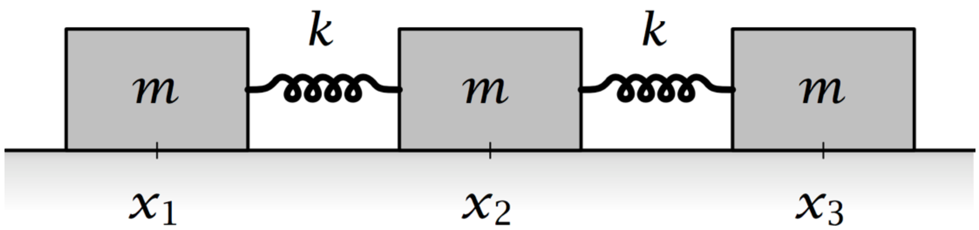
Activity 7: Three blocks connected by springs

Three identical blocks connected by springs slide along a frictionless horizontal surface as shown below. The mass of each block is m, the spring constant of both springs is k, and their rest length is l.



a. Find the Lagrangian in terms of $(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3)$. Does it have any cyclic coordinates?
b. Do you think that there are any conserved quantities (constants of the motion) for this problem? If so, what are they physically, and what are their formulas in terms of the given coordinates?
c. Find a new set of three variables so that if you express the Lagrangian in these variables there is at least one cyclic variable. Find the corresponding conserved quantity.

a. $L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) + \frac{k}{2}[(x_2 - x_1 - l)^2 + (x_3 - x_2 - l)^2)]$
All three coordinates appear in the Lagrangian, so there are no cyclic coordinates
b. This system has translational symmetry along the x axis because it doesn't change if you move it as a whole along the x axis. So I expect that the total momentum of the system is a constant of motion. So let's give the location of x_2 in order to specify where the system is located along the x axis, and then express the position of masses 1 and 3 by giving the distance to mass 2. Because the behavior of the system si only affected by how far the masses are from each other, not by where along the x axis the system is located, I would expect x_2 to be cyclic. Let's see …
Well define the generalized coordinates X_1, X_1, X_3 in the following way:
$X_2 = x_2 - x_1 \quad X_3 = x_3 - x_2 \quad X_1 = x_2$
The coordinate transformation equations are:
$x_1 = x_2 - X_2 = X_1 - X_2$
$x_2 = X_1$
$x_3 = X_3 + x_2 = X_3 + X_1$
$\dot{x}_1^2 = \dot{X}_1^2 + \dot{X}_2^2 - 2\dot{X}_1\dot{X}_2$
$\dot{x}_2^2 = \dot{X}_1^2$
$\dot{x}_3^2 = \dot{X}_1^2 + \dot{X}_3^2 + 2\dot{X}_1\dot{X}_3$

Plugging these transformations into the Lagrangian we get:
c. $L = \frac{m}{2}\left(\dot{X}_1^2 + \dot{X}_2^2 - 2\dot{X}_1\dot{X}_2 + \dot{X}_1^2 + \dot{X}_1^2 + \dot{X}_3^2 + 2\dot{X}_1\dot{X}_3\right) + \frac{k}{2}[(X_1 - X_1 - X_2 - l)^2 +$
$(X_3 + X_1 - X_1 - l)^2)]$

$$L = \frac{m}{2}\left[3\dot{X}_1^2 + \dot{X}_2^2 + \dot{X}_3^2 + 2\dot{X}_1\left(\dot{X}_3 - \dot{X}_2\right)\right] + \frac{k}{2}\left[(X_2 - l)^2 + (X_3 - l)^2\right]$$

As expected, the variable that specifies the location of the system along the horizontal axis, X_1, does not appear in the Lagrangian and is therefore cyclic or ignorable. The corresponding conserved quantity is

$$p_x = \frac{\partial L}{\partial \dot{X}_1} = \frac{m}{2}\left(6\dot{X}_1 - 2\dot{X}_2 + 2\dot{X}_3\right) = m(3\dot{x}_2 - \dot{x}_2 + \dot{x}_1 + \dot{x}_3 - \dot{x}_2) = m(\dot{x}_1 + \dot{x}_2 + \dot{x}_3)$$

This is the total momentum of the system.