Physics 460—Superposition of Hydrogen States Spring 2020

At time $t = 0$ a hydrogen atom is in the superposition state

$$|\Psi_0\rangle = \frac{1}{\sqrt{14}}\Big[|2,1,1\rangle - 2|3,2,-1\rangle + 3i|3,2,2\rangle\Big].$$

(1) If you measured the energy of the atom, what values could you measure and with what probabilities? What is the expected value of the energy?

(2) If you measured the magnitude of the angular momentum of the atom, what values could you measure and with what probabilities? What is the expected value of the magnitude of the angular momentum?

(3) If you measured the z-component of the angular momentum of the atom, what values could you measure and with what probabilities? What is the expected value of the z-component of the angular momentum?

(4) Find the radial probability density, $|\psi(r)|^2$, for this state. To find this, you start with the full probability density, $|\psi(r,\theta,\phi)|^2$, and integrate over the angles:

$$|\psi(r)|^2 = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, |\psi(r,\theta,\phi)|^2.$$

You can then use the orthonormality of the spherical harmonics to help evaluate these integrals.

(5) What is $\langle r\rangle$ for this state?

1 E_2 with probability $\frac{1}{14}$, E_3 with probability $\frac{13}{14}$

$\langle E\rangle = \dfrac{E_2 + 13E_3}{14} = ?$

2 $|L| = \sqrt{\ell(\ell+1)}\,\hbar$

$\sqrt{2}\,\hbar$ with probability $\frac{1}{14}$, $\sqrt{6}\,\hbar$ with probability $\frac{13}{14}$

3 $\hbar$ with probability $1/14$

$-\hbar$ with probability $4/14$

$2\hbar$ with probability $9/14$

4) $$\psi = \frac{1}{\sqrt{14}}\left(\psi_{211} + 2\psi_{32-1} + 3i\psi_{322}\right)$$

$$|\psi|^2 = \frac{1}{14}\left|\psi_{211} + 2\psi_{32-1} + 3i\psi_{322}\right|^2$$

$$|\psi(r,\theta,\phi)|^2 = \frac{1}{14}\Big(|\psi_{211}|^2 + 4|\psi_{32-1}|^2 + 9|\psi_{322}|^2$$
$$+ 2\left(\cancel{\psi^*_{211}\psi_{32-1}} + \cancel{\psi_{211}\psi^*_{32-1}}\right)$$
$$+ 3i\left(\cancel{\psi^*_{211}\psi_{322}} - \cancel{\psi_{211}\psi^*_{322}}\right)$$
$$+ 6i\left(\cancel{\psi^*_{32-1}\psi_{322}} - \cancel{\psi_{32-1}\psi_{322}}\right)\Big)$$

To find $|\psi(r)|^2$, we integrate $|\psi|^2$ over the angles:

$$|\psi(r)|^2 = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, |\psi(r,\theta,\phi)|^2$$

Note that $$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, Y_{\ell_1}^{m_1 *}(\theta,\phi) Y_{\ell_2}^{m_2}(\theta,\phi) = \delta_{\ell_1 \ell_2}\delta_{m_1 m_2}$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, |\psi_{211}|^2 = |R_{21}(r)|^2$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, |\psi_{32-1}|^2 = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, |\psi_{322}|^2 = |R_{32}(r)|^2$$

All others zero!

$$|\psi(r)|^2 = \frac{1}{14}\left(|R_{21}(r)|^2 + 13\,|R_{32}(r)|^2\right)$$

$$= \frac{1}{14}\left(\underline{\frac{r^2}{96\,a_z^5}\,e^{-r/a_z}} + \underline{13\,\frac{4}{81^2}\,\frac{2}{15a_z^3}\,\frac{r^4}{81a_z^4}e^{-2r/3a_z}}\right)$$

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$$\langle r\rangle = \int_0^\infty r^2\,dr\left(r\,|\psi(r)|^2\right)$$

$$= \frac{1}{14}\left(\frac{1}{96a_z^5}\int_0^\infty r^5 e^{-r/a_z}\,dr\right.$$

$$\left. + 13\,\frac{4}{81^3}\,\frac{2}{15a_z^7}\int_0^\infty r^7 e^{-2r/3a_z}\,dr\right)$$

$$\langle r\rangle = \frac{1}{14}\left(\frac{1}{96\,a_z^5}\left(120\,a_z^6\right) + \frac{8\cdot 13}{15\cdot 81^3 a_z^7}\left(\frac{2066715\,a_z^8}{16}\right)\right)$$

$$\langle r\rangle = \frac{1}{14}\left(\frac{5\,a_z}{4} + \frac{13\cdot 56}{27}\,a_z\right) = \frac{1}{14}\,\frac{3047}{108}\,a_z = 2.02\,a_z$$