Homework 2

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1 BD 1.1.1

1. Example (a)

   (a) Here let X be a R.V. indicating the diameter of a pebble and $Y = log(X)$. The logarithm of the diameter is normally distributed, so:

   $$P_Y(Y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{y-\mu}{\sigma})^2}$$

   To find the distribution of X, we can do a simple transformation using $\frac{d}{dx}Y = \frac{1}{X}$ and see that

   $$P_X(X) = \frac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{log(x)-\mu}{\sigma})^2}$$

   (b) Pebble diameters must be $X > 0$, so $log(X) \in \mathbb{R}$. Because we are assuming $log(X) \sim \mathcal{N}(\mu, \sigma^2)$, $\mu \in \mathbb{R}$ and $\sigma > 0$.

   (c) This is a parametric model because we are assuming a specific distribution for the pebble diameters.

2. Example (b)

   (a) For this example we have the model $X_i = \mu + \epsilon_i$, for $1 \leq i \leq n$ and $\epsilon \sim \mathcal{N}(0.1, \sigma^2)$. Therefore

   $$X_i \sim \mathcal{N}(\mu + 0.1, \sigma^2)$$

   and

   $$P_X(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu+0.1}{\sigma})^2}$$

   (b) In this case the variance of the errors is known, so the parameter space is $\mu \in \mathbb{R}$.

   (c) This is also a parametric model because we are assuming a distribution for the errors.

3. Example (c)

   (a) This is similar to the model above, but this time $X_i = \mu + \epsilon_i$, for $1 \leq i \leq n$ and $\epsilon \sim \mathcal{N}(\theta, \sigma^2)$. Therefore

   $$X_i \sim \mathcal{N}(\mu + \theta, \sigma^2)$$

and

$$P_X(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu+\theta}{\sigma})^2}$$

(b) The variance of the errors is still known, but this time we are only able to estimate the parameter $\mu + \theta \in \mathbb{R}$ as the model is unidentifiable for μ or θ alone.

(c) This is still a parametic model because we assume a distribution of the errors.

4. Example (d)

   (a) Let $X =$ the number of eggs laid by an insect, which follows a Poisson distribution:

   $$P_X(X) = \frac{e^{-\lambda}\lambda^x}{x!}$$

   for $x = 0, 1, ...$ and $\lambda > 0$. If $Y =$ the number of eggs that hatch assuming each egg hatches with probability p, then Y follows a binomial distribution given the number of eggs laid:

   $$P_Y(Y|n = x) = \binom{x}{y} p^y (1-p)^{x-y}$$

   (b)

   $$\lambda > 0$$
   $$Y = 0, 1, ...$$
   $$0 \leq p \leq 1$$

   (c) This is also a parametric model because we are assuming distributions for X and $Y|X$.

1.1 BD 1.1.2

1. Problem 1.1.1(c): It is possible to estimate the parameter $\mu+\theta$, but it is not possible to estimate μ or θ separately because there are many possible values of μ and θ that would produce the same $\mu + \theta$. For example, $(\mu = 2, \theta = 2)$ and $(\mu = 3, \theta = 1)$.

2. The parameterization of 1.1.1(d) is identifiable because the entomologist is collecting the number of eggs laid by each insect, which allows for estimation of λ. They are also collecting the number of eggs hatching, which makes it possible to estimate p. See end of homework for additional details.

3. Unlike the case above, if the entomologist is only collecting data on the number of eggs hatched, the model would be unindentifiable. The current parameterization assumes that n is known, so that if the entomologist records for example 6 eggs hatching out of a total of 36 eggs laid, they can estimate $\hat{p} = \frac{1}{6}$. However, if the number of eggs is unknown, then 6 hatchings could imply that $\hat{p}_1 = \frac{6}{10}$, $\hat{p}_1 = \frac{6}{6}$, etc. because the denominator is unknown. Therefore, $P_{\theta_1} = P_{\theta_2}$ does not imply $\theta_1 = \theta_2$.

1.2 BD 1.2.7

Example 1.1.1: Let X represent the number of defective items in a random sampling inspection where $X(k) = k$ for $k = 0, 1, ..., n$. If θ represents the number of defective items in the population, then

$$p(X = k) = \frac{\binom{\theta}{k}\binom{N-\theta}{n-k}}{\binom{N}{n}}$$

Assume θ has a $\mathcal{B}(N, \pi_0)$ distribution:

$$\pi(\theta) = \binom{N}{\theta}\pi_0^\theta(1-\pi_0)^{N-\theta}$$

Then we have that the posterior distribution of θ given $X = k$:

$$\pi(\theta|X=k) = \frac{\pi(\theta)p(x|\theta)}{c} \propto \binom{N}{\theta}\pi_0^\theta(1-\pi_0)^{N-\theta}\frac{\binom{\theta}{k}\binom{N-\theta}{n-k}}{\binom{N}{n}}$$

$$\binom{N}{\theta}\pi_0^\theta(1-\pi_0)^{N-\theta}\frac{\binom{\theta}{k}\binom{N-\theta}{n-k}}{\binom{N}{n}}$$

Which equals:

$$\frac{N!}{\theta!(N-\theta)!}\frac{(N-\theta)!}{(n-k)!(N-\theta-(n-k))!}\frac{N!}{\theta!(N-\theta)!}\frac{n!(N-n)!}{N!}\frac{\theta!}{k!(t-k)!}\pi_0^\theta(1-\pi_0)^{N-\theta}$$

Several terms cancel, leaving us with:

$$\frac{n!(N-n)!}{k!(n-k)!(\theta-k)!(N-n-(\theta-k))!}\pi_0^\theta(1-\pi_0)^{N-\theta}$$

This can be written as:

$$\binom{n}{k}\binom{N-n}{\theta-k}\pi_0^\theta(1-\pi_0)^{N-\theta}$$

Next we have

$$c = \sum_{t=k}^{N-n+k}\pi(t)p(x|t) = \sum_{t=k}^{N-n+k}\binom{n}{k}\binom{N-n}{t-k}\pi_0^t(1-\pi_0)^{N-t}$$

Multiplying by $\frac{\pi_0^k(1-\pi_0)^{n-k}}{\pi_0^k(1-\pi_0)^{n-k}}$ results in:

$$\binom{n}{k}\pi_0^k(1-\pi_0)^{n-k}\sum_{t=k}^{N-n+k}\binom{N-n}{t-k}\pi_0^{t-k}(1-\pi_0)^{N-t-n+k}$$

The sum term here is the pmf of a $\mathcal{B}(N-n, t-k)$ distribution, so it sums to 1 and the posterior reduces to:

$$\pi(\theta|X=k) = \frac{\binom{n}{k}\binom{N-n}{\theta-k}\pi_0^\theta(1-\pi_0)^{N-\theta}}{\binom{n}{k}\pi_0^k(1-\pi_0)^{n-k}} = \binom{N-n}{\theta-k}\pi_0^{\theta-k}(1-\pi_0)^{N-n-\theta+k}$$

Using a simple change of variable $Z = \theta - k$, we see

$$\pi(Z|X=k) = \binom{N-n}{z}\pi_0^z(1-\pi_0)^{N-n-z}$$

which is a $\mathcal{B}(N-n, \pi_0)$ distribution.

1.3 BD 1.2.12

1. Given $X_1, ..., X_n$ iid $\mathcal{N}(\mu_0, \frac{1}{\theta})$ variables, the joint density $p(x|\theta)$ is:

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \theta^{\frac{1}{2}} e^{-\frac{1}{2}\theta(x_i-\mu_0)^2} = \sqrt{2\pi}^{-n} \theta^{\frac{1}{2}n} e^{-\frac{n\theta}{2}\sum_{i=1}^{n}(x_i-\mu_0)^2}$$

Letting $t = \sum_{i=1}^{n}(x_i - \mu_0)^2$, this density is proportional to:

$$\theta^{\frac{1}{2}n} e^{-\frac{1}{2}\theta t}$$

2. If $\pi(\theta) \propto \theta^{\frac{1}{2}(\lambda-2)} e^{-\frac{1}{2}\nu\theta}$, then the posterior distribution

$$\pi(\theta|x) \propto \theta^{\frac{1}{2}(\lambda-2)} e^{-\frac{1}{2}\nu\theta} \theta^{\frac{1}{2}n} e^{-\frac{1}{2}\theta t}$$

by 1.2.10. This can be simplified to

$$\theta^{\frac{1}{2}(n+\lambda-2)} e^{-\frac{1}{2}\theta(\nu+t)} = \theta^{\frac{n+\lambda}{2}-1} e^{-\frac{\theta(\nu+t)}{2}}$$

which is the kernel of a

$$Gamma(\frac{n+\lambda}{2}, \frac{2}{\nu+t}) = \frac{1}{\Gamma(\frac{n+\lambda}{2})(\frac{2}{\nu+t})^{\frac{n+\lambda}{2}}} \theta^{\frac{n+\lambda}{2}-1} e^{-\frac{\theta(\nu+t)}{2}}$$

Using a simple change of variables where $a = \theta(\nu + t)$, this becomes:

$$\frac{1}{\Gamma(\frac{n+\lambda}{2}) 2^{\frac{n+\lambda}{2}}} a^{\frac{n+\lambda}{2}-1} e^{-\frac{a}{2}}$$

So $a \sim \chi^2_{n+\lambda}$.

3. We can find the distribution of σ by plugging it into the posterior density with another change of variables $\sigma = \theta^{-\frac{1}{2}}$ and $\frac{d}{d\theta}\sigma = \frac{-2}{\sigma^3}$:

$$p(\sigma|x) = \frac{1}{\Gamma(\frac{n+\lambda}{2})(\frac{2}{\nu+t})^{\frac{n+\lambda}{2}}} (\frac{2}{\sigma^3})(\frac{1}{\sigma^2})^{\frac{n+\lambda}{2}-1} e^{-\frac{\nu+t}{2\sigma^2}}$$

1.4 BD 1.3.8

1. To show that s^2 is an unbiased estimator, we find its expected value:

$$E[s^2] = E[\frac{1}{n-1}\sum_{i=1}^{n}(X_i^2 - n\bar{X}^2)] = \frac{1}{n-1}(nE[X_1^2] - nE[\bar{X}^2])$$

because the X_i are sampled from the same population.

$$\frac{1}{n-1}(nE[X_1^2] - nE[\bar{X}^2]) = \frac{1}{n-1}(n(\sigma^2+\mu^2) - n(\frac{\sigma^2}{n}+\mu^2)) =$$

$$\frac{1}{n-1}(n\sigma^2 - \sigma^2) = \frac{\sigma^2(n-1)}{n-1}$$

This shows that $E[s^2] = \sigma^2$ and it is therefore an unbiased estimator.

2. Because s^2 is an unbiased estimator, the MSE is $Var(s^2)$. Using the fact that:

$$\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$$

it's obvious that

$$Var(\frac{(n-1)s^2}{\sigma^2}) = Var(\chi^2_{n-1}) = 2(n-1)$$

Rearranging this gives:

$$Var(s^2) = \frac{2\sigma^4(n-1)}{(n-1)^2} = \frac{2\sigma^4}{n-1}$$

3. If $\hat{\sigma_c^2} = c\sum_{i=1}^{n}(X_i - \bar{X})^2$, then $\hat{\sigma_c^2} = c(n-1)s^2$.

So, $Var(\hat{\sigma_c^2}) = c^2(n-1)2\sigma^4$. The bias of $\hat{\sigma_c^2}$ is $c(n-1)\sigma^2 - \sigma^2$, so:

$$MSE(\hat{\sigma_c^2}) = c^2(n-1)2\sigma^4 + (c(n-1)\sigma^2 - \sigma^2)^2$$

Which expands to:

$$c^2(n-1)2\sigma^4 + c^2(n-1)^2\sigma^4 - 2c(n-1)\sigma^4 + \sigma^4$$

Taking the derivative with respect to c and setting equal to 0 gives:

$$4c(n-1)\sigma^4 + 2c(n-1)^2\sigma^4 - 2(n-1)\sigma^4 = 0$$

Dividing both sides by $\sigma^4(n-1)$ results in:

$$4c + 2c(n-1) - 2 = 0$$

So

$$2c + c(n-1) = 1$$

and

$$c = \frac{1}{n+1}$$

To check that this is a minimum take the second derivative of the MSE:

$$\frac{d^2}{dc^2}MSE = 4(n-1)\sigma^4 + 2(n-1)^2\sigma^4 = 2\sigma^4(n-1)(n+1)$$

This function is positive for $n > 1$, so $c = \frac{1}{n+1}$ minimizes MSE.