Lecture - 11

Review

Normal Dist'n (Gaussian Dist'n)

Normal(μ, σ^2)

pdf	$f(x\|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$
mean and variance	$\mathrm{E}X = \mu, \quad \mathrm{Var}\, X = \sigma^2$
mgf	$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$
notes	Sometimes called the *Gaussian* distribution.

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} I_{(-\infty,\infty)}(x)$$

Standard normal:

$$f(x|\mu=0,\sigma^2=1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} I_{(-\infty,\infty)}(x)$$

If $Z \sim N(0,1)$ $\quad E[Z] = \int_{-\infty}^{\infty} z \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = \int \frac{e^{-u}}{\sqrt{2\pi}} du \qquad u = z^2/2, \; du = z$

$$= \frac{(-1)}{\sqrt{2\pi}} e^{-u}\Big| = \frac{(-1)}{\sqrt{2\pi}} e^{-z^2/2}\Big|_{-\infty}^{\infty} = \frac{-1}{\sqrt{2\pi}}(0-0) = 0$$

If $X = \mu + \sigma Z \quad E[X] = \mu + \sigma E[Z] = \mu$

$$\mathrm{Var}[X] = \mathrm{Var}(\mu + \sigma Z) = \sigma^2 \mathrm{Var}(Z) = \sigma^2$$

Next Pages...

Show $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1$

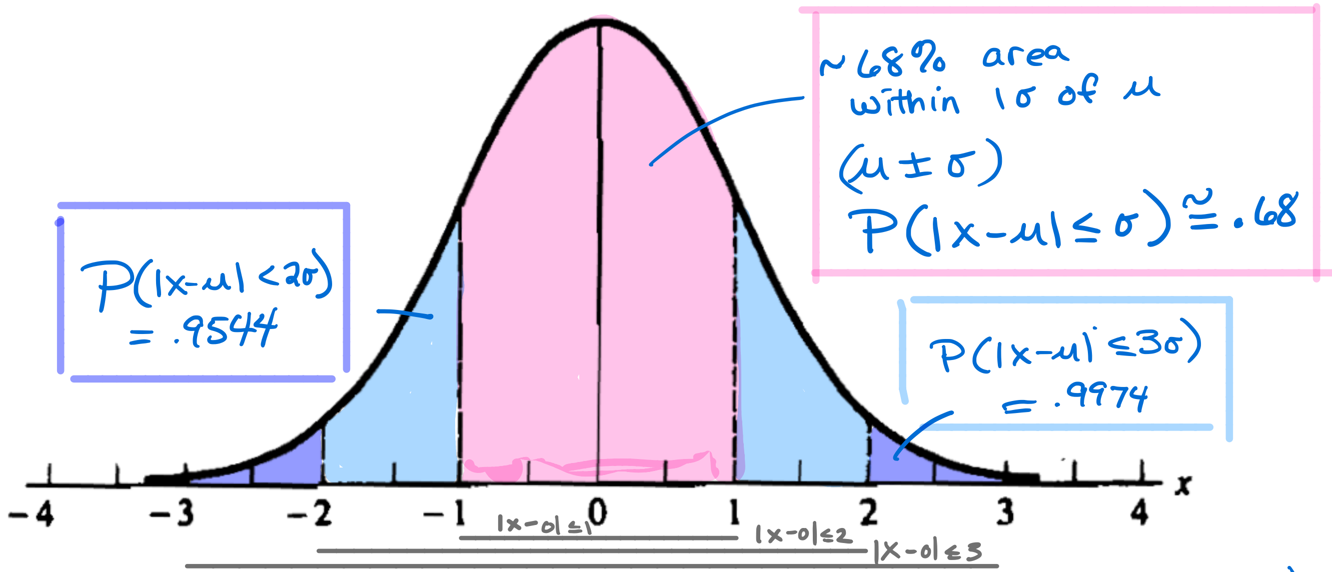


Figure 3.3.1. *Standard normal density* $(\mu = 0, \sigma = 1)$

Review

Use mgf to get $E[X]$, $E[X^2]$ and $Var[X]$

$$mgf = e^{\mu t + \sigma^2 t^2/2}$$

$$\frac{d}{dt} M_x(t) = (\mu + \sigma^2 t)\, e^{\mu t + \sigma^2 t^2/2} \Big|_{t=0} = \mu$$

$$\frac{d^2}{dt^2} M_x(t) = \sigma^2 e^{\mu t + \sigma^2 t^2/2} + (\mu + \sigma^2 t)^2 e^{\mu t + \sigma^2 t^2/2} \Big|_{t=0} = \sigma^2 + \mu^2$$

$$Var(X) = E[X^2] - (E[X])^2 = \sigma^2 + \cancel{\mu^2} - \cancel{\mu^2} = \sigma^2$$

$$\boxed{\text{Show } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1}$$

← Adding few more details to C & B's version.

Calc Review

Change of variables in integrals

$$\int_0^2 x \cos(x^2)\, dx$$

let $u = x^2$
$du = 2x\,dx$
$\to x\,dx = \frac{du}{2}$

$x = 0 \to u = 0$
$x = 2 \to u = 4$

- change variable
- change "dx"
- change bounds

$$= \int_0^4 \cos(u) \frac{du}{2} = \frac{1}{2} (\sin(u)) \Big|_0^4 = \frac{1}{2}\sin(4) - \frac{1}{2}\sin(0) = \frac{1}{2}\sin(4)$$

Jacobians

Let $x = g(u,v)$ and $y = h(u,v)$ be a transformation of the plane. Then the Jacobian of this transform is defined by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \underbrace{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}_{\text{determinant}}$$

Change of variables in multiple integrals

Let $\begin{Bmatrix} x = g(u,v) \\ y = h(u,v) \end{Bmatrix}$ be a transformation of the plane that is one-to-one from a region S in the (u,v) plane to a region R in the (x,y) plane. If g and h have continuous partial derivatives such that the Jacobian is never zero, then

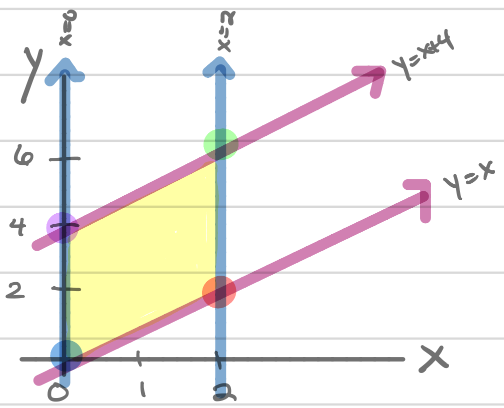
$$\iint_R f(x,y)\,dx\,dy = \iint_S f[g(u,v), h(u,v)] \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \partial u\, \partial v$$

absolute value

Change of variables in double integral example.

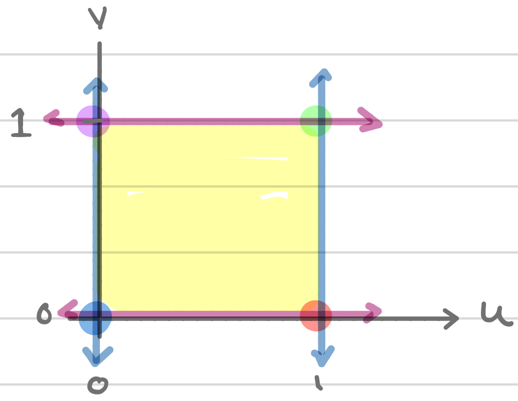
New coordinant system

$$\iint_R xy\, dA \quad \text{where } R: 0 \le x \le 2 \text{ and } x \le y \le x+4 \quad \text{let } \begin{array}{l} x = 2u \\ y = 4v + 2u \end{array} \qquad \begin{array}{l} u = \frac{x}{2} \\ v = \frac{y-x}{4} \end{array}$$



x,y coordinant system

vertices (x,y)		vertices (u,v)
(0,0)	→	(0,0)
(0,4)		(0,1)
(2,2)		(1,0)
(2,6)		(1,1)



u,v coordinate system

$$\text{Jacobian: } \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial(2u)}{\partial u} & \frac{\partial(2u)}{\partial v} \\ \frac{\partial(4v+u)}{\partial u} & \frac{\partial(4v+u)}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8 - 0$$

$$\iint_R xy\, dA = \int_0^1\int_0^1 (2u)(4v+2u)8\, dv\, du = \iint (8uv + 4u^2)8\, dv\, du$$

$$= 32 \int_0^1\int_0^1 (2uv + u^2)\, dv\, du = 32\int_0^1 \left(\frac{2uv^2}{2} + u^2 v \Big|_0^1\right) du$$

$$32\int_0^1 u + u^2\, du = 32\left(\frac{u^2}{2} + \frac{u^3}{3}\right) = 32\left(\frac{1}{2} + \frac{1}{3}\right) - 0 = 32\left(\frac{5}{6}\right) = \boxed{\frac{80}{3}}$$

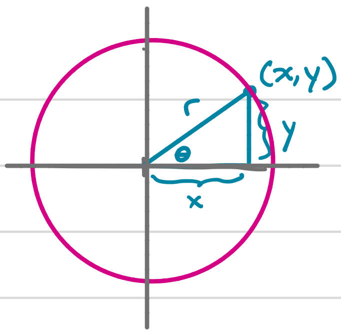
Check

$$\int_0^2\int_x^{x+4} xy\, dy\, dx = \int_0^2 \left[\frac{xy^2}{2}\Big|_x^{x+4}\right] dx = \int_0^2 \left[\frac{x(x+4)^2}{2} - \frac{x(x^2)}{2}\right] dx$$

$$= \int_0^2 \frac{x(\cancel{x^2} + 8x + 16)}{2} - \cancel{\frac{x^3}{2}}\, dx = \int_0^2 4x^2 + 8x\, dx$$

$$= \frac{4x^3}{3} + \frac{8x^2}{2}\Big|_0^2 = \frac{32}{3} + \frac{32}{2} = 32\left(\frac{1}{3} + \frac{1}{2}\right) = \boxed{\frac{80}{3}}$$

Geometry review



$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r} \qquad x = r\cos\theta$

Also remember: $\sin^2\theta + \cos^2\theta = 1$; $\underbrace{0 < \theta < 2\pi \ ; \ 0 < r < \infty}_{\text{covers entire } x,y \text{ plane}}$;

$$\boxed{\text{Show } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1} \quad \boxed{\text{or } \underline{\text{better}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1}$$

Show $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$

Let $Q = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$ Let $u = \left(\frac{x-\mu}{\sigma}\right)$ and $du = \frac{dx}{\sigma} \rightarrow dx = \sigma du$

$x \rightarrow -\infty \quad u \rightarrow -\infty$

$x \rightarrow \infty \quad u \rightarrow \infty$

$$Q = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-\frac{1}{2}u^2} \cancel{\sigma} du$$

u,v place holders

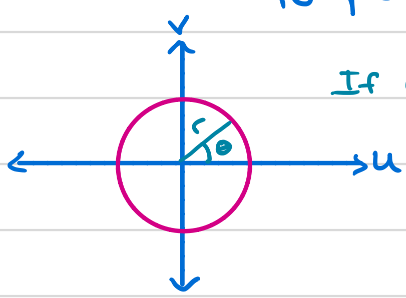
Easier calculate $Q^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} dv\right)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(u^2+v^2)} du\,dv$$

Change of variables to polar coordinates

let $u = r\cos\theta$

$v = r\sin\theta$



If $0 \le \theta \le 2\pi$ & $0 < r < \infty$ cover entire cartesian plane ($-\infty < u < \infty$ and $-\infty < v < \infty$).

$$Q^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}(r^2\cos^2\theta + r^2\sin^2\theta)} |J|\, dr\, d\theta$$

where $|J| = \begin{vmatrix} \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial r} \end{vmatrix}$

$$|J| = \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} = \left|-r\sin^2\theta - r\cos^2\theta\right| = \left|-r(\sin^2\theta+\cos^2\theta)\right| = r$$

$$Q^2 = \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(r^2(\cos^2\theta + \sin^2\theta))} r\,dr\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}r^2} r\,dr\,d\theta$$

Let $w = r^2$ $\quad dw = 2r\,dr$
$r\,dr = \frac{dw}{2}$

$r=0 \to w=0$
$r=\infty \to w=\infty$

$$Q^2 = \int_0^{2\pi}\left(\int_0^{\infty} \frac{1}{2\pi} e^{-w/2} \frac{dw}{2}\right) d\theta$$

$$Q^2 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\cancel{2}}\left(-\cancel{2}e^{-\frac{1}{2}w}\right)\Big|_0^{\infty} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}[-0+1]\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} 1\,d\theta = \frac{1}{2\pi}\,\theta\Big|_0^{2\pi} = \frac{1}{2\pi}[2\pi - 0] = 1$$

//
Ta Da!
(QED)

Beta Dist'n

Beta(α, β) — bounded support $\rangle$ used to model proportions

pdf	$f(x\mid\alpha,\beta) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \quad \underline{0 \le x \le 1}, \quad \alpha > 0, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}\,X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
mgf	$M_X(t) = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$
notes	The constant in the beta pdf can be defined in terms of gamma functions, $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. Equation (3.2.18) gives a general expression for the moments.

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

$$E[X^n] = \frac{\Gamma(\alpha+\beta)\,\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(\alpha+n+\beta)} \leftarrow \text{details lecture-10}$$

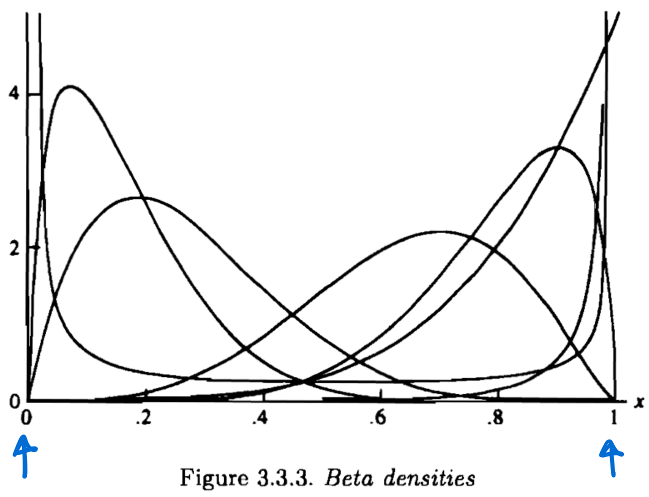


Figure 3.3.3. *Beta densities*

strictly increasing $\alpha>1, \beta=1$
Strictly decreasing $\alpha=1, \beta>1$
U shaped $(\alpha<1, \beta<1)$
Unimodal $(\alpha>1, \beta>1)$

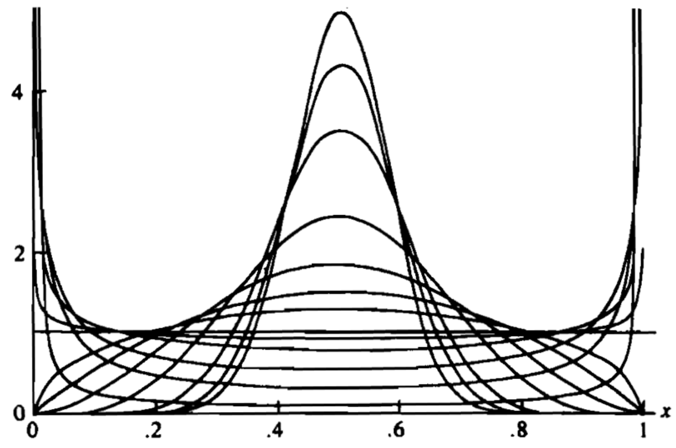


Figure 3.3.4. *Symmetric beta densities*

$\alpha=\beta$ symmetric around $1/2$.
mean $= \frac{\alpha}{\alpha+\alpha} = \frac{1}{2}$

Cauchy Dist'n

Cauchy(θ, σ)

pdf	$f(x\|\theta,\sigma) = \frac{1}{\pi\sigma}\frac{1}{1+\left(\frac{x-\theta}{\sigma}\right)^2}, \quad -\infty < x < \infty; \quad -\infty < \theta < \infty, \quad \sigma > 0$
mean and variance	do not exist
mgf	does not exist
notes	Special case of Student's t, when degrees of freedom $= 1$. Also, if X and Y are independent $\mathrm{n}(0,1)$, X/Y is Cauchy.

Caution:
Ratio of Normals is Cauchy.

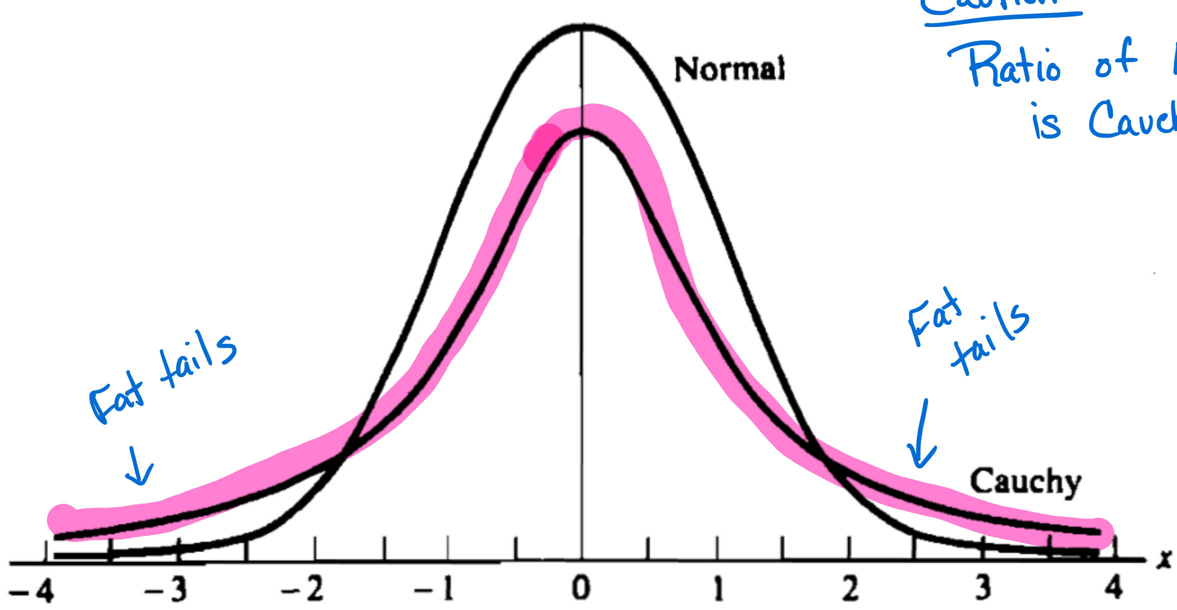


Figure 3.3.5. *Standard normal density and Cauchy density*

MS Theory I

Lecture-11

7

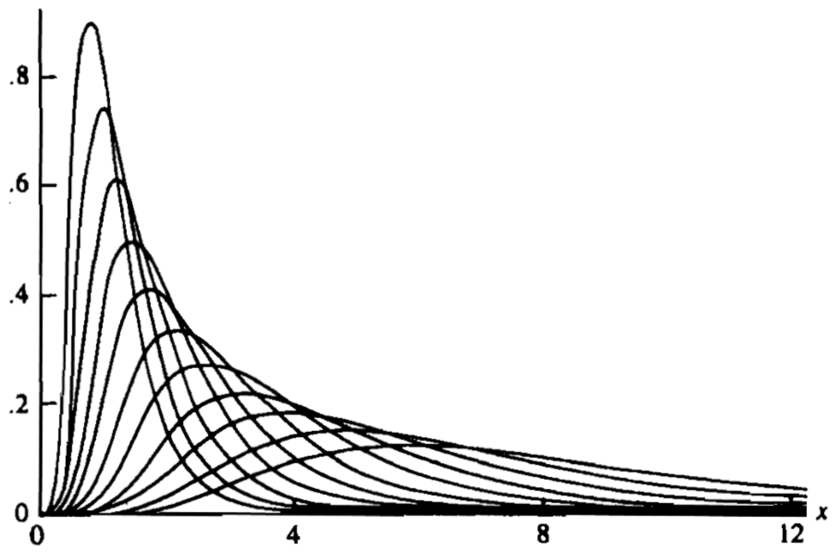
Lognormal(μ, σ^2)

pdf	$f(x\|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\frac{e^{-(\log x-\mu)^2/(2\sigma^2)}}{x}, \quad 0 \le x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$
mean and variance	$\mathrm{E}X = e^{\mu+(\sigma^2/2)}, \quad \mathrm{Var}\, X = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$
moments (mgf does not exist)	$\mathrm{E}X^n = e^{n\mu+n^2\sigma^2/2}$
notes	Example 2.3.5 gives another distribution with the same moments.

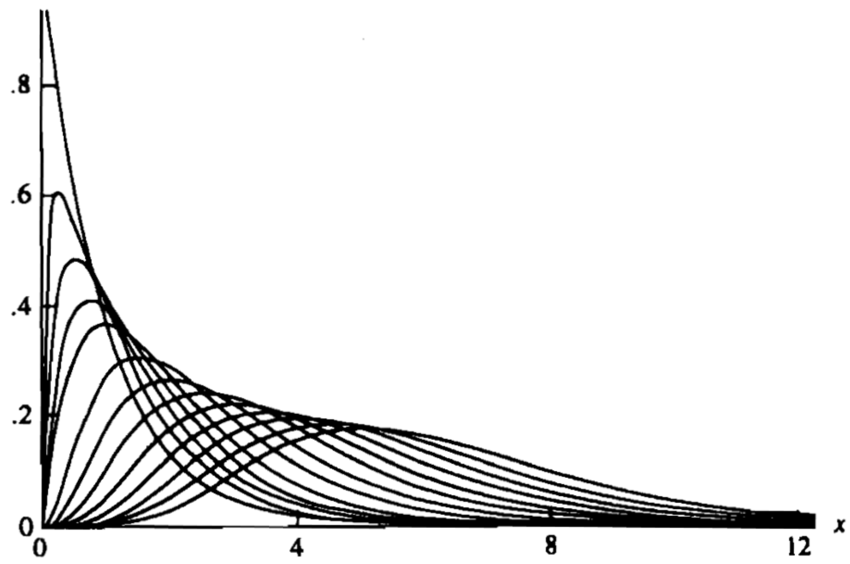
Homework #4 mgf

Double exponential(μ, σ)

pdf	$f(x\|\mu,\sigma) = \frac{1}{2\sigma}e^{-\|x-\mu\|/\sigma}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$
mean and variance	$\mathrm{E}X = \mu, \quad \mathrm{Var}\, X = 2\sigma^2$
mgf	$M_X(t) = \frac{e^{\mu t}}{1-(\sigma t)^2}, \quad \|t\| < \frac{1}{\sigma}$
notes	Also known as the *Laplace* distribution.



a.



b.

Figure 3.3.6. (a) *Some lognormal densities;* (b) *some gamma densities*

§3.4 Exponential Families (valuable tool!)

A pdf or pmf is exponential family if can be expressed in form:
cont. discrete

Sam's notation $\underset{\sim}{\theta}$ (bold in C&B could be vector).

$$f(x|\underset{\sim}{\theta}) = h(x)c(\underset{\sim}{\theta})\exp\left(\sum_{i=1}^{k} w_i(\underset{\sim}{\theta})t_i(x)\right)$$

- $\underset{\sim}{\theta}$ represents parameter
- $h(x) \geq 0$ ← can include indicator of support/sample space
- $t_1(x), t_2(x) \ldots t_k(x)$ ft'ns of x - don't depend on θ.
- $c(\underset{\sim}{\theta}) \geq 0$
- $w_1(\theta), \ldots, w_k(\theta)$ ft'ns of θ - don't depend on x

Take $f(x|\underset{\sim}{\theta})$ & identify $h(x), c(\underset{\sim}{\theta}), w_i(\underset{\sim}{\theta})$ & $t_i(x)$
- ideally k as small as possible!

Example 3.4.1 Binomial Dist'n.

$$f(x|p) = \binom{n}{x} p^x (1-p)^{n-x} I_{[0,1,2,\ldots n]}(x)$$

$$I_{[0,1,2\ldots n]}(x) = \begin{cases} 1 & x \in [0,1,2,\ldots n] \\ 0 & \text{else} \end{cases}$$

Note: not a ft'n of p.

$$= \binom{n}{x}(1-p)^n \left(\frac{p}{1-p}\right)^x * I_{[0,1,2\ldots n]}(x)$$

$$= \binom{n}{x} * I_{[0,1,2\ldots n]}(x) * (1-p)^n * \exp\left[\log\left(\frac{p}{1-p}\right) * x\right]$$

$h(x) = \binom{n}{x} * I_{[0,1,2,\ldots n]}(x)$ $\qquad$ $t_1(x) = x$

$c(\theta) = (1-p)^n$ $\qquad$ $w_1(\theta) = \log\left(\frac{p}{1-p}\right)$

$$f(x|p) = h(x)c(p)\exp[w_1(\theta)t_1(x)] \quad //$$

Normal dist'n $\sim N(\mu, \sigma^2)$

$$f(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) * I_{(-\infty,\infty)}(x) \qquad \begin{matrix} -\infty < \mu < \infty \\ 0 < \sigma < \infty \end{matrix} \qquad \underline{\theta} = (\mu, \sigma)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\frac{(x^2 - 2\mu x + \mu^2)}{\sigma^2}\right] * I_{(-\infty,\infty)}(x)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left[\frac{-\mu^2}{2\sigma^2}\right]}_{c(\mu,\sigma^2)} \exp\left[-\frac{x^2}{2\sigma^2} + \frac{\cancel{2}\mu x}{\cancel{2}\sigma^2}\right] * \underbrace{I_{(-\infty,\infty)}(x)}_{h(x)}$$

sample space not ftn of $\underline{\theta}$.

$w_1(\theta) = -1/2\sigma^2$ $\quad$ $w_2(\theta) = \frac{\mu}{\sigma^2}$

$t_1(x) = x^2$ $\quad$ $t_2(x) = x$

Note: constants $\frac{1}{\sqrt{2\pi}}$ could go w/ $h(x)$ instead of $c(\theta)$

$-1/2$ could go w/ $t_1(x)$ instead of $w_1(\theta)$

Definition 3.4.5 The *indicator function* of a set A, most often denoted by $I_A(x)$, is the function

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

An alternative notation is $I(x \in A)$.

Theorem 3.4.2 *If X is a random variable with pdf or pmf of the form (3.4.1), then*

$$\text{(3.4.4)} \quad \mathrm{E}\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\frac{\partial}{\partial \theta_j} \log c(\boldsymbol{\theta});$$

← proof $k=1$ homework.

$$\text{(3.4.5)} \quad \mathrm{Var}\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\frac{\partial^2}{\partial \theta_j^2} \log c(\boldsymbol{\theta}) - \mathrm{E}\left(\sum_{i=1}^{k} \frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(X)\right).$$

Binomial $\quad w_1(p) = \log\left(\frac{p}{1-p}\right); \quad c(p) = (1-p)^n; \quad t_1(x) = X$

$$\frac{d}{dp} w_1(p) = \frac{d}{dp}\log\left(\frac{p}{1-p}\right) = \frac{1}{\frac{p}{(1-p)}}\left[\frac{p}{1-p}\right]'$$

$$= \frac{(1-p)}{p} * \left[\frac{(1-p) - (-1)p}{(1-p)^2}\right] = \frac{1}{p(1-p)}$$

$$\frac{d}{dp}\log c(p) = \frac{d}{dp} n \log(1-p) = \frac{n}{(1-p)}(-1)$$

$$E\left[\left(\frac{d}{dp} w_1(p)\right) t_1(x)\right] = E\left[\frac{1}{p(1-p)} \cdot X\right] = \frac{n}{1-p} = \frac{d}{dp}\log(c(p))$$

$$\frac{1}{p\cancel{(1-p)}} E[X] = \frac{n}{\cancel{(1-p)}} \rightarrow E[X] = np$$

Uniform(a, b)

pdf	$f(x\|a,b) = \frac{1}{b-a}, \quad a \le x \le b$
mean and variance	$\mathrm{E}X = \frac{b+a}{2}, \quad \mathrm{Var}\,X = \frac{(b-a)^2}{12}$
mgf	$M_X(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$
notes	If $a = 0$ and $b = 1$, this is a special case of the beta $(\alpha = \beta = 1)$.

sample space is ft'n of parameter!

$X \sim U(0,\theta) \qquad f_X(x|\theta) = \frac{1}{\theta} I_{[0,\theta]}^{(x)} \qquad I_{[0,\theta]}^{(x)} = \begin{cases} 1 & 0 \le x \le \theta \\ 0 & \text{else} \end{cases}$

try to get in exponential form?

$h(x) = 1$

$c(\theta) = \frac{1}{\theta}$

where does $I_{[0,\theta]}^{(x)}$ go!?!

Can't separate θ & x

Not exponential family ☹

Definition 3.4.7 A *curved exponential family* is a family of densities of the form (3.4.1) for which the dimension of the vector $\boldsymbol{\theta}$ is equal to $d < k$. If $d = k$, the family is a *full exponential family*. (See also Miscellanea 3.8.3.)

$X \sim N(\mu, \mu^2) \qquad f_X(x) = \frac{1}{\sqrt{2\pi\mu^2}} \exp\left(-\frac{(x-\mu)^2}{2\mu^2}\right) I_{(-\infty,\infty)}^{(x)}$

$\vdots$

$= \frac{1}{\sqrt{2\pi\mu^2}} \exp\left(-\frac{1}{2}\right) * I_{(-\infty,\infty)}^{(x)} \exp\left[-\frac{x^2}{2\mu^2} + \frac{x}{\mu}\right]$

$\cdots \quad k=2 \quad \dim(\theta) = 1$,,

Natural Parameter Exponential

An exponential family is sometimes reparameterized as

$$f(x|\eta) = h(x)c^*(\eta)\exp\left(\sum_{i=1}^{k}\eta_i t_i(x)\right). \tag{3.4.7}$$

Here the $h(x)$ and $t_i(x)$ functions are the same as in the original parameterization (3.4.1). The set $\mathcal{H} = \left\{\eta = (\eta_1, \ldots, \eta_k) \colon \int_{-\infty}^{\infty} h(x)\exp\left(\sum_{i=1}^{k}\eta_i t_i(x)\right) dx < \infty\right\}$ is called the *natural parameter space* for the family. (The integral is replaced by a sum over the values of x for which $h(x) > 0$ if X is discrete.) For the values of $\eta \in \mathcal{H}$, we must have $c^*(\eta) = \left[\int_{-\infty}^{\infty} h(x)\exp\left(\sum_{i=1}^{k}\eta_i t_i(x)\right) dx\right]^{-1}$ to ensure that the pdf integrates to 1. Since the original $f(x|\boldsymbol{\theta})$ in (3.4.1) is a pdf or pmf, the set $\{\eta = (w_1(\boldsymbol{\theta}), \ldots, w_k(\boldsymbol{\theta})) \colon \boldsymbol{\theta} \in \Theta\}$ must be a subset of the natural parameter space. But there may be other values of $\eta \in \mathcal{H}$ also. The natural parameterization and the natural parameter space have many useful mathematical properties. For example, $\mathcal{H}$ is convex.

$$f(x|\eta) = h(x)c^*(\eta)\exp(\eta_1 t_1(x))$$

one parameter natural parameterization