Physics 460—Angular Momentum States Spring 2020

A number of very useful results about the angular momentum operators and their eigenstates are given in the course notes without proof. It's excellent practice to derive these from the fundamental relationships between the operators. Here's your chance.

The fundamental properties of angular momentum are determined by the commutation relations between the operators:

$$\left[J_x, J_y\right] = i\hbar J_z, \quad \left[J_y, J_z\right] = i\hbar J_x, \quad \left[J_z, J_x\right] = i\hbar J_y.$$

The total angular momentum is $J^2 = {J_x}^2 + {J_y}^2 + {J_z}^2$, and the raising and lowering operators are defined by $J_\pm = J_x \pm iJ_y$.

(1) Complete the missing calculation of Eq. (1.37) on page 10 of the course notes to show that J_- is a lowering operator.

(2) Complete the "similar calculation" for Eq. (1.54) on page 12 of the course notes to show that

$$C_- = \sqrt{j(j+1) - m(m-1)}\hbar.$$

(3) It says on page 8 of the course notes that $\left[J^2, J_\pm\right] = 0$.

$$\overset{\text{operator}}{A}|a\rangle \neq \underset{\pm}{a}|a\rangle$$

(a) Verify these commutators.

(b) The states $|j, m\rangle$ are eigenstates of J^2. Are they eigenstates of $J_\pm$? Explain.

(c) I thought that operators that commute share eigenstates. Is your answer to part (b) consistent with this? Why or why not?

(1) Want to show:

$$J_z\left(J_-|\lambda,\mu\rangle\right) = (\mu - \hbar)\left(J_-|\lambda,\mu\rangle\right)$$

Given:

$$J_z|\lambda,\mu\rangle = \mu|\lambda,\mu\rangle, \quad [J_z, J_-] = -\hbar J_-$$

$$\Rightarrow J_zJ_- - J_-J_z = -\hbar J_-$$

$$\Rightarrow J_zJ_- = J_-J_z - \hbar J_-$$

$$J_zJ_-|\lambda,\mu\rangle = (J_-J_z - \hbar J_-)|\lambda,\mu\rangle$$

$$= J_-J_z|\lambda,\mu\rangle - \hbar J_-|\lambda,\mu\rangle$$

$$= J_-\mu|\lambda,\mu\rangle - J_-\hbar|\lambda,\mu\rangle$$

$$J_z\left(J_-|\lambda,\mu\rangle\right) = J_-(\mu - \hbar)|\lambda,\mu\rangle$$

(2) Want to show $J_-|j,m\rangle = c_-|j,m-1\rangle$

where $c_- = \sqrt{j(j+1) - m(m-1)}\,\hbar$

Know $\langle j,m|j,m\rangle = 1$, $\langle j,m-1|j,m-1\rangle = 1$

$\langle\psi| = \langle j,m|J_-^+$, $J_-^+ = J_+$

What about the state $|\psi\rangle = J_-|j,m\rangle$? Is it normalized?

Let's check! $\langle\psi|\psi\rangle = \langle j,m|J_+J_-|j,m\rangle$

Eq (1.30): $J^2 = J_+J_- + J_z(J_z - \hbar I)$

$\Rightarrow J_+J_- = J^2 - J_z(J_z - \hbar I)$

$\langle j,m|\left(J^2 - J_z(J_z - \hbar I)\right)|j,m\rangle$

$J_z|j,m\rangle = m\hbar|j,m\rangle$

$J^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$

$\langle j,m|\left(j(j+1)\hbar^2 - m\hbar(m\hbar - \hbar)\right)|j,m\rangle$

$$\left(j(j+1)\hbar^2 - m\hbar(m\hbar - \hbar)\right)\underbrace{\langle j,m|j,m\rangle}_{=1} = \langle j,m-1|c_-^*c_-|j,m-1\rangle$$

$$= |c_-|^2\underbrace{\langle j,m-1|j,m-1\rangle}_{=1}$$

$$\boxed{c_- = \sqrt{j(j+1) - m(m-1)}\,\hbar}$$

③

ⓐ Since $[J^2, J_x] = 0$ & $[J^2, J_y] = 0$

$$\Rightarrow [J^2, J_\pm] = [J^2, J_x \pm iJ_y] = [J^2, J_x] \pm i[J^2, J_y]$$
$$= 0 \pm 0 = 0$$

ⓑ No! $J_+ |j,m\rangle = C_+ |j,m+1\rangle \neq \lambda |j,m\rangle$

$$J_- |j,m\rangle = C_- |j,m-1\rangle \neq \lambda |j,m\rangle$$

ⓒ If two operators commute, it is possible, but not mandatory, that they share all eigenstates.