In-class exam October 8

Hamill chapter 1

- Generalized coordinates, coordinate transformations
- Degrees of freedom, constraints, holonomic constraints
- Principle of virtual work
- Deriving equations of motion
- Cyclic coordinates and constants of motion

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Hamill chapter 2

- Functionals
- Euler-Lagrange equation
- Minimizing the integral of a functional
- Lagrange multipliers (probably not on the first exam)

Line element ds in different coordinate systems

In Cartesian coordinates: $ds^2 = dx^2 + dy^2 + dz^2$

In figure 2.2 and solution 2.1 the author uses the differential arc length on the surface of a sphere

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta \, d\phi^2$$

to derive the functional for a geodesic. Where does this expression for ds^2 come from?

Derive the expression for the differential element of path in cylindrical and spherical coordinates!

Differential line element

$$ds^2 = dx^2 + dy^2 + dz^2$$

spherical coordinates:

$x(r, \phi, \theta) = r \cos\phi \sin\theta$

$y = \cdots \quad z = \cdots$

$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \phi} d\phi + \frac{\partial x}{\partial \theta} d\theta$

($dr = 0$ for surface of a sphere)

Cylindrical coordinates:

$x(r, \theta) = r \cos\theta \quad \mathrm{y} = \cdots \quad z = \cdots$

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Example: Shortest curve between two points in a plane the calculus of variations

Use Cartesian coordinates (b/c the points are in a plane):

$$ds = \sqrt{dx^2 + dy^2}$$

Length of the curve we want to minimize:

$$I = \int_i^f ds = \int_i^f \sqrt{dx^2 + dy^2} = \int_i^f \sqrt{1 + \frac{dy^2}{dx^2}}\, dx = \int_i^f \sqrt{1 + y'^2}\, dx$$

$$\Phi(y) = \sqrt{1 + y'^2}$$

Is called a functional. It's a function of a function. Our goal is to find a function $y = y(x)$ so that the integral (the length of the curve) is minimized.

The calculus of variations tells us that for the integral to be minimized, the functional must satisfy the Euler-Lagrange equation:

$$\frac{\partial\Phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial\Phi}{\partial y'}\right) = 0$$

Independent variable x

Function of the independent variable x

Short side track: There are both partial and total derivatives in the Euler-Lagrange equation because of the integration by parts on page 50. What is the difference between partial and total derivatives?

Example: $f(x(t), y(t), t) = x(t)^2 + y(t)^2 - c^2 t^2$

Partial derivative $\partial f / \partial t$: We probe how f changes if we vary t but keep $x(t)$ and $y(t)$ constant:

$$\frac{\partial f}{\partial t} = -2c^2 t$$

Total derivative: The rate of change of f without holding $x(t)$ and $y(t)$ constant. f changes not just because t changes, but also because x and y change with time:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = -2c^2 t + 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Back to calculating the shortest distance between two points in a plane:

$$\frac{\partial \Phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial y'}\right) = 0$$

Plugging in the functional $\Phi(y) = \sqrt{1 + y'^2}$:

$$\frac{d}{dx}\left(\frac{\partial\left(\sqrt{1 + y'^2}\right)}{\partial y'}\right) = 0$$

because y does not appear in the functional, so $\frac{\partial \Phi}{\partial y} = 0$. This means the partial derivative with respect to y' is constant:

$$\frac{\partial\left(\sqrt{1 + y'^2}\right)}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}} = c$$

$$\frac{y'^2}{1+y'^2} = c^2$$

$$y'^2 = c^2 + c^2 y'^2$$

$$(1-c^2)y'^2 = c^2$$

$$y' = \sqrt{\frac{c^2}{1-c^2}} = \alpha$$

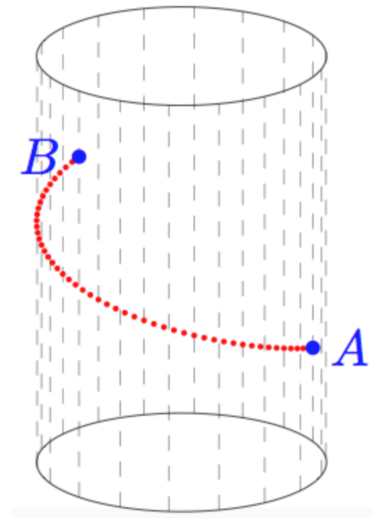
Integrate:

$$y = \alpha x + \beta$$

A straight line!

Now try this yourself: What's the shortest distance between two points on the surface of a cyclinder?

Activity 8:
Geodesic on a cylinder



Show that the formula for the shortest path between two points on a cylinder of radius R is $\theta(z) = \alpha z + \beta$.

1. Derive the differential line element $ds(\theta, z)$ in cylindrical coordinates for $R = const$.
2. Derive the functional $\Phi(\theta, \theta', z)$
3. Use the Euler-Lagrange equation $\frac{\partial \Phi}{\partial \theta} - \frac{d}{dz}\frac{\partial \Phi}{\partial \theta'} = 0$ to drive an expression for $\theta'$and integrate this expression to show that $\theta(z) = \alpha z + \beta$