Physics 460—Quantum Rigid Rotor **Spring 2020**

The quantum rigid rotor is described by the Hamiltonian

$$H = \frac{L^2}{2I},$$

where I is the moment of inertia of the rotor and L^2 is the total angular momentum operator. Clearly, the energy eigenstates of the rotor are the eigenstates of L^2, the states $|l, m\rangle$, represented in position space by the spherical harmonics, $Y_l^m(\theta, \phi)$.

(1) At time $t = 0$ the wave function of a particular rigid rotor is $\psi(\theta, \phi, 0) = A \sin^2\theta (1 - \cos 2\phi)$.

(a) Write this state as a superposition of spherical harmonics.

(b) Find the normalization constant A.

(c) Find the state and its position-space wave function at a later time t.

(d) If you measure the total angular momentum at time t, what results could you obtain and with what probabilities?

(e) If you measure the z-component of the angular momentum at time t, what results could you obtain and with what probabilities?

(f) At time $t = 0$ you measure the orientation of the rotor. What is the probability that you will find it in the range $\theta = [0, \pi/2]$ and $\phi = [0, \pi]$?

(a)

$$\psi = A\sin^2\theta\left(1 - \cos 2\phi\right) = \underbrace{A\sin^2\theta}_{Y_2^0,\, Y_0^0} - \underbrace{A\sin^2\theta\cos 2\phi}_{Y_2^{\pm 2}}$$

Compare to spherical harmonics...

Looks like $Y_2^{\pm 2}$, Y_0^0, Y_2^0

$$\langle 0,0|\psi\rangle = A\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \times \sqrt{\frac{1}{4\pi}} \sin^2\theta\left(1 - \cos 2\phi\right)$$

$$= A\sqrt{\frac{1}{4\pi}} \int_0^{2\pi}\left(1 - \cos 2\phi\right) d\phi \int_0^{\pi} \sin^3\theta\, d\theta$$

$$= A\sqrt{\frac{1}{4\pi}}\left(2\pi\right)\left(\frac{4}{3}\right) = A\,\frac{8\pi}{3}\sqrt{\frac{1}{4\pi}}$$

$$\langle 2,0|\psi\rangle = A\sqrt{\frac{5}{16\pi}}\left(2\pi\right)\left(-\frac{8}{15}\right) = -A\,\frac{16\pi}{15}\sqrt{\frac{5}{16\pi}}$$

$$\langle 2,\pm 2|\psi\rangle = A\sqrt{\frac{15}{32\pi}}\left(-\pi\right)\left(\frac{16}{15}\right) = -A\,\frac{16\pi}{15}\sqrt{\frac{15}{32\pi}}$$

$$|\psi\rangle = A\pi\sqrt{\frac{1}{4\pi}}\left(\frac{40}{15}|0,0\rangle - \frac{16}{15}\sqrt{\frac{5}{4}}\,|2,0\rangle - \frac{16}{15}\sqrt{\frac{15}{8}}\,|2,2\rangle - \frac{16}{15}\sqrt{\frac{15}{8}}\,|2,-2\rangle\right)$$

$$|\psi\rangle = \underbrace{A\pi\sqrt{\frac{1}{4\pi}}\,\frac{8}{15}}_{C}\left(5\,|0,0\rangle - 2\sqrt{\frac{5}{4}}\,|2,0\rangle - 2\sqrt{\frac{15}{8}}\,|2,2\rangle - 2\sqrt{\frac{15}{8}}\,|2,-2\rangle\right)$$

$$|\psi\rangle = C\left(5\,|0,0\rangle - \sqrt{5}\,|2,0\rangle - \sqrt{\frac{15}{2}}\,|2,2\rangle - \sqrt{\frac{15}{2}}\,|2,-2\rangle\right)$$

$$c^2\left(25 + 5 + \frac{15}{2} + \frac{15}{2}\right) = 1 = c^2\left(60\right)$$

$$c = \frac{1}{\sqrt{60}}$$

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{60}}\left(5\,|0,0\rangle - \sqrt{5}\,|2,0\rangle - \sqrt{\frac{15}{2}}\,|2,2\rangle - \sqrt{\frac{15}{2}}\,|2,-2\rangle\right)}$$

(c) $H = \dfrac{L^2}{2I}$

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi\rangle$$

$$L^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle$$

$$L^2|0,0\rangle = 0, \quad L^2|2,m\rangle = 6\hbar^2|l,m\rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{60}}\Big(5|0,0\rangle - \sqrt{5}\,e^{-6\hbar t/2I}|2,0\rangle - \sqrt{\tfrac{15}{2}}\,e^{-6\hbar t/2I}|2,2\rangle - \sqrt{\tfrac{15}{2}}\,e^{-6\hbar t/2I}|2,-2\rangle\Big)$$

(d) measure $L^2 = 0$ with probability

$$\frac{25}{60} = \frac{5}{12}$$

measure $L^2 = 6\hbar^2$ with probability

$$\frac{35}{60} = \frac{7}{12}$$

(e) measure $L_z = +2\hbar$ with probability

$$\frac{15}{120} = \frac{5}{40} = \frac{1}{8}$$

same fo $L_z = -2\hbar$

measure $L_z = 0$ with probability

$\frac{3}{4}$

(f)

$$P(\theta < \pi/2,\ 0 < \phi < \pi) = \int_0^{\pi} d\phi \int_0^{\pi/2} \sin\theta \, d\theta \, |A|^2 \left(\sin^2\theta(1-\cos 2\phi)\right)^2$$

we need A to continue...

$$C = \frac{1}{\sqrt{60}} = A\pi \sqrt{\frac{1}{4\pi}} \frac{8}{15} \Rightarrow A = \frac{1}{\sqrt{60}} \frac{15}{8\pi} \sqrt{4\pi}$$

$$A = \frac{\sqrt{15}}{8\sqrt{\pi}}$$

Using WolframAlpha,

$$\int_0^{\pi} d\phi \int_0^{\pi/2} \sin\theta \, d\theta \left(\sin^2\theta(1-\cos 2\phi)\right)^2$$

$$= \int_0^{\pi} d\phi \, (1-\cos 2\phi)^2 \int_0^{\pi/2} \sin^5\theta \, d\theta = \left(\frac{3\pi}{2}\right)\left(\frac{8}{15}\right)$$

$$\therefore \ P = \frac{\cancel{15}}{64\cancel{\pi}}\left(\frac{3\cancel{\pi}}{2}\right)\left(\frac{8}{\cancel{15}}\right) = \frac{3}{16} \qquad \text{whew!}$$