Lecture 1

Course Overview

C&B Chapters:

1: Probability Theory
2: Transformations / Expectations
3: Common Families of Distributions
4: Multiple Random Variables
5: Properties of a Random Sample
6: Principles of Data Reduction
7: Point Estimation
8: Hypothesis Testing
9: Interval Estimation
10: Asymptotic Evaluations

Foundation — Gaining Tools (chapters 1–5); Applying Tools (chapters 6–10)

1st Semester (chapters 1–5); 2nd Semester (chapters 6–10)

§1.1 Set Theory

- Goal: Make inferences or draw conclusions about a population.
  Conduct experiment
  Identify all possible outcomes = Sample Space

Definition 1.1.1 The set, S, of all possible outcomes of a particular experiment is called the *sample space* for the experiment.

- Toss a coin $S = \{H, T\}$
- Post-surgery 30 day mortality $S = \{\text{Alive}, \text{Dead}\}$
- Systolic Blood Pressure (integer)
  <120 = 'Normal'
  >180 = Hypertensive Crisis
  <90 = 'Low'
  $S = \{0, 1, \ldots 90, \ldots 180, \ldots, 300^*\}$ * range of sphygmomanometer

countable (sample space put in 1-1 correspondence with subset integers)

- Experimental reaction time
  $S = (0, \infty)$

uncountable

↳ if round to nearest second
$S = \{0, 1, 2, 3, \ldots\}$
countable

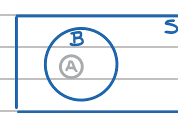
Once Sample Space defined consider possible outcomes:

Definition 1.1.2 An *event* is any collection of possible outcomes of an experiment, that is, any subset of S (including S itself).

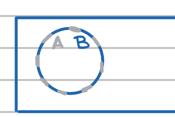
A = Event, subset of S.
Event A occurs if outcome of experiment is in set A.

- <u>Containment</u> $A \subset B \Leftrightarrow x \in A \Rightarrow x \in B$

$A \subset B \underset{\text{if and only if}}{\overset{\text{implies}}{\rightleftarrows}} \underbrace{x \text{ in } A \overset{\text{implies}}{\Rightarrow} x \text{ in } B}$ events A, B



- <u>Equality</u> $A = B \Leftrightarrow \underline{A \subset B \text{ and } B \subset A}$

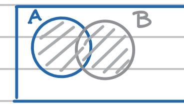


Operations

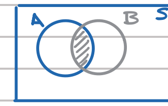
Union: The union of events (or sets) A and B written $A \cup B$, is set of elements that belongs to A or B or both.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

all elements x such that x in A or x in B



Intersection: The intersection of A and B, written $A \cap B$ or AB, is the set of elements that belong to both A and B.

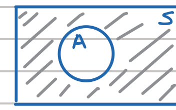


$$A \cap B = AB = \{x : x \in A \text{ and } x \in B\}$$

Complementation: The complement of A, A^c, is the set of all elements that are not in A:

$$A^c = \{x : x \notin A\}$$

all elements of x such that x not in A



Example: Design a clinical trial with equal numbers Blacks = B, Whites = W, Hispanics = H

The sample space is $S = \{B, W, H\}$

Possible events $A = \{B, W\}$
$B = \{W, H\}$

Then
$A \cup B = \{B, W, H\}$, $A \cap B = \{W\}$, $A^c = \{H\}$

Further
Since $A \cup B = S$ $(A \cup B)^c = \emptyset$ the null set.

The operations (union, intersection, complementation) can be combined to provide the following:

Theorem 1.1.4 *For any three events, A, B, and C, defined on a sample space S,*

a. Commutativity $A \cup B = B \cup A$,
$A \cap B = B \cap A$;

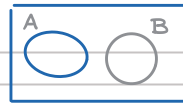
b. Associativity $A \cup (B \cup C) = (A \cup B) \cup C$,
$A \cap (B \cap C) = (A \cap B) \cap C$;

c. Distributive Laws $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

d. DeMorgan's Laws $(A \cup B)^c = A^c \cap B^c$,
$(A \cap B)^c = A^c \cup B^c$.

Theorem 1.1.4 can be visualized using Venn diagrams; although C&B note this is not a formal proof.

Definition 1.1.5 Two events A and B are *disjoint* (or *mutually exclusive*) if $A \cap B = \emptyset$. The events $A_1, A_2, \ldots$ are *pairwise disjoint* (or *mutually exclusive*) if $A_i \cap A_j = \emptyset$ for all $i \neq j$.



Definition 1.1.6 If $A_1, A_2, \ldots$ are pairwise disjoint and $\cup_{i=1}^{\infty} A_i = S$, then the collection $A_1, A_2, \ldots$ forms a *partition* of S.

S

A_1	A_2	A_3	A_4
A_5	A_6	A_7	A_8
A_9	A_{10}	A_{11}	A_{12}

§1.2 Basics of Probability Theory

§1.2.1 Axiomatic Foundations

Event A in sample space S, want to associate with A : $0 \leq Pr(A) \leq 1$
where $Pr(A)$ = probability of event A.

Consider subsets of S:

Definition 1.2.1 A collection of subsets of S is called a *sigma algebra* (or *Borel field*), denoted by $\mathcal{B}$, if it satisfies the following three properties:

a. $\emptyset \in \mathcal{B}$ (the empty set is an element of $\mathcal{B}$).

b. If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$ ($\mathcal{B}$ is closed under complementation).

c. If $A_1, A_2, \ldots \in \mathcal{B}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$ ($\mathcal{B}$ is closed under countable unions).

a) + b) $\Rightarrow$ if $\emptyset \in \mathcal{B}$ then $S \in \mathcal{B}$

b + c) $\Rightarrow$ if $A_1, A_2 \in \mathcal{B}$ then $A_1^c, A_2^c \in \mathcal{B}$ and $A_1^c \cup A_2^c \in \mathcal{B}$

by De Morgan's law $A_1^c \cup A_2^c = (A_1 \cap A_2)^c$

$(A_1^c \cup A_2^c)^c = (A_1 \cap A_2)$

Also closed under countable intersection

<u>Example</u>: "If S is finite or countable, then these technicalities really do not arise."

Define : $\mathcal{B} = \{$all subsets of S, including S itself$\}$.

If S has n elements 2^n sets in $\mathcal{B}$

If $S = \{1,2,3\}$ then $2^3 = 8$ sets :

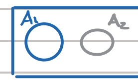
$\{1\}$ $\{1,2\}$ $\{1,2,3\}$
$\{2\}$ $\{1,3\}$ $\{\emptyset\}$
$\{3\}$ $\{2,3\}$

Probability Function (Axioms of Probability or Kolmogorov Axioms)

Definition 1.2.4 Given a sample space S and an associated sigma algebra $\mathcal{B}$, a *probability function* is a function P with domain $\mathcal{B}$ that satisfies

1. $P(A) \geq 0$ for all $A \in \mathcal{B}$.
2. $P(S) = 1$.
3. If $A_1, A_2, \ldots \in \mathcal{B}$ are pairwise disjoint, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Axiom of Finite Additivity
(Simplified Axiom-3)
A_1 & A_2 are disjoint



$Pr(A_1 \cup A_2) = P(A_1) + P(A_2)$

Theorem 1.2.6 *Let $S = \{s_1, \ldots, s_n\}$ be a finite set. Let $\mathcal{B}$ be any sigma algebra of subsets of S. Let $p_1, \ldots, p_n$ be nonnegative numbers that sum to 1. For any $A \in \mathcal{B}$, define $P(A)$ by*

$$P(A) = \sum_{\{i: s_i \in A\}} p_i.$$

(The sum over an empty set is defined to be 0.) Then P is a probability function on $\mathcal{B}$. This remains true if $S = \{s_1, s_2, \ldots\}$ is a countable set.

$S = \{1,2,3\}$ then $\mathcal{B} = \{1\} \{2\} \{3\} \{1,2\} \{1,3\} \{2,3\} \{\emptyset\} \{1,2,3\}$ (σ-algebra or Borel field)

$p_1 = Pr(1)$ $p_2 = Pr(2)$, $p_3 = P(3)$
If $A = \{2,3\}$ $P(A) = p_2 + p_3$

3-sided die (or dice).

§ 1.2.2 Calculus of Probabilities

From axioms of probability (def'n 1.2.4) we can derive:

Theorem 1.2.8 *If P is a probability function and A is any set in $\mathcal{B}$, then*

a. $P(\emptyset) = 0$, *where $\emptyset$ is the empty set;*
b. $P(A) \leq 1$;
c. $P(A^c) = 1 - P(A)$.

Proof:

(c) A & A^c partition sample space: $S = A \cup A^c$. $\rightarrow$ $P(A \cup A^c) = \underbrace{P(S) = 1}_{\text{Axiom-2}}$,

A & A^c are disjoint by Axiom-3 $P(A \cup A^c) = P(A) + P(A^c)$

$\therefore$ (Therefore) $P(A^c) = 1 - P(A)$

(b) Since $P(A^c) \geq 0$ (Axiom 1); from (c) $P(A) = 1 - P(A^c) \leq 1$

(a) Use $S = S \cup \emptyset$ (both S and $\emptyset$ in $\mathcal{B}$); Further S and $\emptyset$ are disjoint
$1 = P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset)$
$\therefore$ $P(\emptyset) = 0$

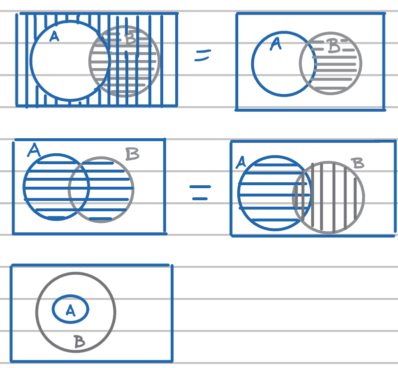
Theorem 1.2.9 *If P is a probability function and A and B are any sets in $\mathcal{B}$, then*

a. $P(B \cap A^c) = P(B) - P(A \cap B)$;

b. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;

c. *If* $A \subset B$, *then* $P(A) \leq P(B)$.

a) b) c)



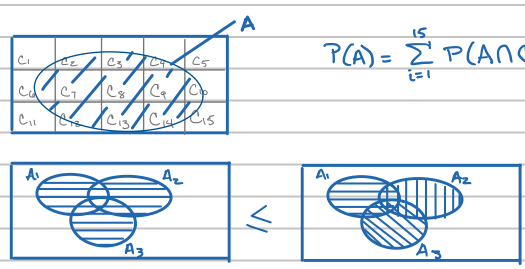
Venn diagrams not formal proof.

Theorem 1.2.11 *If P is a probability function, then*

a. $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$ *for any partition* $C_1, C_2, \ldots$;

b. $P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$ *for any sets* $A_1, A_2, \ldots$. *(Boole's Inequality)*

a) b)



$P(A) = \sum_{i=1}^{15} P(A \cap C_i)$ for partition $C_1, C_2 \ldots C_{15}$

$P(A_1 \cup A_2 \cup A_3) \leq P(A_1) + P(A_2) + P(A_3)$

§1.2.3 Counting

Example: Assume
- 6 marbles labeled 1,2,3,4,5,6 in a pottery jar
- Wish to draw 2 marbles from jar

Ordered, with replacement

6 possible for 1^{st} draw
6 possible for 2^{nd} draw $\Big\}$ $6 \times 6 = 6^2 = 36$

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Ordered without replacement

6 possible for 1^{st} draw
5 possible for 2^{nd} draw $\Big\}$ $6 \times 5 = 30$

-	1,2	1,3	1,4	1,5	1,6
2,1	-	2,3	2,4	2,5	2,6
3,1	3,2	-	3,4	3,5	3,6
4,1	4,2	4,3	-	4,5	4,6
5,1	5,2	5,3	5,4	-	5,6
6,1	6,2	6,3	6,4	6,5	-

doubles not possible

Unordered without replacement

6 possible for 1^{st} draw
5 possible for 2^{nd} draw
- half of outcomes are duplicates $\Big\}$ $\frac{6 \times 5}{2} = 15$

1,2	1,3	1,4	1,5	1,6
-	2,3	2,4	2,5	2,6
-	-	3,4	3,5	3,6
-	-	-	4,5	4,6
-	-	-	-	5,6
-	-	-	-	-

no doubles
no duplicates

Unordered with replacement

6 marbles, 2 draws
- no duplicates
- doubles ok $\Big\}$ $\frac{6 \times 5}{2} + 6 = 21$

1,1	1,2	1,3	1,4	1,5	1,6
-	2,2	2,3	2,4	2,5	2,6
-	-	3,3	3,4	3,5	3,6
-	-	-	4,4	4,5	4,6
-	-	-	-	5,5	5,6
-	-	-	-	-	6,6

doubles ok
no duplicates

Theorem 1.2.14 *If a job consists of k separate tasks, the ith of which can be done in n_i ways, $i = 1, \dots, k$, then the entire job can be done in $n_1 \times n_2 \times \cdots \times n_k$ ways.*

Ordered

Marble Example:

- Ordered with replacement: $6*6$
- Ordered without replacement: $6*5$

General case (n marbles, r draws):

- Ordered with replacement: n^r
- Ordered without replacement: $n*(n-1)*(n-2)*\dots*(n-r+1)$
  (assume $n \geq r$)

  $= \frac{n!}{(n-r)!}$ (next page)

Definition 1.2.16 For a positive integer n, $n!$ (read n factorial) is the product of all of the positive integers less than or equal to n. That is,

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

Furthermore, we define $0! = 1$.

Definition 1.2.17 For nonnegative integers n and r, where $n \geq r$, we define the symbol $\binom{n}{r}$, read n *choose* r, as

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

Unordered

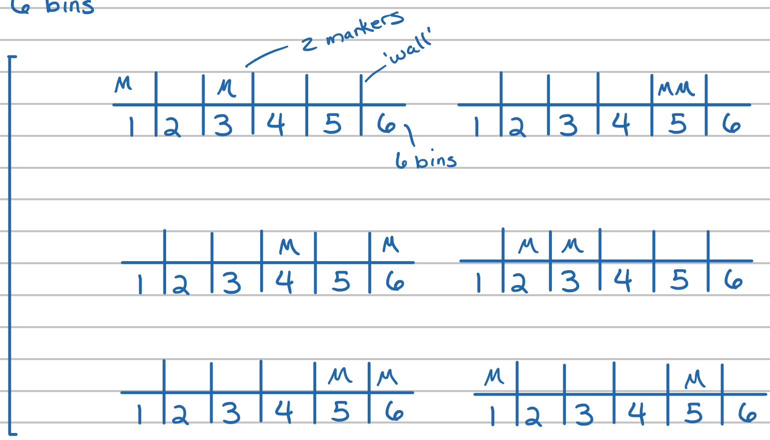
Marble example

Unordered without replacement: $\binom{6}{2} = 6!/(2!\times 4!) = 6\times 5/2$

Unordered with replacement: $\binom{6+2-1}{2} = \frac{7!}{2!\,5!} = \frac{7\cdot 6}{2} = 21$

└ arrange 2 markers (or chips) on 6 bins

Unordered with replace examples



Keep track of 5 walls (don't need outer walls) & 2 markers

$5+2 = 7$ objects to track

eliminate redundant orderings (unordered)
divide by $2!$ (Number markers or draws)
$5!$ (Number walls)

$$\binom{6+2-1}{2} = \frac{7!}{5!\,2!} = \frac{7\cdot 6}{2} = 21$$

General Case unordered (n marbles, r draws):

Unordered without replacement: $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$

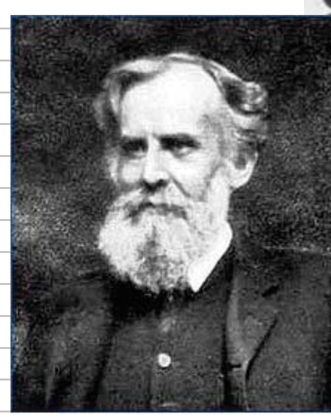
Unordered with replacement: $\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!\,(n-1)!}$

Table 1.2.1. *Number of possible arrangements of size r from n objects*

	Without replacement	With replacement
Ordered	$\frac{n!}{(n-r)!}$	n^r
Unordered*	$\binom{n}{r}$	$\binom{n+r-1}{r}$

* For this class unordered will be most important.

Andrey Kolmogorov
Russian Mathematician
(1903 - 1987)
Kolmogorov's Axioms



John Venn
English logician and philosopher
1834 - 1923

Cambridge England
Stained glass window at Gonville and Caius college commemorating Venn and Venn diagram.

Source: Wikipedia accessed 8/7/18