Last time we started to explore some of the consequences of

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{R_o^2 a^2} \qquad (1)$$

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0 \qquad (2)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) \qquad (3)$$

with $P = w\,\epsilon$

To start to build some physical intuition of the effect of each energy density component, we'll now explore *toy* universes composed of single components. We'll then put it all together to give us a picture of the universe. What do we mean by single component? **Epochs:**

$$\begin{array}{cc} \Omega_r & \Omega_m \\ \Omega_\Lambda & \Omega_{\mathrm{other}} \end{array}$$

Current status: $\frac{\Omega_{\Lambda,o}}{\Omega_{m,o}} = 2.3$ so universe is currently dominated by cosmological constant. But this was not always the case

Do question (1) on the worksheet and S T O P

(1) $\frac{\epsilon_\Lambda(a)}{\epsilon_m(a)} = \frac{\epsilon_{\Lambda,o}}{\epsilon_{m,o}/a^3} = a^3\frac{\epsilon_{\Lambda,o}}{\epsilon_{m,o}}$ When, $\epsilon_\Lambda(a) = \epsilon_m(a)$ then $\frac{\epsilon_\Lambda(a)}{\epsilon_m(a)} = 1 = \frac{\epsilon_{\Lambda,o}}{\epsilon_{m,o}/a^3} = a^3\frac{\epsilon_{\Lambda,o}}{\epsilon_{m,o}}$ $\Rightarrow a_{m=\Lambda} \approx 0.75$

Doing the same analysis for the matter/radiation terms gives

$$\frac{\epsilon_m(a)}{\epsilon_r(a)} = 1 = \frac{\epsilon_{m,o}}{\epsilon_{m,o}/a} = a\frac{\epsilon_{m,o}}{\epsilon_{r,o}} \quad \Rightarrow a_{m=r} \approx 2.8 \times 10^{-4}$$

Recall that $a = \frac{1}{1+z}$ So that we can solve for *z* and determine that radiation was dominant when $z \approx 3600$ and dark energy becomes dominant when $z \approx 1/3$

We'll now work our way through several *toy* universes. From each of these we'll get important insights as to how the universe evolves.

We'll begin with what appears to be a silly example, a universe with nothing in it.

Do question (2) on the worksheet and S T O P

(2a) When *k = 0,* then $\dot{a} = 0$ and in an empty universe a static, spatially flat universe is permitted.

(2b) *k>0,* positively curved, as this would require and imaginary scale factor, *a*.

(2c) $\dot{a} = \pm\frac{c}{R_o} \Rightarrow a(t) = \pm\frac{c}{R_o}t$ so that an empty universe must either expand or contract. We'll look at the expanding case.

(2d) We have $a(t) = \frac{t}{t_o} \quad (t_o \equiv R_o/c)$ so that $1 + z = \frac{1}{a(t_e)} = \frac{t_o}{t_e}$

and

$$H_o = \frac{\dot{a}}{a} = \frac{1/t_o}{t_e/t_o} = \frac{1+z}{t_o}$$

(2e) Actually already done in (2d). We get that $1 + z = \frac{1}{a(t_e)} = \frac{t_o}{t_e} \Rightarrow t_e = \frac{t_o}{1+z}$

(2f)

$$\begin{aligned} d_p(t_o) &= c\int_{t_e}^{t_o} \frac{dt}{a(t)} \\ &= c\int_{t_e}^{t_o} dt\frac{t_o}{t_e} \\ &= ct_o \ln\left(\frac{t_o}{t_e}\right) \quad \text{or equivalently} \\ d_p(t_o) &= \frac{c}{H_o}\ln(1+z) \end{aligned}$$

(2g) $d_p(t_e) = \frac{c}{H_o}\frac{\ln(1+z)}{1+z}$

Next we'll add *stuff* to the universe and explore the consequences.
However, we'll first look at universes with no curvature, i.e. *k = 0*

In this case, the Friedmann equation is:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{R_o^2 a^2} \Rightarrow \boxed{\dot{a}^2 = \frac{8\pi G\epsilon_o}{3c^2}a^{-1+3w}}$$

where we have used that $\epsilon_w = -\epsilon_{w,o}a^{-3(1+w)}$

Tricky to solve.

Let *a(t) = t* ^{q}. Substituting into the Friedmann equation then results in

$$\begin{aligned}
\dot{a}^2 &= \frac{8\pi G\epsilon_o}{3c^2}a^{-(1+3w)} \\
\left(qt^{q-1}\right)^2 &= Kt^{-q(1+3w)} \text{ Now note that} \\
\left(qt^{q-1}\right)^2 &= q^2t^{2q-2} \propto t^{2q-2} \text{ and substituting above gives} \\
t^{2q-2} &\propto t^{-q(1+3w)} \text{ and this can hold only if exponents are equal so} \\
-2+\frac{2}{q} &= 1+3w \\
q &= \frac{2}{3+3w} \\
&\text{or} \\
a(t) &= \left(\frac{t}{t_o}\right)^{2/(3+3w)} \quad w \neq -1;\ t_o = \frac{1}{1+w}\left(\frac{c^2}{6\pi G\epsilon_o}\right)
\end{aligned}$$

Do question (3) on worksheet
and S T O P

(3a) $H_o = \frac{1}{t_o}\frac{2}{3(1+3w)}$

(3b) $t_e = \frac{t_o}{(1+z)^{3(1+3w)/2}} = \frac{2}{3(1+w)H_o}\frac{1}{(1+z)^{3(1+w)/2}}$

(3c) $d_p(t_o) = c\int_{t_e}^{t_o}\frac{dt}{a(t)} = ct_o\frac{3(1+w)}{1+3w}\left[1-(t_e/t_o)^{(1+3w)/(3+3w)}\right]$

(3d) $d_{hor}(t_o) = ct_o\frac{3(1+w)}{1+3w} = \frac{c}{H_o}\frac{2}{(1+3w)}$

To study the behavior of single component universe is now just a matter of using the appropriate *w*.

Do question (4) on the worksheet and S T O P

(4) Matter, *w = 0*

$$t_0 = \frac{2}{3H_o} \qquad d_p(t_o) = \frac{2c}{H_o}\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

$$d_p(t_e) = \frac{2c}{H_o(1+z)}\left[1 - \frac{1}{\sqrt{1+z}}\right] \qquad d_{hor} = \frac{2c}{H_o}$$

Radiation, *w* = 1/3

$$t_0 = \frac{1}{2H_o} \qquad d_p(t_o) = \frac{c}{H_o}\frac{z}{1+z}$$

$$d_p(t_e) = \frac{c}{H_o}\frac{z}{(1+z)^2} \qquad d_{hor} = \frac{c}{H_o}$$

Λ, *w* = -1

$$d_p(t_o) = \frac{c}{H_o}z; \quad d_p(t_e) = \frac{c}{H_o}\,\frac{z}{1+z}$$

Do question (5) on the worksheet