13

Representations of Functions

We consider four different ways to represent a function:

• verbally		(by a description in words)
• numerically		(by a table of values)
• visually		(by a graph)
• algebraically		(by an explicit formula)

If a single function can be represented in all four ways, it's often useful to go from one representation to another to gain additional insight into the function. (In Example 2, for instance, we started with algebraic formulas and then obtained graphs.) But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the above.

12   **A.** The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A = \pi r^2$ or, in function notation, $A(r) = \pi r^2$. It is also possible to compile a table of values or sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$ and the range is also $(0, \infty)$.

B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t. Let's measure t so that $t = 0$ corresponds to the year 1900. Table 2 provides a convenient representation of this function. If we plot the ordered pairs in the table, we get the graph (called a *scatter plot*) in Figure 9. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t. But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, using methods explained in Section 1.2, we obtain an approximation for the population P:

$$P(t) \approx f(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

Figure 10 shows that it is a reasonably good "fit." The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

Table 2 World Population

t (years since 1900)	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300

t (years since 1900)	Population (millions)
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

Figure 9

▶ Details

Figure 10

5 × 10⁹

0   20   40   60   80   100   120 t
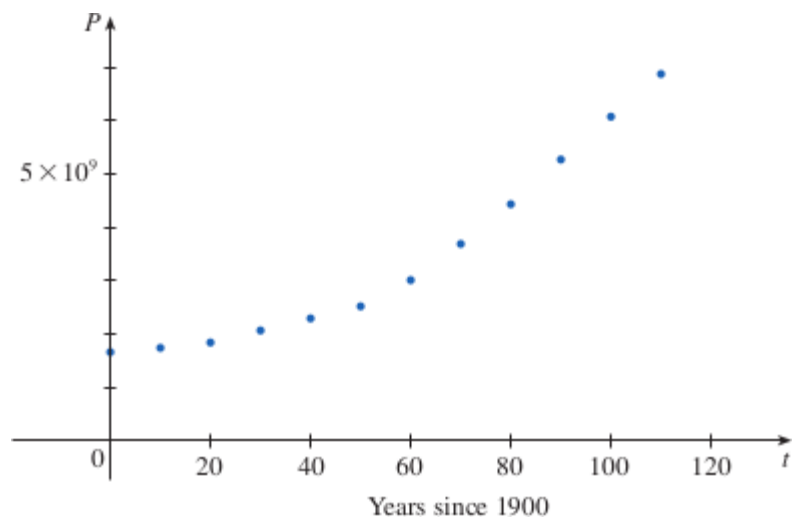
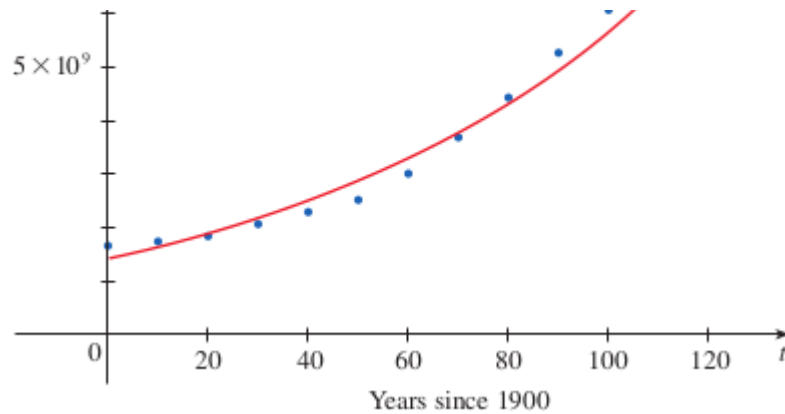Years since 1900

▸ Details

The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.

C. Again, the function is described in words: Let $C(w)$ be the cost of mailing a large envelope with weight w. The rule that the US Postal Service used as of 2019 is as follows: The cost is 1 dollar for up to 1 oz, plus 15 cents for each additional ounce (or less) up to 13 oz. A table of values is the most convenient representation for this function (see Table 3), though it is possible to sketch a graph (see Example 10).

Table 3

w (ounces)	$C(w)$ (dollars)
$0 < w \leqslant 1$	1.00
$1 < w \leqslant 2$	1.15

w (ounces)	$C(w)$ (dollars)
$2 < w \leqslant 3$	1.30
$3 < w \leqslant 4$	1.45
$4 < w \leqslant 5$	1.60
$\vdots$	$\vdots$

A function defined by a table of values is called a *tabular* function.

D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

In the next example we sketch the graph of a function that is defined verbally.

Example 4

When you turn on a hot-water faucet that is connected to a hot-water tank, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

Solution The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of

the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t shown in [Figure 11](#).

Figure 11
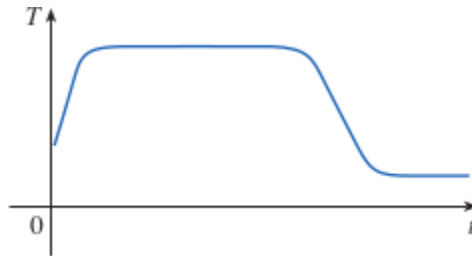
▶ Details

In the following example we start with a verbal description of a function in a physical situation and obtain an explicit algebraic formula. The ability to do this is a useful skill in solving calculus problems that ask for the maximum or minimum values of quantities.

Example 5

A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs $\$10$ per square meter; material for the sides costs $\$6$ per square meter. Express the cost of materials as a function of the width of the base.

Solution We draw a diagram as in [Figure 12](#) and introduce notation by letting w and $2w$ be the width and length of the base, respectively, and h be the height.

Figure 12
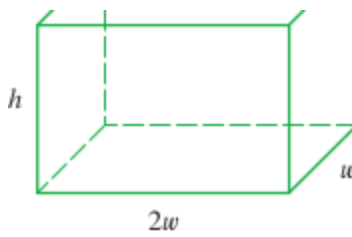
The area of the base is $(2w)w = 2w^2$, so the cost, in dollars, of the material for the base is $10(2w^2)$. Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$. The total cost is therefore

$$C = 10(2w^2) + 6\left[2\left(wh\right) + 2\left(2wh\right)\right] = 20w^2 + 36wh$$

To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is 10 m^3 . Thus

$$w\left(2w\right)h = 10$$

which gives

$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for C , we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

Therefore the equation

$$C\left(w\right) = 20w^2 + \frac{180}{w} \qquad w > 0$$

expresses C as a function of w .

PS In setting up applied functions as in Example 5, it may be useful to review the principles of problem solving at the end of this chapter, particularly Step 1: Understand the Problem.

In the next example we find the domain of a function that is defined algebraically. If a function is given by a formula and the domain is not stated explicitly, we use the following **domain convention**: the domain of the function is the set of all inputs for which the formula makes sense and gives a real-number output.

Example 6

Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$

(b) $g(x) = \dfrac{1}{x^2 - x}$

Solution

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geqslant 0$. This is equivalent to $x \geqslant -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x-1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. So the domain of g is

$$\{x \mid x \neq 0, x \neq 1\}$$

which could also be written in interval notation as

$$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$$