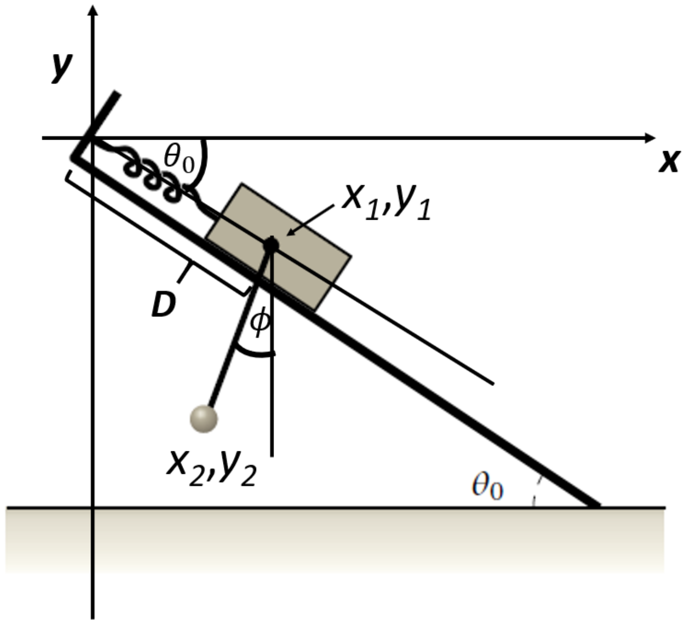
A block of mass *M* slides along a frictionless surface that makes an angle of θ_0 with the horizontal as shown below. The block is attached to a spring of rest length *L* and spring constant *k*. Hanging from the block is a pendulum of length *l* and mass *m*.



a. Define a set of Cartesian coordinates and express the position of the block and the mass in these coordinates.
b. Identify the constraints and the equations of constraint for this system. Are the constraints holonomic? How many degrees of freedom does this system have?
c. Write down the kinetic and potential energy for the system in the Cartesian coordinates you've chosen.
d. Pick a set of generalized coordinates (that obey the constraints) and express the kinetic and potential energy in the generalized coordinates.
e. Find the Lagrangian for the system and the equations of motion for the block and the pendulum bob using your chosen generalized coordinates.

Sample Solution:

a. See figure
b. Constraint 1: Block constraint to the surface: $\frac{y_1}{x_1} + \tan\theta_0 = 0$

   Constraint 2: Length of the pendulum is constant, so $(x_2 - x_1)^2 + (y_2 - y_1)^2 - l = 0$
   Both constraints are holonomic. Without constraints the number of degrees of freedom would be 4 (the x and y positions of masses M and m. There are two holonomic constraints, so the number of degrees of freedom is 4-2=2. The system can be fully described by specifying the length of the spring and the pendulum angle ϕ.
c. $T = \frac{1}{2}M(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$
   The potential energy has a spring and a gravitational component. The length of the spring

is $D = \sqrt{x_1^2 + y_1^2}$, so the spring potential energy is $V_{spring} = \frac{1}{2}k\left(\sqrt{x_1^2 + y_1^2} - l\right)^2$. The gravitational potential energy of the block and pendulum bob are $V_{grav} = Mgy_1 + mgy_2$, so

$$V = \frac{1}{2}k\left(\sqrt{x_1^2 + y_1^2} - l\right)^2 + Mgy_1 + mgy_2$$

d. We have two holonomic constraints so we have two degrees of freedom and need two independent variables to describe the system. Using the length of the spring *D* and the pendulum angle ϕ as generalized coordinates is consistent with the two constraints, meaning varying these two coordinates doesn't violate the constraints. To derive the kinetic and potential energies we will first express the Cartesian coordinates and velocities in terms of *D* and ϕ and then plug these terms into the equations for *T* and *V* above.

$$x_1 = D\cos\theta_0 \qquad y_1 = -D\sin\theta_0$$
$$x_2 = x_1 + l\sin\phi = D\cos\theta_0 + l\sin\phi$$
$$y_2 = y_1 - l\cos\phi = -D\sin\theta_0 - l\cos\phi$$

Taking the time derivatives of these transformation equations:

$$\dot{x}_1 = \dot{D}\cos\theta_0 \qquad \dot{y}_1 = -\dot{D}\sin\theta_0$$
$$\dot{x}_2 = \dot{D}\cos\theta_0 + l\dot{\phi}\cos\phi \qquad \dot{y}_2 = -\dot{D}\sin\theta_0 + l\dot{\phi}\sin\phi$$

Plugging the equations for the generalized velocities into the equation for the kinetic energy we get:

$$T = \frac{1}{2}M(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \frac{1}{2}M\left(\dot{D}^2\cos^2\theta_0 + \dot{D}^2\sin^2\theta_0\right) + \frac{1}{2}m\left[\left(\dot{D}\cos\theta_0 + l\dot{\phi}\cos\phi\right)^2 + \left(-\dot{D}\sin\theta_0 + l\dot{\phi}\sin\phi\right)^2\right] =$$
$$\frac{m+M}{2}\dot{D}^2 + \frac{ml^2}{2}\dot{\phi}^2 + ml\dot{D}\dot{\phi}(\cos\theta_0\cos\phi - \sin\theta_0\sin\phi)$$

$$= \frac{m+M}{2}\dot{D}^2 + \frac{ml^2}{2}\dot{\phi}^2 + ml\dot{D}\dot{\phi}\cos(\phi + \theta_0)$$

For the last two steps I used the trigonometric identities $\sin^2\alpha + \cos^2\alpha = 1$ and $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha + \beta)$

The potential energy is

$$V = \frac{1}{2}k\,(D - L)^2 - MgD\sin\theta_0 - mg\,(D\sin\theta_0 + l\cos\phi)$$
$$= \frac{1}{2}k\,(D - L)^2 - (M + m)gD\sin\theta_0 - mgl\cos\phi$$

The Lagrangian is

$$L = T - V = \frac{m+M}{2}\dot{D}^2 + \frac{ml^2}{2}\dot{\phi}^2 + ml\dot{D}\dot{\phi}\cos(\phi + \theta_0) - \frac{1}{2}k\,(D - L)^2 + (M + m)gD\sin\theta_0 + mgl\cos\phi$$

Plug this Lagrangian into the Lagrange equations to get the equations of motion for D and ϕ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{D}} = \frac{\partial L}{\partial D} \Rightarrow$$

$$(m+M)\ddot{D} + ml\frac{d}{dt}\left[\dot{\phi}^2\cos(\phi+\theta_0)\right] = -k(D-L) + (M+m)g\sin\theta_0$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \Rightarrow$$

$$ml^2\ddot{\phi} + ml\frac{d}{dt}\left[\dot{D}\cos(\phi+\theta_0)\right] = -ml\dot{D}\dot{\phi}\sin(\phi+\theta_0) - mgl\sin\phi$$

$$ml^2\ddot{\phi} + ml\ddot{D}\cos(\phi+\theta_0) - ml\dot{D}\dot{\phi}\sin(\phi+\theta_0) = -ml\dot{D}\dot{\phi}\sin(\phi+\theta_0) - mgl\sin\phi$$

$$\ddot{\phi} + \frac{\ddot{D}}{l}\cos(\phi+\theta_0) + \frac{g}{l}\sin\phi = 0$$