Physics 460—Homework Report 2
Due Tuesday, Apr. 14, 1 pm **Name:** ______________________________

Complete all the problems on the accompanying assignment.

List all the problems you worked on in the space below. Circle the ones you fully completed:

Please place the problems into the following categories:

- These problems helped me understand the concepts better: ____________________
- I found these problems fairly easy: ____________________
- I found these problems very challenging: ____________________

In the space below, show your work (even if not complete) for any problems you still have questions about. Indicate where in your work the question(s) arose, and ask specific questions that I can answer.

Use the back of this sheet or attach additional paper, if necessary.

If you have no remaining questions about this homework assignment, use this space for one of the following:

- Write one or two of your solutions here so that I can give you feedback on its clarity.
- Explain how you checked that your work is correct.

(1) A particle with positive electric charge and spin11/2 is placed in a uniform magnetic field $\vec{B} = B_0(\hat{\imath}+\hat{k})/\sqrt{2}$. The Hamiltonian is then $H = -\gamma\vec{S}\cdot\vec{B}$, where γ is a positive constant called the gyromagnetic ratio.

(a) Working in the $\{|+_z\rangle, |-_z\rangle\}$ basis, find the matrix representation of the Hamiltonian H. [Continue working explicitly in this representation for parts (b)–(d).]

(b) Find the eigenvalues and eigenstates of H.

(c) At time $t = 0$, the particle is in state $|\Psi_0\rangle = |-_z\rangle$. If you measure the energy of the particle, what values can you measure, and what is the probability of measuring each value?

(d) Find the state vector at time t, and calculate $\langle S_x\rangle$ at time t.

My Answer:

(a) If $\vec{B} = B_0(\hat{\imath}+\hat{k})/\sqrt{2}$ then the Hamiltonian is $H = -\gamma B_0(S_x + S_z)/\sqrt{2}$. Written out as a matrix in the $\{|+_z\rangle, |-_z\rangle\}$ basis, this is

$$H \leftrightarrow -\frac{\gamma B_0}{\sqrt{2}}\frac{\hbar}{2}\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right) = -\frac{\Omega_0\hbar}{2\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}.$$

Here I've used $\Omega_0 = |\gamma|B_0$ as in the course notes.

(b) The energy eigenstates are given by the equation $H|E\rangle = E|E\rangle$. We can find the eigenvalues in the usual way. Letting $\alpha = \Omega_0\hbar/2\sqrt{2}$, we have

$$\det\begin{bmatrix}-\alpha - E & -\alpha\\ -\alpha & \alpha - E\end{bmatrix} = 0 = -(\alpha^2 - E^2) - \alpha^2.$$

Therefore, the eigenvalues are $E_\pm = \pm\sqrt{2}\alpha$, or $E_\pm = \pm\Omega_0\hbar/2$. This makes complete sense: if the strength of the magnetic field is B_0 and the spin of the particle is $\pm\hbar/2$, then the energy must be $\pm\gamma B_0\hbar/2$.

If we call the state associated with positive energy $|E_+\rangle$ and the state associated with negative energy $|E_-\rangle$, then $H|E_+\rangle = E_+|E_+\rangle$ and $H|E_-\rangle = E_-|E_-\rangle$. Written in the $\{|+_z\rangle, |-_z\rangle\}$ basis, this is

$$-\frac{E_+}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = E_+\begin{bmatrix}a\\ b\end{bmatrix} \quad\Rightarrow\quad -\frac{a+b}{\sqrt{2}} = a \quad\text{and}\quad -\frac{a-b}{\sqrt{2}} = b.$$

The second of these implies that $a = (1-\sqrt{2})b$. Picking $b = 1$ and normalizing, I find that

$$|E_+\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix}1-\sqrt{2}\\ 1\end{bmatrix}.$$

Similarly, $|E_-\rangle$ can be found through

$$\frac{E_-}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = E_-\begin{bmatrix}a\\ b\end{bmatrix} \quad\Rightarrow\quad \frac{a+b}{\sqrt{2}} = a \quad\text{and}\quad \frac{a-b}{\sqrt{2}} = b.$$

The second of these implies that $a = (1+\sqrt{2})b$. Picking $b = 1$ and normalizing, I find that

$$|E_-\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1+\sqrt{2}\\ 1\end{bmatrix}.$$

(c) At time $t = 0$ the state of the system is $|\Psi\rangle = |-_z\rangle$. This can be expressed in terms of the energy eigenstates as $|\Psi_0\rangle = c_+|E_+\rangle + c_-|E_-\rangle$:

$$|\Psi_0\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{c_+}{\sqrt{4-2\sqrt{2}}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} + \frac{c_-}{\sqrt{4+2\sqrt{2}}} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}.$$

Because our states are orthonormal, it's fairly easy to find c_+ and c_-:

$$c_+ = \langle E_+|\Psi_0\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{bmatrix} 1-\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{4-2\sqrt{2}}}$$

$$c_- = \langle E_-|\Psi_0\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{bmatrix} 1+\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{4+2\sqrt{2}}}.$$

Therefore, you can measure $E_+ = \hbar\Omega_0/2$ with probability

$$\mathcal{P}_+ = |c_+|^2 = \frac{1}{4-2\sqrt{2}},$$

and $E_- = -\hbar\Omega_0/2$ with probability

$$\mathcal{P}_- = |c_-|^2 = \frac{1}{4+2\sqrt{2}}.$$

As a partial check, I'll verify that these probabilities sum to unity:

$$\mathcal{P}_+ + \mathcal{P}_- = \frac{1}{4-2\sqrt{2}} + \frac{1}{4+2\sqrt{2}} = \frac{8}{16-8} = 1.$$

(d) From above, the initial state is

$$\begin{aligned} |\Psi_0\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{c_+}{\sqrt{4-2\sqrt{2}}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} + \frac{c_-}{\sqrt{4+2\sqrt{2}}} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} \\ &\leftrightarrow \frac{1+\sqrt{2}}{2\sqrt{2}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} - \frac{1-\sqrt{2}}{2\sqrt{2}} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}. \end{aligned}$$

Therefore, at time t the state is

$$\begin{aligned} |\Psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht/\hbar}|\Psi_0\rangle &\leftrightarrow \frac{1+\sqrt{2}}{2\sqrt{2}} \mathrm{e}^{-\mathrm{i}\Omega_0 t/2} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} - \frac{1-\sqrt{2}}{2\sqrt{2}} \mathrm{e}^{\mathrm{i}\Omega_0 t/2} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} \\ &\leftrightarrow \frac{1}{2\sqrt{2}} \left(\mathrm{e}^{-\mathrm{i}\Omega_0 t/2} \begin{bmatrix} -1 \\ 1+\sqrt{2} \end{bmatrix} - \mathrm{e}^{\mathrm{i}\Omega_0 t/2} \begin{bmatrix} -1 \\ 1-\sqrt{2} \end{bmatrix} \right) \\ &\leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} \mathrm{i}\sin\frac{\Omega_0 t}{2} \\ -\mathrm{i}\sin\frac{\Omega_0 t}{2} + \sqrt{2}\cos\frac{\Omega_0 t}{2} \end{bmatrix}. \end{aligned}$$

Note that the state is still normalized, of course.

The expectation value of S_x is

$$\begin{aligned} \langle S_x \rangle &= \frac{1}{2} \begin{bmatrix} \mathrm{i}\sin\frac{\Omega_0 t}{2} & -\mathrm{i}\sin\frac{\Omega_0 t}{2} + \sqrt{2}\cos\frac{\Omega_0 t}{2} \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{i}\sin\frac{\Omega_0 t}{2} \\ -\mathrm{i}\sin\frac{\Omega_0 t}{2} + \sqrt{2}\cos\frac{\Omega_0 t}{2} \end{bmatrix} \\ &= \frac{\hbar}{4} \begin{bmatrix} \mathrm{i}\sin\frac{\Omega_0 t}{2} & -\mathrm{i}\sin\frac{\Omega_0 t}{2} + \sqrt{2}\cos\frac{\Omega_0 t}{2} \end{bmatrix} \begin{bmatrix} -\mathrm{i}\sin\frac{\Omega_0 t}{2} + \sqrt{2}\cos\frac{\Omega_0 t}{2} \\ \mathrm{i}\sin\frac{\Omega_0 t}{2} \end{bmatrix} \\ &= -\frac{\hbar}{2}\sin^2\frac{\Omega_0 t}{2}. \end{aligned}$$

(2) Consider a spin-1/2 particle in a rotating magnetic field

$$\vec{B} = B_1 \cos \omega t\, \hat{\imath} - B_1 \sin \omega t\, \hat{\jmath} + B_0\, \hat{k}.$$

This magnetic field has a constant z-component and a component that rotates in the x–y plane with frequency ω. If the initial state of the particle is $|+_z\rangle$, it can be shown that the state of the system at a later time t is

$$|\Psi(t)\rangle = \left(\cos \frac{\Omega t}{2} + \mathrm{i}\frac{\Omega_0 - \omega}{\Omega} \sin \frac{\Omega t}{2}\right) e^{\mathrm{i}\omega t/2}|+_z\rangle + \frac{\mathrm{i}\Omega_1}{\Omega} \sin \frac{\Omega t}{2}\, e^{-\mathrm{i}\omega t/2}|-_z\rangle,$$

where

$$\Omega_0 = \gamma B_0, \qquad \Omega_1 = \gamma B_1, \qquad \text{and} \qquad \Omega = \sqrt{\Omega_1{}^2 + (\Omega_0 - \omega)^2}.$$

(a) Explain how you would verify that $|\Psi(t)\rangle$ given above is the state of the system at a later time t. You don't have to carry out all the algebra (unless you want to!), but explain what you would have to do to verify that this $|\Psi(t)\rangle$ is correct.

(b) The situation of *paramagnetic resonance* occurs when $\omega = \Omega_0$. How does the state $|\Psi(t)\rangle$ simplify in this case? Interpret your result in terms of the spin of the particle.

(c) Suppose that in the paramagnetic case the rotating field is applied to the particle for a finite time τ such that

$$\Omega_1 \tau = \frac{\pi}{2}.$$

This field is called a 90° pulse. Why do you think it has this name?

(d) Suppose that in the paramagnetic case the rotating field is applied to the particle for a finite time τ such that

$$\Omega_1 \tau = \pi.$$

This field is called a 180° pulse. Why do you think it has this name?

My Answer:

(a) For $|\Psi(t)\rangle$ to be the state of the system at a later time t, it must be a solution of the Schrödinger equation with initial state $|\Psi(0)\rangle = |+_z\rangle$. If we evaluate $|\Psi(t)\rangle$ at time $t = 0$, we do in fact get $|+_z\rangle$. So that's good. The other thing that we would have to check is that

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = H|\Psi(t)\rangle.$$

This isn't hard to do in principle. We would have to take the derivative with respect to time of this state and compare it to the result of acting with H on the state. We know how to do both those things; it's just a lot of tedious algebra to take the derivative and to evaluate the right-hand side.

(b) If $\omega = \Omega_0$, then $\Omega = \Omega_1$, and the term that contains $\Omega_0 - \omega$ disappears. The state then becomes

$$|\Psi(t)\rangle = \cos \frac{\Omega_1 t}{2}\, e^{\mathrm{i}\omega t/2}|+_z\rangle + \mathrm{i} \sin \frac{\Omega_1 t}{2} e^{-\mathrm{i}\omega t/2}|-_z\rangle.$$

To interpret this, we can look at the probability of measuring, for example, the spin along the z axis. The possible results are $+\hbar/2$ and $-\hbar/2$, of course, and the probabilities are

$$\mathcal{P}(+_z) = |\langle +_z|\Psi(t)\rangle|^2 = \left|\cos \frac{\Omega_1 t}{2}\, e^{\mathrm{i}\omega t/2}\right|^2 = \cos^2 \frac{\Omega_1 t}{2},$$

$$\mathcal{P}(-_z) = |\langle -_z|\Psi(t)\rangle|^2 = \left|\sin \frac{\Omega_1 t}{2}\, e^{-\mathrm{i}\omega t/2}\right|^2 = \sin^2 \frac{\Omega_1 t}{2}.$$

The effect of these fields on the spin in the z direction is to cause the particle to "rotate" with frequency $\Omega_1/2$.

A similar calculation for the spin along the x axis leads to (if I have done my algebra correctly)

$$\mathcal{P}(+_x) = |\langle +_x|\Psi(t)\rangle|^2 = \frac{1}{2}\left(\cos^2\frac{(\Omega_1-\omega)t}{2} + \sin^2\frac{(\Omega_1+\omega)t}{2}\right),$$
$$\mathcal{P}(-_x) = |\langle -_x|\Psi(t)\rangle|^2 = \frac{1}{2}\left(\sin^2\frac{(\Omega_1-\omega)t}{2} + \cos^2\frac{(\Omega_1+\omega)t}{2}\right).$$

This result is a little more complicated, but it is similar. The particle is rotating and there is a beating phenomenon between the two frequencies Ω_1 and ω.

(c) The particle is initially in the state $|+_z\rangle$, so its spin is initially aligned along the positive z axis. If we apply the field for a time $\Omega_1\tau = \pi/2$, then the probabilities of spin up or spin down along the z axis are

$$\mathcal{P}(+_z) = \cos^2\frac{\Omega_1\tau}{2} = \cos^2\frac{\pi}{4} = \frac{1}{2},$$
$$\mathcal{P}(-_z) = \sin^2\frac{\Omega_1\tau}{2} = \sin^2\frac{\pi}{4} = \frac{1}{2}.$$

These probabilities are equal. The interpretation of this is that the spin of the particle lies in the x–y plane, so the spin has been rotated $90°$ from its initial orientation.

(d) If we apply the field for a time $\Omega_1\tau = \pi$, then the probabilities of spin up or spin down along the z axis are

$$\mathcal{P}(+_z) = \cos^2\frac{\Omega_1\tau}{2} = \cos^2\frac{\pi}{2} = 0,$$
$$\mathcal{P}(-_z) = \sin^2\frac{\Omega_1\tau}{2} = \sin^2\frac{\pi}{2} = 1.$$

The interpretation of this is that the spin of the particle is now aligned with the negative z axis, so the spin has been rotated $180°$ from its initial orientation.