F-1 problem 2

(2)

$$|\psi\rangle = 1/3\left(2|2,1,1\rangle - (2-i)|2,1,-1\rangle\right)$$

$$= 1/3\left(2\psi_{2,1,1}(r,\theta,\varphi) - (2-i)\psi_{2,1,-1}(r,\theta,\varphi)\right)$$

(a) $P(r = 4a_0 \to \infty)$

$$P(r,\theta,\varphi) = \psi^*\psi = |\psi(r,\theta,\varphi)|^2$$

$$= 1/3\left[2\psi^*_{2,1,1}(r,\theta,\varphi) - (2+i)\psi^*_{2,1,-1}(r,\theta,\varphi)\right]$$

$$\cdot\, 1/3\left[2\psi_{2,1,1}(r,\theta,\varphi) - (2-i)\psi_{2,1,-1}(r,\theta,\varphi)\right]$$

$$= 1/9\left[\left[2R^*_{21}(r)Y_1^{1*}(\theta,\varphi) - (2+i)R^*_{21}(r)Y_1^{-1*}(\theta,\varphi)\right]\right.$$

$$\left.\cdot\left[2R_{21}(r)Y_1^1(\theta,\varphi) - (2-i)R_{21}(r)Y_1^{-1}(\theta,\varphi)\right]\right]$$

$$= 1/9\left[4|R_{21}|^2\cdot|Y_1^1|^2 - (4-2i)|R_{21}|^2 Y_1^1 Y_1^{-1} - (4+2i)|R_{21}|^2 Y_1^1 Y_1^{-1}\right.$$

$$\left. + 5|R_{21}|^2|Y_1^{-1}|^2\right]$$

$$= 1/9\left[4|R_{21}|^2\cdot|Y_1^1|^2 - 8|R_{21}|^2 Y_1^1 Y_1^{-1} + 5|R_{21}|^2|Y_1^{-1}|^2\right]$$

Using

$$\langle l_1, m_1 | l_2, m_2\rangle = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\left[Y_{l_1}^{m_1*} Y_{l_2}^{m_2}\right] = \delta_{l_1,l_2}\delta_{m_1,m_2}$$

Will reduce the expression we simplified above. Our new state will have all the same Y terms go to $\to 1$ and unlike Y terms go to $\to 0$.

$$= 1/9\left[4|R_{21}|^2\cdot \underbrace{|Y_1^1|^2}_{\to 1} \cancel{- 8|R_{21}|^2 \underbrace{Y_1^1 Y_1^{-1}}_{\to 0}} + 5|R_{21}|^2 \underbrace{|Y_1^{-1}|^2}_{\to 1}\right]$$

Therefore we are only left with the r terms where

$$P(4a_0 \le r \le \infty) = \int_{4a_0}^{\infty} \frac{r^2}{9}\left[4|R_{21}|^2 + 5|R_{21}|^2\right]dr = \int_{4a_0}^{\infty} \frac{r^2}{\cancel{9}}\left[\cancel{9}|R_{21}|^2\right]dr$$

$P(4a_0 \le r \le \infty) = \int^{\infty} \; e^{-r/a} \; r^4 \, dr \; 103 \approx$

$$P(4a_0 \le r \le \infty) = \int_{4a_0} \frac{[illegible]}{3(2a_0)^3}\,\frac{[illegible]}{a_0^2}\,dr = \frac{[illegible]}{3e^4} \approx 0.[illegible] \sim \boxed{[illegible]\%}$$

(B) Bounds with range φ and θ $\;0 \le \varphi \le \pi/2$ and $0 \le \theta \le \pi/2$
With no bound restriction on r.

Using

$$\langle l_1, m_1 | l_2, m_2 \rangle = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \left[Y_{l_1}^{m_1 *} Y_{l_2}^{m_2} \right] = \delta_{l_1, l_2} \delta_{m_1, m_2}$$

will no longer work. This is because for this to work we need the bounds of $0 \le \varphi \le 2\pi$ and $0 \le \theta \le \pi$. But this is no longer true. The new Bounds are $0 \le \varphi \le \pi/2$ and $0 \le \theta \le \pi/2$. Therefore, going back to the State

$$= 1/9 \left[4|R_{21}|^2 \cdot |Y_1^1|^2 - 8|R_{21}|^2 Y_1^1 Y_1^{-1} + 5|R_{21}|^2 |Y_1^{-1}|^2 \right]$$

we can find the probability by integrating over

$$P = \left[\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\infty} 1/9 \left[4|R_{21}|^2 \cdot |Y_1^1|^2 - 8|R_{21}|^2 Y_1^1 Y_1^{-1} + 5|R_{21}|^2 |Y_1^{-1}|^2 \right] \sin\theta \, r^2 \, d\theta \, d\varphi \, dr \right]$$

$$P = \left[\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\infty} \frac{1}{9} \left[4|R_{21}|^2 \cdot |Y_1^1|^2 - 8|R_{21}|^2 Y_1^1 Y_1^{-1} + 5|R_{21}|^2 |Y_1^{-1}|^2 \right] \sin\theta \, r^2 \, d\theta \, d\varphi \, dr \right]$$

$$= \frac{1}{9} \left[\left[\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\infty} \frac{4e^{-r/a}}{3(2a_0)^3} \frac{r^4}{a_0^2} \cdot \frac{3}{8\pi} \sin^3\theta \, \overbrace{\cancel{e^{-i\varphi} e^{i\varphi}}}^{\text{Complex conj}} \, d\varphi \, d\theta \, dr \right. \right.$$

$$- \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\infty} \frac{8e^{-r/a}}{3(2a_0)^3} \frac{r^4}{a_0^2} \cdot \left(-\frac{3}{8\pi} \sin^3\theta \right) d\theta \, d\varphi \, dr$$

$$+ \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\infty} \frac{5e^{-r/a}}{3(2a_0)^3} \frac{r^4}{a_0^2} \cdot \frac{3}{8\pi} \sin^3\theta \, \overbrace{\cancel{e^{-i\varphi} e^{i\varphi}}}^{\text{Complex conj}} \, d\theta \, d\varphi \, dr$$

$$\int_0^{\pi/2} \sin^3(x) \, dx = 2/3$$

Note: $\int_0 \sin(x)dx = /3$

$$= \frac{1}{9}\left[\int_0^\infty \frac{4e^{-r/a}}{3(2a_0)^3}\frac{r^4}{a_0^2}\cdot\frac{3}{8\pi}\cdot\frac{2}{3}\frac{\pi}{2}\right.$$

$$- \int_0^\infty \frac{8e^{-r/a}}{3(2a_0)^3}\frac{r^4}{a_0^2}\cdot\frac{-3}{8\pi}\cdot\frac{2}{3}\cdot\frac{\pi}{2}\,dr$$

$$\left.+ \int_0^\infty \frac{5e^{-r/a}}{3(2a_0)^3}\frac{r^4}{a_0^2}\cdot\frac{3}{8\pi}\cdot\frac{2}{3}\frac{\pi}{2}\,dr\right]$$

$$= \frac{1}{9}\left[\int_0^\infty \frac{e^{-r/a_0}r^4}{48a_0^5}dr - \int_0^\infty \frac{e^{-r/a_0}r^4}{24a_0^5}dr + \int_0^\infty \frac{5e^{-r/a_0}r^4}{192a_0^5}dr\right.$$

$$= \frac{1}{9}\left[\frac{1}{8} + \frac{1}{4} + \frac{5}{32}\right] \boxed{= \frac{17}{288} \approx 0.059 \approx 5.9\%}$$

(C) $\Psi_{n\ell m}$

Magnitude

$|L| = \sqrt{\ell(\ell+1)}\,\hbar$ one ℓ value $\ell = 1$

$|L| = \sqrt{1(1+1)} = \boxed{\sqrt{2}\,\hbar}$ $P = 1$

Z-Component

$L_z = m\hbar$

$L_z = 1\cdot\hbar = \boxed{\hbar}$ $P_\hbar = |\langle 1|\psi\rangle|^2 = \boxed{4/9}$

$L_z = -1\cdot\hbar = \boxed{-\hbar}$ $P_{-\hbar} = |\langle -1|\psi\rangle|^2 = \boxed{5/9}$

$m_s = 1/2$
$S = 1/2$

(D) $|\psi\rangle = 1/3\left(2|2,1,1\rangle - (2-i)|2,1,-1\rangle\right)\otimes|+\rangle$

$n=2,\ \ell=1,\ m_\ell=1$ $n=2,\ \ell=1,\ m_\ell=-1$

$m_\ell + m_s = 3/2$ $m_\ell + m_s = -1/2$