PHY 420 **Spring 2021**

Class Summary—Week 2, Day 2—Thursday, Apr 8

Laplace Equation in Cylindrical Coordinates

In PHY 411, we discussed solutions to the Laplace equation $\nabla^2\Phi = 0$ in rectangular and spherical geometries, and I had mentioned at the time that we would leave the cylindrical case for this quarter.

In cylindrical coordinates (ρ, ϕ, z), **Laplace's equation**, $\nabla^2\Phi = 0$, is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

By differentiating the first term, we can put this into the form in Jackson:

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{3.71}$$

as you demonstrated in Question 1(a) of today's worksheet.

Separate variables by setting

$$\Phi(\rho, \phi, z) = R(\rho)\, Q(\phi)\, Z(z) \tag{3.72}$$

On today's worksheet, you showed in Question 1(b) that this leads to the three ordinary differential equations (assuming non-periodic boundary condition on z; we'll deal with the case of periodic boundary condition on z later):

$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \tag{3.73}$$

$$\frac{d^2Q}{d\phi^2} + \nu^2 Q = 0 \tag{3.74}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0 \tag{3.75}$$

The solutions to equations (3.73) and (3.74) are, respectively

$$\begin{aligned} Z(z) &= e^{\pm kz} \\ Q(\phi) &= e^{\pm i\nu\phi} \end{aligned} \tag{3.76}$$

as you verified by explicit substitution in Question 2(a) of today's worksheet. Notice here that the z-solution is dictated by our choice of constant after separation of variables. Later, we will also examine cases for which $Z(z) = e^{\pm ikz}$ (i.e., examples with periodic boundary conditions in z).

For the potential to be single-valued when the full azimuthal span is allowed, ν must be an integer. However, the parameter k is arbitrary although, as always, it might be constrained by some boundary condition requirement in the z-direction. For now, assume that k is real and positive.

The radial equation in (3.75) can be put into a standard form for which we know the solutions from mathematics. To do so, divide by k^2 and carry out the following rearrangement:

$$\frac{d^2R}{d(k\rho)^2} + \frac{1}{k\rho}\,\frac{dR}{d(k\rho)} + \left(1 - \frac{\nu^2}{[k\rho]^2}\right) R = 0$$

as you did in Question 2(b) on today's worksheet. Then, change variables and put $x = k\rho$ in the radial equation. This gives

$$\frac{d^2R}{dx^2} + \frac{1}{x}\,\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0 \tag{3.77}$$

Equation (3.77) is known as the **Bessel equation** and its solutions are known as the ***Bessel functions*** of order ν.

Assuming a power series solution of the form

$$R(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j \tag{3.78}$$

allows us to find that

$$\alpha = \pm\nu \tag{3.79}$$

and

$$a_{2j} = -\frac{1}{4j(j+\alpha)}\,a_{2j-2} \tag{3.80}$$

for $j = 1, 2, 3, \ldots$, implying that all odd powers of x^j have vanishing coefficients. You will demonstrate both these results in homework.

To get the Bessel function solutions, we iterate the recursion formula (3.80) in steps of $(j-1)$ down to the first term:

$$\begin{aligned}
a_{2j} &= -\frac{1}{4j(j+\alpha)}\,a_{2j-2} \\
&= -\frac{1}{4j(j+\alpha)}\left[-\frac{1}{4(j-1)\,(j-1+\alpha)}\,a_{2j-4}\right] \\
&= +\frac{1}{4j(j+\alpha)}\,\frac{1}{4(j-1)\,(j-1+\alpha)}\left[-\frac{1}{4(j-2)\,(j-2+\alpha)}\,a_{2j-6}\right] \\
&= \ldots
\end{aligned}$$

We can see some part of the pattern already: each step down contributes a factor of 2^2, so that by the time you get to $a_{2j-2j} \equiv a_0$, which should take j steps, we would get a factor 2^{2j}.

Also, every step down involves an alternating sign: a_{2j} to a_{2j-2} is a $-$, a_{2j} to a_{2j-4} is a $(-)(-) = +$, a_{2j} to a_{2j-4} is a $(-)(-) = +$, and so on. In general, we can represent this by writing $(-1)^j$.

And, in the first step, $a_{2j-2} \equiv a_{2(j-1)}$ on the right hand side has $j(j+\alpha)$ with it, so the final step with $a_0 \equiv a_{2(0)}$ on the right hand side should have $1(1+\alpha)$ with it.

On the next page, we will rearrange the terms in the expression for a_{2j} with these patterns in mind.

Rearranging the terms in the expression for a_{2j} by using the patterns written at the bottom of the previous page, we get:

$$a_{2j} = \frac{(-1)^j}{2^{2j}} \left[\frac{1}{j(j-1)(j-2)\,\ldots\,1}\right] \left[\frac{1}{(\alpha+j)(\alpha+j-1)(\alpha+j-2)\,\ldots\,(\alpha+1)}\right] a_0$$

or

$$a_{2j} = \frac{(-1)^j}{2^{2j}} \left[\frac{1}{j!}\right] \left[\frac{1}{(\alpha+j)(\alpha+j-1)(\alpha+j-2)\,\ldots\,(\alpha+1)}\right] a_0$$

as you found in Question 3(a) of today's worksheet.

In Question 3(b) of today's worksheet, you showed that the expression above can be written in a more compact form:

$$a_{2j} = \frac{(-1)^j}{2^{2j}\, j!} \frac{\Gamma(\alpha+1)}{\Gamma(j+\alpha+1)}\, a_0 \tag{3.81}$$

where the **Gamma function**, Γ, is defined as

$$\Gamma(p) = (p-1)!$$

The series in equation (3.81) doesn't need to be terminated, unlike the Legendre polynomials. Instead, by convention, we choose

$$a_0 = \frac{1}{2^\alpha\, \Gamma(\alpha+1)}$$

and putting this expression for a_0 in equation (3.81), we get

$$a_{2j} = \frac{(-1)^j}{2^{2j}\, j!} \frac{\Gamma(\alpha+1)}{\Gamma(j+\alpha+1)} \left[\frac{1}{2^\alpha\, \Gamma(\alpha+1)}\right]$$

or

$$a_{2j} = \frac{(-1)^j}{2^{2j}\, j!} \frac{1}{\Gamma(j+\alpha+1)} \left[\frac{1}{2^\alpha}\right]$$

Putting this expression for a_{2j} in the power series solution for $R(x)$ written in equation (3.78), but written now with $2j$ instead of j because we know that only the even powers are present, we get

$$\begin{aligned} R(x) &= x^\alpha \sum_{j=0}^{\infty} a_{2j}\, x^{2j} \\ &= x^\alpha \sum_{j=0}^{\infty} \left[\frac{(-1)^j}{2^{2j}\, j!} \frac{1}{\Gamma(j+\alpha+1)} \left(\frac{1}{2^\alpha}\right)\right] x^{2j} \\ &= \left(\frac{x}{2}\right)^\alpha \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\; \Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j} \end{aligned}$$

Since $\alpha = \pm\nu$ from equation (3.79), this means that we will get two power series for $R(x)$, one for ν and the other for $-\nu$.

With $\alpha = \pm\nu$ from equation (3.79), the solution for $R(x)$ that we derived on the previous page can be written as

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\;\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j} \tag{3.82}$$

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\;\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j} \tag{3.83}$$

The solutions $J_\nu(x)$ and $J_{-\nu}(x)$ are called **Bessel functions of the first kind** of order $\pm\nu$.

If ν is not an integer, these two solutions $J_{\pm\nu}(x)$ form a pair of linearly independent solutions to the second order Bessel equation.

On the other hand, if ν is an integer, the solutions are linearly dependent; for $\nu = m$, an integer, we get

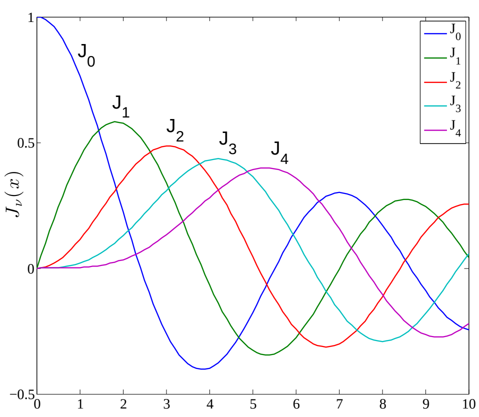
$$J_{-m}(x) = (-1)^m J_m(x) \tag{3.84}$$

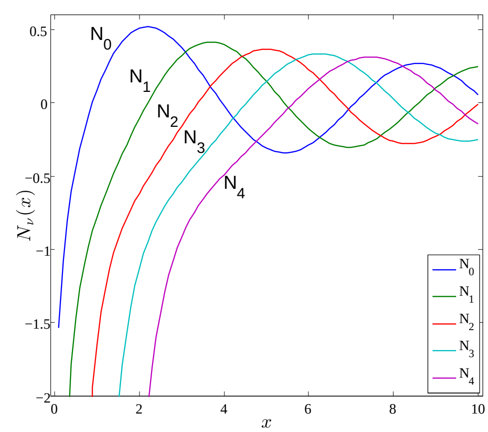
So, if ν is an integer, we need to find another linearly independent solution.

In such cases, and more generally even if ν is not an integer, it is customary to replace the pair $J_{\pm\nu}(x)$ by $J_\nu(x)$ and $N_\nu(x)$, the so-called **Neumann function**, or **Bessel function of the second kind**:

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \tag{3.85}$$

Plots of $J_\nu(x)$ and $N_\nu(x)$ are shown below.





From the point of view of using them in electrodynamics, it is important to remember that Bessel functions of the second kind $N_\nu(x)$ diverge at $x = 0$. So, in problems where the geometry includes the origin, we cannot have $N_\nu(x)$ in the solution.