Review

$$\boxed{\text{Show } \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1}$$

Let $Q = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$ Let $u = \left(\frac{x-\mu}{\sigma}\right)$ and $du = \frac{dx}{\sigma} \rightarrow dx = \sigma du$

$x \rightarrow -\infty \quad u \rightarrow -\infty$
$x \rightarrow \infty \quad u \rightarrow \infty$

$$Q = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-\frac{1}{2}u^2} \cancel{\sigma} du$$

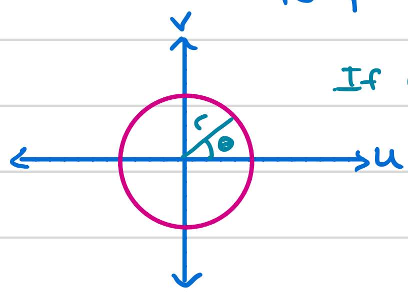
u,v place holders

Easier calculate $Q^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} dv\right)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(u^2+v^2)} du\,dv$$

Change of variables to polar coordinates

let $u = r\cos\theta$
$v = r\sin\theta$



If $0 \le \theta \le 2\pi$ & $0 < r < \infty$ cover entire cartesian plane ($-\infty < u < \infty$ and $-\infty < v < \infty$).

$$Q^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}(r^2\cos^2\theta + r^2\sin^2\theta)} |J|\, dr\, d\theta$$

where $|J| = \begin{vmatrix} \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial r} \end{vmatrix}$

$$|J| = \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} = \left|-r\sin^2\theta - r\cos^2\theta\right| = \left|-r(\sin^2\theta+\cos^2\theta)\right| = r$$

$$Q^2 = \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(r^2(\cos^2\theta+\sin^2\theta))} r\, dr\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}r^2} r\,dr\,d\theta$$

$$Q^2 = \int_0^{2\pi}\left(\int_0^{\infty} \frac{1}{2\pi} e^{-w/2} \frac{dw}{2}\right) d\theta$$

Let $w = r^2 \quad dw = 2r\,dr$
$r\,dr = \frac{dw}{2}$

$r=0 \rightarrow w=0$
$r=\infty \rightarrow w=\infty$

$$Q^2 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\cancel{2}}\left(-\cancel{2}e^{-\frac{1}{2}w}\right)\Big|_0^{\infty} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}[-0+1]\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} 1\,d\theta = \frac{1}{2\pi}\theta\Big|_0^{2\pi} = \frac{1}{2\pi}[2\pi - 0] = 1$$

$Q = +\sqrt{Q^2} = 1 \quad \left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} > 0 \;\forall x\right)$

↑ must be positive root.

// Ta Da! (QED)

Review

§3.4 Exponential Families (valuable tool!)

A pdf or pmf is exponential family if can be expressed in form:
(pdf = cont., pmf = discrete)

Sam's notation $\underline{\theta}$ (bold in C&B could be vector).

$$f(x|\underline{\theta}) = h(x)c(\underline{\theta})\exp\left(\sum_{i=1}^{k} w_i(\underline{\theta})t_i(x)\right)$$

- $\underline{\theta}$ represents parameter
- $h(x) \geq 0$ ← can include indicator of support/sample space
- $t_1(x), t_2(x) \dots t_k(x)$ ft'ns of x - don't depend on θ.
- $c(\underline{\theta}) \geq 0$
- $w_1(\theta), \dots, w_k(\theta)$ ft'ns of θ - don't depend on x

Take $f(x|\underline{\theta})$ & identify $h(x), c(\underline{\theta}), w_i(\underline{\theta})$ & $t_i(x)$
- ideally k as small as possible!

→ Note: If sample space is ft'n of θ, then it won't fit w/ $h(x)$ or $c(\theta)$.

→ We can interchange $\int_{-\infty}^{\infty}$ & $\frac{d}{d\theta}$ for exponential families ☺
- Needed for a few important Theorem proofs!

Normal dist'n $\sim N(\mu, \sigma^2)$

$$f(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) * I_{(-\infty,\infty)}(x)$$

$-\infty < \mu < \infty$, $0 < \sigma < \infty$, $\underline{\theta} = (\mu, \sigma)$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2}\frac{(x^2 - 2\mu x + \mu^2)}{\sigma^2}\right] * I_{(-\infty,\infty)}(x)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left[\frac{-\mu^2}{2\sigma^2}\right]}_{c(\mu,\sigma^2)}\exp\left[\frac{-x^2}{2\sigma^2} + \frac{\cancel{2}\mu x}{\cancel{2}\sigma^2}\right] * \underbrace{I_{(-\infty,\infty)}(x)}_{h(x)}$$

← sample space not ft'n of $\underline{\theta}$.

$w_1(\theta) = -1/2\sigma^2$ $\quad$ $w_2(\theta) = \frac{\mu}{\sigma^2}$

$t_1(x) = x^2$ $\quad$ $t_2(x) = x$

Note: constants $\frac{1}{\sqrt{2\pi}}$ could go w/ $h(x)$ instead of $c(\theta)$
$-1/2$ could go w/ $t_1(x)$ instead of $w_1(\theta)$

exponential family

Theorem 3.4.2 *If X is a random variable with pdf or pmf of the form (3.4.1), then*

$$\text{(3.4.4)} \qquad \mathrm{E}\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\frac{\partial}{\partial \theta_j} \log c(\boldsymbol{\theta});$$

$$\text{(3.4.5)} \quad \mathrm{Var}\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\frac{\partial^2}{\partial \theta_j^2} \log c(\boldsymbol{\theta}) - \mathrm{E}\left(\sum_{i=1}^{k} \frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(X)\right).$$

Find Expectations by taking derivatives! ☺ & w/out mgfs!

Try Expected values for a normal (μ, σ^2) (Notes Page 2)

<u>Calculate derivatives</u>

	$w_1(\theta) = \frac{-1}{2\sigma^2}$	$w_2(\theta) = \frac{\mu}{(\sigma^2)}$	$\log c(\mu,\sigma^2) = \cdots = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{\mu^2}{2\sigma^2}$
$\mu = \theta_1$	$\frac{\partial}{\partial\mu} w_1(\theta) = 0$	$\frac{\partial}{\partial\mu} w_2(\theta) = 1/\sigma^2$	$-\left(\frac{\partial}{\partial\mu}\log c(\mu,\sigma^2)\right) = \cancel{-}\left(\cancel{-}\frac{\cancel{2}\mu}{\cancel{2}\sigma^2}\right) = \frac{\mu}{\sigma^2}$
$\sigma^2 = \theta_2$	$\frac{\partial}{\partial(\sigma^2)} w_1(\theta) = \frac{1}{2(\sigma^2)^2}$	$\frac{\partial}{\partial(\sigma^2)} w_2(\theta) = \frac{-\mu}{(\sigma^2)^2}$	$-\left(\frac{\partial}{\partial\sigma^2}\log c(\mu,\sigma^2)\right) = \cancel{-}\left(-\frac{1}{2}\frac{1}{(2\pi\sigma^2)}(2\pi) \cancel{-} \frac{\mu^2(-1)}{2(\sigma^2)^2}\right) = \frac{1}{2(\sigma^2)^2}\left[\sigma^2 - \mu^2\right]$

$$t_1(x) = x^2 \text{ and } t_2(x) = x$$

For $\theta_1 = \mu$

$$E\left[0\cdot x^2 + \frac{\overset{1}{\text{[redacted]}}}{\sigma^2}x\right] = \frac{\mu}{\sigma^2} \longrightarrow \boxed{E[X] = \mu}$$

For $\theta_2 = (\sigma^2)$

$$\frac{1}{\cancel{2(\sigma^2)^2}}\left[E\left[x^2 - 2\mu x\right]\right] = \frac{1}{\cancel{2(\sigma^2)^2}}\left[\sigma^2 - \mu^2\right]$$

$$E[X^2] - 2\mu^2 = \sigma^2 - \mu^2$$

$$\boxed{E[X^2] = \sigma^2 + \mu^2}$$

Definition 3.4.5 The *indicator function* of a set A, most often denoted by $I_A(x)$, is the function

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

deja vu!

- deja moo!

An alternative notation is $I(x \in A)$.

Sample space (support) depends on θ.

example: $f(x|\theta) = \frac{1}{\theta} e^{(1-x/\theta)} I_{(\theta,\infty)}(x)$ $\quad \theta > 0$

Recall exponential family ($k=1$)

$$f(x|\theta) = h(x)\, c(\theta) \exp(t(x) w(\theta))$$

$$f(x|\theta) = \frac{1}{\theta} e^{1} e^{-x/\theta} I_{(\theta,\infty)}(x)$$

→ if we ignore indicator (sample space)
- it looks like exponential family....

$c(\theta) = 1/\theta \quad h(x) = e^{1} \quad w(\theta) = -1/\theta \quad t(x) = x$

→ Writing with indicator makes it clear
→ Not exponential family!

→ since $I_{(\theta,\infty)}(x)$ doesn't fit w/ $h(x)$ or $c(\theta)$.

→ exponential families, we can interchange $\int$ + $\frac{d}{d\theta}$.

→ if sample space (support) is ft'n of θ, it won't be exponential family!

parameter

C+B tend to use indicator functions only when sample space is function of parameter, but they always state possible values of x.

For example Poisson

support or sample space

$$f_X(x) = \begin{cases} \frac{e^{-\lambda} \lambda^{x}}{x!} & x = 0,1,2,\dots \\ 0 & \text{else} \end{cases}$$

Definition 3.4.7 A *curved exponential family* is a family of densities of the form (3.4.1) for which the dimension of the vector $\boldsymbol{\theta}$ is equal to $d < k$. If $d = k$, the family is a *full exponential family*. (See also Miscellanea 3.8.3.)

Curved exponential

- $\dim(\underline{\theta}) < k$ $\qquad f(x|\underline{\theta}) = h(x)\, c(\underline{\theta}) \exp\left(\sum_{i=1}^{k} w_i(\underline{\theta})\, t_i(x)\right)$

Full exponential ← statistical nirvana → A state of perfect happiness; an ideal or idyllic place

- $\dim(\underline{\theta}) = k$

Example of curved exponential:

$$f(x|\mu) \sim N(\mu, \mu^2)$$

$$f(x|\mu) = \frac{1}{\sqrt{2\pi\mu^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\mu}\right)^2\right) I_{(-\infty,\infty)}(x)$$

$$= \frac{1}{\sqrt{2\pi\mu^2}} \exp\left[-\frac{1}{2}\left(\frac{x^2 - 2\mu x + \mu^2}{\mu^2}\right)\right] I_{(-\infty,\infty)}(x)$$

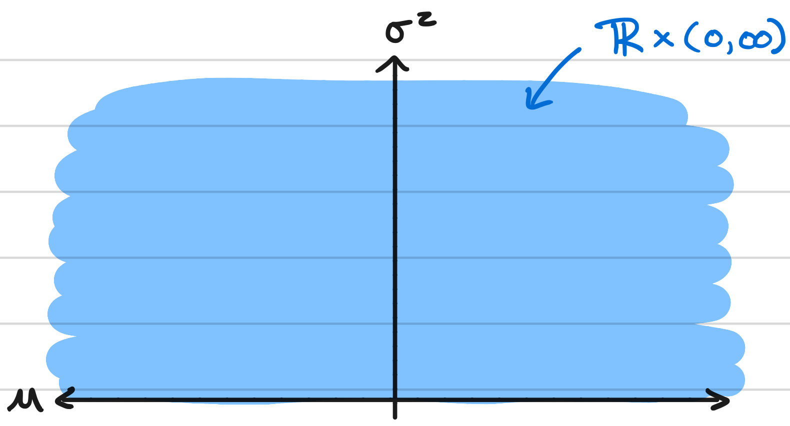
$$= \frac{1}{\sqrt{2\pi\mu^2}}\, e^{-1/2}\, e^{-x^2/2\mu^2 + \frac{x}{\mu}} * I_{(-\infty,\infty)}(x)$$

$$c(\mu) = \frac{1}{\sqrt{2\pi\mu^2}} \qquad h(x) = e^{-1/2} I_{(-\infty,\infty)}(x)$$

$$w_1(\mu) = -1/2\mu^2 \qquad t_1(x) = x^2$$

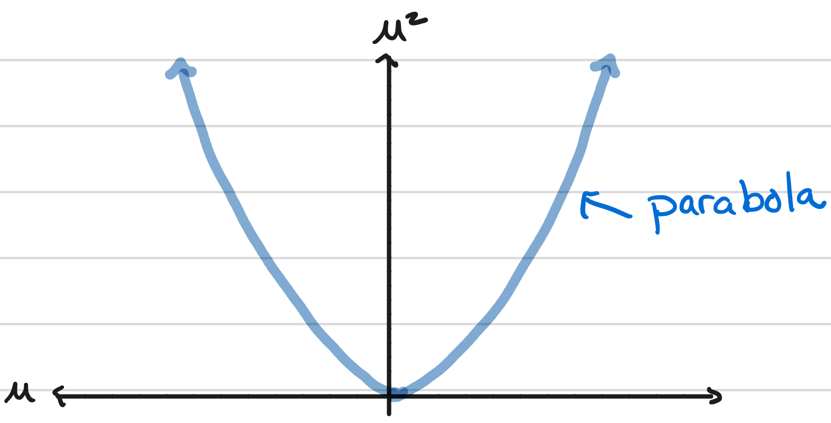
$$w_2(\mu) = 1/\mu \qquad t_2(x) = x$$

Parameter Space for $N(\mu, \sigma^2)$



Full exponential

Parameter Space for $N(\mu, \mu^2)$



Curved Exponential

Curved Exponentials & Preview to Central Limit Theorem (CLT)

Theorem 5.5.14 (Central Limit Theorem) *Let $X_1, X_2, \ldots$ be a sequence of iid random variables whose mgfs exist in a neighborhood of 0 (that is, $M_{X_i}(t)$ exists for $|t| < h$, for some positive h). Let $\mathrm{E}X_i = \mu$ and $\mathrm{Var}\, X_i = \sigma^2 > 0$. (Both μ and σ^2 are finite since the mgf exists.) Define $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$. Let $G_n(x)$ denote the cdf of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$. Then, for any x, $-\infty < x < \infty$,*

$$\lim_{n\to\infty} G_n(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy;$$

that is, $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a limiting standard normal distribution.

Summary: $\left.\begin{array}{l} E[X] = \mu < \infty \\ Var[X] = \sigma^2 < \infty \end{array}\right\}$ mgf exists

$$\bar{X} \xrightarrow{\text{dist'n}} N(\mu, \sigma^2/n)$$

→ If $X_1, X_2, \ldots X_n$ is a sample from the Poisson (λ) pop'n.

For Poisson: $E[X] = \lambda$ $\quad E[\bar{X}] = \lambda$

$Var[X] = \lambda$ $\quad Var[\bar{x}] = \lambda/n$

by CLT as n 'big' $\bar{x} \sim N(\lambda, \lambda/n)$ ← curved exponential

$k=2$, $\dim(\theta) = 1$

→ Similarly, if $X_1, X_2 \ldots X_n$ is a sample from a Bernoulli dist'n. In the normal binomial approximation, then

$\bar{X} \sim N\left(p, \frac{p(1-p)}{n}\right)$ ← curved exponential

$k=2$, $\dim(\theta) = 1$

Curved exponential

- enjoy 'many' of properties of <u>full families</u> ... ← statistical nirvana

such as

$$E\left[\sum_{i=1}^{k} \frac{\partial}{\partial \theta_j} w_i(\theta)\, t_i(x)\right] = -\frac{\partial}{\partial \theta_j} \log c(\theta) \qquad \text{(Thm 3.4.2)}$$

more later...

Natural Parameter Exponential

An exponential family is sometimes reparameterized as

$$f(x|\eta) = h(x)c^*(\eta)\exp\left(\sum_{i=1}^{k} \eta_i t_i(x)\right). \tag{3.4.7}$$

since pdf
$c^*(\eta) = \left[\int_{-\infty}^{\infty} h(x)\exp\left(\sum_{i=1}^{k}\eta_i t_i(x)\right)\right]^{-1}$

$\eta_i = w_i(\theta)$
$i=1,\dots,k$

$$f(x|\eta) = h(x)c^*(\eta)\exp\left(\sum_{i=1}^{k} \eta_i t_i(x)\right).$$

$h(x)$, $t(x)$ same as original parameterization

$$\mathcal{H} = \left\{\eta = (\eta_1,\dots,\eta_k) : \underbrace{\left(\int_{-\infty}^{\infty} h(x)\exp \sum_{i=1}^{k}\eta_i t_i(x)\,dx\right)}_{\frac{1}{c^*(\eta)}} < \infty \right\}$$

integral replaced by sum for discrete

$\mathcal{H}$ = Natural Parameter Space

$\eta = (w_1(\theta),\dots,w_k(\theta)) : \underset{\sim}{\theta} \in \Theta \}$ parameter space for original parameterization

- must be subset of natural param space.
- maybe other values of $\eta \in \mathcal{H}$ also....

→ "many useful properties" (homework & later...)

→ $\mathcal{H}$ is convex.

$$f(x|\eta) = h(x)c^*(\eta)\exp(\eta_1 t_1(x))$$

one parameter natural parameterization

Natural Parameterization

$$f(x|\eta) = h(x)c^*(\eta)\exp\left(\sum_{i=1}^{k}\eta_i t_i(x)\right).$$

Example $N(\mu,\sigma^2)$ assuming: $\sigma > 0$; $-\infty < \mu < \infty$

$h(x) = I^{(x)}_{(-\infty,\infty)}$

$c(\theta) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-\mu^2}{2\sigma^2}\right)$ $\begin{array}{l} -\infty < \mu < \infty \\ \sigma > 0 \end{array}$

$t_1(x) = -x^2/2$ $\qquad$ $t_2(x) = x$

$w_1(\mu,\sigma) = \frac{1}{\sigma^2}\{\sigma > 0$ $\qquad$ $w_2(\mu,\sigma) = \frac{\mu}{\sigma^2}\begin{cases} -\infty < \mu < \infty \\ \sigma > 0 \end{cases}$

$\eta_1 = \frac{1}{\sigma^2}\{\eta_1 > 0$ $\qquad$ $\eta_2 = \frac{\mu}{\sigma^2}\{-\infty < \eta_2 < \infty$

- solve for old in terms of new $\qquad$ plug into $c(\theta) = c(\mu,\sigma^2) \Rightarrow c^*(\eta)$

$\sigma = \frac{1}{\sqrt{\eta_1}}$; $\eta_2 = \mu\cdot\eta_1$ $\quad \mu = \frac{\eta_2}{\eta_1}$

$c^*(\eta) = \frac{\sqrt{\eta_1}}{\sqrt{2\pi}}\exp\left(\frac{-1}{2}\left(\frac{\eta_2}{\eta_1}\right)^2\eta_1\right) = \frac{\sqrt{\eta_1}}{\sqrt{2\pi}}\exp\left(\frac{-\eta_2^2}{2\eta_1}\right)$

$$f(x|\eta_1,\eta_2) = \frac{\sqrt{\eta_1}}{\sqrt{2\pi}}\exp\left(\frac{-\eta_2^2}{2\eta_1}\right)\exp\left(\eta_1\left(\frac{-x^2}{2}\right) + \eta_2 x\right) * I^{(x)}_{(-\infty,\infty)}$$

- $\eta_1 = \frac{1}{\sigma^2} \Rightarrow \eta_1 > 0$

- also η_1 must be > 0 for $c^*(\eta)$ to be finite $\left[\exp\left(\frac{\eta_1}{2}(-x^2)\right)\right]$

  $\lim_{x\to\pm\infty}\exp\left(\frac{\eta_1}{2}(-x^2)\right) = 0$ if $\eta_1 > 0$

- if $\eta_1 > 0$ can have $-\infty < \eta_2 = \frac{\mu}{\sigma^2} < \infty$

- $\exp(\underbrace{-\eta_1 x^2/2 + \eta_2 x}_{\text{order }(x^2)})$ finite $\begin{array}{l}\eta_1 > 0 \\ -\infty < \eta_2 < \infty\end{array}$

- "Although natural parameterizations provide a convenient mathematical formulation, they sometimes lack simple interpretations like mean and variance."

§ 3.5 Location and Scale Families

- Location families: $f(x-\mu) \leftarrow$ where ever x in pdf subtract μ.

example: $N(\mu,1)$ $\quad f(x|\mu,\sigma=1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-\mu)^2\right) * I_{(-\infty,\infty)}(x)$

- Scale families: $\frac{1}{\sigma} f(x/\sigma) \leftarrow$ whereever x in pdf divided by σ.
+ extra $\left(\frac{1}{\sigma}\right)$ out front.

example: $N(0,\sigma^2)$ $\quad f(x|\mu=0,\sigma^2) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right) * I_{(-\infty,\infty)}(x)$

- Location - Scale families: $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) \leftarrow$ x in pdf, μ subtract σ divide
$\frac{1}{\sigma}$ out front.

example: $N(\mu,\sigma^2)$ $\quad f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) * I_{(-\infty,\infty)}(x)$

- Location, Scale, Location-Scale generated from a standard pdf

example: $f(x|\mu=0,\sigma^2=1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$\uparrow$
$\mu=0, \sigma=1$
location shift — scale shift

Theorem 3.5.1 *Let $f(x)$ be any pdf and let μ and $\sigma > 0$ be any given constants. Then the function*

$$g(x|\mu,\sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

is a pdf.

Proof C+B: (page 116). show $g(x|\mu,\sigma) \geq 0 \ \forall\ x,\mu,\sigma$
$\int_{-\infty}^{\infty} g(x|\mu,\sigma)\,dx = 1$
$\Big\}$ assuming $f_X(x)$ is pdf.

Location Family: $f(x-\mu)$

- start with any pdf $f(x)$ and generate a family of pdfs by introducing a location parameter.

Definition 3.5.2 Let $f(x)$ be any pdf. Then the family of pdfs $f(x-\mu)$, indexed by the parameter μ, $-\infty < \mu < \infty$, is called the *location family with standard pdf* $f(x)$ and μ is called the *location parameter* for the family.

$\underbrace{f_X(x)}_{\text{standard pdf}}$ & μ $\left\{ \begin{array}{l} \text{location parameter (constant)} \\ -\infty < \mu < \infty \end{array} \right.$ $\rightarrow$ $f(x-\mu)$ $\left\{ \text{location family} \right.$

$f(x|\mu=0) = f(x-\mu|\mu)$

standard pdf ($\mu=0$) →

← location family ($\mu=2$) (shaped unchanged).

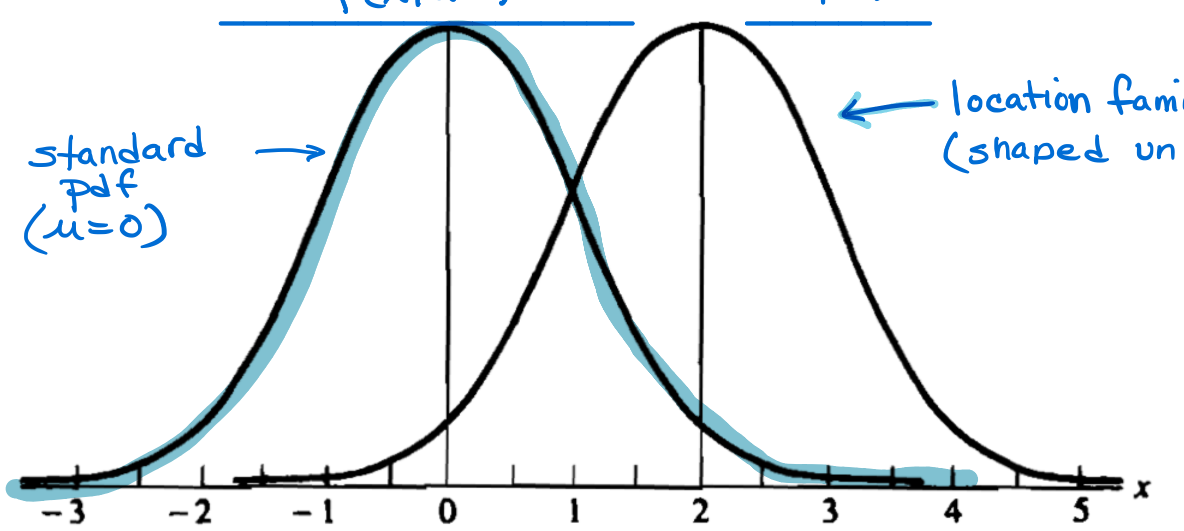


Figure 3.5.1. *Two members of the same location family: means at 0 and 2*
(same shape – shifted right by $\mu=2$)
$f(x-\mu) = f(x)$

Assume we know $f(x)$, want to get pdf of 'new' pdf with a location shift of μ.

new dist'n			standard
x	$f(x-\mu)$		$f(x)$
μ	$f(\mu-\mu)$	$=$	$f(0)$
$\mu+1$	$f((\mu+1)-\mu)$	$=$	$f(1)$
$\mu+2$	$f((\mu+2)-\mu)$	$=$	$f(2)$

<u>Similarly areas:</u> $P(\mu-1 \le X \le \mu+2 \,|\, \mu)$ ← location family (μ)
$= P(-1 \le X \le 2 \,|\, \mu=0)$ ← standard pdf

Location Family cont.

Cauchy dist'n :

$$f(x|\theta,\sigma) = \frac{1}{\pi\sigma}\frac{1}{1+\left(\frac{x-\theta}{\sigma}\right)^2} I_{(-\infty,\infty)}(x) \qquad \begin{matrix} -\infty<\theta<\infty \\ \sigma>0 \end{matrix}$$

θ is a location parameter

Double exponential :

μ is location parameter

$$f(x|\mu,\sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma} I_{(-\infty,\infty)}(x) \qquad \begin{matrix} -\infty<\mu<\infty \\ \sigma>0 \end{matrix}$$

standard pdf

If $X \sim f(x-\mu)$ we can represent $X = Z+\mu$, $Z \sim f(z)$

Example (Exponential location family):

standard pdf : $f(x) = e^{-x} I_{[0,\infty)}(x)$

location family: $f(x-\mu) = e^{-(x-\mu)} I_{[\mu,\infty)}(x)$

$$I_{[\mu,\infty)}(x) = \begin{cases} 1 & x-\mu \geq 0 \quad (x \geq \mu) \\ 0 & \text{else} \end{cases}$$

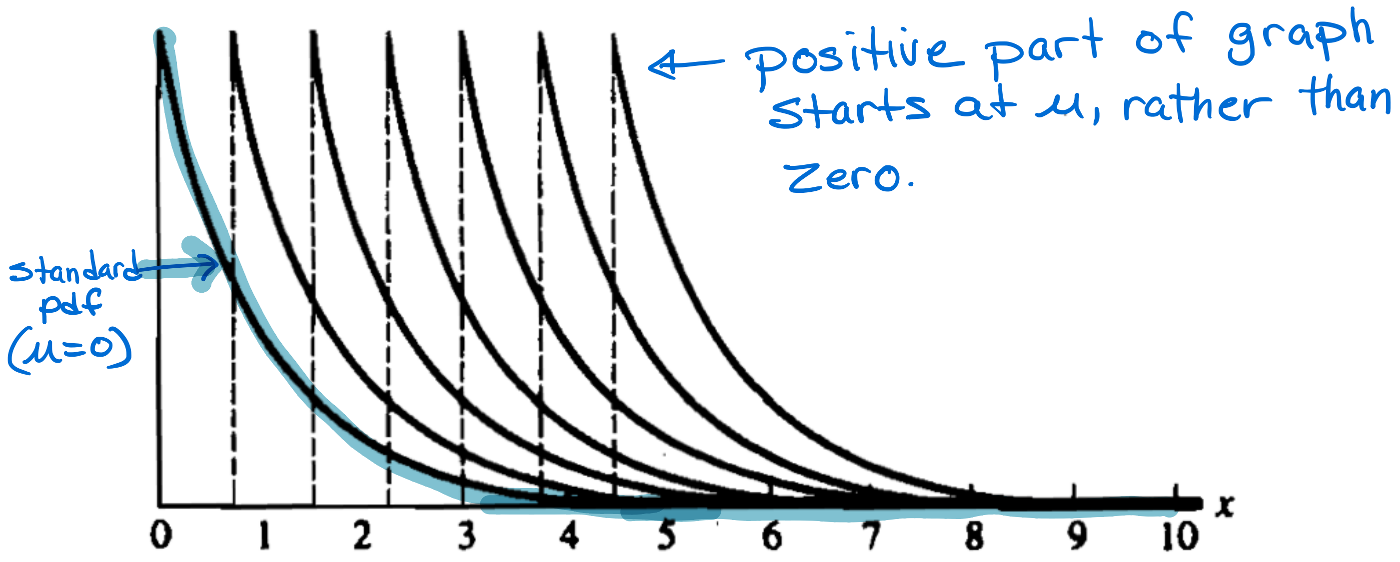


Figure 3.5.2. *Exponential location densities*

Next Lecture

Scale family $\frac{1}{\sigma}f(\frac{x}{\sigma})$

Definition 3.5.4 Let $f(x)$ be any pdf. Then for any $\sigma > 0$, the family of pdfs $(1/\sigma)f(x/\sigma)$, indexed by the parameter σ, is called the *scale family with standard pdf* $f(x)$ and σ is called the *scale parameter* of the family.

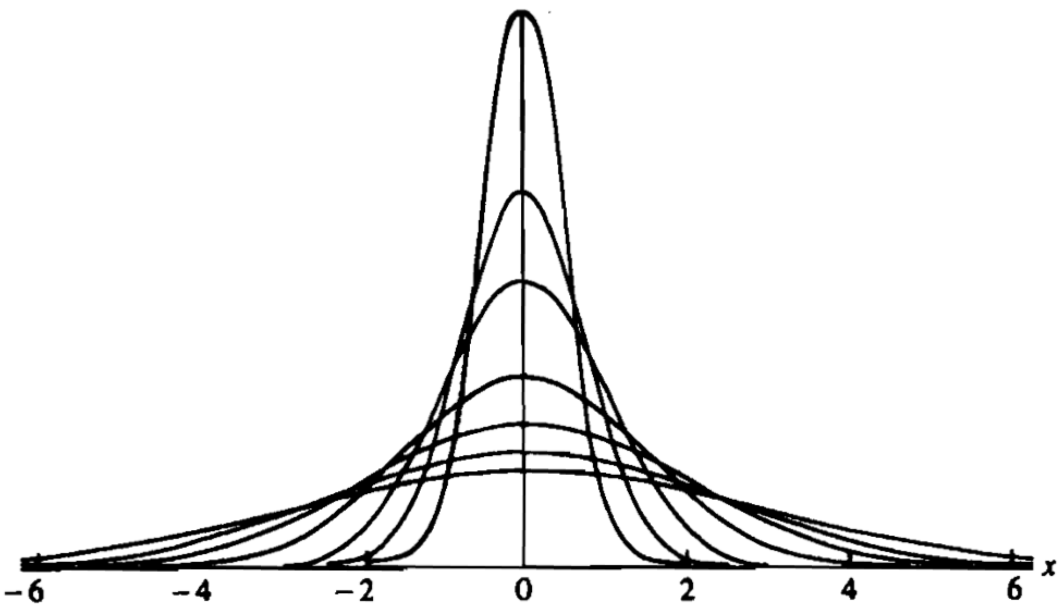


Figure 3.5.3. *Members of the same scale family*

Location-scale family $\frac{1}{\sigma}f(\frac{x-\mu}{\sigma})$

Definition 3.5.5 Let $f(x)$ be any pdf. Then for any μ, $-\infty < \mu < \infty$, and any $\sigma > 0$, the family of pdfs $(1/\sigma)f((x-\mu)/\sigma)$, indexed by the parameter (μ, σ), is called the *location–scale family with standard pdf* $f(x)$; μ is called the *location parameter* and σ is called the *scale parameter.*

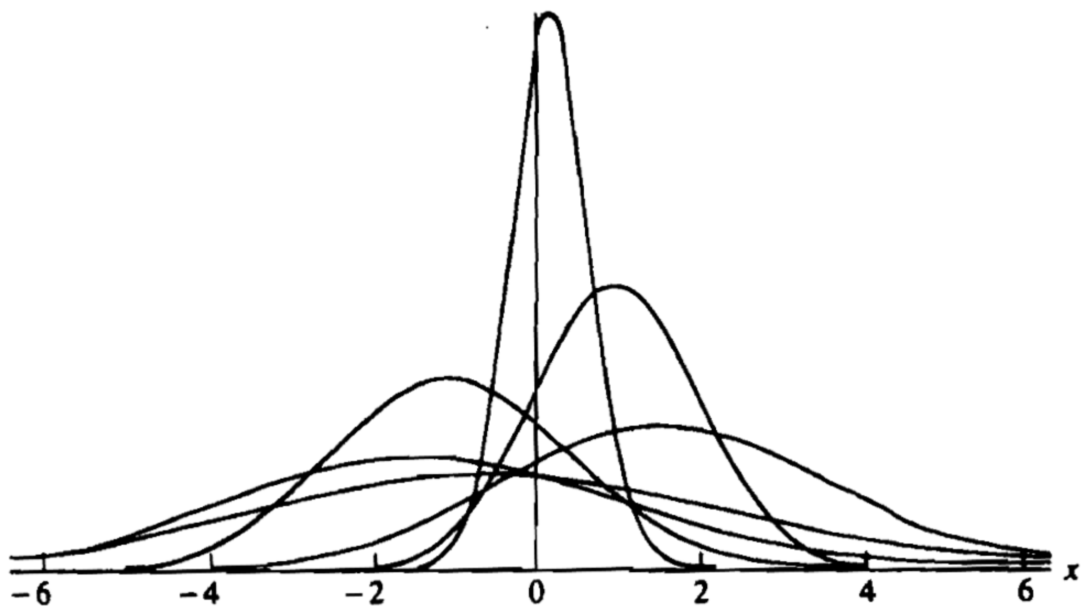


Figure 3.5.4. *Members of the same location–scale family*