Physics 460—Practice S–4 (Due May 1, 1 pm) Name: Timothy Holmes

S–4: I can analyze three dimensional systems with spherically symmetric potentials.

Unsatisfactory Progressing Acceptable Polished

The normalized energy eigenstates for the hydrogen atom are

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi), \quad \text{with energies} \quad E_n = -\frac{e^2}{2a_0 n^2}.$$

The first few normalized radial and angular wave functions are

$$R_{10}(r) = \frac{2}{\sqrt{a_0}^3}e^{-r/a_0}, \quad R_{20}(r) = \frac{2}{\sqrt{(2a_0)^3}}\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}, \quad R_{21}(r) = \frac{1}{\sqrt{3(2a_0)^3}}\frac{r}{a_0}e^{-r/2a_0},$$

and

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_1^{\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}.$$

(1) A hydrogen atom is prepared in the state

$$\psi(r,\theta,\phi) = A\Big[3\psi_{210}(r,\theta,\phi) - \psi_{211}(r,\theta,\phi)\Big].$$

(a) Find A and explain why you don't have to evaluate any integrals to do so.

(b) If you measured the energy of the electron, what values could you obtain and with what probabilities?

(c) If you measured the total orbital angular momentum and z-component of the orbital angular momentum of the electron, what values could you obtain and with what probabilities?

(d) If you made many measurements of the distance of the electron from the nucleus, what would be the average value of these measurements?

Practice Assessment 4

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$$\psi_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)Y_\ell^m(\theta,\varphi), \text{ with energies } E_n = -\frac{e^2}{2a_0 n^2}$$

The first few normalized radial and angular wave functions are

$$R_{10}(r) = \frac{2}{\sqrt{a_0^3}} e^{-r/a_0}, \quad R_{20}(r) = \frac{2}{\sqrt{(2a_0)^3}}\left(1 - \frac{r}{2a_0}\right)e^{-r/2a_0}$$

$$R_{21}(r) = \frac{1}{\sqrt{3(2a_0)^3}}\frac{r}{a_0}e^{-r/a_0}$$

and

$$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}, \quad Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta,$$

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(1) A hydrogen atom is prepared in the state

$$\psi(r,\theta,\varphi) = A\left[3\psi_{210}(r,\theta,\varphi) - \psi_{211}(r,\theta,\varphi)\right]$$

(a) Find A and explain why you don't have to evaluate any integrals to do so.

We can write the state ψ in bracket notation such that

$$|\psi_{r,\theta,\varphi}\rangle = A\left[3|2,1,0\rangle - |2,1,1\rangle\right]$$

To normalize the state we can do the following, $A = 1/\sqrt{\langle\psi|\psi\rangle}$. So the state becomes

$$1 = \langle \psi | \psi \rangle = A^2 \left[\left(3\langle 2,1,0| - \langle 2,1,1| \right) \left(3|2,1,0\rangle - |2,1,1\rangle \right) \right]$$

$$1 = A^2 \left[9 \cancel{\langle 2,1,0|2,1,0\rangle}^{1} + 1 \cancel{\langle 2,1,1|2,1,1\rangle}^{1} \right] \Rightarrow 1 = A^2 [10]$$

$$A^2 10 = 1 \rightarrow A^2 = 1/10 \rightarrow \boxed{A = 1/\sqrt{10}}$$

$$\boxed{|\psi_{r,\theta,\varphi}\rangle = \frac{1}{\sqrt{10}} \left[3|2,1,0\rangle - |2,1,1\rangle \right]}$$

(B)

$$E_n = -\frac{13.6 eV}{n^2} \rightarrow E_2 = -\frac{13.6 eV}{2^2} = -3.4 eV$$

$$P_{E2} = \sum_{l=0}^{1} \sum_{m=1}^{1} |\langle 2,l,m|\psi\rangle|^2 = |\langle 2,1,0|\psi\rangle|^2 + |\langle 2,1,1|\psi\rangle|^2$$

$$= \left| \frac{3}{\sqrt{10}} \right|^2 + \left| \frac{1}{\sqrt{10}} \right|^2 = 1$$

(C) $|L| = \sqrt{1(1+1)}\, \hbar = \sqrt{2}\, \hbar$

$$P_{\sqrt{2}\hbar} = \sum |\langle n,1,m|\psi\rangle|^2 = |\langle 2,1,0|\psi\rangle|^2 + |\langle 2,1,1|\psi\rangle|^2$$

$$= \left| \frac{3}{\sqrt{10}} \right|^2 + \left| \frac{-1}{\sqrt{10}} \right|^2 = 1$$

$L_z = 0 \quad P_0 = |\langle 2,1,0|\psi\rangle|^2 = \left| \frac{3}{\sqrt{10}} \right|^2 = 9/10$

$L_z = \hbar \quad P_\hbar = |\langle 2,1,1|\psi\rangle|^2 = \left| \frac{1}{\sqrt{10}} \right|^2 = 1/10$

(D) $P(r,\theta,\varphi) = |\psi(r,\theta,\varphi)|^2$

$$= \frac{1}{\sqrt{\,}} \left[3\psi^*_{210}(r,\theta,\varphi) - \psi^*_{211}(r,\theta,\varphi) \right] \frac{1}{\sqrt{\,}} \left[3\psi_{210}(r,\theta,\varphi) \right.$$

$$- \psi_{211}(r,\theta,\varphi) \Big]$$

$$= \frac{1}{10} R_{21}(r)^* R_{21}(r) \left[3Y_1^{0*}(\theta,\varphi) - Y_1^{1*}(\theta,\varphi)\right]\left[3Y_1^0(\theta,\varphi - Y_1^1(\theta,\varphi)\right]$$

$$= \frac{1}{10}\left[|R_{21}(r)|^2\right]\left[9|Y_1^0(\theta,\varphi)|^2 - Y_1^{0*}(\theta,\varphi)Y_1^1(\theta,\varphi) - 3Y_1^{1*}(\theta,\varphi)Y_1^0(\theta,\varphi) + |Y_1^1(\theta,\varphi)|^2\right]$$

$$\langle r\rangle = \int_0^\infty \frac{1}{10} r^3 |R_{21}|^2 dr \int_0^{2\pi}\int_0^\pi \left[9|Y_1^0|^2 - \cancel{3Y_1^{0*}Y_1^1} - \cancel{3Y_1^{1*}Y_1^0} + |Y_1^1|^2\right] \sin\theta \, d\theta \, d\varphi$$

$$\langle r\rangle = \int_0^\infty r^3 |R_{21}(r)|^2 dr$$

$$= \int_0^\infty r^3 \left|\frac{1}{\sqrt{3(2a_0)^3}} \frac{r}{a_0} e^{-r/2a_0}\right|^2 dr = \int_0^\infty \frac{r^5}{24a_0^5} e^{-r/a_0}$$

Wolfram

$$\boxed{\rightarrow 5a_0}$$