1 Numerical Solutions to ODEs–Second order ODEs and Finite Differencing

Second order ODEs are ubiquitous in physics. For example, Newton's second law in the case of constant mass can be expressed as

$$\frac{d^2x}{dt^2} = \frac{\boldsymbol{F}}{m}$$

where x is the position of the object and $\boldsymbol{F}$ the force acting on the object.

The Runge-Kutta methods as we've developed them do not seem to lend them to second order ODEs. Thus we must either modify Runge-Kutta or modify the second order ODE. The latter proves to be the right approach.

(1) Consider the general second order ODE of the form

$$\frac{d^2\boldsymbol{Y}}{dt^2} = \boldsymbol{f}(t, \boldsymbol{Y}) \rightarrow F \tag{1}$$

Define a new variable, $\boldsymbol{Z} = d\boldsymbol{Y}/dt$ and write Eq. (1) in terms of this new variable.

$$Z = \frac{dY}{dt}$$

$$F = \frac{dZ}{dt}$$

(2) There are now, two, first order ODEs. Write these two as one single vector ODE equation. Discuss how you would now solve the original second order ODE numerically. How or what would you need to to modify existing code.

turn a second order ODE into a first order.

→ just add a vector

(3) Modify your Runge-Kutta code to address second order ODEs as you discussed in (2) and use it solve the following second order ODE,

$$\frac{d^2y}{dt^2} = -4y, \quad y(0) = 1, \dot{y} = 0.$$

Solve it for $0 \le t \le 2\pi$.

$$X = \frac{dy}{dt}$$

$$-4y = \frac{dx}{dt}$$

We are now ready to generalize 2nd order ODEs to multiple dimensions.

(4) Convert the set of second order ODEs

$$X = \frac{dY}{dt}, \qquad \frac{dX}{dt} = f(X, Y, t)$$

$$\begin{aligned} \frac{d^2\boldsymbol{Y}}{dt^2} &= \boldsymbol{f}(\boldsymbol{X}, \boldsymbol{Y}, t) \\ \frac{d^2\boldsymbol{X}}{dt^2} &= \boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}, t) \end{aligned} \tag{2}$$

$$Y = \frac{dX}{dt}, \qquad \frac{dY}{dt} = g(X, Y, t)$$

into a single, first order, vector ODE.

$$X = g(X, Y, t) \qquad y = f(x, y, z)$$

(5) Modify your code for second order ODEs to accommodate 2-dimensional second order ODEs.

(6) Begin homework