PHY 474/374 **Spring 2021**

Week 2—Monday, April 5—Discussion Worksheet

The Physics of Stars

Although the physics of stellar interiors can be very complex, there is one simplifying assumption that enables us to learn about it. This is the assumption of ***thermodynamic equilibrium***.

1. The existence of thermodynamic equilibrium depends on the temperatures in the following distributions being the same. Let us figure out what they are.

$$\text{(I)} \qquad f(v) = 4\pi \left(\frac{m}{2\pi k T_{\mathrm{I}}}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT_{\mathrm{I}}}\right) \qquad (3.31)$$

$$\text{(II)} \qquad \frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left(-\frac{E_2 - E_1}{kT_{\mathrm{II}}}\right)$$

$$\text{(III)} \qquad B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT_{\mathrm{III}}) - 1}$$

There is also a fourth (Saha ionization formula), but we will work with the three written above.

(47) (a) What is the name of the distribution in (I) above? What does $f(v)\,dv$ tell you? What is the name given to T_{I}?

- Maxwell distribution
- Probability of finding the particle in the range of the speed between $v + dv$.
- T_1 = kinetic Temperature ($T_I \to T_{KE}$)

(b) What is the name of the expression in (II) above? What does it tell you? What is the name given to T_{II}?

($T_{II} \to T_{Ex}$)

- Boltzmann equation
- Ratio of the number of atoms in the n_1 level with the number of atoms in all levels (T_{II}) excitation Temperature
- (g_i) Statistical weight, (E_i) Energy levels, (k) Boltzmann constant

(c) What is the name of the expression in (III) above? What does it tell you? What is the name given to T_{III}?

- Planck's law of black-body radiation ($T_{III} \to T$)
- $B_\nu(T)$ is the spectral radiance density of freq ν, per unit frequency at thermal equilibrium at temp (T_{III}) temperature
- (h) Planck's constant, (c) speed of light, (k) Boltzmanns constants, (ν) freq. of EM radiation, (T) Radiation Temp

2. The starting point in learning about the physics of stellar interiors is the ***Ideal Gas Law***.

(a) Starting from the Ideal Gas Law, $PV = NkT$, where N is the number of particles, show that

$$P = \frac{\rho k T}{\mu m_p}$$

where ρ is the (mass) density, and μ is the (dimensionless) atomic mass. Note that I'll be writing k for the Boltzmann constant instead of k_B (in *Dalsgaard*). I'll also use the proton mass m_p for m_u; since 1 m_u (or amu) = 1/12 mass of ^{12}C, it is equal to m_p for all practical purposes.

$\rho = (\text{mass})/(\text{Volume}) \rightarrow V = m/\rho$

$$N = \frac{m}{\mu m_u}$$

$$P = \frac{NkT}{V} \rightarrow \frac{\cancel{m}}{\mu m_u} \cdot \frac{\rho}{\cancel{m}} kT \rightarrow P = \frac{\rho k T}{\mu m_u}$$

(b) For a monatomic ideal gas (what your text means by no internal degrees of freedom) for which the mean internal energy per particle is $\frac{3}{2}kT$, show that the *internal energy per unit volume* is

$$u = \frac{3}{2} P$$

Internal energy Per particle: $3/2\, kT$

Multiply by amount of substance

(number of moles)

$\frac{3}{2} \underbrace{kT(n)}_{P}$, when $P = nkT$

$$u = \frac{3}{2} P$$

3. The physics of stellar interiors is governed by changes in gas properties, and is described by the ***first law of thermodynamics***:

$$dQ = dU + PdV$$

which says that the added energy dQ (as heat) goes partly into changing the internal energy U of the gas, and partly into work PdV to change the volume of the gas.

(a) If we define quantities per unit mass, so that $U = u/\rho$, show that

$$c_V = \frac{3}{2}\frac{k}{\mu m_p}$$

where c_V is the ***specific heat at constant volume***, the amount of heat that has to be added per unit mass to raise the temperature of the gas by one degree.

$$U = \frac{u}{\rho} \rightarrow \frac{(3/2)P}{\rho} \rightarrow \frac{3}{2\cancel{\rho}}\left(\frac{\cancel{\rho} kT}{\mu m_u}\right)$$

$$dQ = \frac{3}{2}\left(\frac{k}{\mu m_u}\right)dT + 0$$

$$C_v = \frac{3}{2}\frac{k}{\mu m_u}$$

(b) For an adiabatic process, one that occurs without any transfer of heat so that $dQ = 0$, show that

$$\frac{dP}{P} = \gamma\frac{d\rho}{\rho}$$

where $\gamma = c_P/c_V$ is the ratio of specific heats, and since we have defined quantities per unit mass, $V = 1/\rho$, where ρ is the (mass) density.

$$C_p = \frac{5}{2}\frac{k_B}{\mu m_u}$$

$dQ = dU + PdV$ IF $dQ = 0$; $dU = -PdV$

$$\rightarrow C_v\, dT = -PdV$$

$$\rightarrow C_v\left(\frac{dP}{P} + \frac{dV}{V}\right) \rightarrow -\frac{k}{\mu m_u}\frac{dV}{V} = (C_v - C_p)\frac{dV}{V}$$

$$\rightarrow \frac{dP}{P} = -\frac{C_p}{C_v}\frac{dV}{V} = \frac{dP}{P} = \gamma\frac{d\rho}{\rho}$$

In practice, stellar matter is comprised of a mixture of different elements, the atoms of which are largely ionized. The total pressure in the gas is then

$$P = \sum_i P_i = \sum_i n_i kT \quad (3.22)$$

where n_i are the number densities of each type of particle.

4. Consider a mixture of atoms of different elements, all of which are assumed to be fully ionized. Denote atomic number of element j by Z_j, its atomic mass byA_j, and its mass fraction by X_j. When fully ionized, each atom contributes $(Z_j + 1)$ particles (Z_j electrons and one nucleus).

(a) Show that P written above can then be put in the form in Question 2(a), with the ***mean molecular weight*** μ given by

$$\mu^{-1} = \sum_j X_j \frac{Z_j + 1}{A_j} \quad (3.25)$$

Number of atoms per unit volume: $\rho X_j/(A_j m_u)$

Total # of atoms per unit volume: $\rho X_j (Z_j+1)/(A_j m_u)$

$$P = \sum_j \rho X_j \frac{Z_j+1}{A_j m_u} kT \rightarrow \frac{\rho k T}{\mu m_u}$$

$$\rightarrow \mu^{-1} = \sum_j X_j \frac{Z_j+1}{A_j}$$

(b) If we denote the mass fractions of H and He by X and Y respectively, and the mass fraction of the remaining (so-called) heavy elements by Z, so that $X + Y + Z = 1$, and $Z \ll X, Y$ in all cases, and take $A_1 = 1$ for hydrogen, $A_2 = 4$ for He, and approximate $(Z_j + 1)/A_j$ by $\frac{1}{2}$for the heavy elements, then show that

$$\mu = \frac{4}{3 + 5X - Z} \quad (3.27)$$

1/2 for heavy elements → Z heavy element → 1/2 Z

$X = X$, Z

H: $X \frac{(1+1)}{1} \rightarrow 2X$

A_1

$X = y$, $Z = 2$

He: $y \frac{(1+2)}{4} \rightarrow \frac{3}{4} y$

A_2

$$\mu^{-1} = 2X + \frac{3}{4} Y + \frac{1}{2} Z$$