PHY 474/374 **Spring 2021**

Week 3—Wednesday, April 14—Discussion Worksheet

Stellar Atmospheres

Computing models of stellar atmospheres is a complicated problem. Fortunately, we can make the reasonable approximation of an isothermal atmosphere with T a little below T_{eff}. The bottom of the atmosphere is called the **photosphere** and corresponds roughly to the visible surface of the star.

1. Let's begin by exploring some of the differences between the stellar interior and the stellar atmosphere to understand why handling the stellar atmosphere is more difficult.

(a) First, there is a question of **scale**. Using numbers for our Sun, show that the atmosphere is just a narrow surface layer, $\sim 10^{-3}\ R$.

Note: $R_\odot = 6.96 \times 10^8$ m. The Sun's atmosphere is about 2000 km in extent (including only the chromosphere, and not the solar corona which extends out to distances $\gg R_\odot$).

$$\frac{R_{atmos}}{R_\odot} \sim \frac{2000 \times 10^3\ m}{6.96 \times 10^8\ m} = 3 \times 10^{-3} \sim 10^{-3}$$

$$R_{atmos} \sim 10^{-3} R_\odot \Rightarrow R_{atmos} \sim 10^{-3} R$$

(b) The **mean free path** λ is also different between the interior and the atmosphere. In the interior, λ is a few cm, whereas in the atmosphere it is of the order of the scale height H of the atmosphere (see Question 2), given by

$$H = \frac{kT}{\mu m_p g} \qquad \text{where} \qquad g = \frac{Gm}{r^2} \tag{1}$$

Find the scale height in the Sun's atmosphere. Use $T = 5500$ K, and $m = M_\odot, r = R_\odot$, where $M_\odot = 1.99 \times 10^{30}$ kg, $R_\odot = 6.96 \times 10^8$ m, and the mean molecular weight $\mu = 0.6$.

$$g = \frac{(6.67 \times 10^{-11}\ m^3\ kg^{-1})(1.99 \times 10^{30}\ kg)}{(6.96 \times 10^8\ m)^2} = 274\ m$$

$$H = \frac{(1.38 \times 10^{-23}\ m^2\ kg\ s^{-2}\ K^{-1})(5500\ K)}{(0.6)(1.67 \times 10^{-27}\ kg)(274\ m)} = 276{,}454\ m$$

or 276 km

2. On the previous page, we introduced the term ***scale height*** of a stellar atmosphere. We will now learn where this comes from.

Recall that we discussed in a previous class how there exist simple solutions to the equation of hydrostatic equilibrium. One such example is for the case of an isothermal gas, which is a valid assumption in a stellar atmosphere. Assume also that the atmosphere is small in extent compared to the radius of the star, so that the gravitational acceleration $g = GM/R^2$ can be assumed to be constant.

(a) Starting from the equation of stellar structure (*Dalsgaard*, eq. 4.4), given by $\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$, show that we can write

$$\frac{dP}{dr} = -\frac{P}{H} \tag{2}$$

where $P = \rho kT/\mu m_p$ from the ideal gas law, and the scale height H is given in equation (1).

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} = -\left[\frac{\rho kT}{\mu m_p}\right]\left[\frac{\mu m_p}{\rho kT}\right]\frac{Gm\rho}{r^2}$$ mult/Div by $P = \rho kT/\mu m_p$

if $\frac{dP}{dr} = -\frac{P}{H}$ then $H = \frac{kT}{\mu m_p g}$ where $g = \frac{Gm}{r^2}$

(b) Integrate equation (2) and show that

$$P = P_0 \exp\left(-\frac{h}{H}\right)$$

where $h = r - r_0$, with r_0 defining some arbitrary reference level in the atmosphere, and P_0 is the value of P at $h = 0$.

$$\frac{dP}{P} = -\frac{1}{H}dr$$

$$\ln P = -r/H + C$$

At $r = r_0$, $P = P_0$ (b/c $h = 0$, $r - r_0 = 0$)

$$\ln P_0 = -\frac{r_0}{H} + c$$

$$\ln P = -\frac{r}{H} + \left[\ln P_0 + \frac{r_0}{H}\right]$$

So, $\ln\left(\frac{P}{P_0}\right) = -\frac{(r - r_0)}{H} = -\frac{h}{H}$, thus $P = P_0 \exp\left(-\frac{h}{H}\right)$

(c) Use the equation in part (b) to interpret the scale height?

For each ↑ in height by the scale height H, the pressure drops by a factor $1/e = 0.368$

3. Even though we have been using zero pressure at the stellar surface in our equations, there is some pressure at the bottom of the star's atmosphere (on the photosphere), $P_{\rm ph}$, given by

$$P_{\rm ph} = \left[\frac{GM(a+1)}{R^2\kappa_0}\right]^{1/(a+1)} \left[\frac{k_B}{\mu m_p}\right]^{a/(a+1)} T_{\rm eff}^{(a-b)/(a+1)}$$

where the opacity, κ, has been expressed as a power law

$$\kappa = \kappa_0\,\rho^a T^b$$

and I've written the Boltzmann constant as k_B to distinguish it from κ; note that the a in this equation should not be confused with its use for the radiation constant in the previous class (i.e., a here is different, the exponent to the density in the expression for the opacity).

(a) Calculate the pressure on the Sun's photosphere, for which

$$\kappa_0 = 1.6\times10^{-33}, \qquad a = 0.4 \qquad b = 9.3$$

Recall that the temperature at the Sun's photosphere is $T_{\rm eff} = 5800$ K, and $\mu = 0.6$.

$$\left[\frac{(6.67\times10^{-11}\,m^3\,kg^{-1})(1.99\times10^{30}\,kg)(1.4)}{(6.96\times10^{8}\,m)^2\,(1.6\times10^{-33})}\right]^{(1/1.4)} \left[\frac{1.38\times10^{-23}\,m^2\,kg\,s^{-2}\,K^{-1}}{(0.6)(1.67\times10^{-27}\,kg)}\right]$$

$$(5800\,K)^{(-8.9)/1.4}$$

$$P_{ph} = 338\ N/m^2$$

(b) Show that, assuming an ideal gas, the density in the Sun's photosphere is given by

$$\rho_{\rm ph} = \left[\frac{GM(a+1)}{R^2\kappa_0}\right]^{1/(a+1)} \left[\frac{k_B}{\mu m_p}\right]^{-1/(a+1)} \left[T_{\rm eff}\right]^{-(b+1)/(a+1)}$$

$$\rho_{ph} = \underbrace{P_{ph}\left[\frac{kT}{\mu m_p}\right]^{-1} = \left[\frac{GM(a+1)}{R^2 k_0}\right]^{1/(a+1)}}_{\text{Remains the same}} \left[\frac{k}{\mu m_p}\right]^{\frac{a}{a+1}-1} T_{eff}^{\frac{a-b}{a+1}-1}$$

Becomes $\dfrac{a-(a+1)}{a+1}$ $\qquad$ $\dfrac{a-b-(a+1)}{a+1}$

$$\rho_{ph} = 4.2\times10^{-6}\ N/m^2$$

The Energy Equation

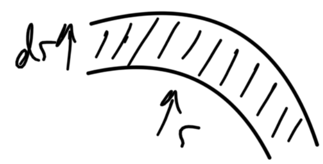
In the previous class, we derived an equation for the temperature gradient, dT/dr, in a star. We also need an equation for the luminosity as a function of r.

4. During most of a star's lifetime, the energy is produced by nuclear reactions.

(a) If ϵ is the rate of energy production per unit mass by nuclear reactions, show that

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon$$

Hint: Consider a spherical shell of radius r and thickness dr, and find the energy per unit time produced in it; this should lead to an increase dL in the luminosity.



$\epsilon \rightarrow$ rate
of Energy produced per unit mass

Energy per unit time $\epsilon\, dm$

where $dm = \rho\, dV = \rho(4\pi r^2 dr)$

$dL = \epsilon\, dm = \epsilon \rho 4\pi r^2 dr$ $\qquad \frac{dL}{dr} = 4\pi r^2 \rho \epsilon$

(b) The equation in part (a) does not describe the liberation of energy during gravitational contraction, and hence cannot be used in the early phases of stellar evolution. A more complete description can be obtained by starting from the first law of thermodynamics, and we get

$$\frac{dL}{dr} = 4\pi r^2 \left[\rho\epsilon - \rho\frac{d}{dt}\left(\frac{u}{\rho}\right) + \frac{P}{\rho}\frac{d\rho}{dt}\right] \qquad dQ = dU + PdV$$

Interpret this equation.

Energy radiated comes from energy produced by nuclear reactions ($\rho\epsilon$) minus the energy needed to raise the internal energy of the gas $[-\rho\, d/dt\,(u/\rho)]$ (from dU) and minus the energy needed to do work on gas (PdV)

5. In the equation for dL/dr in Question 4(b), the second and third term are essentially negligible during the normal nuclear burning phases. To see this, consider the Kelvin-Helmholtz timescale that we discussed earlier, which can be written in the present context as

$$t_{\rm KH} = \frac{U_{\rm tot}}{L_s}$$

where $U_{\rm tot} \simeq R^3 u$ is the total internal energy of the star. You may recall from previous calculations that $t_{\rm KH} \sim 10^7$ yr, where the nuclear burning timescale is $t_{\rm nuc} \sim 10^{10}$ yr for our Sun. By replacing dL/dr by L_s/R, and du/dt by $u/t_{\rm nuc}$, show that the magnitude of the second term is

$$\frac{\left|4\pi r^2 \rho \dfrac{d}{dt}\left(\dfrac{u}{\rho}\right)\right|}{\dfrac{dL}{ddr}} \simeq \frac{t_{\rm KH}}{t_{\rm nuc}} \ll 1$$

$4\pi r^2 \rho \, d/dt\,(u/\rho)$, assuming ρ constant

goes $R^2 \cancel{\rho}\, 1/\cancel{\rho}\, (du/dt) \sim R^2 \left[\frac{u}{t_{\rm nuc}}\right]$

$$\frac{\left|4\pi r^2 \rho \, d/dt\,(u/\rho)\right|}{dL/dr} = \frac{R^2\left[\frac{u}{t_{\rm nuc}}\right]}{L_s/R}$$

$$= \frac{R^3 u/t_{\rm nuc}}{L_s} = \frac{U_{\rm tot}/t_{\rm nuc}}{L_s}$$

$$= \frac{t_{\rm KH}}{t_{\rm nuc}} = \frac{10^7\,\text{yr}}{10^{10}\,\text{yr}} \sim 10^{-3} \ll 1$$