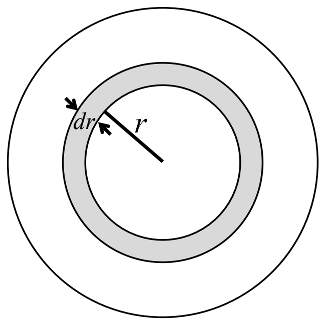
PHY 474/374 **Spring 2021**

Class Summary—Week 2, Day 2—Wednesday, April 7

Hydrostatic Equilibrium

Today, we will discuss the balance of forces inside a star, known as **hydrostatic equilibrium**. We already know from our discussion of the dynamical timescale that the forces inside a star must be perfectly balanced. *Dalsgaard* considers a spherical shell of radius r and thickness dr in the star, as shown in the figure below.



By considering that this shell is subject to the gravitational force and pressure, *Dalsgaard* obtains the equation of motion

$$\rho \frac{d^2 r}{dt^2} = -\rho \frac{Gm}{r^2} - \frac{dP}{dr} \tag{4.3}$$

where ρ is the density, and $m = m(r)$ is the mass interior to the shell. Note that equations are numbered to match those in *Dalsgaard*, for easy reference. I won't go into the details of the derivation here, since they are given in reasonably good detail in **Section 4.1** in *Dalsgaard* (**pages 53-54**).

Another useful relation, obtained by considering the mass in the shell, is

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{4.5}$$

Using equation (4.3), together with the Ideal Gas Law from the previous class, we can show (*Dalsgaard*, **pages 55-56**) that the central pressure and temperature of a star with mass M and radius R are given by

$$P_c \simeq \frac{GM^2}{R^4} \qquad \text{and} \qquad T_c \simeq \frac{\mu_c m_p}{k} \frac{GM}{R}$$

respectively, where the mean molecular weight μ_c at the center of the star can be found by using the expression from the previous class for the mean molecular weight: $\mu = 4/(3 + 5X - Z)$.

On Question 1 of today's worksheet, you calculated P_c and T_c at the center of our Sun using the relations above. You used $X = 0.35, Z = 0.02$, because nuclear fusion has reduced the hydrogen mass fraction at the center to 0.35 from its usual value of $\approx$0.7 near the surface and in interstellar clouds. These estimates were reasonably accurate. More realistic models of the Sun find that $P_c = 2.4 \times 10^{16}$ N/m^2, and $T_c = 15$ million K.

Hydrostatic equilibrium requires that the left hand side of equation (4.3) be equal to zero, so that

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2} \tag{4.4}$$

On Question 2(a) of today's worksheet, you used the expression for hydrostatic equilibrium written above, together with $dm/dr = 4\pi r^2 \rho$ written in equation (4.5), to show that

$$\frac{d}{dr}\left[P + \frac{Gm^2}{8\pi r^4}\right] = -\frac{Gm^2}{2\pi r^5} \tag{4.13}$$

This allowed you to set a lower limit on the central pressure.

- Since the quantity on the right hand side of equation (4.13) above is negative, we conclude that the quantity $\Psi(r) = P + Gm^2/8\pi r^4$ is a decreasing function of r.
- Since mass is density times volume, and volume goes as r^3, the quantity

  $$\frac{Gm^2}{8\pi r^4} \propto \frac{(r^3)^2}{r^4} = r^2$$

  which implies that at small r near the center of the star, the term $Gm^2/8\pi r^4$ will go to zero. Thus, at the center of the star where $r = 0$, we will get

  $$\Psi(0) = P_c$$

- Meanwhile, at the surface of the star, P is essentially zero so that

  $$\Psi(R) = 0 + \frac{GM^2}{8\pi R^4}$$

  since $m(r) = M$, the entire mass of the star if we take $r = R$.
- Now, we concluded from equation (4.13) that $\Psi(r)$ is a decreasing function of r, so we must have

  $$\Psi(0) > \Psi(R) = \frac{GM^2}{8\pi R^4}$$

  and since $\Psi(0) = P_c$, we obtain that the limit to the central pressure is

  $$P_c > \frac{GM^2}{8\pi R^4}$$

We can set a stronger limit on P_c if, in addition to hydrostatic equilibrium, we assume that the mean density $\rho(r)$ inside r decreases with increasing r. The procedure is described in *Dalsgaard*(**page 59**), and gives

$$P_c > \frac{3}{8\pi}\frac{GM^2}{R^4}$$

Thus, by adding the constraint that the mean density is a decreasing function of r, the lower bound on the central pressure of the star has been increased by a factor of 3.

The Virial Theorem

The **Virial Theorem** can be stated in many different ways, but essentially it is a powerful statement that describes a system in equilibrium. It has been widely applied in astronomy, from computing the velocities of dust particles in dark nebulae (Bart Bok), to investigating the period of a pulsating star (Ledoux), to finding the energy balance in star-forming clouds.

We will be using a simple version of the theorem that states that

$$\Omega + 2U = 0$$

where Ω is the gravitational potential energy and U is the (total) internal energy, also designated as the (total) thermal energy; think of U as the total kinetic energy of all the particles constituting the star (relative to its center of mass).

In its most generalized form, the theorem may be expressed as follows (Huang 1954): with the origin at the center of mass of a system of particles (molecules in an interstellar cloud or a star), let $\vec{r}_i$ be the coordinates of a particle of mass m_i, $\vec{F}_i$ the force acting on m_i, U the total kinetic energy of the motions of the particles relative to the center of gravity, Ω the gravitational potential energy of the system of particles, and I the moment of inertia of the system of particles about its center of mass. The virial theorem requires that

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2U + \sum_i (\vec{r}_i \cdot \vec{F}_i)$$

The forces acting on m_i could be due to attractions of other particles inside the system ($\vec{F}_{i,\text{int}}$) or it could be due to some agency external to the system ($\vec{F}_{i,\text{ext}}$). We can ignore the latter for stars, and since

$$\sum_i (\vec{r}_i \cdot \vec{F}_{i,\text{int}}) = -\sum_{i\neq j} \frac{Gm_im_j}{r_{ij}} = \Omega$$

the equation above becomes

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2U + \Omega$$

In a steady state, the moment of inertia does not vary with time, and we get the form of the virial theorem written above

$$\Omega + 2U = 0$$

Also of interest is that the total energy of the star is then

$$E = \Omega + U = (-2U) + U = -U$$

Thus, the total energy is negative, indicating that the star is stable.

As an example of its applicability (and power), you used the Virial Theorem to find an expression for the *average temperature* of the Sun *in Question 3 of today's worksheet.* The power of the Virial Theorem is certainly revealed by this application; we were able to figure the internal temperature of the Sun as a realistic approximation to its actual value without going into any details of its inner workings, but merely by using information about its bulk characteristics.

Solutions to Hydrostatic Equilibrium Equations

Although solutions to the equation of hydrostatic equilibrium require more material to be discussed later in the quarter, there are two exceptions, one when ρ is a known function of r (an example of which you'll do on the homework), and the other when ρ is a known function of P, a particular example of which is a relation of the form

$$P(r) = K\Big[\rho(r)\Big]^{\gamma} \tag{4.36}$$

where K and *gamma* are constants. This is called a **polytropic relation** and the resulting models are called polytropic models.

Details of the polytropic relation and polytropic models are in **Section 4.6** (*Dalsgaard*, **pages 64-69**), but I'll sketch some of the major points here.

To obtain the equation satisfied by polytropic models, you used the hydrostatic equilibrium equation (4.4) together with the expression for dm/dr in equation (4.5), and equation (4.36) written above to show that

$$K\gamma \frac{d}{dr}\left(r^2 \rho^{\gamma-2}\frac{d\rho}{dr}\right) = -4\pi G\rho r^2 \tag{4.37}$$

in Question 4 of today's worksheet.

You then introduced a dimensionless measure θ of the density, so that $\rho = \rho_c\,\theta^n$, where ρ_c is the central density, and the polytropic index $n = 1/(\gamma-1)$, so that $\gamma = 1+1/n$. With these replacements, you showed *in Question 5(a) of today's worksheet* that equation (4.37) above becomes

$$\frac{(n+1)\,K\,\rho_c^{1/n}}{4\pi G\rho_c}\,\frac{1}{r^2}\,\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n \tag{4.40}$$

In Question 5(b) of today's worksheet, you simplified equation (4.40) further by introducing a dimensionless measure ξ of the distance to the center which is given by $r = \alpha\xi$, where the quantity $\alpha^2 = (n+1)\,K\,\rho_c^{1/n}/4\pi G\rho_c$, and obtained

$$\frac{1}{\xi^2}\,\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

which is known as the **Lane-Emden equation**.

The solution to this equation, $\theta = \theta_n(\xi)$, is called the Lane-Emden function. Since $\rho = \rho_c\theta^n$, solutions must satisfy the boundary condition $\theta_n(0) = 1$ for $\xi = 0$. Analytical solutions to the Lane-Emden function exist only for $n = 0, 1, 5$; the rest require numerical solutions.