PHY 411 **Winter 2021**

Class Summary—Week 7, Day 1—Tuesday, Feb 16

Laplace equation in spherical coordinates (contd.)

The Laplace equation in spherical coordinates is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{3.1}$$

In the previous class, we assumed a product form for the potential:

$$\Phi = \frac{U(r)}{r}\,P(\theta)Q(\phi) \tag{3.2}$$

and separated variables to obtain:

$$r^2 \sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{P\,r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right)\right] + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0 \tag{3.3}$$

The last term on the left hand side is a function of ϕ only, so we set

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2 \tag{3.4}$$

and obtained the solution

$$Q = e^{\pm im\phi} \tag{3.5}$$

In general, m need not be an integer, but we will choose m to be an integer so that we can run through the full azimuthal range and still get single-valued Q.

We then separated the r and θ terms (*on the worksheet for the previous class*) and obtained

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) + \left[l\,(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0 \tag{3.6}$$

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0 \tag{3.7}$$

Note that, at this stage, we have placed no constraint on $l\,(l+1)$; all we're saying is that it is a real number. If you're wondering why we chose such a weird-looking relation (e.g., why not just L instead), it is because we know that writing it in this way will produce a standard-looking equation whose solutions are already available from mathematicians. I have seen some treatments that use an L and then go through a series of steps to show that $L = l\,(l+1)$ is inevitable, but we'll stick to Jackson's approach.

Using a power solution method with $U = r^a$ (*on the worksheet for the previous class*), we found the solution for the radial equation (3.7) to be

$$U = A\,r^{l+1} + B\,r^{-l} \tag{3.8}$$

where A and B are constants that will ultimately be determined from the boundary conditions, and again l is, as yet, undetermined, but will be determined from consideration of the θ-equation.

It is customary to write the θ-equation in terms of $x = cos\,\theta$ instead of θ itself. Then, we get

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right]+\left[l\,(l+1)-\frac{m^2}{1-x^2}\right]\,P=0 \tag{3.9}$$

as you showed on today's worksheet. This equation is called the **generalized Legendre equation** and its solutions are the **associated Legendre functions**.

Judging from the looks of the equation, things will be complicated! So, let's do the simple case of $m = 0$ first, for which equation (3.9) reduces to the **Legendre equation**:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right]+l\,(l+1)\,P=0 \tag{3.10}$$

We assume that the whole range of cos θ, including the north and south poles (at cos $\theta = \pm 1$) is in the region of interest. The desired solution should then be single valued, finite and continuous on the interval $-1 \leq x \leq 1$ in order to represent a physical potential.

Let's try a power series of the form:

$$P(x)=x^{\alpha}\sum_{j=0}^{\infty}a_j x^j \tag{3.11}$$

where α is a parameter to be determined.

By substituting this power series and its derivative into equation (3.10), it is easy to show that we would obtain the series

$$\sum_{j=0}^{\infty}\left[(\alpha+j)(\alpha+j-1)\,a_j x^{\alpha+j-2}-\left\{(\alpha+j)(\alpha+j+1)-l\,(l+1)\right\}a_j x^{\alpha+j}\right]=0 \tag{3.12}$$

This must hold for all values of x, so the coefficient of each power of x must vanish separately. So, let's start with $j = 0$ and work our way through the summation term by term.

The $j = 0$ term in the series is

$$\alpha(\alpha-1)\,a_0\,x^{\alpha-2}-\left\{\alpha\,(\alpha+1)-l\,(l+1)\right\}a_0\,x^{\alpha}$$

The $j = 1$ term in the series is

$$(\alpha+1)\,\alpha\,a_1 x^{\alpha-1}-\left\{(\alpha+1)(\alpha+2)-l\,(l+1)\right\}a_1 x^{\alpha+1}\Big]$$

The $j = 2$ term in the series is

$$(\alpha+2)\,(\alpha+1)\,a_2 x^{\alpha}-\left\{(\alpha+2)(\alpha+3)-l\,(l+1)\right\}a_2 x^{\alpha+2}\Big]$$

The $j = 3$ term in the series is

$$(\alpha+3)\,(\alpha+2)\,a_3 x^{\alpha+1}-\left\{(\alpha+3)(\alpha+4)-l\,(l+1)\right\}a_3 x^{\alpha+3}\Big]$$

So, if the coefficient of each power of x has to vanish separately in equation (3.12), we must have the following:

The lowest power of x is $x^{\alpha-2}$, and there is only one term in the summation with this power, so for its coefficient to vanish, we must have

$$\alpha(\alpha-1)\,a_0 = 0$$

The next higher power is $x^{\alpha-1}$, and again there is only one term in the summation with this power, so for its coefficient to vanish, we must have

$$\alpha\,(\alpha+1)\,a_1 = 0$$

The next higher power is x^{α}, but now there are two terms in the summation with this power, so for its coefficient to vanish, we must have

$$(\alpha+2)\,(\alpha+1)\,a_2 - \Big\{\alpha\,(\alpha+1) - l\,(l+1)\Big\}\,a_0 = 0$$

which connects a_2 to a_0.

Continuing this, the next higher power is $x^{\alpha+1}$, and again, there are two terms in the summation with this power, so for its coefficient to vanish, we must have

$$(\alpha+3)\,(\alpha+2)\,a_3 - \Big\{(\alpha+1)(\alpha+2) - l\,(l+1)\Big\}\,a_1$$

which connects a_3 to a_1.

So, we see a pattern emerging — the lowest powers in the series constrain the parameter α, and higher values are connected by a recurrence relation.

Furthermore, the above relations tell us that

$$\begin{aligned} &\text{if } a_0 \neq 0, \text{ then } \alpha\,(\alpha-1) = 0 \\ &\text{if } a_1 \neq 0, \text{ then } \alpha\,(\alpha+1) = 0 \end{aligned} \tag{3.13}$$

whereas, for a general j value, we get the recurrence relation

$$a_{j+2} = \left[\frac{(\alpha+j)(\alpha+j+1) - l\,(l+1)}{(\alpha+j+1)(\alpha+j+2)}\right]\,a_j \tag{3.14}$$

Now, it is easy to see that the two relations in equation (3.13) are *equivalent*; you can check this out for yourself by writing the terms out explicitly. So, we need to choose either $a_0 \neq 0$ or $a_1 \neq 0$, but not both.

Let's choose $a_0 \neq 0$. Then $\alpha = 0$ or $\alpha = 1$ from equation (3.15). The $\alpha = 0$ case generates even powers of x and the $\alpha = 1$ case generates odd powers of x. Now, consider the following properties:

- The series converges for $x^2 < 1$, regardless of the value of l.
- The series diverges at $x = \pm 1$, unless it terminates.

Remember, though, that the whole range of $x = \cos\theta$, including the north and south poles (at $\cos\theta = \pm 1$) is in the region of interest for us. The desired solution should be single valued, finite and continuous on the interval $-1 \leq x \leq 1$ in order to represent a physical potential.

That means we need to *demand* that the series terminate. Since α and j are positive integers or zero, the recurrence relation equation (3.14) will terminate only if l is *zero or a positive integer.*

Even with this choice of l as zero or an integer, only one of the two series converges at $x = \pm 1$.

- If l is even, then only the $\alpha = 0$ series terminates.

  The polynomial will then have x^l as the highest power of x, the next highest being x^{l-2}, and so on, down to x^0. For example, $P_4(x)$, for which $l = 4$, should have powers of $x^4, x^2, x^0 = 1$, as indeed it does (see below).

- If l is odd, then only the $\alpha = 1$ series terminates.

  Again, the polynomial will then have x^l as the highest power, the next highest being x^{l-2}, and so on, but this time the lowest will be x^1. For example, $P_5(x)$ for which $l = 5$, should have powers of $x^5, x^3, x^1 = x$, as indeed it does (see below).

Next, note that a_0 is still undetermined. So, we define each polynomial to have the value 1 when its argument is 1. That is, we define

- $P_0(x) = 1$ at $x = +1$
- $P_1(x) = 1$ at $x = +1$
- $P_2(x) = 1$ at $x = +1$
- $P_3(x) = 1$ at $x = +1$
- $P_4(x) = 1$ at $x = +1$, etc.

This is what Jackson means when he says that "by convention, these polynomials are normalized to have the value unity at $x = +1$" on page 97.

These $P_l(x)$ are called the **Legendre polynomials of order l**. Let's work out the first few.

- $P_0(x)$: Here, $l = 0$, so the highest power of x is 0. Since $P_l(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$, we must have

  $$P_0(x) = x^0 \Big[a_0 x^0 \Big] = a_0$$

  To determine a_0, use the normalization

  $$P_0(+1) = 1 = a_0$$

  which means that $a_0 = 1$, and so we get

  $$P_0(x) = 1$$

- $P_1(x)$: Here, $l = 1$, so the highest power of x is 1. Again, using $P_l(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$, we get

$$P_1(x) = x^1 \left[a_0 x^0\right] = a_0 x$$

that is, we terminate the series at $j = 0$ because the next term $a_1 x^1$ in the summation would produce a term $a_1 x^2$ which we cannot have, because the highest power of x is 1. Note that we are now using the $\alpha = 1$ series for odd l (see discussion above). To determine a_0, again use the normalization

$$P_1(+1) = 1 = a_0(1) = a_0$$

which means that $a_0 = 1$, and so we get

$$P_1(x) = x$$

- $P_2(x)$: Here, $l = 2$, so the highest power of x is 2. This is an even series, so we use $\alpha = 0$. So, this time we have

$$P_2(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j = x^0 \left[a_0 x^0 + a_2 x^2\right] = a_0 + a_2 x^2$$

Use the recurrence relation in equation (3.14), with $j = 0, l = 2$ to express a_2 in terms of a_0:

$$a_{0+2} = \left[\frac{(0)(1) - 2\,(2+1)}{(1)(2)}\right] a_0$$

which means $a_2 = -3a_0$. So, we get

$$P_2(x) = -3a_0 x^2 + a_0$$

where I've swapped the terms to write them in decreasing powers of x. To determine a_0, use the normalization

$$P_2(+1) = 1 = -3a_0 + a_0 = -2a_0$$

which means that $a_0 = -1/2$, and so we get

$$P_2(x) = -3\left(-\frac{1}{2}\right) x^2 + \left(-\frac{1}{2}\right)$$

or

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

- $P_3(x)$: Here, $l = 3$, so the highest power of x is 3. Meanwhile, $\alpha = 1$ because this is an odd series. So, we must have

$$P_3(x) = x^1 \left[a_0 + a_2 x^2\right] = a_0 x + a_2 x^3$$

Using the recurrence relation with $\alpha = 1$ (because this is an odd series), $j = 0, l = 3$

$$a_2 = \left[\frac{(1)(2) - 3\,(3+1)}{(2)(3)}\right] a_0 = -\frac{5}{3}\, a_0$$

and using the normalization to determine a_0:

$$P_3(+1) = 1 = -\frac{5}{3}\, a_0 + a_0 = -\frac{2}{3}\, a_0$$

so that $a_0 = -3/2$. Therefore

$$P_3(x) = -\frac{5}{3}\left(-\frac{3}{2}\right) x^3 - \frac{3}{2}\, x = \frac{1}{2}\left(5x^3 - 3x\right)$$

Therefore, the first few Legendre polynomials are

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \\
P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right) \\
P_4(x) &= \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)
\end{aligned} \tag{3.15}$$

Remember that in all of these, $x = \cos\theta$.

Try deriving the last one as an exercise.

Legendre polynomials have odd or even symmetry about the origin:

$$P_l(-x) = (-1)^l \, P_l(x)$$

A compact generating function for Legendre polynomials, known as the Rodrigues' formula, is

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} \left(x^2 - 1\right)^l \tag{3.16}$$

The Legendre polynomials form a complete orthogonal set of functions on the interval $-1 \le x \le 1$, and the orthogonality condition is

$$\int_{-1}^{1} P_{l'}(x) \, P_l(x) \, dx = \frac{2}{2l+1} \, \delta_{l'l} \tag{3.21}$$

If you remember the orthogonal functions and expansions discussion from the previous chapter, then you'll know that since the Legendre polynomials form a complete set of orthogonal functions, any function $f(x)$ in the interval $-1 \le x \le 1$ can be expanded in terms of these polynomials:

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \tag{3.23}$$

where

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) \, P_l(x) \, dx \tag{3.24}$$

Now let's apply all this to finding the potential in an electrostatic problem.

Boundary value problems with azimuthal symmetry

The general solution to the Laplace equation $\nabla^2\Phi = 0$ in spherical coordinates (r, θ, ϕ) for a problem possessing azimuthal symmetry ($m = 0$) is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\theta) \tag{3.33}$$

where the coefficients A_l and B_l are determined from boundary conditions. Note that azimuthal symmetry means that the problem includes the entire azimuthal range, with ϕ going from 0 to 2π. The boundary conditions must not depend on the azimuthal angle.

Consider, for example, a boundary value problem in which the potential is specified to be $V(\theta)$ on the surface of a sphere of radius a. Our objective is to find the potential inside the sphere.

First, since the problem has azimuthal symmetry, the solution for the potential inside the sphere would be given by equation (3.33). To complete the problem, we need to determine A_l and B_l. To do so, consider that the potential must be finite at the origin (as there are no charges there), so since the B_l's would all blow up at the origin, we cannot have them in our solution. So, we set $B_l = 0$ for all l. This means that the potential inside the sphere will be given by

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l \, P_l(\cos\theta)$$

We are given that the potential on the surface of the sphere (i.e., at $r = a$) is $V(\theta)$, so the above equation becomes

$$\Phi(a,\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = V(\theta) \tag{3.34}$$

This is just a Legendre series of the form in equation (3.23) that we wrote moments ago, so the coefficients A_l can be determined by invoking the orthogonality of the Legendre polynomials. So multiply equation (3.34) on both sides by $P_{l'}(\cos\theta)$ and apply the orthogonality condition:

$$\sum_{l=0}^{\infty} A_l a^l \int_0^{\pi} P_l(\cos\theta)\, P_{l'}(\cos\theta)\, \sin\theta\, d\theta = \int_0^{\pi} V(\theta) P_{l'}(\cos\theta)\, \sin\theta\, d\theta$$

Orthogonality dictates that only the $l = l'$ term survives, which takes out the summation, and we get

$$A_l\, a^l \left[\frac{2}{2l+1} \right] = \int_0^{\pi} V(\theta)\, P_l(\cos\theta)\, \sin\theta\, d\theta$$

so that

$$A_l = \frac{2l+1}{2a^l} \int_0^{\pi} V(\theta)\, P_l(\cos\theta)\, \sin\theta\, d\theta \tag{3.35}$$

We'll have to leave the solution in closed integral form like this, unless we're given the form that V takes on the surface, in which case we could actually carry out the integrations. Now we'll move on to problems that don't have azimuthal symmetry, in which case we have to consider the generalized Legendre equation equation (3.9).