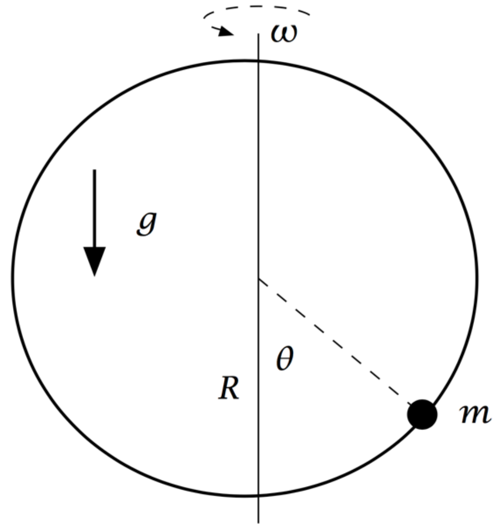
A massless, frictionless vertical hoop rotates with constant angular frequency ω about a diameter, as shown in the figure below. A particle of mass m is free to move along the hoop. The angle θ measures the angle of the particle relative to the axis of rotation.

a. Find the Lagrangian for this system.
b. Find the equations of motion.
c. Find the equilibrium solutions and their stability. You should find that the number of equilibrium solutions change as ω changes. Explain physically why this happens.



a. Cartesian coordinates:

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \quad V = mgz$

Coordinate transformation from Cartesian to spherical coordinates:

$x = R\cos\phi\sin\theta\,, \qquad y = R\sin\phi\sin\theta\,, \qquad z = -R\cos\theta$

Assume that the hoop is in the $x - z$ plane at $t = 0$, then $\phi = \omega t$

$x = R\cos\omega t\sin\theta\,, \qquad y = R\sin\omega t\sin\theta$

$\dot{x} = -\omega R\sin\omega t\sin\theta + R\dot{\theta}\cos\omega t\cos\theta$

$\dot{y} = \omega R\cos\omega t\sin\theta + R\dot{\theta}\sin\omega t\cos\theta$

$\dot{z} = R\dot{\theta}\sin\theta$

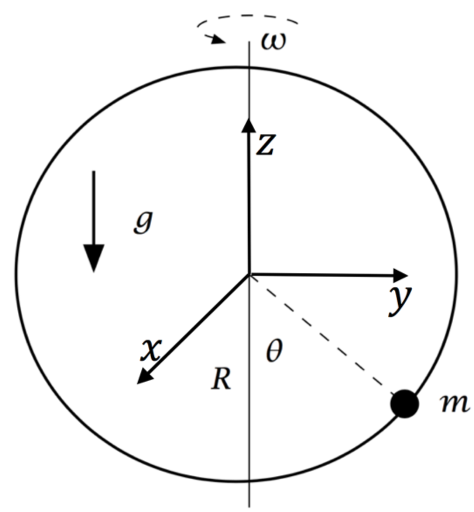
$$\begin{aligned}
\dot{x}^2 + \dot{y}^2 &= \omega^2R^2\sin^2\omega t\sin^2\theta + R^2\dot{\theta}^2\cos^2\omega t\cos^2\theta \\
&\quad + \omega^2R^2\cos^2\omega t\sin^2\theta \\
&\quad + R^2\dot{\theta}^2\sin^2\omega t\cos^2\theta \\
&\quad - \cancel{2\omega R\sin\omega t\sin\theta\, R\dot{\theta}\cos\omega t\cos\theta} \\
&\quad + \cancel{2\omega R\cos\omega t\sin\theta\, R\dot{\theta}\sin\omega t\cos\theta}
\end{aligned}$$

$= \omega^2R^2\sin^2\theta + R^2\dot{\theta}^2\cos^2\theta$

$\dot{z}^2 = R^2\dot{\theta}^2\sin^2\theta$

$T = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \omega^2R^2\sin^2\theta\right), \quad V = -mgR\cos\theta$

$L = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + mgR\cos\theta$



b. Lagrange's equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial q} = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mR^2\ddot{\theta}$$
$$\frac{\partial L}{\partial \theta} = mR^2\omega^2 \sin\theta \cos\theta - mgR\sin\theta$$
$$mR^2\ddot{\theta} = mR^2\omega \sin\theta\cos\theta - mgR\sin\theta$$
$$\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta$$

c. If we place the particle at an equilibrium position the acceleration on the particle is zero ($\ddot{\theta} = 0$), so that the particle stays at this position. Thus:

$$\ddot{\theta} = 0 = \left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta$$

This is satisfied either when $\sin\theta = 0$ or when $\cos\theta = \frac{g}{R\omega^2}$. The first option tells us that the particle is in equilibrium at the top and bottom of the hoop $\theta_1 = 0, \theta_2 = \pi$. The second option can only be satisfied when the square of angular velocity of the hoop is at least $\frac{g}{R}$ $\left(\omega^2 \geq \frac{g}{R}\right)$. If that is the case, then we get two additional equilibrium solutions: $\theta_{3,4} = \pm\cos^{-1}\left(\frac{g}{\omega^2 R}\right)$. When the angular velocity has increased to $\omega^2 = \frac{g}{R}$ the argument of the inverse cosine is 1, so $\theta_3 = \theta_4 = 0$. For larger ω the equilibrium point at the bottom of the hoop splits in two (bifurcates). The larger ω the further the two equilibrium points are from the bottom of the hoop. For $\omega \to \infty$, $\frac{g}{\omega^2 R} \to 0$, so $\theta_{3,4} \to \pm\frac{\pi}{2}$.

Stability:

Let's look at the equation of motion to see which of the four equilibrium points is stable.

- For $\omega^2 < \frac{g}{R}$ the term in parenthesis $\left(\omega^2\cos\theta - \frac{g}{R}\right)$ is always negative. For small positive perturbations at the bottom of the hoop $\theta_1 = 0 + \epsilon$ the acceleration is therefore negative and for small negative perturbations $\theta_1 = 0 - \epsilon$ it is positive, making the equilibrium point at the bottom stable. For small positive perturbations at the top of the hoop $\theta_2 = \pi + \epsilon$ the acceleration is positive and for small negative perturbations $\theta_2 = \pi - \epsilon$ it is negative, making the equilibrium point at the top unstable.
- For $\omega^2 \geq \frac{g}{R}$ the term in parenthesis $\left(\omega^2\cos\theta - \frac{g}{R}\right)$ is positive at bottom of the hoop and negative at the top. That means both θ_1 and θ_2 are now unstable (positive perturbations lead to positive acceleration and vice versa). At the two equilibrium points with $\theta_{3,4} = \pm\cos^{-1}\left(\frac{g}{\omega^2 R}\right)$ the term in parentheses is 0. If we increase θ_3 a little this term becomes negative (because $\cos\theta$ decreases), but $\sin\theta$ stays positive. So the acceleration is negative, pushing the particle back to the equilibrium points. The reverse happens if we decrease θ_3 a little. If we decrease θ_4 a little the term in parentheses again becomes negative, and $\sin\theta$ remains negative, making the acceleration positive and pushing the particle back to the equilibrium point. The reverse happens if θ_4 is increased a little. Thus, both θ_3 and θ_4 are stable.