PHY 420 Spring 2021

Class Summary—Week 9, Day 2—Thursday, May 27

Transformation of Electromagnetic Fields (continued)

For the specific Lorentz transformation corresponding to a boost along the x^1 axis with speed $c\beta$ from the unprimed frame to the primed frame, the explicit **equations of transformation** are

$$\begin{aligned} E_1' &= E_1 & B_1' &= B_1 \\ E_2' &= \gamma\left(E_2 - \beta B_3\right) & B_2' &= \gamma\left(B_2 + \beta E_3\right) \\ E_3' &= \gamma\left(E_3 + \beta B_2\right) & B_3' &= \gamma\left(B_3 - \beta E_2\right) \end{aligned} \tag{11.148}$$

In the previous class, you verified the transformation relations for E_1' and E_2'.

In Question 1(a) of today's worksheet, you wrote down the matrices corresponding to $F' = AF\tilde{A}$. Then, *in Questions 1(b), 2, 3(a)*, you verified the transformation relations for E_3', B_1', B_2', and B_3'.

The inverse of equation (11.148) is found, as usual, by interchanging primed and unprimed quantities and putting $\beta \to -\beta$. As *you showed in Question 3(b) on today's worksheet*, you should get

$$\begin{aligned} E_1 &= E_1' & B_1 &= B_1' \\ E_2 &= \gamma\left(E_2' + \beta B_3'\right) & B_2 &= \gamma\left(B_2' - \beta E_3'\right) \\ E_3 &= \gamma\left(E_3' - \beta B_2'\right) & B_3 &= \gamma\left(B_3' + \beta E_2'\right) \end{aligned}$$

For a general Lorentz transformation from K to a frame K' moving with velocity $\vec{v}$ relative to K, the transformation of the fields is

$$\begin{aligned} \vec{E}' &= \gamma\left(\vec{E} + \vec{\beta} \times \vec{B}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right) \\ \vec{B}' &= \gamma\left(\vec{B} - \vec{\beta} \times \vec{E}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right) \end{aligned} \tag{11.149}$$

where $\vec{\beta} = \vec{v}/c$. The transformations in equation (11.149) demonstrate clearly that $\vec{E}$ and $\vec{B}$ have no independent existence. We will discuss this in more detail on the next page.

For now, it is useful to write the inverse of the relations in equation (11.149), which we'll need for proofs later. Recall that the inverse transformation can be written by swapping the primes and setting $\beta \to -\beta$. Since β occurs twice in the second term on the right hand, the sign remains unchanged for the second term, and we get

$$\begin{aligned} \vec{E} &= \gamma\left(\vec{E}' - \vec{\beta} \times \vec{B}'\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{E}'\right) \\ \vec{B} &= \gamma\left(\vec{B}' + \vec{\beta} \times \vec{E}'\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{B}'\right) \end{aligned}$$

The transformations in equation (11.149) demonstrate clearly that $\vec{E}$ and $\vec{B}$ have no independent existence.

- A purely electric or magnetic field in one coordinate system will appear as a mixture of electric and magnetic fields in another coordinate frame.
- Certain restrictions, apply, of course — as you will show on the homework, a purely electrostatic field in one coordinate system cannot be transformed into a purely magnetostatic field in another.
- In summary, the fields are interrelated, and one should speak of the electromagnetic field $F^{\alpha\beta}$, rather than $\vec{E}$ and $\vec{B}$ separately.

Now, consider the case when **there is only an electric field, and no magnetic field exists in a certain frame** K'. This could be the case if, for example, there are one or more point charges at rest in K'. For such an instance, that is when no magnetic field exists in a certain frame K', the inverse of equation (11.149) written on the previous page shows that in the frame K, the magnetic field $\vec{B}$ and electric field $\vec{E}$ are linked by the simple relation, *as you demonstrated in Question 5 on today's worksheet*:

$$\vec{B} = \vec{\beta} \times \vec{E} \tag{11.150}$$

where you should take note that $\vec{E}$ is not the electrostatic field in K', but instead the field transformed from K' to K.

I'm going to write down the derivation of equation (11.150) here even though you've already demonstrated it on the worksheet; it is very important. Since $\vec{B}\,' = 0$, we get from the second equation for $\vec{B}$ written at the bottom of the previous page that

$$\vec{B} = \gamma\left(0 + \vec{\beta} \times \vec{E}\,'\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot 0\right)$$

or

$$\vec{B} = \gamma\left(\vec{\beta} \times \vec{E}\,'\right)$$

Meanwhile from the first equation of the two written at the bottom of the previous page, we get for $\vec{B}\,' = 0$ that

$$\vec{E} = \gamma\left(\vec{E}\,' - 0\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot \vec{E}\,'\right)$$

so that, upon taking the cross products of both sides with $\vec{\beta}$, we get

$$\vec{\beta} \times \vec{E} = \gamma\left(\vec{\beta} \times \vec{E}\,'\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta} \times \vec{\beta}\left(\vec{\beta}\cdot \vec{E}\,'\right)$$

Since $\vec{\beta} \times \vec{\beta}$ must be zero, this reduces to

$$\vec{\beta} \times \vec{E} = \gamma\left(\vec{\beta} \times \vec{E}\,'\right) = \vec{B}$$

where I've used the relation above (in the equation written in purple font), that $\vec{B} = \gamma\left(\vec{\beta} \times \vec{E}\,'\right)$. Therefore, we have derived that

$$\vec{B} = \vec{\beta} \times \vec{E}$$

as written in equation (11.150)