PHY 411 Winter 2021

Week 1—Tuesday, Jan 5—Discussion Worksheet

Math You Should Know

We will begin by reviewing some of the math we'll be using in this class. You will be working with vectors quite a bit, and it is worth deriving some expressions explicitly, just so you are comfortable with vectors.

1. Carry out the following explicitly using the unit vector relationships from equations (W1.13)-(W1.16) as needed, and show that:

$$\vec{A} \times \vec{B} = \Big(A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}\Big) \times \Big(B_x\,\hat{x} + B_y\,\hat{y} + B_z\,\hat{z}\Big)$$

$$= \Big(A_yB_z - A_zB_y\Big)\,\hat{x} + \Big(A_zB_x - A_xB_z\Big)\,\hat{y} + \Big(A_xB_y - A_yB_x\Big)\,\hat{z}$$

$$\begin{aligned}\vec{A} \times \vec{B} = \Big(& \overset{0}{\cancel{A_x\hat{x} \times B_x\hat{x}}} + A_x\hat{x} \times B_y\hat{y} + A_x\hat{x} \times B_z\hat{z} + \\ & A_y\hat{y} \times B_x\hat{x} + \overset{0}{\cancel{A_y\hat{y} \times B_y\hat{y}}} + A_y\hat{y} \times B_z\hat{z} + \\ & A_z\hat{z} \times B_x\hat{x} + A_z\hat{z} \times B_y\hat{y} + \overset{0}{\cancel{A_z\hat{z} \times B_z\hat{z}}} \Big)\end{aligned}$$

$$\begin{aligned} = \Big(& 0 + A_xB_y\hat{z} - A_xB_z\hat{y} - \\ & A_yB_x\hat{z} + 0 + A_yB_z\hat{x} + \\ & A_zB_x\hat{y} - A_zB_y\hat{x} + 0 \Big)\end{aligned}$$

$$\boxed{= (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}}$$

Remember

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

2. Some things are worth doing explicitly at least once, just so you get to see how it works. Therefore, ***without*** using any conveniences like Levi-Civita, prove by explicit evaluation that

$$\vec{\nabla} = \hat{x}\left(\frac{\partial}{\partial x}\right) + \hat{y}\left(\frac{\partial}{\partial y}\right) + \hat{z}\left(\frac{\partial}{\partial z}\right) \qquad \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right) = 0$$

where $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$ is any arbitrary vector.

$$\vec{\nabla}\times\vec{A} = \left(\hat{x}\left(\frac{\partial}{\partial x}\right) + \hat{y}\left(\frac{\partial}{\partial y}\right) + \hat{z}\left(\frac{\partial}{\partial z}\right)\right) \times \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)$$

$$= \Big(\cancel{\hat{x}\left(\frac{\partial}{\partial x}\right)\times A_x\hat{x}}^{\,0} + \hat{x}\left(\frac{\partial}{\partial x}\right)\times A_y\hat{y} + \hat{x}\left(\frac{\partial}{\partial x}\right)\times A_z\hat{z} +$$

$$\hat{y}\left(\frac{\partial}{\partial y}\right)\times A_x\hat{x} + \cancel{\hat{y}\left(\frac{\partial}{\partial y}\right)\times A_y\hat{y}}^{\,0} + \hat{y}\left(\frac{\partial}{\partial y}\right)\times A_z\hat{z} +$$

$$\hat{z}\left(\frac{\partial}{\partial z}\right)\times A_x\hat{x} + \hat{z}\left(\frac{\partial}{\partial z}\right)\times A_y\hat{y} + \cancel{\hat{z}\left(\frac{\partial}{\partial z}\right)\times A_z\hat{z}}^{\,0} \Big)$$

$$= A_y\left(\frac{\partial}{\partial x}\right)\hat{z} - A_z\left(\frac{\partial}{\partial x}\right)\hat{y}$$

$$- A_x\left(\frac{\partial}{\partial y}\right)\hat{z} + A_z\left(\frac{\partial}{\partial y}\right)\hat{x}$$

$$- A_x\left(\frac{\partial}{\partial z}\right)\hat{y} - A_y\left(\frac{\partial}{\partial z}\right)\hat{x}$$

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right) = \left(\hat{x}\left(\frac{\partial}{\partial x}\right) + \hat{y}\left(\frac{\partial}{\partial y}\right) + \hat{z}\left(\frac{\partial}{\partial z}\right)\right)$$

$$\cdot \left(A_z\left(\frac{\partial}{\partial y}\right) - A_y\left(\frac{\partial}{\partial z}\right)\right)\hat{x} - \left(A_z\left(\frac{\partial}{\partial x}\right) + A_x\left(\frac{\partial}{\partial z}\right)\right)\hat{y}$$

$$+ \left(A_y\left(\frac{\partial}{\partial x}\right) - A_x\left(\frac{\partial}{\partial y}\right)\right)\hat{z}$$

$$= \cancel{\left(\frac{\partial}{\partial x}\right)}\left(\cancel{A_z\left(\frac{\partial}{\partial y}\right)} - \cancel{A_y\left(\frac{\partial}{\partial z}\right)}\right) - \cancel{\left(\frac{\partial}{\partial y}\right)}\left(\cancel{A_z\left(\frac{\partial}{\partial x}\right)} + \cancel{A_x\left(\frac{\partial}{\partial z}\right)}\right)$$

$$+ \cancel{\left(\frac{\partial}{\partial z}\right)}\left(\cancel{A_y\left(\frac{\partial}{\partial x}\right)} - \cancel{A_x\left(\frac{\partial}{\partial y}\right)}\right) = 0$$

3. By explicit evaluation (i.e., ***without*** using any conveniences like Levi-Civita), show that

$$\vec{\nabla} \times (f\vec{A}) = \vec{\nabla} f \times \vec{A} + f\,\vec{\nabla} \times \vec{A}$$

where f is a scalar, and $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$ is any arbitrary vector.

$$\vec{\nabla} \times (f\vec{A}) = \left(\hat{x}\left(\frac{\partial}{\partial x}\right) + \hat{y}\left(\frac{\partial}{\partial y}\right) + \hat{z}\left(\frac{\partial}{\partial z}\right)\right) \times \left(fA_x\hat{x} + fA_y\hat{y} + fA_z\hat{z}\right)$$

$$\begin{aligned}
= \Big(& \cancelto{0}{\hat{x}\left(\frac{\partial}{\partial x}\right) \times fA_x\hat{x}} + \hat{x}\left(\frac{\partial}{\partial x}\right) \times fA_y\hat{y} + \hat{x}\left(\frac{\partial}{\partial x}\right) \times fA_z\hat{z} + \\
& \hat{y}\left(\frac{\partial}{\partial y}\right) \times fA_x\hat{x} + \cancelto{0}{\hat{y}\left(\frac{\partial}{\partial y}\right) \times fA_y\hat{y}} + \hat{y}\left(\frac{\partial}{\partial y}\right) \times fA_z\hat{z} + \\
& \hat{z}\left(\frac{\partial}{\partial z}\right) \times fA_x\hat{x} + \hat{z}\left(\frac{\partial}{\partial z}\right) \times fA_y\hat{y} + \cancelto{0}{\hat{z}\left(\frac{\partial}{\partial z}\right) \times fA_z\hat{z}} \Big)
\end{aligned}$$

4. By explicit evaluation, show that

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

where $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$, and $\vec{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$ are arbitrary vectors. *Explicit evaluation* means that you shouldn't use any conveniences like Levi-Civita, etc.

From problem 1 we know that

$$\vec{A} \times \vec{B} = (A_y B_z - A_x B_y)\hat{x} + (A_z B_z - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

5. Use of the Levi-Civita tensor often makes steps more compact.

(a) The $k = 1$ (i.e., $k = x$) cross product term (see equation W1.12) is

$$(\vec{A} \times \vec{B})_x = A_y B_z - A_z B_y \tag{1}$$

The cross-product, written using the Levi-Civita tensor defined in equation (W1.19), is

$$(\vec{A} \times \vec{B})_k = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \epsilon_{ijk}$$

Write out all the 9 terms in this summation (for $k = x$), determine which ones become zero and why, and then show that what is left looks like the expression written in equation (1) above.

$$(\vec{A} \times \vec{B})_{k=x} = A_x B_x \overset{0}{\cancel{\epsilon_{xxx}}} + A_x B_y \cancel{\epsilon_{xyx}} + A_x B_z \cancel{\epsilon_{xzx}} \quad \Big] \; i = 1 \to x$$

$$0 + 0 + A_y B_z \epsilon_{yzx} \quad \Big] \; i = 2 = y$$

$$0 + A_z B_y \epsilon_{zyx} + 0 \quad \Big] \; i = 3 \to z$$

$$(\vec{A} \times \vec{B})_{k=x} = A_y B_z \epsilon_{yzx} + A_z B_y \epsilon_{zyx}$$

$$= A_y B_x (+1) + A_z B_y (-1)$$

(b) Show that the cross product anti-commutes, that is

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$