PHY 420 Spring 2021
Week 9—Tuesday, May 25—Discussion Worksheet

At this time, we've written several important equations of electrodynamics in ***covariant*** form. By covariance, we mean *invariance in form*, meaning that the form of the equations does not change when we transform from one (inertial) frame to another.

The key to writing electrodynamics equations in covariant form is to write all the quantities as 4-vectors. For example, by putting together the charge density ρ and the current density $\vec{J}$ into a 4-vector J^α, given by

$$J^\alpha = (c\rho, \vec{J}) \tag{11.128}$$

we can write the continuity equation $\dfrac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ in the ***covariant form***

$$\partial_\alpha J^\alpha = 0 \tag{11.129}$$

where we are using the notation from equation (11.76) that $\partial_\alpha \equiv \dfrac{\partial}{\partial x^\alpha} = \left(\dfrac{\partial}{\partial x^0}, \vec{\nabla}\right)$

We will now put the Lorentz force equation into covariant form. For a particle of charge q, the Lorentz force is given by

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{u}}{c}\times\vec{B}\right) \tag{11.124}$$

Recall that we wrote in an earlier class the 4-momentum (or energy-momentum 4-vector):

$$p^\alpha = (p_0, \vec{p}) = (E/c, \vec{p}) = m\,(U_0, \vec{U}) = m\,(\gamma_u c, \gamma_u \vec{u})$$

where $\gamma_u = (1 - u^2/c^2)^{-1/2}$. Also recall that $dt = \gamma_u\, d\tau$, where $d\tau$ is the proper time.

1. Show that for the space component $\vec{p}$ of the 4-momentum p^α, we get

$$\frac{d\vec{p}}{d\tau} = \frac{q}{c}\left(U_0\,\vec{E} + \vec{U}\times\vec{B}\right) \tag{11.125}$$

$$\frac{d\vec{P}}{d\tau} = \frac{d\vec{P}}{dt}\frac{dt}{d\tau}$$

$$= \gamma_u \frac{d\vec{P}}{dt}$$

$$= q\left(\vec{E} + \frac{\vec{U}}{c}\times\vec{B}\right)\gamma_u$$

$$= q/c\left[(\gamma_u c)\vec{E} + \gamma_u\vec{U}\times\vec{B}\right]$$

$$= q/c\left(U_0\vec{E} + \vec{U}\times\vec{B}\right)$$

2. We are writing the Lorentz force equation in its covariant form.

(a) Show that the corresponding time component of the 4-momentum is just the time rate of change of energy of the particle, which can be written as

$$\frac{dp_0}{d\tau} = \frac{q}{c}\vec{U}\cdot\vec{E} \tag{11.126}$$

Note: A useful result that you will need to use here is $\frac{dE_0}{dt} = q\vec{u}\cdot\vec{E}$.

$dt = \gamma d\tau$

$$\frac{dP_0}{d\tau} = \frac{1}{c}\underbrace{\frac{dE_0}{dt}}\underbrace{\frac{dt}{d\tau}}$$

$$= \frac{q}{c}\vec{u}\cdot\vec{E} \quad \gamma_u$$

$$= \frac{q}{c}\vec{U}\cdot\vec{E}$$

(b) Show that the Lorentz force equation then takes the covariant form

$$\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta \tag{11.144}$$

by proving equation (11.144) explicitly for $\alpha = 0$. **Hint:** Start from the right hand side.

$$= q/c \; F^{\alpha\beta}U_\beta$$

$$= q/c\left[F^{00}U_0 + F^{01}U_1 + F^{02}U_2 + F^{03}U_3\right]$$

$$= q/c\left[(0)U_0 + (-E_1)U_1 + (-E_2)U_2 + (-E_3)U_3\right]$$

$$= q/c\left[0 + E^1U_1 + E^2U_2 + E^3U_3\right]$$

$$= q/c\left[U_1E^1 + U_2E^2 + U_3E^3\right]$$

$$= q/c\;\vec{U}\cdot\vec{E} = dp_0/d\tau = dp^0/d\tau$$

You should try $\alpha = 1, 2, 3$ on your own using the result from Question 1; they work very similarly to part (b) above, but you'll need to be careful with the minus signs.

Transformation of Electromagnetic Fields

Since $\vec{E}$ and $\vec{B}$ are the elements of a second-rank tensor $F^{\alpha\beta}$, their values in one inertial frame K' can be expressed in terms of the values in another inertial frame K according to

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \tag{11.146}$$

In the matrix notation that we have adopted (taken from Section 11.7 of Jackson), and as you showed explicitly by matrix multiplication in the previous class, this can be written as

$$F' = AF\tilde{A} \tag{11.147}$$

where F and F' are 4×4 matrices as written in equation (11.137):

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

and A is the Lorentz transformation matrix of equation (11.93):

$$A = e^{-\vec{\omega}\cdot\vec{S}-\vec{\zeta}\cdot\vec{K}}$$

where $\vec{\omega}$ and $\vec{\zeta}$ are constant 3-vectors, their three components (each) corresponding to the six parameters of the transformation (three rotations and three Lorentz boosts); the six fundamental matrices S_1, S_2, S_3, and K_1, K_2, K_3 are written in equation (11.91).

Jackson notes that the subscripts $1, 2, 3$ in $F^{\alpha\beta}$ above, and in the discussion to follow, represent ordinary Cartesian spatial components and not covariant indices. Remember, however, that according to his treatment in Section 11.3, he set x_1 equal to z, but in writing the elements of $F^{\alpha\beta}$ above, I've written E_x as E_1. None of this should matter, as long as you keep using x_0, x_1, x_2, x_3 and pay attention to the fact that the boost is along, e.g., the x_1 axis, which is the case we will begin discussing on the next page.

3. We will now focus on a particular situation in which there is a Lorentz boost along the x_1 axis, so that $\vec{\omega} = 0$, and $\vec{\zeta} = \zeta\,\hat{e}_1$. We showed last week that the matrix A then takes the form

$$A = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rewrite this matrix in terms of γ, β and/or $\gamma\beta$, as appropriate, where

$$\gamma = \cosh\zeta, \qquad \beta = \tanh\zeta, \qquad \gamma\beta = \sinh\zeta$$

$$A = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. For the simple situation in which $\vec{\omega} = 0$, and $\vec{\zeta} = \zeta\,\hat{e}_1$, corresponding to a Lorentz boost along the x_1 axis with speed $c\beta$ from the unprimed frame to the primed frame, use the matrix A you obtained on the previous page to evaluate by explicit matrix multiplication the quantity

$$A F \tilde{A}$$

where elements $F^{\alpha\beta}$ of matrix F are written on the previous page, and $\tilde{A}$ is the transpose of A.

$$AF = \begin{pmatrix} \gamma & -\gamma B & 0 & 0 \\ -\gamma B & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -B_1E_1\gamma & -E_1\gamma & BB_3\gamma - E_2\gamma & -E_3\gamma - BB_2\gamma \\ E_1\gamma & BE_1\gamma & BE_2\gamma - B_3\gamma & BE_3\gamma + B_2\gamma \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$AF\tilde{A} =$$

$$\begin{pmatrix} -B_1E_1\gamma & -E_1\gamma & BB_3\gamma - E_2\gamma & -E_3\gamma - BB_2\gamma \\ E_1\gamma & BE_1\gamma & BE_2\gamma - B_3\gamma & BE_3\gamma + B_2\gamma \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma B & 0 & 0 \\ -\gamma B & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{-B_1E_1\gamma^2 + B_1E_1\gamma^2}\ 0 & B^2E_1\gamma^2 - E_1\gamma^2 & BB_3\gamma - E_2\gamma & -E_3\gamma - BB_2\gamma \\ E_1\gamma^2 - B^2E_1\gamma^2 & \cancel{-BE_1\gamma^2 + BE_1\gamma^2}\ 0 & BE_2\gamma - B_3\gamma & BE_3\gamma + B_2\gamma \\ E_2\gamma - BB_3\gamma & B_3\gamma - BE_2\gamma & 0 & -B_1 \\ E_3\gamma + BB_2\gamma & -BE_3\gamma - B_2\gamma & B_1 & 0 \end{pmatrix}$$

5. We are working on the **transformation of electromagnetic fields** from one frame to another, for the simple situation in which $\vec{\omega} = 0$, and $\vec{\zeta} = \zeta\,\hat{e}_1$, corresponding to a Lorentz boost along the x_1 axis with speed $c\beta$ from the unprimed frame to the primed frame.

(a) Set the result you computed on the previous page equal to F', which has the same elements as F but with its field components primed. Use it to show that $E_1' = E_1$.

$$\begin{pmatrix} 0 & \beta^2 E_1 \gamma^2 - E_1 \gamma^2 & \beta B_3 \gamma - E_2 \gamma & -E_3 \gamma - \beta B_2 \gamma \\ E_1' \gamma^2 - \beta^2 E_1' \gamma^2 & 0 & \beta E_2 \gamma - B_3 \gamma & \beta E_3 \gamma + B_2 \gamma \\ E_2' \gamma - \beta B_3 \gamma & B_3 \gamma - \beta E_2 \gamma & 0 & -B_1 \\ E_3 \gamma + \beta B_2 \gamma & -\beta E_3 \gamma - B_2 \gamma & B_1 & 0 \end{pmatrix}$$

$$E_1 = E_1' \left(\gamma^2 - \beta^2 \gamma^2 \right)$$
$$= E_1' \left(\cosh^2 - \sinh^2 \right)$$
$$E_1 = E_1'$$

(b) Use $F' = AF\tilde{A}$ from the previous part to derive the transformation

$$E_2' = \gamma\left(E_2 - \beta B_3\right)$$

1st column, 3rd row elements

$$E_2' = \gamma E_2 - \gamma \beta B_3$$
$$= \gamma \left(E_2 - \beta B_3 \right) \checkmark$$

Do the other components if you have time, but they are important enough that we will work them out explicitly in the next class.