PHY 411 Winter 2021

Homework 6 solutions

1. In class, we learned how to solve Laplace's equation in different geometries. Here is a chance to work with its counterpart, Poisson's equation: do **Problem 1.5** in Jackson (*page 51*).

Solution: Poisson's equation, $\nabla^2\Phi = -\rho/\epsilon_0$ will need to be evaluated in spherical coordinates. We need only the r-part since $\Phi(r)$ is a function only of r, and so we'll need only

$$-\frac{\rho}{\epsilon_0} = \nabla_r^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left[\frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)\right]\right)$$

where q is the magnitude of the electronic charge, and $\alpha = 2/a_0$, with a_0 being the Bohr radius. Rearranging, we get

$$-\frac{\rho}{\epsilon_0} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left[\frac{e^{-\alpha r}}{r}+\frac{\alpha}{2}e^{-\alpha r}\right]\right)$$

so that

$$-\frac{\rho}{\epsilon_0} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[\frac{-\alpha\, e^{-\alpha r}}{r}+e^{-\alpha r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\frac{\alpha^2}{2}e^{-\alpha r}\right]\right)$$

We'll carry the derivative of $1/r$ through because we'll need to handle it differently at $r = 0$ (where it blows up) and $r \neq 0$. Thus

$$-\frac{\rho}{\epsilon_0} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(-\left[\alpha r - r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)+\frac{\alpha^2 r^2}{2}\right]e^{-\alpha r}\right)$$

so that, after canceling a minus with that on the right hand side and differentiating, we get

$$\frac{\rho}{\epsilon_0} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\left(\left[\alpha - \frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right\}+\alpha^2 r\right]e^{-\alpha r}+\left[\alpha r - r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)+\frac{\alpha^2r^2}{2}\right]\left[-\alpha e^{-\alpha r}\right]\right)$$

I'll skip some steps here that I only wrote out by hand to save time, but after differentiating, canceling terms that are identical but of opposite sign, and rearranging, we arrive at

$$\frac{\rho}{\epsilon_0} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\left(-\frac{\alpha^3 r^2}{2}e^{-\alpha r} - e^{-\alpha r}\frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right\}\right)$$

so that

$$\rho = -\frac{q}{8\pi}\alpha^3 e^{-\alpha r} - \frac{q}{4\pi}e^{-\alpha r}\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right\}$$

For $r > 0$, differentiation of the second term just gives zero, but for $r = 0$, we must handle it differently; notice that the term in purple is just the ∇_r^2 written on the first line of the solution, which can be written as ∇^2 since it is operating only on $1/r$. Meanwhile, from the bottom of page 35 in Jackson, we have that $\nabla^2(1/r) = -4\pi\delta(r)$, and since $e^{-\alpha r} = 1$ at $r = 0$, we get that

$$\rho = -\frac{q}{8\pi}\alpha^3 e^{-\alpha r} \quad \text{for } r > 0, \qquad \text{and} \qquad \rho = -\frac{q}{8\pi}\alpha^3(1) + q\,(1)\delta(r) \quad \text{for } r = 0$$

where I've put (1) in place of $e^{-\alpha r}$ at $r = 0$. We can write the solutions separately as above, or combine them into one, since $\delta(r) = 0$ for $r \neq 0$. Therefore, if we want to write one solution for all r, we can write

$$\rho = -\frac{q}{8\pi}\alpha^3 e^{-\alpha r} + q\,\delta(r)$$

This is what Jackson means by finding both continuous and discrete parts of the charge distribution. The second term on the right, $q\delta(r)$, is a discrete charge at the origin (the nucleus of the hydrogen atom), and the first term is a continuous contribution that decays exponentially with radial distance away from the center, which can be classically pictured as the cloud of electrons around the nucleus.

2. The two-dimensional Laplace equation, in which the potential can be assumed to be independent of one of the coordinates, has applications in, e.g., a long uniform transmission line. Consider such a case in which the potential is independent of z, so that the Laplace equation is

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{1}$$

with boundary conditions $\Phi = 0$ at $x = 0$ and $x = a$, $\Phi = V$ at $y = 0$ for $0 \leq x \leq a$ and $\Phi \to 0$ for large y (e.g., see Figure 2.10 on page 73 in Jackson).

By the method of separation of variables, show that the potential in $0 \leq x \leq a, y \geq 0$ is given by

$$\Phi(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{-n\pi y/a}$$

where A_n is a constant.

Solution: To separate variables, write $\Phi(x, y) = X(x)\,Y(y)$, substitute in equation (1) above, and divide by XY to get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \tag{2}$$

The first term on the left hand side of equation (2) is a function of x only, and the second term is a function of y only. Therefore, they must be equal to the same, but oppositely signed constant, in order for their sum to be equal to zero. So, set

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2 \qquad \text{and} \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = +\alpha^2$$

in order to get a $\sin(\dots)$ solution for the x-equation, since the boundary conditions are periodic along x. Cross multiplying, we get

$$\frac{d^2X}{dx^2} = -\alpha^2 X \qquad \text{and} \qquad \frac{d^2Y}{dy^2} = +\alpha^2 Y$$

These are well known differential equations, with solutions

$$X(x) = \sin(\alpha x) \qquad \text{and} \qquad Y(y) = e^{-\alpha y} \tag{3}$$

From the boundary condition that $X = 0$ at $x = a$, we get $0 = \sin(\alpha a)$, and since $\sin(n\pi) = 0$, this gives

$$\alpha = \frac{n\pi}{a}, \qquad n = 1, 2, 3, \dots$$

Therefore, equation (3) is actually an array of solutions $X_n(x)$ and $Y_n(y)$, indexed by $n = 1, 2, 3, \dots$, and thus, $\Phi(x, y)$ is the superposition of all these solutions bundled together with appropriate constants A_n, so that

$$\Phi(x, y) = \sum_{n=1}^{\infty} A_n\, X_n(x)\, Y_n(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{-n\pi y/a} \tag{4}$$

This is the solution we are asked to derive.

3. Use the appropriate boundary condition(s) to evaluate A_n in the previous problem in the usual way, and show that

$$A_n = \begin{cases} \dfrac{4V}{\pi n} & \text{for } n \text{ odd} \\ \\ 0 & \text{for } n \text{ even} \end{cases}$$

Solution: At $y = 0$, $\Phi = V$, so equation (4) gives

$$V = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{-n\pi(0)/a}$$

so that

$$V = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \tag{5}$$

To determine the A_n's, we will make use of the orthogonality of the sine functions.

Multiply on both sides of equation (5) by $\sin\left(\dfrac{n'\pi x}{a}\right)$, and integrate from 0 to a:

$$\sum_{n=1}^{\infty} A_n \int_0^a \sin\left(\frac{n'\pi x}{a}\right)\left(\frac{n\pi x}{a}\right) dx = \int_0^a V \sin\left(\frac{n'\pi x}{a}\right) dx \tag{6}$$

By the orthogonality of the sine function, we know that

$$\int_0^a \sin\left(\frac{n'\pi x}{a}\right)\left(\frac{n\pi x}{a}\right) dx = \left(\frac{a}{2}\right) \delta_{n'n}$$

On the left hand side of equation (6), therefore, the orthogonality condition makes every term with $n' \neq n$ in the summation go to zero, and we are left with only one term from the entire summation, namely the one for which $n' = n$, and we get

$$A_n\left(\frac{a}{2}\right) = V\left[\frac{-\cos(n\pi x/a)}{n\pi/a}\right]_0^a = -\frac{aV}{n\pi}\Big\{\cos(n\pi) - \cos(0)\Big\}$$

Now, recall that $\cos(n\pi)$ is equal to 1 if n is even, and equal to -1 if n is odd. Mathematically, this can be written as $(-1)^n$. Therefore, we get

$$A_n = -\frac{2V}{n\pi}\Big\{(-1)^n - 1\Big\} = \begin{cases} -\dfrac{2V}{n\pi}\Big\{-1-1\Big\} = \dfrac{4V}{n\pi} & \text{for } n \text{ odd} \\ -\dfrac{2V}{n\pi}\Big\{+1-1\Big\} = 0 & \text{for } n \text{ even} \end{cases}$$

which is the condition that we are required to prove.

4. A compact representation of Legendre polynomials is given by *Rodrigues' formula*:

$$P_l(x) = \frac{1}{2^l\, l!}\, \frac{d^l}{dx^l}\, (x^2 - 1)^l$$

Using Rodrigues' formula, derive that

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)\, P_l = 0$$

Solution: It's a matter of personal taste where you choose to start. I'll begin with $\dfrac{dP_{l+1}}{dx}$.

$$\begin{aligned}
\frac{dP_{l+1}}{dx} &= \frac{1}{2^{l+1}\,(l+1)!}\, \frac{d}{dx}\, \left[\frac{d^{l+1}}{dx^{l+1}}(x^2-1)^{l+1}\right] \\
&= \frac{1}{2^{l+1}\,(l+1)!}\, \frac{d^l}{dx^l}\, \left[\frac{d^2}{dx^2}(x^2-1)^{l+1}\right] \qquad \text{since we need at least } d^2/dx^2 \\
&= \frac{1}{(2)\,2^l\, l!(l+1)}\, \frac{d^l}{dx^l}\, \left[\frac{d}{dx}\left\{(l+1)\,(x^2-1)^l\,(2x-0)\right\}\right] \\
&= \frac{1}{2^l\, l!}\, \frac{d^l}{dx^l}\, \left[\frac{d}{dx}\left\{x\,(x^2-1)^l\right\}\right] \\
&= \frac{1}{2^l\, l!}\, \frac{d^l}{dx^l}\, \left[(x^2-1)^l + x\, l(x^2-1)^{l-1}\,(2x-0)\right] \\
&= \frac{1}{2^l\, l!}\, \frac{d^l}{dx^l}\, (x^2-1)^l + \frac{2l}{2^l\, l!}\, \frac{d^l}{dx^l}\, x^2\,(x^2-1)^{l-1}
\end{aligned}$$

The first term is $P_l(x)$, so the second term must work out to $\dfrac{dP_{l-1}}{dx} + 2lP_l$.

So, let us write $\dfrac{dP_{l-1}}{dx}$:

$$\frac{dP_{l-1}}{dx} = \frac{1}{2^{l-1}\,(l-1)!}\, \frac{d}{dx}\, \left[\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^{l-1}\right]$$

and multiply and divide by $2l$ to get

$$\frac{dP_{l-1}}{dx} = \frac{2l}{(2)2^{l-1}\,(l-1)!(l)}\, \frac{d}{dx}\, \left[\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^{l-1}\right] = \frac{2l}{2^l\, l!}\, \frac{d^l}{dx^l}\, (x^2-1)^{l-1}$$

Subtract this from the expression for $\dfrac{dP_{l+1}}{dx}$ above to get

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} = \frac{1}{2^l\, l!}\, \frac{d^l}{dx^l}\, (x^2-1)^l + \frac{2l}{2^l\, l!}\, \frac{d^l}{dx^l}\, x^2\,(x^2-1)^{l-1} - \frac{2l}{2^l\, l!}\, \frac{d^l}{dx^l}\, (x^2-1)^{l-1}$$

Proceeding, we get

$$\begin{aligned}
\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} &= \frac{1}{2^l\, l!}\,\frac{d^l}{dx^l}\,(x^2-1)^l + \frac{2l}{2^l\, l!}\,\frac{d^l}{dx^l}\,x^2\,(x^2-1)^{l-1} - \frac{2l}{2^l\, l!}\,\frac{d^l}{dx^l}\,(x^2-1)^{l-1} \\
&= \frac{1}{2^l\, l!}\,\frac{d^l}{dx^l}\,(x^2-1)^l + \frac{2l}{2^l\, l!}\,\frac{d^l}{dx^l}\,\Big[x^2\,(x^2-1)^{l-1} - (x^2-1)^{l-1}\Big] \\
&= \frac{1}{2^l\, l!}\,\frac{d^l}{dx^l}\,(x^2-1)^l + \frac{2l}{2^l\, l!}\,\frac{d^l}{dx^l}\,\Big[(x^2-1)\,(x^2-1)^{l-1}\Big] \\
&= \frac{1}{2^l\, l!}\,\frac{d^l}{dx^l}\,(x^2-1)^l + \frac{2l}{2^l\, l!}\,\frac{d^l}{dx^l}\,(x^2-1)^l \\
&= \Big(2l+1\Big)\,\frac{1}{2^l\, l!}\,\frac{d^l}{dx^l}\,(x^2-1)^l \\
&= \Big(2l+1\Big)\,P_l(x)
\end{aligned}$$

Moving all terms to the left hand side, we get

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)\,P_l = 0$$

which is the expression we were asked to derive.