PHY 411 **Winter 2021**

Class Summary—Week 3, Day 2—Thursday, Jan 21

Reflection and Refraction: Dynamic Properties

Starting from the four boundary conditions (I.17)-(I.20) in Jackson, we derived the boundary conditions for reflection and refraction at a plane interface in the previous class; they are

$$\left[\epsilon\left(\vec{E}_0 + \vec{E}_0''\right) - \epsilon'\,\vec{E}_0'\right]\cdot\hat{n} = 0 \tag{7.37.a}$$

$$\left[\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0'' - \vec{k}'\times\vec{E}_0'\right]\cdot\hat{n} = 0 \tag{7.37.b}$$

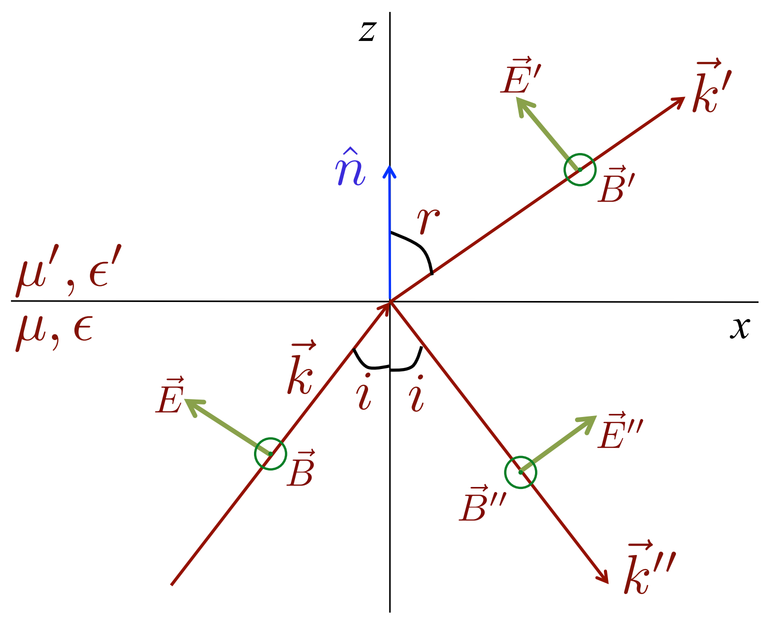
$$\left[\vec{E}_0 + \vec{E}_0'' - \vec{E}_0'\right]\times\hat{n} = 0 \tag{7.37.c}$$

$$\left[\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0''\right) - \frac{1}{\mu'}\left(\vec{k}'\times\vec{E}_0'\right)\right]\times\hat{n} = 0 \tag{7.37.d}$$

In the previous class, we applied the boundary conditions to derive expressions for the case when the $\vec{E}$-fields are perpendicular to the plane of incidence. Today, we will study the situation when the $\vec{E}$-fields are parallel to the plane of incidence.

$\vec{E}$ parallel to the plane of incidence

The situation is shown in Figure 7.6(b) on page 305 in Jackson, and reproduced in the figure below.



This time, the $\vec{E}$-fields are in the plane of the page, and the $\vec{B}$-fields are pointing out of the page (so that $\vec{E}\times\vec{B}$ is in the direction of propagation of the waves).

Since the $\vec{B}$-fields are now perpendicular to the plane of incidence or, equivalently, the $\vec{B}$-fields are parallel to the boundary surface, we have $\vec{B} \cdot \hat{n} = 0$ and $\vec{B} \times \hat{n} \neq 0$. So, equation (7.37.b) is of no use, whereas equation (7.37.d) is the one to pursue. We have a choice of either of the other two equations — since the $\vec{E}$-fields are at an angle to the surface, both give useful relations. It turns out, though, that equation (7.37.a) just gives the same relation as equation (7.37.d) after Snell's law is applied to it. So, we'll just use equation (7.37.c).

Using equation (7.37.c), we get

$$\cos i \left(E_0 - E_0''\right) - \cos r \, E_0' = 0 \tag{7.40.a}$$

Next, using equation (7.37.d), we get

$$\sqrt{\frac{\epsilon}{\mu}} \left(E_0 + E_0''\right) - \sqrt{\frac{\epsilon'}{\mu'}} \, E_0' = 0 \tag{7.40.b}$$

Together, equation (7.40.a) and equation (7.40.b) are written as equation (7.40) in Jackson; I've split up the numbering here for convenience. You should work out how to derive these on your own. I decided not to assign them on the worksheet since the steps would be very similar to what you did for the perpendicular case.

The relative amplitudes of the refracted and reflected waves are then

$$\begin{aligned} \frac{E_0'}{E_0} &= \frac{2nn' \cos i}{\dfrac{\mu}{\mu'} n'^{\,2} \cos i + n\sqrt{n'^{\,2} - n^2 \sin^2 i}} \\[2ex] \frac{E_0''}{E_0} &= \frac{\dfrac{\mu}{\mu'} n'^{\,2} \cos i - n\sqrt{n'^{\,2} - n^2 \sin^2 i}}{\dfrac{\mu}{\mu'} n'^{\,2} \cos i + n\sqrt{n'^{\,2} - n^2 \sin^2 i}} \end{aligned} \tag{7.41}$$

As for equation (7.40) above, you should make sure you can work out equation (7.41) on your own. The steps are very similar to the perpendicular case from the previous class, which is why I decided not to assign them again on the worksheet.

Again, just as the ratios in equation (7.39) provided a complete description for $\vec{E}$-fields perpendicular to the plane of incidence, the equations in (7.41) above provide a complete description of the problem, this time for $\vec{E}$-fields parallel to the plane of incidence. On the next page, we will write down how we can use the ratios in equation (7.39) and equation (7.41) to find the energy flow by using the time-averaged Poynting vector.

In closing this part of the discussion, it is worth noting that while we worked out the general case for $\vec{E}$ perpendicular to and parallel to the plane of incidence respectively, in most problems you will be able to pick the geometry so that the math simplifies considerably (as you'll see on the homework). On the other hand, if you pick the wrong geometry, the problem might prove impossible to solve. Therefore, give some thought to the geometry before starting on a problem.

Finally, to the part where we'll use the equations we derived in equation (7.39) and equation (7.41) to find the energy flow.

You will frequently be asked to calculate transmission and reflection coefficients. This is the fraction of energy (per unit area per unit time) in the transmitted wave and reflected wave respectively, as a fraction of the energy in the incident wave.

Computing these requires writing the Poynting vector $\vec{S}$ and then computing $\vec{S} \cdot \hat{n}$ through the surface (for the incident, refracted, and reflected waves). **Be careful when you look at equation (7.13) for $\vec{S}$ — the unit vector $\hat{n}$ written in that section is not the unit vector $\hat{n}$ being considered here** — in the current section, $\hat{n}$ is a unit vector directed perpendicular to the boundary (e.g., see Figure 7.5 on page 303). So, let's rewrite equation (7.13) with the direction of $\vec{k}$, etc., as appropriate:

$$
\begin{aligned}
\vec{S} &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left|E_0\right|^2 \hat{k} \\
\vec{S}' &= \frac{1}{2}\sqrt{\frac{\epsilon'}{\mu'}}\left|E_0'\right|^2 \hat{k}' \\
\vec{S}'' &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left|E_0''\right|^2 \hat{k}''
\end{aligned}
\tag{7.13.a}
$$

and then find $\vec{S} \cdot \hat{n}$, etc., where $\hat{n}$ is a unit vector directed perpendicular to the boundary (e.g., see Figure 7.5 on page 303, or see the figure on page 1 of this class summary):

$$
\begin{aligned}
\vec{S} \cdot \hat{n} &= = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left|E_0\right|^2 \cos\, i \\
\vec{S}' \cdot \hat{n} &= \frac{1}{2}\sqrt{\frac{\epsilon'}{\mu'}}\left|E_0'\right|^2 \cos\, r \\
\vec{S}'' \cdot \hat{n} &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left|E_0''\right|^2 \cos\, r'
\end{aligned}
\tag{7.13.b}
$$

You derived the second relation above on Question 5 of the worksheet for today; it is also worked out on the last page of this class summary, where we'll need the result to prove that energy does not transmit across the boundary when a wave is totally internally reflected.

The **transmission coefficient** is then defined as

$$
T = \frac{\vec{S}' \cdot \hat{n}}{\vec{S} \cdot \hat{n}} \tag{7.13.c}
$$

and the **reflection coefficient** is

$$
R = \frac{\vec{S}'' \cdot \hat{n}}{\vec{S} \cdot \hat{n}} \tag{7.13.d}
$$

We will now discuss some important topics related to reflection and transmission of electromagnetic waves.

Brewster's Angle

For a wave polarized *parallel* to the plane of incidence (i.e., for $\vec{E}$ parallel to the plane of incidence that we just studied), there is a curious result that you are likely well aware of from freshman physics. Let us consider the amplitude of the reflected wave which is given by the second equation in (7.41):

$$\frac{E_0''}{E_0} = \frac{\dfrac{\mu}{\mu'}\, n'^{\,2} \cos i - n\sqrt{n'^{\,2} - n^2 \sin^2 i}}{\dfrac{\mu}{\mu'}\, n'^{\,2} \cos i + n\sqrt{n'^{\,2} - n^2 \sin^2 i}}$$

For those angles for which the numerator of the above expression is zero, that is, if

$$\frac{\mu}{\mu'}\, n'^{\,2} \cos i = n\sqrt{n'^{\,2} - n^2 \sin^2 i}$$

the amplitude of the reflected wave will be zero.

Taking $\mu = \mu'$ to simplify the algebra (and this is true at optical frequencies), you should be able to show (*as you did on Question 2 of the worksheet for today*) that the angle of incidence for which the amplitude of the reflected wave is zero is given by

$$i_B = \arctan\left(\frac{n'}{n}\right) \tag{7.43}$$

From freshman physics, you might remember that this is called **Brewster's angle** — it is the angle of incidence for a wave polarized parallel to the plane of incidence for which there is no reflected wave.

The discussion above also implies that if an unpolarized plane wave is incident on a plane surface at the Brewster angle, the reflected wave is predominantly plane-polarized perpendicular to the plane of incidence. On the homework, you will draw graphs of T and R and verify that that the Brewster angle shows up in these graphs at the predicted value.

Space left blank for student notes; class summary continues on next page.

Total Internal Reflection

From freshman physics, you know that if the incident ray is in a medium of larger refractive index than the refracted ray, then Snell's law says there is an angle of incidence i_0 for which the angle of refraction will be 90°.

Mathematically, Snell's law ($n \sin i = n' \sin r$) implies that $r > i$ if $n > n'$. Therefore, for some value of the angle of incidence which we will designate as i_0, we will get $r = 90°$, so that using Snell's law

$$\sin i_0 = \left(\frac{n'}{n}\right) \tag{7.44}$$

In other words, for waves incident at the angle of incidence $i = i_0$, the refracted wave is propagated parallel to the interface of the two media, e.g., if light was traveling in a glass block and $i = i_0$, the light will travel (along the glass surface) parallel to the glass-air interface. There can be no energy flow across the surface.

What happens if $i > i_0$?

For $i > i_0$, we get $\sin r > 1$. This is easy to demonstrate (*as you did in Question 3(a) on today's worksheet*) because $n' \sin r = n \sin i$ from Snell's law gives

$$\sin r = \frac{n \sin i}{n'} = \frac{\sin i}{n'/n} = \frac{\sin i}{\sin i_0}$$

We will need this result to derive equation (7.46) on the next page.

Meanwhile, because $i > i_0$, we get that

$$\cos r = \sqrt{1 - \sin^2 r} = i\sqrt{\sin^2 r - 1} = i\sqrt{\left(\frac{n}{n'}\right)^2 \sin^2 i - 1}$$

Be careful you don't confuse i outside the square root (where it denotes the imaginary number) with the symbol for the angle of incidence i under the square root — I've put the imaginary i in a different colored font.

Replace n/n' with $1/\sin i_0$ from equation (7.44) to obtain

$$\cos r = i\sqrt{\left(\frac{\sin i}{\sin i_0}\right)^2 - 1} \tag{7.45}$$

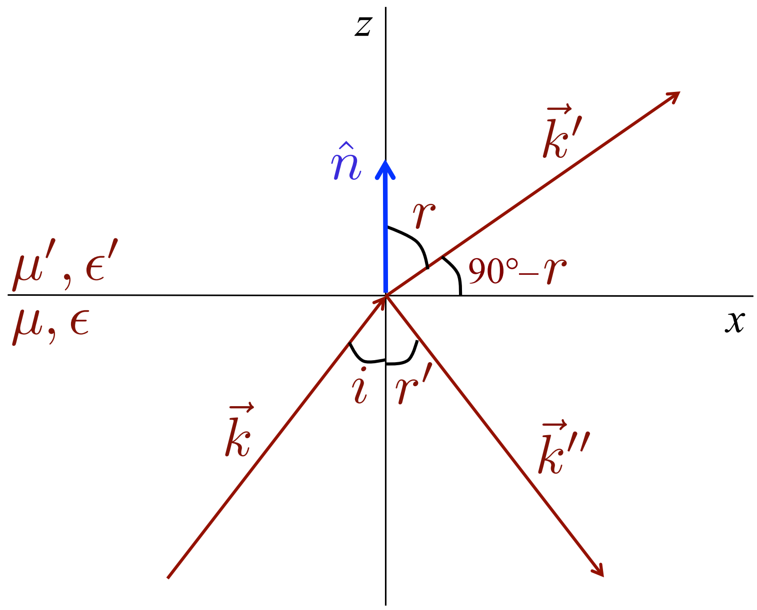
as you did on Question 3(b) of today's worksheet. This means that r is a complex angle with a purely imaginary cosine. Once again, be careful you don't confuse i outside the square root (where it denotes the imaginary number) with the symbol for the angle of incidence i under the square root — I've put the imaginary i in a this purple colored font (for handwritten materials, I usually write it as i_{im}, like I did on the worksheets in class).

Space left blank for student notes; class summary continues on next page.

To understand the implications of complex quantities like equation (7.45), consider the spatial part of the phase factor for the refracted wave $e^{i\vec{k}'\cdot\vec{x}}$. We can write

$$\vec{k}'\cdot\vec{x} = k'\hat{k}'\cdot(x\hat{x}+y\hat{y}+z\hat{z})$$

where I've defined $\hat{k}'$ to be a unit vector along the direction of $\vec{k}'$.



Now consider the figure above, which we introduced in the previous class (*Jackson*, Figure 7.5), and which I've annotated with the required angles. It is clear from the figure that the dot products between the unit vectors in the relation for $\vec{k}'\cdot\vec{x}$ are

$$\hat{k}'\cdot\hat{x} = \cos(90^\circ - r) = \sin r; \qquad \hat{k}'\cdot\hat{y} = \cos 90^\circ = 0; \qquad \hat{k}'\cdot\hat{z} = \cos r$$

So the spatial part of the phase factor for the refracted wave $e^{i\vec{k}'\cdot\vec{x}}$ becomes

$$e^{i\vec{k}'\cdot\vec{x}} = e^{ik'(x\sin r + z\cos r)} = e^{-k'\left(\sqrt{(\sin i/\sin i_0)^2-1}\right)z}\, e^{ik'(\sin i/\sin i_0)\,x} \tag{7.46}$$

as you showed in Question 4 on today's worksheet.

Notice again that I'm writing the angle of incidence i in black-colored font, and the imaginary number i (where $i^2 = -1$) in this purple-colored font.

Equation (7.46) shows that, for $i > i_0$, the refracted wave is attenuated exponentially beyond the interface $z = 0$. The attenuation occurs within a very few wavelengths of the boundary, except for $i \simeq i_0$.

Even though fields exist on the other side of the surface, there is no energy flow through the surface. To verify this, we need to work with the Poynting vector; since it is in the medium with (μ', ϵ'), I'll use the notation $\vec{S}'$ (note that this is different from Jackson's careless use of $\vec{S}$ here). We will use the second relation in equation (7.13.a), which I'll derive on the next page first.

We will now demonstrate that even though fields exist on the other side of the surface ($z > 0$), there is no energy flow through the surface when $i > i_0$.

Since $\hat{n}$ points perpendicular to the interface from the medium (μ, ϵ) in $z < 0$ to the medium (μ', ϵ') in $z > 0$ (see the figure on the first page of this class summary), the time-averaged normal component of $\vec{S}'$ just inside the surface ($z > 0$) is

$$\vec{S}' \cdot \hat{n} = \frac{1}{2}\mathrm{Re}\Big[\hat{n} \cdot (\vec{E}' \times \vec{H}'^{*})\Big] \tag{7.47}$$

From equation (7.31), written in the previous class summary, we know that

$$\vec{B}' = \sqrt{\mu'\epsilon'}\,\frac{\vec{k}' \times \vec{E}'}{k'}$$

from which we obtain

$$\vec{H}' = \frac{\vec{B}'}{\mu'} = \frac{\sqrt{\mu'\epsilon'}}{\mu'}\,\frac{\vec{k}' \times \vec{E}'}{k'} = \frac{\vec{k}' \times \vec{E}'}{\mu'\omega}$$

since $\vec{k}' = \omega\sqrt{\mu'\epsilon'}$. Substituting this into equation (7.47), we get

$$\vec{S}' \cdot \hat{n} = \frac{1}{2\mu'\omega}\mathrm{Re}\Big[\hat{n} \cdot (\vec{E}' \times \vec{k}' \times \vec{E}'^{*})\Big]$$

where we've written $\vec{k}'$ rather than $\vec{k}'^{*}$ because the wave vector is real.

Now, we know that for an electromagnetic wave $\vec{E}', \vec{B}'$ and $\vec{k}'$ are mutually perpendicular in a right-handed sense, so $(\vec{k}' \times \vec{E}'^{*})$ gives a vector of magnitude $|\vec{k}'\vec{E}'^{*}|$ pointing along the direction of $\vec{B}$, and so we can write

$$\vec{S}' \cdot \hat{n} = \frac{1}{2\mu'\omega}\mathrm{Re}\Big[\hat{n} \cdot \Big(\vec{E}' \times \hat{b}'\Big|\vec{k}'\vec{E}'^{*}\Big|\Big)\Big] = \frac{1}{2\mu'\omega}\mathrm{Re}\Big[\hat{n} \cdot \Big(\vec{E}' \times \hat{b}'\Big)k'\Big|\vec{E}'^{*}\Big|\Big]$$

where I'm using $\hat{b}'$ to indicate a vector pointing along $\vec{B}'$.

Next, $\vec{E}' \times \hat{b}'$ is just a vector with magnitude $|\vec{E}'|$ pointing along the direction of $\vec{k}'$, so we have

$$\begin{aligned}
\vec{S}' \cdot \hat{n} &= \frac{1}{2\mu'\omega}\mathrm{Re}\Big[\hat{n} \cdot \Big(\Big|\vec{E}'\Big|\hat{k}'\Big)k'\Big|\vec{E}'^{*}\Big|\Big] \\
&= \frac{1}{2\mu'\omega}\mathrm{Re}\Big[\Big(\hat{n} \cdot k'\hat{k}'\Big)\Big|\vec{E}'\vec{E}'^{*}\Big|\Big] \\
&= \frac{1}{2\mu'\omega}\mathrm{Re}\Big[\Big(\hat{n} \cdot \vec{k}'\Big)\Big|\vec{E}'\vec{E}'^{*}\Big|\Big] \qquad (7.47.\mathrm{a})
\end{aligned}$$

Further simplification comes from writing explicitly the electric field from equation (7.30):

$$\Big|\vec{E}'\vec{E}'^{*}\Big| = \Big|\vec{E}_0' e^{i(\vec{k}'\cdot\vec{x}-\omega t)}\vec{E}_0'^{*} e^{-i(\vec{k}'\cdot\vec{x}-\omega t)}\Big| = \Big|\vec{E}_0'\vec{E}_0'^{*}\Big| = \Big|E_0'\Big|^2$$

so that equation (7.47.a) above changes to

$$\vec{S}' \cdot \hat{n} = \frac{1}{2\mu'\omega}\mathrm{Re}\Big[\Big(\hat{n} \cdot \vec{k}'\Big)\Big|E_0'\Big|^2\Big] \tag{7.47.b}$$

To continue, begin with equation (7.47.b) written on the previous page

$$\vec{S}' \cdot \hat{n} = \frac{1}{2\mu'\omega}\mathrm{Re}\left[\left(\hat{n}\cdot\vec{k}'\right)\left|E_0'\right|^2\right]$$

From the figure on page 6 of this class summary, we see that since the angle between $\hat{n}$ and $\vec{k}'$ is just the angle of refraction r, the dot product in parentheses above is just

$$\hat{n}\cdot\vec{k}' = k'\cos\, r$$

so that

$$\vec{S}' \cdot \hat{n} = \frac{1}{2\mu'\omega}\mathrm{Re}\left[\left(k'\cos\, r\right)\left|E_0'\right|^2\right] \tag{7.47.c}$$

as you showed in Question 5 on today's worksheet.

This expression is enough for what we're trying to prove, but in passing, I'll note that we've also verified the second relation in equation (7.13.b), which we obtain by putting back $k' = \omega\sqrt{\mu'\epsilon'}$, so that

$$\vec{S}' \cdot \hat{n} = \frac{1}{2\mu'\omega}\left[\left(\omega\sqrt{\mu'\epsilon'}\cos\, r\right)\left|E_0'\right|^2\right] = \frac{\sqrt{\mu'\epsilon'}}{2\mu'}\left[\left|E_0'\right|^2\cos\, r\right)\right]$$

where I've left out the explicit specification that the quantity in square brackets is real, because we don't need it for general procedures like in equation (7.13.b). We get finally

$$\vec{S}' \cdot \hat{n} = \frac{1}{2}\sqrt{\frac{\epsilon'}{\mu'}}\left|E_0'\right|^2\cos\, r$$

which proves equation (7.13.b).

Returning now to the matter in the current section, consider again equation (7.47.c):

$$\vec{S}' \cdot \hat{n} = \frac{1}{2\mu'\omega}\mathrm{Re}\left[\left(k'\cos\, r\right)\left|E_0'\right|^2\right]$$

We know from equation (7.45) that $\cos\, r$ is purely imaginary for the case $i > i_0$ that we're discussing here, so the above expression tells us that $\vec{S}' \cdot \hat{n} = 0$.

With $\vec{S}' \cdot \hat{n} = 0$, the transmission coefficient T in equation (7.13.c) becomes zero, and since we must always have the sum of the transmission and reflection coefficients equal to 1 (i.e., $T + R = 1$), this means that the reflection coefficient $R = 1$. So, for $i > i_0$, the wave undergoes total internal reflection, and is reflected back into the incident medium of higher refractive index.