23-24. Categorical Predictors and Testing General Linear Hypotheses

Readings: Kleinbaum, Kupper, Nizam, and Rosenberg (KKNR): Ch. 12

SAS: PROC REG

Homework: Homework 10 due by midnight on December 3
Final Project due by midnight on December 7

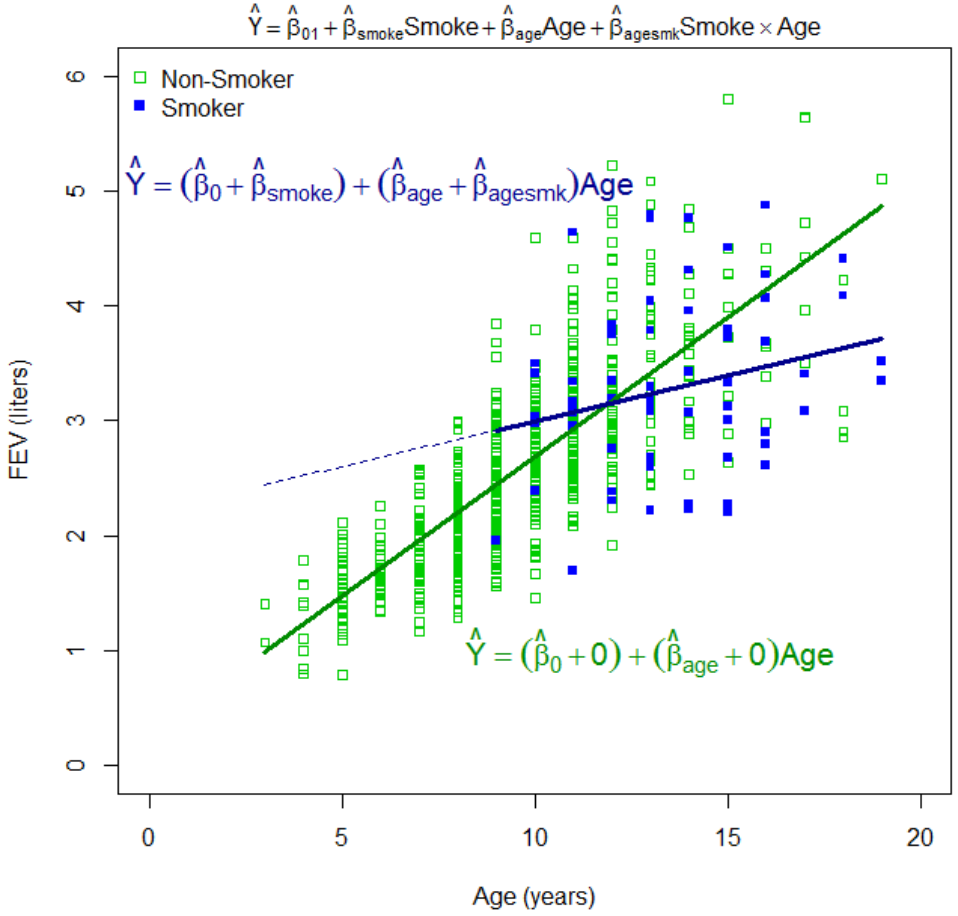
Overview

A) Re/Preview of Topics
B) Categorical Predictors with >2 Categories
C) Tests of General Linear Hypotheses
D) Linear Contrasts
E) Orthogonal Polynomials
F) Equivalence of Reference Cell Coding and Cell Means Coding for Orthogonal Contrasts
G) Equivalence of Reference Cell Coding and Cell Means Coding for Orthogonal Polynomials
H) ANOVA Table and Degrees of Freedom Summary

A. Review (Lecture 22)/Current (Lecture 23-24)/ Preview (Lecture 25)

Lecture 22:

- Effect Modification (Interactions):
  - $E[FEV_i] = \beta_0 + \beta_{age} Age_i + \beta_{smoke} Smoke_i + \beta_{agesmk} Age_i \times Smoke_i$
  - Allows for different slopes (FEV vs. age) for smokers and non-smokers
- MLE vs LSE (same β's, different variance)



Lectures 23-24:

- Categorical Predictors
  - Indicator variables
- Test of general linear hypothesis
- Linear contrasts
- Orthogonal polynomials

Lecture 25:

- Polynomial Regression: quadratic, cubic, quartic
- Other remedies for non-linearity

B. Categorical Explanatory Variables: More Than 2 Categories

Motivating Example: An investigator is interested in studying the relationship between infant birthweight (pounds) and smoking status of the mother during the first trimester. The investigator chose **five pregnant women from each of four smoking categories** (X) (*never*, *former*, *light*, and *heavy* smokers, with X coded 0, 1, 2, 3) from a larger study. The table below provides the birthweight (Y) of each baby and the average birthweight for each smoking category.

	Never Smokers (X=0)	*Former Smokers (X=1)*	*Light Smokers (X=2)*	*Heavy Smokers (X=3)*
	7.50	5.80	5.90	6.20
	6.20	7.30	6.20	6.80
	6.90	8.20	5.80	5.70
	7.40	7.10	4.70	4.90
	9.20	7.80	8.30	6.20
$\overline{Y}\|X_i$	7.44	7.24	6.18	5.96
$S^2{}_{Y\|X_i}$	1.233	0.833	1.727	0.503

```
proc import
datafile="~/birthweight_smoking_5per
group_dataset.csv"
     out=bwt5 /* name for data set
for SAS to reference */
     dbms=csv /* identify file as
csv */
     replace; /* overwrite BWT if
already present */
     getnames=yes; /* take first row
as column names from data */
run;
```

Potential Scientific Questions:

- Is there an association between smoking status and birthweight?
- Is there a difference in birthweight between never smokers and former smokers?
- Is there a difference in birthweight between non-smokers and current smokers?
- Is there an association between smoking and birthweight adjusting for weight of the mother?

To address these scientific questions in a regression model, you can create a different indicator variable or "dummy variable" for each of the categories:

$$\text{never}=\begin{cases}1 & \text{if smoke}=0.\\ 0 & \text{if smoke}=1,2,3.\end{cases} \qquad \text{former}=\begin{cases}1 & \text{if smoke}=1.\\ 0 & \text{if smoke}=0,2,3.\end{cases}$$

$$\text{light}=\begin{cases}1 & \text{if smoke}=2.\\ 0 & \text{if smoke}=0,1,3.\end{cases} \qquad \text{heavy}=\begin{cases}1 & \text{if smoke}=3.\\ 0 & \text{if smoke}=0,1,2.\end{cases}$$

Any *three* of these indicator variables can be used in the model if an intercept is included.

```
/* create dummy variables */
DATA bwt5;
     set bwt5;

     *** Create dummy variables ****;
     IF momsmoke = 'Never' THEN Never = 1; ELSE Never = 0;
     IF momsmoke = 'Former' THEN Former = 1; ELSE Former = 0;
     IF momsmoke = 'Light' THEN Light = 1; ELSE Light = 0;
     IF momsmoke = 'Heavy' THEN Heavy = 1; ELSE Heavy = 0;

     *** Create variable for two groups with current status ****;
     IF momsmoke = 'Never' THEN group = 0;
     IF momsmoke = 'Former' THEN group = 1;
     IF momsmoke = 'Light' THEN group = 2;
     IF momsmoke = 'Heavy' THEN group = 3;

     non = (group = 0 or group = 1);
     smoke = (group = 2 or group = 3);

RUN;
```

Notes on Using Indicator Variables

The reference category is the category associated with the indicator variable left out of the model (if specifying a model with an intercept).

Using never smoker as the reference category:

$$E[\text{birthweight}] = \beta_0 + \beta_{former} I_{former} + \beta_{light} I_{light} + \beta_{heavy} I_{heavy}$$

From this regression equation we can still determine the estimated mean for each group:

$$E[\text{birthweight}|\text{never}] = \beta_0 = \mu_{never}$$
$$E[\text{birthweight}|\text{former}] = \beta_0 + \beta_{former} = \mu_{former}$$
$$E[\text{birthweight}|\text{light}] = \beta_0 + \beta_{light} = \mu_{light}$$
$$E[\text{birthweight}|\text{heavy}] = \beta_0 + \beta_{heavy} = \mu_{heavy}$$

The β's can be used to estimate the difference between the mean of any two groups:

$$E[\text{birthweight}|\text{light}] - E[\text{birthweight}|\text{never}] = (\beta_0 + \beta_{light}) - \beta_0 = \beta_{light}$$
$$E[\text{birthweight}|\text{heavy}] - E[\text{birthweight}|\text{never}] = (\beta_0 + \beta_{heavy}) - \beta_0 = \beta_{heavy}$$
$$E[\text{birthweight}|\text{light}] - E[\text{birthweight}|\text{heavy}] = (\beta_0 + \beta_{light}) - (\beta_0 + \beta_{heavy}) = \beta_{light} - \beta_{heavy}$$

Notes on Using Indicator Variables (cont.)

From our regression equation, we can conduct the **Overall F-test** and make the direct connection to the one-way ANOVA:

$H_0: \beta_{former} = \beta_{light} = \beta_{heavy} = 0$ (Step 1: add β_0 to the H_0)

$H_0: \beta_{former} + \beta_0 = \beta_{light} + \beta_0 = \beta_{heavy} + \beta_0 = \beta_0$ (Step 2: substitute in definition for μ_x)

$H_0: \mu_{former} = \mu_{light} = \mu_{heavy} = \mu_{never}$

The intercept represents the level of the outcome in the reference category:

$$E[\text{birthweight}] = \beta_0 + \beta_{former} I_{former} + \beta_{light} I_{light} + \beta_{heavy} I_{heavy} \Rightarrow E[\text{birthweight} \mid \text{never}] = \beta_0$$

You can choose a different reference category by selecting which indicator variables are included in the model. For example, if we made heavy smoking mothers our reference category:

$$E[\text{birthweight}] = \beta^*{}_0 + \beta^*_{\boldsymbol{never}} I_{\boldsymbol{never}} + \beta^*_{former} I_{former} + \beta^*_{light} I_{light} \Rightarrow E[\text{birthweight} \mid \text{heavy}] = \beta^*_0$$

Notes on Using Indicator Variables – Testing A Category's Coefficient

The test of one category's coefficient is conceptually equivalent a *t*-test of that category against the reference category, *but* it isn't mathematically identical.

- Because we are using a "pooled variance" from all four categories/groups
- Not just the two groups we are comparing

The parameter estimates and some of the p-values for the parameter estimates will change if the reference category is changed.

The *F* test or partial *F* test can be used to test the overall significance of the categorical variable (this does not depend on the reference category).

- The *F* test and partial *F* test will **not** change if the reference category is changed.

Association between smoking and birthweight (reference group: never smokers)

```
/* REFERENCE CELL MODEL (reference group: never smokers) */
PROC REG DATA=bwt5;
    MODEL birthwt = former light heavy;
RUN;
```

$$E[\text{birthweight}] = \beta_0 + \beta_{former} I_{former} + \beta_{light} I_{light} + \beta_{heavy} I_{heavy}$$

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	3	**8.28550**	2.76183	2.57	0.0904
Error	16	17.18400	1.07400		
Corrected Total	19	25.46950			

SS explained by smoking status.

Parameter Estimates					
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Intercept	1	7.44000	0.46347	16.05	<.0001
Former	1	-0.20000	0.65544	-0.31	**0.7642**
Light	1	-1.26000	0.65544	-1.92	**0.0725**
Heavy	1	-1.48000	0.65544	-2.26	**0.0383**

$H_0: \beta_{former}=\beta_{light}=\beta_{heavy}=0$

or

$H_0: \beta_{former}+\beta_0=\beta_{light}+\beta_0=\beta_{heavy}+\beta_0 = \beta_0$

or

$H_0: \mu_{former}=\mu_{light}=\mu_{heavy}=\mu_{never}$

Global Hypotheses vs Multiple Comparisons

If you **reject** the null hypothesis H_0: $\mu_{former}=\mu_{light}=\mu_{heavy}=\mu_{never}$ (which we did **not**, p=0.0904)

AND you want to perform 6 additional tests, *then*:

Correct the alpha level, ***if*** you perform the additional tests (**Review Lectures 14-15**)

1. H_0: $\mu_{former} = \mu_{never}$	4. H_0: $\mu_{former} = \mu_{light}$
2. H_0: $\mu_{light} = \mu_{never}$	5. H_0: $\mu_{former} = \mu_{heavy}$
3. H_0: $\mu_{heavy} = \mu_{never}$	6. H_0: $\mu_{light} = \mu_{heavy}$

Parameter Estimates						
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**	
Intercept	1	7.44000	0.46347	16.05	<.0001	
Former	1	-0.20000	0.65544	-0.31	**0.7642**	$H_0: \mu_{former} = \mu_{never}$
Light	1	-1.26000	0.65544	-1.92	**0.0725**	$H_0: \mu_{light} = \mu_{never}$
Heavy	1	-1.48000	0.65544	-2.26	**0.0383**	$H_0: \mu_{heavy} = \mu_{never}$

This is the explanation as to why H_0: $\mu_{heavy} = \mu_{never}$ is rejected at alpha=0.05, *but* the overall F-test for H_0: $\mu_{former}=\mu_{light}=\mu_{heavy}=\mu_{never}$ is not significant.

Note: no multiple comparison correction is needed for the null hypothesis of all means are equal (H_0: $\mu_{former}=\mu_{light}=\mu_{heavy}=\mu_{never}$), because it is only 1 test.

Association between smoking and birthweight (reference group: never smokers)

```
PROC REG DATA=bwt5;
    MODEL birthwt = former light heavy / covb;
RUN;
```

$$E[\text{birthweight}] = \beta_0 + \beta_{former} I_{former} + \beta_{light} I_{light} + \beta_{heavy} I_{heavy}$$

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	3	**8.28550**	2.76183	2.57	0.0904
Error	16	17.18400	1.07400		
Corrected Total	19	25.46950			

SS explained by smoking status.

Parameter Estimates					
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Intercept	1	7.44000	0.46347	16.05	<.0001
Former	1	-0.20000	0.65544	-0.31	**0.7642**
Light	1	-1.26000	0.65544	-1.92	**0.0725**
Heavy	1	-1.48000	0.65544	-2.26	**0.0383**

H_0: $\beta_{former}=\beta_{light}=\beta_{heavy}=0$

or

H_0: $\beta_{former}+\beta_0=\beta_{light}+\beta_0=\beta_{heavy}+\beta_0=\beta_0$

or

H_0: $\mu_{former}=\mu_{light}=\mu_{heavy}=\mu_{never}$

Overall test: Does smoking status (the *entire set* of indicator variables) contribute significantly to the prediction of birthweight?

H_0: $\beta_{former}=\beta_{light}=\beta_{heavy}=0$; No, because F=2.57, p=0.0904.

REDUCED MODEL for Partial *F*

```
PROC REG DATA=bwt5;
    MODEL birthwt = ;
RUN;
```

$E[\text{birthweight}] = \beta_0 = \bar{Y}$

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	0	0	.	.	.
Error	19	25.46950	1.34050		
Corrected Total	19	25.46950			

Root MSE	1.15780	R-Square	0.0000
Dependent Mean	6.70500	Adj R-Sq	0.0000
Coeff Var	17.26771		

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	6.70500	0.25889	25.90	<.0001

Overall *F* test (using Partial *F* test):

$$\frac{[SS_{model}(full) - SS_{model}(reduced)]/k}{MS_{error}(full)} = \frac{[SS_{model}(full) - 0]/k}{MS_{error}(full)} = \frac{MS_{model}(full)}{MS_{error}(full)} = \frac{2.76183}{1.07400} = \mathbf{2.57}$$

Association between smoking and birthweight (reference group: heavy smokers)

```
PROC REG DATA=bwt5;
    MODEL birthwt = never former light;
RUN;
```

$$E[\text{birthweight}] = \beta_0 + \beta_{never} I_{never} + \beta_{former} I_{former} + \beta_{light} I_{light}$$

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	3	8.28550	2.76183	2.57	0.0904
Error	16	17.18400	1.07400		
Corrected Total	19	25.46950			

Parameter Estimates					
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Intercept	1	5.96000	0.46347	12.86	<.0001
Never	1	1.48000	0.65544	2.26	0.0383
Former	1	1.28000	0.65544	1.95	0.0686
Light	1	0.22000	0.65544	0.34	0.7415

$H_0: \beta_{never}=\beta_{former}=\beta_{light}=0$
or
$H_0: \mu_{former}=\mu_{light}=\mu_{heavy}=\mu_{never}$

The overall F test does not depend on the choice of reference group used in the model. The parameter estimates table *does* depend on the choice of indicator variables. Note that the ANOVA table is identical to the results on slide 10, but the parameter estimates table has changed.

Tests of Individual Coefficients (reference group: never smokers)

```
PROC REG DATA=bwt5;
    MODEL birthwt = former light heavy / covb;
RUN;
```

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	8.28550	2.76183	2.57	0.0904
Error	16	17.18400	1.07400		
Corrected Total	19	25.46950			

$H_0: \beta_{former}=\beta_{light}=\beta_{heavy}=0$

or

$H_0: \mu_{former}=\mu_{light}=\mu_{heavy}=\mu_{never}$

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	7.44000	0.46347	16.05	<.0001
Former	1	-0.20000	0.65544	-0.31	0.7642
Light	1	-1.26000	0.65544	-1.92	0.0725
Heavy	1	-1.48000	0.65544	-2.26	0.0383

Covariance of Estimates				
Variable	Intercept	Former	Light	Heavy
Intercept	0.2148	-0.2148	-0.2148	-0.2148
Former	-0.2148	0.4296	0.2148	0.2148
Light	-0.2148	0.2148	0.4296	0.2148
Heavy	-0.2148	0.2148	0.2148	0.4296

$$\boldsymbol{\Sigma} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\hat{\sigma}^2_{Y|X}$$

$$\hat{Y} = 7.44 + (-0.20) \times \text{former} + (-1.26) \times \text{light} + (-1.48) \times \text{heavy}$$

$$\hat{Y} = 7.44 + (-0.20) \times \text{former} + (-1.26) \times \text{light} + (-1.48) \times \text{heavy}$$

What is the interpretation of the intercept?

This is the expected mean birthweight for the reference group (non-smokers) or expected birthweight for an individual baby born to a non-smoking mother.

$\hat{Y}$ = 7.44 + (-0.20)×0 +(-1.26)×0 + (-1.48)×0 = 7.44 lbs

What is the expected birthweight for former smokers?

$\hat{Y}$ = 7.440 + (-0.20)×**1** +(-1.26)×0 + (-1.48)×0 = 7.24 lbs

What is the expected birthweight for heavy smokers?

$\hat{Y}$ = 7.44 + (-0.20)×0 +(-1.26)×0 + (-1.48)×**1** = 5.96 lbs

What is the difference in expected birthweight between heavy smokers and <u>never</u> smokers?

E[birthweight|heavy]-E[birthweight|never] = $\left(\beta_0 + \beta_{heavy}\right) - \beta_0 = \beta_{heavy}$
t = $\hat{\beta}_{heavy} / SE\left(\hat{\beta}_{heavy}\right)$ = -1.48/0.65544 = -2.26, p = 0.0383

Why isn't this mathematically the same as an independent samples t-test?

Because we are using a "pooled variance" from all four smoking categories/groups, not just the two groups we are comparing

Form of the Variance Covariance Matrix

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$Var(\hat{\beta}) = \hat{\sigma}^2_{Y|X}(X^T X)^{-1}$$

$$X^T X = \begin{bmatrix} 20 & 5 & 5 & 5 \\ 5 & 5 & 0 & 0 \\ 5 & 0 & 5 & 0 \\ 5 & 0 & 0 & 5 \end{bmatrix}$$

$$(X^T X)^{-1} = \begin{bmatrix} 0.2 & -0.2 & -0.2 & -0.2 \\ -0.2 & 0.4 & 0.2 & 0.2 \\ -0.2 & 0.2 & 0.4 & 0.2 \\ -0.2 & 0.2 & 0.2 & 0.4 \end{bmatrix}$$

$$E[\text{birthweight}] = \beta_0 + \beta_{former} I_{former} + \beta_{light} I_{light} + \beta_{heavy} I_{heavy}$$

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	7.44000	0.46347	16.05	<.0001
Former	1	-0.20000	0.65544	-0.31	0.7642
Light	1	-1.26000	0.65544	-1.92	0.0725
Heavy	1	-1.48000	0.65544	-2.26	0.0383

Covariance of Estimates				
Variable	Intercept	Former	Light	Heavy
Intercept	0.2148	-0.2148	-0.2148	-0.2148
Former	-0.2148	0.4296	0.2148	0.2148
Light	-0.2148	0.2148	0.4296	0.2148
Heavy	-0.2148	0.2148	0.2148	0.4296

What is the difference in average birthweight between heavy smokers and light smokers?

E[birthweight|heavy]-E[birthweight|light] = $\left(\beta_0 + \beta_{\text{heavy}}\right) - \left(\beta_0 + \beta_{\text{light}}\right) = \beta_{\text{heavy}} - \beta_{\text{light}}$

Then $\hat{\beta}_{\text{heavy}}$-$\hat{\beta}_{\text{light}}$= -1.48 – (-1.26) = -0.22

Is this difference significantly different from zero?

$$t = \frac{\hat{\beta}_{heavy} - \hat{\beta}_{light}}{SE\left(\hat{\beta}_{heavy} - \hat{\beta}_{light}\right)} = \frac{\hat{\beta}_{heavy} - \hat{\beta}_{light}}{\sqrt{Var\left(\hat{\beta}_{heavy}\right) + Var\left(\hat{\beta}_{light}\right) - 2Cov\left(\hat{\beta}_{heavy}, \hat{\beta}_{light}\right)}}$$

$$= \frac{-1.48 - (-1.26)}{\sqrt{\mathbf{0.4296} + \mathbf{0.4296} - 2 * \mathbf{0.2148}}} = \frac{-0.22}{\sqrt{0.4296}} = 0.336 \sim t_{16}; p = 0.742$$

Model treating Smoking Status as a *continuous* variable (no dummy codes):

```
PROC REG DATA=bwt5;
    MODEL birthwt = group;
RUN;
```

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	1	7.56250	7.56250	7.60	0.0130
Error	18	17.90700	0.99483		
Corrected Total	19	25.46950			

H_0: $\beta_{smkgroup} = 0$

Parameter Estimates					
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Intercept	1	7.53000	0.37320	20.18	<.0001
group	1	-0.55000	0.19948	-2.76	0.0130

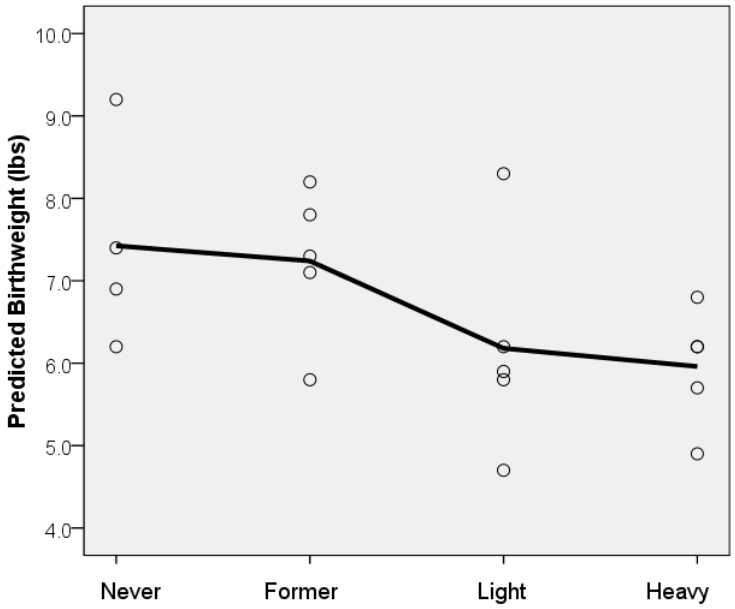
$$\hat{Y} = 7.53 + (-0.55) \times Group$$

$$\hat{Y} = 7.53 + (-0.55) \times Group$$

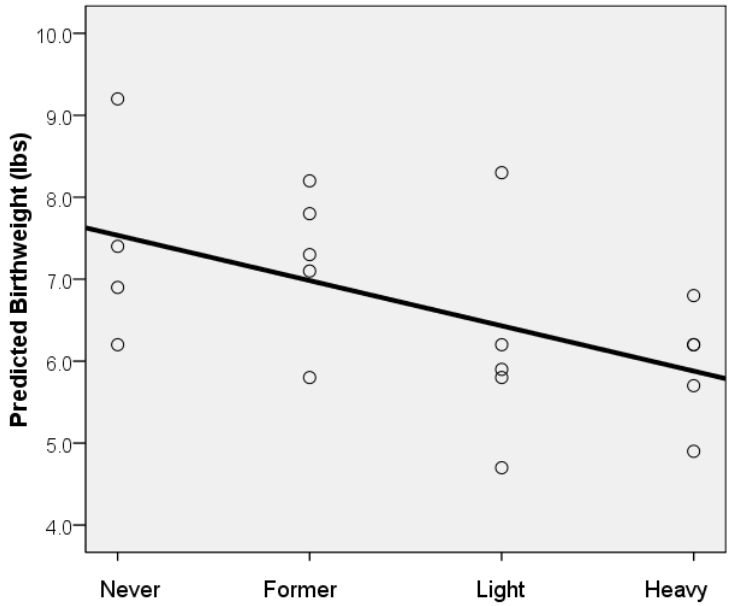
Interpretation of the intercept? Expected birthweight for a non-smoking mother is 7.53 lbs.
$E[\text{birthweight}] = \beta_0 + \beta_{group} Group \Rightarrow E[\text{birthweight} \mid \text{never}] = \beta_0$

Interpretation of $\hat{\beta}_1$? On average, birthweight decreases by 0.55 pounds for every category increase in smoking status (assumed to be the same increase between all adjacent categories).

Predicted Model using Dummy Coding (using 4 degrees of freedom)



Predicted Model using Continuous Variable (using 2 degrees of freedom)



C. Tests of General Linear Hypotheses

Tests on individual regression parameters and on subsets of parameters can be put into a more general framework that allows much more flexibility by the use of the general linear hypothesis:

$$H_0: \mathbf{c}\beta = \mathbf{d}$$
$$H_1: \mathbf{c}\beta \neq \mathbf{d}$$

The matrix **c** is an $r \times p^*$ matrix that is of rank r and $r \leq p^*$, where

- $p^* = p$ when an intercept is included in the model
- $p^* = p$-1 for a no intercept model.

In other words, we can postulate $r \leq p^*$ non-redundant and non-contradictory statements about the parameters.

We can use this framework to test a single parameter:

$$H_0: (0 \quad 0 \quad 1)\begin{pmatrix}\beta_0\\ \beta_1\\ \beta_2\end{pmatrix} = 0 \Rightarrow H_0: \beta_2 = 0$$

We can use this framework to compare two or more parameters:

$$H_0: (0 \quad 1 \quad -1)\begin{pmatrix}\beta_0\\ \beta_1\\ \beta_2\end{pmatrix} = 0 \Rightarrow H_0: \beta_1 - \beta_2 = 0 \text{ or } H_0: \beta_1 = \beta_2$$

We can use this framework for simultaneous hypothesis tests:

$$H_0: \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\beta_0\\ \beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow H_0: \begin{matrix}\beta_2 = 0\\ \beta_1 = 0\end{matrix}$$

$$H_0: \begin{pmatrix}0 & 1 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix}\begin{pmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow H_0: \begin{matrix}\beta_1 - \beta_2 = 0\\ \beta_1 - \beta_3 = 0\end{matrix} \text{ or } H_0: \begin{matrix}\beta_1 = \beta_2\\ \beta_1 = \beta_3\end{matrix}$$

The *F*-test can be used to test our general linear hypotheses:

$$F = \frac{[(\boldsymbol{c\hat{\beta}} - \boldsymbol{d})'(\boldsymbol{c}(\boldsymbol{X'X})^{-1}\boldsymbol{c}')^{-1}(\boldsymbol{c\hat{\beta}} - \boldsymbol{d})/r]}{\hat{\sigma}^2_{Y|X}} \sim F_{r,n-p-1}$$

where *r* is the number of linear combinations of parameters we wish to test (which is equal to the number of rows in **c**).

This reduces to our Partial *F* test for testing a group of variables, since:

$$(\boldsymbol{c\hat{\beta}})'(\boldsymbol{c}(\boldsymbol{X'X})^{-1}\boldsymbol{c}')^{-1}(\boldsymbol{c\hat{\beta}}) = SS_{model}(full) - SS_{model}(reduced).$$

And reduces to our *t* test for a single parameter (or test of a single linear hypothesis):

$$t = \frac{\boldsymbol{c\hat{\beta}}}{\sqrt{\boldsymbol{c}(\boldsymbol{X'X})^{-1}\boldsymbol{c}'\hat{\sigma}^2_{Y|X}}}.$$

Recall:
$(X`X)^{-1}\hat{\sigma}^2_{Y|X} = \Sigma$

In terms of the covariance matrix of $\boldsymbol{\beta}$, $\boldsymbol{\Sigma}$, the *F* and *t* tests become:

$$F = (\boldsymbol{c\hat{\beta}} - \boldsymbol{d})'(\boldsymbol{c}\sum\boldsymbol{c}')^{-1}(\boldsymbol{c\hat{\beta}} - \boldsymbol{d})/r \sim F_{r,n-1-p}$$

$$t = (\boldsymbol{c\hat{\beta}})\left(\sqrt{\boldsymbol{c}\sum\boldsymbol{c}'}\right)^{-1} \sim t_{n-1-p}$$

Tests of General Linear Hypotheses: Example

$$E[\text{birthweight}] = \beta_0 + \beta_{former} I_{former} + \beta_{light} I_{light} + \beta_{heavy} I_{heavy}$$

We want to test the hypothesis: H_0: $\beta_{heavy} = \beta_{light}$, or equivalently H_0: $\beta_{heavy} - \beta_{light} = 0$

Which can also be written as: $H_0: (0 \quad 0 \quad -1 \quad 1) \begin{pmatrix} \beta_0 \\ \beta_{former} \\ \beta_{light} \\ \beta_{heavy} \end{pmatrix} = 0$

$$\boldsymbol{b} = \widehat{\boldsymbol{\beta}} = \begin{pmatrix} 7.440 \\ -0.200 \\ -1.260 \\ -1.480 \end{pmatrix} \quad \boldsymbol{\Sigma} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\hat{\sigma}^2_{Y|X} = \begin{pmatrix} 0.2148 & -0.2148 & -0.2148 & -0.2148 \\ -0.2148 & 0.4296 & 0.2148 & 0.2148 \\ -0.2148 & 0.2148 & 0.4296 & 0.2148 \\ -0.2148 & 0.2148 & 0.2148 & 0.4296 \end{pmatrix}$$

$$t = (\boldsymbol{cb})\left(\sqrt{\boldsymbol{c\Sigma c}'}\right)^{-1} \quad t = (0 \quad 0 \quad -1 \quad 1) \begin{pmatrix} 7.440 \\ -0.200 \\ -1.260 \\ -1.480 \end{pmatrix} \left(\sqrt{(0 \quad 0 \quad -1 \quad 1) \begin{pmatrix} 0.2148 & -0.2148 & -0.2148 & -0.2148 \\ -0.2148 & 0.4296 & 0.2148 & 0.2148 \\ -0.2148 & 0.2148 & 0.4296 & 0.2148 \\ -0.2148 & 0.2148 & 0.2148 & 0.4296 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}} \right)^{-1}$$

$$t = \left(-1.260 - (-1.480)\right)\left(\sqrt{(0.4296+0.4296-2 \times 0.2148)}\right)^{-1} = \frac{0.22}{0.6554} = 0.336 \sim t_{16}$$

Testing Generalized Linear Hypotheses in SAS with Matrices using PROC IML

```
PROC IML;
 beta = {7.44000, -0.20000,  -1.26000,  -1.48000};

 sigma ={0.2148        -0.2148        -0.2148        -0.2148,
        -0.2148         0.4296         0.2148         0.2148,
        -0.2148         0.2148         0.4296         0.2148,
        -0.2148         0.2148         0.2148         0.4296};

PRINT "t statistic for b(heavy)=b(light)";
c = {0 0 -1 1};
t = (c*beta)*INV(SQRT(c*sigma*c`));
PRINT t;

PRINT "F statistic for b(former)=b(heavy)=b(light)= 0";
c = {0 1 0 0, 0 0 1 0, 0 0 0 1};
F = (c*beta)`*INV(c*sigma*c`)*(c*beta)/NROW(c);
PRINT F;
```

t
-0.335653

F
2.5715394

Testing General Linear Hypotheses Directly in SAS

```
PROC REG DATA=bwt5;
    MODEL birthwt = former light heavy;

    /* these 3 statement request the equivalent test */
    TEST light = heavy;
    TEST light-heavy;
    TEST light-heavy=0;

    /* these 3 statement request the equivalent test */
    TEST former=light=heavy=0;
    TEST former,light,heavy;
    test former=0, light=0, heavy=0;
RUN;
```

Test 1 Results for Dependent Variable birthwt

Source	DF	Mean Square	F Value	Pr > F
Numerator	1	0.12100	0.11	0.7415
Denominator	16	1.07400		

Test 4 Results for Dependent Variable birthwt

Source	DF	Mean Square	F Value	Pr > F
Numerator	3	2.76183	2.57	0.0904
Denominator	16	1.07400		

Note: Tests 2 and 3 results are identical to Test 1 and are not shown here. Similarly, Tests 5 and 6 are identical to Test 4 and are not shown here.

Compare Test 4's F- and p-value to ANOVA table on Slides 8/10.

D. Linear Contrasts

A linear contrast (L) is any linear combination of the parameters such that the linear coefficients add up to 0. Specifically,

$$L = \sum_{i=1}^{k} c_i \mu_i \quad \text{where} \quad \sum_{i=1}^{k} c_i = 0$$

Our contrast is estimated from our sample means and we can estimate its variability:

$$\hat{L} = \sum_{i=1}^{k} c_i \bar{y}_i \quad \text{and} \quad Var(\hat{L}) = \hat{\sigma}^2_{Y|X} \sum_{i=1}^{k} c_i^2 / n_i$$

Different coding schemes can be used:

- Reference Cell (Dummy codes)
- Cell Means (No Intercept)
- Effect Coding (Design coding)
- Orthogonal Polynomial Coding (Lectures 23-24)

Linear contrasts are most often used to test linear combinations of group means in a Cell Means Model (a model which includes a dummy code for each category/group and specifies no intercept in the model).

A *t*-statistic can be used to test a single linear contrast, and an *F*-statistic can be used for testing several linear contrasts simultaneously: $t = \frac{\hat{L}}{SE(\hat{L})}$

We can show Var(L) from the previous slide applying the various properties we have learned throughout the semester:

$$Var(\hat{L}) = Var\left(\sum_{i=1}^{k} c_i \bar{y}_i\right)$$

Independent means: Covariances are 0

$$= c_1^2\, var(\bar{y}_1) + c_2^2\, var(\bar{y}_2) + \ldots + c_k^2\, var(\bar{y}_k) + 2c_1c_2\, cov(\bar{y}_1, \bar{y}_2) + \ldots + 2c_{k-1}c_k\, cov(\bar{y}_{k-1}, \bar{y}_k)$$

$$= c_1^2\, var(\bar{y}_1) + c_2^2\, var(\bar{y}_2) + \ldots + c_k^2\, var(\bar{y}_k)$$

$$= \frac{c_1^2}{n_1} var(y_1) + \frac{c_2^2}{n_2} var(y_2) + \ldots + \frac{c_k^2}{n_k} var(y_k)$$

$$= \frac{c_1^2}{n_1} var(y|x=1) + \frac{c_2^2}{n_2} var(y|x=2) + \ldots + \frac{c_k^2}{n_k} var(y|x=3)$$

Assume all variances are equal

$$= var(y|x) \sum_{i=1}^{k} \left(c_i^2/n_i\right)$$

$$= \hat{\sigma}^2_{Y|X} \sum_{i=1}^{k} \left(c_i^2/n_i\right)$$

Linear Contrasts (cont.)

Orthogonal contrasts: Two contrasts, L_A and L_B, are orthogonal to one another if:

$$\sum_{i=1}^{k} \frac{c_{Ai}c_{Bi}}{n_i} = 0 \qquad \text{or} \qquad \sum_{i=1}^{k} c_{Ai}c_{Bi} = 0 \text{ (when the } n_i\text{'s are equal.)}$$

Orthogonality is a desirable property because the Model sums of squares can then be partitioned into statistically independent sums of squares, where the sums of squares for a given contrast, *L*, is given by:

$$SS(\hat{L}) = \frac{(\hat{L})^2}{\sum_{i=1}^{k} c_i^2/n_i}$$

$$\frac{SS(\hat{L})}{MSE} \sim F_{1,n-k}$$

For a cell means model, the number of orthogonal contrasts cannot exceed the group degrees of freedom (i.e., the number of groups minus 1).

Benefits of Orthogonal Contrasts

A priori (pre-planned) orthogonal contrasts are extremely powerful, because they do not need correction for multiple comparisons like *post-hoc* tests do.

This benefit for orthogonal contrasts is because we can partition our model sum of squares into the meaningful components associated with specific comparisons of interest (note, these are subjectively defined by the researchers and may be different for each person).

Assume we have defined *t* pairwise orthogonal contrasts, then our partition of the SS_{Model} is:

$$SS_{Model} = SS(\hat{L}_1) + \cdots SS(\hat{L}_t) + SS_{Remainder}$$

For our categorical variable context, if *t* is equal to the number of groups minus 1, then $SS_{Remainder}$ equals 0. Otherwise, if *t* is less than the number of groups minus 1 (perhaps we don't have more comparisons of interest), then

$$SS_{Remainder} = SS_{Model} - \left[SS(\hat{L}_1) + \cdots SS(\hat{L}_t)\right]$$

However, if the contrasts are not orthogonal, we cannot partition our SS_{Model} correctly and the results will be built on incorrect assumptions and, consequently, incorrect interpretations.

Cell Means Model Example: Mother's Smoking Status and Birthweight

```
PROC REG DATA=bwt5;
    MODEL birthwt = never former light heavy / noint;
RUN;
```

NOTE: No intercept in model. R-Square is redefined. It uses the uncorrected sum of squares and is not meaningful to compare to the R^2 from models which include an intercept.

$$E[birthweight] = \beta_{never}I_{never} + \beta_{former}I_{former} + \beta_{light}I_{light} + \beta_{heavy}I_{heavy}$$

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	4	907.42600	226.85650	211.23	<.0001
Error	16	17.18400	1.07400		
Uncorrected Total	20	924.61000			

H_0: $\beta_{never}=\beta_{former}=\beta_{light}=\beta_{heavy}=0$

or

H_0: $\mu_{never}=\mu_{former}=\mu_{light}=\mu_{heavy}=0$

Parameter Estimates					
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Never	1	7.44000	0.46347	16.05	<.0001
Former	1	7.24000	0.46347	15.62	<.0001
Light	1	6.18000	0.46347	13.33	<.0001
Heavy	1	5.96000	0.46347	12.86	<.0001

Group Means

Notes on Using Indicator Variables – No Intercept

Model **without** an intercept:

$$E[\text{birthweight}] = \beta_{never} I_{never} + \beta_{former} I_{former} + \beta_{light} I_{light} + \beta_{heavy} I_{heavy}$$

$$E[\text{birthweight} \mid \text{never}] = \beta_{never} = \mu_{never}$$
$$E[\text{birthweight} \mid \text{former}] = \beta_{former} = \mu_{former}$$
$$E[\text{birthweight} \mid \text{light}] = \beta_{light} = \mu_{light}$$
$$E[\text{birthweight} \mid \text{heavy}] = \beta_{heavy} = \mu_{heavy}$$

$$E[\text{birthweight} \mid \text{former}] - E[\text{birthweight} \mid \text{never}] = \beta_{former} - \beta_{never}$$
$$E[\text{birthweight} \mid \text{light}] - E[\text{birthweight} \mid \text{never}] = \beta_{light} - \beta_{never}$$
$$E[\text{birthweight} \mid \text{heavy}] - E[\text{birthweight} \mid \text{never}] = \beta_{heavy} - \beta_{never}$$

Overall F-test for the model without an intercept:

$$H_0: \beta_{never} = \beta_{former} = \beta_{light} = \beta_{heavy} = 0 \Rightarrow H_0: \mu_{former} = \mu_{light} = \mu_{heavy} = \mu_{never} = 0$$

Orthogonal Contrasts: Examples

For each set of three linear contrasts, what hypotheses are being tested? Are the contrasts orthogonal?

1. $$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \beta_{never} \\ \beta_{former} \\ \beta_{light} \\ \beta_{heavy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \beta_{never} - \beta_{former} \\ \beta_{never} - \beta_{light} \\ \beta_{never} - \beta_{heavy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Not Orthogonal [$1 \times 1 + (-1) \times 0 + 0 \times (-1) + 0 \times 0 = 1$ (row 1 and row2)]

2. $$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \beta_{never} \\ \beta_{former} \\ \beta_{light} \\ \beta_{heavy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \beta_{never} - \beta_{former} \\ \beta_{light} - \beta_{heavy} \\ \beta_{never} + \beta_{former} - \beta_{light} - \beta_{heavy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Orthogonal

2b. $$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0.5 & 0.5 & -0.5 & -0.5 \end{pmatrix} \begin{pmatrix} \beta_{never} \\ \beta_{former} \\ \beta_{light} \\ \beta_{heavy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \beta_{never} - \beta_{former} \\ \beta_{light} - \beta_{heavy} \\ \frac{\beta_{never}+\beta_{former}}{2} - \frac{\beta_{light}+\beta_{heavy}}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Orthogonal

Orthogonal Contrasts: Examples

3. $$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \beta_{never} \\ \beta_{former} \\ \beta_{light} \\ \beta_{heavy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \beta_{former} - \beta_{never} \\ \beta_{light} - \beta_{former} \\ \beta_{heavy} - \beta_{light} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Not Orthogonal

4. $$\begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \beta_{never} \\ \beta_{former} \\ \beta_{light} \\ \beta_{heavy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -3\beta_{never} + \beta_{former} + \beta_{light} + \beta_{heavy} \\ -2\beta_{former} + \beta_{light} + \beta_{heavy} \\ -\beta_{light} + \beta_{heavy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Orthogonal

5. $$\begin{pmatrix} -3 & -1 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} \beta_{never} \\ \beta_{former} \\ \beta_{light} \\ \beta_{heavy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -3\beta_{never} - 1\beta_{former} + 1\beta_{light} + 3\beta_{heavy} \\ 1\beta_{never} - 1\beta_{former} - 1\beta_{light} + 1\beta_{heavy} \\ -1\beta_{never} + 3\beta_{former} - 3\beta_{light} + 1\beta_{heavy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Orthogonal {Orthogonal Polynomials}

NOTE: These three contrasts are providing the same information as the reference cell model, comparing each smoking group to the never smokers (which we will see again in two slides).

$$1.\begin{pmatrix}1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 1 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\beta_{never}\\ \beta_{former}\\ \beta_{light}\\ \beta_{heavy}\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$ **Not orthogonal**

```
PROC REG DATA=bwt5;
    MODEL birthwt=never former light heavy/noint;
    TEST never-former; * row 1 ;
    TEST never-light;  * row 2 ;
    TEST never-heavy;  * row 3 ;
    TEST never-former, never-light, never-heavy;
RUN;
```

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	907.42600	226.85650	211.23	<.0001
Error	16	17.18400	1.07400		
Uncorrected Total	20	924.61000			

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Never	1	7.44000	0.46347	16.05	<.0001
Former	1	7.24000	0.46347	15.62	<.0001
Light	1	6.18000	0.46347	13.33	<.0001
Heavy	1	5.96000	0.46347	12.86	<.0001

ANOVA $H_0: \beta_{heavy} = \beta_{former} = \beta_{light} = \beta_{never} = 0$

$$E[\text{birthweight}] = \beta_{never}I_{never} + \beta_{former}I_{former} + \beta_{light}I_{light} + \beta_{heavy}I_{heavy}$$

Test 1. `never-former`, $H_0: \mu_{never} = \mu_{former}$

Test 1 Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	0.10000	0.09	0.7642
Denominator	16	1.07400		

SS(contrast)
= MS(contrast)*df
= MS(contrast)*1
= MS(contrast)

Test 2. `never-light`, $H_0: \mu_{never} = \mu_{light}$

Test 2 Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	3.96900	3.70	0.0725
Denominator	16	1.07400		

Compare to t tests of betas on next page

Test 3. `never-heavy`, $H_0: \mu_{never} = \mu_{heavy}$

Test 3 Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	5.47600	5.10	0.0383
Denominator	16	1.07400		

$$\sum SS(contrast) = 0.1 \times 1 + 3.969 \times 1 + 5.476 \times 1 = 9.545 \neq 8.2855$$

Test 4. `never-former, never-light, never-heavy`

$H_0: \beta_{never} - \beta_{former} = \beta_{never} - \beta_{light} = \beta_{never} - \beta_{heavy} = 0 \Rightarrow H_0: \mu_{never} = \mu_{former} = \mu_{light} = \mu_{heavy}$

Test 4 Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	3	2.76183	2.57	0.0904
Denominator	16	1.07400		

$$\sum SS(contrast) = 2.76183 \times 3 = 8.2855$$

8.2855 is the SS explained by smoking status.

Reference Cell Model Comparison with Cell Means Model for Orthogonal Contrast Example 1

```
PROC REG DATA=bwt5;
 MODEL weight = former light heavy;
RUN;
```

$$E[\text{birthweight}] = \beta_0 + \beta_{former} I_{former} + \beta_{light} I_{light} + \beta_{heavy} I_{heavy}$$

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	8.28550	2.76183	2.57	0.0904
Error	16	17.18400	1.07400		
Corrected Total	19	25.46950			

SS explained by smoking status. Compare to previous page.

H_0: $\beta_{former}=\beta_{light}=\beta_{heavy}=0$

or

H_0: $\mu_{former}=\mu_{light}=\mu_{heavy}=\mu_{never}$

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	7.44000	0.46347	16.05	<.0001
Former	1	-0.20000	0.65544	-0.31	**0.7642**
Light	1	-1.26000	0.65544	-1.92	**0.0725**
Heavy	1	-1.48000	0.65544	-2.26	**0.0383**

Compare to F tests on previous page.

$$2.\begin{pmatrix}1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 1 & 1 & -1 & -1\end{pmatrix}\begin{pmatrix}\beta_{never}\\ \beta_{former}\\ \beta_{light}\\ \beta_{heavy}\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$ **Orthogonal**

```
PROC REG DATA=bwt5;
    MODEL birthwt=never former light heavy/noint;
    TEST never-former; * row 1 ;
    TEST light-heavy;  * row 2 ;
    TEST never+former-light-heavy;  * row 3 ;
    TEST never-former, light-heavy, never+former-light-heavy;
RUN;
```

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	907.42600	226.85650	211.23	<.0001
Error	16	17.18400	1.07400		
Uncorrected Total	20	924.61000			

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Never	1	7.44000	0.46347	16.05	<.0001
Former	1	7.24000	0.46347	15.62	<.0001
Light	1	6.18000	0.46347	13.33	<.0001
Heavy	1	5.96000	0.46347	12.86	<.0001

$$E[\text{birthweight}] = \beta_{never}I_{never} + \beta_{former}I_{former} + \beta_{light}I_{light} + \beta_{heavy}I_{heavy}$$

Test 1. `never-former`, $H_0: \mu_{never} = \mu_{former}$

Test 1 Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	0.10000	0.09	0.7642
Denominator	16	1.07400		

SS(contrast)
= MS(contrast)*df
= MS(contrast)*1
= MS(contrast)

Test 2. `light-heavy`, $H_0: \mu_{light} = \mu_{heavy}$

Test 2 Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	0.12100	0.11	0.7415
Denominator	16	1.07400		

Test 3. `never+former-light-heavy`, $H_0: \mu_{never} + \mu_{former} = \mu_{light} + \mu_{heavy}$

Test 3 Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	8.06450	7.51	0.0145
Denominator	16	1.07400		

$$\sum SS(contrast) = 0.1 \times 1 + 0.121 \times 1 + 8.0645 \times 1 = 8.2855$$

Test 4. `never-former, light-heavy, never+former-light-heavy`

$H_0: \mu_{never} - \mu_{former} = \mu_{light} - \mu_{heavy} = \mu_{never} + \mu_{former} - \mu_{light} - \mu_{heavy}$

Test 4 Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	3	2.76183	2.57	0.0904
Denominator	16	1.07400		

$$\sum SS(contrast) = 2.76183 \times 3 = 8.2855$$

Brief Interlude: Why Test 4 is identical for the Non-Orthogonal Example Contrast 1 (page 34) and the Orthogonal Example Contrast 2 (page 37).

From page 21, note we can calculate the F-statistic from the matrices directly:

$$F = (\boldsymbol{c\hat{\beta} - d})'(\boldsymbol{c\sum c'})^{-1}(\boldsymbol{c\hat{\beta} - d})/r \sim F_{r,n-1-p}$$

Here we define $\hat{\beta} = \begin{pmatrix} 7.44 \\ 7.24 \\ 6.18 \\ 5.96 \end{pmatrix}$, $c_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$, $c_2 \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$, $d = \mathbf{0}$, and

$\boldsymbol{\Sigma} = \begin{pmatrix} 0.2148 & 0 & 0 & 0 \\ 0 & 0.2148 & 0 & 0 \\ 0 & 0 & 0.2148 & 0 \\ 0 & 0 & 0 & 0.2148 \end{pmatrix}$ [from COVB specified for cell means model]

For contrast 1: $F = \begin{pmatrix} 0.47 & 0.51 & 2.96 \end{pmatrix} \begin{pmatrix} 0.20 \\ 0.22 \\ 2.54 \end{pmatrix} / 3 = 2.57$

For contrast 2: $F = \begin{pmatrix} 0.47 & -2.44 & 5.91 \end{pmatrix} \begin{pmatrix} 0.20 \\ 0.22 \\ 1.38 \end{pmatrix} / 3 = 2.57$

Why does this happen? Because if we define the maximum number of independent contrasts (in our case this is the number of groups minus 1), any new contrast can be determined as some linear combination of the existing contrasts.

Calculate the value of a contrast by hand

3rd contrast from example 2: $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \beta_{never} \\ \beta_{former} \\ \beta_{light} \\ \beta_{heavy} \end{pmatrix}$

L = (1)×7.44 + (1)×7.24 + (-1)×6.18 + (-1)×5.96 = 2.540

$Var(L) = \hat{\sigma}^2_{Y|X} \sum_{i=1}^{k} c_i^2/n_i$ (from slide 25)

Var(L) = 1.074 × [$(1)^2/5$ + $(1)^2/5$ +$(-1)^2/5$ +$(-1)^2/5$] = 0.8592

t = 2.540 / √(0.8592) = 2.540/0.92693 = 2.7402 ~ t_{16}

F = 2.7402^2 = 7.509; p = 0.0145

Alternatively, we can directly calculate the F statistic from our formula on slide 27:

$$SS(\hat{L}) = \frac{(\hat{L})^2}{\sum_{i=1}^{k} c_i^2/n_i} = \frac{2.54^2}{[(1)^2/5 + (1)^2/5 +(-1)^2/5 +(-1)^2/5]} = \frac{2.54^2}{0.8} = 8.0645$$

$$\frac{SS(\hat{L})}{MSE} = \frac{8.0645}{1.074} = 7.509 \sim F_{1,16}$$

What is the null hypothesis being tested by the third contrast:

$$(1 \quad 1 \quad -1 \quad -1)\begin{pmatrix} \beta_{never} \\ \beta_{former} \\ \beta_{light} \\ \beta_{heavy} \end{pmatrix}$$

1. In terms of the βs? $\beta_{never} + \beta_{former} - \beta_{light} - \beta_{heavy} = 0$
   $\beta_{never} + \beta_{former} = \beta_{light} + \beta_{heavy}$
   $\frac{1}{2}(\beta_{never} + \beta_{former}) = \frac{1}{2}(\beta_{light} + \beta_{heavy})$

2. In terms of the 4 population means? $\mu_{never} + \mu_{former} - \mu_{light} - \mu_{heavy} = 0$
   $\frac{1}{2}(\mu_{never} + \mu_{former}) = \frac{1}{2}(\mu_{light} + \mu_{heavy})$

TEST: $\frac{1}{2}(\beta_{never} + \beta_{former}) = \frac{1}{2}(\beta_{light} + \beta_{heavy})$

$$L = (0.5)\times 7.44 + (0.5)\times 7.24 + (-0.5)\times 6.18 + (-0.5)\times 5.96 = 1.27 \text{ lbs}$$

$$\text{Var}(L) = 1.074 \times [(0.5)^2/5 + (0.5)^2/5 + (-0.5)^2/5 + (-0.5)^2/5] = 0.2148$$

$$t = 1.27 / \sqrt{(0.2148)} = 2.7402 \sim t_{16}$$

$p = 0.0145$ (equivalent to previous results)

3. Can the null hypothesis for this contrast be written in terms of 2 population means (non-smokers and current smokers)? What assumptions are being made?

$$\mu_{non} = \mu_{smoker}$$

We are assuming that the sample of non-smokers (never plus former) is representative of the population of non-smokers.

But since we the investigator didn't randomly select non-smokers (the investigator chose 5 never and 5 former smokers or 50% of each in our contrast) the observed average ($\bar{y}_{non}$) for the non-smokers probably isn't equal to the population mean.

Now test the linear contrast, assuming 25% of non-smokers in the population are former smokers and 50% of current smokers in the population are heavy smokers:

$$L = (0.75)\times 7.44 + (0.25)\times 7.24 + (-0.5)\times 6.18 + (-0.5)\times 5.96 = 1.32 \text{ lbs}$$

$$\text{Var}(L) = 1.074 \times [(-0.75)^2/5 + (-0.25)^2/5 + (0.5)^2/5 + (0.5)^2/5] = 1.074*0.225$$

$$t = 1.32 / \sqrt{(0.24165)} = 2.6852 \sim t_{16}$$

$$F = 2.6852^2 = 7.2104 \text{ and } p = 0.0163$$

```
PROC REG DATA=bwt5;
    MODEL birthwt=never former light heavy/noint;
    TEST .75*never + .25*former - .5*light - .5*heavy;
RUN;
```

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	907.42600	226.85650	211.23	<.0001
Error	16	17.18400	1.07400		
Uncorrected Total	20	924.61000			

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Never	1	7.44000	0.46347	16.05	<.0001
Former	1	7.24000	0.46347	15.62	<.0001
Light	1	6.18000	0.46347	13.33	<.0001
Heavy	1	5.96000	0.46347	12.86	<.0001

Test 1 Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	7.74400	7.21	0.0163
Denominator	16	1.07400		

What is the difference between fitting a cell means model and testing H_0: $\mu_{never} + \mu_{former} - \mu_{light} - \mu_{heavy} = 0$, and testing H_0: $\mu_{non} - \mu_{smoke} = 0$ by estimating the following regression model:

$$Y = \beta_0 + \beta_1 \times non + \varepsilon$$

where *non* is coded 1 for non-smokers (never or former) and 0 for current smokers (light or heavy)?

```
PROC REG DATA=bwt5;
    MODEL birthwt=non;
RUN;
```

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	1	8.06450	8.06450	8.34	0.0098
Error	18	17.40500	0.96694		
Corrected Total	19	25.46950			

NOTE: Different MSE than cell means model which fit all four group means. The SSE must stay the same or increase (it increased slightly in this example) when combining groups, but the degrees of freedom also increase, and thus the MSE in this example is actually smaller than the cell means model. In practice, the MSE usually increases when combining groups.

Parameter Estimates					
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Intercept	1	6.07000	0.31096	19.52	<.0001
non	1	1.27000	0.43976	2.89	0.0098

NOTE: Same point estimate as the linear contrast on page 38, but different SE due to the different MSE.

How can we replicate the F-Value and p-value from the contrast statement on slide 37 [(1 1 -1 -1) *replicated below*]?

Test 3 Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	8.06450	7.51	0.0145
Denominator	16	1.07400		

$$SE\left(\hat{\beta}_{non}\right) = \sqrt{\frac{\hat{\sigma}^2_{Y|X}}{\sum_{i=1}^{n}(X_i - \bar{X})^2}} = \sqrt{\frac{0.96694}{\sum_{i=1}^{n}(X_i - 0.5)^2}} = \sqrt{\frac{0.96694}{5}} = 0.43976$$

$$t = \frac{\hat{\beta}_{non}}{\sqrt{\frac{MSE(full)}{MSE(reduced)}\left[SE\left(\hat{\beta}_{non}\right)\right]^2}} = \frac{1.27}{\sqrt{\frac{1.074}{0.96694}(0.43976)^2}} = 2.74022$$

$$F = t^2 = 7.5088$$

Unequal Sample Sizes

Finally, note that if we had unequal sample sizes in our groups, then we would also get a different SS(L) and a different mean difference by testing:

$$H_0: 0.5\mu_{never} + 0.5\mu_{former} - 0.5\mu_{light} - 0.5\mu_{heavy} = 0$$

versus

$$H_0: \mu_{non} - \mu_{smoke} = 0$$

i	*Never*	*Former*	*Light*	*Heavy*
1	7.50	5.80	5.90	6.20
2	6.20	7.30	6.20	6.80
3	6.90	8.20	5.80	5.70
4	7.40	7.10	4.70	4.90
5	9.20	7.80	8.30	6.20
6	8.30		7.20	7.10
7	7.60		6.20	5.80
8				5.40
$\bar{Y}_j$=	7.586	7.240	6.329	6.013

What linear contrast would give us the same SS(L) as testing: $H_0: \mu_{non} - \mu_{smoke} = 0$ (although with a different MSE)?:

$$H_0: \frac{7}{12}\mu_{never} + \frac{5}{12}\mu_{former} - \frac{7}{15}\mu_{light} - \frac{8}{15}\mu_{heavy} = 0$$

E. Orthogonal Polynomials

Orthogonal polynomials are a new set of independent variables that are defined in terms of the simple polynomials (e.g., X, X^2, X^3; natural polynomials will be discussed in a future lecture) but have more complicated structures.

The orthogonal polynomial variables *contain exactly the same information* as the simple polynomial variables, but unlike the simple polynomial variables, the orthogonal polynomial variables are uncorrelated with each other. Therefore, they avoid the serious collinearity inherent in using natural polynomials.

As the order increases, computational accuracy may decease with the simple polynomial variables due to collinearity. However, the orthogonal polynomial variables are not impacted because they are uncorrelated. ***One of the main motivations for using orthogonal polynomial variables is to avoid the serious collinearity of simple polynomial variables in determining what higher order, if any, is needed.***

Because orthogonal polynomial variables contain the same information as the simple polynomial variables, the overall regression F-test and multiple R^2 values will be identical, even though the β's will be different and have different interpretations.

Because these special contrasts are still orthogonal, we can still partition the Model Sums of Squares into statistically independent sums of squares for each polynomial contrast (linear, quadratic, etc.) and take advantage of more powerful *a priori* tests.

The orthogonal polynomials can also be used to perform linear contrasts in a cell means model by defining the TEST statement using the contrast matrix values.

Table A7 of KKNR provides the orthogonal polynomial coefficients for equally spaced predictor values with the same number of replicates at each value.

Example:

k=4	**X**				
	1	**2**	**3**	**4**	**Σp_i^2**
Linear	-3	-1	1	3	20
Quadratic	1	-1	-1	1	4
Cubic	-1	3	-3	1	20

The assumption of equally spaced predictor values may not make intuitive sense in cases with nominal or ordinal groups (e.g., assuming the “space” between smoking statuses is equal). However, in contexts where groups are based on interval values (e.g., different dose levels being study in a trial) this assumption is more straightforward.

Example (Orthogonal Polynomial Contrasts, EQUAL N):

	Never Smokers (X=0)	*Former Smokers (X=1)*	*Light Smokers (X=2)*	*Heavy Smokers (X=3)*
	7.50	5.80	5.90	6.20
	6.20	7.30	6.20	6.80
	6.90	8.20	5.80	5.70
	7.40	7.10	4.70	4.90
	9.20	7.80	8.30	6.20
$\overline{Y}\|X$	7.44	7.24	6.18	5.96

```
PROC REG DATA=bwt5;
    MODEL birthwt=never former light heavy/noint;
    Overall:    TEST never=former=light=heavy;
    Linear:     TEST -3*never -1*former +1*light +3*heavy=0;
    Quadratic:  TEST  1*never -1*former -1*light +1*heavy=0;
    Cubic:      TEST -1*never +3*former -3*light +1*heavy=0;
RUN;
```

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	4	907.42600	226.85650	211.23	<.0001
Error	16	17.18400	1.07400		
Uncorrected Total	20	924.61000			

$H_0: \beta_{former}=\beta_{light}=\beta_{heavy}=\beta_{never}=0$

or

$H_0: \mu_{former}=\mu_{light}=\mu_{heavy}=\mu_{never}=0$

Root MSE	1.03634	**R-Square**	0.9814
Dependent Mean	6.70500	**Adj R-Sq**	0.9768
Coeff Var	15.45622		

Parameter Estimates					
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Never	1	7.44000	0.46347	16.05	<.0001
Former	1	7.24000	0.46347	15.62	<.0001
Light	1	6.18000	0.46347	13.33	<.0001
Heavy	1	5.96000	0.46347	12.86	<.0001

Test Overall Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	3	**2.76183**	2.57	0.0904
Denominator	16	1.07400		

3 x 2.76183 = 8.28550

Sums of Squares Due to Smoking

H_0: $\mu_{never}=\mu_{former}=\mu_{light}=\mu_{heavy}$

Test Linear Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	**7.56250**	7.04	0.0173
Denominator	16	1.07400		

Test Quadratic Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	**0.00050000**	0.00	0.9831
Denominator	16	1.07400		

Sum the linear, quadratic, and cubic contrast SS:

7.56250
+0.00050
+0.72250
Σ=8.28550

Test Cubic Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	**0.72250**	0.67	0.4242
Denominator	16	1.07400		

Example (Orthogonal Polynomial Contrasts Using Data Step):

```
data bwt5;
    set bwt5;
    IF group = 0 THEN DO;
        linear = -3;
        quad   =  1;
        cubic  = -1;
    END;
    IF group = 1 THEN DO;
        linear = -1;
        quad   = -1;
        cubic  =  3;
    END;
    IF group = 2 THEN DO;
        linear =  1;
        quad   = -1;
        cubic  = -3;
    END;
    IF group = 3 THEN DO;
        linear =  3;
        quad   =  1;
        cubic  =  1;
    END;
RUN;

PROC REG data=bwt5;
    MODEL birthwt = linear quad cubic;
RUN;
```

Group	Variable Coding		
	linear	quad	cubic
0=Non	-3	1	-1
1=Former	-1	-1	3
2=Light	1	-1	-3
3=Heavy	3	1	1

PROC REG OUTPUT:

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	3	8.28550	2.76183	2.57	0.0904
Error	16	17.18400	1.07400		
Corrected Total	19	25.46950			

Parameter Estimates					
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Intercept	1	6.70500	0.23173	28.93	<.0001
linear	1	-0.27500	0.10363	-2.65	0.0173
quad	1	-0.00500	0.23173	-0.02	0.9831
cubic	1	0.08500	0.10363	0.82	0.4242

Example (Orthogonal Polynomial Contrasts, UNEQUAL N)

Note: KKNR orthogonal polynomial contrasts are for equal N's:

-3(1) + -1(1) + 1(-1) + 3(1) = 0, but -3(1)/7 + -1(1)/5 + 1(-1)/7 + 3(1)/8 ≠ 0

```
PROC REG DATA=bwt;
    MODEL birthwt=never former light heavy/noint;
    Overall:   TEST never=former=light=heavy;
    Linear:    TEST -3*never -1*former +1*light +3*heavy=0;
    Quadratic: TEST  1*never -1*former -1*light +1*heavy=0;
    Cubic:     TEST -1*never +3*former -3*light +1*heavy=0;
RUN;
```

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	1234.44639	308.61160	349.60	<.0001
Error	23	20.30361	0.88277		
Uncorrected Total	27	1254.75000			

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Never	1	7.58571	0.35512	21.36	<.0001
Former	1	7.24000	0.42018	17.23	<.0001
Light	1	6.32857	0.35512	17.82	<.0001
Heavy	1	6.01250	0.33218	18.10	<.0001

Test Overall Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	3	**3.89090**	4.41	0.0137
Denominator	23	0.88277		

3 x 3.89090 = 11.6727

Sums of Squares
Due to Smoking

Test Linear Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	**11.51558**	13.04	0.0015
Denominator	23	0.88277		

Test Quadratic Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	**0.00144**	0.00	0.9681
Denominator	23	0.88277		

Sum the linear, quadratic, and cubic contrast SS:
11.51558
+0.00144
+0.40199
Σ=11.9190
≠ 11.6727

Test Cubic Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	**0.40199**	0.46	0.5065
Denominator	23	0.88277		

NOTE: The Contrast SS DO NOT add up to the Model SS due to unequal N's across groups.

Example (Orthogonal Polynomial Contrasts: Adjusting for unequal n)

```
PROC IML;
    N = {7,5,7,8};
    X = {0,1,2,3};
    op = ORPOL(X,3,N);
PRINT op;

DATA bwt;
    set bwt;
    IF group = 0 THEN DO;
        o1 = -0.263541;
        o2 = 0.1740137;
        o3 = -0.07801;
    END;
    IF group = 1 THEN DO;
        o1 = -0.098062;
        o2 = -0.214473;
        o3 = 0.3276404;
    END;
    IF group = 2 THEN DO;
        o1 = 0.0674175;
        o2 = -0.215651;
        o3 = -0.234029;
    END;
    IF group = 3 THEN DO;
        o1 =  0.2328967;
        o2 = 0.1704784;
        o3 = 0.0682584;
    END;
```

Group	Variable Coding		
	o1	o2	o3
0=Non	-.263541	0.174013	-0.07801
1=Former	-.098062	-.214473	0.327640
2=Light	0.067418	-.215651	-.234029
3=Heavy	0.232897	0.170479	0.068259

```
PROC REG DATA=bwt;
    MODEL birthwt = o1 o2 o3;
    Linear:    TEST o1;
    Quadratic: TEST o2;
    Cubic:     TEST o3;
RUN;
```

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	11.67269	3.89090	4.41	0.0137
Error	23	20.30361	0.88277		
Corrected Total	26	31.97630			

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	6.72963	0.18082	37.22	<.0001
o1	1	-3.35494	0.93956	-3.57	0.0016
o2	1	0.12287	0.93956	0.13	0.8971
o3	1	0.63402	0.93956	0.67	0.5065

Test Linear Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	11.25561	12.75	0.0016
Denominator	23	0.88277		

Test Quadratic Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	0.01510	0.02	0.8971
Denominator	23	0.88277		

Test Cubic Results for Dependent Variable birthwt				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	1	0.40198	0.46	0.5065
Denominator	23	0.88277		

Sum the linear, quadratic, and cubic contrast SS:

11.25561
+0.01510
+0.40198
Σ=11.67269

(matches slide 51)

F. Equivalence of Orthogonal Contrasts for Reference Cell and Cell Means Models

```
PROC REG DATA=birthsmk2;/* Reference Cell Coding Model*/
 MODEL weight = former light heavy;
/*Algebraic Translation of Orthogonal Contrast Matrix*/
 REFortha: TEST Intercept- Intercept-former = 0,
 Intercept+light - Intercept-heavy = 0,
 Intercept + Intercept+former - Intercept-light –
 Intercept-heavy = 0;
 REForths: TEST -former=0, light-heavy=0, former-light-
 heavy=0; /*Simplified Algebraic*/

 REForth1: TEST -former=0; /* Never vs. Former */
 REForth2: TEST light-heavy=0;
 REForth3: TEST former-light-heavy=0; /* Never+Former - (Light+Heavy) */
RUN;
```

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	8.28550	2.76183	2.57	0.0904
Error	16	17.18400	1.07400		
Corrected Total	19	25.46950			

Root MSE	1.03634	R-Square	0.3253
Dependent Mean	6.70500	Adj R-Sq	0.1988
Coeff Var	15.45622		

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	7.44000	0.46347	16.05	<.0001
Former	1	-0.20000	0.65544	-0.31	0.7642
Light	1	-1.26000	0.65544	-1.92	0.0725
Heavy	1	-1.48000	0.65544	-2.26	0.0383

```
PROC REG DATA= birthsmk2; /* Cell Means Coding Model */
 MODEL weight = never former light heavy / noint;

/* Orthogonal Contrast Matrix, 3 rows */
CMortha: TEST never-former=0, light-heavy=0, never+former-light-heavy=0;

 CMorth1: TEST never-former=0;
 CMorth2: TEST light-heavy=0;
 CMorth3: TEST never+former-light-heavy=0;
RUN;
```

NOTE: No intercept in model. R-Square is redefined.

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	907.42600	226.85650	211.23	<.0001
Error	16	17.18400	1.07400		
Uncorrected Total	20	924.61000			

Root MSE	1.03634	R-Square	0.9814
Dependent Mean	6.70500	Adj R-Sq	0.9768
Coeff Var	15.45622		

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Never	1	7.44000	0.46347	16.05	<.0001
Former	1	7.24000	0.46347	15.62	<.0001
Light	1	6.18000	0.46347	13.33	<.0001
Heavy	1	5.96000	0.46347	12.86	<.0001

Test REFortha Results for Dependent Variable weight

Test REFortha Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	3	2.76183	2.57	0.0904
Denominator	16	1.07400		

Test REForths Results for Dependent Variable weight

Test REForths Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	3	2.76183	2.57	0.0904
Denominator	16	1.07400		

Test REForth1 Results for Dependent Variable weight

Test REForth1 Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	0.10000	0.09	0.7642
Denominator	16	1.07400		

Test CMortha Results for Dependent Variable weight

Test CMortha Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	3	2.76183	2.57	0.0904
Denominator	16	1.07400		

Test CMorth1 Results for Dependent Variable weight

Test CMorth1 Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	0.10000	0.09	0.7642
Denominator	16	1.07400		

Test REForth2 Results for Dependent Variable weight

Test REForth2 Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	0.12100	0.11	0.7415
Denominator	16	1.07400		

Test REForth3 Results for Dependent Variable weight

Test REForth3 Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	8.06450	7.51	0.0145
Denominator	16	1.07400		

Test CMorth2 Results for Dependent Variable weight

Test CMorth2 Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	0.12100	0.11	0.7415
Denominator	16	1.07400		

Test CMorth3 Results for Dependent Variable weight

Test CMorth3 Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	8.06450	7.51	0.0145
Denominator	16	1.07400		

G. Equivalence of Orthogonal Polynomials for Reference Cell and Cell Means Models

```
PROC REG DATA= birthsmk2;/*Reference Cell Coding Model*/
 MODEL weight = linear quad cubic;



 OverOrth: TEST linear=0, quad=0, cubic=0;


RUN;
```

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	8.28550	2.76183	2.57	0.0904
Error	16	17.18400	1.07400		
Corrected Total	19	25.46950			

Root MSE	1.03634	R-Square	0.3253
Dependent Mean	6.70500	Adj R-Sq	0.1988
Coeff Var	15.45622		

```
PROC REG DATA= birthsmk2; /* Cell Means Coding Model */
 MODEL weight=never former light heavy/noint;
 Overall:   TEST never=former=light=heavy;
 Linear:    TEST -3*never -1*former +1*light +3*heavy=0;
 Quadratic: TEST  1*never -1*former -1*light +1*heavy=0;
 Cubic:     TEST -1*never +3*former -3*light +1*heavy=0;
 OverOrth:  TEST -3*never -1*former +1*light +3*heavy=0,
              1*never -1*former -1*light +1*heavy=0,
             -1*never +3*former -3*light +1*heavy=0;
RUN;
```

NOTE: No intercept in model. R-Square is redefined.

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	907.42600	226.85650	211.23	<.0001
Error	16	17.18400	1.07400		
Uncorrected Total	20	924.61000			

Root MSE	1.03634	R-Square	0.9814
Dependent Mean	6.70500	Adj R-Sq	0.9768
Coeff Var	15.45622		

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Never	1	7.44000	0.46347	16.05	<.0001
Former	1	7.24000	0.46347	15.62	<.0001
Light	1	6.18000	0.46347	13.33	<.0001
Heavy	1	5.96000	0.46347	12.86	<.0001

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	6.70500	0.23173	28.93	<.0001
linear	1	-0.27500	0.10363	-2.65	0.0173
quad	1	-0.00500	0.23173	-0.02	0.9831
cubic	1	0.08500	0.10363	0.82	0.4242

Test Overall Results for Dependent Variable weight

Test Overall Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	3	2.76183	2.57	0.0904
Denominator	16	1.07400		

Test Linear Results for Dependent Variable weight

Test Linear Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	7.56250	7.04	0.0173
Denominator	16	1.07400		

Test Quadratic Results for Dependent Variable weight

Test Quadratic Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	0.00050000	0.00	0.9831
Denominator	16	1.07400		

Test Cubic Results for Dependent Variable weight

Test Cubic Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	1	0.72250	0.67	0.4242
Denominator	16	1.07400		

Test OverOrth Results for Dependent Variable weight

Test OverOrth Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	3	2.76183	2.57	0.0904
Denominator	16	1.07400		

Test OverOrth Results for Dependent Variable weight

Test OverOrth Results for Dependent Variable birthwt				
Source	DF	Mean Square	F Value	Pr > F
Numerator	3	2.76183	2.57	0.0904
Denominator	16	1.07400		

H. ANOVA Table and Degrees of Freedom Summary

Now that we have two approaches for regression modeling (the reference cell/group model *with* an intercept **versus** the cell means model *without* an intercept) we can summarize the similarities and differences between the two approaches.

Let N be the overall sample size and assume that we do not necessarily have the same X's between models (X versus X^*). For the given regression equation, the ANOVA tables will be:

Reference Cell Model: $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \cdots + \hat{\beta}_p X_p$ (note there are **p+1** beta coefficients)

SAS ANOVA Table for Reference Cell Model (Includes Intercept)					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model (Explained Variability)	p	SS_{Model}	SS_{Model} / p	MS_{Model}/MS_{Error}	Compare to $F_{p,N-p-1}$ distribution
Error (Unexplained Variability)	N-p-1	SS_{Error}	SS_{Error} / (N-p-1)		
Corrected Total	N-1	SS_{Total}			

Cell Means Model: $\hat{Y}^* = \hat{\beta}_1 X_1^* + \cdots + \hat{\beta}_k X_k^*$ (note there are **k** beta coefficients)

SAS ANOVA Table for Cell Means Model (Does NOT Include Intercept)					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model (Explained Variability)	k	SS_{Model}	SS_{Model} / k	MS_{Model}/MS_{Error}	Compare to $F_{k,N-k}$ distribution
Error (Unexplained Variability)	N-k	SS_{Error}	SS_{Error} / (N-k)		
Uncorrected Total	N	SS_{Total}			

Note: If we fit a reference cell model with 3 dummy variables for a 4-category variable, the corresponding cell means model has 4 dummy variables. In this context p=3 for the reference cell model and k=p+1=4 for our cell means model.

Why the different degrees of freedom between the two modeling approaches?

First, let us revisit the type of variability described by our ANOVA table sources:

- ***Model:*** the variation from our observed outcome explained by the regression model (i.e., the variation explained by the X's we included in our model)
- ***Error:*** the variation from our observed outcome that is not explained by the regression model (i.e., it is unlikely that we have all relevant X's in our model and there will be some variation from our outcome that is explained by variables that are not included in the regression model)
- ***Total:*** the variation of our outcome (i.e., the difference between each observed outcome and the mean of the outcome)

The error degrees of freedom for the reference cell model is **N-p-1**, which accounts for the estimation of our intercept term plus all p slope beta coefficients.

However, when we do not estimate an intercept we are essentially fixing this value at 0 (i.e., $\beta_0 = 0$). By not estimating this term, we have gained 1 degree of freedom since we have one fewer parameter to estimate for the mean. However, it does not change the model degrees of freedom because there is no variable (X) associated with the intercept. The intercept is a constant that applies to every observation equally in our linear regression models.

How do we determine the degrees of freedom to use for the different approaches to statistical inference for regression that we have covered so far? (Lecture 19)

Overall F Test for the Entire Set of Independent Variables

If we are interested in testing the null hypothesis that the entire set of beta coefficients for our independent variables is equal to 0 (i.e., H_0: $\beta_1 = \cdots = \beta_p = 0$), the degrees of freedom for the numerator of our F distribution is p and for the denominator is:

- N-p-1 for the reference cell model ($F_{p,N-p-1}$)
- N-k for the cell means model ($F_{k,N-k}$)

Testing Addition of a Single Variable

If we are interested in testing if *one* particular independent variable adds significantly to the prediction of the outcome over and above that achieved by the other independent variables already present in the model, then we use:

- N-p-1 degrees of freedom for the reference cell model (t_{N-p-1})
- N-k for the cell means model (t_{N-k})

The reason we use N-p-1 or N-k degrees of freedom is because we need to account for the estimation of the other beta coefficients (including the intercept for the reference cell model).

But wait, why does the "Parameter Estimates" table provide a degrees of freedom column with the "wrong" degrees of freedom!?

Indeed, this is one of the potentially misleading aspects of the PROC REG output. Consider the parameter estimates table from Slide 8 (never smoker used as the reference category):

Parameter Estimates					
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Intercept	1	7.44000	0.46347	16.05	<.0001
Former	1	-0.20000	0.65544	-0.31	**0.7642**
Light	1	-1.26000	0.65544	-1.92	**0.0725**
Heavy	1	-1.48000	0.65544	-2.26	**0.0383**

Here the DF column does not correspond to the degrees of freedom used in the "Pr > |t|" column, but it represents the degrees of freedom contributed by that specific variable. Unless the model is not full rank (e.g., if we included an intercept *and* all 4 smoker categories), the DF column will always list 1.

Given that we used the dummy variable representation so that the included Former, Light, and Heavy all contribute 1 degree of freedom, we can also note that the overall categorical variable of Smoking Status contributes the number of groups minus 1 degree of freedom, or simply the sum of the DF in our table: 4-1 = 1+1+1 = 3.