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Homework #1

1. a) Let $\mathcal{A}$ denote a class of subsets of the sample space Ω, to which we can assign probabilities. $\mathcal{A}$ is taken to be a σ-field, so

$$A \in \mathcal{A},\ A^c \in \mathcal{A},\ \text{and } \emptyset \in \mathcal{A}.$$

Also, $P(\mathcal{A}) = P(\Omega) = 1$. A and A^c are pairwise disjoint by definition, so $P(A) + P(A^c) = 1$, which means $P(A^c) = 1 - P(A)$.

b) Take the identity $A \cup B = A \cup (B \cap A^c)$ (Casella & Berger 1.2.7). A and A^c are disjoint by definition, so A and $(B \cap A^c)$ are also disjoint. Therefore,

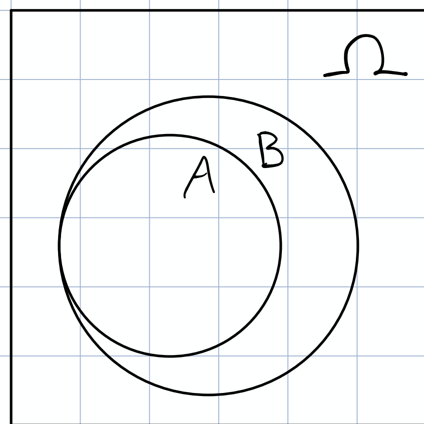
$$P(A \cup B) = P(A) + P(B \cap A^c) = P(A) + P(B) - P(A \cap B)$$

If $A \subset B$, $A \cap B = A$, so

$$P(A \cap B) = P(A) \quad \text{and}$$
$$P(B \cap A^c) = P(B) - P(A)$$

Because $P(A^c \cap B)$ must be ≥ 0, we see that $0 \le P(A^c \cap B) = P(B) - P(A)$ and

$\therefore \; P(A) \le P(B)$.



I know Venn diagrams aren't a formal proof, but they help me think about this stuff.

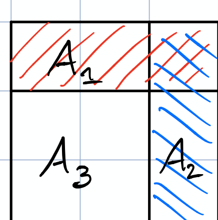
c) $P\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} P(A_n)$

First construct a disjoint set A_n^* such that $\bigcup_{n=1}^{\infty} A_n^* = \bigcup_{n=1}^{\infty} A_n$ with

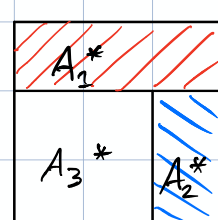
$A_1^* = A_1$ and for $i = 2, 3, \dots$ $A_i^* = A_i \setminus \bigcup_{j=1}^{i-1} A_j$

In other words, a set that covers the same space as $\bigcup_{n=1}^{\infty} A_n$ but is pairwise disjoint.

A_n :



A_n^* :



Therefore $P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} A_n^*\right) = \sum_{n=1}^{\infty} P(A_n^*)$

by the properties of probability models.
By construction, $P(A_n^*) \leq P(A_n)$

$$\therefore \quad \sum_{n=1}^{\infty} P(A_n^*) = P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$$

d) The approach for this is essentially the same as c) above, but applied to the complement. So,

$$P\left(\bigcup_{i=1}^{K} A_i^c\right) \leq \sum_{i=1}^{K} P(A_i^c) \quad \text{by A.2.5.}$$

From this we can rearrange each side:

$$\sum_{i=1}^{K} P(A_i^c) = \sum_{i=1}^{K} \left(1 - P(A_i)\right) = K - \sum_{i=1}^{K} P(A_i)$$

and

$$P\left(\bigcup_{i=1}^{K} A_i^c\right) = P\left(\bigcap_{i=1}^{K} A_i\right)^c = 1 - P\left(\bigcap_{i=1}^{K} A_i\right)$$

$$\therefore \quad 1 - P\left(\bigcap_{i=1}^{K} A_i\right) \leq K - \sum_{i=1}^{K} P(A_i^c)$$

$$P\left(\bigcap_{i=1}^{K} A_i\right) \geq 1 - \sum_{i=1}^{K} P(A_i^c)$$

2. a) $f_X(x) = \frac{x^2 e^{-x}}{2}$ for $x \in (0,\infty)$

$= \frac{x^2}{2e^x}$ which is a positive

Function because $x > 0$. Next integrate over x:

$$\int_0^\infty \frac{x^2 e^{-x}}{2} dx = \frac{1}{2} \int_0^\infty x^2 e^{-x}\, dx$$

$u = x^2$ $\quad$ $v = -e^{-x}$

$du = 2x\,dx$ $\quad$ $dv = e^{-x}\,dx$

$$\frac{1}{2}\int_0^\infty x^2 e^{-x}\,dx = \frac{-x^2 e^{-x}}{2}\Big|_0^\infty + \int_0^\infty x e^{-x}\,dx = \int_0^\infty x e^{-x} dx$$

$$\lim_{x\to\infty} \frac{-x^2}{e^{-x}} = 0$$

$$\int_0^\infty x e^{-x}\,dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx = \int_0^\infty e^{-x} dx$$

$$\lim_{x\to\infty} -xe^{-x} = 0$$

$$\int_0^{\infty} e^{-x}\,dx = -e^{-x}\Big|_0^{\infty} = 0-(-1) = 1$$

$$\therefore \int_0^{\infty} \frac{x^2e^{-x}}{2}\,dx = 1$$ and $f_X(x)$ is a density function.

b) $$F(x) = \int_0^x \frac{t^2e^{-t}}{2}\,dt = \frac{-t^2e^{-t}}{2}\Big|_0^x + \int_0^x te^{-t}\,dt$$

$$= \frac{-x^2e^{-x}}{2} + \int_0^x te^{-t}\,dt$$

$$\int_0^x te^{-t}\,dt = -te^{-t}\Big|_0^x + \int_0^x e^{-t}\,dt$$

$$= -xe^{-x} + -e^{-t}\Big|_0^x$$

$$-e^{-t}\Big|_0^x = -e^{-x} + 1$$

$$\therefore F(x) = \frac{-x^2e^{-x}}{2} - xe^{-x} - e^{-x} + 1$$

$$= \frac{-e^{-x}(x^2+2x+2)}{2}$$

c) $F(2) = \dfrac{-e^{-2}(4+4+2)}{2} = \dfrac{10}{2e^2} = \dfrac{5}{e^2} \simeq 0.323$

3. a) $\Gamma(\alpha) = \displaystyle\int_0^\infty x^{\alpha-1} e^{-x}\, dx \qquad \alpha > 0$

$$\Gamma(\alpha+1) = \int_0^\infty x^{(\alpha-1+1)} e^{-x}\, dx = \int_0^\infty x^\alpha e^{-x} dx$$

$u = x^\alpha \qquad v = -e^{-x}$

$du = \alpha x^{\alpha-1}\, dx \qquad dv = e^{-x} dx$

$$\int_0^\infty x^\alpha e^{-x} dx = -x^\alpha e^{-x} \Big|_0^\infty + \int_0^\infty e^{-x}\, \alpha x^{\alpha-1}\, dx$$

$$-x^\alpha e^{-x} \Big|_0^\infty = 0$$

$$\therefore \Gamma(\alpha+1) = \int_0^\infty e^{-x}\, \alpha x^{\alpha-1} dx = \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx =$$

$\alpha\, \Gamma(\alpha).$

$$\Gamma(\alpha) = \int^\infty x^{\alpha-1} e^{-x}\, dx = -x^{\alpha-1} e^{-x} \Big|^\infty + (\alpha-1) \int^\infty x^{\alpha-2} e^{-x} dx$$

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$$

$$\therefore \Gamma(\alpha+1) = (\alpha)(\alpha-1)\ldots\Gamma(1) = \alpha!$$

4) $X \sim N(0,1)$, Find the mean and covariance of $(X, I(x>c))$.

The mean of X is 0 because it is a standard normal distribution. $I(x>c)$ is a Bernoulli with $p = 1 - \Phi(c)$. Call this Function Y, so $Y \sim \text{Bernoulli}(p)$ and $E[Y] = p = 1 - \Phi(c)$

$$\text{Mean vector} = \{0, 1-\Phi(c)\}$$

The variance of X is 1, again because it's a standard normal-distributed RV.

The variance of Y is $p(1-p) =$

$$(1-\Phi(c))(1-(1-\Phi(c))) = \Phi(c)(1-\Phi(c))$$

The covariance of X and Y is:

$$E[XY] - E[X]E[Y] = E[XY] - 0 * E[Y]$$

$$E[XY] = E[X\,I(X>c)] = \int_X x\,I(x>c)\,P_x(x)$$

$$= E[X \mid X>c]$$

$$= \int_c^\infty x\,P_x(x)\,dx = \int_c^\infty \frac{x}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_c^\infty x e^{-x^2/2}\,dx = \frac{e^{-c^2/2}}{\sqrt{2\pi}}$$

Variance-covariance : $\begin{bmatrix} 1 & \frac{e^{-c^2/2}}{\sqrt{2\pi}} \\ \frac{e^{-c^2/2}}{\sqrt{2\pi}} & \left(\Phi(c)(1-\Phi(c))\right) \end{bmatrix}$

5. $T \sim$ exponential (β) $\quad f_x(x|\beta) = \frac{1}{\beta} e^{-x/\beta} \quad 0 \le x < \infty$

$\beta > 0$

$U \sim$ uniform $(0, T)$ $\quad f_x(x|a,b) = \frac{1}{b-a} \quad a \le x \le b$

By the conditional expectation theorem :

$$E[U] = E[E[U|T]]$$

$$E[U|T] = \frac{T+0}{2}$$

$$E[E[U|T]] = \frac{E[T]}{2} = \frac{\beta}{2}$$

By the conditional variance identity:

$$\begin{aligned}Var[U] &= E[Var[U|T]] + Var[E[U|T]] \\ &= E[T^2/12] + Var[T/2] \\ &= \frac{E[T^2]}{12} + \frac{Var[T]}{2}\end{aligned}$$

$$\begin{aligned}E[T^2] &= Var[T] + E[X]^2 \\ &= \beta^2 + \beta^2\end{aligned}$$

$$Var[U] = \frac{2\beta^2}{12} + \frac{\beta^2}{2} = \frac{\beta^2}{6} + \frac{3\beta^2}{6} = \frac{2\beta^2}{3}$$

6. a) $Cov(X, E[Y|X]) = E[X\,E[Y|X]] - E[X]E[E[Y|X]]$

$$\begin{aligned}&= E[XY|X] - E[X]E[Y] \\ &= E[XY] - E[X]E[Y] = Cov(X,Y)\end{aligned}$$

b) $Cov(X, Y - E[Y|X]) =$

$$E[X(Y - E[Y|X])] - E[X]\,E[Y - E[Y|X]]$$

$$= E[XY - XE[Y|X]] - E[X]E[Y - E[Y|X]]$$
$$= E[XY] - E[XE[Y|X]] - E[X](E[Y] - E[E[Y|X]])$$
$$= E[XY] - E[XY] - E[X](E[Y] - E[Y])$$
$$= 0 - 0 = 0$$

$$\text{Correlation}(X, Y - E[Y|X]) = \frac{\text{Cov}(X, Y - E[Y|X])}{\sigma_X \; \sigma_{Y - E[Y|X]}}$$

$$\therefore \; \text{Correlation}(X, Y - E[Y|X]) = 0$$