Longitudinal Homework 1

Tim Vigers

12 September 2019

2. A first-order autoregressive process

a. Expected value

First, expand the expected value:

$$E(\epsilon_t) = E(Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + ...)$$

Because ϕ is a constant and we assume the Zs to be independent, this becomes:

$$E(\epsilon_t) = E(Z_t) + \phi E(Z_{t-1}) + \phi^2 E(Z_{t-2}) + ...) = 0 + 0 + ... + 0$$

An infinite sum of zeroes is zero, so $E(\epsilon_t) = 0$

b. Covariance

$Cov(\epsilon_t, \epsilon_{t+h}) = E(\epsilon_t \epsilon_{t+h}) - E(\epsilon_t)E(\epsilon_{t+h}) = E(\epsilon_t \epsilon_{t+h}) E(\epsilon_t \epsilon_{t+h}) = E((Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + ...)(Z_{t+h} + \phi Z_{t+h-1} + \phi^2 Z_{t+h-2} + ...$

As long as the indices are different, the Z terms are independent, and the expected value of each Z is 0. So, using h = 1 you get:

$$E(\epsilon_t \epsilon_{t+1}) = E(Z_t Z_{t+1} + \phi Z_t Z_t + \phi^2 Z_t Z_{t-1} + \phi Z_{t-1} Z_{t+h} + ...) = \phi Z_t Z_t + \phi^3 Z_{t-1} Z_{t-1} + ... = \phi \sum_{i=0}^{\infty} (\phi^2)^i Z_{t-i}^2$$

And h=2 gives you:

$$E(\epsilon_t \epsilon_{t+2}) = \phi^2 Z_t Z_t + \phi^4 Z_{t-1} Z_{t-1} + ... = \phi^2 \sum_{i=0}^{\infty} (\phi^2)^i Z_{t-i}^2$$

And so on, giving you:

$$\phi^h \sum_{i=0}^{\infty} (\phi^2)^i Z_{t-i}^2$$

The Zs are identically distributed, so we only need to calculate $E(Z_t^2)$ and plug that value in:

$$Var(Z_t) = E(Z_t^2) - E(Z_t)^2 E(Z_t)^2 = 0 E(Z_t^2) = Var(Z_t) = \sigma^2$$

So, using the geometric series:

$$\phi^h \sum_{i=0}^{\infty} (\phi^2)^i Z_{t-i}^2 = \frac{\phi^h \sigma^2}{1 - \phi^2}$$

c. Correlation

$\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}\rho(\epsilon_t, \epsilon_{t+h}) = \dfrac{Cov(\epsilon_t, \epsilon_{t+h})}{\sqrt{Var(\epsilon)Var(\epsilon_{t+h})}}Var(\epsilon_t) = E(\epsilon_t^2) - E(\epsilon_t)^2 = E(\epsilon_t^2)E(\epsilon_t^2) = E((Z_t + \phi Z_{t-1} + \phi^2 Z_t$

$Var(\epsilon_{t+h})$ will be the same, so

$$\sqrt{Var(\epsilon)Var(\epsilon_{t+h})} = \frac{\sigma^2}{1-\phi^2}$$

.

Therefore:

$$\rho(\epsilon_t, \epsilon_{t+h}) = \frac{\phi^h \sigma^2}{1-\phi^2} * \frac{1-\phi^2}{\sigma^2} = \phi^h$$