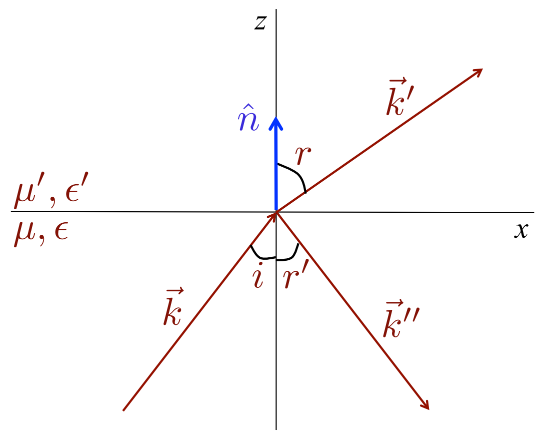
PHY 411 **Winter 2021**

Week 3—Tuesday, Jan 19—Discussion Worksheet

Reflection and Refraction of Electromagnetic Waves

Having learned the fundamentals of propagating plane waves obtained from the Maxwell equations, we will now study some well known phenomena in optics.

Consider a plane wave that is incident on the plane surface $z = 0$. The media below the plane $z = 0$ has permeability μ and permittivity ϵ, whereas that above this plane has permeability μ' and permittivity ϵ'. The geometry of the problem is shown in the figure on the right. Note that the unit normal $\hat{n}$ points from medium (μ, ϵ) toward medium (μ', ϵ') as shown in the figure. Take care you remember that **this is different from what we've been doing so far**, in that $\hat{n}$ was along the direction of propagation up to now, and that is no longer the case.



1. The (electric and magnetic fields of the) incident wave are given below. Write down the fields for the refracted and reflected waves (using the same format), then prove explicitly that $k = k''$, where k is the magnitude of the wave vector $\vec{k}$ of the incident wave, and k'' is the magnitude of the wave vector $\vec{k}''$ of the reflected wave.

Incident Wave	**Refracted Wave**	**Reflected Wave**
$\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$	$\vec{E}' = E_0'\, e^{ik\cdot x - i\omega t}$	$\vec{E}'' = E_0'' e^{ik\cdot x - \omega t}$
$\vec{B} = \sqrt{\mu\epsilon}\,\dfrac{\vec{k}\times\vec{E}}{k}$	$\vec{B}' = \sqrt{\mu'\epsilon'}\,\dfrac{k'\times E'}{k'}$	$\vec{B}'' = \sqrt{\mu\epsilon}\,\dfrac{k''\times E''}{k}$

With $k = k''$, we get equation (7.33) in Jackson:

$$\left|\vec{k}\right| = \left|\vec{k}''\right| = k = \omega\sqrt{\mu\epsilon}$$

$$\left|\vec{k}'\right| = k' = \omega\sqrt{\mu'\epsilon'} \qquad (7.33)$$

Kinematic Properties

2. Because the boundary conditions must be satisfied at all points on the plane $z = 0$ at all times, the spatial (and time) variation of all the fields that you wrote in Question 1 must be the same at $z = 0$. This means that the phase factors of all the three waves must be equal at $z = 0$, and so

$$(\vec{k} \cdot \vec{x})_{z=0} = (\vec{k}' \cdot \vec{x})_{z=0} = (\vec{k}'' \cdot \vec{x})_{z=0} \tag{7.34}$$

$n \Rightarrow$ Index of Refraction

Note:

(a) Starting from equation (7.34) above, derive that

$$k \sin i = k' \sin r = k'' \sin r' \tag{7.35}$$

$\vec{x}$ in Jackson is the position vector also known as $\vec{r}$

$$(\vec{k} \cdot \vec{x})_{z=0} = \left[k\hat{k} \cdot (x\hat{x} + y\hat{y} + z\hat{z})\right]_{z=0}$$

$$= k\left[\hat{k} \cdot (x\hat{x}) + \hat{k} \cdot (y\hat{y}) + \hat{k} \cdot (z\hat{z})\right]_{z=0}$$

$$= k\left[x|\hat{k}||\hat{x}|\cos(90-i) + y|\hat{k}||\hat{x}|\cancel{\cos 90^\circ}^{\,0} + z|\hat{k}||\hat{z}|\cancel{\cos(i)}^{\,0}\right]$$

$$= kx \sin i$$

(b) How does equation (7.35) prove the law of reflection that $i = r'$, i.e., the angle of incidence (i) is equal to the angle of reflection (r')?

$k = k'$

$k \sin i = k'' \sin r' \Rightarrow \sin i = \sin r'$

$i = r'$

(c) Starting from equation (7.35) written above, derive Snell's law for refraction: $n \sin i = n' \sin r$.

Note: From equation (7.5), we have the index of refraction, $n = \sqrt{\dfrac{\mu\epsilon}{\mu_0\epsilon_0}}$.

$$\frac{\sin i}{\sin r} = \frac{k'}{k} = \frac{\omega\sqrt{\mu'\epsilon'}}{\omega\sqrt{\mu\epsilon}} = \frac{\sqrt{\mu'\epsilon'/\mu_0\epsilon_0}}{\sqrt{\mu\epsilon/\mu_0\epsilon_0}} = \frac{n'}{n}$$

thus,

$$n \sin i = n' \sin r$$

Dynamic Properties

To discuss dynamic properties, we will begin from the boundary conditions. If the fields in medium 1 are $\vec{E}_1, \vec{D}_1, \vec{B}_1, \vec{H}_1$ and those in medium 2 are $\vec{E}_2, \vec{D}_2, \vec{B}_2, \vec{H}_2$, then we have the boundary conditions.

$$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = \sigma \qquad \text{(I.17)} \qquad\qquad \hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \qquad \text{(I.19)}$$

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0 \qquad \text{(I.18)} \qquad\qquad \hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K} \qquad \text{(I.20)}$$

where σ is the surface charge density, $\vec{K}$ is the surface current density, and $\hat{n}$ is a unit vector pointing from medium 1 to medium 2. In studying reflection and refraction, of course, there is no surface charge or surface current at the interface.

Our objective is to obtain the following four equations:

$\vec{D}_1 = \vec{D}_0 + \vec{D}_0''$

$$\left[\epsilon\left(\vec{E}_0 + \vec{E}_0''\right) - \epsilon' \vec{E}_0'\right] \cdot \hat{n} = 0 \qquad \text{(7.37.a)}$$

$$\left[\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0'' - \vec{k}' \times \vec{E}_0'\right] \cdot \hat{n} = 0 \qquad \text{(7.37.b)}$$

$$\left[\vec{E}_0 + \vec{E}_0'' - \vec{E}_0'\right] \times \hat{n} = 0 \qquad \text{(7.37.c)}$$

$$\left[\frac{1}{\mu}\left(\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0''\right) - \frac{1}{\mu'}\left(\vec{k}' \times \vec{E}_0'\right)\right] \times \hat{n} = 0 \qquad \text{(7.37.d)}$$

3. Deriving equation (7.37.a) starting from equation (I.17) is easy, and was demonstrated in the mini lecture. Starting from equation (I.18) above, $(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0$, derive the second of the four equations above, that is, **derive equation (7.37.b)**.

$$\left[\left(\vec{D}_0 + \vec{D}_0''\right) - \vec{D}_0'\right] \cdot \hat{n} = 0 \rightarrow \left[\left(\epsilon \vec{E}_0 + \epsilon \vec{E}_0''\right) - \epsilon' \vec{E}_0'\right] \cdot \hat{n} = 0$$

$\vec{D}_0' = \epsilon' \vec{E}_0'$

$\vec{D}_0'' = \epsilon \vec{E}_0''$

$\vec{D}_0 = \epsilon \vec{E}_0$

$$\rightarrow \left[\epsilon\left(\vec{E}_0 + \vec{E}_0''\right) - \epsilon' \vec{E}_0'\right] \cdot \hat{n} = 0$$

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0 \Rightarrow \left[\vec{B}_0' - \left(\vec{B}_0 + \vec{B}_0''\right)\right] \cdot \hat{n} = 0$$

$$\left[\left(\vec{B}_0 + \vec{B}_0''\right) - \vec{B}_0'\right] \cdot \hat{n} = 0$$

$$\Rightarrow \left[\left(\sqrt{\mu\epsilon}\,\frac{\vec{k} \times \vec{E}_0}{k} + \sqrt{\mu\epsilon}\,\frac{\vec{k}'' \times \vec{E}_0''}{k''}\right) - \sqrt{\mu'\epsilon'}\,\frac{\vec{k}' \times \vec{E}_0'}{k'}\right] \cdot \hat{n} = 0$$

$$\Rightarrow \left[\left(\cancel{\sqrt{\mu\epsilon}}\,\frac{\vec{k} \times \vec{E}_0}{\omega\cancel{\sqrt{\mu\epsilon}}} + \cancel{\sqrt{\mu\epsilon}}\,\frac{\vec{k}'' \times \vec{E}_0''}{\omega\cancel{\sqrt{\mu\epsilon}}}\right) - \cancel{\sqrt{\mu'\epsilon'}}\,\frac{\vec{k}' \times \vec{E}_0'}{\cancel{\sqrt{\mu'\epsilon'}}}\right] \cdot \hat{n} = 0$$

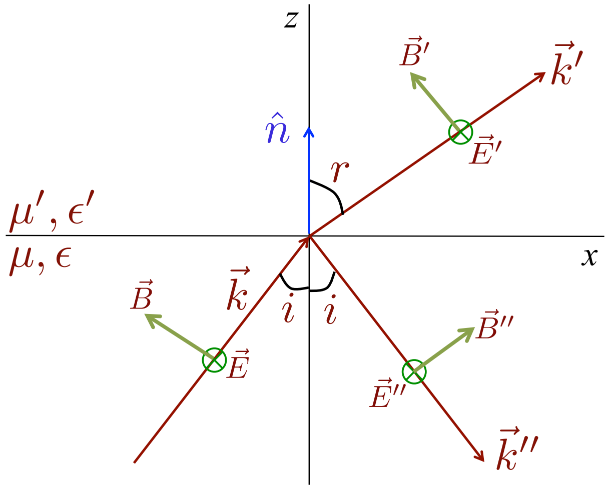
$$\left[\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0'' - \vec{k}' \times \vec{E}_0'\right] \cdot \hat{n} = 0$$

The derivation of equations (7.37.c) and (7.37.d) are in the posted class summary for today.

$\vec{E}$ perpendicular to the plane of incidence

Since a glance at the four equations in (7.37) indicates very messy algebra, we're going to break up the problem into two parts: first consider the case in which the incident plane wave is linearly polarized with its **polarization vector (i.e., $\vec{E}$-field) perpendicular to the plane of incidence**.

The geometry of the problem is shown in Figure 7.6(a) on page 305 in Jackson, and reproduced on the right. The plane of the page is the plane of incidence, and the electric fields are into the page, directed away from us (hence represented by the tail end of an arrow: $\otimes$). The $\vec{B}$-fields are directed as shown, so that when the fingers of the right hand are curled from $\vec{E}$ to $\vec{B}$, the energy flow is in the direction of the $\vec{k}$-vectors.



For the case of the $\vec{E}$-fields perpendicular to the plane of incidence, the equations of interest are the third and fourth boundary conditions in equation (7.37).

4. Consider first the third equation in equation (7.37):

$$\left[\vec{E}_0 + \vec{E}_0'' - \vec{E}_0'\right] \times \hat{n} = 0 \tag{7.37.c}$$

Show that equation (7.37.c) above reduces (in magnitude) to

$$E_0 + E_0'' - E_0' = 0 \tag{7.38.a}$$

$$\vec{E}_0 \times \hat{n} + \vec{E}_0'' \times \hat{n} - \vec{E}_0' \times \hat{n} = 0$$

$$|\vec{E}_0 \times \hat{n}| = |\vec{E}_0||\hat{n}| \sin 90° = E_0(1)(1) = E_0$$

$$|\vec{E}_0'' \times \hat{n}| = E_0''$$

$$|\vec{E}' \times \hat{n}| = E_0'$$

$$E_0 + E'' - E_0' = 0$$

5. Also of interest for the case of the $\vec{E}$-fields perpendicular to the plane of incidence introduced on the previous page is the fourth boundary condition in equation (7.37), given by

$$\left[\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0''\right)-\frac{1}{\mu'}\left(\vec{k}'\times\vec{E}_0'\right)\right]\times\hat{n}=0 \tag{7.37.d}$$

(a) Show that equation (7.37.d) above reduces (in magnitude) to

$$\sqrt{\frac{\epsilon}{\mu}}\,E_0\cos i-\sqrt{\frac{\epsilon}{\mu}}\,E_0''\cos i-\sqrt{\frac{\epsilon'}{\mu'}}\,E_0'\cos r=0 \tag{7.38.b}$$

Proved on HW3

6. Show that equation (7.38.a) and equation (7.38.b) together yield

$$\begin{aligned} \frac{E_0'}{E_0} &= \frac{2n \cos i}{n \cos i + \dfrac{\mu}{\mu'} \sqrt{n'^{\,2} - n^2 \sin^2 i}} \\ \frac{E_0''}{E_0} &= \frac{n \cos i - \dfrac{\mu}{\mu'} \sqrt{n'^{\,2} - n^2 \sin^2 i}}{n \cos i + \dfrac{\mu}{\mu'} \sqrt{n'^{\,2} - n^2 \sin^2 i}} \end{aligned} \tag{7.39}$$

Proved on HW3