Solutions to Homework 1
BIOS 7731

1. **Using the properties of a probability measure, show (BD pg. 443)**

   (a) **A.2.2** For some set A,

   $$\begin{aligned} \Omega &= A + A^c \\ 1 &= P(\Omega) = P(A) + P(A^c). \end{aligned}$$

   Therefore, $P(A^c) = 1 - P(A)$.
   If $A = \emptyset$, then $A^c = \Omega$ and

   $$\begin{aligned} P(\Omega) &= 1 - P(\emptyset) \\ 1 &= 1 - P(\emptyset) \\ P(\emptyset) &= 0. \end{aligned}$$

   (b) **A.2.3** If $A \subset B$, then $B = A \cup (A^c \cap B)$. Since A and $A^c \cap B$ are disjoint, then

   $$\begin{aligned} P(B) &= P(A \cup (A^c \cap B) \\ &= P(A) + P(A^c \cap B). \end{aligned}$$

   Since $P(A^c \cap B) \geq 0$, this implies $P(B) \geq P(A)$.

   (c) **A.2.5** Prove $P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \Sigma_{n=1}^{\infty} P\left(A_n\right)$.

   The key idea is to construct sets B_i which consist of everything in A_i that hasn't been included in the previous sets $A_1, \ldots, A_{i-1}$. Now we show that $B_i \subseteq A_i$, B_i's are pairwise disjoint and that they have the same union as the A_i's. Then use property (ii) of probability distribution on p. 442 on $\bigcup_{i=1}^{\infty} B_i$ to get the desired result.

   Define a series of sets B_i such that, $B_1 = A_1$, and $B_i = A_i \bigcap_{k=1}^{i-1} A_k^c$ for $i > 1$. By defenition $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$ and since they are pairwise disjoint, $P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \Sigma_{i=1}^{\infty} P(B_i)$. However, since each $B_i \subseteq A_i$, then $P(B_i) \leq P(A_i)$.
   Therefore, $P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \Sigma_{i=1}^{\infty} P(B_i) \leq \Sigma_{i=1}^{\infty} P(A_i)$.

   (d) **A.2.7** Prove Bonferroni's inequality $P\left(\bigcap_{i=1}^{k} A_i\right) \geq 1 - \Sigma_{i=1}^{k} P(A_i^c)$.

Recall by DeMorgan's Laws $\left(\bigcap_{\gamma\in\mathbb{F}} A_\gamma\right)^c = \bigcup_{\gamma\in\mathbb{F}} A_\gamma^c$.
So $P\left(\bigcap_{i=1}^k A_i\right) = 1 - P\left(\left(\bigcap_{i=1}^k A_i\right)^c\right) = 1 - P\left(\bigcup_{i=1}^k A_i^c\right)$.
Applying A.2.5, $P\left(\bigcup_{i=1}^k A_i^c\right) \leq \Sigma_{i=1}^k P(A_i^c)$.
$-P\left(\bigcup_{i=1}^k A_i^c\right) \geq -\Sigma_{i=1}^k P(A_i^c) \Rightarrow 1 - P\left(\bigcup_{i=1}^k A_i^c\right) \geq 1 - \Sigma_{i=1}^k P(A_i^c)$
Therefore, $P\left(\bigcap_{i=1}^k A_i\right) \geq 1 - \Sigma_{i=1}^k P(A_i^c)$.

2. **Consider $f_X(x) = \frac{x^2e^{-x}}{2}$ for $x \in (0, \infty)$, and zero otherwise.**

(a) **Show this is a density function by verifying (BD p. 449) A.7.6.**
First, we need to show the density $f_X(x)$ is nonnegative. When $x > 0$, $x^2 > 0$ and $e^{-x} > 0$. Thus $f_X(x) = \frac{x^2e^{-x}}{2} > 0$ for $x \in (0, \infty)$.
Next, we need to show that the density integrates to 1 in $x \in (0, \infty)$. This integral is,

$$\int_0^\infty \frac{x^2e^{-x}}{2}dx = \lim_{b\to\infty}\int_0^b \frac{x^2e^{-x}}{2}dx.$$

Solving by integration by parts, let $u = \frac{x^2}{2}$ and $dv = e^{-x}dx$. Then $du = xdx$, $v = -e^{-x}$ and solution is in the form $uv - \int vdu$

$$= \lim_{b\to\infty} \frac{-x^2e^{-x}}{2}\Big|_0^b - \int_0^b -xe^{-x}dx.$$

Again solve the second integral using integration by parts. Let $u = x$ and $dv = e^{-x}dx$ Then $du = dx$ and $v = -e^{-x}$.

$$\begin{aligned}
&= \lim_{b\to\infty}\left(-\frac{x^2e^{-x}}{2} - xe^{-x}\right)\Big|_0^b + \int_0^b e^{-x}dx \\
&= \lim_{b\to\infty}\left(-\frac{x^2}{2e^x} - \frac{x}{e^x} - \frac{1}{e^x}\right)\Big|_0^b \\
&= \lim_{b\to\infty}\left(-\frac{b^2}{2e^b} - \frac{b}{e^b} - \frac{1}{e^b}\right) - \left(0 - 0 - e^0\right) \\
&= 0 - 0 - 0 + 1. \\
&= 1
\end{aligned}$$

Note: The $\lim_{b\to\infty} \frac{p(b)}{e^b} = 0$ using L'Hôpital's rule.

(b) **Find the distribution function, F(x).** The distribution function is defined as

$$\begin{aligned}
F(x) &= P(X \le x) \\
&= \int_0^x \frac{y^2e^{-y}}{2}dy.
\end{aligned}$$

Repeating integration by parts steps in 3a),

$$\begin{aligned}
F(x) &= \left(-\frac{y^2}{2e^y} - \frac{y}{e^y} - \frac{1}{e^y}\right)\Big|_0^x \\
&= \left(-\frac{x^2}{2e^x} - \frac{x}{e^x} - \frac{1}{e^x}\right) - \left(0 - 0 - \frac{1}{e^0}\right) \\
&= 1 - \frac{1}{e^x}\left(1 + x + \frac{x^2}{2}\right) \quad x \in (0, \infty).
\end{aligned}$$

(c) **Find F(2).** Plugging in 2 from part b),

$$\begin{aligned} F(2) &= \frac{1}{e^2}\Big(-\frac{2^2}{2} - 2 - 1\Big) + 1 \\ &= 1 - \frac{5}{e^2} \approxeq 0.3233. \end{aligned}$$

3. **The Gamma Distribution**

(a) **The gamma function is defined for $\alpha > 0$ by**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$. Show that $\Gamma(x+1) = x!$ for $x = 0, 1, \ldots$.

By integrating e^{-x} and differentiating x^α, integration by parts gives

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^\infty x^\alpha e^{-x} dx \\ &= -x^\alpha e^{-x}|_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x} dx \\ &= \alpha\Gamma(\alpha).\end{aligned}$$

For $x = 0, 1, \ldots$, we can show that $\Gamma(x+1) = x!$ by induction. For $x = 0$, we have

$$\Gamma(0+1) = \Gamma(1) = \int_0^\infty e^{-t} dt = 1.$$

Assume that $\Gamma(n+1) = n!$, then for $x = n+1$

$$\Gamma((n+1)+1) = (n+1)\Gamma(n+1) = (n+1)!$$

Therefore, this holds for all natural numbers n.

(b) **Show that the function**

$$p(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0$$

is a probability density function when $\alpha > 0$ and $\beta > 0$. This density is called the gamma density with parameters α and β. The corresponding probablity distribution function is denoted $\Gamma(\alpha, \beta)$.

Based on a change of variables $u = x/\beta$, the density integrates to 1 over the range of x,

$$\int p(x)dx = \int_0^\infty \beta p(\beta u) du = \int_0^\infty \frac{1}{\Gamma(\alpha)} u^{\alpha-1} e^{-u} du = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1.$$

$p(x)$ is also non-negative since $x^{\alpha-1} > 0$ for $x > 0$, $e^{-x/\beta} > 0$ and the constants $\Gamma(\alpha)$ and β are greater than 0.

(c) **Show that if $X \sim \Gamma(\alpha, \beta)$, then $E[X^r] = \beta^r \Gamma(\alpha + r)/\Gamma(\alpha)$. Use this formula to derive the mean and variance of X.**

Using change of variables again, $u = x/\beta$ and

$$E[X^r] = \int x^r p(x) dx = \int_0^\infty \beta^{r+1} u^r p(\beta u) du$$

$$= \frac{\beta^r}{\Gamma(\alpha)} \int_0^\infty u^{\alpha+r-1} e^{-u} du = \frac{\beta^r \Gamma(\alpha + r)}{\Gamma(\alpha)}.$$

Therefore,

$$E[X] = \beta\Gamma(\alpha + 1)/\Gamma(\alpha) = \beta\alpha,$$

and

$$E[X^2] = \beta^2 \Gamma(\alpha + 2)/\Gamma(\alpha) = \beta^2(\alpha + 1)\alpha.$$

Finally, $Var(X) = E[X^2] - E[X]^2 = \alpha\beta^2$.

4. **Suppose $X \sim N(0,1)$, find the mean and covariance of the random vector $(X, I\{X > c\})$.**

   The variable $I\{X > c\}$ has a Bernoulli distribution with success probability $P(X > c) = 1 - \Phi(c)$, where Φ is the CDF of the normal distribution. Therefore, its mean and variance are $1 - \Phi(c)$ and $\Phi(c)(1 - \Phi(c))$ respectively. The mean and variance of X are 0 and 1 respectively. Finally,

$$Cov(X, I\{X > c\}) = EXI\{X > c\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_c^\infty xe^{-x^2/2} dx = \frac{e^{-c^2/2}}{\sqrt{2\pi}} = \phi(c).$$

   Since we have a random vector, the mean in vector notation is,

$$E \begin{pmatrix} X \\ I\{X > c\} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 - \Phi(c) \end{pmatrix}$$

   and the covariance in matrix notation is

$$Cov \begin{pmatrix} X \\ I\{X > c\} \end{pmatrix} = \begin{pmatrix} 1 & \phi(c) \\ \phi(c) & \Phi(c)(1 - \Phi(c)) \end{pmatrix}.$$

5. **Let T be an exponential random variable and conditional on T, let U be uniform on $[0, T]$. Find the unconditional mean and variance of U.**
Since $T \sim exp(\lambda)$, then $f_T(t) = \lambda e^{-\lambda t}$ for $t > 0$ and $E[T] = \lambda^{-1}$ and $Var(T) = \lambda^{-2}$.
Since $U|T \sim unif[0, T]$, then $E[U|T] = T/2$ and $Var(U|T) = T^2/12$.

   Using the Conditional Expectation Theorem,

$$\begin{aligned} E[U] &= E[E[U|T]] \\ &= E[T/2] = 1/(2\lambda). \end{aligned}$$

   Using the Conditional Variance Identity,

$$\begin{aligned} Var[U] &= E[Var(U|T)] + Var(E[U|T]) \\ &= E[T^2/12] + Var(T/2) \\ &= \frac{1}{12}(Var(T^2) + E[T]^2) + \frac{1}{4\lambda^2} \\ &= \frac{1}{12}(\frac{1}{\lambda^2} + \frac{1}{\lambda^2}) + \frac{1}{4\lambda^2} \\ &= \frac{5}{12\lambda^2}. \end{aligned}$$

6. **For any two random variables X and Y with finite variances, prove that**

   (a) $Cov(X, Y) = Cov(X, E[Y|X])$**.**

$$\begin{aligned} Cov(X, E[Y|X]) &= E[XE[Y|X]] - E[X]E[E[Y|X]] \\ &= E[E[XY|X]] - E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \\ &= Cov(X, Y), \end{aligned}$$

   which follows from the Double Expectation Theorem.

   (b) X **and** $Y - E[Y|X]$ **are uncorrelated.**

$$\begin{aligned} Corr(X, Y - E[Y|X]) &= \frac{Cov(X, Y - E[Y|X])}{\sqrt{Var(X)}\sqrt{Var(Y - E[Y|X])}} \\ &= \frac{Cov(X, Y) + Cov(X, -E[Y|X])}{\sqrt{Var(X)}\sqrt{Var(Y - E[Y|X])}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{Cov(X,Y) - Cov(X, E[Y|X])}{\sqrt{Var(X)}\sqrt{Var(Y - E[Y|X])}} \\
&= \frac{Cov(X,Y) - Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y - E[Y|X])}} \\
&= 0,
\end{aligned}
$$

where line 3 follows from part a) .