Physics 412—The Dual Space Winter 2020

In quantum mechanics every ket $|\Psi\rangle$ has a corresponding bra $\langle\Psi|$ in the dual space. For R^2, the vectors in the dual space are represented by row vectors, not column vectors. So if a vector $\vec{A}$ has components A_x and A_y, the representations are

$$\vec{A} \leftrightarrow \begin{bmatrix} A_x \\ A_y \end{bmatrix}, \quad \text{dual of } \vec{A} \leftrightarrow \begin{bmatrix} A_x & A_y \end{bmatrix}.$$

As in quantum mechanics, when we operator on a dual vector with an operator, we do so from the left, and the answer is a new dual vector.

(1) Are the operators R_{30} and T_{45} Hermitian? If so, prove it, by calculations using the representations of vectors and operators. If not, find the representations of the adjoints (Hermitian conjugates) of the operators R_{30} and T_{45}.

Let $\underset{\sim}{A} \equiv$ dual of $\vec{A} \longleftrightarrow [A_x \; A_y]$

$$R_{30}\vec{A} \longleftrightarrow \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} A_x \\ A_y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3}A_x - A_y \\ A_x + \sqrt{3}A_y \end{bmatrix}$$

not dual!

$$\underset{\sim}{A}\, R_{30} \longleftrightarrow \begin{bmatrix} A_x & A_y \end{bmatrix}\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3}A_x + A_y, & -A_x + \sqrt{3}A_y \end{bmatrix}$$

R_{30} is not Hermitian

But T_{45} is.

(2) Using your matrix representations, find the eigenvalues and eigenvectors of R_{30} and T_{45}.

$\det(A - \lambda I) = 0$:

For R_{30}, $\det\begin{bmatrix} \sqrt{3}/2 - \lambda & -1/2 \\ 1/2 & \sqrt{3}/2 - \lambda \end{bmatrix} = 0 \Rightarrow \left(\frac{\sqrt{3}}{2} - \lambda\right)^2 + \frac{1}{4} = 0$

$\Rightarrow \lambda^2 + \sqrt{3}\lambda + 1 = 0$

$\Rightarrow \lambda = \dfrac{-\sqrt{3} \pm \sqrt{3-4}}{2}$

$\Rightarrow \lambda = -\dfrac{\sqrt{3}}{2} \pm \dfrac{i}{2}$

complex eigenvalues, so no real eigenvectors

T_{45} : $\det\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\underline{\underline{\lambda = +1}}$ $\quad T_{45} \vec{A} = +1 \vec{A} \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}$

$$\Rightarrow A_x = A_y \Rightarrow \vec{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\underline{\lambda = -1}}$ $\quad T_{45} \vec{A} = -1 A \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix} = - \begin{bmatrix} A_x \\ A_y \end{bmatrix}$

$$A_x = -A_y \Rightarrow \vec{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$