PHY 420 **Spring 2021**

Homework 7—due by 9:00 PM, Friday, May 28

You may write by hand and scan as a single PDF, or write in latex (using the template file provided) or Word, and generate PDF. Please submit one PDF file only. Only questions and sub-parts that are numbered clearly, with numbers corresponding to those in this document, will be graded. See the syllabus for more detailed rules.

Submit late homework into the late D2L dropbox for reduced credit (see syllabus); late submissions will be accepted until 8 AM on Monday (May 31). Emailed or paper copies of homework are never accepted; in particular, do not attach homework to email to make an end-run around the D2L deadline or late deadline; such emails are automatically deleted and do not count as submissions.

1. In class, we learned that finding a group of linear transformations that leaves $x \cdot x$ invariant is equivalent to finding all square 4×4 matrices A which, when they transform the coordinates as $x' = Ax$, will leave the norm (x, gx) invariant, that is, they will ensure that $x' \cdot x' = x \cdot x$. Using the six fundamental matrices S_i and K_i $(i = 1, 2, 3)$ in equation (11.91), Jackson constructed the matrix A as

$$A = e^{-\vec{\omega}\cdot\vec{S}-\vec{\zeta}\cdot\vec{K}}$$

where $\vec{\omega}$ and $\vec{\zeta}$ are constant 3-vectors whose components correspond to the six parameters of the transformation. In class, you constructed A for the case of no rotation and a boost along the x^1 axis. In this problem, you will do another example.

For the case of **rotation about the x^3 axis without any boost**, we have: $\vec{\omega} = \omega\hat{\epsilon}_3, \vec{\zeta} = 0$. By running through steps similar to those on Questions 5 and 6 in the Discussion Worksheet for Week 8—Tue, May 18, show that you get the expected matrix for A; that is, show that you get the matrix you wrote in Question 2(b) for rotations about the x^3 axis on that worksheet.

2. Express the Lorentz scalar $F^{\alpha\beta}F_{\alpha\beta}$ in terms of $\vec{E}$ and $\vec{B}$, where $F^{\alpha\beta}$ is given by

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{1}$$

and $F_{\alpha\beta}$ can be obtained from $F^{\alpha\beta}$ by the procedure you worked out on the class worksheet (i.e., by putting $E_i \to -E_i$, and leaving B_i unchanged).

3. Consider the fundamental matrices $S_1, S_2, S_3, K_1, K_2, K_3$ written in equation (11.91) in Jackson. By explicit matrix multiplication, find the commutators

$$[S_2, S_3], \qquad [S_2, K_3], \qquad \text{and} \qquad [K_2, K_3]$$

4. In class, we wrote the field-strength tensor $F^{\alpha\beta}$ starting from the Maxwell equations. You will now derive the elements of $F^{\alpha\beta}$ by writing $\vec{E}$ and $\vec{B}$ in terms of Φ and $\vec{A}$.

Recall that the fields $\vec{E}$ and $\vec{B}$ can be expressed in terms of the potentials as

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi \qquad \text{and} \qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{2}$$

(a) Write down ***all the components*** of $\vec{E}$ and $\vec{B}$ using the ∂^α notation. To do so, first explicitly **show that** the x-components of $\vec{E}$ and $\vec{B}$ are, respectively

$$E_x = -\left(\partial^0 A^1 - \partial^1 A^0\right)$$

and

$$B_x = -\left(\partial^2 A^3 - \partial^3 A^2\right)$$

where

$$\partial^\alpha = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right)$$

and ***then write down by analogy*** E_y, E_z, B_y, and B_z.

(b) Show that the components of $\vec{E}$ and $\vec{B}$ you obtained above are the elements of the field tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{3}$$

by using equation (3) to explicitly generate ***all*** elements of $F^{\alpha\beta}$, and comparing your results to the expressions you obtained in part (a) and referencing equation (1) written on the previous page.

For example, when you write F^{01} using equation (3) above and compare to the expressions you wrote in part (a), you should find that it looks like $-E_x$. Now look in equation (1) on the previous page at the location of F^{01}; it is the element in the first row and the second column, and verify that it is indeed $-E_x$. Do this for all elements of $F^{\alpha\beta}$.