Masters'
Theory III

Lecture 10/1

7 Point Estimation

7.2 Methods of finding estimators

Method of Moments $\tilde{\theta}$

Maximum Likelihood Estimators (MLEs) $\hat{\theta}$

Bayes $\hat{\theta}_B$

EM Algorithm - later

7.3 Methods of Evaluating Estimators

Mean Squared Error (MSE)

Best Unbiased Estimators (BUEs)

Sufficiency & Unbiasedness

Consistency - Asymptotics Chapter 10

Loss Function Optimality - Skip

Definition 7.1.1 A *point estimator* is any function $W(X_1, \ldots, X_n)$ of a sample; that is, any statistic is a point estimator.

- Assume we are sampling from a pop'n described by pdf/pmf $f(\underset{\sim}{x}|\theta)$
- Know $\theta \rightarrow$ information about entire pop'n.
- Want to find a good point estimator of θ
- θ or $\tau(\theta)$ [a ft'n of θ] may be of interest

"Our" def'n of a point estimator, $W(X_1, \ldots, X_n)$, may seem "unnecessarily vague"; ... don't want to "eliminate any candidates from consideration".

Note: Estimat**or** is a ft'n of the sample $X_1, \ldots, X_n$
Estimat**e** is the realized value of estimator ft'n of $\underset{\sim}{x}$.

Masters' Theory III

Lecture 10/2

- Many cases there is an obvious or natural candidate for a point estimator (i.e sample mean = estimator of pop'n mean)

- Leave simple case: "intuition may not only desert us, it may also lead us astray"!

Methods of Finding Estimators

Method of Moments (Karl Pearson late 1800's)

+ quite simple
+ almost always yields an estimate
- Unfortunately, yields estimates that may be improved on.
+ Good place to start when other methods are intractable.

Let $X_1, \ldots, X_n$ sample from a pop'n w/ pdf/pmf $f(\underline{x} \mid \theta_1, \ldots \theta_K)$ [K = # parameters, n = sample size]

First K sample moments

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i^1, \quad \mu'_1 = EX^1,$$

$$m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2, \quad \mu'_2 = EX^2,$$

$$\vdots$$

$$m_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k, \quad \mu'_k = EX^k.$$

First K pop'n moments. Typically ft'n of $\theta_1, \ldots, \theta_K$ say $\mu_j(\theta_1, \ldots, \theta_K)$

Method of Moments: Equate first k sample moments
to first k pop'n moments
Solve resulting system of eq'ns.
$\boxed{(k \text{ eq'ns}; k \text{ unknowns})}$

that is, the Method of Moments estimator $(\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_k)$ for $(\theta_1, \ldots, \theta_k)$ is obtained by solving the eq'ns:

$$\left.\begin{aligned} m_1 &= \mu_1(\theta_1, \theta_2, \ldots \theta_k) \\ m_2 &= \mu_2(\theta_1, \ldots \theta_k) \\ &\vdots \\ m_k &= \mu_k(\theta_1, \ldots \theta_k) \end{aligned}\right\} \text{solve for } (\theta_1, \ldots \theta_k) \text{ in terms of } (m_1, \ldots, m_k).$$

Example 7.2.1

Suppose $X_1, \ldots, X_n$ iid $N(\theta, \sigma^2)$ s.t. $\boxed{\theta_1 = \theta, \ \theta_2 = \sigma^2}$

sample	pop'n
$m_1 = \bar{X}$	$\mu_1 = \theta$
$m_2 = \frac{1}{n}\Sigma X_i^2$	$\mu_2 = \theta^2 + \sigma^2$

equate sample & pop'n moments $\rightarrow$

$$\bar{X} = \theta$$
$$\Sigma X_i^2/n = \theta^2 + \sigma^2$$

solve

$$\tilde{\theta} = \bar{X}$$
$$\tilde{\sigma}^2 = \Sigma X_i^2/n - \bar{X}^2 = \Sigma(X_i - \bar{X})^2/n$$

$\boxed{\text{Method of Moments Estimators coincide with intuition.}}$

Master's
Theory III

Lecture 10/4

Example 7.2.2

Let $X_1, \ldots, X_n$ iid binomial (k, p) $\quad P(X_i = x \mid p) = \binom{k}{x} p^x (1-p)^{k-x}$

Assume p and k unknown $\quad x = 0, 1, \ldots k;\ 0 \le p \le 1$

(application: crime rates (p) for crimes that often go unreported)

Method of Moments → Equate sample & pop'n moments ($k=2$)

$$\bar{X} = kp$$

$$\frac{\sum x_i^2}{n} = kp(1-p) + k^2 p^2$$

$$\vdots$$

$$\tilde{k} = \frac{\bar{X}^2}{\bar{X} - (1/n)\sum(x_i - \bar{x})^2} \qquad \tilde{p} = \bar{X}/\tilde{k}$$

Note: If sample mean, $\bar{X}$ < sample variance $\frac{1}{n}\sum(x_i - \bar{x})^2$

→ $\tilde{k} < 0$ & $\tilde{p} < 0$!?!

Negative $\tilde{k}$ & Negative $\tilde{p}$. The range of the estimator does not coincide with the range of the parameter it is estimating.

— Only negative estimates if $\bar{X} < 1/n \sum(x_i - \bar{x})^2$

→ large amounts of variability in data.

Example: $X_1, \ldots X_n$ iid Poisson (λ) wish to estimate λ

$$m_1 = E[X] = \lambda \quad \rightarrow \quad \tilde{\lambda} = \bar{X}$$

Nothing unique about MoM

$\sum(x_i - \bar{x})^2/n$ also has mean λ. Which moment do we use?

Masters' Theory III

Lecture 10/5

Method of Moments

Advantages

- easy to find
- $\tilde{m}_r \xrightarrow{P} m_r$ by WLLN
- $E[\tilde{m}_r] = m_r$
- good starting point for more sophisticated methods

Disadvantages

- Not always 'best' estimate

i.e. $X_1, \ldots, X_n \sim U(0, \theta)$

$m_1(\theta) = E[X_i] = \theta/2$

$\theta = 2m_1 = 2\bar{X}$

what if $\bar{X} = 3$

$X_{(n)} = 7$?

Methods of Finding Estimators

Maximum Likelihood Estimators (MLEs)

First cover likelihoods functions (§ 6.3.1)

Master's
Theory · III

Lecture 10/6

6.3.1 The Likelihood Function

Definition 6.3.1 Let $f(\mathbf{x}|\theta)$ denote the joint pdf or pmf of the sample $\mathbf{X} = (X_1, \ldots, X_n)$. Then, given that $\mathbf{X} = \mathbf{x}$ is observed, the function of θ defined by

$$L(\theta|\mathbf{x}) = f(\mathbf{x}|\theta)$$

is called the *likelihood function.*

Discrete

If $\underset{\sim}{X}$ is a discrete Random Vector ($\underset{\sim}{RV}$) then

$$L(\theta|\underset{\sim}{x}) = P_\theta(\underset{\sim}{X} = \underset{\sim}{x})$$

If we compare the likelihood function at two parameter points and find that

$$P_{\theta_1}(\underset{\sim}{X} = \underset{\sim}{x}) = L(\theta_1|\underset{\sim}{x}) > L(\theta_2|\underset{\sim}{x}) = P_{\theta_2}(\underset{\sim}{X} = \underset{\sim}{x})$$

then the sample we observed is "more likely" to have occurred if $\theta = \theta_1$ than if $\theta = \theta_2$.

Continuous

If $\underset{\sim}{X}$ is a continuous, realvalued random variable + if the pdf of X is continuous in x, then, for small ε

$$\frac{P_{\theta_1}(x-\varepsilon < X < x+\varepsilon)}{P_{\theta_2}(x-\varepsilon < X < x+\varepsilon)} \approx \frac{L(\theta_1|x)}{L(\theta_2|x)}$$

Compare likelihood function at 2 parameter values gives an approx. comparison of the probability of observed sample, $\underset{\sim}{x}$.

pdf or pmf $f(\underset{\sim}{x}|\theta) \longrightarrow$ θ fixed, $\underset{\sim}{x}$ variable

Likelihood ft'n $L(\theta|\underset{\sim}{x}) \longrightarrow$ $\underset{\sim}{x}$ observed sample point
θ varying over all possible parameter values

Example 6.3.2

$X \sim$ negative binomial $f(x|r,p) = \binom{r+x-1}{x} p^r (1-p)^x \begin{cases} x = 0,1,2,\dots \\ 0 \le p \le 1 \end{cases}$

$x = \#$ failures to r^{th} success

assume $r=3$, $x=2$ & $p = pr(\text{success})$ is unknown

$$L(p|2) = P_p(X=2) = \binom{4}{2} p^3 (1-p)^2$$

In general, $X = x$ $\quad L(p|x) = \binom{3+x-1}{x} p^3 (1-p)^x$

LIKELIHOOD PRINCIPLE: If $\mathbf{x}$ and $\mathbf{y}$ are two sample points such that $L(\theta|\mathbf{x})$ is proportional to $L(\theta|\mathbf{y})$, that is, there exists a constant $C(\mathbf{x}, \mathbf{y})$ such that

(6.3.1) $$L(\theta|\mathbf{x}) = C(\mathbf{x}, \mathbf{y})L(\theta|\mathbf{y}) \quad \text{for all } \theta,$$

then the conclusions drawn from $\mathbf{x}$ and $\mathbf{y}$ should be identical.

where $C(\underset{\sim}{x},\underset{\sim}{y})$ may be different for different $(\underset{\sim}{x},\underset{\sim}{y})$ pairs, but $C(\underset{\sim}{x},\underset{\sim}{y})$ does not depend on θ.

$$\frac{L(\theta|\underset{\sim}{x})}{L(\theta|\underset{\sim}{y})} = C(\underset{\sim}{x},\underset{\sim}{y}) \quad \text{for all } \theta.$$

In special case $C(\underset{\sim}{x},\underset{\sim}{y}) = 1 \Rightarrow$ same likelihood, same info about θ.

Masters'
Theory - III

Lecture 10/8

$$\frac{L(\theta|\underset{\sim}{x})}{L(\theta|\underset{\sim}{y})} = C(\underset{\sim}{x},\underset{\sim}{y}) \text{ for all } \theta.$$

→ 2 samples ($\underset{\sim}{x}$ & $\underset{\sim}{y}$) contain the same info about θ if they have proportional likelihoods for all θ.

if $L(\theta_2|\underset{\sim}{x}) = 2L(\theta_1|\underset{\sim}{x})$ θ_2 is twice as 'plausible' as θ_1

if $L(\theta_2|\underset{\sim}{y}) = 2L(\theta_1|\underset{\sim}{y})$ θ_2 is twice as 'plausible' as θ_1

$$\frac{L(\theta_2|\underset{\sim}{x})}{L(\theta_2|\underset{\sim}{y})} = \frac{2L(\theta_1|\underset{\sim}{x})}{2L(\theta_1|\underset{\sim}{y})} = \frac{L(\theta_1|\underset{\sim}{x})}{L(\theta_1|\underset{\sim}{y})} = C(\underset{\sim}{x},\underset{\sim}{y})$$

proportional for θ_1 + θ_2

Whether we observe $\underset{\sim}{x}$ or $\underset{\sim}{y}$ we conclude that θ_2 is twice as 'plausible' as θ_1.

"We carefully use the word 'plausible' rather than 'probable' in the preceding paragraph because we often think of θ as fixed (albeit unknown) value. Furthermore, although $f(\underset{\sim}{x}|\theta)$, as a ft'n of $\underset{\sim}{x}$, is a pdf, there is no guarantee that $L(\theta|\underset{\sim}{x})$, as a ft'n of θ is a pdf."

Master's
Theory

Lecture 10/9

Methods of Finding Estimators

- Maximum Likelihood Estimators (MLE)
  - "the most popular technique for deriving estimators."

Recall that if $X_1, \ldots, X_n$ are an iid sample from a population with pdf or pmf $f(x|\theta_1, \ldots, \theta_k)$, the likelihood function is defined by

$$L(\theta|x) = L(\theta_1, \ldots, \theta_k|x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i|\theta_1, \ldots, \theta_k). \quad (7.2.3)$$

DEFINITION 7.2.4: For each sample point x, let $\hat{\theta}(x)$ be a parameter value at which $L(\theta|x)$ attains its maximum as a function of θ, with x held fixed. A *maximum likelihood estimator* (MLE) of the parameter θ based on a sample X is $\hat{\theta}(X)$.

Note: Range of mle coincides with range of the parameter

Example: Suppose that for a discrete experiment, there are 3 possible outcomes: z_1, z_2, z_3 and there are 3 possible values for θ: $\theta_1, \theta_2, \theta_3$

$f(z|\theta_1)$ $f(z|\theta_2)$ $f(z|\theta_3)$ ← 3 probability ft'ns columns

	θ_1	θ_2	θ_3	
$L(\theta\|z_1)$	.4	.6	.2	$\neq 1$ ← Probability of z_1 under 3 different models.
$L(\theta\|z_2)$	.2	.3	.1	$\neq 1$
$L(\theta\|z_3)$	.4	.1	.7	$\neq 1$
	1	1	1	

rows, 3 likelihood functions →

↑ Probability of z_1, z_2, z_3 under model $\theta = \theta_1$

Master's Theory

Lecture 10/10

proportions

	θ_1	θ_2	θ_3				
$L(\theta\|z_1)$	2	3	1	=	$.4/.2$	$.6/.2$	$.2/.2$
$L(\theta\|z_2)$	2	3	1		$.2/.1$	$.3/.1$	$.1/.1$
$L(\theta\|z_3)$	4	1	7		$.4/.1$	$.1/.1$	$.7/.1$

→ If we observe z_1, we would estimate $\hat{\theta} = \theta_2$

$L(\theta_2|z_1) : L(\theta_1|z_1) = 3:2$ (θ_2 1.5 times more likely than θ_1 given z_1)

$L(\theta_2|z_1) : L(\theta_3|z_1) = 3:1$ (θ_2 is 3 times more likely than θ_3 given z_1).

→ If we observe z_2 we would draw identical conclusions ($L(\theta|z_1) = 2 \cdot L(\theta|z_2)$ for all θ)

→ If we observe z_3, we would estimate $\hat{\theta} = \theta_3$.

Intuitively, MLE is a reasonable choice
= parameter point for which the observed sample is most likely.
(Many good properties ... later)

Drawbacks :
1) Finding global maximum + verifying that it is a global max.
2) Numerical Sensitivity – may be sensitive to small changes in the data?
Slightly different samples → vastly different mle's.

Finding MLEs:

- First assume we can find the 1^{st} & 2^{nd} derivatives...

$$\frac{\partial}{\partial \theta_i} L(\underset{\sim}{\theta} | \underset{\sim}{x}) = 0 \quad i = 1, \ldots, k$$

$$1^{st} \text{ derivative } = 0 \Rightarrow \begin{cases} \text{local or global } \begin{cases} \text{minima} \\ \text{maxima} \end{cases} \\ \text{Inflection point} \end{cases}$$

But, may have extrema occur at the boundary... must check boundaries separately.

example 7.2.4: Let $X_1, \ldots, X_n$ iid $N(\theta, 1)$

$$L(\theta | \underset{\sim}{x}) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}(x_i - \theta)^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \theta)^2}$$

$$\vdots$$

$$\frac{d}{d\theta} L(\theta | \underset{\sim}{x}) = 0 \Rightarrow \sum_{i=1}^{n} (x_i - \theta) = 0$$

$$\Sigma x_i - n\theta = 0$$

$$\Rightarrow \hat{\theta} = \frac{\Sigma x_i}{n} = \bar{X}$$

$\bar{X}$ = "candidate for mle"

could be local/global min, max, inflection point

ICBST $\frac{d^2}{d\theta^2} L(\theta | \underset{\sim}{x}) \Big|_{\theta = \bar{x}} < 0$

$\therefore$ $\bar{X}$ = only extreme point in the interior and is a maximum.

Check boundaries ($\pm\infty$) to verify global max.

Take limits $\theta \Rightarrow \pm\infty \Rightarrow L(\theta | \underset{\sim}{x}) = 0 \Big|_{\theta \Rightarrow \pm\infty}$

$\therefore$ $\bar{x}$ = global maximum

Note: in this case we could avoid checking boundaries since $\bar{X}$ is a unique interior extremum & is a max. There can be no max at $\pm\infty$. (If there were there would have to be an interior minimum which contradicts uniqueness.)

Master's Theory

Lecture 10/12

Note: If use differentiation, usually easier to work with natural log. (extrema coincide - log strictly increasing on $(0,\infty)$)

example: $X_1,\ldots,X_n$ iid Bernoulli(p).

$$L(p|\underline{x}) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{y}(1-p)^{n-y} \qquad y = \sum x_i$$

$$0 \le y \le n$$

$\boxed{\text{easier to work with log likelihood}}$

$$\log[L(p|x)] = y\log p + (n-y)\log(1-p)$$

for $0 < y < n$ $\quad \frac{d}{dp} L(p|x) = \frac{y}{p} + \frac{(n-y)}{(1-p)}(-1) = 0$

$$\rightarrow \frac{y}{p} = \frac{n-y}{(1-p)} \rightarrow y(1-p) = p(n-y)$$

$$y - \cancel{yp} = pn - \cancel{yp}$$

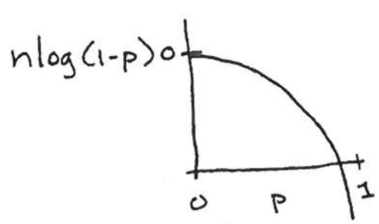
$$\boxed{\hat{p} = \frac{y}{n}}$$

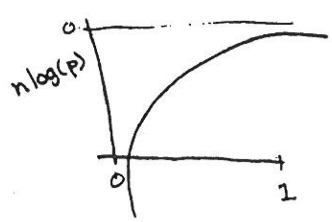
$y=0 \Rightarrow$ loglike $= n\log(1-p)$ monotone ft'n of $p \Rightarrow \hat{p} = y/n = 0$

$y=n \Rightarrow$ loglike $= n\log p$ monotone ft'n of $p \Rightarrow \hat{p} = y/n = 1$

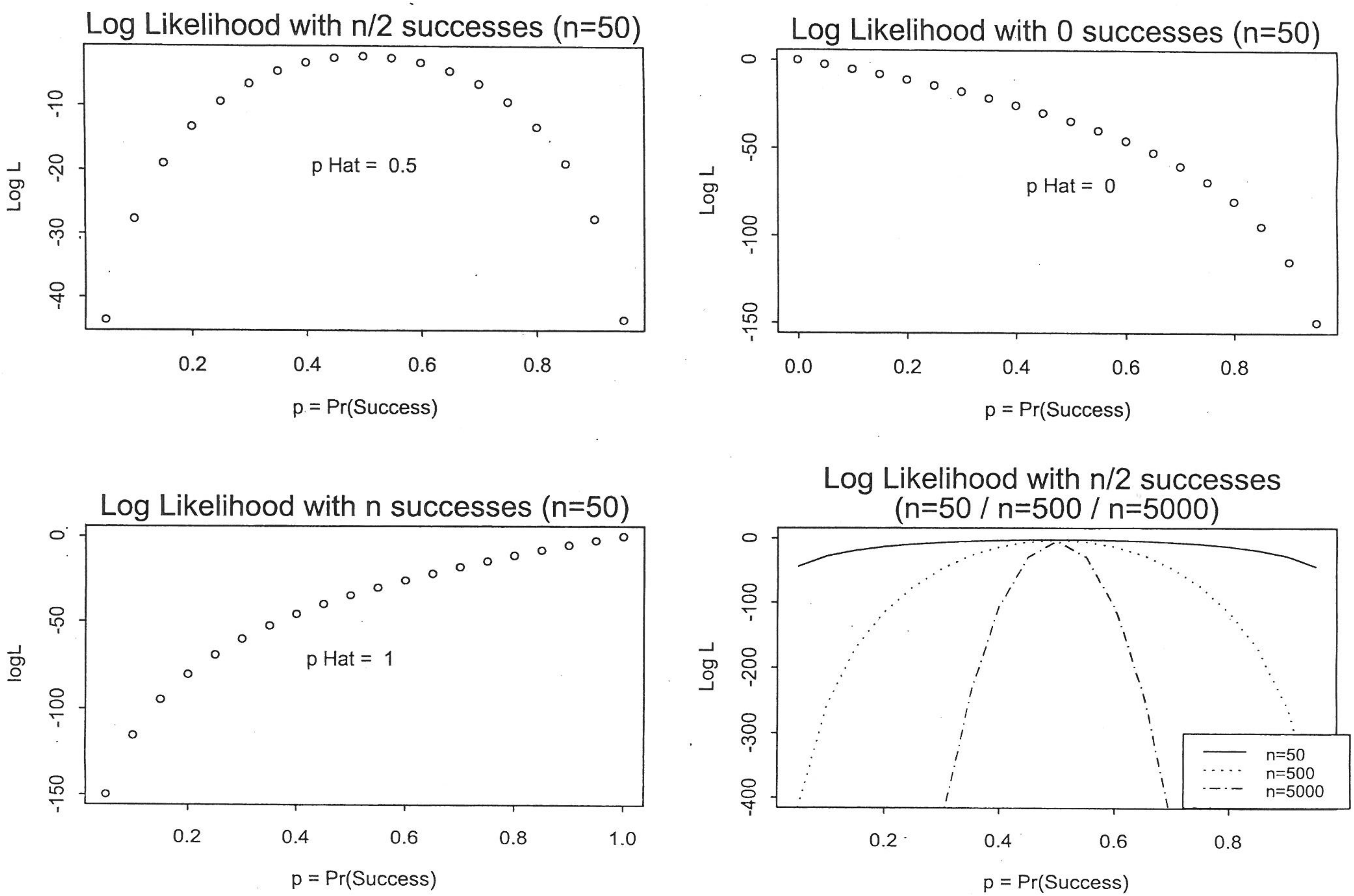
For $0 \le y \le n \Rightarrow \hat{p} = y/n = $ MLE

(Note: must check that $\hat{p}$ is a global max. (PYOR))





Binomial Log Likelihoods



```
# ---------------------------------------------------------------------------
# program:      binom<-boundary.s
# original:     2/28/02
# updated:      3/2/03 (dbinom), 2/27/04 (ind.maxL); 2/22/06 (R compatible)
# for:          master's theory iii lecture-1
# ---------------------------------------------------------------------------
#  Data Input file: NONE
# ---------------------------------------------------------------------------
#  Goal: plot LogLikelihoods: binomial 0, n, n/2 successes
# ---------------------------------------------------------------------------
# clean out directory
#       remove(ls())  # doesn't work in R!
# ---------------------------------------------------------------------------
# shut off graphics
        graphics.off()
# ---------------------------------------------------------------------------
# Libraries
#       library(Hmisc,T)
#       library(Design,T)
# ---------------------------------------------------------------------------
# files

        # use paste() command to attach file name to current directory
        # paste() creates a character string by combining text and/or numeric values

        dir<-"C:/sam/teaching/splus_theory_06/"

        file.out<-paste(dir,"binom_boundary_out.txt",sep="")
        file.run<-paste(dir,"binom_boundary.s",sep="")

        sink(file.out)          # sink sends the output to file.out
# ---------------------------------------------------------------------------
# program / date / time

        cat("\n\n\n")
        cat("Program:")
        cat(file.run)
        cat("\n\n")
        cat("Program to plot Binomial Log Likelihoods for 0, n/2 and n successes: Masters Theory\n\n")
        print(date())
        cat("\n\n")
# ---------------------------------------------------------------------------
# Program

        # allow 4 graphs on one page
        par(mfrow=c(2,2),oma=c(0,0,4,0))

        # create vector of p=Pr(success): 0,.05, .1, ... 1
        p<-seq(0,1,by=.05)

        # define n
        n<-50

        # -----------------------------------------------------------------
        #sum xi = 25

        cat("\n\n-----------------------------------------------------------")
        cat("\n----------------------  n/2 Successes -------------------------\n")
        logL<-log(dbinom(25,50,p))
        plot(p,logL,main="\nLog Likelihood with n/2 successes (n=50)",
        xlab="p = Pr(Success)",ylab="Log L",cex=.7)
        cat("\nLog Likelihood for p=0,.05,...1, with n/2 successes\n")
        print(logL)

        #----------------------
        # find the value of p that maximizes the log(like)
        ind.maxL<-logL==max(logL)     # indicator vector for maximum
        mlep<-p[ind.maxL==T]          # choose p that corresponds to logL==max(logL)
        cat("\nLogL,  p,  indicator of maximum T=TRUE, F=FALSE (numeric 1=TRUE, 0=FALSE) for n/2
        successes\n")
        print(cbind(logL, p, ind.maxL))       # cbind converts ind.maxL to a numeric
```

```
                                # indicator T=1, F=0

        #------------------------
        # another method to find value of p that maximizes logL

        # maxlike<--1e99999    # large negative to begin
        # NN<-length(p)
        # for (i in 1:NN) {
        #       if(logL[i]>maxlike) {
        #               maxlike<-logL[i]
        #               mlep<-p[i]
        #       }
        #}
        #------------------------

cat("\n Value of p that maximized logL with n/2 successes\n")
print(mlep)

mytxt<-paste("p Hat = ",mlep)
text(.5,median(logL)*2,mytxt,cex=.7)  # add text to graph

# ------------------------------------------------------------------
#sum xi =0
cat("\n\n-----------------------------------------------------------------")
cat("\n------------------------  0   Successes -------------------------\n")
logL<-log(dbinom(0,50,p))

plot(p,logL,main="\nLog Likelihood with 0 successes (n=50)",
xlab="p = Pr(Success)",ylab="Log L",cex=.7)
cat("\n\nLog Likelihood for p=0,.05,...1, with 0 successes\n")
print(logL)

#-----------------------
# find the value of p that maximizes the log(like)
ind.maxL<-logL==max(logL)      # indicator vector for maximum
mlep<-p[ind.maxL==T]           # choose p that corresponds to logL==max(logL)
cat("\nLogL,  p,  indicator of maximum T=TRUE, F=FALSE (numeric 1=TRUE, 0=FALSE) for 0
successes\n")
print(cbind(logL, p, ind.maxL))

cat("\n Value of p that maximized logL with 0 successes\n")
print(mlep)

mytxt<-paste("p Hat = ",mlep)
text(.5,median(logL)*2,mytxt,cex=.7)  # add text to graph

# ------------------------------------------------------------------
# sum xi =n
cat("\n\n-----------------------------------------------------------------")
cat("\n------------------------  n   Successes -------------------------\n")
logL<-log(dbinom(50,50,p))
plot(p,logL,main="\nLog Likelihood with n successes (n=50)",
xlab="p = Pr(Success)",cex=.7)
cat("\n\nLog Likelihood for p=0,.05,...1, with n successes\n")
print(logL)

#-----------------------
# find the value of p that maximizes the log(like)
ind.maxL<-logL==max(logL)      # indicator vector for maximum
mlep<-p[ind.maxL==T]           # choose p that corresponds to logL==max(logL)
cat("\nLogL,  p,  indicator of maximum T=TRUE, F=FALSE (numeric 1=TRUE, 0=FALSE) for n
successes\n")
print(cbind(logL, p, ind.maxL))

cat("\n Value of p that maximized logL with n successes\n")
print(mlep)

mytxt<-paste("p Hat = ",mlep)
text(.5,median(logL)*2,mytxt,cex=.7)  # add text to graph

# ------------------------------------------------------------------
```

```
# compare loglikelihoods for n=50, n=500, n=5000, number successes = n/2

logL1<-log(dbinom(25,50,p))
logL2<-log(dbinom(250,500,p))
logL3<-log(dbinom(2500,5000,p))
yrange<-c(-400,0)
plot(p,logL1,main="Log Likelihood with n/2 successes \n(n=50 / n=500 / n=5000)",
xlab="p = Pr(Success)",ylab="Log L",cex=.7,type="l",ylim=yrange)
lines(p,logL2,lty=2)
lines(p,logL3,lty=3)
lgd<-c("n=50","n=500","n=5000")
legend(.72,-300,lgd,lty=c(1:3),cex=.6)

# overall label on graph
mtext("Binomial Log Likelihoods",outer=T,cex=1.25)

sink()                          # close output file
```

#--
OUTPUT

```
Program:C:/sam/teaching/splus_theory_06/binom_boundary.s

Program to plot Binomial Log Likelihoods for 0, n/2 and n successes: Masters Theory

[1] "Wed Feb 22 10:17:01 2006"


-----------------------------------------------------------------
---------------------- n/2 Successes ------------------------

Log Likelihood for p=0,.05,...1, with n/2 successes
 [1]       -Inf -43.705083 -27.728084 -19.020416 -13.343980  -9.378854
 [7]  -6.545637  -4.544570  -3.207352  -2.438061  -2.186803  -2.438061
[13]  -3.207352  -4.544570  -6.545637  -9.378854 -13.343980 -19.020416
[19] -27.728084 -43.705083       -Inf

LogL,  p,  indicator of maximum T=TRUE, F=FALSE (numeric 1=TRUE, 0=FALSE) for n/2 successes
           logL    p ind.maxL
 [1,]      -Inf 0.00        0
 [2,] -43.705083 0.05        0
 [3,] -27.728084 0.10        0
 [4,] -19.020416 0.15        0
 [5,] -13.343980 0.20        0
 [6,]  -9.378854 0.25        0
 [7,]  -6.545637 0.30        0
 [8,]  -4.544570 0.35        0
 [9,]  -3.207352 0.40        0
[10,]  -2.438061 0.45        0
[11,]  -2.186803 0.50        1
[12,]  -2.438061 0.55        0
[13,]  -3.207352 0.60        0
[14,]  -4.544570 0.65        0
[15,]  -6.545637 0.70        0
[16,]  -9.378854 0.75        0
[17,] -13.343980 0.80        0
[18,] -19.020416 0.85        0
[19,] -27.728084 0.90        0
[20,] -43.705083 0.95        0
[21,]      -Inf 1.00        0

 Value of p that maximized logL with n/2 successes
[1] 0.5
```

```
------------------------------------------------------------------
----------------------  0   Successes ------------------------

 Log Likelihood for p=0,.05,...1, with 0 successes
  [1]    0.000000   -2.564665   -5.268026   -8.125946  -11.157178  -14.384104
  [7]  -17.833747  -21.539146  -25.541281  -29.891850  -34.657359  -39.925385
 [13]  -45.814537  -52.491106  -60.198640  -69.314718  -80.471896  -94.855999
 [19] -115.129255 -149.786614        -Inf

 LogL,  p,  indicator of maximum T=TRUE, F=FALSE (numeric 1=TRUE, 0=FALSE) for 0 successes
             logL    p ind.maxL
  [1,]    0.000000 0.00        1
  [2,]   -2.564665 0.05        0
  [3,]   -5.268026 0.10        0
  [4,]   -8.125946 0.15        0
  [5,]  -11.157178 0.20        0
  [6,]  -14.384104 0.25        0
  [7,]  -17.833747 0.30        0
  [8,]  -21.539146 0.35        0
  [9,]  -25.541281 0.40        0
 [10,]  -29.891850 0.45        0
 [11,]  -34.657359 0.50        0
 [12,]  -39.925385 0.55        0
 [13,]  -45.814537 0.60        0
 [14,]  -52.491106 0.65        0
 [15,]  -60.198640 0.70        0
 [16,]  -69.314718 0.75        0
 [17,]  -80.471896 0.80        0
 [18,]  -94.855999 0.85        0
 [19,] -115.129255 0.90        0
 [20,] -149.786614 0.95        0
 [21,]        -Inf 1.00        0

 Value of p that maximized logL with 0 successes
[1] 0

------------------------------------------------------------------
----------------------  n   Successes ------------------------
Log Likelihood for p=0,.05,...1, with n successes
 [1]        -Inf -149.786614 -115.129255  -94.855999  -80.471896  -69.314718
 [7]  -60.198640  -52.491106  -45.814537  -39.925385  -34.657359  -29.891850
[13]  -25.541281  -21.539146  -17.833747  -14.384104  -11.157178   -8.125946
[19]   -5.268026   -2.564665    0.000000

LogL,  p,  indicator of maximum T=TRUE, F=FALSE (numeric 1=TRUE, 0=FALSE) for n successes
            logL    p ind.maxL
 [1,]        -Inf 0.00        0
 [2,] -149.786614 0.05        0
 [3,] -115.129255 0.10        0
 [4,]  -94.855999 0.15        0
 [5,]  -80.471896 0.20        0
 [6,]  -69.314718 0.25        0
 [7,]  -60.198640 0.30        0
 [8,]  -52.491106 0.35        0
 [9,]  -45.814537 0.40        0
[10,]  -39.925385 0.45        0
[11,]  -34.657359 0.50        0
[12,]  -29.891850 0.55        0
[13,]  -25.541281 0.60        0
[14,]  -21.539146 0.65        0
[15,]  -17.833747 0.70        0
[16,]  -14.384104 0.75        0
[17,]  -11.157178 0.80        0
[18,]   -8.125946 0.85        0
[19,]   -5.268026 0.90        0
[20,]   -2.564665 0.95        0
[21,]    0.000000 1.00        1

 Value of p that maximized logL with n successes
[1] 1
```