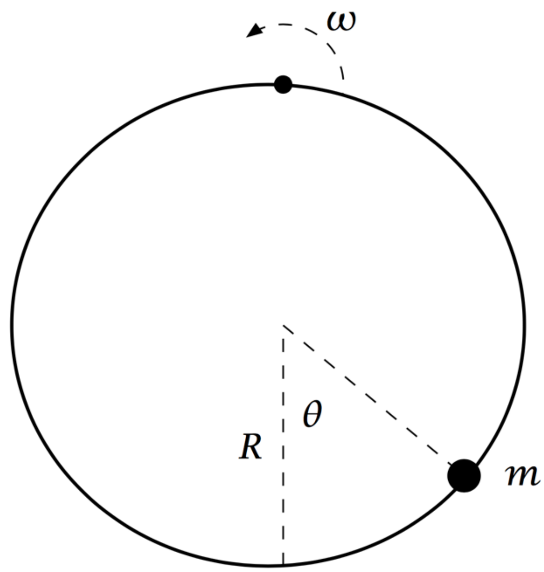
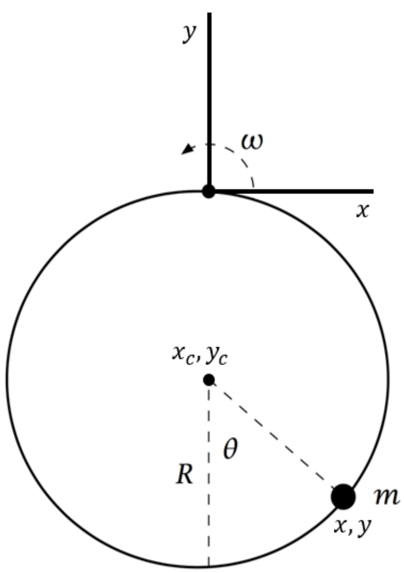
Physics 440, Autumn 2019 Activity 10: Bead on a Horizontal Rotating Hoop

A massless, frictionless hoop lying flat on a table rotates with constant angular frequency ω about one point on the rim, as shown in the figure below. A particle of mass m is free to move along the hoop. The angle θ measures the angle of the particle relative to a fixed reference (the vertical direction in the figure).



a. Show that the Lagrangian for this system can be written as
$L = \frac{mR^2}{2}\left[\omega^2 + \dot{\theta}^2 + 2\omega\dot{\theta}\cos(\theta - \omega t)\right]$
Note that the first term in brackets, ω, is a constant, so it won't affect the equations of motion.
b. Show that the change of variables $\phi = \theta - \omega t$ results in a Lagrangian that does not have an explicit time dependence. What is the physical meaning of the variable ϕ?
c. Working now with this new Lagrangian, find the equations of motion.
d. Find the equilibrium solutions and their stability.

a. I'll define the Cartesian coordinate system, the position of the center of the hoop (x_c, y_c) and the position of m (x, y) as shown below.

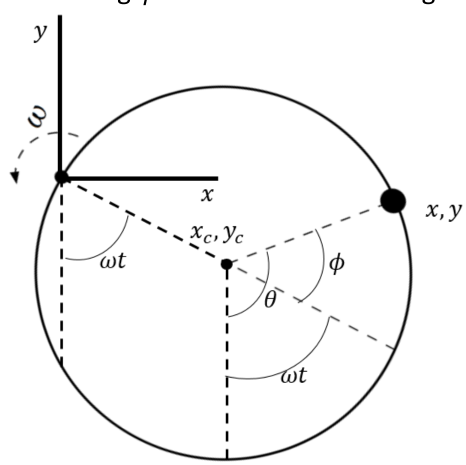


Because the hoop is horizontal we do not need to include potential energy in the Lagrangian.
$L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$
We need to express the position of m in terms time t and the angle θ, then take the derivative to determine the velocity needed for the Lagrangian:
At $t = 0$ the center of the hoop is at $x_c = 0, y_c = -R$
Then, as the hoop rotates counterclockwise around the z −axis:

$x_c(t) = R\sin\omega t\,, y_c(t) = -R\cos\omega t$
The position of m is
$x = x_c + R\sin\theta = R\sin\omega t + R\sin\theta$
$y = y_c - R\cos\theta = -R\cos\omega t - R\cos\theta$
$\dot{x} = R\omega\cos\omega t + R\dot{\theta}\cos\theta$
$\dot{y} = R\omega\sin\omega t + R\dot{\theta}\sin\theta$

$$\dot{x}^2 + \dot{y}^2 = \left(R\omega\cos\omega t + R\dot{\theta}\cos\theta\right)^2 + \left(R\omega\sin\omega t + R\dot{\theta}\sin\theta\right)^2$$
$$= R^2\omega^2\cos^2\omega t + R^2\dot{\theta}^2\cos^2\theta + R^2\omega^2\sin^2\omega t + R^2\dot{\theta}^2\sin^2\theta + 2R^2\omega\dot{\theta}\cos\omega t\cos\theta + 2R^2\omega\dot{\theta}\sin\omega t\sin\theta$$
$$= R^2\left[\omega^2 + \dot{\theta}^2 + 2\omega\dot{\theta}(\cos\theta\cos\omega t + \sin\theta\sin\omega t)\right] = R^2[\omega^2 + \dot{\theta}^2 + 2\omega\dot{\theta}\cos(\theta - \omega t)]$$
For the last step I used the trigonometric identity $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$
$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{mR^2}{2}[\omega^2 + \dot{\theta}^2 + 2\omega\dot{\theta}\cos(\theta - \omega t)]$$
which is what we needed to show

b. The figure blow shows the relationship between the rotation angle ωt of the hoop about the point on the rim, the angle θ of the particle relative to the vertical direction, and the new variable $\phi = \theta - \omega t$. ϕ is the angle of the particle relative to the rotating reference of the spinning hoop. Transforming the Lagrangian to $\phi = \theta - \omega t$ eliminates the explicit time dependence of the Lagrangian because the time dependence is now implicitly included in ϕ by measuring ϕ relative to the rotating reference frame.



$$\phi = \theta - \omega t \Rightarrow \theta = \phi + \omega t, \dot{\theta} = \dot{\phi} + \omega$$

$$L = \frac{mR^2}{2}\left[\omega^2 + \dot{\theta}^2 + 2\omega\dot{\theta}\cos(\theta - \omega t)\right] = \frac{mR^2}{2}\left[\omega^2 + (\dot{\phi} + \omega)^2 + 2\omega(\dot{\phi} + \omega)\cos(\phi)\right]$$
Note that while the Lagrangian still depends implicitly on time because $\phi = \phi(t)$, there is no longer an explicit time dependence.

c. $\frac{\partial L}{\partial \dot{\phi}} = mR^2\left[(\dot{\phi} + \omega) + \omega\cos\phi\right], \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = mR^2\ddot{\phi} - mR^2\omega\dot{\phi}\sin\phi$

$$\frac{\partial L}{\partial \phi} = -mR^2\omega(\dot{\phi}+\omega)\sin\phi$$
$$mR^2\ddot{\phi} - mR^2\omega\dot{\phi}\sin\phi = -mR^2\omega(\dot{\phi}+\omega)\sin\phi$$
$$\ddot{\phi} = -\omega^2\sin\phi$$

This is the equation of motion of a pendulum. The rotation of the hoop generates centrifugal acceleration on the particle $a = \omega^2 l$ that that acts on the particle in the same way gravity would if the pendulum were vertical and not rotating.

We can see from the equation of motion $\ddot{\phi} = -\omega^2\sin\phi$ that we get equilibrium solutions $\ddot{\phi} = 0$ for $\phi = 0$ and $\phi = \pi$. If we perturb the pendulum from $\phi = 0$ to a small positive (negative) angle, $\sin\phi$ becomes positive (negative) and the acceleration of the particle is negative (positive), back toward the equilibrium. This solution $\phi = 0$ is therefore stable. If we perturb the pendulum from $\phi = \pi$ in positive (negative) direction, so $\phi > \pi$ $(\phi < \pi)$, $\sin\phi$ becomes negative (positive) and the acceleration of the particle is positive (negative), pushing the particle away from equilibrium. This solution is therefore unstable.

Another way to analyze the stability of the equilibrium solutions (which we did not discuss, and you are not responsible for) is to write the second order equation of motion as a pair of first-order differential equations by defining a new variable to be the angular velocity $v = \dot{\phi}$. This is similar to problem 3d in the second homework assignment (the pendulum in the accelerating elevator).

$$\dot{\phi} = v = f(\phi, v)$$
$$\dot{v} = -\omega^2\sin\phi = g(\phi, v)$$

The equilibrium occurs when $\phi_1 = 0$ and $\phi_2 = \pi$, and the stability can be found from the eigenvalues of the matrix A that describes the linearized system:

$$A = \begin{pmatrix} \frac{\partial f}{\partial \phi} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial \phi} & \frac{\partial g}{\partial v} \end{pmatrix}_{\phi=0,\pi} = \begin{pmatrix} 0 & 1 \\ -\omega^2\cos\phi_{1,2} & 0 \end{pmatrix}$$

For the eigenvalue of this matrix λ the determinant of $\lambda I - A$ is 0:

$$|\lambda I - A| = \left|\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -\omega^2\cos\phi_{1,2} & 0\end{pmatrix}\right| = \begin{vmatrix} \lambda & -1 \\ \omega^2\cos\phi_{1,2} & \lambda \end{vmatrix} = 0$$

Where I is the identity matrix. So

$$\lambda^2 + (\omega^2\cos\phi_{1,2}) = 0 \Rightarrow \lambda^2 = -\omega^2\cos\phi_{1,2} \Rightarrow \lambda = \pm\sqrt{-\omega^2\cos\phi_{1,2}} = \pm i\omega$$

(for $\phi = \phi_1 = 0$)

and $\lambda = \pm\sqrt{-\omega^2\cos\phi_{1,2}} = \pm\omega$ (for $\phi = \phi_2 = \pi$)

The eigenvalue for $\phi = 0$ is imaginary, and this equilibrium is stable. The eigenvalue for $\phi = \pi$ is real, and this equilibrium is unstable.