PHY 420 **Spring 2021**

Week 1—Tuesday, Mar 30—Discussion Worksheet

Energy and Energy Density

Given the importance of energy, we will begin this quarter with a unifying focus on the idea of energy in the electrostatic, magnetic, and electromagnetic fields.

Electrostatic Potential Energy and Energy Density

Recall that the scalar potential has a physical interpretation when we consider the work done on a test charge q in transporting it from a point A to another point B in the presence of an electric field $\vec{E}(\vec{x})$. This is discussed by Jackson on pages 30-31; the *key conclusion from that discussion* is that $q\Phi$ can be interpreted as the potential energy of the test charge in the electrostatic field. You should also remember that we take the zero of potential energy to be when the charge is located at infinite distance from the system under consideration.

1. If a point charge q_i is brought from infinity to a point $\vec{x}_i$ in a region of localized electric fields described by the scalar potential Φ (which vanishes at infinity), the work done on the charge (and hence its potential energy) is given by

$$W_i = q_i\Phi(\vec{x}_i) \tag{1.47}$$

If the potential Φ is produced by an array of $(n-1)$ charges q_j located at positions $\vec{x}_j$, where $j = 1, 2, 3, \ldots, (n-1)$, then show that

$$W = \frac{1}{8\pi\epsilon_0}\sum_i\sum_j \frac{q_iq_j}{|\vec{x}_i - \vec{x}_j|} \tag{1.51}$$

where it is understood that the $(i = j)$ terms are omitted in the double sum, because the $(i = j)$ terms would correspond to infinite self-energy of a charge interacting with itself.

$$\phi(\vec{X}_i) = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{n-1}\frac{q_i}{|\vec{X}_i - \vec{X}_j|} \quad (1.48)$$

Sub 1.48 into 1.47

$$W_i = q_i/4\pi\epsilon_0 \sum_{j=1}^{n-1}\frac{q_i}{|\vec{X}_i - \vec{X}_j|}$$

Adding n charge

$$W = \sum_{i=1}^{n} W_i = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_iq_j}{|\vec{X}_i - \vec{X}_j|}$$

$$\rightarrow W = \frac{1}{8\pi\epsilon_0}\sum_i\sum_j\frac{q_i - q_j}{|\vec{X}_i - X_j|}$$

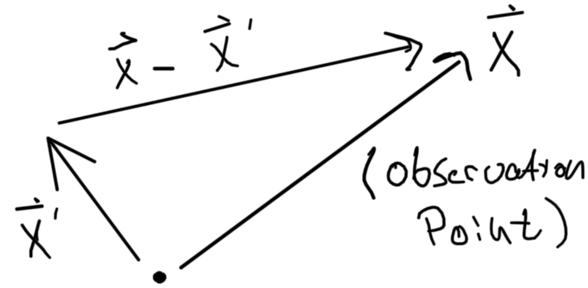
2. For a continuous charge distribution, we can put $q_i \to \rho(\vec{x})d^3x$ and $q_j \to \rho(\vec{x}\,')\,d^3x'$.

(a) With this, and appropriate replacements in equation (1.51), show that we get

$$W = \frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x \tag{1.53}$$

$$W = \frac{1}{2}\int d^3x \underbrace{\int d^3x' \frac{\rho(\vec{x})\,\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} \cdot \overset{\frac{1}{4\pi\epsilon_0}}{C}}_{\varphi(\vec{x})}$$

$$= \frac{1}{2}\int \rho(\vec{x})\,\varphi(\vec{x})\,d^3x$$



(b) Equations (1.51) and (1.53) express the electromagnetic potential energy *in terms of the charges*, emphasizing the interactions between them via Coulomb forces. An alternative approach is to emphasize the electric field and to interpret the energy as being stored in the electric field surrounding the charges. Using the fact that the potential Φ must satisfy the Poisson equation equation (1.28):

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \to \rho = -\epsilon_0$$

show that you obtain

$$W = -\frac{\epsilon_0}{2}\int \Phi\nabla^2\Phi\,d^3x$$

$$W = \frac{1}{2}\int \rho(\vec{x})\,\varphi(\vec{x})\,d^3x$$

where $\rho = -\epsilon_0 \nabla^2 \varphi$

$$= \frac{1}{2}\int -\epsilon_0\,\varphi\,\nabla^2\varphi\,d^3x$$

$$W = -\frac{\epsilon_0}{2}\int \varphi\,\nabla^2\varphi\,d^3x$$

3. The expression in Question 2(b) allows us to write the potential energy in terms of the electric field. Using

$$\vec{\nabla} \cdot \left(\Phi \vec{\nabla} \Phi\right) = \left|\vec{\nabla} \Phi\right|^2 + \Phi \nabla^2 \Phi$$

show that

$$\vec{E} = - \nabla \varphi$$

$$W = \frac{\epsilon_0}{2} \int \left|\vec{E}\right|^2 d^3x \tag{1.54}$$

where the integration is over all space.

$$W = -\frac{\epsilon_0}{2} \int \varphi \nabla^2 \varphi \, d^3x \quad (2b)$$

$$\rightarrow \frac{\epsilon_0}{2} \int \vec{\nabla} \cdot (\varphi \vec{\nabla} \varphi) + |\vec{\nabla} \varphi|^2 \, d^3x$$

$$\rightarrow \nabla \cdot \vec{E} = 0 \text{ and } \vec{\nabla} \varphi = \vec{E}$$

$$\rightarrow \frac{\epsilon_0}{2} \int |\vec{E}|^2 \, d^3x$$

In equation (1.54), all explicit reference to charges has gone, and the potential energy is expressed as an integral of the square of the electric field over all space. This leads naturally to the identification of the integrand as an energy density w:

$$w = \frac{\epsilon_0}{2} \left|\vec{E}\right|^2 \tag{1.55}$$

This is intuitively reasonable, since regions of high field must contain considerable energy. Note one puzzling aspect of equation (1.55), however — the energy density is positive definite. Consequently, its volume integral is nonnegative. This seems to contradict our expression in equation (1.51) that the potential energy of two charges of opposite sign is negative. The reason for this apparent contradiction is that equation (1.54) and equation (1.55) contain "self-energy" contributions to the energy density, whereas the double sum in equation (1.51) does not.

Electrostatic Energy in Dielectric Media

In the previous section, we discussed the energy of a system of charges in free space, and obtained the result

$$W = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x \tag{4.83}$$

for the energy due to a charge density $\rho(\vec{x})$ and a potential $\Phi(\vec{x})$. This cannot be directly taken over to macroscopic dielectric media, because we assembled the elemental charges bit by bit by bringing each one in from infinitely far away against the action of the then existing electric field. With dielectric media, work is done not only to bring real (macroscopic) charge into position, but also to produce a certain state of polarization in the medium.

Therefore, consider a small change in the energy δW due to some sort of change $\delta\rho$ in the macroscopic charge density ρ that exists in all space. The work done to accomplish this change is

$$\delta W = \int \delta\rho(\vec{x})\, \Phi(\vec{x})\, d^3x \tag{4.84}$$

where $\Phi(\vec{x})$ is the potential due to the charge density $\rho(\vec{x})$ already present. Since $\vec{\nabla} \cdot \vec{D} = \rho$, we can relate the change $\delta\rho$ to a change in the displacement of $\delta\vec{D}$:

$$\delta\rho = \vec{\nabla} \cdot (\delta\vec{D}) \tag{4.85}$$

4. By bringing $\vec{D}$ from an initial value $\vec{D} = 0$ to its final value $\vec{D}$, show that we can write a formal expression for the total electrostatic energy:

$$W = \int d^3x \int_0^D \vec{E} \cdot \delta\vec{D} \tag{4.87}$$

$$W = \frac{1}{2}\int \rho(\vec{x})\,\varphi(\vec{x})\, d^3x \rightarrow \delta W = \int \underbrace{\delta\rho(\vec{x})}_{\vec{\nabla}\cdot(\delta\vec{D})}\varphi(\vec{x})\, d^3x$$

$$\rightarrow \delta W = \int \underbrace{\vec{\nabla}\cdot\varphi}_{\vec{E}}\,(\delta\vec{D})\, d^3x$$

$$W = \int d^3x \int_0^D \vec{E}\cdot\delta\vec{D}$$

If the medium is linear, so that

$$\vec{E} \cdot \delta\vec{D} = \frac{1}{2}\,\delta\left(\vec{E} \cdot \vec{D}\right) \tag{4.88}$$

then it is straightforward using equation (4.87) to show that the total electrostatic energy is

$$W = \frac{1}{2} \int \vec{E} \cdot \vec{D}\, d^3x \tag{4.89}$$

A problem of considerable interest is the change in energy when a dielectric object is placed in an electric field whose sources are fixed. Suppose initially that there exists an electric field $\vec{E}_0$ due to a distribution of charges $\rho_0(\vec{x})$ in a medium with ϵ_0, which may be a function of position. The initial electrostatic energy is $W_0 = \frac{1}{2}\int \vec{E}_0 \cdot \vec{D}_0 \, d^3x$, where $\vec{D}_0 = \epsilon_0 \vec{E}_0$. Then, with the sources fixed in position, a dielectric object of volume V_1 is introduced into the field, changing the field from $\vec{E}_0$ to $\vec{E}$. Then $\epsilon(\vec{x})$ has the value ϵ_1 inside V_1 and ϵ_0 outside V_1. The electrostatic energy is then $W_1 = \frac{1}{2}\int \vec{E} \cdot \vec{D} \, d^3x$. where $\vec{D} = \epsilon_1 \vec{E}$. The change in the energy, $W = W_1 - W_0$, is given by

$$W = \frac{1}{2}\int \left(\vec{E} \cdot \vec{D} - \vec{E}_0 \cdot \vec{D}_0\right) d^3x \tag{4.90}$$

On Homework 1, you will show that this can be written as

$$W = \frac{1}{2}\int_{V_1} \left(\vec{E} \cdot \vec{D}_0 - \vec{D} \cdot \vec{E}_0\right) d^3x \tag{4.91}$$

where we need to carry out the integration only over the volume V_1 of the object, since, outside V_1, we have $\vec{D} = \epsilon_0 \vec{E}$ (meaning that the contribution to the integral is zero outside V_1).

5. If the medium surrounding the dielectric body is free space, then using the definition of polarization $\vec{P} = (\epsilon_1 - \epsilon_0)\,\vec{E}$ from equation (4.57), show that we can write equation (4.91) as

$$W = -\frac{1}{2}\int_{V_1} \vec{P} \cdot \vec{E}_0 \, d^3x \tag{4.93}$$

where $\vec{P}$ is the (induced) polarization of the dielectric that has been introduced into the field.

$$W = 1/2 \int_{V_1} \left(\vec{E} \cdot \vec{D}_0 - \vec{D} \cdot \vec{E}_0 \right) d^3x$$

$\vec{D}_0 = \epsilon_0 \vec{E}_0$ and $\vec{D} = \epsilon_1 \vec{E}$ inside volume V_1

$$W = 1/2 \int_{V_1} \left[\vec{E} \cdot (\epsilon_0 \vec{E}_0) - (\epsilon_1 \vec{E}) \cdot \vec{E}_0 \right] d^3x$$

$$= 1/2 \int_{V_1} \underbrace{(\epsilon_0 - \epsilon_1)\vec{E}}_{= -\vec{P}} \cdot \vec{E}_0 \, d^3x$$

Therefore,

$$W = -1/2 \int_{V_1} \vec{P} \cdot \vec{E}_0 \, d^3x$$

Equation (4.93) shows that the energy density of a dielectric placed in a field $\vec{E}_0$ whose sources are fixed is given by

$$w = -\frac{1}{2}\vec{P} \cdot \vec{E}_0 \tag{4.94}$$

This tells us that the energy goes down if the object moves into a region with a larger electric field E_0, provided $\epsilon_1 > \epsilon_0$. That is, a dielectric object will tend to move toward regions of increasing $\vec{E}_0$, provided $\epsilon_1 > \epsilon_0$. What, however, if *the potentials are kept fixed*?

On the previous page, we discussed the change in energy when a dielectric object is placed in an electric field whose sources are fixed. What, however, if *the potentials are kept fixed*? This is usually true in practical situations involving the motion of dielectrics, where the electric fields are often produced by a configuration of electrodes held at fixed potentials by connecting to an external source such as a battery.

The case when the potentials are kept fixed is ***discussed in the Class Summary for today and assigned on the homework***, but briefly, we can see that to maintain the potentials constant as the distribution of dielectric varies, charge will flow to or from the battery to the electrodes. This means that energy is being supplied from the external source, and we can compare the energy supplied in that way with the energy change we calculated on the previous page for fixed sources of the field. You will demonstrate on the homework that

$$\delta W_V = -\delta W_Q \tag{4.101}$$

where the subscript denotes the quantity being held fixed. Equation (4.101) tells us that the change in energy at fixed potentials is the negative of the energy change at fixed charges.

Energy in the Magnetic Field

In discussing steady-state magnetic fields in Chapter 5 last quarter, we avoided the question of field energy and energy density, because the creation of a steady-state configuration of currents and associated magnetic fields involves an initial transient period during which the currents and fields are brought from zero to the final values. Time-varying fields imply induced electromotive forces, via Faraday's law, that cause the sources of current to do work. Since the energy in the field is, by definition, the total work done to establish the fields from a state of zero field, we must consider these contributions. The development closely follows the treatment of dielectrics that we just discussed, so I'm going to save time by having you read it on your own in the posted class summary and assigning problems on the homework.