PHY 474 Spring 2021

Homework 2—due by 5:00 PM, Friday, Apr 16

You may write by hand and scan as a single PDF, or write in latex (using the template file provided) or Word, and generate PDF. Please submit one PDF file only. Only questions and sub-parts that are numbered clearly, with numbers corresponding to those in this document, will be graded. See the syllabus for more detailed rules.

Submit late homework into the late D2L dropbox for reduced credit (see syllabus); late submissions will be accepted until 8 AM on Monday (Apr 19). Emailed or paper copies of homework are never accepted; in particular, do not attach homework to email to make an end-run around the D2L deadline or late deadline; such emails are automatically deleted and do not count as submissions.

1. The distribution of speeds in a gas is given by the Maxwell distribution

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right)$$

(a) The most probable speed, v_p, of this distribution can be found by doing

$$\frac{df(v)}{dv} = 0$$

By doing this explicit differentiation, show that

$$v_p = \sqrt{\frac{2kT}{m}}$$

(b) The average speed, v_{avg}, of this distribution can be found by doing $\int v\, f(v)\, dv$.

By doing this integral, show that

$$v_{\text{avg}} = \sqrt{\frac{8kT}{\pi m}}$$

Note: You may need the standard integral

$$\int_0^\infty x^3 \exp(-ax^2)\, dx = \frac{1}{2a^2}$$

(c) Explain in words why v_p and v_{avg} are different for a Maxwell distribution.

Question 2 begins on the next page.

2. *Dalsgaard* mentions that a simple solution to the equation of hydrostatic equilibrium can be obtained when ρ is a known function of r. Consider a linear density model

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right)$$

where ρ_c is the central density, and R is the radius of the star.

(a) By substituting this expression for ρ into equation (4.5) for dm/dr in *Dalsgaard*, find the total mass M of the star and hence show that the central density is given by

$$\rho_c = \frac{3M}{\pi R^3}$$

(b) Show that the mass interior to radius r is given by

$$m = M\left(4x^3 - 3x^4\right)$$

where M is the total mass of the star, and $x = r/R$.

3. Consider again the linear density model in Question 2 above.

(a) Assuming $P = 0$ at the surface $r = R$, show that the pressure is given by

$$P = \frac{5}{4\pi}\frac{GM^2}{R^4}\left(1 - \frac{24}{5}x^2 + \frac{28}{5}x^3 - \frac{9}{5}x^4\right)$$

where, again, $x = r/R$.

(b) Plot P *vs.* x. Plot with P in units of $(5/4\pi)\, GM^2/R^4$ for convenience.

You may use Matlab or Python (or equivalent), but *you will get zero points if you use an online calculator like Desmos.* Please submit your program if you want full credit.

4. The Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n$$

where symbols are explained in *Dalsgaard* (and also the posted class summary), is derived from a polytropic relation, $P = K\rho^\gamma$, where the polytropic index $n = 1/(\gamma - 1)$.

(a) Show that the pressure in such a polytropic model is given by

$$P = P_c\,\theta^{n+1}$$

where P_c is the central pressure.

(b) The Lane-Emden equation has analytical solutions only for $n = 0, 1, 5$. Although the $n = 0$ solution is technically a singularity, it is useful to illustrate properties of polytropes. Show that the solution for $n = 0$ is

$$\theta = 1 - \frac{\xi^2}{6} \qquad \text{where} \qquad \xi_1 = \sqrt{6}$$

Recall that the surface is defined by the point $\xi = \xi_1$ where $\theta = 0$ (reflecting the fact that the pressure P is zero at the surface of the star).