Homework 3

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BD 1.4.5

Give an example in which the best linear predictor of Y given Z is a constant (has no predictive value) whereas the best predictor of Y given Z predicts Y perfectly.

The unique best linear predictor $\mu_L(Z) = E[Y] - \frac{Cov(Z,Y)}{Var(Z)}E[Z] + \frac{Cov(Z,Y)}{Var(Z)}Z$ and the best MSPE predictor of Y given Z is $E[Y|Z]$. First, take $Y = Z^2$ and calculate the covariance of Y and Z:

$$Cov(Z,Y) = E[ZY] - E[Z]E[Y] = E[Z^3] - E[Z]E[Z^2]$$

If we restrict Z so that $E[Z^3]$ and $E[Z]$ both equal 0, then:

$$\mu_L(Z) = E[Y] - 0 + 0 = E[Y]$$

The expected value of Y is a constant, so this satisfies the first part of the question. Next, check the best MSPE predictor of Y given Z:

$$\mu(Z) = E[Y|Z] = E[Z^2|Z] = Z^2 = Y$$

Thus, $\mu(Z)$ perfectly predicts Y and this satisfies the second part of the question.

BD 1.4.14

Let Z_1 and Z_2 be independent and have exponential distributions with density $\lambda e^{-\lambda z}$, $z > 0$. Define $Z = Z_2$ and $Y = Z_1 + Z_1 Z_2$. Find:

a)

The best MSPE predictor $E[Y|Z = z]$ of Y given $Z = z$:

First find $E[Y|Z = z] = E[Z_1 + Z_1 Z_2 | Z_2 = z]$. Because Z_1 and Z_2 are independent, this simplifies to $E[Z_1] + E[Z_1]E[Z_2|Z_2 = z]$, which is $\frac{1}{\lambda} + \frac{1}{\lambda}z = \frac{z+1}{\lambda}$.

b)

$E[E[Y|Z]]$:

First find $E[Y|Z] = E[Z_1 + Z_1 Z_2 | Z_2] = \frac{Z+1}{\lambda}$ (see above). This contains the random variable Z, so take the expectation again:

$$E[\frac{Z+1}{\lambda}] = \frac{E[Z]+1}{\lambda} = \frac{\frac{1}{\lambda}+1}{\lambda} = \frac{1}{\lambda^2} + \frac{1}{\lambda}$$

c)

$Var(E[Y|Z])$:

From above we know that $E[Y|Z] = \frac{Z+1}{\lambda}$. So, we find the variance of this using $Var(\frac{Z+1}{\lambda}) = \frac{Var(Z+1)}{\lambda^2}$. Because the variance of a RV plus a constant is the same as the variance of the RV, this simplifies to $\frac{Var(Z)}{\lambda^2} = \frac{\frac{1}{\lambda^2}}{\lambda^2} = \frac{1}{\lambda^4}$

d)

$Var(Y|Z = z)$:

First we write Y in terms of Z_1 and Z_2 to get $Var(Y|Z = z) = Var(Z_1 + Z_1 Z_2|Z = z)$. Then we can plug in $Z_2 = z$ to get $Var(Y|Z = z) = Var(Z_1 + Z_1 z)$ and rearrange and simplify to get $Var((z + 1)Z_1) = (z + 1)^2 Var(Z_1)$. So, $Var(Y|Z = z) = (\frac{z+1}{\lambda})^2$.

e)

$E[Var(Y|Z)]$:

From above we know that $Var(Y|Z) = Var(Z_1 + Z_1 Z|Z) = (Z + 1)^2 Var(Z_1) = \frac{(Z+1)^2}{\lambda^2}$. By expanding the numerator we get $E[Var(Y|Z)] = E[\frac{Z^2+2Z+1}{\lambda^2}]$. To find $E[Z^2]$ we rearrange the formula for variance to get $E[Z^2] = Var(Z) + E[Z]^2 = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{2}{\lambda^2}$. So, plugging this back in we get:

$$E[Var(Y|Z)] = E[\frac{Z^2 + 2Z + 1}{\lambda^2}] = \frac{E[Z^2] + E[2Z] + 1}{\lambda^2} = \frac{\frac{2}{\lambda^2} + \frac{2}{\lambda} + 1}{\lambda^2}$$

This could be further rearranged, but I kind of like this form.

f)

The best linear MSPE predictor of Y based on $Z = z$:

Given $Z = z$, $Cov(Z, Y) = 0$ because z is a constant. Therefore, the best linear predictor $\mu_L(Z) = E[Y|Z = z]$ (see equations in problem 1). So this is the same as part a).

BD 1.6.4

Which of the following families of distributions are exponential families? (Prove or disprove.)

b)

$p(x, \theta) = exp[-2log\theta + log(2x)]1[x \in (0, \theta)]$

This is not an exponential family because the indicator function depends on both x and θ, so the support depends on the parameter.

d)

$\mathcal{N}(\theta, \theta^2)$

See scanned pages for the remainder of these problems.