(1) In the lecture we saw various ways one can treat data whose length is not a power of 2. For the following data,

$$f(t) = (1 \quad 2 \quad 9 \quad 3 \quad 6 \quad 4 \quad 8 \quad 8 \quad 7)$$

do the extensions discussed in class.

$\boxed{2\ 4\ 6\ 8\ 8}$ $\boxed{1\ 3\ 9\ 7}$

$\boxed{2\ 6}$ $\boxed{4\ 8}$ $\boxed{1\ 9}$ $\boxed{3\ 7}$

$\boxed{2}\boxed{6}$ $\boxed{4}\boxed{8}$ $\boxed{1}\boxed{9}$ $\boxed{3}\boxed{7}$

(2) Download the file `MySamplingProblem.m` and `MySpec.m` from the Teams page. The first program just samples the data as required in homework problem 5, and the second computes the spectrum. Make sure you understand what the codes are doing. To get full credit on the problem, you need to interpret what you are seeing and answer the question about what the sampling rate must be in order to pick up the true frequency of the original data.

# 1 More Fourier

There are two mathematical operations that are of great use in signal processing that use the Fourier transform to ease the calculation. These are *convolution* and *correlation*

Mathematically convolution is defined as

$$p \otimes q \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(\tau) q(t-\tau) d\tau \tag{1}$$

and correlation is defined as

$$p \odot q \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^*(\tau) q(t+\tau) d\tau \tag{2}$$

Physically, convolution describes how a physical system responds to a given input. We'll talk about what exactly this means in just a bit. Correlation is useful in extracting a signal from a noisy background. Again we'll see how in just a moment.

The practical usefulness of the quantities comes about after applying the Fourier transform. Applying the Fourier transform to Eq.(1) give

$$\mathcal{F}(p \otimes q) = \mathcal{F}(p)\mathcal{F}(q) \tag{3}$$

while

$$\mathcal{F}(p \odot q) = \mathcal{F}(p)^*\mathcal{F}(q) \tag{4}$$

**Convolution.** We begin by exploring Eqs (1) and (3).

Suppose that we know $\mathcal{F}(p)\mathcal{F}(q)$ and we want $p(t)$. We can do a little algebra and re-write Eq(3) as

$$\mathcal{F}(p) = \frac{\mathcal{F}(p \otimes q)}{\mathcal{F}(q)}$$

or that

$$p(t) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(p \otimes q)}{\mathcal{F}(q)}\right)$$

This situation arises almost anytime a measurement takes place. The real and pure signal is *convolved* with signal arising from the measuring instrument. Consider an schematic case shown in figure 1.

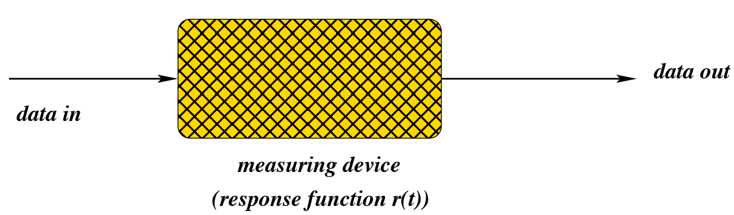


Figure 1: Idealized measurement of a signal.

A signal enters the device on the left, is measured, and exits on the right. The device is not perfect, so it adds does not perfectly reproduce the input.

In addition the device does not immediately react to the presence of the signal, nor does it stop after the signal has passed. These effects we group into a single function called a *response function*, $r(t)$. Figures 2 and 3 show the effects of an exponential like response function. Figure 2 shows the *pure signal*, $s(t)$ figure 3 the convolved signal, $s(t) \otimes r(t)$.

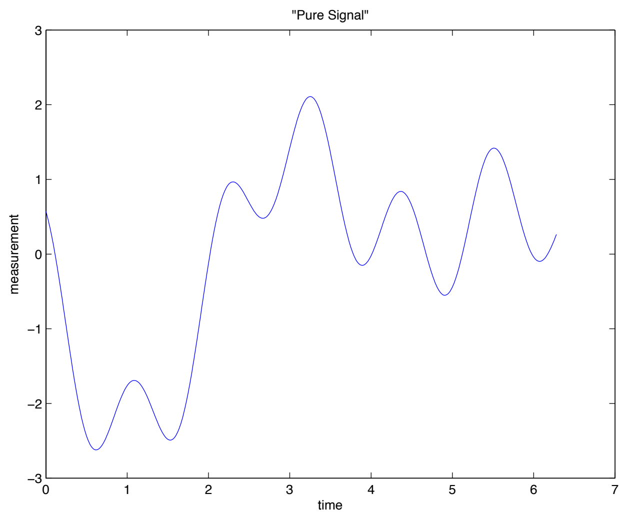


Figure 2: The *pure* signal.

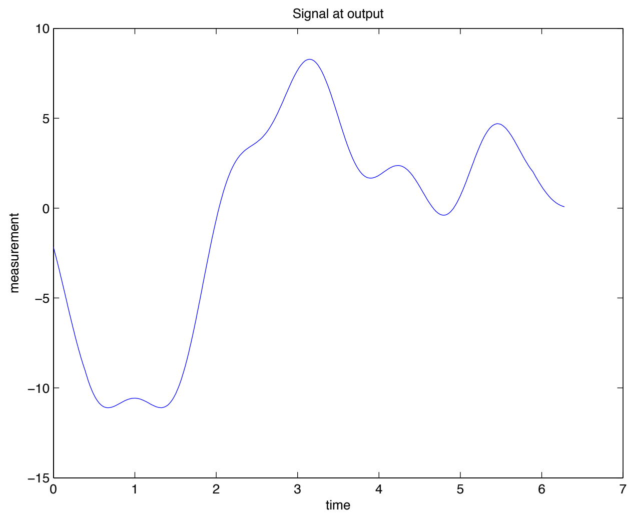


Figure 3: The signal after output from the measuring device.

Eq. (3) is important here because the response function $r(t)$ is often a function of frequency. That is, the response of the device to an input signal will differ depending on the frequency.

(3) Begin homework problem 7.

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**Correlation.** Eqs (2) and (4) give the mathmatical definitions of correlation. This quantity measures how much one signal in similar to another. In particular, the *autocorrelation*,

$$p \odot p = \int p^*(t)\, p(t+\tau) d\tau$$

describes how well a signal is like itself at some *lag time* $\tau$.

To see why this is important, consider the case were we want to observe a signal, $s(t)$. In any realistic case, what one will measure instead is a

$p(t) = s(t) + n(t)$ where $n(t)$ is *random noise*. If we then autocorrelate $p$ we get that,

$$p \odot p = s \odot s + s \odot n + n \odot s + n \odot n.$$

For sufficiently long lag times, the signal and noise are uncorrelated. This means that the terms $s \odot n + n \odot s \to 0$. Thus we have that

$$p \odot p = s \odot s + n \odot n = s \odot s + |\langle n(t) \rangle|^2.$$

This result is now very useful. The autocorrelation of $s$ is embedded in a *constant* noise term, $|\langle n(t) \rangle|^2$.

(4) Download the MATLAB file called `MyAutoCorrelation.m` from the dropbox. Look over the program and understand what each piece does. Then run the program and find the autocorrelation. Change the function to $t^{2+noise}$ and repeat.