PHY 411 **Winter 2021**

Week 8—Tuesday, Feb 23—Discussion Worksheet

Multipole Expansion of $\Phi(\vec{x})$

We will now study the potential due to a localized charge distribution and its expansion in multipoles. The larger aim is to prepare us for the expansion of the vector potential that we will need for learning about radiating systems in Chapter 9.

1. Consider a localized distribution of charge described by the charge density $\rho(\vec{x}\,')$ that is contained within a sphere of radius R around some origin. Note that the sphere of radius R is an arbitrary conceptual device employed merely to divide space into regions with and without charge.

Outside the sphere of radius R, the potential can be written as an expansion in spherical harmonics:

$$\Phi(\vec{x}) \equiv \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}\,r^{l} + \frac{B_{lm}}{r^{l+1}}\right] Y_{lm}(\theta,\phi) \tag{3.61}$$

where $Y_{lm}(\theta,\phi)$ are the *spherical harmonics*, given by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi} \tag{3.53}$$

(a) What must we do in equation (3.61) for the problem under consideration, remembering that we want to be far away from the charge distribution but we will *not* be anywhere near $r = 0$?

We need to drop $A_{lm} r^l$

$A_{lm} = 0$, for all l

Since we want to be far away from the charge Distribution (otherwise $A_{lm} r^l$ will blow up the solution at large r)

(b) Based on your answer above, write down the new form for the potential.

$$\Phi(\vec{X}) = \Phi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{B_{lm}}{r^{l+1}}\right] Y_{lm}(\theta,\varphi)$$

Please STOP here! Do not proceed until told to do so.

2. To proceed, we will write the B_{lm}'s in the form

$$B_{lm} = \frac{1}{4\pi\epsilon_0}\,\frac{4\pi}{2l+1}\,q_{lm}$$

to match Jackson's "particular choice of constant coefficients ... made for later convenience" and you will work out the form of q_{lm} below.

(a) Write down the expansion for the potential after writing B_{lm} in the manner above.

$$\Phi(\vec{x}) = \Phi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{4\pi\epsilon_0}\,\frac{4\pi}{(2l+1)}\,q_{lm}\right] Y_{lm}(\theta,\varphi) \tag{4.1}$$

Equation (4.1) for $\Phi(\vec{x})$ is called a **multipole expansion**; the $l = 0$ term is called the monopole term, $l = 1$ are the dipole terms, etc. — the reason for these names will become clear shortly.

We must now determine the quantities q_{lm} in terms of the properties of the charge density $\rho(\vec{x}\,')$ in order to obtain to fully solve the problem. To do so, let us look at the integral form for the potential that we wrote previously in equation (1.17):

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x'$$

and substitute in it equation (3.70) for $1/|\vec{x}-\vec{x}\,'|$ that we wrote previously:

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}}\right)Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi) \tag{3.70}$$

(b) Since we're interested in the potential outside the charge distribution, what should we put for $r_<$ and $r_>$ in equation (3.70) above?

$r_< = r'$ and $r_> = r$

(c) After making the appropriate choices for $r_<$ and $r_>$ in equation (3.70), substitute it into $\Phi(\vec{x})$ above and show that

$$\Phi(\vec{x}) = \frac{1}{\epsilon_0}\sum_{l,m}\frac{1}{2l+1}\,q_{lm}\,\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \qquad \text{where} \qquad q_{lm} = \int Y^*_{lm}(\theta',\varphi')(r')\,\rho(\vec{x}\,')\,d^3x'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}\,')\left[4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\,\frac{(r')^l}{r^{l+1}}\,Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)\right]d^3x'$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\underbrace{\left[\int Y^*_{lm}(\theta',\varphi')(r')^l\,\rho(\vec{x}\,')\,d^3x'\right]}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\,\boxed{q_{lm}}\,\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

3. The coefficients q_{lm}, given by

$$q_{lm} = \int Y^*_{lm}(\theta', \phi')\, r'^{\,l}\, \rho(\vec{x}\,')\, d^3x' \tag{4.3}$$

are called the **multipole moments**. To see their physical significance, let's express the first few explicitly in terms of Cartesian coordinates.

(a) Show that $q_{00} = \dfrac{q}{\sqrt{4\pi}}$.

$$\underset{l\ m}{q_{00}} = \int \underbrace{Y^*_{00}(\theta', \varphi')}_{\downarrow} (r')^0\, \rho(\vec{x}\,')\, d^3x'$$

$$= \frac{1}{\sqrt{4\pi}} \int \rho(\vec{x}\,')\, d^3x'$$

$$= q/\sqrt{4\pi}$$

(b) Find $\Phi(\vec{x})|_{l=0}$, the potential taking only the $l = 0$ term into account.

$$\Phi(\vec{x})\Big|_{l=0} = \frac{1}{\cancel{4\pi}\epsilon_0} \sum_l \sum_m \frac{\cancel{4\pi}}{2l+1}\, q_{lm}\, \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}}$$

$$= \frac{1}{\epsilon_0}\, \frac{1}{2(0)+1}\, q_{00}\, \frac{Y_{00}(\theta, \varphi)}{r^{0+1}}$$

$$= \frac{1}{\epsilon_0}\, \frac{1}{1} \left[\frac{q}{\sqrt{4\pi}}\right] \frac{1/\sqrt{4\pi}}{r} \Rightarrow \Phi(\vec{x})\Big|_{l=0} = \frac{q}{4\pi\epsilon_0 r}$$

(c) Discuss whether your result in part (b) makes sense. Remember that you're far away from the charge distribution.

If you are far away from a total charge q, it appears like a point charge (to lowest order), and potential ~~of~~ a point charge is $q/4\pi\epsilon_0 r$. This result also explains the designation of q_{00} as the monopole moment.

On Homework 7, you will demonstrate that the $l = 1$ terms (q_{11}, q_{10}) are proportional to the components of the electric dipole moment $\vec{p}$. Meanwhile, the $l = 2$ terms (q_{22}, q_{21}, q_{20}) are proportional to the quadrupole moments Q_{ij}.

4. Now, consider again: $$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1}\, q_{lm}\, \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

Since $\vec{E} = -\vec{\nabla}\Phi$, show by direct differentiation of $\Phi(\vec{x})$ that **for a particular** (l, m), the radial component of the electric field is given by

$$E_r = \frac{(l+1)}{(2l+1)\,\epsilon_0}\, q_{lm}\, \frac{Y_{lm}(\theta,\phi)}{r^{l+2}}$$

For a particular value of $(l,m) \Rightarrow$ no summations

$$\Phi = \frac{1}{\cancel{4\pi}\epsilon_0} \frac{\cancel{4\pi}}{2l+1}\, q_{lm}\, \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

and

$$E_r = -\nabla_r \Phi = -\left[\frac{\partial \Phi}{\partial r}\right]$$

Thus,

$$E_r = -\frac{1}{(2l+1)\epsilon_0}\, q_{lm} Y_{lm}(\theta,\varphi)\, \frac{\partial}{\partial r}\left[\frac{1}{r^{l+1}}\right]$$

$$= -\frac{q_{lm} Y_{lm}(\theta,\varphi)}{(2l+1)\epsilon_0} \frac{\partial}{\partial r}\left[r^{-(l+1)}\right]$$

$$= -\frac{q_{lm} Y_{lm}(\theta,\varphi)}{(2l+1)\epsilon_0} \frac{\partial}{\partial r}\left[r^{-(l+1)}\right]$$

Thus,

$$E_r = \frac{l+1}{(2l+1)\epsilon_0}\, q_{lm}\, \frac{Y_{lm}(\theta,\varphi)}{r^{l+2}}$$

5. We can also find the components E_θ and E_ϕ of the field by direct differentiation of the potential Φ given on the previous page.

(a) Show by direct differentiation of $\Phi(\vec{x})$ that **for a particular** (l, m), the θ-component of the electric field is given by

$$E_\theta = -\frac{1}{(2l+1)\,\epsilon_0}\,q_{lm}\,\frac{1}{r^{l+2}}\,\frac{\partial}{\partial\theta}\,Y_{lm}(\theta,\phi)$$

$$E_\theta = -\vec{\nabla}_\theta \Phi \Rightarrow E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta}$$

$$E_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{(2l+1)\epsilon_0} q_{lm} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\right]$$

thus,

$$E_\theta = -\frac{1}{(2l+1)\epsilon_0} q_{lm} \frac{1}{r^{l+2}} \frac{\partial}{\partial\theta} Y_{lm}(\theta,\varphi)$$

(b) Show by direct differentiation of $\Phi(\vec{x})$ that **for a particular** (l, m), the ϕ-component of the electric field is given by

$$E_\phi = \frac{1}{(2l+1)\,\epsilon_0}\,q_{lm}\,\frac{1}{r^{l+2}}\,\frac{im}{\sin\,\theta}\,Y_{lm}(\theta,\phi)$$

$$E_\varphi = -\vec{\nabla}_\varphi \Phi$$

$$= -\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \varphi}$$

$$= -\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left[\frac{1}{(2l+1)\epsilon_0} q_{lm} \frac{1}{r^{l+1}}\left(\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}\right)\right.$$

$$E_\varphi = -\frac{1}{(2l+1)\epsilon_0} q_{lm} \frac{1}{r^{l+2}} \frac{im}{\sin\theta} Y_{lm}(\theta,\varphi)$$