

## Class Summary—Week 1, Day 2—Wednesday, March 31

## Spectral Analysis

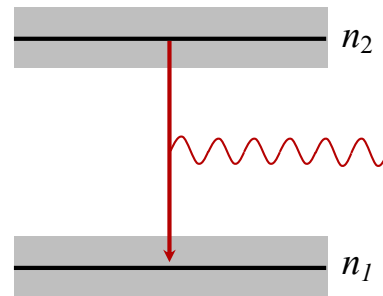
A reminder that class summaries for this course will be very concise because we are using a text, and will usually refer you to the relevant section of the text, unless I feel a need to write more detailed notes on a particular topic than what the text has in it.

In the previous class, we learned about the brightness and the magnitude scale (among other things), but we need a way to get more information about stars. After all, stars are subject to significant amounts of **interstellar absorption**; see **Section 2.4** in *Dalgaard* (page 20). Thus begins our journey into the world of stellar spectra.

There are three types of spectra (see the PowerPoint slides in the Video Discussion): **Continuum, Emission Line, and Absorption Line Spectra**. Of these, stellar atmospheres produce primarily absorption line spectra and some emission line spectra (again, see the PowerPoint slides for examples of the solar spectrum, and the spectrum of a Sun-like star).

You are surely aware that spectral lines are formed when an atom or electron or molecule, as the case may be, jumps from an upper level to a lower level, as shown in the figure on the right. The frequency of this transition is given by

$$\nu = \frac{E_2 - E_1}{h}$$



If each energy level were a sharply defined value, then each spectral line would be a  $\delta$ -function spike at a particular frequency. That, however, is not the case. Even if we were to disregard all motion and any interaction, there is still Heisenberg's Uncertainty Principle, which states that the uncertainty in energy multiplied by the uncertainty in time is of the order of  $\hbar$  (Planck's constant  $h$  divided by  $2\pi$ ). Since we cannot know how long the atom or electron will remain in its upper state, but only assign a probability (usually expressed by the so-called Einstein  $A$ -coefficient for spontaneous emission), we will have a spread in the energy of each level (which I've depicted using the light gray rectangles in the figure above). This leads to a broadening of the line, known as **natural broadening**.

Consider again a two-state system, with 2 being the upper level and 1 being the lower level. Since the Einstein  $A$ -coefficient,  $A_{21}$ , gives the probability that the system will transition out of its upper state in a certain period of time,  $1/A_{21}$  is the time the system will spend in the upper state. Thus,  $1/A_{21}$  is also the uncertainty in the transition time,  $\Delta t$ , since the system will transition at some point in this time. *On Question 1(a) of today's worksheet*, you used this to show that the spread in wavelength due to natural broadening,  $\Delta\lambda$ , is given by

$$\Delta\lambda = \frac{\lambda^2}{2\pi c} \frac{1}{\Delta t}$$

where  $c$  is the speed of light.

In computing  $\Delta\lambda$  written on the previous page, one should consider both  $\Delta t_u$  for the upper level and  $\Delta t_l$  for the lower level, in which case  $\Delta\lambda$  is just a sum of the two:

$$\Delta\lambda = \frac{\lambda^2}{2\pi c} \left[ \frac{1}{\Delta t_u} + \frac{1}{\Delta t_l} \right]$$

On Question 1(b) of today's worksheet, however, you were able to get by with only  $\Delta t_u$  because the Lyman- $\alpha$  transition is to the ground state, for which  $\Delta t_l = \infty$ . You found that  $\Delta\lambda$  is tiny, and in practice, rarely ever measurable.

In general, an atom that is in an excited state  $n$  can drop down to any lower state  $n'$ . If the spontaneous decay of an atomic state  $n$  (to all lower energy levels  $n'$ ) proceeds at the rate

$$\gamma_n = \sum_{n'} A_{nn'}$$

then the probability of an emitted photon having a frequency  $\nu$  is given by the distribution function

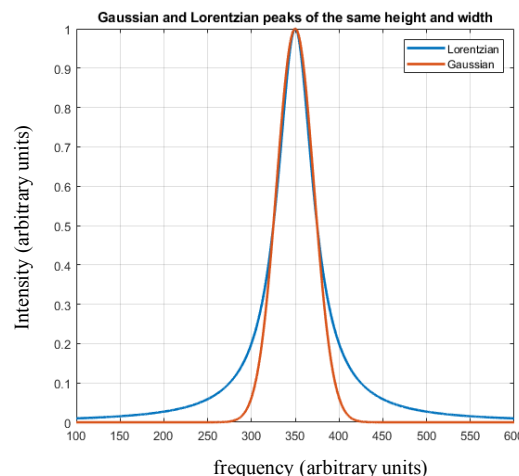
$$\phi(\nu) d\nu = \frac{\gamma_n/4\pi}{(\nu - \nu_0)^2 + (\gamma_n/4\pi)^2} \frac{d\nu}{\pi}$$

where  $\nu_0$  is the center (peak) frequency of the line. This is known as a **Lorentzian profile**. An example of a Lorentzian profile is shown in the figure below (source: [umd.edu](http://umd.edu)), along with a Gaussian profile of the same width and height for comparison. Notice that the Lorentzian profile is higher in the wings than the Gaussian profile.

In Question 2(a) of today's worksheet, you showed that the maximum height of the Lorentzian profile, which occurs at  $\nu = \nu_0$ , is given by

$$\phi(\nu_0) = \frac{4}{\gamma_n}$$

Thus, the higher the value of  $\gamma_n$  (that is, the shorter the lifetime of the energy level  $n$ ), the lower the peak value. Note that  $\gamma_n$  is also called the damping factor, which has its origin in the model used to generate this profile function  $\phi(\nu)$ , which is that of a damped harmonic oscillator.



In Question 3(a) of today's worksheet, you determined the width of a Lorentzian distribution by finding an expression for  $\Delta\lambda/\lambda$  at the point where  $\phi(\nu)$  drops to half its peak value. You determined that

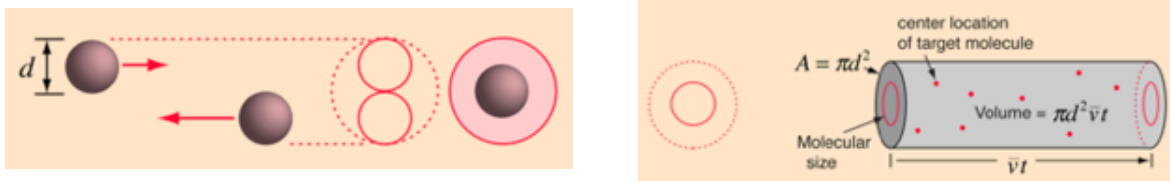
$$\frac{\Delta\lambda}{\lambda} = \frac{\gamma_n}{4\pi\nu} = \frac{\gamma_n \lambda}{4\pi c}$$

Thus, the higher the value of  $\gamma_n$  (that is, the shorter the lifetime of the energy level  $n$ ), the broader is the Lorentzian distribution.

**Collisional or pressure broadening** results from the fact that atoms are not isolated. Instead, they interact with their neighbors. They will collide directly with some (neutral) neighbors, and positive ions will also feel the electric fields of nearby (charged) particles (e.g., electrons). Both of these effects also produce a Lorentzian profile. We can use the same expression for the broadening that we developed in Question 1(a) on today's worksheet (and written on the first page of this summary), except now  $\Delta t$  is the average time between collisions, given by

$$\Delta t = \frac{l}{v}$$

where  $l$  is the mean free path. To find an expression for the mean free path  $l$ , the average distance between collisions, consider the figures below (source: hyperphysics).



If the molecules have diameter  $d$ , then the effective cross section for collisions can be modeled by putting two molecules end-to-end as shown in the figure on the left, forming a circle of diameter  $2d$ . This circle then has radius  $d$ , so the effective collision area will be  $A = \pi d^2$ . Take care to avoid being confused here; the effective collision area is formed by putting two molecules (each of diameter  $d$ ) end to end, so the effective collision area has diameter  $2d$  (see figure on the left above), thus radius  $d$ , and its area is  $A = \pi d^2$ . We will use symbol  $\sigma$  for the collision cross section, so  $\sigma = \pi d^2$ .

If the molecule is moving at speed  $v$ , then in time  $t$ , it will have covered a distance  $vt$ . Thus, as time passes, the circle of diameter  $2d$  sweeps out the volume shown in the figure above on the right, with the volume equal to  $A(vt)$ , or  $\sigma vt$ , in our notation. If  $n$  is the particle density (the number of atoms per unit volume), then in the volume swept out by the cylinder in time  $t$ , the number of collisions will be the volume times  $n$ , that is,  $(\sigma vt) n$ .

The mean free path  $l$ , the average distance between collisions, can then be found by dividing the length of the path by the number of collisions, so that

$$l = \frac{\text{length of path}}{\text{number of collisions}} = \frac{vt}{(\sigma vt) n}$$

Thus, we obtain

$$l = \frac{1}{n\sigma}$$

where  $n$  is the particle density, and  $\sigma$  is the collisional cross section. Assuming that the gas particles are described by a Maxwell-Boltzmann distribution, we can use for  $v$  the most probable velocity given by  $\sqrt{2kT/m}$ , and so

$$\Delta t = \frac{l}{v} = \frac{1}{n\sigma} \frac{1}{\sqrt{2kT/m}}$$

Thus

$$\Delta\lambda = \frac{\lambda^2}{2\pi c} \frac{1}{\Delta t} = \frac{\lambda^2}{2\pi c} n\sigma \sqrt{\frac{2kT}{m}}$$

We see that pressure broadening is directly proportional to the number density  $n$ , implying that it will be significant in higher density regions.

**Doppler broadening** arises because the atoms in a gas move around randomly with a distribution of speeds that is described by the Maxwell-Boltzmann distribution. Along any one given direction, and thus along the line of sight, the distribution of velocities follows a Gaussian distribution with dispersion  $\sigma = \sqrt{kT/m}$ , so in the nonrelativistic case the wavelengths of light emitted by atoms in the gas are Doppler-shifted by  $\Delta\lambda/\lambda = \sigma/c$ .

On Question 5(a) in today's worksheet, you calculated  $\Delta\lambda/\lambda$  due to Doppler broadening for hydrogen atoms in the Sun's photosphere, where  $T = 5800$  K, and found it to be

$$\frac{\Delta\lambda}{\lambda} = 2.3 \times 10^{-5}$$

Thus, the Doppler broadening of the  $H\alpha$  line ( $\lambda = 656.3$  nm) in the Sun's photosphere will be 0.015 nm, as you found in Question 5(b) on today's worksheet. The  $H\alpha$  (or Balmer- $\alpha$ ) line is produced by a jump from the  $n = 3$  (second excited state) to the  $n = 2$  (first excited state) of hydrogen. Note that some texts multiply this by an additional factor of  $\sqrt{2}$  because they use the most probable value for the velocity instead. For comparison, you calculated the natural broadening of the  $H\alpha$  line in Question 5(c) on today's worksheet, and found it to be much smaller than the Doppler broadening of the  $H\alpha$  line.

Observing spectral lines allows us to classify the spectra of stars, a task carried out by a talented team of women at Harvard in the late 1800's and early 1900's hired by its director Edward Pickering, a team that included Annie Jump Cannon who eventually published a lot of these data. Based on low resolution spectra, stars were classified into several spectral classes like A, B, and so on. It was only later based on the work of Meghnad Saha that the connection to temperature was established.

Today, the major spectral classes are O, B, A, F, G, K, M, with the hottest being O stars, and M being the coolest. The connection to temperature allows us to figure why certain lines occur only in some classes but not in others. See the posted video for more details.