Which Rules Define Functions?

Not every equation defines a function. The equation $y = x^2$ defines y as a function of x because the equation determines exactly one value of y for each value of x. However, the equation $y^2 = x$ does *not* define y as a function of x because some input values x correspond to more than one output y; for instance, for the input $x = 4$ the equation gives the outputs $y = 2$ and $y = -2$.

Similarly, not every table defines a function. Table 3 defined C as a function of w—each package weight w corresponds to exactly one mailing cost. On the other hand, Table 4 does *not* define y as a function of x because some input values x in the table correspond to more than one output y; for instance, the input $x = 5$ gives the outputs $y = 7$ and $y = 8$.

Table 4

x	2	4	5	5	6
y	3	6	7	8	9

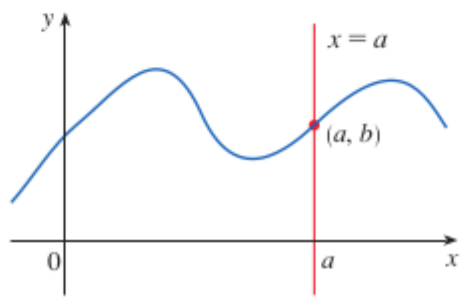
What about curves drawn in the xy-plane? Which curves are graphs of functions? The following test gives an answer.

The Vertical Line Test

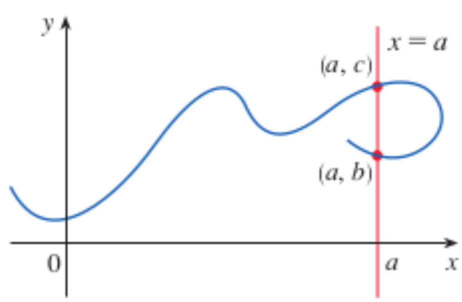
A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 13. If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one function value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve can't represent a function because a function can't assign two different values to a.

Figure 13



(a) This curve represents a function.

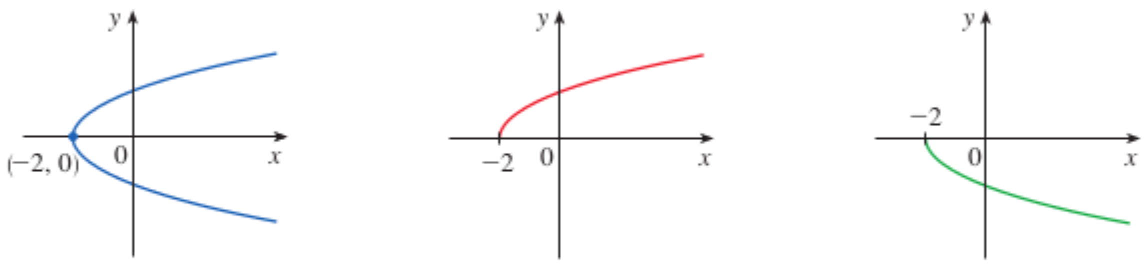


(b) This curve doesn't represent a function.

▸ Details

For example, the parabola $x = y^2 - 2$ shown in Figure 14(a) is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of x. Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x + 2}$. Thus the upper and lower halves of the parabola are the graphs of the functions $f(x) = \sqrt{x + 2}$ [from Example 6(a)] and $g(x) = -\sqrt{x+2}$. [See Figures 14(b) and (c).]

Figure 14



(a) $x = y^2 - 2$ (b) $y = \sqrt{x+2}$ (c) $y = -\sqrt{x+2}$

Details

We observe that if we reverse the roles of x and y , then the equation $x = h\left(y\right) = y^2 - 2$ *does* define x as a function of y (with y as the independent variable and x as the dependent variable). The graph of the function h is the parabola in Figure 14(a).