PHY 411 Winter 2021
Homework 7 solutions

1. In class, we showed that the potential of a localized distribution of charge described by the charge density $\rho(\vec{x}\,')$ is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1}\, q_{lm}\, \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

where the *multipole moments* q_{lm} are given by

$$q_{lm} = \int Y^*_{lm}(\theta',\phi')\, r'^{\,l}\, \rho(\vec{x}\,')\, d^3x' \tag{1}$$

Explicitly evaluate q_{11} and q_{10} and show that

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\Big(p_x - ip_y\Big) \qquad \text{and} \qquad q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z$$

where p_x, p_y, p_z are the components of the electric dipole moment: $\vec{p} = \displaystyle\int \vec{x}\,'\, \rho(\vec{x}\,')\, d^3x'$.

Solution: For q_{11}, we have from equation (1) above that

$$q_{11} = \int Y^*_{11}(\theta',\phi')\, r'^{\,1}\, \rho(\vec{x}\,')\, d^3x'$$

and since (from page 109 in *Jackson*)

$$Y_{11}(\theta',\phi') = -\sqrt{\frac{3}{8\pi}}\, \sin\,\theta'\, e^{i\phi'} \qquad \text{so that} \qquad Y^*_{11} = -\sqrt{\frac{3}{8\pi}}\, \sin\,\theta'\, e^{-i\phi'}$$

this becomes

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int \sin\,\theta'\, e^{-i\phi'}\, r'\, \rho(\vec{x}\,')\, d^3x' \tag{2}$$

Since

$$e^{-i\phi'} = \cos\,\phi' - i\sin\,\phi'$$

equation (2) becomes

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int \Big[r'\sin\,\theta'\,\cos\,\phi' - ir'\sin\,\theta'\,\sin\,\phi'\Big]\, \rho(\vec{x}\,')\, d^3x' \tag{3}$$

But, in the spherical coordinate system

$$x' = r'\sin\,\theta'\,\cos\,\phi'$$
$$y' = r'\sin\,\theta'\,\sin\,\phi'$$

so that equation (3) becomes

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int \Big[x' - iy'\Big]\, \rho(\vec{x}\,')\, d^3x' \tag{4}$$

and since from the expression given above for $\vec{p}$, we can write

$$p_x = \int x'\, \rho(\vec{x}\,')\, d^3x' \qquad \text{and} \qquad p_y = \int y'\, \rho(\vec{x}\,')\, d^3x'$$

equation (4) becomes finally the desired relation

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\Big(p_x - ip_y\Big)$$

For q_{10}, we have from equation (1) on the previous page that

$$q_{10} = \int Y^*_{10}(\theta', \phi')\, r'^{\,1}\, \rho(\vec{x}\,')\, d^3x' \tag{5}$$

and since (from page 109 in *Jackson*)

$$Y^*_{10}(\theta', \phi') = \sqrt{\frac{3}{4\pi}}\, \cos\,\theta'$$

equation (5) above becomes

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int \cos\,\theta'\, r'\, \rho(\vec{x}\,')\, d^3x'$$

or, upon rearranging

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int \Big[r'\, \cos\,\theta'\Big]\, \rho(\vec{x}\,')\, d^3x' \tag{6}$$

But, in the spherical coordinate system

$$z' = r'\, \cos\,\theta'$$

so that equation (6) above becomes

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int z' \rho(\vec{x}\,')\, d^3x' \tag{7}$$

and since from the expression given in the question on the previous page for $\vec{p}$, we can write

$$p_z = \int z'\, \rho(\vec{x}\,')\, d^3x'$$

equation (7) becomes finally

$$q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z$$

which is the desired relation.

Question 2 begins on the next page.

2. Also of interest are the quadrupole moments q_{22}, q_{21}, and q_{20}, for which the algebra is more tedious. Therefore, we will limit ourselves to one example. Show that

$$q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\Big(Q_{13} - i\,Q_{23}\Big)$$

where Q_{ij} is the quadrupole moment tensor given by

$$Q_{ij} = \int \Big(3x'_i x'_j - r'^{\,2}\,\delta_{ij}\Big)\,\rho(\vec{x}\,')\,d^3x' \tag{8}$$

Solution: We have from equation (1) written in the previous question that

$$q_{21} = \int Y^*_{21}(\theta', \phi')\,r'^{\,2}\,\rho(\vec{x}\,')\,d^3x' \tag{9}$$

and since (from page 109 in *Jackson*)

$$Y^*_{21}(\theta', \phi') = -\sqrt{\frac{15}{8\pi}}\,\sin\,\theta'\,\cos\,\theta'\,e^{-i\phi'} \tag{10}$$

equation (9) becomes

$$q_{21} = -\sqrt{\frac{15}{8\pi}}\int \sin\,\theta'\,\cos\,\theta'\,e^{-i\phi'}\,r'^{\,2}\,\rho(\vec{x}\,')\,d^3x' \tag{11}$$

Let's leave it this way for now, and work with the right hand side; it's easier to do.

Using equation (8) with $x'_1 = x', x'_2 = y'$, and $x'_3 = z'$, we get

$$Q_{13} - i\,Q_{23} = \int \Big(3x'z'\Big)\,\rho(\vec{x}\,')\,d^3x' - i\int \Big(3y'z'\Big)\,\rho(\vec{x}\,')\,d^3x' \tag{12}$$

In the spherical coordinate system

$$\begin{aligned} x' &= r'\sin\,\theta'\,\cos\,\phi' \\ y' &= r'\sin\,\theta'\,\sin\,\phi' \\ z' &= r'\cos\,\theta' \end{aligned}$$

so equation (12) becomes

$$Q_{13} - i\,Q_{23} = 3\int \Big(r'\sin\,\theta'\,\cos\,\phi'\,\{r'\,\cos\,\theta'\} - ir'\sin\,\theta'\,\sin\,\phi'\{r'\,\cos\,\theta'\}\Big)\rho(\vec{x}\,')\,d^3x'$$

from which we obtain

$$Q_{13} - i\,Q_{23} = 3\int r'^{\,2}\sin\theta'\,\cos\theta'\,\Big(\cos\,\phi' - i\sin\,\phi'\Big)\rho(\vec{x}\,')\,d^3x'$$

But $\cos\,\phi' - i\sin\,\phi' = e^{-i\phi'}$, so the above equation becomes

$$Q_{13} - i\,Q_{23} = 3\int r'^{\,2}\sin\theta'\,\cos\theta'\,e^{-i\phi'}\,\rho(\vec{x}\,')\,d^3x' \tag{13}$$

From equation (11) and equation (13), we get

$$q_{21} = -\sqrt{\frac{15}{8\pi}}\,\frac{1}{3}\,(Q_{13} - i\,Q_{23})$$

so that, finally, we get the desired relation

$$q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\Big(Q_{13} - i\,Q_{23}\Big)$$

3. In class, you obtained by direct differentiation that the coordinates of the electric field E_r, E_θ, and E_ϕ are given by

$$E_r = \frac{(l+1)}{(2l+1)\,\epsilon_0}\, q_{lm}\, \frac{Y_{lm}(\theta,\phi)}{r^{l+2}}$$

$$E_\theta = -\frac{1}{(2l+1)\,\epsilon_0}\, q_{lm}\, \frac{1}{r^{l+2}}\, \frac{\partial}{\partial\theta} Y_{lm}(\theta,\phi)$$

$$E_\phi = \frac{1}{(2l+1)\,\epsilon_0}\, q_{lm}\, \frac{1}{r^{l+2}}\, \frac{im}{\sin\,\theta} Y_{lm}(\theta,\phi)$$

For a dipole $\vec{p}$ along the z-axis, show that the fields above reduce to:

$$E_r = \frac{2p\,\cos\,\theta}{4\pi\epsilon_0 r^3} \qquad\qquad E_\theta = \frac{p\,\sin\,\theta}{4\pi\epsilon_0 r^3} \qquad\qquad E_\phi = 0$$

Solution: In Question 1, we demonstrated that

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\left(p_x - ip_y\right) \qquad \text{and} \qquad q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z \tag{14}$$

Since the dipole $\vec{p}$ in this question is along the z-axis, let's write $\vec{p} = p\hat{z}$, so that

$$p_x = 0, \qquad p_y = 0 \qquad \text{and} \qquad p_z = p \tag{15}$$

Then, we have from equation (14) that

$$q_{11} = 0 \qquad \text{and} \qquad q_{10} = \sqrt{\frac{3}{4\pi}}\, p \tag{16}$$

We can now work with the E-field components given in the question above; we only need $l = 0, m = 0$, since equation (16) tells us that the other components are zero.

So, with $l = 1, m = 0$, we get

$$E_r = \frac{(1+1)}{(2\{1\}+1)\,\epsilon_0}\, q_{10}\, \frac{Y_{10}(\theta,\phi)}{r^{1+2}}$$

$$E_\theta = -\frac{1}{(2\{1\}+1)\,\epsilon_0}\, q_{10}\, \frac{1}{r^{1+2}}\, \frac{\partial}{\partial\theta} Y_{10}(\theta,\phi)$$

$$E_\phi = \frac{1}{(2\{1\}+1)\,\epsilon_0}\, q_{10}\, \frac{1}{r^{1+2}}\, \frac{i(0)}{\sin\,\theta} Y_{10}(\theta,\phi)$$

which shows that $E_\phi = 0$.

On the next page, we will simplify E_r and E_θ.

On the previous page, we obtained that

$$E_r = \frac{(1+1)}{(2\{1\}+1)\,\epsilon_0}\, q_{10}\, \frac{Y_{10}(\theta,\phi)}{r^{1+2}}$$

$$E_\theta = -\frac{1}{(2\{1\}+1)\,\epsilon_0}\, q_{10}\, \frac{1}{r^{1+2}}\, \frac{\partial}{\partial\theta}\, Y_{10}(\theta,\phi)$$

$$E_\phi = 0$$

Simplifying E_r and E_θ, we get

$$E_r = \frac{2}{3\,\epsilon_0}\left[\sqrt{\frac{3}{4\pi}}\; p\right]\frac{1}{r^3}\left[\sqrt{\frac{3}{4\pi}}\;\cos\theta\right]$$

$$E_\theta = -\frac{1}{3\,\epsilon_0}\left[\sqrt{\frac{3}{4\pi}}\; p\right]\frac{1}{r^3}\frac{\partial}{\partial\theta}\left[\sqrt{\frac{3}{4\pi}}\;\cos\theta\right]$$

$$E_\phi = 0$$

so that

$$E_r = \frac{2}{3\,\epsilon_0}\left[\frac{3}{4\pi}\right]\frac{p}{r^3}\,\cos\theta$$

$$E_\theta = -\frac{1}{3\,\epsilon_0}\left[\frac{3}{4\pi}\right]\frac{p}{r^3}\frac{\partial}{\partial\theta}\left[\cos\theta\right]$$

$$E_\phi = 0$$

and thus

$$E_r = \frac{2}{\epsilon_0}\left[\frac{1}{4\pi}\right]\frac{p}{r^3}\,\cos\theta$$

$$E_\theta = -\frac{1}{\epsilon_0}\left[\frac{1}{4\pi}\right]\frac{p}{r^3}\left[-\sin\theta\right]$$

$$E_\phi = 0$$

which gives us the desired result that for a dipole $\vec{p}$ along the z-axis, the fields reduce to

$$E_r = \frac{2p\,\cos\,\theta}{4\pi\epsilon_0 r^3} \qquad E_\theta = \frac{p\,\sin\,\theta}{4\pi\epsilon_0 r^3} \qquad E_\phi = 0$$

4. Suppose that we have a uniform magnetic field $\vec{B}_0 = B_0\hat{z}$, where B_0 is a constant.

(a) Examine whether

$$\vec{A} = \frac{\vec{B}_0}{2} \times \vec{x}$$

is an appropriate vector potential for this given field.

Solution: In order to show that $\vec{A}$ is an appropriate vector potential, we will have to show that $\vec{B}_0 = \vec{\nabla} \times \vec{A}$, where $\vec{B}_0 = B_0\hat{z}$.

Now,

$$\begin{aligned}
\vec{\nabla} \times \vec{A} &= \vec{\nabla} \times \left(\frac{\vec{B}_0}{2} \times \vec{x}\right) \\
&= \vec{\nabla} \times \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{B_0}{2} \\ x & y & z \end{vmatrix} \\
&= \vec{\nabla} \times \left[\hat{x}\left(0 - \frac{yB_0}{2}\right) + \hat{y}\left(\frac{xB_0}{2} - 0\right) + \hat{z}\left(0 - 0\right)\right] \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{yB_0}{2} & \frac{xB_0}{2} & 0 \end{vmatrix} \\
&= \hat{x}\left[0 - \frac{B_0}{2}\frac{\partial x}{\partial z}\right] + \hat{y}\left[-\frac{B_0}{2}\frac{\partial y}{\partial z} - 0\right] + \hat{z}\left[\frac{B_0}{2}\frac{\partial x}{\partial x} - \left(-\frac{B_0}{2}\right)\frac{\partial y}{\partial y}\right] \\
&= \hat{x}\,(0) + \hat{y}\,(0) + \hat{z}\left[\frac{B_0}{2} + \frac{B_0}{2}\right]
\end{aligned}$$

Therefore, we get

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\frac{\vec{B}_0}{2} \times \vec{x}\right) = \hat{z}\left[B_0\right]$$

meaning that we have demonstrated that

$$\vec{\nabla} \times \vec{A} = B_0\hat{z} = \vec{B}_0$$

implying that $\vec{A}$ is an appropriate vector potential for the given field $\vec{B} = B_0\,\hat{z}$.

(b) Does this vector potential satisfy the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$?

Solution: Let us evaluate explicitly.

$$\begin{aligned}
\vec{\nabla} \cdot \vec{A} &= \left[\hat{x}\,\frac{\partial}{\partial x} + \hat{y}\,\frac{\partial}{\partial y} + \hat{z}\,\frac{\partial}{\partial z}\right] \cdot \left[\hat{x}\,A_x + \hat{y}\,A_y + \hat{z}\,A_z\right] \\
&= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad (17)
\end{aligned}$$

In part (a), we obtained that

$$A_x = \left(\frac{\vec{B}_0}{2} \times \vec{x}\right)_x = \hat{x}\left(-\frac{yB_0}{2}\right) \qquad (18)$$

$$A_y = \left(\frac{\vec{B}_0}{2} \times \vec{x}\right)_y = \hat{y}\left(\frac{xB_0}{2}\right) \qquad (19)$$

$$A_z = \left(\frac{\vec{B}_0}{2} \times \vec{x}\right)_z = \hat{z}\Big(0\Big) \qquad (20)$$

Substituting these expressions in equation (17), we get

$$\begin{aligned}
\vec{\nabla} \cdot \vec{A} &= \frac{\partial}{\partial x}\left[-\frac{yB_0}{2}\right] + \frac{\partial}{\partial y}\left[\frac{xB_0}{2}\right] + \frac{\partial}{\partial z}\Big[0\Big] \\
&= 0 + 0 + 0
\end{aligned}$$

Therefore, we find that

$$\vec{\nabla} \cdot \vec{A} = 0$$

which means that the given vector potential $\vec{A}(\vec{x})$ does satisfy the Coulomb gauge.