PHY 411 **Winter 2021**

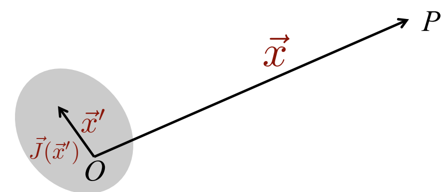
Week 9—Tuesday, Mar 2—Discussion Worksheet

Magnetic Fields of a Localized Current Distribution

In the previous class, we used $\vec{B} = \vec{\nabla} \times \vec{A}$, and the form of $\vec{B}$ in equation (5.16), to write the *vector potential* $\vec{A}(\vec{x})$ as

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x' \tag{5.32}$$

Consider a localized current distribution as shown in the figure on the right (taken from Figure 5.6 on page 185 in Jackson). Assuming that the distance to the observation point $|\vec{x}| \gg |\vec{x}\,'|$, the dimensions of the source, we will expand the denominator of equation (5.32) in powers of $\vec{x}\,'$ measured relative to a suitable origin in the localized current distribution:



$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{(|\vec{x}|^2 - 2\vec{x}\cdot\vec{x}\,' + |\vec{x}\,'|^2)^{1/2}} = \frac{1}{|\vec{x}|}\left[1 - \frac{2\vec{x}\cdot\vec{x}\,'}{|\vec{x}|^2} + \frac{|\vec{x}\,'|^2}{|\vec{x}|^2} + \dots\right]^{-1/2} \approx \frac{1}{|\vec{x}|}\left[1 + \frac{\vec{x}\cdot\vec{x}\,'}{|\vec{x}|^2} + \dots\right]$$

so that

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}\,'}{|\vec{x}|^3} + \dots \tag{5.50}$$

Then, the ith component of the vector potential will have the expansion

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{1}{|\vec{x}|}\int J_i(\vec{x}\,')\, d^3x' + \frac{\vec{x}}{|\vec{x}|^3}\cdot\int J_i(\vec{x}\,')\,\vec{x}\,'\, d^3x' + \dots\right] \tag{5.51}$$

1. By referring back to the electrostatic expansion in Chapter 4 that we discussed last week, you can figure that the first term in equation (5.51) corresponds to the magnetic field generated by a monopole. However, there are no magnetic monopoles, so this term should be zero. Prove this mathematically by showing that

$$\int J_i(\vec{x}\,')\, d^3x' = 0$$

Note: Jackson goes about this in a complicated manner on page 185. Use a simpler method by writing $\vec{\nabla}\,' \cdot [x_i'\,\vec{J}(\vec{x}\,')]$ as two terms, and then applying the physics.

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{1}{|\vec{x}|}\int J_i(\vec{x}\,')\, d^3x' + \frac{\vec{x}}{|\vec{x}|^3}\cdot\int J_i(\vec{x}\,')\, d^3x' + \dots\right]$$

$$\vec{\nabla}\cdot(a\vec{J}) = \vec{\nabla}a\cdot\vec{J} + a\vec{\nabla}\cdot\vec{J} = \vec{\nabla}a\cdot\vec{J}$$

$$\int d^3x'\, \vec{\nabla}\cdot(x_i\vec{J}(\vec{x}\,')) = \int d\vec{x}\,'\, x_i' \cdot \vec{J}(\vec{x}\,') = \boxed{\int d^3x'\, J_i(\vec{x}\,') = 0}$$

$$\vec{\nabla}\cdot\vec{J} = 0$$

2. Without looking at Jackson, show that the integral in the second term of equation (5.51) can be written as

$$\vec{x} \cdot \int \vec{x}\,' J_i(\vec{x}\,')\, d^3x' = -\frac{1}{2}\left[\vec{x} \times \int \left(\vec{x}\,' \times \vec{J}\right) d^3x'\right]_i$$

Note: You will need the following:

$$\int \left(x_i' J_j + x_j' J_i\right) d^3x' = 0$$

$$= \sum_{j=1}^{3} \vec{x}_j \int \left(\vec{x}_j{}' J_i(\vec{x}\,')\right) d^3x'$$

where $\int d^3x'\, x_j' J_i = -\int d^3x'\, x_i' J_j$

$$= \frac{1}{2} \sum_{j=1}^{3} x_j \int d^3x' \left(x_i' J_j + x_j' J_i\right)$$

$$= -\frac{1}{2} \sum_{j=1}^{3} x_j \sum_{k=1}^{3} \varepsilon_{ijk} \int \left(\vec{x}\,' \times \vec{J}\right)_k d^3x'$$

$$= -\frac{1}{2}\left[\vec{x} \times \int d^3x' \left(x' \times \vec{J}\right) d^3x'\right]_i$$

If we keep only the second term in the expansion in equation (5.51) and drop all higher terms, and put back all three components $i = x, y, z$ to write $\vec{A}(\vec{x})$, the vector potential is

$$\vec{A}(\vec{x}) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\left\{\vec{x}\times\int\left(\vec{x}'\times\vec{J}\right)d^3x'\right\} = \frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\left[\int\left(\vec{x}'\times\vec{J}\right)d^3x'\right]\times\vec{x}$$

3. Now, recall equation (4.10) in Chapter 4 where the dipole term in the multipole expansion for the electrostatic field is written as

$$\Big[\Phi(\vec{x})\Big]_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{|\vec{x}|^3}$$

We can write a similar expression for the vector potential $\vec{A}(\vec{x})$ if we define the magnetic dipole moment $\vec{m}$:

$$\vec{m} = \frac{1}{2}\int\vec{x}'\times\vec{J}(\vec{x}')\,d^3x' \tag{5.54}$$

(a) Given $\vec{m}$ above, write down $\vec{A}(\vec{x})$.

$$\vec{A}(\vec{x}) = \frac{\mu_0(\vec{m}\times\vec{x})}{4\pi|\vec{x}|^3} = \frac{\mu_0}{4\pi}\frac{(\vec{m}\times\vec{x})}{\vec{x}^3}$$

Note that your answer is the lowest non vanishing term in the expansion of $\vec{A}$ for a localized steady-state current distribution. By evaluating directly the curl of $\vec{A}(\vec{x})$, we can show that $\vec{B}$ is given by

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{n}(\hat{n}\cdot\vec{m})-\vec{m}}{|\vec{x}|^3}\right] \tag{5.56}$$

where $\hat{n}$ is a unit vector in the direction of $\vec{x}$. $\vec{B}$ has exactly the form of the field of a dipole. Far away from any localized current distribution, $\vec{B}$ is that of a magnetic dipole with dipole moment given by equation (5.54).

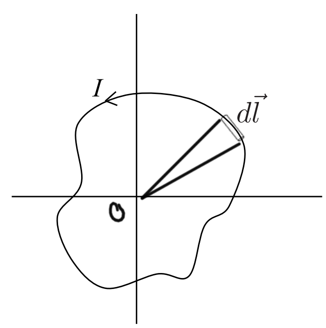
(b) If current I flows in a closed circuit loop whose line element is $d\vec{l}$ (as shown in the figure below), show that equation (5.54) becomes

$$\vec{m} = \frac{I}{2}\oint\vec{x}'\times d\vec{l}'$$

Current density over area

where $da = 1/2\,\vec{x}'\times d\vec{l}'$

I · area of loop



(c) Discuss that the magnetic moment then has magnitude

$$\left|\vec{m}\right| = I\times(\text{area of the loop}) \tag{5.57}$$

$\frac{1}{2}(\vec{x}\times d\vec{l})$ is the area "da" of the shaded portion, so integrating over the loop gives: $|\vec{m}| = I\times$ (area of the loop).

Macroscopic Equations

So far, we have assumed that the current density $\vec{J}$ is a known function of position, but in macroscopic problems this is often not true. The electrons in atoms in matter often cause effective atomic currents, and their current density is a rapidly fluctuating quantity. Only its average over a macroscopic volume is known or pertinent. Furthermore, atomic electrons contribute intrinsic magnetic moments in addition to those from their orbital motion; these moments can give rise to dipole fields that vary appreciably on atomic scales.

Just as in electrostatics, the averaging of the microscopic equation $\vec{\nabla} \cdot \vec{B}_{\text{micro}} = 0$ leads to the same equation

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{5.75}$$

for the macroscopic $\vec{B}$. This is good, because it allows us to use the concept of a vector potential $\vec{A}(\vec{x})$ whose curl gives $\vec{B}$.

The large number of atoms or molecules per unit volume, each with its molecular magnetic moment $\vec{m}_i$ gives rise to an average macroscopic magnetization or magnetic moment density

$$\vec{M}(\vec{x}) = \sum_i N_i \langle \vec{m}_i \rangle \tag{5.76}$$

where N_i is the average number per unit volume of molecules of type i and $\langle \vec{m}_i \rangle$ is the average molecular moment in a small volume at the point $\vec{x}$. Such a magnetization may be induced, or it may already exist if we're dealing with permanent magnets. The freshman-level picture is to think of numerous small regions of magnetic moments that are called domains. If we're not dealing with permanent magnets, then the magnetic moments of these domains point in random directions and cancel each other out on average. However, when an external magnetic field is applied, or a current density creates an applied magnetic field, then the domains feel a torque and align, creating an average magnetization.

In addition to the bulk magnetization, we assume that there is a macroscopic current density $\vec{J}(\vec{x})$ from the flow of free charges in the medium. Then the vector potential from a small volume ΔV at the point $\vec{x}\,'$ will be

$$\Delta \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \left[\underbrace{\frac{\vec{J}(\vec{x}\,')\,\Delta V}{|\vec{x} - \vec{x}\,'|}} + \underbrace{\frac{\vec{M}(\vec{x}\,') \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3}\,\Delta V} \right]$$

4. After reading the content of this page, write down the cause (or source) of the first and second terms in the expression for $\vec{A}(\vec{x})$ above.

First term is due to the macroscopic current density $\vec{J}$ due to the flow of free charges in the medium.

The second term is the dipole vector potential, due to the existence of an average macroscopic magnetization

If we let ΔV become the macroscopically infinitesimal d^3x', then the total vector potential at $\vec{x}$ can be written as the integral over all space:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \left[\frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} + \frac{\vec{M}(\vec{x}\,') \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3} \right] d^3x' \tag{5.77}$$

5. Show that equation (5.77) can be written as

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,') + \vec{\nabla}' \times \vec{M}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x' \tag{5.78}$$

Hint: Use $\vec{\nabla}'\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3}$ (no, there is no typo; $\vec{\nabla}(\ldots)$ had a minus; $\vec{\nabla}'(\ldots)$ is $+$.)

$$\rightarrow \frac{\mu_0}{4\pi}\int \left[\frac{J(\vec{x}')}{|\vec{x} - \vec{x}'|} + \vec{M}(\vec{x}') \times \underbrace{\frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}}_{\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|}} \right] d^3x'$$

$$\rightarrow \frac{\mu_0}{4\pi}\int \left[\frac{J(\vec{x}')}{|\vec{x} - \vec{x}'|} + \vec{M}(\vec{x}') \times \vec{\nabla} \frac{1}{|\vec{x}-\vec{x}'|}\right.$$

$$\rightarrow \frac{\mu_0}{4\pi}\int \left[\frac{J(\vec{x}') + \vec{\nabla} \times \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right] d^3x'$$

From equation (5.78), we see that the magnetization is contributing an effective current density

$$\vec{J}_M = \vec{\nabla} \times \vec{M} \tag{5.79}$$

so that the macroscopic equivalent of $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ is

$$\vec{\nabla} \times \vec{B} = \mu_0 \Big[\vec{J} + \vec{\nabla} \times \vec{M} \Big] \tag{5.80}$$

implying that $(\vec{J} + \vec{J}_M)$ plays the role of the current density in the macroscopic equivalent.

6. The $\vec{\nabla} \times \vec{M}$ term in equation (5.80) can be combined with $\vec{B}$ to define a new macroscopic field $\vec{H}$, often called the magnetic field

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \tag{5.81}$$

To complete the description, there must be a constitutive relation between $\vec{B}$ and $\vec{H}$, analogous to electrostatics. For isotropic diamagnetic and paramagnetic substances, there is a simple linear relation

$$\vec{B} = \mu\vec{H} \tag{5.84}$$

where μ is called the magnetic permeability, and is a characteristic parameter of the medium.

(a) Derive an expression for $\vec{\nabla} \times \vec{H}$.

$$\nabla \times B = \mu_0 [J + \nabla \times M]$$

$$\rightarrow \nabla \times \frac{B}{\mu_0} = J + \nabla \times M$$

$$\rightarrow \nabla \times \frac{B}{\mu_0} - \nabla \times M = J$$

$$\nabla \times \underbrace{\left(\frac{B}{\mu_0} - M\right)}_{H} = J$$

$$\boxed{\nabla \times H = J}$$

(b) Show that

$$\vec{M} = \left[\frac{\mu}{\mu_0} - 1\right]\vec{H}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\rightarrow H = \frac{\mu\vec{H}}{\mu_0} - \vec{M}$$

$$\rightarrow \vec{M} = \frac{\mu\vec{H}}{\mu_0} - \vec{H}$$

$$\rightarrow \boxed{\vec{M} = \left(\frac{\mu}{\mu_0} - 1\right)\vec{H}}$$

(c) In paramagnetic materials, $\mu > \mu_0$. What does this tell us about the direction of $\vec{M}$ in relation to the direction of $\vec{H}$?

$\vec{M} = +\vec{H}$

Same direction

(d) In diamagnetic materials, $\mu < \mu_0$. What does this tell us about the direction of $\vec{M}$ in relation to the direction of $\vec{H}$?

$\vec{M} = -\vec{H}$

opposite direction