{7.3}$$

as you did on Question 1 of today's worksheet. The expression involves the opacity, κ. Let's look at this quantity in more detail.

Opacity parametrizes the microscopic interaction between radiation and matter. Recall that we described this earlier in terms of the mean free path, λ. Opacity and mean free path convey the same information, but opacity is the preferred quantity in the literature. Recall that the mean free path is given by

$$\lambda = \frac{1}{n\sigma}$$

where n is the number of particles (e.g., atoms or molecules) per unit volume, and σ is the cross section that describes the microscopic interaction between radiation and matter; roughly speaking, a photon will interact with an atom if it passes within an area σ of the atom.

Recall that the particle density, $n = \rho/(\mu m_p)$. Thus, since n is proportional to the mass density ρ, we would like to replace n with ρ in the equation for λ above. To do so, we can write

$$\lambda = \frac{1}{n\sigma} = \frac{1}{\rho\kappa}$$

as you did in Question 2(a) of today's worksheet. This tells us that the opacity κ can then be defined as

$$\kappa = \frac{\sigma n}{\rho}$$

Let's check the units to see what we have here. Since the cross section σ is in m^2, the particle density n is in cm^{-3}, and the mass density ρ is in g cm^{-3}, the units of the opacity κ will be

$$[\kappa] = \frac{\text{cm}^2\ \text{cm}^{-3}}{\text{g cm}^{-3}} = \text{cm}^2\ \text{g}^{-1}$$

and thus, we can define κ as the **cross section per unit mass**.

Contributions to Opacity

Since opacity parametrizes the interaction between radiation and matter, we need to figure out the processes that contribute to the opacity. There are four processes I will discuss here. They are:

- Electron scattering
- Bound-free (bf) absorption
- Free-free (ff) absorption
- Bound-bound (bb) absorption

Let's discuss these in more detail. In general, Thomson scattering is the process whereby photons scatter off charged particles. Thus, one of the contributions to opacity comes from photons scattering off free electrons. This can be quantified by saying that photons scatter off free electrons when the free electrons have thermal motions that are much smaller than their rest mass energy, that is,

$$kT \ll m_e c^2$$

On Question 2(b) of today's worksheet, you calculated $m_e c^2/k$ and verified that $T \ll 5.9 \times 10^9$ K; thus electron scattering can certainly take place in stars.

If the gas is completely ionized (a reasonable assumption in a star), the opacity due to electron scattering is independent of density and temperature. To demonstrate this, we begin with the opacity for free electrons, which is given by

$$\kappa_e = \frac{\sigma_e \, n_e}{\rho}$$

The Thomson cross-section of the electron, σ_e, is given **in cgs units** by

$$\sigma_e = \frac{8\pi}{3} \left(\frac{e^2}{m_e c^2} \right)^2$$

where the electron charge in cgs units is $e = 4.8 \times 10^{-10}$ esu, whereas $m_e = 9.11 \times 10^{-28}$ g. Note that you **cannot** substitute for all the quantities in this formula in SI units and get the correct answer for σ_e; since this is an electrodynamic quantity, the formula itself will need to be changed (if you're interested, include $4\pi\epsilon_0$ in the denominator inside the parentheses to get the correct form in SI units). *On Question 3(a) of today's worksheet*, you calculated that

$$\sigma_e = 6.6 \times 10^{-25} \text{ cm}^2$$

Next, the electron density, $n_e = \rho/\mu_e m_p$, where the mean molecular weight per electron can be obtained by modifying the expression for μ from a previous class to

$$\frac{1}{\mu_e} = \sum_j \frac{X_j Z_j}{A_j}$$

By substituting appropriate numbers to find μ_e, and then n_e, *you showed in Question 3b on today's worksheet* that the opacity due to electron scattering is

$$\kappa_e = 0.2\,(1 + X) \text{ cm}^2\,\text{g}^{-1}$$

which is independent of the density and temperature.

In general, though, the opacity depends on the density and temperature, and can be approximated by a power law

$$\kappa \simeq \kappa_0\, \rho^{\lambda} T^{-\nu} \tag{7.4}$$

where κ_0 is a constant that depends on the chemical composition of the star.

Consider, for example, the other contributors to the opacity listed on the previous page. In the process of **bound-free absorption** (also known as photoionization), an electron that is initially bound to an atom is ejected by the interaction with a photon of sufficient energy; the freed electron then has a kinetic energy equal to the difference between the energy of the photon and the ionization potential (the minimum energy required to free the electron from the atom). The opacity due to bound-free absorption processes can be approximated by

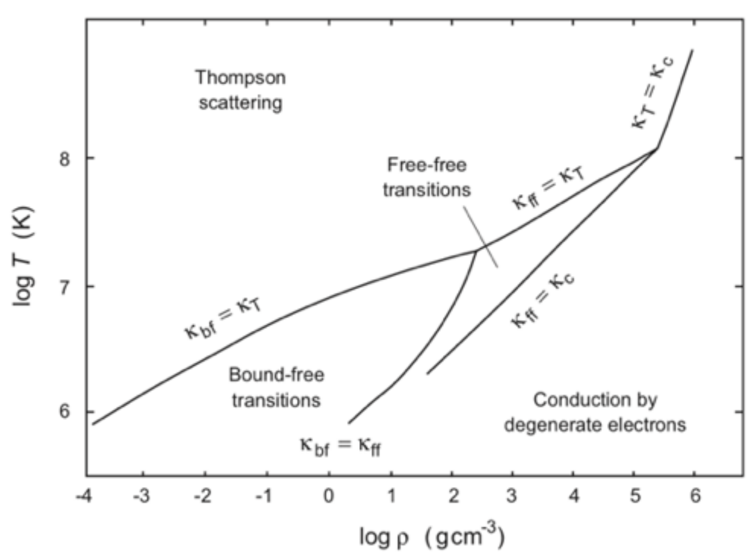
$$\kappa = \kappa_0\, \rho\, T^{-3.5}$$

which is known as Kramer's law. Comparing to equation (7.4), we see that in Kramer's law, $\lambda = 1, \nu = 3.5$.

Another contributor to the opacity is the process of **free-free absorption** (also known as inverse bremsstrahlung), in which a free electron in the vicinity of an ion absorbs energy from a photon and becomes a free electron with greater energy. Free-free absorption also follows Kramer's law.

Finally, there is the process of **bound-bound absorption**, in which electrons bound to a neutral or partially ionized atom absorb energy from a photon and are excited to a higher energy state, but one which is still bound. At one time, it was believed that bound-bound processes would not play much of a role in the interiors of stars, where even partially ionized atoms would be rare, but careful calculation of bound-bound opacities has revised opacity estimates in the solar interior by significant factors, leading to a better understanding of processes in the Sun.

A graph of temperature *vs.* density is shown below to highlight where each process dominates. At high temperatures and low densities (upper left of the graph), we see that electron scattering dominates. At high density and low temperature, conduction by degenerate electrons dominates; we will discuss this process later when we learn about white dwarfs. In between, the other processes discussed above (bound-free, free-free, and bound-bound) processes dominate.



Having discussed the opacity term, let us return to the luminosity expression in equation (7.3) on page 1. This luminosity, together with the power law expression for κ in equation (7.4) on the previous page, then gives for the surface luminosity (as you will demonstrate on the homework) that

$$L_s \simeq \frac{ac}{\kappa_0}\left(\frac{G\mu m_p}{k_B}\right)^{4+\nu} R^{3\lambda-\nu}\, M^{3+\nu-\lambda} \tag{7.5}$$

There are two distinct cases we can consider.

For relatively low mass stars (including our Sun), the opacity is dominated by atomic processes, in particular bound-free transitions, and can be approximated by the so-called Kramer's law, as we've already noted:

$$\kappa = \kappa_0\, \rho\, T^{-3.5}$$

For bound-free transitions, the constant in the expression is given by

$$\kappa_0 \simeq 4\times 10^{25}\, Z\,(1+X) \qquad \text{in cgs units}$$

On Question 4 of today's worksheet, you used equation (7.5) together with the expression for κ from Kramer's law and the constant κ_0 written above to calculate the surface luminosity of our Sun.

Comparing the expression written above for κ from Kramer's law to the power law approximation of the opacity in equation (7.4):

$$\kappa \simeq \kappa_0\, \rho^{\lambda} T^{-\nu}$$

we see that

$$\lambda = 1 \qquad \text{and} \qquad \nu = 3.5$$

Thus, equation (7.5) for the surface luminosity changes to

$$L_s \simeq \frac{ac}{(4\times 10^{25})\, Z\,(1+X)}\left(\frac{G\mu m_p}{k_B}\right)^{(4+3.5)} R^{3(1)-3.5}\, M^{3+3.5-1}$$

so that

$$L_s \simeq \frac{ac}{(4\times 10^{25})\, Z\,(1+X)}\left(\frac{G\mu m_p}{k_B}\right)^{7.5} R^{-0.5}\, M^{5.5}$$

In this expression, a is the radiation density constant from the expression for the radiation energy density $u = aT^4$ and has the value $a = 7.565\times 10^{-15}$ erg cm^{-3} K^{-4}.

Since we are working in cgs units, the mass of a proton is $m_p = 1.67\times 10^{-24}$ g, the Gravitational constant is $G = 6.67\times 10^{-8}$ dyne cm^2 g^{-2}, and Boltzmann's constant is $k_B = 1.38\times 10^{-16}$ erg K^{-1}. We will also need the radius of the Sun, $R_\odot = 6.96\times 10^{10}$ cm, and the mass of the Sun, $M_\odot = 1.99\times 10^{33}$ g.

Finally, use the hydrogen fraction $X = 0.7$, and metal fraction $Z = 0.02$, both in the expression for $kappa_0$, and to find the mean molecular weight $\mu = 4/(3+5X-Z) = 0.62$.

You found that your estimate is higher than the actual solar luminosity of $L_\odot = 3.846\times 10^{33}$ erg/s, not surprising in view of the approximations made, but remarkable in that the result is quite close to the actual value.

Meanwhile, for **high mass stars**, the temperature is higher and so the opacity is dominated by electron scattering, so that $\kappa_e = 0.2\,(1+X)$ in cgs units. The surface luminosity of high mass stars is thus given by

$$L_s \simeq 3 \times 10^{35} \left(\frac{M}{M_\odot}\right)^3 \left(\frac{\mu}{0.62}\right)^4 \frac{1.7}{1+X}\,\text{erg s}^{-1}$$

Take care you understand how to use this expression. If you want to find the surface luminosity of a 10 $M_\odot$ star, then you should substitute $M = 10\ M_\odot$, so that the parentheses essentially becomes $(10)^3$. In other words, M must be substituted in units of $M_\odot$. Likewise, μ is in units of 0.62, so any value for μ must be cast in terms of "units" of 0.62; rather confusing, but useful to compare to solar values.

On Question 5 of today's worksheet, you calculated the surface luminosity of a 20 $M_\odot$ star in erg s^{-1}, assuming that the star has the same chemical composition and hydrogen mass fraction as the Sun. Thus, $\mu = 0.62$, and $X = 0.7$, and we get

$$L_s \simeq 3 \times 10^{35} \left(\frac{20\,M_\odot}{M_\odot}\right)^3 \left(\frac{0.62}{0.62}\right)^4 \frac{1.7}{1+0.7}\,\text{erg s}^{-1}$$

so that

$$L_s \simeq 3 \times 10^{35} \left(20\right)^3 = 2.4 \times 10^{39}\ \text{ergs s}^{-1}$$

Upon expressing your answer in terms of the luminosity of our Sun, where $L_\odot = 3.846 \times 10^{33}$ erg/s, you determined that such a 20 $M_\odot$ star has $L_s = 6 \times 10^5\ L_\odot$, a little over half a million times more luminous than our Sun!