PHY 411 — Winter 2021

Homework 7—due by 9:00 PM, Tuesday, March 2

Late submissions will be accepted until 8 AM on Saturday (March 6).

1. In class, we showed that the potential of a localized distribution of charge described by the charge density $\rho(\vec{x}\,')$ is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1}\, q_{lm}\, \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

where the *multipole moments* q_{lm} are given by

$$q_{lm} = \int Y^*_{lm}(\theta',\phi')\, r'^{\,l}\, \rho(\vec{x}\,')\, d^3x'$$

Explicitly evaluate q_{11} and q_{10} and show that

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\left(p_x - ip_y\right) \qquad \text{and} \qquad q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z$$

where p_x, p_y, p_z are the components of the electric dipole moment: $\vec{p} = \int \vec{x}\,'\, \rho(\vec{x}\,')\, d^3x'$.

2. Also of interest are the quadrupole moments q_{22}, q_{21}, and q_{20}, for which the algebra is more tedious. Therefore, we will limit ourselves to one example. Show that

$$q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\left(Q_{13} - i\,Q_{23}\right)$$

where Q_{ij} is the quadrupole moment tensor given by

$$Q_{ij} = \int \left(3x'_i x'_j - r'^{\,2}\,\delta_{ij}\right) \rho(\vec{x}\,')\, d^3x'$$

3. In class, you obtained by direct differentiation that the coordinates of the electric field E_r, E_θ, and E_ϕ are given by

$$E_r = \frac{(l+1)}{(2l+1)\,\epsilon_0}\, q_{lm}\, \frac{Y_{lm}(\theta,\phi)}{r^{l+2}}$$

$$E_\theta = -\frac{1}{(2l+1)\,\epsilon_0}\, q_{lm}\, \frac{1}{r^{l+2}}\, \frac{\partial}{\partial\theta} Y_{lm}(\theta,\phi)$$

$$E_\phi = \frac{1}{(2l+1)\,\epsilon_0}\, q_{lm}\, \frac{1}{r^{l+2}}\, \frac{im}{\sin\,\theta} Y_{lm}(\theta,\phi)$$

For a dipole $\vec{p}$ along the z-axis, show that the fields above reduce to:

$$E_r = \frac{2p\,\cos\,\theta}{4\pi\epsilon_0 r^3} \qquad\qquad E_\theta = \frac{p\,\sin\,\theta}{4\pi\epsilon_0 r^3} \qquad\qquad E_\phi = 0$$

4. Suppose that we have a uniform magnetic field $\vec{B}_0 = B_0\hat{z}$, where B_0 is a constant.

(a) Examine whether

$$\vec{A} = \frac{\vec{B}_0}{2} \times \vec{x}$$

is an appropriate vector potential for this given field.

(b) Does this vector potential satisfy the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$?

You may write by hand and scan as a single PDF, or write in latex (using the template file provided) or Word, and generate PDF. Please submit one PDF file only. Only questions and sub-parts that are numbered clearly, with numbers corresponding to those in this document, will be graded. See the syllabus for more detailed rules.

Submit late homework into the late D2L dropbox for reduced credit (see syllabus); late submissions will be accepted by the deadline specified on the previous page. Emailed or paper copies of homework are never accepted; in particular, do not attach homework to email to make an end-run around the D2L deadline or late deadline; such emails are automatically deleted and do not count as submissions.