1 Fourier Series

The value of Taylor expansions should be clear, they allow one to expand a continuous function as an infinite series of polynomials. We can they examine the behavior of the function by how close it *matches* each polynomial in the series.

There are at least three situations for which Taylor series are inadequate. The first is when a function is periodic. The function replicates itself every period, and Taylor expansions 'miss' this point. Second, if a function has discontinuities, Taylor expansions fail since the derivatives $\to \infty$. Lastly, Information is encoded in different ways by different expansions. Some information is more conveniently extracted using other expansions.

To address these points, we introduce *Fourier Series*. We will see that this leads to a whole array of powerful techniques to extract information from a function.

(1) We begin our study of Fourier series by looking at a problem that seems to bear no relationship at all to Fourier series: ordinary vectors in a plane.

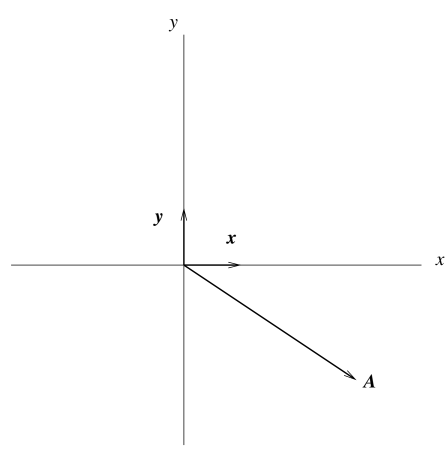

Figure 1: A vector on the plane.

Express the general vector $\boldsymbol{A}$ using the *unit vectors* labeled $\boldsymbol{x}$ and $\boldsymbol{y}$.

(2) Can a general vector in a plane be represented by just one of the unit vectors?

Can be written as a linear Combination

$$\vec{A} = A_x \hat{x} + A_y \hat{y}$$

(3) Does a general vector in the plane need more than the two unit vectors you used in part (1) to be adequately represented.

The minimum number of vectors needed to express a general vector is called completeness

Any more is redundent

(4) Can any vector in the plane be represented by the unit vectors, $\boldsymbol{x}$ and $\boldsymbol{y}$?

The vectors $\hat{x}, \hat{y}$ are orthogonal unit vectors

(5) Items (2) through (4) show a property of the unit vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ called *completness*. These vectors are also said to *span the space*. Using what you learned in (2) – (4), write down your definition of what it means for a set of vectors to span a space..

a set of vectors span the space iff the vectors are linear independent

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In parts (1) – (5) you were introduced to the idea of a basis for a finite dimensional space. In the lecture,we saw that there are basis that are particularly useful, orthogonal basis.

Beginning in the early 1800's, an extension of the idea of basis was developed. In particular the idea of basis was extended to the *infinite* dimensional realm of functions that are piecewise continuous on some finite interval, $[a, b]$. As we go through the following steps, keep in mind what we did in (1) – (5) and use these to help guide you. Note also that this is not a course in Fourier series, so much theoretical development will be omitted.

(6) In parts (1) – (5) we found that an orthogonal basis for a 2-D vector space were the two cartesian unit vectors $\boldsymbol{x}, \boldsymbol{y}$. We are now interested in *nice* functions on a symmetric interval, $[-\pi, \pi]$ (we will relax this later). It was found that the set

$$\{1, \cos(nx), \sin(mx)\}, \quad n, m \text{ integers} \tag{1}$$

formed an orthogonal basis. Write the general *nice function* ,$f(x)$, in terms of the basis given in Eq.(1).

$$f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

(7) It turns out that the set given by Eq. (1) is also orthogonal. How would you show that this set is orthogonal?

$$\int_{-\pi}^{\pi} \sin(mt) \sin(nt)\, dt \qquad \int_{-\pi}^{\pi} \sin(mt) \cos(nt)\, dt = 0$$

$$\int_{-\pi}^{\pi} \cos(mt) \cos(nt)\, dt$$

(8) Using the orthogonality conditions, show how to find the expansion coefficients of expansion, a_n, b_n.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt$$

(9) Now recap. Relate what you did in parts (1) – (5) to what you have uncovered in parts (6) – (8). Compare and constrast the basis vectors used in 2-D and the basis vectors used in the space of piece–wise smooth functions.

(10) It is often more convenient to write the trigonometric functions as complex exponential using Euler's relation,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Using these relations, the Fourier expansion becomes,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}.$$

Do you think the definition for the inner product we used above will work here? If not, discuss why not and suggest an alternative definition.