PHY 411 **Winter 2021**

Class Summary—Week 6, Day 1—Tuesday, Feb 9

Covering Chapters 1-5

Jackson starts from the basics, by relating the electric force $\vec{F}$ on a charge q to the electric field $\vec{E}$ at the location of the charge, and we've already discussed in a previous class this quarter how to set up the coordinate system and write the electric field. As a reminder, the electric field at the position $\vec{x}$ due to a charge density $\rho(\vec{x}\,')$ is given by

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}\,') \frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3} \, d^3x' \tag{1.5}$$

where $d^3x' = dx'dy'dz'$ is the 3-dimensional volume element at $\vec{x}\,'$.

We spent some time in the previous class deriving these vector expressions (as part of our discussion of Green functions in electrostatics), so if you've forgotten where they come from, please refer back to your notes, or come and see me during office hours or ask for a meeting. Needless to say, you have to become very familiar and comfortable with this vector representation, since we'll be working with it a lot over the next few weeks.

Chapter 1 begins on page 24. I recommend a quick read of pages 24-26 to remind yourselves of all the undergrad-level concepts leading up to equation (1.5) above. Page 26 also contains a discussion of the Dirac δ-function that we discussed during the first week of classes this quarter.

On pages 27-29 are a discussion of Gauss' law, which we've already covered while introducing Maxwell's equations. If you're planning on going for a Ph.D., you should not only remember Gauss' law in differential form, equation (1.13) on page 29, but also the integral form:

$$\oint_S \vec{E} \cdot \hat{n} \, da = \frac{1}{\epsilon_0} \int_V \rho(\vec{x}) \, d^3x \tag{1.11}$$

as many universities have a B-level qualifier where they test students on their undergraduate preparation, and working out Gauss' law problems in different geometries is a favorite.

On page 29, Jackson introduces the electrostatics version of the curl of the electric field, which we covered this quarter while introducing Faraday's law. I would recommend remembering the relation

$$\frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{x} - \vec{x}\,'|}\right)$$

because it helps us express equation (1.5) above in the form

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[-\vec{\nabla} \int \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \, d^3x'\right] \tag{1.15}$$

which leads to the definition of the scalar potential Φ:

$$\vec{E} = -\vec{\nabla}\Phi \tag{1.16}$$

and allows us to define the scalar potential in terms of the charge density by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \, d^3x' \tag{1.17}$$

It goes without saying that you should have, by the end of your undergrad career, understood that Φ has a physical interpretation — if you didn't, I recommend reading over Jackson's discussion at the bottom of page 30 and appreciating that $q\Phi$ can be interpreted as the potential energy of a test charge q in the electrostatic field.

Section 1.6 on pages 31-34 is a discussion of the determination of electric field or potential due to a given surface distribution of charges. It's worth a quick read, especially because Jackson introduces a Taylor series expansion for the first time on page 33 to discuss the potential due to a dipole layer distribution.

On page 34, Jackson introduces the Poisson and Laplace equations. Specifically, Gauss' law in differential form $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$, and equation (1.16): $\vec{E} = -\vec{\nabla}\Phi$, can be combined into

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \tag{1.28}$$

which is called the **Poisson equation**.

In regions of space where the charge density is zero, the scalar potential satisfies the **Laplace equation**

$$\nabla^2 \Phi = 0 \tag{1.29}$$

We will now discuss the solution of Laplace's equation in different geometries, but before that, let's talk about what else is discussed in Chapter 1.

- On page 35, Jackson applies the Laplacian ∇^2 to equation (1.17) for Φ to verify directly that it satisfies the Poisson equation. I recommend looking at the procedure because out of it comes the expression of the Laplacian in terms of the Dirac δ-function

  $$\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}\,'|} \right) = -4\pi\delta(\vec{x} - \vec{x}\,') \tag{1.31}$$

  which was our starting point for the discussion of Green functions a few weeks ago.

- To handle boundary conditions, Jackson introduces (on page 36 in Section 1.8) Green's first identity and Green's second identity or Green's theorem.

- Next, Jackson asks what boundary conditions are appropriate for the Poisson or Laplace equation to ensure that a unique and physically reasonable solution will exist inside the bounded region. There are two kinds and a third mixed type:

  1. Specify the potential Φ on a closed surface, e.g., a system of conductors held at different potentials. This is called a Dirichlet problem, or Dirichlet boundary condition.

  2. Specify the normal derivative of the potential $\partial\Phi/\partial n$ (i.e., the electric field) everywhere on a surface, corresponding to a given charge density. This is called a Neumann boundary condition.

  3. Specify both Φ and $\partial\Phi/\partial n$ on a boundary. This is called a Cauchy boundary condition.

- On pages 37 and 38, Jackson then discusses the uniqueness of the solution of the Poisson equation inside a volume V subject to either Dirichlet or Neumann boundary conditions on a closed bounding surface S. We won't go into the details, except to note his conclusion that the solution is unique for Dirichlet boundary conditions, and also for Neumann boundary conditions. Likewise, there is also a unique solution to a problem with mixed boundary conditions (i.e., Dirichlet over part of the surface, Neumann over the remaining part). However, a solution to the Poisson equation with both Φ and $\partial\Phi/\partial n$ specified arbitrarily on a closed boundary (Cauchy boundary condition) does not exist, since there are unique solutions for Dirichlet and Neumann conditions separately, and these will in general not be consistent. The conclusion on page 38, then, is that electrostatic problems are specified only by Dirichlet or Neumann boundary conditions on a closed surface.

- The solution of the Poisson or Laplace equation in a finite volume V with either Dirichlet or Neumann boundary conditions on the bounding surface S can be obtained with Green's theorem or Green functions. We've already discussed some of this when we introduced Green functions, except that we only wrote the part pertaining to the volume (which was enough for us to use as an example and move on to the Green function for the wave equation) and left out the part pertaining to the boundary surface. We will defer discussion of this, and the remainder of Chapter 1 (pages 39-50), to a future date so that we can move on to the study of Laplace's equation in different geometries.

We will now study several techniques that are favorites on qualifier exams and upper level courses in electrodynamics that involve working with Laplace's equation.

First, though, let us understand why we're doing this by going over Jackson's introduction to Chapter 2. As we discussed moments ago, many problems in electrostatics involve boundary surfaces on which Dirichlet or Neumann conditions are specified. The formal solution of such problems involves Green functions, discussed by Jackson in Chapter 1. In practical situations, or even idealized approximations to practical situations, however, it might be difficult sometimes to build the Green function. That's why a number of approaches to the boundary value problems have been developed in electrostatics. One of them is the method of images which you'll discuss as part of your projects, and which is closely related to the Green function. A second method is the expansion in orthogonal functions directly through the differential equation and rather remote from the direct construction of Green functions. The third is by numerical methods. Our discussion below is about the second method of differential equations.

Specifically, we'll be discussing solutions to the Laplace equation in rectangular, spherical, and cylindrical coordinates. The Laplace equation is relevant when we have a charge-free region of space bounded by surfaces on which the potential is known.

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Laplace's equation in rectangular coordinates

We will begin by using the *separation of variables* method of solving partial differential equations to solve Laplace's equation. Let's do it first in rectangular coordinates, the easiest case.

The Laplace equation in rectangular coordinates is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{2.48}$$

The separation of variables method, which you'll hopefully remember from your undergrad course on partial differential equations, involves assuming that the function, in this case the potential $\Phi(\vec{x})$, can be represented by a product of three functions, one for each coordinate:

$$\Phi(\vec{x}) = X(x)Y(y)Z(z) \tag{2.49}$$

Substitute this in equation (2.48):

$$\frac{\partial^2}{\partial x^2}\Big[X(x)Y(y)Z(z)\Big] + \frac{\partial^2}{\partial y^2}\Big[X(x)Y(y)Z(z)\Big] + \frac{\partial^2}{\partial z^2}\Big[X(x)Y(y)Z(z)\Big] = 0$$

to get

$$YZ\,\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\,\frac{\partial^2 Z}{\partial z^2} = 0$$

and divide by XYZ:

$$\frac{1}{X}\,\frac{d^2 X}{dx^2} + \frac{1}{Y}\,\frac{d^2 Y}{dy^2} + \frac{1}{Z}\,\frac{d^2 Z}{dz^2} = 0 \tag{2.50}$$

where we've replaced partial derivatives with total derivatives, since each term involves a function of one variable only.

As is usual in the method of separation of variables, the next steps involves understanding that if equation (2.50) is to hold for arbitrary values of the independent coordinates, then each of the three terms must be separately constant, this giving us three ordinary differential equations.

$$\begin{aligned} \frac{1}{X}\,\frac{d^2 X}{dx^2} &= -\alpha^2 \\ \frac{1}{Y}\,\frac{d^2 Y}{dy^2} &= -\beta^2 \\ \frac{1}{Z}\,\frac{d^2 Z}{dz^2} &= \gamma^2 \end{aligned} \tag{2.51}$$

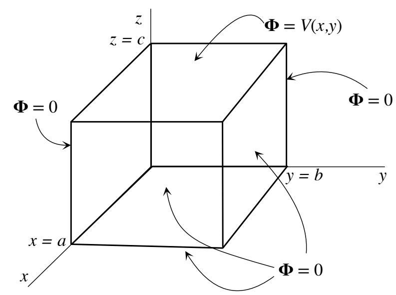
The choice of the minus and plus signs might seem arbitrary in equation (2.51), but it is not — we usually choose the minus sign when we realize there are periodic boundary conditions to be imposed on certain surfaces (e.g., the potential Φ goes to zero at $x = 0$ and $x = a$, the two boundaries along the x-direction, and likewise for the y-direction). However, once you've chosen the signs for, e.g., α^2 and β^2 to be negative, the sign of γ^2 is constrained by equation (2.50), so we must have

$$\alpha^2 + \beta^2 = \gamma^2$$

We can then solve these ordinary differential equations in equation (2.51) for $X(x)$, $Y(y)$, and $Z(z)$ respectively, and substitute the solutions into equation (2.49) to get the solution for the potential $\Phi(x)$. The values of the constants α, β, and γ come from the specific boundary conditions.

Before looking at a specific example, though, a word about boundary conditions. Recall that if we have an equation with a first derivative (e.g., $dX/dx = \ldots$), then we'll need a boundary condition in order to get a unique solution. Likewise, $d^2X/dx^2 = \ldots$ will need two boundary conditions. With the Laplace equation in three dimensions, we have a second derivative for each of three coordinates (x, y, z), so that means we'll need six boundary conditions. That's why, in the problem we're about to do, you need the potential specified on each face of the three dimensional box.

As an example, **you solved the following problem on the worksheet:** Consider a rectangular box with dimensions (a, b, c) along the (x, y, z) directions, as shown in the figure on the right. All surfaces of the box are kept at zero potential except the surface $z = c$, which is kept at a potential $V(x, y)$. Find the potential everywhere inside the box.



I solved this problem in the video, so I won't repeat the solution here. Since we aren't given an explicit form for $V(x, y)$, we have to leave the answer in the form of a closed integral, as written in equation (2.58) in Jackson.

To get a sense of orthogonal functions and expansions in general, I recommend looking over Jackson's summary in Section 2.8 (pages 67-69). In particular, you should understand:

- Orthonormality of a set of real or complex functions $U_n(\xi)$ — equations (2.28) and (2.29)
- Expanding an arbitrary function $f(\xi)$ in a series of orthonormal functions $U_n(\xi)$ — equations (2.30) and (2.33)
- Evaluating the coefficients in the expansion of $f(\xi)$ — equation (2.32)
- Being aware of the most famous orthogonal functions, the sines and cosines — equations (2.36) and (2.37)
- Generalization to higher dimensions — equations (2.38) and (2.39)
- Expansion of a continuum of functions through the Fourier integral — equations (2.40) up to (2.46)

Laplace's equation in spherical coordinates

In spherical coordinates (r, θ, ϕ), the Laplace equation can be written as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{3.1}$$

As a reminder, in the spherical coordinate system, r is the radial distance from the origin to the observation point, θ is the (polar) angle made with the z-axis by a line from the origin to the observation point, and ϕ is the (azimuthal) angle (see figure below).

Again, to separate variables, assume a product form for the potential:

$$\Phi = \frac{U(r)}{r}\,P(\theta)Q(\phi) \tag{3.2}$$

The extra factor of r is included here to force the same dimensionality into each factor; eventually, this will simplify the mathematics.

Substitute this into equation (3.1) to get

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\left[\frac{UPQ}{r}\right]\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\left[\frac{UPQ}{r}\right]\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left[\frac{UPQ}{r}\right] = 0$$

from which we obtain

$$\frac{PQ}{r}\frac{d^2U}{dr^2} + \frac{UQ}{r^3\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) + \frac{UP}{r^3\sin^2\theta}\frac{d^2Q}{d\phi^2} = 0$$

Unlike Laplace's equation in rectangular coordinates, the spherical coordinate case is more complicated. We can't separate all the variables at one. Let's multiply by $r^3\sin^2\theta/UPQ$first; we get

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{P\,r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right)\right] + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0 \tag{3.3}$$

We see that this step has allowed the ϕ-dependence to be isolated in the last term; as in the rectangular case, a ϕ-dependent term cannot depend on r or θ, so we can set it equal to a constant:

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2 \tag{3.4}$$

with solutions

$$Q = e^{\pm im\phi} \tag{3.5}$$

From the math point of view, m is not necessarily an integer. However, in order to allow the full azimuthal range of values, we must make m an integer to keep the solution Q single-valued.

Why did we set it equal to $-m^2$ and not $+m^2$? It's because we can anticipate the physics, more than anything else. Many problems have azimuthal symmetry, so it's good to make the solution be harmonic in that coordinate. For the same reason, we'll set the r and θ part equal to $l(l+1)$ because it puts the θ equation into a well known form that we can handle.

We will continue discussing Laplace's equation in spherical coordinates in the next class.