PHY 474 **Spring 2021**

Homework 2 solutions

1. The distribution of speeds in a gas is given by the Maxwell distribution

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right)$$

(a) The most probable speed, v_p, of this distribution can be found by doing

$$\frac{df(v)}{dv} = 0$$

By doing this explicit differentiation, show that

$$v_p = \sqrt{\frac{2kT}{m}}$$

Solution: Differentiating $f(v)$ with respect to v and setting it equal to zero, we get

$$\frac{df(v)}{dv} = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{d}{dv}\left[v^2 \exp\left(-\frac{mv^2}{2kT}\right)\right] = 0$$

where I've taken out constants or variables that don't need to be differentiated.

Differentiating as the product of two functions, this gives

$$\frac{df(v)}{dv} = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left[2v \exp\left(-\frac{mv^2}{2kT}\right) + v^2 \left(-\frac{m}{2kT}\right) \exp\left(-\frac{mv^2}{2kT}\right) \left(2v\right)\right] = 0$$

or

$$4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} 2v \left[1 + v^2 \left(-\frac{m}{2kT}\right)\right] \exp\left(-\frac{mv^2}{2kT}\right) = 0$$

Since none of the quantities outside the square brackets can be zero, we can solve for v by just setting the quantity in square brackets equal to zero, upon doing which we get

$$1 + v_p^2 \left(-\frac{m}{2kT}\right) = 0$$

where I've introduced explicitly a subscript into the symbol for the most probable speed.

Moving the second term on the left to the right hand side, we get

$$1 = v_p^2 \left(\frac{m}{2kT}\right)$$

so that

$$\frac{2kT}{m} = v_p^2$$

and, therefore, we get for the most probable speed that

$$v_p = \sqrt{\frac{2kT}{m}}$$

(b) The average speed, v_{avg}, of this distribution can be found by doing $\displaystyle\int v\, f(v)\, dv$.

By doing this integral, show that

$$v_{\text{avg}} = \sqrt{\frac{8kT}{\pi m}}$$

Solution: We have

$$v_{\text{avg}} = \int v\, f(v)\, dv = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty v \left[v^2 \exp\left(-\frac{mv^2}{2kT}\right)\right] dv$$

where I've taken quantities that don't need to be part of the integration outside the integral.

Thus, we have

$$v_{\text{avg}} = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 \exp\left(-\frac{mv^2}{2kT}\right) dv$$

We can use the standard integral

$$\int_0^\infty x^3 \exp(-ax^2)\, dx = \frac{1}{2a^2}$$

with $a = m/2kT$ to get

$$v_{\text{avg}} = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{1}{2\,(m/2kT)^2}$$

so that

$$v_{\text{avg}} = 2\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left(\frac{2kT}{m}\right)^2$$

or

$$v_{\text{avg}} = 2\pi \sqrt{\frac{1}{\pi^3}} \left(\frac{2kT}{m}\right)^{2-3/2}$$

Thus

$$v_{\text{avg}} = \sqrt{\frac{4\pi^2}{\pi^3}} \left(\frac{2kT}{m}\right)^{1/2} = \sqrt{\frac{4}{\pi}} \sqrt{\frac{2kT}{m}}$$

Therefore, the average speed is given by

$$v_{\text{avg}} = \sqrt{\frac{8kT}{\pi m}}$$

(c) Explain in words why v_p and v_{avg} are different for a Maxwell distribution.

Solution: The Maxwell distribution, as you can see from the plot in the posted class summary, is ***not*** symmetric about its peak. Instead, it has an extended tail on the higher velocity side of the distribution. This shifts the average speed to the right of the peak of the distribution. Since the most probable speed is at the peak of the distribution, we get that v_{avg} is different from v_p, and consistent with the previous parts (since $8/\pi = 2.5 > 2$), we must have $v_{\text{avg}} > v_p$.

2. *Dalsgaard* mentions that a simple solution to the equation of hydrostatic equilibrium can be obtained when ρ is a known function of r. Consider a linear density model

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right)$$

where ρ_c is the central density, and R is the radius of the star.

(a) By substituting this expression for ρ into equation (4.5) for dm/dr in *Dalsgaard*, find the total mass M of the star and hence show that the central density is given by

$$\rho_c = \frac{3M}{\pi R^3}$$

Solution: Starting from equation (4.5) in *Dalsgaard*:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

and integrating, we get

$$\int_0^M dm = \int_0^R 4\pi r^2 \, \rho \, dr = \int_0^R 4\pi r^2 \left[\rho_c \left(1 - \frac{r}{R}\right)\right] dr$$

where, in the expression on the extreme right, I've substituted for ρ using the linear density model given above. Then

$$m\Big|_0^M = 4\pi\rho_c \int_0^R r^2 \left(1 - \frac{r}{R}\right) dr$$

so that

$$M = 4\pi\rho_c \int_0^R \left(r^2 - \frac{r^3}{R}\right) dr$$

Integrating the right hand side, we get

$$M = 4\pi\rho_c \left[\frac{r^3}{3} - \frac{1}{R}\frac{r^4}{4}\right]_0^R$$

so that

$$M = 4\pi\rho_c \left[\left(\frac{R^3}{3} - 0\right) - \frac{1}{R}\left(\frac{R^4}{4} - 0\right)\right]$$

or

$$M = 4\pi\rho_c \left[\frac{R^3}{3} - \frac{R^3}{4}\right]$$

which gives

$$M = 4\pi\rho_c \left[\frac{4R^3 - 3R^3}{12}\right]$$

so that finally

$$M = \pi\rho_c \left[\frac{R^3}{3}\right]$$

Therefore, the central density ρ_c is given by

$$\rho_c = \frac{3M}{\pi R^3}$$

which is the desired result.

(b) Show that the mass interior to radius r is given by

$$m = M\left(4x^3 - 3x^4\right)$$

where M is the total mass of the star, and $x = r/R$.

Solution: Proceed similarly to part (a), but integrate from 0 to r instead of up to the full radius R of the star. Thus, starting again from $dm/dr = 4\pi r^2 \rho$, and substituting the linear relation for the density and integrating, we get

$$\int_0^{m(r)} dm = \int_0^r 4\pi r^2\, \rho\, dr = \int_0^r 4\pi r^2 \left[\rho_c \left(1 - \frac{r}{R}\right)\right] dr$$

so that

$$m(r) = 4\pi\rho_c \int_0^r \left(r^2 - \frac{r^3}{R}\right) dr$$

Again, integrating the right hand side, we get

$$m(r) = 4\pi\rho_c \left[\frac{r^3}{3} - \frac{1}{R}\frac{r^4}{4}\right]_0^r$$

so that

$$m(r) = 4\pi\rho_c \left[\frac{r^3}{3} - \frac{r^4}{4R}\right]$$

At this point, we need to proceed a little differently from part (a). Substitute $\rho_c = 3M/\pi R^3$ from the result in part (a) so that the expression above becomes

$$m(r) = 4\pi \left(\frac{3M}{\pi R^3}\right) \left[\frac{r^3}{3} - \frac{r^4}{4R}\right]$$

then move R^3 inside the square brackets to get

$$m(r) = 4\pi \left(\frac{3M}{\pi}\right) \left[\frac{r^3}{3R^3} - \frac{r^4}{4R^4}\right]$$

Putting $x = r/R$, this becomes

$$m(r) = 4\pi \left(\frac{3M}{\pi}\right) \left[\frac{x^3}{3} - \frac{x^4}{4}\right]$$

then working with the terms outside the square brackets, we get

$$m(r) = 12M \left[\frac{x^3}{3} - \frac{x^4}{4}\right]$$

so that

$$m(r) = M \left[\frac{12x^3}{3} - \frac{12x^4}{4}\right]$$

Therefore, we get finally that

$$m = M\left(4x^3 - 3x^4\right)$$

which is the desired result.

3. Consider again the linear density model in Question 2 above.

(a) Assuming $P = 0$ at the surface $r = R$, show that the pressure is given by

$$P = \frac{5}{4\pi}\frac{GM^2}{R^4}\left(1 - \frac{24}{5}x^2 + \frac{28}{5}x^3 - \frac{9}{5}x^4\right)$$

where, again, $x = r/R$.

Solution: Begin from equation (4.4) of stellar structure that says the pressure gradient must be given by

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}$$

and substitute for $m(r)$ from Question 2(b) above, and the linear density model given in Question 2(a), to get

$$\frac{dP}{dr} = -\frac{G}{r^2}\left[M\left(4x^3 - 3x^4\right)\right]\left[\frac{3M}{\pi R^3}\left(1 - x\right)\right]$$

where I've also substituted $\rho_c = 3M/\pi R^3$ from the result in Question 3(a), and put $x = r/R$. Thus

$$dP = -\frac{3GM^2}{\pi R^3}\left[\frac{\left(4x^3 - 3x^4\right)\left(1 - x\right)}{r^2}\right]dr$$

Now, multiply and divide r^2 in the denominator by R^2 so it can be written it as x^2. Thus

$$dP = -\frac{3GM^2}{\pi R^3}\left[\frac{\left(4x^3 - 3x^4\right)\left(1 - x\right)}{R^2\,(r/R)^2}\right]dr$$

or

$$dP = -\frac{3GM^2}{\pi R^5}\left[\frac{\left(4x^3 - 3x^4\right)\left(1 - x\right)}{x^2}\right]dr$$

which simplifies to

$$dP = -\frac{3GM^2}{\pi R^5}\left[\left(4x - 3x^2\right)\left(1 - x\right)\right]dr$$

or

$$dP = -\frac{3GM^2}{\pi R^5}\left[4x - 7x^2 + 3x^3\right]dr$$

Let's move an R from R^5 in the denominator to dr so that we can write it as dx. Thus

$$dP = -\frac{3GM^2}{\pi R^4}\left[4x - 7x^2 + 3x^3\right]\frac{dr}{R}$$

and since, $x = r/R$, we have $dx = dr/R$, so that

$$dP = -\frac{3GM^2}{\pi R^4}\left[4x - 7x^2 + 3x^3\right]dx$$

We are ready to integrate this expression.

Integrating the expression at the bottom of the previous page, we get

$$P = -\frac{3GM^2}{\pi R^4}\left[4\left(\frac{x^2}{2}\right) - 7\left(\frac{x^3}{3}\right) + 3\left(\frac{x^4}{4}\right)\right] + C$$

where C is a constant of integration.

At $r = R$, where $x = 1$, we must have $P = 0$, so that

$$0 = -\frac{3GM^2}{\pi R^4}\left[4\left(\frac{1}{2}\right) - 7\left(\frac{1}{3}\right) + 3\left(\frac{1}{4}\right)\right] + C$$

from which we get that

$$C = \frac{5}{4}\,\frac{GM^2}{\pi R^4}$$

Substituting this value for C and moving -3 into the square brackets, we get

$$P = \frac{GM^2}{\pi R^4}\left[-6\,x^2 + 7\,x^3 - \frac{9x^4}{4}\right] + \frac{5}{4}\,\frac{GM^2}{\pi R^4}$$

or

$$P = \frac{GM^2}{\pi R^4}\left[\frac{5}{4} - 6\,x^2 + 7\,x^3 - \frac{9x^4}{4}\right]$$

or

$$P = \frac{GM^2}{\pi R^4}\left[\frac{5 - 24\,x^2 + 28\,x^3 - 9x^4}{4}\right]$$

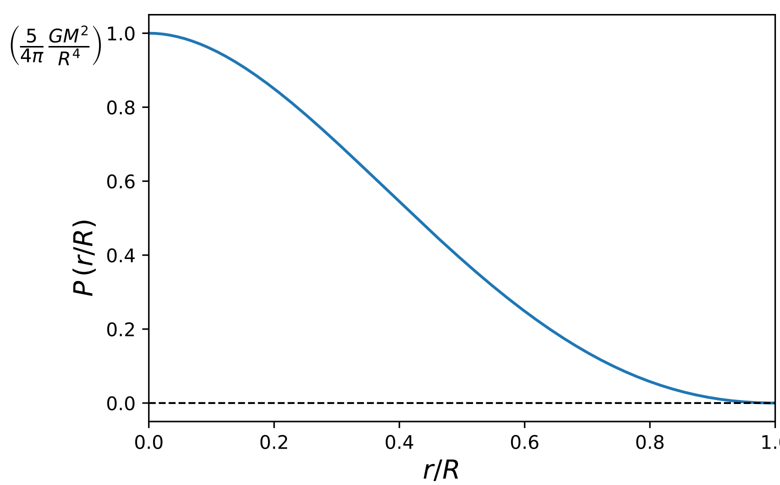
Therefore

$$P = \frac{5}{4\pi}\,\frac{GM^2}{R^4}\left(1 - \frac{24}{5}\,x^2 + \frac{28}{5}\,x^3 - \frac{9}{5}\,x^4\right)$$

which is the desired expression.

(b) Plot P *vs.* x. Plot with P in units of $(5/4\pi)\,GM^2/R^4$ for convenience.

Solution: The plot is shown below. The pressure shows the expected behavior, starting from its maximum value at the center of the star and going to zero at the surface.



4. The Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

where symbols are explained in *Dalsgaard* (and also the posted class summary), is derived from a polytropic relation, $P = K\rho^\gamma$, where the polytropic index $n = 1/(\gamma - 1)$.

(a) Show that the pressure in such a polytropic model is given by

$$P = P_c\,\theta^{n+1}$$

where P_c is the central pressure.

Solution: Substituting $\rho = \rho_c\,\theta^n$ in the polytropic relation, $P = K\rho^\gamma$, we get

$$P = K\rho^\gamma = K\Big[\rho_c\,\theta^n\Big]^\gamma = K\rho_c^\gamma\Big[\theta^n\Big]^\gamma = K\rho_c^\gamma\Big[\theta^n\Big]^{(n+1)/n} = K\rho_c^\gamma\,\theta^{n+1}$$

where I've put $\gamma = 1 + 1/n = (n+1)/n$. Now $K\rho_c^\gamma = P_c$, the central pressure, so we get

$$P = P_c\,\theta^{n+1}$$

which is the desired relation.

(b) The Lane-Emden equation has analytical solutions only for $n = 0, 1, 5$. Although the $n = 0$ solution is technically a singularity, it is useful to illustrate properties of polytropes. Show that the solution for $n = 0$ is

$$\theta = 1 - \frac{\xi^2}{6} \qquad \text{where} \qquad \xi_1 = \sqrt{6}$$

Recall that the surface is defined by the point $\xi = \xi_1$ where $\theta = 0$ (reflecting the fact that the pressure P is zero at the surface of the star).

Solution: For $n = 0$, the right hand side of the Lane-Emden equation is just -1, so after some rearrangement, we get

$$d\left(\xi^2\frac{d\theta}{d\xi}\right) = -\xi^2\,d\xi \qquad \text{which, upon integration, gives} \qquad \xi^2\frac{d\theta}{d\xi} = -\frac{\xi^2}{3} + c_1$$

where c_1 is a constant of integration. Because $\xi = \xi_1$ at $\theta = 0$, we have the boundary condition that $d\theta/d\xi = 0$ at $\xi = 0$, so we must have $c_1 = 0$. Thus, we now have

$$\frac{d\theta}{d\xi} = -\frac{\xi}{3} \qquad \text{so that} \qquad d\theta = -\frac{\xi}{3}\,d\xi$$

Integrating, we get

$$\theta = -\frac{1}{3}\,\frac{\xi^2}{2} + c_2 = c_2 - \frac{\xi^2}{6}$$

Since another boundary condition is that $\theta = 1$ at $\xi = 0$, we get that $c_2 = 1$, and thus

$$\theta = 1 - \frac{\xi^2}{6}$$

which is the desired relation.

Finally, since $\xi = \xi_1$ where $\theta = 0$, this also gives

$$0 = 1 - \frac{\xi_1^2}{6}$$

so that

$$\xi_1 = \sqrt{6}$$