16 Piecewise Defined Functions

The functions in the following four examples are defined by different formulas in different parts of their domains. Such functions are called **piecewise defined functions**.

Example 7

A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

Solution Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x . If it happens that $x \leqslant -1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > -1$, then the value of $f(x)$ is x^2 . Note that even though two different formulas are used, f is *one* function, not two.

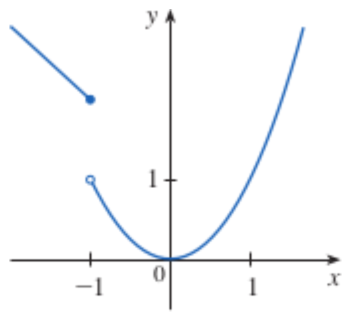
$$\text{Since } -2 \leqslant -1, \text{ we have } f(-2) = 1 - (-2) = 3.$$

$$\text{Since } -1 \leqslant -1, \text{ we have } f(-1) = 1 - (-1) = 2.$$

$$\text{Since } 0 > -1, \text{ we have } f(0) = 0^2 = 0.$$

How do we draw the graph of f ? We observe that if $x \leqslant -1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = -1$ must coincide with the line $y = 1 - x$, which has slope -1 and y-intercept 1 . If $x > -1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 15. The solid dot indicates that the point $(-1, 2)$ is included on the graph; the open dot indicates that the point $(-1, 1)$ is excluded from the graph.

Figure 15



Details

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a , denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0 , so we have

$$|a| \geqslant 0 \quad \text{for every number } a$$

For a more extensive review of absolute values, see Appendix A.

For example,

$$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1 \quad |3 - \pi| = \pi - 3$$

In general, we have

$$|a| = a \quad \text{if } a \geqslant 0$$

$$|a| = -a \quad \text{if } a < 0$$

(Remember that if a is negative, then $-a$ is positive.)

Example 8

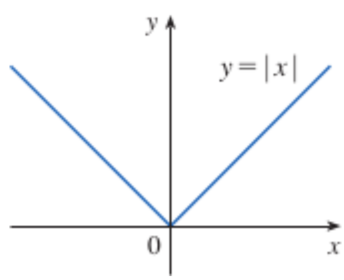
Sketch the graph of the absolute value function $f(x) = |x|$.

Solution From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 7, we see that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 16).

Figure 16

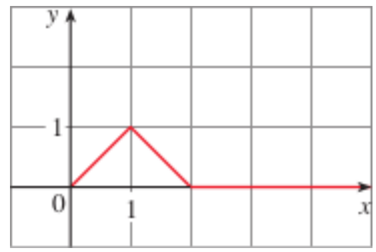


Details

Example 9

Find a formula for the function f graphed in Figure 17.

Figure 17



Details

Solution The line through $(0, 0)$ and $(1, 1)$ has slope $m = 1$ and y-intercept $b = 0$, so its equation is $y = x$. Thus, for the part of the graph of f that joins $(0, 0)$ to $(1, 1)$, we have

$$f(x) = x \qquad \text{if } 0 \leqslant x \leqslant 1$$

The line through $(1, 1)$ and $(2, 0)$ has slope $m = -1$, so its point-slope form is

$$y - 0 = (-1)(x - 2)$$

or

$$y = 2 - x$$

So we have

$$f(x) = 2 - x \qquad \text{if } 1 < x \leqslant 2$$

We also see that the graph of f coincides with the x-axis for $x > 2$. Putting this information together, we have the following three-piece formula for f:

$$f(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } 1 < x \leqslant 2 \\ 0 & \text{if } x > 2 \end{cases}$$

The point-slope form of the equation of a line is $y - y_1 = m(x - x_1)$.

See Appendix B.

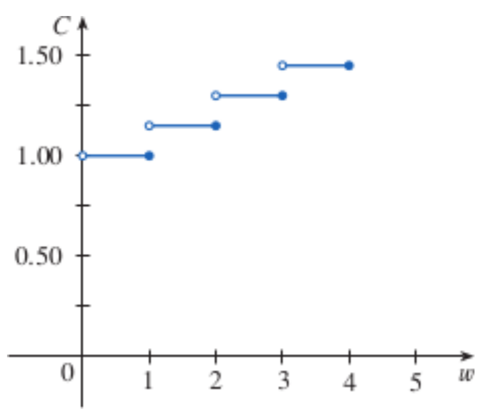
Example 10

In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a large envelope with weight w. In effect, this is a piecewise defined function because, from Table 3, we have

$$C(w) = \begin{cases} 1.00 & \text{if } 0 < w \leqslant 1 \\ 1.15 & \text{if } 1 < w \leqslant 2 \\ 1.30 & \text{if } 2 < w \leqslant 3 \\ 1.45 & \text{if } 3 < w \leqslant 4 \\ \vdots & \end{cases}$$

The graph is shown in Figure 18.

Figure 18



▸ Details

Looking at Figure 18, you can see why a function like the one in Example 10 is called a **step function**.