Multi-component universes

(1) We begin to look at multicomponent universes by first coming up with some general results. You'll start by writing the Friedmann equation,

$$H(t)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{kc^2}{R_o^2 a(t)^2} \tag{1}$$

in a different form.

(a) Recalling that

$$\frac{k}{R_o^2} = \frac{H_o^2}{c^2}\left(\Omega_o - 1\right)$$

rewrite Eq. (1) without the curvature, k, or radius, R_o.

(b) Now divide your result in (a) by H_o^2 and use,

$$\epsilon_{c,o} = \frac{3c^2 H_o^2}{8\pi G}$$

to rewrite Eq. (1)

(c) Previously we defined the density parameter as

$$\Omega_w = \frac{\epsilon_w(t)}{\epsilon_{c,o}}$$

where the subscript indicates the different components. Now recall that

$$\begin{aligned} \epsilon(t)_{\text{matter}} &\rightarrow \frac{\epsilon_{\text{matter,o}}}{a^3} \\ \epsilon(t)_{\text{rad}} &\rightarrow \frac{\epsilon_{\text{rad,o}}}{a^4} \\ \epsilon(t)_{\Lambda} &\rightarrow \epsilon_{\Lambda,o} \end{aligned}$$

Use these results and the fact that $\epsilon_{\text{total}} = \sum_w \epsilon_w$, to rewrite your result in (b) in terms of the critical densities, Ω_w and scale factor, a.

(2) We'll first explore a universe that contains non-relativistic matter and curvature only. Recall that the key results from our analysis in the lecture were

$$\frac{H(t)^2}{H_o^2} = \frac{\Omega_{r,o}}{a^4} + \frac{\Omega_{m,o}}{a^3} + \Omega_{\Lambda,o} + \frac{1-\Omega_o}{a^2} \tag{2}$$

$$\frac{\dot{a}}{H_o} = \left[\frac{\Omega_{r,o}}{a^2} + \frac{\Omega_{m,o}}{a} + \Omega_{\Lambda,o}a^2 + (1-\Omega_o)\right]^{1/2} \tag{3}$$

$$t = \frac{1}{H_o}\int_0^a \frac{da'}{\left[\Omega_{r,o}/a^2 + \Omega_{m,o}/a + \Omega_{\Lambda,o}a^2 + (1-\Omega_o)\right]^{1/2}} \tag{4}$$

(a) Consider a universe that consists of non-relativistic matter and curvature only. Write the Friedmann equation (Eq (2) above) for this universe.

(b) Suppose this kind of universe that is initially expanding. Determine if this universe will contiue to expand forever. What is (are) the condition(s) for this universe to cease expanding. Write the Friedmann equation for this condition. (*Hint: Think about* $H(t)$). Reason what kind of curvature this universe must have.

(c) Find the maximum scale factor, a_{max} when this universe stops expanding.

(d) Using the Friedmann equation, determine the fate of this universe (i.e. matter + curvature) if now $\Omega_o < 1$, and $k = -1$.

(e) Use Eq. (4) above to set up the integrals that determine the age of the universe at given scale factor a. Note the integrals can be challenging to do.

(f) Recap what you have learned about a universe consisting of matter and curvature.

(3) We now look at a universe that is very close to the one we seem to live in: a spatially flat universe with matter and Lambda as components. Recall that for a spatially flat universe,

$$\Omega_{r,o} + \Omega_{m,o} + \Omega_\Lambda = 1.$$

(a) By taking advantage of the flat universe condition, find Ω_Λ in terms of $\Omega_{m,o}$ only.

(b) Rewrite the Friedmann equation (Eq. (2) on page 2) in terms of $\Omega_{m,o}$.

(c) Consider the case in which $\Omega_\Lambda < 0$, with the universe initially expanding. By referring to the Friedmann equation, describe the behavior of this universe.

(d) Determine the maximum scale factor for a universe with $\Omega_\Lambda < 0$

(e) In the lecture, we saw that

$$H_o t = \frac{2}{3\sqrt{\Omega_{m,o} - 1}} \ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1 + \left(\frac{a}{a_{m\Lambda}}\right)^3}\right]$$

where

$$a_{m\Lambda} = \left(\frac{\Omega_{m,o}}{1 - \Omega_{m,o}}\right)^{1/3}.$$

Use these results to find a general expression for the current age of the universe (i.e. when $a = 1$.)

(f) Using $\Omega_{m,o} = 0.31, \Omega_\Lambda = 0.69$, and $H_o = 68\,\mathrm{km\ s^{-1} Mpc^{-1}}$ calculate a specific age for this universe.

(g) Find the age at which matter and the cosmological constant had equal density (i.e., $a = a_{m\Lambda}$).

(4) Recap the important topics covered today. Compare and contrast your results with others at your table.

Homework 02–Due Friday Jan 31

1. Problem 4.4
2. Problem 4.5
3. Problem 5.1
4. Problem 5.3

Additional Grad Student Problem(s)

5. Consider a flat universe universe with a single component characterized by the equation of state parameter, $w = -1$.

   (a) Show that in such a universe the Friedmann equation takes the form
   $$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon_\Lambda a^2 \tag{5}$$

   (b) Show that the Hubble constant in a Λ-dominated universe is given by
   $$H_o \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_o} = \left(\frac{8\pi G\epsilon_\Lambda}{3c^2}\right)^{1/2} \tag{6}$$

   (c) Solve Eq. (1) and show that the dependence of the scale factor on time is given by
   $$a(t) = e^{H_o(t-t_o)}$$