PHY 474/374 **Spring 2021**

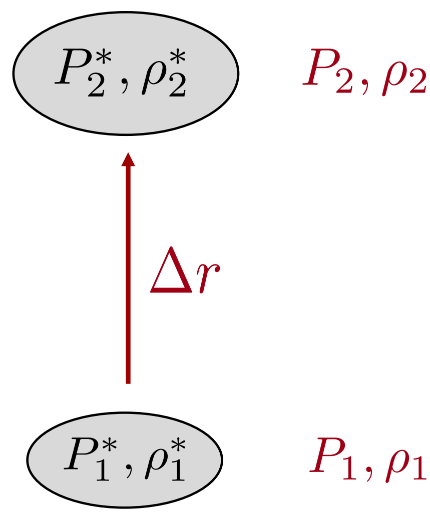
Class Summary—Week 4, Day 1—Monday, April 19

Energy Transport by Convection

We need to consider energy transport by convection because models with radiation only are, in general, unstable.

Read **Section 6.1** (*Dalsgaard*, **pages 81-82**) to understand what is meant by an instability, in particular, a **convective instability**. We will now discuss the instability criterion and how it leads to the need for convection.

Consider a parcel of gas that is moved outward by a distance Δr, as shown in the figure below.



Before the motion, the pressure and density inside this parcel are P_1^* and ρ_1^* respectively, whereas outside the parcel, they are P_1 and ρ_1. After the motion, the pressure and density are P_2^* and ρ_2^* inside the parcel, and P_2 and ρ_2 outside.

Initially, the parcel of gas is identical to its surroundings, so

$$P_1^* = P_1 \qquad \text{and} \qquad \rho_1^* = \rho_1$$

The buoyant force on the parcel of gas is then given by

$$f_{\text{buoy}} = -g\,\Delta\rho \tag{6.1}$$

where $\Delta\rho = \rho_2^* - \rho_2$, *as you showed in Question 1 on today's worksheet.*

If $f_{\text{buoy}} > 0$, then equation (6.1) tells us that $\Delta\rho$ must be negative (since g is positive). This means that $\rho_2 > \rho_2^*$, and so the force on the parcel of gas is upward (outward), and this leads to an instability.

On the other hand, if $f_{\text{buoy}} < 0$, the parcel of gas has a tendency to return to its original position, so the situation is stable.

In order to determine $\Delta\rho$ and decide between stability and instability, we make the following assumptions:

- **Assumption 1:** The motion is slow enough that there is pressure balance between the element and the surroundings.
- **Assumption 2:** The motion is fast enough that there is no heat loss to the surroundings.

Are these assumptions justified? Pressure balance is established on the dynamical timescale, $t_{\rm dyn}$, whereas a characteristic timescale for heat loss is the Kelvin-Helmholtz timescale, $t_{\rm KH}$, given by respectively

$$t_{\rm dyn} = \left(\frac{GM}{R^3}\right)^{-1/2} \qquad \text{and} \qquad t_{\rm KH} = \frac{U_{\rm tot}}{L_s}$$

both of which we've discussed earlier in earlier classes; $U_{\rm tot}$ is the total internal energy of the star, and L_s is the surface luminosity. For the Sun, $t_{\rm dyn} \simeq 1$ hr, and $t_{\rm KH} \simeq 10^7$ yr; thus, the conditions listed above are likely justified.

From Assumption 1, we have that

$$P_2^* = P_2$$

whereas Assumption 2 tells us that the motion must proceed adiabatically, so that

$$\frac{d\rho^*}{\rho^*} = \frac{1}{\Gamma_1}\frac{dP^*}{P^*} \tag{6.4}$$

where the adiabatic exponent Γ_1 was defined in equation (3.18) as

$$\left(\frac{\partial \ln P}{\partial \ln \rho}\right)_s = \Gamma_1$$

where s stands for entropy (if you know about entropy, good; if you don't, we won't worry about it for now).

We can use these two assumptions to obtain an equation for the density gradient *as you did in Question 2 on today's worksheet.* To do so, begin with

$$\Delta\rho = \rho_2^* - \rho_2$$

Now, since we are looking for a gradient, let's connect to position 1 by adding and subtracting ρ_1, so that

$$\Delta\rho = \rho_2^* - \rho_2 + \rho_1 - \rho_1$$

which can be written as

$$\Delta\rho = (\rho_2^* - \rho_1) - (\rho_2 - \rho_1)$$

Now, since $\rho_1^* = \rho_1$ (see previous page), we can replace the ρ_1 that we grouped with ρ_2^* to get

$$\Delta\rho = (\rho_2^* - \rho_1^*) - (\rho_2 - \rho_1)$$

so that

$$\Delta\rho = d\rho^* - (\rho_2 - \rho_1)$$

Replacing $d\rho^*$ from equation (6.4), we get

$$\Delta\rho = \frac{\rho^*}{\Gamma_1}\frac{dP^*}{P^*} - (\rho_2 - \rho_1)$$

Writing both terms on the right hand side of the equation at the bottom of the previous page in terms of gradients, we get

$$\Delta\rho = \frac{\rho}{P}\frac{1}{\Gamma_1}\frac{dP}{dr}\Delta r - \frac{d\rho}{dr}\Delta r$$

where I've dropped all mention of * since we don't need to refer any more to the conditions interior to the parcel of gas (i.e., the equations are now written in terms of gradients); alternatively, you could think of writing $P_1^* = P_1, \rho_1^* = \rho_1$ as we mentioned earlier and then describe the change from position 1 to position 2 in terms of the gradients in density and pressure. Finally, therefore, we can write

$$\Delta\rho = \left[\left(\frac{d\rho}{dr}\right)_{\rm ad} - \frac{d\rho}{dr}\right]\Delta r \tag{6.5}$$

where we have introduced the **density gradient** resulting from adiabatic motion in the given pressure gradient:

$$\left(\frac{d\rho}{dr}\right)_{\rm ad} \equiv \frac{1}{\Gamma_1}\frac{\rho}{P}\frac{dP}{dr} \tag{6.6}$$

Now, recall the condition for instability that $f_{\rm buoy} > 0$. From equation (6.1), we know that $f_{\rm buoy}$ goes as $-\Delta\rho$, and thus the condition for instability will be $\Delta\rho < 0$, so that

$$\left(\frac{d\rho}{dr}\right)_{\rm ad} < \left(\frac{d\rho}{dr}\right) \tag{6.7}$$

This condition may also be expressed as

$$\frac{d\ln\rho}{d\ln P} < \frac{1}{\Gamma_1} \tag{6.8}$$

For a completely ionized ideal gas, $1/\Gamma_1 = 3/5$.

Equation (6.8) tells us that instability occurs when the density does not decrease sufficiently rapidly, or even increases, toward the surface of the star. In practice, we can ignore the latter criterion (that density increases) since density decreases with decreasing pressure toward the surface of the star.

The instability criterion is usually expressed in terms of the **temperature gradient** rather than the gradient in density. Using the ideal gas law in the form $P = \rho kT/\mu m_p$ that we derived earlier this quarter, so that

$$\rho = \frac{\mu m_p P}{kT} \tag{6.9}$$

and assuming that chemical composition is independent of position and also that the gas is fully ionized so that μ is constant, we obtain by differentiation that

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{P}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr} \tag{6.10}$$

as you showed in Question 3(a) on today's worksheet.

From this, we obtain that

$$\left(\frac{d\rho}{dr}\right)_{\rm ad} - \frac{d\rho}{dr} = -\frac{\Gamma_1 - 1}{\Gamma_1}\frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr} \tag{6.11}$$

as you showed in Question 3(b) on today's worksheet.

A correct thermodynamical treatment that takes into account partial ionization and departures from the ideal gas law will show that Γ_1 in equation (6.11) must be replaced by Γ_2, the adiabatic exponent for the relation between P and T, defined in equation (3.19) as

$$\left(\frac{\partial \ln P}{\partial \ln T}\right)_s = \frac{\Gamma_2}{\Gamma_2 - 1}$$

Therefore, the instability condition becomes (*as you showed in Question 4(a) on today's worksheet*)

$$\left(\frac{dT}{dr}\right)_{\rm ad} > \frac{dT}{dr} \tag{6.12}$$

where

$$\left(\frac{dT}{dr}\right)_{\rm ad} \equiv \frac{\Gamma_2 - 1}{\Gamma_2}\,\frac{T}{P}\,\frac{dP}{dr} \tag{6.13}$$

is the adiabatic temperature gradient.

In analogy with equation (6.8), we can also write equation (6.12) as

$$\frac{d\ln T}{d\ln P} > \frac{\Gamma_2 - 1}{\Gamma_2} \tag{6.14}$$

as you showed in Question 4(b) on today's worksheet. This equation shows that instability occurs if the temperature decreases too rapidly out through the star. This is in line with the conceptual discussion in Section 6.1 (*Dalsgaard*).

Note that it is easier to demonstrate the result in equation (6.14) by beginning from it and writing

$$\frac{\frac{1}{T}\,dT}{\frac{1}{P}\,dP} = \frac{P}{T}\,\frac{dT/dr}{dP/dr} > \frac{\Gamma_2 - 1}{\Gamma_2}$$

Next, we must multiply by dP/dr on both sides to get the result, but upon doing this, we must flip the direction of the inequality because dP/dr is a negative quantity.

Finally, it is convenient to introduce the notation

$$\nabla = \frac{d\ln T}{d\ln P} \qquad \text{and} \qquad \nabla_{\rm ad} \equiv \frac{\Gamma_2 - 1}{\Gamma_2} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_{\rm ad} \tag{6.15}$$

so that the instability condition in equation (6.14) can now be written compactly as

$$\nabla > \nabla_{\rm ad} \tag{6.16}$$

Since equation (6.16) essentially tells us that the temperature gradient must be larger than the adiabatic temperature gradient, the usual terminology for the condition written in this equation is that the temperature gradient is superadiabatic. For a fully ionized ideal gas, $\nabla_{\rm ad} = 2/5$.

Note that our assumption of constant chemical composition in deriving equation (6.12) and equation (6.13) is not correct in regions where nuclear fusion is taking place. In such regions, the hydrogen abundance increases outward, and hence μ decreases outward. The derivation of equation (6.8) for P and ρ is correct in spite of this complication, however. Thus, equation (6.14) should be corrected by including a term containing the gradient of μ.

Where does convection occur?

In order to determine the circumstances under which to expect convection to occur, *Dalsgaard* examines the stability of a model in which energy transport takes place through radiation.

We will need equation (5.8) from the previous week:

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi acr^2T^3} \tag{6.17}$$

Together with the equation (4.4) for hydrostatic equilibrium and the ideal gas law, given by respectively

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2} \qquad \text{and} \qquad P = \frac{\rho k_B T}{\mu m_p}$$

where I've written the Boltzmann constant as k_B because it will be in the same equation with the opacity κ, we obtain for the temperature gradient in the form written in equation (6.15) that

$$\nabla \equiv \nabla_R \equiv \frac{d\ln T}{d\ln P} = \frac{3k_B}{16\pi acGm_p}\frac{\kappa}{\mu}\frac{L(r)}{m(r)}\frac{\rho}{T^3} \tag{6.18}$$

where ∇_R is the **radiative temperature gradient**, the gradient required to transport the entire luminosity by radiation. The condition for instability can then be written as

$$\nabla_\mathrm{R} > \nabla_\mathrm{ad} \tag{6.19}$$

When this condition is satisfied, energy transport by radiation requires too steep a temperature gradient, and thus convection must take place.

From equation (6.18) and equation (6.19), we may expect convection under the following circumstances.

- The ratio $L(r)/m(r)$ is large, *as you concluded in Question 5(a) of today's worksheet.* That is, the average rate of energy generation per unit mass within the radius r is large. This is typically the case in the interiors of high mass stars. We will learn later how the energy generation in such stars is a rapidly increasing function of temperature and hence is strongly concentrated toward the center of the star. Therefore, L/m is large in massive stars, and they have a convective core.

- The opacity κ is large, *as you concluded in Question 5(b) of today's worksheet.* This condition is satisfied in the outer parts of lower mass stars or, more generally, in stars with low surface temperatures where the temperature in the outer parts of the star is low and the opacity consequently high. A further contribution to the high opacity in these regions comes from the ionization of hydrogen.

- The quantity ρ/T^3 is large, *as you concluded in Question 5(b) of today's worksheet.* This is typically satisfied in the outer parts of relatively cool stars. In the photosphere (the base of the atmosphere, or the visible surface of the star), ρ/T^3 increases rapidly with decreasing effective temperature.

- The adiabatic temperature gradient, ∇_ad, is small. This is satisfied in the ionization zone of hydrogen, again in the outer parts of cool stars.

The first condition written above predicts that the cores of massive stars will be convective, and the remaining conditions predict convection in the outer parts of cool stars (e.g., low mass stars on the main sequence, and the so-called red giants which we will learn about later in the quarter); see Figure 6.2 in *Dalsgaard* (page 86).