Homework 5

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Problem 1

$$\vec{J}(\vec{x})e^{-i\omega t} = \begin{cases} Isin(kz)\delta(x)\delta(y)e^{-i\omega t}\hat{z} & \text{if } \; -\frac{d}{2} < z < \frac{d}{2} \\ 0 & \text{if } \; |z| > \frac{d}{2} \end{cases}$$

The vector potential $\vec{A}(\vec{x})$ is gave by

$$A(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int J(\vec{x}')d'x.$$

Therefore, if we substitute the current density equation, with the proper bounds, into the vector potential equation we get

$$A(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int_{-\infty}^{\infty}\delta(x)\;dx\int_{-\infty}^{\infty}\delta(y)\;dy\;\int_{-d/2}^{d/2}Isin(kz)\;\hat{z}\;dz.$$

Since we are in the far zone and the equation is difficult set up this way, we can rewrite it as

$$A(\vec{x}) = \frac{iI\mu_0}{4\pi}\frac{e^{ikr}}{r}\int_0^{d/2}2sin(kz)sin(kzcos\theta)\;\hat{z}\;dz$$

Integrating the equation above will result in

$$A(\vec{x}) = \frac{iI\mu_0}{4\pi}\frac{e^{ikr}}{kr}\left[\frac{1}{1-cos\theta}sin((1-cos\theta)kz) - \frac{1}{1+cos\theta}sin((1+cos\theta)kz)\right]_0^{d/2}$$

and entering the bounds on integration gives

$$\vec{A}(\vec{x}) = \frac{\mu_0 I}{2\pi}\frac{e^{ikr}}{ikr}\left[\frac{sin(\pi\ cos\theta))}{sin^2\theta}\right]\hat{z}.$$

Problem 2

Use your expression for $\vec{A}(\vec{x})$ to find $\vec{B}$ and $\vec{E}$ in the radiation zone.

We can now work in spherical coordinates, therefore $\hat{z} = \hat{r}cos\theta - \hat{\theta}sin\theta$. The vector potential can now be expressed as

$$\vec{A}(\vec{x}) = \frac{\mu_0 I}{2\pi}\frac{e^{ikr}}{ikr}\left[\frac{sin(\pi cos\theta))}{sin^2\theta}\right](\hat{r}cos\theta - \hat{\theta}sin\theta).$$

The magnetic field is gave by

$$\begin{aligned} H &= \frac{ik}{\mu_0}\hat{n} \times \vec{A} \\ &= \frac{ik}{\mu_0}\hat{r} \times \frac{\mu_0 I}{2\pi}\frac{e^{ikr}}{ikr}\left[\frac{sin(\pi cos\theta))}{sin^2\theta}\right](\hat{r}cos\theta - \hat{\theta}sin\theta) \\ &= -\hat{\phi}\frac{Ie^{ikr}}{2\pi r}\frac{sin(\pi cos\theta)}{sin\theta} \end{aligned}$$

or

$$\vec{B} = -\hat{\phi}\frac{Ie^{ikr}\mu_0}{2\pi r}\frac{sin(\pi cos\theta)}{sin\theta}$$

The electric field is gave by

$$\begin{aligned} E &= ikz_0(\hat{n} \times \vec{A}) \times \hat{n} \\ &= ikz_0\hat{r} \times \frac{\mu_0 I}{2\pi}\frac{e^{ikr}}{ikr}\left[\frac{sin(\pi cos\theta))}{sin^2\theta}\right](\hat{r}cos\theta - \hat{\theta}sin\theta) \times \hat{r} \\ &= -\hat{\phi}\frac{Ie^{ikr}}{2\pi r}\frac{sin(\pi cos\theta)}{sin\theta} \times \hat{r} \\ &= -\hat{\theta}\frac{Iz_0\mu_0 e^{ikr}}{2\pi r}\frac{sin(\pi cos\theta)}{sin\theta} \end{aligned}$$

Problem 3

Calculate $\frac{dP}{d\Omega}$, the power radiated per unit solid angle.

The time averaged power radiated per unit solid angle by the oscillating dipole moment is gave by

$$\frac{dP}{d\Omega} = \frac{1}{2} Re[r^2 n \cdot E \times H^*]$$

By rearranging and eliminating some terms, this equation can also be wrote as

$$\frac{dP}{d\Omega} = \frac{r^2}{2} \mu_0 \epsilon_0 |\vec{H}|^2$$

The magnitude of $\vec{H}$ can be found by

$$|\vec{H}| = k \; sin\theta \; |\vec{A}_z| / \mu_0$$

Therefore, we have the power radiated per unit solid angle as

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2} \left| \frac{sin(\pi cos\theta)}{sin\theta} \right|^2 .$$

Problem 4

My top picks for the project go by:

1. Synchrotron Radiation: Bridging Particle Accelerators and Astrophysics
    - My prior work with Dr. Gonzalez and current thesis work (as well as prior work) with Dr. Landahl at the APS make this topic my top choice. While understanding the idea of Synchrotron Radiation, I have yet to work through this topic mathematically.
2. Thomas Precession
    - This topic is of interest since it will allow me to expand my knowledge of relativity.
3. Green Function for the sphere
    - This topic interests me by allowing me to expand my mathematical knowledge.