1.1.3

(a) $X_1, \dots, X_p$ ind. $X_i \sim N(\alpha_i + \nu, \sigma^2)$

$$\theta = (\alpha_1, \alpha_2, \dots, \alpha_p, \nu, \sigma^2)$$

$$\theta_1 = (\alpha_1 + \nu, \dots, \alpha_p + \nu, 0, \sigma^2) \neq \theta_2 = (\alpha_1, \dots, \alpha_p, \nu, \sigma^2)$$

but $P_{\theta_1} = P_{\theta_2} = N_p(\mu, \Sigma)$

$$\mu = \begin{pmatrix} \alpha_1 + \nu \\ \vdots \\ \alpha_p + \nu \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{pmatrix}$$

unidentifiable

(b) $\theta_1 = (\alpha_1, \dots, \alpha_p, \nu, \sigma^2)$

$\theta_2 = (\alpha_1', \dots, \alpha_p', \nu', \sigma'^2)$

$\theta_1 \neq \theta_2$

$P_{\theta_1} = N_p(\mu, \Sigma)$ $\quad P_{\theta_2} = N_p(\mu', \Sigma')$

$$\mu = \begin{pmatrix} \alpha_1 + \nu \\ \vdots \\ \alpha_p + \nu \end{pmatrix} \quad \mu' = \begin{pmatrix} \alpha_1' + \nu' \\ \vdots \\ \alpha_p' + \nu' \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sigma'^2 & & 0 \\ & \ddots & \\ 0 & & \sigma'^2 \end{pmatrix}$$

Suppose $P_{\theta_1} = P_{\theta_2}$

Then $\sigma^2 = \sigma'^2$

and $\sum_{i=1}^p \alpha_i + p\nu = \sum_{i=1}^p \alpha_i' + p\nu'$

or (since $\sum_{i=1}^p \alpha_i = \sum_{i=1}^p \alpha_i' = 0$) $\quad \nu = \nu'$

and therefore

$\alpha_1 = \alpha_1'$

$\vdots$

$\alpha_p = \alpha_p'$ $\quad$ yes, identifiable

(c) $Y - X$ $\quad X, Y$ independent

Видимо предполагается что σ^2 просто известное число, иначе это - не параметр задачи.

$\theta = (\mu_1, \mu_2)$

$$X \sim \mathcal{N}(\mu_1, \sigma^2)$$
$$Y \sim \mathcal{N}(\mu_2, \sigma^2)$$

$$Y - X \sim \mathcal{N}(\mu_2 - \mu_1, 2\sigma^2)$$

$$\theta_1 \neq \theta_2 \qquad \boxed{\begin{array}{c}\cancel{\mu_1 \neq \mu_1'} \\ \mu_2 \neq \mu_2'\end{array}} \qquad \text{unidentifiable}$$

μ_1, μ_2

$\mu_1' = \mu_1 + a$

$\mu_2' = \mu_2 + a$

(d) X_{ij}, $i = 1, \dots, p$; $j = 1, \dots, b$ independent

$$X_{ij} \sim \mathcal{N}(\mu_{ij}, \sigma^2)$$

$$\mu_{ij} = \nu + \alpha_i + \lambda_j$$

$$\cancel{\theta = (\alpha_1 + \dots + \alpha_p}$$

$$\theta = (\alpha_1, \dots, \alpha_p, \lambda_1, \dots, \lambda_b, \nu, \sigma^2)$$

$$(X_{11}, \dots, X_{pb}) \sim P_\theta$$

$$\cancel{\theta_1 = (\alpha_1 + \nu}$$

$$X_{11}, \dots, X_{pb} \sim \mathcal{N}(\mu, \Sigma)$$

$$\mu = \begin{pmatrix} \nu + \alpha_1 + \lambda_1 \\ \nu + \alpha_1 + \lambda_2 \\ \nu + \alpha_1 + \lambda_b \\ \nu + \alpha_2 + \lambda_1 \\ \dots \\ \nu + \alpha_p + \lambda_b \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{pmatrix}$$

$$\theta_1 = (\alpha_1 + \nu, \dots, \alpha_p + \nu, \lambda_1, \dots, \lambda_b, 0, \sigma^2)$$

$$\theta_2 = (\alpha_1, \dots, \alpha_p, \lambda_1, \dots, \lambda_b, \nu, \sigma^2)$$

$\theta_1 \neq \theta_2$ but $P_{\theta_1} = P_{\theta_2}$

unidentifiable

1.1.3 e)

Let $P_{\theta_1} = P_{\theta_2}$

$\theta_1 = (\alpha_1, \dots, \alpha_p, \lambda_1, \dots, \lambda_b, \vartheta, \sigma^2)$ (3)

$\theta_2 = (\alpha_1', \dots, \alpha_p', \lambda_1', \dots, \lambda_b', \vartheta', \sigma'^2)$

(obviously $\sigma^2 = \sigma'^2$)

Then

$$\left.\begin{aligned} \vartheta + \alpha_1 + \lambda_1 &= \vartheta' + \alpha_1' + \lambda_1' \\ \vartheta + \alpha_1 + \lambda_2 &= \vartheta' + \alpha_1' + \lambda_2' \\ &\dots \\ \vartheta + \alpha_1 + \lambda_b &= \vartheta' + \alpha_1' + \lambda_b' \end{aligned}\right\} \Rightarrow [\text{add}] \Rightarrow$$

$$\left.\begin{aligned} \Rightarrow b\vartheta + b\alpha_1 &= b\vartheta' + b\alpha_1' \\ \text{similarly} \\ b\vartheta + b\alpha_2 &= b\vartheta' + b\alpha_2' \\ &\dots \\ b\vartheta + b\alpha_p &+ b\vartheta' + b\alpha_p' \end{aligned}\right\} \Rightarrow [\text{add}] \Rightarrow$$

$$\Rightarrow \vartheta = \vartheta' \Rightarrow \alpha_i = \alpha_i', \; i = 1, \dots, p \Rightarrow$$

$$\Rightarrow \lambda_i = \lambda_j', \; j = 1, \dots, b \Rightarrow \theta_1 = \theta_2$$

identifiable

1.1.6 (a) yes (just by definition)

(b) yes (also, by definition)

(c) No, ~~the~~ distribution of Y is a mixture: $P(Y=1)>0$ and distribution of Y contains an absolutely continuous component.

(d) [As far as I understand a treatment response is either

$$0.1\theta, \quad 0.2\theta, \ldots \quad 0.9\theta$$

with probab $p(0.1), \ldots \qquad p(0.9)$

or

$$0.1+\theta, \quad 0.2+\theta, \ldots \quad 0.9+\theta$$

with probab]

In both cases – no

1.1.9 $\quad Y_i = \sum_{j=1}^{p} z_{ij}\beta_j + \varepsilon_i \;,\; \varepsilon_i \sim N(0,\sigma^2)$ iid $, 1 \le i \le n,$

$$\vec{z}_j = \begin{pmatrix} z_{1j} \\ \vdots \\ z_{nj} \end{pmatrix} \qquad Y = (Y_1, \dots, Y_n)$$

$\beta_1, \dots, \beta_p$

$$Y_i \sim N\left(\sum_{j=1}^{p} z_{ij}\beta_j \,,\, \sigma^2\right) \qquad \cancel{\theta = (\beta_1}$$

$$Y \sim N_n(\mu, \Sigma)$$

$$\mu = \begin{pmatrix} \sum_{j=1}^{p} z_{1j}\beta_j \\ \vdots \\ \sum_{j=1}^{p} z_{nj}\beta_j \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

(a) $\theta_1 = (\beta_1, \dots, \beta_p)$ $\qquad P_{\theta_1} = P_{\theta_2} \Rightarrow$

$\theta_2 = (\beta_1', \dots, \beta_p')$

$$\sum_{j=1}^{p} z_{1j}\beta_j = \sum_{j=1}^{p} z_{1j}\beta_j'$$

$$\dots$$

$$\sum_{j=1}^{p} z_{nj}\beta_j = \sum_{j=1}^{p} z_{nj}\beta_j'$$

$$\begin{cases} \sum_{j=1}^{p} (\beta_j - \beta_j') z_{1j} = 0 \\ \dots \\ \sum_{j=1}^{p} (\beta_j - \beta_j') z_{nj} = 0 \end{cases} \Rightarrow \begin{matrix} \beta_1 = \beta_1' \\ \beta_p = \beta_p' \end{matrix} \Rightarrow \theta_1 = \theta_2 \qquad \begin{pmatrix} z_{11} & \dots & z_{1p} \\ \vdots & & \\ z_{n1} & \dots & z_{np} \end{pmatrix}$$

$n < p$

(b) $n < p$.

1.1.11. $P(Y \le t) = P(\delta X \le t) \quad \forall t > 0$ or

$Y \sim G(t)$ $\quad$ ~~$G(t) = F(t/\delta) \quad \delta > 0, t > 0$~~

$X \sim F(t)$

[shift model

$G(t) = F(t - \Delta) \quad P(Y \le t) = P(X \le t - \Delta) = P(X + \Delta \le t)$]

(a) $\ln Y \sim G_e(t) \qquad \ln X \sim F_e(t)$

$G_e(t) = P(\ln Y \le t) = P(Y \le e^t) = P(\delta X \le e^t) =$

$= P(X \le \frac{1}{\delta} e^t) = P(X \le e^{t - \ln \delta}) = P(\ln X \le t - \ln \delta) =$

$= F_e(t - \ln \delta)$ ok

(b) $P(Y \le t) = P(X + \Delta \le t)$

$\Downarrow$

$(t>0)$ $P(e^Y \le t) = P(Y \le \ln t) = P(X \le \ln t - \Delta) =$

$= P(e^X \le e^{\ln t - \Delta}) = P(e^X \le t e^{-\Delta}) =$

$= P(e^\Delta e^X \le t)$ ok

(c) $P(Y' \le t) = P(Y^c \le t) = P(Y \le t^{\frac{1}{c}}) = P(\delta X \le t^{\frac{1}{c}}) =$

$= P(\delta^c X^c \le t) = P(\delta^c X' \le t)$

Yes with parameter δ^c

$P(\ln Y' \le t) = P(\ln Y^c \le t) = P(c \ln Y \le t) =$

$= P(\ln Y \le t/c) = P(Y \le e^{t/c}) = P(\delta X \le e^{t/c}) =$

$= P(X \le \frac{1}{\delta} e^{t/c}) = P(X^c \le (\frac{1}{\delta})^c e^t) =$

$= P(\ln X^c \le c \ln \frac{1}{\delta} + t) = P(\ln X' \le t - c \ln \delta)$

Yes with parameter $c \ln \delta$

1.2.2.

$$p(x|\theta) = \frac{2x}{\theta^2} I_{(0,\theta)}(x) = \frac{2x}{\theta^2} I_{(x,\infty)}(\theta)$$

(a) $\pi(\theta|x) \propto p(x|\theta)\pi(\theta) = \frac{2x}{\theta^2} I_{(x,\infty)}(\theta) I_{[0,1]}(\theta)$

$$= \frac{2x}{\theta^2} I_{(x,\infty)}(\theta) I_{[0,1]}(\theta) \propto \frac{1}{\theta^2} I_{(x,1]}(\theta)$$

$$\pi(\theta|x) = \frac{c}{\theta^2} I_{(x,1]}(\theta)$$

$$c\int_x^1 \frac{d\theta}{\theta^2} = 1 \qquad \int_x^1 \frac{d\theta}{\theta^2} = \frac{1}{x} - 1 = \frac{1-x}{x}$$

$$\pi(\theta|x) = \frac{x}{1-x} \frac{1}{\theta^2} I_{(x,1]}(\theta)$$

(b) $\pi(\theta|x) \propto \frac{2x}{\theta^2} I_{(x,\infty)}(\theta)\, 3\theta^2 I_{[0,1]}(\theta) \propto$

$$\propto I_{(x,1]}(\theta) \Rightarrow \pi(\theta|x) = \frac{1}{1-x} I_{(x,1]}(\theta)$$

(uniform)

(c) for (a) $E(\theta|x) = -\frac{x \ln x}{1-x}$ for (b) $E(\theta|x) = \frac{x+1}{2}$

(d) $p(x_1, \dots, x_n|\theta) = \prod_{i=1}^n p(x_i|\theta) =$

$$= 2^n \frac{x_1 \cdot \ldots \cdot x_n}{\theta^{2n}} \prod_{i=1}^n I_{(0,\theta)}(x_i) =$$

$$= 2^n \frac{x_1 \cdot \ldots \cdot x_n}{\theta^{2n}} I_{(0,\theta)}(\max\{x_i\}) \propto \frac{1}{\theta^{2n}} I_{(\max\{x_i\},\infty)}(\theta)$$

$$\pi(\theta|\vec{x}) = \frac{1}{\theta^{2n}} I_{(\max\{x_i\},1]}(\theta)$$

$$\text{const} = (2n-1)\frac{(\max\{x_i\})^{2n-1}}{1-(\max\{x_i\})^{2n-1}}$$

1.2.11. $p(x|\theta) = e^{-(x-\theta)} I_{(0,x)}(\theta)$

$\pi(\theta|x) \propto e^{+\theta} I_{(0,x)}(\theta)\, e^{-2\theta} = e^{-\theta} I_{(0,x)}(\theta)$

i.e.

$$\pi(\theta|x) = \frac{e^{-\theta}}{1-e^{-x}} I_{(0,x)}(\theta)$$

1.3.14

Let

$$\delta_1 = \begin{cases} \delta_{11} \text{ with probab. } p_1 \\ \vdots \\ \delta_{1n} \text{ —〃— } p_n \end{cases}$$

$$\delta_2 = \begin{cases} \delta_{21} \text{ —〃— } q_1 \\ \vdots \\ \delta_{2m} \text{ —〃— } q_m \end{cases}$$

Then $R(\theta, \delta_1) = \sum_{i=1}^{n} p_i R(\theta, \delta_{1i})$

$$R(\theta, \delta_2) = \sum_{j=1}^{m} q_j R(\theta, \delta_{2j})$$

Put

$$\delta_3 = \begin{cases} \delta_{11} \text{ —〃— } \alpha p_1 \\ \vdots \\ \delta_{1n} \text{ —〃— } \alpha p_n \\ \delta_{21} \text{ —〃— } (1-\alpha) q_1 \\ \vdots \\ \delta_{2m} \text{ —〃— } (1-\alpha) q_m \end{cases}$$

Then

$$R(\theta, \delta_3) = \sum_{i=1}^{n} \alpha p_i R(\theta, \delta_{1i}) + \sum_{j=1}^{m} (1-\alpha) q_j R(\theta, \delta_{2j}) =$$

$$= \alpha R(\theta, \delta_1) + (1-\alpha) R(\theta, \delta_2)$$

1.3.15.

Похоже, что задача неверна

Пример

$X \sim \mathrm{Pois}(\theta)$

$\Theta = \{1, 2\}$

$H_0 : \theta = \theta_0 = 1$

$H_1 : \theta = \theta_1 = 2$

$L(\theta, a)$

$\theta \backslash a$	a_0 H_0	a_1 H_1
~~$\theta = \theta_0 = 1$~~	~~0~~	~~0~~
$\theta = \theta_1 = 2$	1	0

Then

$R(\theta_0, \delta(X)) = 0$

$R(\theta_1, \delta(X)) = 1$

Consider $\delta_1(X) \equiv a_1$

$R(\theta_0, \delta_1(X)) = R(\theta_1, \delta_1(X)) = 0$

т. е.

$\delta_1(X)$ better than $\delta(X)$

and $\delta(X)$ is inadmissible

1.4.5 $\quad Z$-symmetric (or just $EZ = EZ^3 = 0$)

$$Y = Z^2$$

Then

$$\operatorname{cov}(Y, Z) = EYZ - EYEZ = EZ^3 - EZ^2 EZ = 0$$

Therefore (best linear predictor)

$$\mu_L(Z) = a + bZ = EY = \text{const}$$

$$a = EY - \frac{\operatorname{cov}(Y, Z)}{\operatorname{Var} Z} EZ = EY$$

$$b = \frac{\operatorname{cov}(Y, Z)}{\operatorname{Var} Z} = 0$$

~~i.e. μ~~ while (best predictor)

$$\mu(Z) = E(Y|Z) = E(Z^2|Z) = Z^2$$

[но вообще в условии следовало бы указать, что „привести „нетривиальный пример" (не const), потому что

$$Y = EZ \quad \forall Z$$

тоже удовлетворяет условию задачи, причём здесь

$$\mu_L(Z) = \mu(Z).$$

Можно, конечно, сказать, что нам не надо знать параметры в случае $Y = Z^2$ а только эту зависимость и наблюдение Z. Впрочем ...]

1.4.6

$$Z = \begin{cases} a & p \\ -a & 1-p \end{cases}$$

$Y = |Z|$

Then Z predicts Y perfectly (evidently) but $Y = \text{const}$ (a.s.), so Y and Z are independent and, therefore, [if X and Y are independent, then $E(X|Y) = EX$]

$$\mathrm{Var}(Z|Y) = E\left[(Z - E(Z|Y))^2 / Y\right] =$$

$$= E(Z - E(Z|Y))^2 = E(Z - EZ)^2 = \mathrm{Var}\, Z$$

[In fact, conditional distribution of Z ~~to~~ given Y coincides with unconditional].

1.5.3. $X_1, \ldots, X_n$

(a)

$$p(x;\theta) = \theta x^{\theta-1}, \quad 0<x<1, \quad \theta>0$$

$$p(x_1,\ldots,x_n;\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n (x_1 \cdots x_n)^{\theta-1}$$

$$T(X) = \prod_{i=1}^{n} X_i \quad \text{or} \quad T_1(X) = \sum_{i=1}^{n} \ln X_i$$

(b) $$p(x_1,\ldots,x_n;\theta) = \prod_{i=1}^{n} \theta a\, x_i^{a-1} e^{-\theta x_i^a} =$$

$$= \underbrace{\theta^n a^n \underbrace{\left(\prod_{i=1}^{n} x_i\right)^{a-1}}_{h(x)} e^{-\theta \sum x_i^a}}_{g(\sum x_i^a;\,\theta)}$$

$$T(X) = \sum_{i=1}^{n} X_i^a$$

(c)

$$p(x_1,\ldots,x_n;\theta) = \prod_{i=1}^{n} \frac{\theta a^\theta}{x_i^{\theta+1}} I_{[a,\infty)}(x_i) =$$

$$= \theta^n a^{n\theta} \frac{1}{(\prod x_i)^{\theta+1}} I_{[a,\infty)}(\min\{x_i\})$$

$$T(X) = \prod_{i=1}^{n} X_i$$

1.5.7.

$$X_1, \dots, X_n \sim \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}} I_{[\mu,\infty)}(x)$$

$$\theta = (\mu, \sigma)$$

$$p(x_1, \dots, x_n; \theta) = \prod_{i=1}^{n} \frac{1}{\sigma} e^{-\frac{(x_i-\mu)}{\sigma}} I_{[\mu,\infty)}(x_i) =$$

$$= \sigma^{-n} e^{-\frac{1}{\sigma}\sum x_i} e^{\frac{\mu}{\sigma}n} I_{[\mu,\infty)}(\min\{x_i\})$$

(a) σ - fixed, then

$$g(\min\{x_i\}, \mu) = e^{\frac{\mu n}{\sigma}} I_{[\mu,\infty)}(\min\{x_i\})$$

ok

(b) evidently $\sum x_i$

(c) $(\min\{X_i\}, \sum X_i)$

1.5.9.

$X_1, \dots, X_n \sim a(\theta)\, h(x)\, I_{(\theta_1,\theta_2)}(x)$

$p(x_1,\dots,x_n;\theta) = [a(\theta)]^n \prod_{i=1}^{n} h(x_i) \prod_{i=1}^{n} I_{(\theta_1,\theta_2)}(x_i) =$

$= \left[I_{(\theta_1,\theta_2)}(x) = I_{(\theta_1,\infty)}(x)\, I_{(-\infty,\theta_2)}(x) \right] =$

$= [a(\theta)]^n \prod_{i=1}^{n} h(x_i) \prod_{i=1}^{n} I_{(\theta_1,\infty)}(x_i) \prod_{i=1}^{n} I_{(-\infty,\theta_2)}(x_i) =$

$= [a(\theta)]^n\, I_{(\theta_1,\infty)}(\min\{x_i\})\, I_{(\theta_2,\infty)}(\max\{x_i\}) \cdot \prod_{i=1}^{n} h(x_i)$

Therefore

$T(X) = (\min\{X_1,\dots,X_n\}, \max\{X_1,\dots,X_n\})$

1.5.15 $T(x) = (x_{(1)}, \dots, x_{(n)})$

~~Criterion~~

$x = (x_1, \dots, x_n)$

$$p(x;\theta) = \prod_{i=1}^{n} \frac{1}{\pi(1+(x_i-\theta)^2)}$$

Criterion: $T(X)$ is ~~suff~~ minimal sufficient iff $\frac{p(x;\theta)}{p(y;\theta)}$ does not depend on $\theta \Leftrightarrow T(x) = T(y)$

"$\Leftarrow$" is ~~...~~ evident. Prove "$\Rightarrow$"

Suppose that "does not depend" but $T(x) \neq T(y)$

$$\frac{p(x;\theta)}{p(y;\theta)} = \frac{[1+(y_1-\theta)^2]\cdot\ldots\cdot[1+(y_n-\theta)^2]}{[1+(x_1-\theta)^2]\cdot\ldots\cdot[1+(x_n-\theta)^2]}$$

without loss of generality we can suppose that $x_i \neq y_j \ \forall i,j = 1,\dots,n$ (otherwise we cancel out equal factors)

Then roots of $p(x;\theta)=0$ (with respect to θ) coincide with those of $p(y;\theta)=0$ (because $p(x;\theta)$ and $p(y;\theta)$ are polinomials). But these roots are respectively

$\theta_{1,2}^{(y)} = y_1 \pm i, \quad \dots \quad, \theta_{2n-1,2n}^{(y)} = y_n \pm i$

$\theta_{1,2}^{(x)} = x_1 \pm i, \quad \dots, \quad \theta_{2n-1,2n}^{(x)} = x_n \pm i$

and due to our asumption $(T(x) \neq T(y))$ we come to a contradiction

1.6.1

1) $N(\mu, \sigma^2)$

a) σ^2 fixed $\quad \theta = \mu$

$$p(x;\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} e^{\frac{\mu}{\sigma^2}x - \frac{\mu^2}{2\sigma^2}}$$

$$h(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}, \quad \eta(\theta) = \frac{\mu}{\sigma^2}, \; T(x) = x, \; B(\theta) = \frac{\mu^2}{2\sigma^2}$$

b) μ fixed $\quad \theta = \sigma$ (or $\theta = \sigma^2$)

$$p(x;\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2 - \ln\sigma}$$

$$h(x) = \frac{1}{\sqrt{2\pi}}, \quad \eta(\theta) = -\frac{1}{2\sigma^2}, \; T(x) = (x-\mu)^2, \; B(\theta) = \ln\sigma$$

2) $\Gamma(p, \lambda)$ $\quad \frac{\lambda^p}{\Gamma(p)} x^{p-1} e^{-\lambda x}$

a) p fixed $\quad \theta = \lambda$

$$p(x;\theta) = \frac{x^{p-1}}{\Gamma(p)} e^{-\lambda x + p\ln\lambda}$$

$$h(x) = \frac{x^{p-1}}{\Gamma(p)}, \quad \eta(\theta) = -\lambda, \; T(x) = x, \; B(\theta) = -p\ln\lambda$$

b) λ fixed $\quad \theta = p$

$$p(x;\theta) = e^{-\lambda x} e^{(p-1)\ln x - [\ln\Gamma(p) - p\ln\lambda]}$$

$$h(x) = e^{-\lambda x}, \quad \eta(\theta) = p-1, \; T(x) = \ln x, \; B(\theta) = \ln\Gamma(p) - p\ln\lambda$$

3) $\beta(r, s)$ $\quad \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}$

a) r fixed, $\theta = s$

$$p(x;\theta) = \frac{1}{\Gamma(r)} e^{(r-1)\ln x} e^{(s-1)\ln(1-x) - [\ln\Gamma(s) - \ln\Gamma(r+s)]}$$

$$\eta(\theta) = s-1 \qquad T(x) = \ln(1-x)$$

b) s fixed, $\theta = r$

$$p(x;\theta) = \frac{1}{\Gamma(s)} e^{(s-1)\ln(1-x)} e^{(r-1)\ln x - [\ln\Gamma(r) - \ln\Gamma(r+s)]}$$

$$\eta(\theta) = r-1, \quad T(x) = \ln x$$

1.6.2. [Достаточно показать для одной сл. величины. Тогда следует для iid и легко найти соответствующие ф-ии]

а)

$$p(x;\theta) = \theta x^{\theta-1} = \frac{1}{x} e^{\theta \ln x + \ln\theta}$$

$$h(x) = \frac{1}{x} \Rightarrow h_n(x) = \prod_{i=1}^{n} \frac{1}{x_i}$$

$$\eta(\theta) = \theta = \eta_n(\theta)$$

$$T(x) = \ln x \Rightarrow T_n(x) = \sum_{i=1}^{n} \ln x_i$$

$$B(\theta) = -\ln\theta \Rightarrow B_n(\theta) = -n\ln\theta$$

в) $p(x;\theta) = a x^{a-1} e^{\ln\theta - \theta x^a}$

$$h(x) = a x^{a-1} \Rightarrow h_n(x) = a^n \prod_{i=1}^{n} x_i^{a-1}$$

$$\eta(\theta) = -\theta \Rightarrow \eta_n(\theta) = -\theta$$

$$T(x) = x^a \Rightarrow T_n(x) = \sum_{i=1}^{n} x_i^a$$

$$B(\theta) = -\ln\theta \Rightarrow B_n(\theta) = -n\ln\theta$$

с) $p(x;\theta) = \frac{1}{x} e^{\ln\theta + \theta\ln a - \theta\ln x}$ $\quad x > a$

$$h(x) = \frac{1}{x} \Rightarrow h_n(x) = \prod_{i=1}^{n} \frac{1}{x_i}$$

$$\eta(\theta) = -\theta \Rightarrow \cancel{h_n(x)}\ \eta_n(\theta) = -\theta$$

$$T(x) = \ln x \Rightarrow T_n(x) = \sum_{i=1}^{n} \ln x_i$$

$$B(\theta) = \ln\theta + \theta\ln a \Rightarrow B_n(\theta) = n(\ln\theta + \theta\ln a)$$

1.6.4 (a–c)

For exponential families

$$p(x;\theta) = h(x)\, e^{\eta(\theta)T(x)+B(\theta)}$$

that implies that the set

$$\{x : p(x;\theta) > 0\}$$

does not depend on θ. ($e^s > 0$)

In all three cases (a–c) this condition is not satisfied, therefore these families are not exponential.

1.6.5

(a) $\theta = (\alpha, \beta)$

$$p(x;\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} =$$

$$= \frac{1}{x(1-x)} e^{\alpha \ln x + \beta \ln(1-x) - \ln\Gamma(\alpha) - \ln\Gamma(\beta) + \ln\Gamma(\alpha+\beta)}$$

$h(x) = \frac{1}{x(1-x)}$, $\eta_1(\theta) = \alpha$, $T_1(x) = \ln x$

$\eta_2(\theta) = \beta$, $T_2(x) = \ln(1-x)$

$B(\theta) = \ln\Gamma(\alpha) + \ln\Gamma(\beta) - \ln\Gamma(\alpha+\beta)$

(b) $\theta = (\alpha, \beta)$

$$p(x;\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} =$$

$$= \frac{1}{x} e^{\alpha \ln\beta - \ln\Gamma(\alpha) + \alpha \ln x - \beta x}$$

$h(x) = \frac{1}{x}$, $\eta_1(\theta) = \alpha$, $T_1(x) = \ln x$

$\eta_2(\theta) = -\beta$, $T_2(x) = x$

$B(\theta) = \ln\Gamma(\alpha) - \alpha\ln\beta$

1.6.7

$\cancel{p(x,}$ $p(x;y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}$

$$\prod_{i=1}^{n} p(x_i, y_i) = \left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)^{-n} \times e^{-\frac{1}{2(1-\rho^2)}\left[\sum_{i=1}^{n}\left(\frac{x_i-\mu_1}{\sigma_1}\right)^2 - 2\rho\sum_{i=1}^{n}\left(\frac{x_i-\mu_1}{\sigma_1}\right)\left(\frac{y_i-\mu_2}{\sigma_2}\right) + \sum_{i=1}^{n}\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}$$

$\sum x_i \quad \sum x_i^2 \quad \sum x_i y_i \quad \sum y_i \quad \sum y_i^2$

$T_1(x,y) \quad T_2(x,y) \quad T_3(x,y) \quad T_4(x,y) \quad T_5(x,y)$

1.6.11 (c)

$$p(x;\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} = \quad x = 0,1,\dots,n$$

$$= \binom{n}{x} e^{x\ln\theta + (n-x)\ln(1-\theta)} =$$

$$= \binom{n}{x} e^{[\ln\theta - \ln(1-\theta)]x + n\ln(1-\theta)}$$

$$\eta = \ln\frac{\theta}{1-\theta} \qquad e^\eta = \frac{\theta}{1-\theta} \qquad \theta = \frac{e^\eta}{1+e^\eta} \quad 1-\theta = \frac{1}{1+e^\eta}$$

$$A(\eta) = -n\ln\frac{1}{1+e^\eta} = n\ln(1+e^\eta)$$

$T(X) = X$

$$M_X(s) = e^{n\ln(1+e^{s+\eta}) - n\ln(1+e^\eta)} =$$

$$= e^{n\ln\frac{1+e^{s+\eta}}{1+e^\eta}} = \left[\frac{1+e^{s+\eta}}{1+e^\eta}\right]^n =$$

$$= \left[\frac{1+e^s\frac{\theta}{1-\theta}}{1+\frac{\theta}{1-\theta}}\right]^n = \left[1-\theta+\theta e^s\right]^n$$

(b) $$p(x;\theta) = \frac{\theta^p}{\Gamma(p)} x^{p-1} e^{-\theta x} =$$

$$= \frac{x^{p-1}}{\Gamma(p)} e^{p\ln\theta - \theta x}$$

$$\eta = -\theta \qquad T(X) = X \qquad A(\eta) = -p\ln(-\eta)$$

$$M_X(s) = e^{A(s+\eta) - A(\eta)} =$$

$$= e^{-p\ln(-s-\eta) + p\ln(-\eta)} =$$

$$= e^{\ln\left[\frac{-\eta}{-s-\eta}\right]^p} = \left(\frac{\eta}{s+\eta}\right)^p = \left(\frac{-\theta}{s-\theta}\right)^p =$$

$$= \left(\frac{\theta}{\theta-s}\right)^p = \left(\frac{1}{1-s/\theta}\right)^p$$

2.1.3.

$\alpha = (\alpha_1, \alpha_2)$ let (α, β)

$$\mu_1 = \frac{\alpha}{\alpha+\beta} \qquad Var = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\mu_2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{\alpha^2}{(\alpha+\beta)^2} =$$

$$= \frac{\alpha\beta + \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha(\alpha+1)\cancel{(\alpha+\beta)}}{(\alpha+\beta)^{\cancel{2}}(\alpha+\beta+1)} =$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1}$$

therefore

$$\begin{cases} \alpha\mu_1 + \beta\mu_1 = \alpha & (1) \\ \mu_2 = \mu_1 \dfrac{\alpha+1}{\alpha+\beta+1} & (2) \end{cases}$$

From (1) $\beta = \dfrac{\alpha - \alpha\mu_1}{\mu_1}$

$\cancel{\alpha\mu_2 + \beta\mu_2 +}$ Решал неправильно

$$\alpha = \frac{\mu_1(\mu_1 - \mu_2)}{\mu_2 - \mu_1^2} \qquad \beta = \frac{(\mu_1 - \mu_2)(1-\mu_1)}{\mu_2 - \mu_1^2}$$

therefore

$$\hat{\alpha} = \frac{\bar{X}(\bar{X} - \frac{1}{n}\sum X_i^2)}{\frac{1}{n}\sum(X_i - \bar{X})^2}$$

$$\hat{\beta} = \frac{(\bar{X} - \frac{1}{n}\sum X_i^2)}{\frac{1}{n}\sum(X_i - \bar{X})^2}(1-\bar{X}$$

2.1.6.

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{[X_i,\infty)}(x) = \begin{cases} 0, & x < X_{(1)} \\ \frac{k}{n}, & X_{(k)} \le x < X_{(k+1)} \\ & k=1,2,\dots,n-1 \\ 1, & x \ge X_{(n)} \end{cases}$$

2.1.9 (a)

$$\mu_2 = \sigma^2$$

$$\widehat{\sigma^2} = \frac{1}{n}\sum X_i^2$$

(b) $$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}$$

(c) $$\mu_k = E|X|^k$$

$$\mu_1(\hat{F}_n) = \frac{1}{n}\sum_{i=1}^{n}|X_i|$$

$$\mu_1 = \sigma\sqrt{2\pi} \qquad \sigma = \frac{1}{\sqrt{2\pi}}\mu_1$$

$$\hat{\sigma} = \frac{1}{\sqrt{2\pi}}\hat{\mu}_1 = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{n}\sum_{i=1}^{n}|X_i|$$

2.2.1

$$Y_i = \frac{\theta}{2} t_i^2 + \varepsilon_i \quad , i = 1, 2, \ldots, n$$

$$Y = (Y_1, \ldots, Y_n)$$

$$E\varepsilon_i = 0$$
$$Var\, \varepsilon_i = \sigma^2$$
$$cov(\varepsilon_i, \varepsilon_j) = 0$$

$$\rho(Y, \theta) = \sum \left(Y_i - \frac{\theta}{2} t_i^2\right)^2 \longrightarrow \min$$

$$\frac{\partial \rho(Y, \theta)}{\partial \theta} = -2 \sum_{i=1}^{n} \left(Y_i - \frac{\theta}{2} t_i^2\right) \frac{t_i^2}{2}$$

$$\sum_{i=1}^{n} \left(Y_i - \frac{\theta}{2} t_i^2\right) t_i^2 = 0$$

$$\sum Y_i t_i^2 = \frac{\theta}{2} \sum t_i^4$$

$$\hat{\theta} = 2 \frac{\sum Y_i t_i^2}{\sum t_i^4}$$

Замечание: Условия (на ε) не нужны, чтобы найти LSE, а только, чтобы она была „reasonable"

2.2.10 a) $\hat{\theta} = \dfrac{1}{\bar{X}}$

(b) $L_X(\theta) = \dfrac{\theta^n e^{n\theta}}{(\prod X_i)^{\theta+1}}$ $\qquad \ln L_X(\theta) = n \ln\theta + n\theta - (\theta+1)\ln(\prod X_i)$

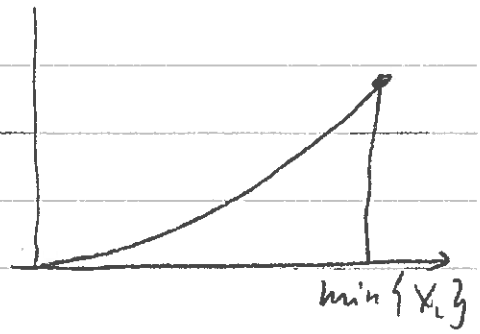
$$\frac{\partial \ln L_X(\theta)}{\partial\theta} = \frac{n}{\theta} + n \ln e - \ln(\prod X_i) = 0$$

$$\hat{\theta} = \frac{1}{-\ln e + \frac{1}{n}\sum_{i=1}^{n} \ln X_i}$$

(c)

$$L_X(\theta) = \frac{c^n \theta^{nc}}{(\prod X_i)^{c+1}} I_{[\theta,\infty)}(\min\{X_i\}) =$$

$$= \frac{c^n \theta^{nc}}{(\prod X_i)^{c+1}} I_{(0,\min\{X_i\})}(\theta)$$



$\hat{\theta} = \min\{X_i\}$

(d) $L_X(\theta) = \theta^{\frac{n}{2}} (\prod X_i)^{\sqrt{\theta}-1}$

$$\ln L_X(\theta) = \frac{n}{2}\ln\theta + (\sqrt{\theta}-1)\ln \prod X_i$$

$$\frac{\partial \ln L_X(\theta)}{\partial\theta} = \frac{n}{2\theta} + \frac{1}{2\sqrt{\theta}} \ln(\prod X_i) = 0$$

$$\frac{n}{\sqrt{\theta}} = -\ln \prod X_i \qquad \hat{\theta} = \left(\frac{n}{-\sum \ln X_i}\right)^2$$

(e) $L_X(\theta) = \theta^{-2n} \prod X_i \, e^{-\frac{1}{2\theta^2}\sum X_i^2}$

$$\frac{\partial \ln L_X(\theta)}{\partial\theta} = -\frac{2n}{\theta} + \frac{1}{\theta^3}\sum X_i^2$$

$$\hat{\theta} = \sqrt{\frac{1}{2n}\sum X_i^2}$$

2.2.10 (continued)

(f) $$L_X(\theta) = \theta^n c^n \left(\prod X_i\right)^{c-1} e^{-\theta \sum X_i^c}$$

$$\ln L_X(\theta) = n \ln\theta + n \ln c + (c-1) \ln \prod X_i -$$

$$- \theta \sum X_i^c$$

$$\frac{\partial \ln L_X(\theta)}{\partial\theta} = \frac{n}{\theta} - \sum X_i^c = 0$$

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} X_i^c}$$

2.2.13

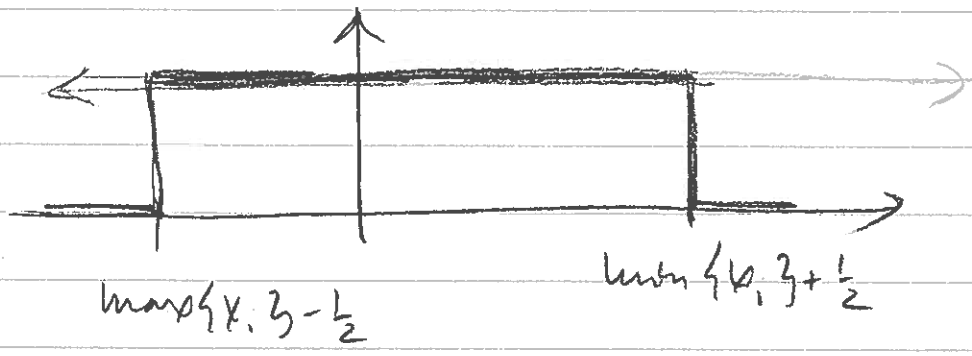
$$L_X(\theta) = \prod_{i=1}^{n} I_{[\theta-\frac{1}{2},\,\theta+\frac{1}{2}]}(X_i) =$$

$$= \prod I_{(-\infty,\theta+\frac{1}{2}]}(X_i)\, I_{[\theta-\frac{1}{2},\infty)}(X_i) =$$

$$= I_{(-\infty,\theta+\frac{1}{2}]}(\max\{X_i\})\, I_{[\theta-\frac{1}{2},\infty)}(\min\{X_i\}) =$$

$$= I_{(-\infty,\theta]}(\max\{X_i\}-\tfrac{1}{2})\, I_{[\theta,\infty)}(\min\{X_i\}+\tfrac{1}{2}) =$$

$$= I_{[\max\{X_i\}-\frac{1}{2},\infty)}(\theta)\, I_{(-\infty,\min\{X_i\}+\frac{1}{2}]}(\theta)$$



Example $X=(X_1,\dots,X_n)$ iid $\sim$ gamma(α,β)

$\theta=(\alpha,\beta)$ $\quad \alpha>0$, $\beta>0$

$$p_0(x;\theta)=\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}=$$

$$=\frac{1}{x}e^{\beta(-x)+\alpha\ln x-[\ln\Gamma(\alpha)-\alpha\ln\beta]}$$

$x=(x_1,\dots,x_n)$

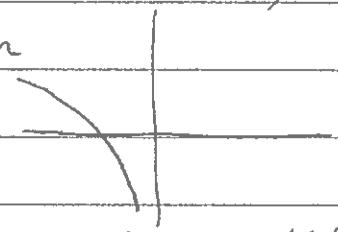
$$p(x;\theta)=\prod_{i=1}^{n}\frac{1}{x_i!}e^{\beta(-\sum x_i)+\alpha\sum\ln x_i-n(\ln\Gamma(\alpha)-\alpha\ln\beta)}$$

$\eta_1=\beta \quad T_1(x)=-\sum x_i$

$\eta_2=\alpha \quad T_2(x)=\sum\ln x_i$

$n=1 \quad T(X_1)=(-X_1,\ \ln X_1)$

Singular



$C_T^0=\emptyset$ and MLE does not exist

$n\geq 2 \quad T(X)=(-\sum X_i,\ \sum\ln X_i)$

absolutely continuous and MLE exists, unique and is solution of the likelihood equation

$$\frac{\partial A(\alpha,\beta)}{\partial\beta}=-\sum X_i$$

$$\frac{\partial A(\alpha,\beta)}{\partial\alpha}=\sum\ln X_i$$

$$A(\alpha,\beta)=n[\ln\Gamma(\alpha)-\alpha\ln\beta]$$

GSM05 2ℳ.2

~~Lectures 9-10~~ (7)

$$\frac{\partial A}{\partial \alpha} = n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - n\ln\beta = \sum \ln X_i$$

$$\frac{\partial A}{\partial \beta} = -\frac{n\alpha}{\beta} = -\sum X_i$$

$$\begin{cases} \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \ln\beta = \frac{1}{n}\sum \ln X_i \\ \frac{\alpha}{\beta} = \overline{X} \end{cases}$$