PHY 411 - Homework 2

1)

given Faraday's law, $\vec{\nabla} \times \vec{E} = -\partial \vec{B} / \partial t$ and Helmholtz wave equation $\vec{E}(\vec{x},t) = \vec{\mathcal{E}} e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$. Then substituting Helmholtz wave equation into Faraday's law gives

$$= \vec{\nabla} \times \vec{\mathcal{E}} e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = -\frac{\partial \vec{B} e^{i(k\hat{n}\cdot\vec{x}-\omega t)}}{\partial t}$$

Using note: $\vec{\nabla} \times \vec{\mathcal{E}} e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = -\vec{\mathcal{E}} \times \vec{\nabla} e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$

$$= -\vec{\mathcal{E}} \times \vec{\nabla} e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = -\frac{\partial \vec{B} e^{i(k\hat{n}\cdot\vec{x}-\omega t)}}{\partial t}$$

$$= -\vec{\mathcal{E}} \times \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) e^{i(k\hat{n}\cdot\hat{x}-\omega t)} = -\frac{\partial \vec{B} e^{i(k\hat{n}\cdot\vec{x}-\omega t)}}{\partial t}$$

$$\frac{\partial}{\partial x} = (ikn_x)\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$

$$\frac{\partial}{\partial y} = (ikn_y)\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$

$$\frac{\partial}{\partial z} = (ikn_z)\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{B}(-i\omega)\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$

all exp() cancel

$$= -\vec{\mathcal{E}} \times (\cancel{i}k(n_x + n_y + n_z)) = \vec{B}(\cancel{-i}\omega)$$

Remember $k = \sqrt{\mu\epsilon}\,\omega$ and note above

$$= \sqrt{\mu\epsilon}\,\cancel{\omega}\,(n_x\hat{x} + n_y\hat{y} + n_z\hat{z}) \times \vec{\mathcal{E}} = \vec{B}\cancel{\omega}$$

$$= \sqrt{\mu\epsilon}\,\hat{n} \times \vec{\mathcal{E}} = \vec{B}$$

Note that $\vec{k} = k\hat{n}$ so $\vec{k}/k = \hat{n}$. Thus

$$\boxed{\vec{B} = \sqrt{\mu\epsilon}\left(\frac{\vec{k} \times \vec{E}}{k}\right)}$$

2. Show $(\nabla^2 + \mu\epsilon\omega^2)\vec{B} = 0$

Helmholz equation for $\vec{B}$

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right)\vec{B}(\vec{x},t) = 0 \quad \text{where} \quad \vec{B}(\vec{x},t) = \vec{B}e^{i(kx-\omega t)}$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + k^2\right)\vec{B}e^{i(kx-\omega t)} = 0$$

$$\frac{\partial e^{i(kx-\omega t)}}{\partial x} = ik\,e^{i(kx-\omega t)}$$

$$\frac{\partial^2 e^{i(kx-\omega t)}}{\partial x^2} = (ik)(ik)e^{i(kx-\omega t)} = -k^2 e^{i(kx-\omega t)}$$

$$\boxed{\left(-k^2 e^{i(kx-\omega t)} + k^2 e^{i(kx-\omega t)}\right) = 0} \Rightarrow \text{proved}$$

3. $$\vec{E} = E_0 \frac{\hat{x} - i\hat{y}}{\sqrt{2}} e^{i(kz-\omega t)}$$

(a) Verify Gauss Law $\vec{\nabla} \cdot \vec{E} = 0$

Remember $\hat{x}\cdot x = 1$, $\hat{y}\cdot\hat{y} = 1$, $\hat{z}\cdot\hat{z} = 1$, $\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0$

$$\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(E_0\frac{\hat{x} - i\hat{y}}{\sqrt{2}} e^{i(kz-\omega t)}\right)$$

$$= \left(\frac{\partial}{\partial x} \frac{1}{\sqrt{2}} + \frac{\partial}{\partial y} \left(-\frac{i}{\sqrt{2}} \right) + 0 \right)$$

$$\frac{\partial}{\partial x} \left(\frac{1}{\sqrt{2}} \right) = 0 \qquad \frac{\partial}{\partial y} \left(-\frac{i}{\sqrt{2}} \right) = 0$$

Thus, $\boxed{\vec{\nabla} \cdot \vec{E} = 0}$

(B) Use Faraday's law to find $\vec{B}$.

$$\vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} =$$

$$\left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \times \left(E_0 \frac{\hat{x} - i\hat{y}}{\sqrt{2}} e^{i(kz - \omega t)} \right)$$

$$= \left(\left(E_0 \frac{0 - i\hat{z}}{\sqrt{2}} e^{i(kz - \omega t)} \left(\frac{\partial}{\partial x} \right) \right) + \left(-E_0 \frac{\hat{z} - i(0)}{\sqrt{2}} e^{i(kz - \omega t)} \left(\frac{\partial}{\partial y} \right) \right) + \left(E_0 \frac{\hat{y} + i\hat{x}}{\sqrt{2}} e^{i(kz - \omega t)} \left(\frac{\partial}{\partial z} \right) \right) \right)$$

$$\frac{\partial}{\partial x} E_0 \frac{i\hat{z}}{\sqrt{2}} e^{i(kz - \omega t)} = 0$$

$$\frac{\partial}{\partial y} E_0 \frac{\hat{z}}{\sqrt{2}} e^{i(kz - \omega t)} = 0$$

$$\frac{\partial}{\partial z} E_0 \frac{\hat{y} + i\hat{x}}{\sqrt{2}} e^{i(kz - \omega t)} = E_0 \frac{k(\hat{y} i - \hat{x})}{\sqrt{2}} e^{i(kz - \omega t)}$$

$$\Rightarrow E_0 \frac{k(\hat{y} i - \hat{x})}{\sqrt{2}} e^{i(kz - \omega t)} = \frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow E_0 \frac{k(\hat{y} i - \hat{x})}{\sqrt{2}} e^{i(kz - \omega t)} = \frac{\partial \vec{B} e^{i(k\vec{n} \cdot \vec{x} - \omega t)}}{\partial t}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{B}(k\hat{n} \cdot \vec{x} - i\omega) e^{i(k\hat{n} \cdot \vec{x} - \omega t)}$$

$$\frac{\partial t}{} = B(k\hat{n}x - \omega t)$$

$$\Rightarrow -\frac{\partial}{\partial t}\left[\hat{x}B_x e^{i(k\hat{n}\vec{x}-\omega t)} + \hat{y}B_y e^{i(k\hat{n}\vec{x}-\omega t)} + \hat{z}B_z e^{i(k\hat{n}\vec{x}-\omega t)}\right]$$

$$\Rightarrow -\left[-\hat{x}B_x i\omega - \hat{y}B_y i\omega - \hat{z}B_z i\omega\right] e^{i(k\hat{n}\vec{x}-\omega t)}$$

$$= E_0 \frac{k(\hat{y}i - \hat{x})}{i\omega\sqrt{2}} e^{(kz - ikn x)} = \vec{B}$$

(C) The State of polarization for this wave is going to be Circular due to the components. The components show that the wave will move in the $\hat{z}$ direction while x shows the Real part of the wave and y shows the imaginary part of the wave

4.a Linear polarized basis

$$\vec{E}(\vec{x},t) = (\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2) e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$E_1 \downarrow a_1 e^{i\delta t} \qquad E_2 \downarrow a_2 e^{i\delta t}$$

Unit Vectors

$$\hat{\epsilon}_1 = \frac{1}{\sqrt{2}}(\hat{\epsilon}_1 + i\hat{\epsilon}_2) \qquad \hat{\epsilon}_2 = \frac{1}{\sqrt{2}}(\hat{\epsilon}_1 - i\hat{\epsilon}_2)$$

From equation 7.27 we know that

$$S_0 = a_1^2 + a_2^2 \qquad S_1 = a_1^2 - a_2^2$$

$$\Rightarrow a_1^2 = S_0 - a_2^2 \xleftarrow[\text{Sub}]{} a_2^2 = a_1^2 - S_1$$

$$\Rightarrow a_1^2 = S_0 - a_1^2 + S_1$$

$\Rightarrow a_1^2 = S_0 - a_1^2 + S_1$

$\Rightarrow a_1^2 + a_1^2 = S_0 + S_1 \rightarrow 2a_1^2 = S_0 + S_1 \rightarrow a_1^2 = (S_0 + S_1)/2$

$\Rightarrow a_1 = \sqrt{(S_0 + S_1)/2}$ Similarly, it can be shown using the same procedure that

$$a_2 = \sqrt{(S_0 - S_1)/2}$$

Since $S_0 = 3$ and $S_1 = -1$ then a_1 can be found by

$$a_1 = \sqrt{\frac{S_0 + S_1}{2}} = \sqrt{\frac{3-1}{2}} = \sqrt{\frac{2}{2}} = 1$$

Also,

$$a_2 = \sqrt{\frac{S_0 - S_1}{2}} = \sqrt{\frac{3+1}{2}} = \sqrt{\frac{4}{2}} = \sqrt{2}$$

Now knowing a_1 and a_2, the following δ_1 and δ_2 can be found for S_2, S_3. Using the equations

$$S_2 = 2\,\mathrm{Re}[(\epsilon_1 \cdot E)^* (\epsilon_2 \cdot E)] = 2a_1 a_2 \cos(\delta_2 - \delta_1)$$
$$S_3 = 2\,\mathrm{Im}[(\epsilon_1 \cdot E)^* (\epsilon_2 \cdot E)] = 2a_1 a_2 \sin(\delta_2 - \delta_1)$$

So,

$$(\delta_2 - \delta_1) = \tan^{-1}(S_3/S_2) \rightarrow \tan^{-1}(-2/2)$$

$$\cos(\delta_2 - \delta_1) = \frac{S_2}{\sqrt{(S_0^2 - S_1^2)}} = \frac{2}{\sqrt{9-1}} \quad \frac{2}{\sqrt{22}} = \frac{1}{\sqrt{2}}$$

Similar to Cos

$$\sin(\delta_2 - \delta_1) = \frac{S_3}{\sqrt{S_0^2 - S_1^2}} = \frac{-2}{\sqrt{9-1}} = -\frac{1}{\sqrt{2}}$$

Combining above and solving E_1

$$\boxed{\vec{E} = \left(\hat{\epsilon}_1 + \frac{1}{2}(1-i)\hat{\epsilon}_2\right) e^{i(k\hat{n}\cdot\vec{x} - \omega t)}}$$

(B) Similar to part a, all a_+, a_-, δ_+, and δ_- values have to be solved for.

So, $S_0 = a_+^2 + a_-^2$ the corresponding equation is S_3, S_0

$$a_+ = \sqrt{\frac{S_0 + S_3}{2}} \rightarrow \sqrt{\frac{3-2}{2}} = \frac{1}{\sqrt{2}}$$

and

$$a_- = \sqrt{\frac{S_0 - S_3}{2}} \rightarrow \sqrt{\frac{3+2}{2}} = \frac{\sqrt{5}}{\sqrt{2}}$$

$$\cos(\delta_- - \delta_+) = \frac{S_1}{\sqrt{S_0^2 - S_3^2}} = \frac{-1}{\sqrt{9-4}} = -\frac{1}{\sqrt{5}}$$

$$\sin(\delta_- - \delta_+) = \frac{S_2}{\sqrt{S_0^2 - S_3^2}} = \frac{2}{\sqrt{9-4}} = \frac{2}{\sqrt{5}}$$

$$\boxed{E = \left(\frac{1}{\sqrt{2}}\epsilon_+ + \left(-\frac{1}{\sqrt{2}} + i\sqrt{2}\,e_-\right)\right) e^{i(k\hat{n}\cdot\vec{x} - \omega t)}}$$