

1.1. Four Ways to Represent a Function

Functions

Functions arise whenever one quantity depends on another. Consider the following four situations.

- A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A , and we say that A is a *function* of r .
- B. The human population of the world P depends on the time t . [Table 1](#) gives estimates of the world population P at time t , for certain years. For instance,

$$P \approx 2,560,000,000 \quad \text{when } t = 1950$$

For each value of the time t there is a corresponding value of P , and we say that P is a function of t .

Table 1 World Population

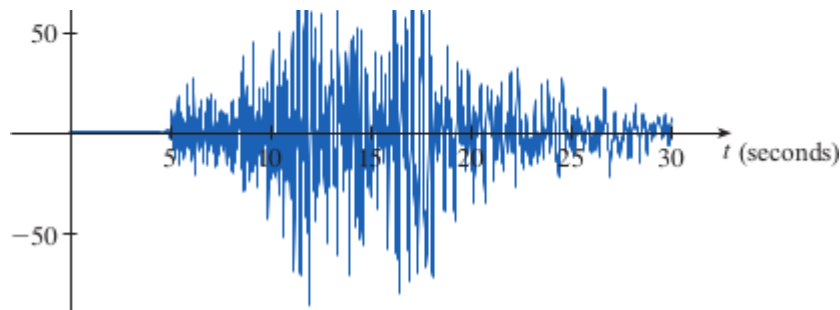
Year	Population (millions)
1900	1650
1910	1750
1920	1860

Year	Population (millions)
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

- C. The cost C of mailing an envelope depends on its weight w . Although there is no simple formula that connects w and C , the post office has a rule for determining C when w is known.
- D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t . [Figure 1](#) shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t , the graph provides a corresponding value of a .

Figure 1





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Vertical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number (r in [Example A](#)), another number (A) is assigned. In each case we say that the second number is a function of the first number. If f represents the rule that connects A to r in [Example A](#), then we express this in **function notation** as $A = f(r)$.

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E .

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read “ f of x .” The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In [Example A](#), for instance, r is the independent variable and A is the dependent variable.

It’s helpful to think of a function as a **machine** (see [Figure 2](#)). If x is in the domain of the function f , then when x enters the machine, it’s accepted as an **input** and the machine produces an **output** $f(x)$ according to the rule of the function. So we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs. The preprogrammed functions in a

calculator are good examples of a function as a machine. For example, if you input a number and press the squaring key, the calculator displays the output, the square of the input.

Figure 2

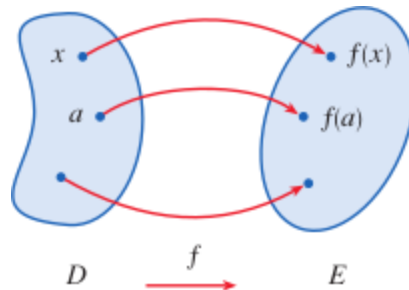


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Machine diagram for a function f

Another way to picture a function is by an **arrow diagram** as in [Figure 3](#). Each arrow connects an element of D to an element of E . The arrow indicates that $f(x)$ is associated with x , $f(a)$ is associated with a , and so on.

Figure 3



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Arrow diagram for f

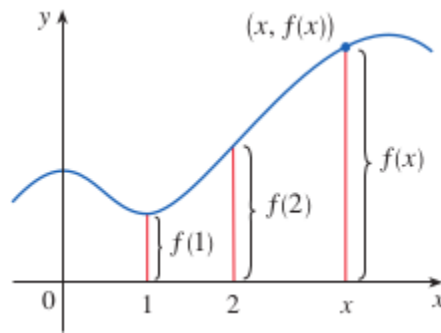
Perhaps the most useful method for visualizing a function is its graph. If f is a function with domain D , then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f .

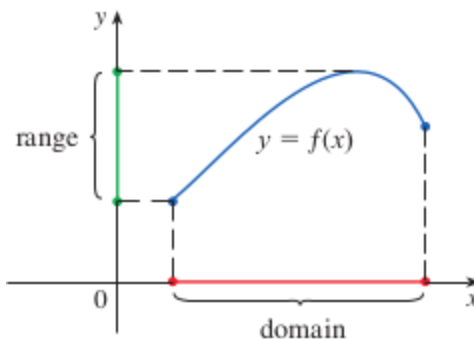
The graph of a function f gives us a useful picture of the behavior or “life history” of a function. Since the y -coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x . (See [Figure 4](#).) The graph of f also allows us to picture the domain of f on the x -axis and its range on the y -axis as in [Figure 5](#).

Figure 4



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Figure 5



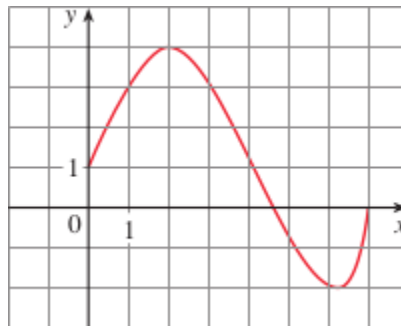
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Example 1

The graph of a function f is shown in [Figure 6](#).

- (a) Find the values of $f(1)$ and $f(5)$.
- (b) What are the domain and range of f ?

Figure 6



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Solution

- (a) We see from [Figure 6](#) that the point $(1, 3)$ lies on the graph of f , so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x -axis.)

When $x = 5$, the graph lies about 0.7 units below the x -axis, so we estimate that $f(5) \approx -0.7$.

- (b) We see that $f(x)$ is defined when $0 \leq x \leq 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4 , so the range of f is

$$\{y \mid -2 \leq y \leq 4\} = [-2, 4]$$

The notation for intervals is given in Appendix A.

In calculus, the most common method of defining a function is by an algebraic

equation. For example, the equation $y = 2x - 1$ defines y as a function of x . We can express this in function notation as $f(x) = 2x - 1$.

Example 2

Sketch the graph and find the domain and range of each function.

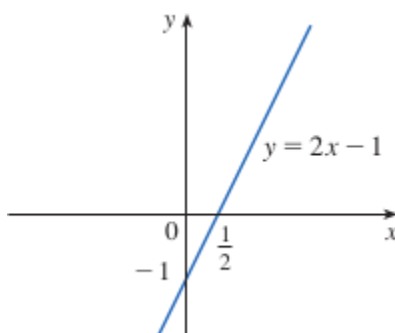
(a) $f(x) = 2x - 1$

(b) $g(x) = x^2$

Solution

(a) The equation of the graph is $y = 2x - 1$, and we recognize this as being the equation of a line with slope 2 and y -intercept -1 . (Recall the slope-intercept form of the equation of a line: $y = mx + b$. See Appendix B.) This enables us to sketch a portion of the graph of f in [Figure 7](#). The expression $2x - 1$ is defined for all real numbers, so the domain of f is the set of all real numbers, which we denote by \mathbb{R} . The graph shows that the range is also \mathbb{R} .

Figure 7

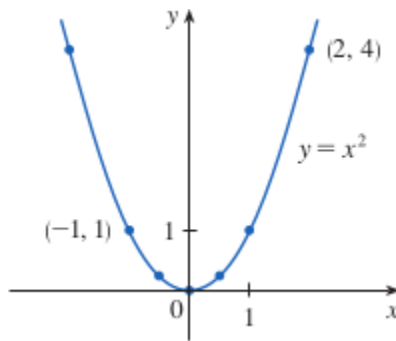


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(b) Since $g(2) = 2^2 = 4$ and $g(-1) = (-1)^2 = 1$, we could plot the points $(2, 4)$ and $(-1, 1)$, together with a few other points on the graph, and join them to produce the graph ([Figure 8](#)). The equation of the graph is $y = x^2$, which represents a parabola (see

Appendix C). The domain of g is \mathbb{R} . The range of g consists of all values of $g(x)$, that is, all numbers of the form x^2 . But $x^2 \geq 0$ for all numbers x and any positive number y is a square. So the range of g is $\{y \mid y \geq 0\} = [0, \infty)$. This can also be seen from [Figure 8](#).

Figure 8



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Example 3

If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\frac{f(a+h) - f(a)}{h}$.

Solution We first evaluate $f(a+h)$ by replacing x by $a+h$ in the expression for $f(x)$:

$$\begin{aligned} f(a+h) &= 2(a+h)^2 - 5(a+h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a+h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\begin{aligned}
 \frac{f(a+h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\
 &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\
 &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5
 \end{aligned}$$

The expression

$$\frac{f(a+h) - f(a)}{h}$$

in [Example 3](#) is called a **difference quotient** and occurs frequently in calculus. As we will see in [Chapter 2](#), it represents the average rate of change of $f(x)$ between $x = a$ and $x = a + h$.