PHY 411 Winter 2021
Homework 2 solutions

1. The plane wave solutions to the Helmholtz wave equation are

$$\vec{E}(\vec{x},t) = \vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$
$$\vec{B}(\vec{x},t) = \vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \tag{H2.1}$$

where $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are constant vectors. By substituting the plane wave solutions written above into Faraday's law, $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$, derive the expression

$$\vec{B} = \sqrt{\mu\epsilon}\left(\frac{\vec{k}\times\vec{E}}{k}\right)$$

Note: You may assume without proof that $\vec{\nabla}\times\vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = -\vec{\mathcal{E}}\times\vec{\nabla}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$.

Solution: Let us begin by putting the expression for $\vec{E}$ and $\vec{B}$ given above in equation (H2.1) into Faraday's law:

$$\vec{\nabla}\times\vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = -\frac{\partial}{\partial t}\left[\vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right] \tag{H2.2}$$

Since $\vec{\mathcal{E}}$ is a constant vector, we can write

$$-\vec{\mathcal{E}}\times\vec{\nabla}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = -\frac{\partial}{\partial t}\left[\vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right] \tag{H2.3}$$

where, as directed in the note above, I've assumed the result on the left hand side without proof.

The right hand side of equation (H2.3) is easy, because the time-derivative just brings out a factor $(-i\omega)$.

$$-\frac{\partial}{\partial t}\left[\vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right] = -\Big(-i\omega\Big)\,\vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \tag{H2.4}$$

The left hand side of equation (H2.3), though, will need some work. Let's find the gradient first:

$$\begin{aligned}\vec{\nabla}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = \vec{\nabla}\left[e^{ik\hat{n}\cdot\vec{x}}e^{-i\omega t}\right] &= \vec{\nabla}\left[e^{ik(\hat{x}n_x+\hat{y}n_y+\hat{z}n_z)\cdot(x\hat{x}+y\hat{y}+z\hat{z})}e^{-i\omega t}\right]\\ &= \vec{\nabla}\left[e^{ik(n_x x+n_y y+n_z z)}e^{-i\omega t}\right]\end{aligned} \tag{H2.5}$$

Note that instead of writing n_x, I could also write $|\hat{n}|\cos\theta_x = \cos\theta_x$, where θ_x is the angle between $\hat{n}$ and the x-axis; both mean the same thing, i.e., $n_x = \cos\theta_x$. I prefer to write the latter (as I did in class) because it makes the point that even though $|\hat{n}| = 1$, n_x, etc., are not equal to 1. In this homework solution, though, I'm going to use n_x, etc., to remind you that they are the same. Finally, note that once we get into incidence at angles $i \neq 0$ like in the reflection and refraction problem, we have to change notation to be consistent with Jackson, and can no longer use $\hat{n}$ as a unit vector along the direction of $\vec{k}$.

Continuing to evaluate the gradient in equation (H2.5), we obtain

$$\begin{aligned}
\vec{\nabla}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} &= \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\left[e^{ik(n_x x+n_y y+n_z z)}e^{-i\omega t}\right] \\
&= \left(\hat{x}ikn_x+\hat{y}ikn_y+\hat{z}ikn_z\right)\left[e^{ik(n_x x+n_y y+n_z z)}e^{-i\omega t}\right] \\
&= ik\left(\hat{x}n_x+\hat{y}n_y+\hat{z}n_z\right)e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \\
&= ik\hat{n}e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \qquad \text{(H2.6)}
\end{aligned}$$

Substituting the gradient in equation (H2.6) into the left hand side of equation (H2.3), we get

$$\begin{aligned}
-\vec{\mathcal{E}}\times\vec{\nabla}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} &= -\vec{\mathcal{E}}\times ik\hat{n}e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \\
&= ik\Big(-\vec{\mathcal{E}}\times\hat{n}\Big)e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \\
&= ik\Big(\hat{n}\times\vec{\mathcal{E}}\Big)e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \\
&= i\Big(k\hat{n}\times\vec{\mathcal{E}}\Big)e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \\
&= i\Big(\vec{k}\times\vec{\mathcal{E}}\Big)e^{i(k\hat{n}\cdot\vec{x}-\omega t)}
\end{aligned}$$

With the expression above substituted into the left hand side of equation (H2.3), that equation becomes

$$i\Big(\vec{k}\times\vec{\mathcal{E}}\Big)e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = -\frac{\partial}{\partial t}\left[\vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right]$$

whereas the right hand side of the equation above can be evaluated according to equation (H2.4), and we get

$$i\Big(\vec{k}\times\vec{\mathcal{E}}\Big)e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = -\Big(-i\omega\Big)\,\vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$

Cleaning up, we obtain

$$\vec{k}\times\vec{\mathcal{E}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)} = \omega\vec{\mathcal{B}}\, e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$

so that

$$\vec{k}\times\vec{E} = \omega\vec{B} \qquad \text{(H2.7)}$$

Now, recall from equation (7.4) in Jackson that $k=\omega\sqrt{\mu\epsilon}$, so we can write $\omega = \dfrac{k}{\sqrt{\mu\epsilon}}$.

Thus, we have from equation (H2.7) that

$$\vec{k}\times\vec{E} = \frac{k}{\sqrt{\mu\epsilon}}\vec{B}$$

and we finally have our desired result

$$\vec{B} = \sqrt{\mu\epsilon}\,\left(\frac{\vec{k}\times\vec{E}}{k}\right)$$

2. In class, we derived the Helmholtz equation for $\vec{E}$. Starting from the Maxwell equations, and following similar procedures, derive the Helmholtz equation for $\vec{B}$:

$$\left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{B} = 0$$

Solution: Since the Helmholtz equation is a traveling wave equation, we must begin in free space where there are no sources (i.e., with $\rho = 0$ and $\vec{J} = 0$). In such regions, Maxwell's equations are

$$\begin{aligned} \vec{\nabla}\cdot\vec{B} &= 0 & \qquad \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{D} &= 0 & \qquad \vec{\nabla}\times\vec{H} &= \frac{\partial\vec{D}}{\partial t} \end{aligned} \tag{H2.8}$$

Let us assume solutions with harmonic time dependence for $\vec{E}, \vec{B}, \vec{D}$, and $\vec{H}$; for example

$$\vec{B}(\vec{x},t) = \vec{B}(\vec{x})\,e^{-i\omega t}, \qquad \vec{D}(\vec{x},t) = \vec{D}(\vec{x})\,e^{-i\omega t} \tag{H2.9}$$

As in the case when we derived the Helmholtz equation for $\vec{E}$, henceforth I will just write $\vec{B}$ instead of $\vec{B}(\vec{x})$ to be consistent with Jackson's notation in this section. With the choice of harmonic time dependence in equation (H2.9), the time derivative for the two equations on the right in equation (H2.8) simply pulls out a factor $-i\omega$, and we get

$$\begin{aligned} \vec{\nabla}\cdot\vec{B} &= 0 & \qquad \vec{\nabla}\times\vec{E} &= -(-i\omega\vec{B}) \\ \vec{\nabla}\cdot\vec{D} &= 0 & \qquad \vec{\nabla}\times\vec{H} &= -i\omega\vec{D} \end{aligned} \tag{H2.10}$$

Assuming uniform, isotropic, linear media for which we have $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$, we can modify the curl equations for $\vec{E}$ and $\vec{H}$ written in equation (H2.10) above to

$$\vec{\nabla}\times\vec{E} = i\omega\vec{B} \qquad \text{and} \qquad \vec{\nabla}\times\vec{B} = -i\omega\mu\epsilon\vec{E} \tag{H2.11}$$

Now, consider the equation for the curl of $\vec{B}$ on the right. Taking the curl of both sides, we get

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) = -i\omega\mu\epsilon\left(\vec{\nabla}\times\vec{E}\right) \tag{H2.12}$$

and using the result from the inside front cover of Jackson for $\vec{\nabla}\times(\vec{\nabla}\times\vec{B})$, this becomes

$$\left[\vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B}\right] = -i\omega\mu\epsilon\left(\vec{\nabla}\times\vec{E}\right) \tag{H2.13}$$

Putting $\vec{\nabla}\cdot\vec{B} = 0$ from equation (H2.8), and $\vec{\nabla}\times\vec{E} = i\omega\vec{B}$ from equation (H2.11), this becomes

$$-\nabla^2 B = -i\omega\mu\epsilon\left(i\omega\vec{B}\right) \tag{H2.14}$$

so that

$$-\nabla^2\vec{B} = -i^2\mu\epsilon\,\omega^2\vec{B} \tag{H2.15}$$

and since $i^2 = -1$, we get

$$-\nabla^2\vec{B} = \mu\epsilon\omega^2\,\vec{B} \tag{H2.16}$$

so that we finally obtain the desired Helmholtz equation for $\vec{B}$:

$$\left(\nabla^2 + \mu\epsilon\omega^2\right)\vec{B} = 0$$

3. Consider the electric field given by

$$\vec{E} = E_0 \, \frac{\hat{x} - i\hat{y}}{\sqrt{2}} \, e^{i(kz - \omega t)}$$

(a) By explicit computation, verify Gauss' law $\vec{\nabla} \cdot \vec{E} = 0$ for this field.

Solution: The components of the field are

$$E_x = \frac{E_0}{\sqrt{2}} \, e^{i(kz-\omega t)} \qquad E_y = -i\frac{E_0}{\sqrt{2}} \, e^{i(kz-\omega t)} \qquad E_z = 0 \tag{H2.17}$$

Clearly, E_x does not have an x-dependent term, E_y does not have an y-dependent term, and $E_z = 0$. So

$$\frac{\partial E_x}{\partial x} = 0, \qquad \frac{\partial E_y}{\partial y} = 0, \qquad \frac{\partial E_z}{\partial z} = 0$$

Therefore

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 + 0 + 0 = 0$$

(b) Use Faraday's law to find $\vec{B}$.

Solution: Faraday's law is

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{H2.18}$$

First evaluate the cross product on the left hand side of equation (H2.18):

$$\begin{aligned}
\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} &= \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \\
&= \hat{x}\left(0 - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - 0\right) + \hat{z}\Big(0 - 0\Big)
\end{aligned} \tag{H2.19}$$

where I've used the dependencies on x, y, z in E_x, E_y, E_z written in equation (H2.17) to set some of the derivatives in equation (H2.19) equal to zero.

Next, let us evaluate the non zero terms in equation (H2.19):

$$\begin{aligned}
\vec{\nabla} \times \vec{E} &= -\hat{x}\frac{\partial}{\partial z}\left[-i\frac{E_0}{\sqrt{2}} \, e^{i(kz-\omega t)}\right] + \hat{y}\frac{\partial}{\partial z}\left[\frac{E_0}{\sqrt{2}} \, e^{i(kz-\omega t)}\right] \\
&= \hat{x}\left(\frac{i^2 k E_0}{\sqrt{2}}\right) e^{i(kz-\omega t)} + \hat{y}\left(\frac{ikE_0}{\sqrt{2}}\right) e^{i(kz-\omega t)}
\end{aligned}$$

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At the bottom of the previous page, we obtained that

$$\vec{\nabla} \times \vec{E} = \hat{x} \left(\frac{i^2 k E_0}{\sqrt{2}} \right) e^{i(kz-\omega t)} + \hat{y} \left(\frac{i k E_0}{\sqrt{2}} \right) e^{i(kz-\omega t)} \tag{H2.20}$$

Since this is equal to $-\dfrac{\partial \vec{B}}{\partial t}$, the easiest way to proceed is to write

$$\vec{B} = \hat{x} B_x \, e^{i(kz-\omega t)} + \hat{y} B_y \; e^{i(kz-\omega t)} \tag{H2.21}$$

having realized that B_z must also be zero. Then

$$\frac{\partial \vec{B}}{\partial t} = \hat{x} \Big(- i\omega B_x \Big) e^{i(kz-\omega t)} + \hat{y} \Big(- i\omega B_y \Big) \; e^{i(kz-\omega t)}$$

and

$$-\frac{\partial \vec{B}}{\partial t} = \hat{x} \Big(i\omega B_x \Big) e^{i(kz-\omega t)} + \hat{y} \Big(i\omega B_y \Big) \; e^{i(kz-\omega t)} \tag{H2.22}$$

Setting the $\hat{x}$ and $\hat{y}$ terms on the right hand side of equation (H2.20) and equation (H2.22) separately equal to each other, we get

$$i\omega B_x = \frac{i^2 k E_0}{\sqrt{2}} \qquad \text{and} \qquad i\omega B_y = \frac{i k E_0}{\sqrt{2}}$$

and so

$$B_x = \frac{i^2 k E_0}{i\omega\sqrt{2}} = \frac{i k E_0}{\omega\sqrt{2}} \qquad \text{and} \qquad B_y = \frac{i k E_0}{i\omega\sqrt{2}} = \frac{k E_0}{\omega\sqrt{2}} \tag{H2.23}$$

Putting B_x and B_y from equation (H2.23) into equation (H2.21), we get

$$\vec{B} = \hat{x} \left(\frac{i k E_0}{\omega\sqrt{2}} \right) e^{i(kz-\omega t)} + \hat{y} \left(\frac{k E_0}{\omega\sqrt{2}} \right) e^{i(kz-\omega t)}$$

Rearranging, we get finally that

$$\vec{B} = \frac{k E_0}{\omega} \left(\frac{i\hat{x} + \hat{y}}{\sqrt{2}} \right) e^{i(kz-\omega t)}$$

(c) What is the state of polarization of this wave? Explain your answer clearly.

Solution: The state of polarization is usually described by the $\vec{E}$-field, as we discussed in class. In this case, since

$$\vec{E} = E_0 \, \frac{\hat{x} - i\hat{y}}{\sqrt{2}} \, e^{i(kz-\omega t)}$$

we see that E_x and E_y have the same amplitude $E_0/\sqrt{2}$, but they are out of phase by $-\pi/2$, since $e^{-\pi/2} = -i$. Therefore, we have here a circularly polarized wave.

That would be enough for the answer, but if you want to go deeper, since we have $\hat{x} - i\hat{y}$, the rotation of the electric vector is clockwise when an observer is looking toward an oncoming wave, and such a wave is said to be right circularly polarized (in the optical convention).

4. In class, we represented the electric and magnetic fields along two orthogonal directions by using unit vectors $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$; this can be termed the *linear polarization basis.*

Another general representation of the polarization may be done in terms of the unit vectors:

$$\hat{\epsilon}_+ = \frac{1}{\sqrt{2}}\Big(\hat{\epsilon}_1 + i\hat{\epsilon}_2\Big) \qquad \text{and} \qquad \hat{\epsilon}_- = \frac{1}{\sqrt{2}}\Big(\hat{\epsilon}_1 - i\hat{\epsilon}_2\Big)$$

and this can be termed the *circular polarization basis.*

A useful way to express the state of polarization is via the four Stokes parameters, s_0, s_1, s_2, and s_3. Suppose these have the values

$$s_0 = 3, \qquad s_1 = -1, \qquad s_2 = 2, \qquad s_3 = -2$$

(a) For the values of s_0, s_1, s_2, and s_3 given above, determine the amplitude of the electric field up to an overall phase in the linear polarization basis.

Solution: We need to go from the Stokes parameters to the fields.

Optional: The following part is optional, and you can begin by assuming equation (7.27) in Jackson without proof. However, since I didn't write it in the class summary, here is some relevant information. In class, we wrote the most general (homogeneous) plane wave propagating in the direction of the wave vector $\vec{k}$ as

$$\vec{E}(\vec{x},t) = (\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{7.19}$$

This definition uses the coordinate system with unit vectors $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ orthogonal to each other, and each orthogonal to the direction of propagation $\hat{z}$. Thus, the scalar products $\hat{\epsilon}_1 \cdot \vec{E}$ and $\hat{\epsilon}_2 \cdot \vec{E}$ give the amplitude of radiation along the $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ directions respectively (e.g., x and y directions). Then, the squares of these amplitudes give a measure of the intensity of each type of polarization. The Stokes parameters are then defined so that these intensities can be related in a useful way to measured characteristics of the wave, e.g., s_0 is defined as

$$s_0 = |\hat{\epsilon}_1 \cdot \vec{E}|^2 + |\hat{\epsilon}_2 \cdot \vec{E}|^2$$

so that it is equal to the total intensity of the wave. Likewise, for s_1, s_2, and s_3.

Next, if we define the *complex* quantities E_1 and E_2 in equation (7.19) in terms of a (real) magnitude times a phase factor:

$$E_1 = a_1 e^{i\delta_1} \qquad \text{and} \qquad E_2 = a_2 e^{i\delta_2}$$

then we get

$$\hat{\epsilon}_1 \cdot \vec{E} = \hat{\epsilon}_1 \cdot (\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} = E_1\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \text{and, likewise} \qquad \hat{\epsilon}_2 \cdot \vec{E} = E_2\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

so that

$$s_0 = |\hat{\epsilon}_1 \cdot \vec{E}|^2 + |\hat{\epsilon}_2 \cdot \vec{E}|^2 = \Big[E_1\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\Big]\Big[E_1^*\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\Big] + \Big[E_2\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\Big]\Big[E_2^*\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\Big]$$

since for complex z, we have $|z|^2 = zz^*$, where z^* is the complex conjugate of z. Thus

$$s_0 = E_1\, E_1^* + E_2\, E_2^* = (a_1 e^{i\delta_1})(a_1 e^{-i\delta_1}) + (a_2 e^{i\delta_2})(a_2 e^{-i\delta_2}) = a_1^2 + a_2^2$$

By similar procedures, we can derive the expressions for s_1, s_2, and s_3 listed in equation (7.27) in Jackson (page 301).

End optional section, begin required part of solution on the next page.

We are given s_0, s_1, s_2, and s_3, and wish to invert equation (7.27) in Jackson (page 301) to find a_1, a_2, δ_1, and δ_2. The first two equations in equation (7.27) are

$$s_0 = a_1^2 + a_2^2 \qquad \text{and} \qquad s_1 = a_1^2 - a_2^2$$

Adding and subtracting these two, we get respectively

$$s_0 + s_1 = 2a_1^2 \qquad \text{and} \qquad s_0 - s_1 = 2a_2^2$$

Thus

$$a_1 = \sqrt{\frac{s_0 + s_1}{2}} \qquad \text{and} \qquad a_2 = \sqrt{\frac{s_0 - s_1}{2}} \tag{H2.24}$$

Having determined a_1, a_2, we seek now to find δ_1 and δ_2. One might be inclined to think, well, there are four equations in equation (7.27) and four unknowns, so I should be able to determine δ_1 and δ_2. Notice, however, that the third and fourth equations both involve $(\delta_2 - \delta_1)$. Thus, we cannot solve for δ_1 and δ_2, but only for $(\delta_2 - \delta_1)$. This is because the Stokes parameters are not independent, and if you look at equation (7.27) carefully, you'll see that

$$s_0^2 = s_1^2 + s_2^2 + s_3^2$$

That's why the question says to find the result up to an overall phase, meaning that we are asked to find $(\delta_2 - \delta_1)$. If we use the third equation in equation (7.27), we get

$$\delta_2 - \delta_1 = \cos^{-1}\left[\frac{s_2}{2a_1 a_2}\right] \tag{H2.25}$$

Rather than write these expressions in general, I'm going to substitute the values given in the problem:

$$s_0 = 3, \qquad s_1 = -1, \qquad s_2 = 2, \qquad s_3 = -2$$

so that, from equation (H2.24), we get

$$a_1 = \sqrt{\frac{s_0 + s_1}{2}} = \sqrt{\frac{3 + (-1)}{2}} = 1 \qquad \text{and} \qquad a_2 = \sqrt{\frac{s_0 - s_1}{2}} = \sqrt{\frac{3 - (-1)}{2}} = \sqrt{2}$$

and from equation (H2.25), we get

$$\delta_2 - \delta_1 = \cos^{-1}\left[\frac{s_2}{2a_1 a_2}\right] = \cos^{-1}\left[\frac{2}{2(1)\sqrt{2}}\right] = \cos^{-1}\left[\frac{1}{\sqrt{2}}\right] = \frac{\pi}{4}$$

The electric field in equation (7.19) is then

$$\vec{E}(\vec{x}, t) = (\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2)\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} = (\hat{\epsilon}_1 a_1 e^{i\delta_1} + \hat{\epsilon}_2 a_2 e^{i\delta_2})\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

and so

$$\vec{E}(\vec{x}, t) = \left[\hat{\epsilon}_1 e^{i\delta_1} + \hat{\epsilon}_2 \sqrt{2} e^{i(\delta_1 + \pi/4)}\right] e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

or, if we were to write only the amplitude of the field, it would be

$$\boxed{\hat{\epsilon}_1 e^{i\delta_1} + \hat{\epsilon}_2 \sqrt{2} e^{i(\delta_1 + \pi/4)}}$$

The answer could be left as above, or cleaned up some more:

$$e^{i\delta_1}\left[\hat{\epsilon}_1 + \hat{\epsilon}_2 \sqrt{2} e^{i\pi/4}\right] = e^{i\delta_1}\left[\hat{\epsilon}_1 + \hat{\epsilon}_2 (1 + i)\right]$$

with the latter really bringing out the amplitude expressed up to an overall phase factor.

(b) For the values of s_0, s_1, s_2, and s_3 given above, determine the amplitude of the electric field up to an overall phase in the circular polarization basis.

Solution: In the circular polarization basis, with unit vectors $\hat{\epsilon}_+$ and $\hat{\epsilon}_-$, the general representation of a (homogeneous) plane wave is given in equation (7.24) in Jackson, and the Stokes parameters are given in equation (7.28). Notice that in equation (7.28), it is the the first and fourth equations that are relevant, and they are

$$s_0 = a_+^2 + a_-^2 \qquad \text{and} \qquad s_3 = a_+^2 - a_-^2$$

Adding and subtracting these two, we get respectively

$$s_0 + s_3 = 2a_+^2 \qquad \text{and} \qquad s_0 - s_3 = 2a_-^2$$

Thus

$$a_+ = \sqrt{\frac{s_0 + s_3}{2}} \qquad \text{and} \qquad a_- = \sqrt{\frac{s_0 - s_3}{2}} \tag{H2.26}$$

Again, the question asks us to find the result up to an overall phase, meaning that we are asked to find $(\delta_- - \delta_+)$. If we use the second equation in equation (7.28), we get

$$\delta_- - \delta_+ = \cos^{-1}\left[\frac{s_1}{2a_+a_-}\right] \tag{H2.27}$$

Once again, rather than write these expressions in general, I'm going to substitute the values given in the problem:

$$s_0 = 3, \qquad s_1 = -1, \qquad s_2 = 2, \qquad s_3 = -2$$

so that, from equation (H2.26), we get

$$a_+ = \sqrt{\frac{s_0 + s_3}{2}} = \sqrt{\frac{3 + (-2)}{2}} = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad a_- = \sqrt{\frac{s_0 - s_3}{2}} = \sqrt{\frac{3 - (-2)}{2}} = \sqrt{\frac{5}{2}}$$

and from equation (H2.27), we get

$$\delta_- - \delta_+ = \cos^{-1}\left[\frac{s_1}{2a_+a_-}\right] = \cos^{-1}\left[\frac{-1}{2(1/\sqrt{2})(\sqrt{5/2})}\right] = \cos^{-1}\left[\frac{-1}{\sqrt{5}}\right] = 0.65\,\pi$$

The electric field in equation (7.24) is then

$$\vec{E}(\vec{x}, t) = (E_+\hat{\epsilon}_+ + E_-\hat{\epsilon}_-)\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} = (\hat{\epsilon}_+ a_+ e^{i\delta_+} + \hat{\epsilon}_- a_- e^{i\delta_-})\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

and so

$$\vec{E}(\vec{x}, t) = \left[\hat{\epsilon}_+ \frac{1}{\sqrt{2}}\, e^{i\delta_+} + \hat{\epsilon}_- \sqrt{\frac{5}{2}} e^{i(\delta_+ +\ 0.65\pi)}\right] e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

or, if we were to write only the amplitude of the field, it would be

$$\boxed{\hat{\epsilon}_+ \frac{1}{\sqrt{2}}\, e^{i\delta_+} + \hat{\epsilon}_- \sqrt{\frac{5}{2}} e^{i(\delta_+ +\ 0.65\pi)}}$$

The answer could be left as above, or cleaned up some more:

$$\frac{e^{i\delta_+}}{\sqrt{2}}\left[\hat{\epsilon}_+ + \hat{\epsilon}_- \sqrt{5} e^{i(0.65\pi)}\right]$$