PHY 420 **Spring 2021**

Week 2—Thursday, Apr 8—Discussion Worksheet

Laplace Equation in Cylindrical Coordinates

In cylindrical coordinates (ρ, ϕ, z), Laplace's equation is

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{3.71}$$

1. The equation above looks a little different from what is given in Jackson.

(a) In order to match Jackson, rewrite equation (3.71) with the first term expanded to two terms.

$$\frac{\partial^2 \varphi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \varphi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \varphi}{\partial \varphi^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

(b) Separate variables by setting

$$\Phi(\rho, \phi, z) = R(\rho)\, Q(\phi)\, Z(z) \tag{3.72}$$

and show that this leads to the three ordinary differential equations:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \tag{3.73}$$

$$\frac{d^2 Q}{d\phi^2} + \nu^2 Q = 0 \tag{3.74}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0 \tag{3.75}$$

(3.72) into (3.71) $\quad QZ\frac{d^2 R}{d\rho^2} + \frac{QZ}{\rho}\frac{dR}{d\rho} + \frac{RZ}{\rho^2}\frac{d^2 Q}{d\varphi^2} + RQ\frac{d^2 Z}{dz^2} = 0$

Divide by RQZ $\quad \underbrace{\frac{1}{R}\frac{d^2 R}{d\rho^2} + \frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{\rho^2 Q}\frac{d^2 Q}{d\varphi^2}}_{} + \underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}}_{+k^2} = 0$

$+k^2 \rightarrow \frac{1}{Z}\frac{dZ}{dz^2} = k^2$

$$\frac{\rho^2}{R}\frac{d^2 R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + \frac{1}{Q}\frac{dQ^2}{d\varphi^2} = -k^2\rho^2$$

$$\rightarrow \underbrace{\frac{\rho^2}{R}\frac{d^2 R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + k^2\rho^2}_{+\nu^2} + \underbrace{\frac{1}{Q}\frac{d^2 Q}{d\varphi^2}}_{-\nu^2} = 0 \rightarrow \frac{1}{Q}\frac{d^2 Q}{d\varphi^2} = -\nu^2 \rightarrow \frac{d^2 Q}{d\varphi^2} + \nu^2 q = 0$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + k^2 R = \nu^2\left(\frac{R}{\rho^2}\right) \Rightarrow \frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0$$

2. The solutions to equations (3.73) and (3.74) are, respectively

$$Z(z) = e^{\pm kz}$$
$$Q(\phi) = e^{\pm i\nu\phi} \tag{3.76}$$

Note that the z-solution is dictated by our choice of constant after separation of variables. Later, we will also examine cases for which $Z(z) = e^{\pm ikz}$. For the potential to be single-valued when the full azimuthal span is allowed, ν must be an integer. However, the parameter k is arbitrary although, as always, it might be constrained by some boundary condition requirement in the z-direction. For the present, we assume that k is real and positive.

(a) Verify by explicit substitution that $Z(z)$ and $Q(\phi)$ in equation (3.76) are indeed solutions to equation (3.73) and (3.74) respectively.

$$Z = e^{+kz} \rightarrow \frac{dZ}{dz} = ke^{kz} \rightarrow \frac{d^2Z}{dz^2} = k^2 e^{kz}$$
$$Z = e^{-kz} \rightarrow \frac{dZ}{dz} = -ke^{-kz} \rightarrow \frac{d^2Z}{dz^2} = +k^2 e^{-kz}$$

$$\frac{d^2Z}{dz} - k^2 Z = k^2 e^{\pm kz} - k^2 e^{\pm kz} = 0$$

$$Q = e^{i\nu\varphi} \rightarrow \frac{dQ}{d\varphi} = i\nu e^{i\nu\varphi} \rightarrow \frac{d^2Q}{d\varphi^2} = -\nu^2 e^{i\nu\varphi}$$
$$Q = e^{-i\nu\varphi} \rightarrow \frac{dQ}{d\varphi} = (-i\nu)e^{-i\nu\varphi} \rightarrow \frac{d^2Q}{d\varphi^2} = -\nu^2 e^{-i\nu\varphi}$$

$$\frac{d^2Q}{d\varphi^2} + \nu^2 Q = -\nu^2 e^{\pm i\nu\varphi} + \nu^2(e^{\pm i\nu\varphi}) = 0$$

(b) The radial equation (3.75) can be put into a standard form for which we know the solutions from mathematics. Show that equation (3.75) can be written in the form

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0 \tag{3.77}$$

where $x =$ ρk. Equation (3.77) is known as the **Bessel equation** and its solutions are the **Bessel functions** of order ν.

From (3.75) and (3.77)

$$\frac{1}{k^2}\frac{d^2R}{d\rho^2} + \frac{1}{k^2\rho}\frac{dR}{d\rho} + \left(1 - \frac{\nu^2}{k^2\rho^2}\right) R = 0$$

$$\rightarrow \frac{d^2R}{d(k\rho)^2} + \frac{1}{k\rho}\frac{dR}{d(k\rho)} + \left[1 - \frac{\nu^2}{(k\rho)^2}\right] R = 0$$

$x = k\rho$

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0$$

Assuming a power series solution of the form

$$R(x) = x^{\alpha} \sum_{j=0}^{\infty} a_j x^j \tag{3.78}$$

allows us to find that

$$\alpha = \pm\nu \tag{3.79}$$

and

$$a_{2j} = -\frac{1}{4j(j+\alpha)}\, a_{2j-2} \tag{3.80}$$

for $j = 1, 2, 3, \ldots$, whereas all odd powers of x^j have vanishing coefficients.

3. To get the Bessel function solutions, we iterate the recursion formula (3.80) in steps of $(j-1)$ down to the first term:

$$\begin{aligned} a_{2j} &= -\frac{1}{4j(j+\alpha)}\, a_{2j-2} \\ &= -\frac{1}{4j(j+\alpha)} \left[-\frac{1}{4(j-1)\,(j-1+\alpha)}\, a_{2j-4}\right] \\ &= +\frac{1}{4j(j+\alpha)} \frac{1}{4(j-1)\,(j-1+\alpha)} \left[-\frac{1}{4(j-2)\,(j-2+\alpha)}\, a_{2j-6}\right] \\ &= \ldots \end{aligned}$$

(a) By examining this pattern, write down a_{2j} in terms of a_0.

Hint: Each step contributes 2^2, there are j steps, and each step has alternating sign.

1^{st} step: $a_{2j-2} = a_{2(j-1)}$ has $j(j+\alpha)$ in the denominator

last step: $a_0 = a_{2(0)} = a_{2(1-1)}$ must hae $1(1+\alpha)$

$$a_{2j} = \frac{(-1)^j}{2^{2j}} \left[\frac{1}{j(j+\alpha)} \frac{1}{(j-1)(j-1+\alpha)} \cdots \frac{1}{1(1+\alpha)} \right] a_{2(0)}$$

$$\rightarrow a_{2j} = \frac{(-1)^j}{2^{2j}} \left[\frac{1}{j(j-1)(j-2)\ldots 1} \right] \left[\frac{1}{(\alpha+j)(\alpha+j-1)\ldots(\alpha+1)} \right] a_0$$

(b) Show that your answer above can be written in the more compact form

$$a_{2j} = \frac{(-1)^j}{2^{2j}\, j!} \frac{\Gamma(\alpha+1)}{\Gamma(j+\alpha+1)}\, a_0 \tag{3.81}$$

where the Γ function is defined as $\Gamma(p) = (p-1)!$.

$$a_{2j} = \frac{(-1)^j}{2^{2j}} \left[\frac{1}{j!} \right] \left[\frac{\alpha(\alpha-1)(\alpha-2)\ldots 1}{(\alpha+j)(\alpha+j-1)\ldots(\alpha+1)\,\alpha(\alpha-1)(\alpha-2)\ldots 1} \right] a_0$$

$$= \frac{(-1)^j}{2^{2j}} \left[\frac{1}{j!} \right] \left[\frac{\alpha!}{(\alpha+j)!} \right] a_0 \rightarrow \begin{cases} \alpha! = (\alpha+1-1)! = \Gamma(\alpha+1) \\ (\alpha+j+1-1)! = \Gamma(\alpha+j+1) \end{cases}$$

Thus,

$$a_{2j} = \frac{(-1)^j}{2^{2j}\, j!} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+j+1)}\, a_0$$

By convention, we choose

$$a_0 = \frac{1}{2^\alpha\,\Gamma(\alpha+1)}$$

Then, with $\alpha = \pm\nu$ from equation (3.79), we get

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\;\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j} \tag{3.82}$$

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\;\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j} \tag{3.83}$$

The solutions $J_\nu(x)$ and $J_{-\nu}(x)$ are called ***Bessel functions of the first kind*** of order $\pm\nu$.

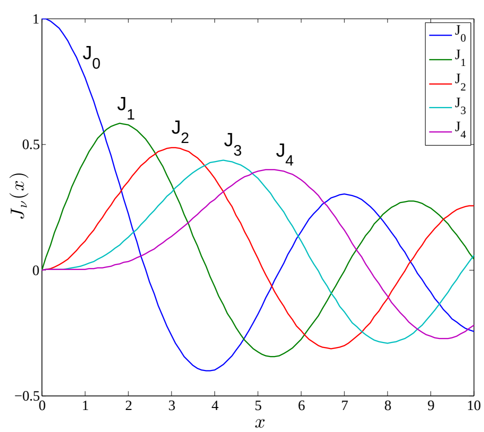
If ν is not an integer, these two solutions $J_{\pm\nu}(x)$ form a pair of linearly independent solutions to the second order Bessel equation. On the other hand, if ν is an integer, the solutions are linearly dependent; for $\nu = m$, an integer, we get

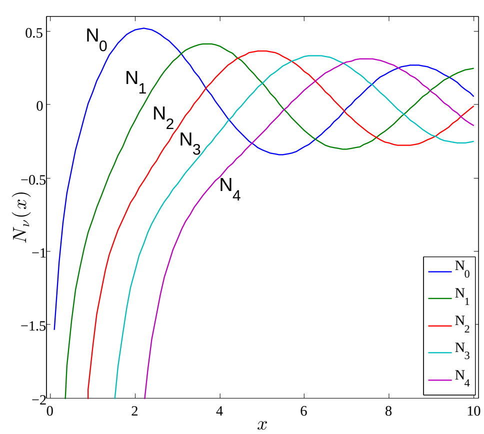
$$J_{-m}(x) = (-1)^m\,J_m(x) \tag{3.84}$$

So, if ν is an integer, we need to find another linearly independent solution. More generally, even if ν is not an integer, it is customary to replace the pair $J_{\pm\nu}(x)$ by $J_\nu(x)$ and $N_\nu(x)$, the so-called **Neumann function**, or ***Bessel function of the second kind***:

$$N_\nu(x) = \frac{J_\nu(x)\,\cos\,\nu\pi - J_{-\nu}(x)}{\sin\,\nu\pi} \tag{3.85}$$

Graphs of $J_\nu(x)$ and $N_\nu(x)$. are shown below.





4. In problems where the geometry includes the origin, can you have $N_\nu(x)$ in the solution? Explain why or why not.

No, $N_\nu(x) \to \infty$, as $x \to 0$, so the solution will diverge at the origin if it includes $N_\nu(x)$.