Review:

§ 3.5 Location and Scale Families

- Location families: $f(x-\mu)$ ← where ever x in pdf subtract μ.

example: $N(\mu,1)$ $f(x|\mu,\sigma=1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-\mu)^2\right) * I_{(-\infty,\infty)}(x)$

- Scale families: $\frac{1}{\sigma} f(x/\sigma)$ ← whereever x in pdf divided by σ. + extra $\left(\frac{1}{\sigma}\right)$ out front.

example: $N(0,\sigma^2)$ $f(x|\mu=0,\sigma^2) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right) * I_{(-\infty,\infty)}(x)$

- Location-Scale families: $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$ ← x in pdf, μ subtract σ divide $\frac{1}{\sigma}$ out front.

example: $N(\mu,\sigma^2)$ $f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) * I_{(-\infty,\infty)}(x)$

- Location, Scale, Location-Scale generated from a standard pdf

↑

$\mu=0$, $\sigma=1$

location shift   scale shift

example: $f(x|\mu=0,\sigma^2=1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Theorem 3.5.1 *Let $f(x)$ be any pdf and let μ and $\sigma > 0$ be any given constants. Then the function*

$$g(x|\mu,\sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

is a pdf.

Review

Location Family: $f(x-\mu)$

Definition 3.5.2 Let $f(x)$ be any pdf. Then the family of pdfs $f(x-\mu)$, indexed by the parameter μ, $-\infty < \mu < \infty$, is called the *location family with standard pdf* $f(x)$ and μ is called the *location parameter* for the family.

$f_X(x)$ & μ {location parameter (constant) $-\infty < \mu < \infty$} $\rightarrow f(x-\mu)$ {location family}

standard pdf

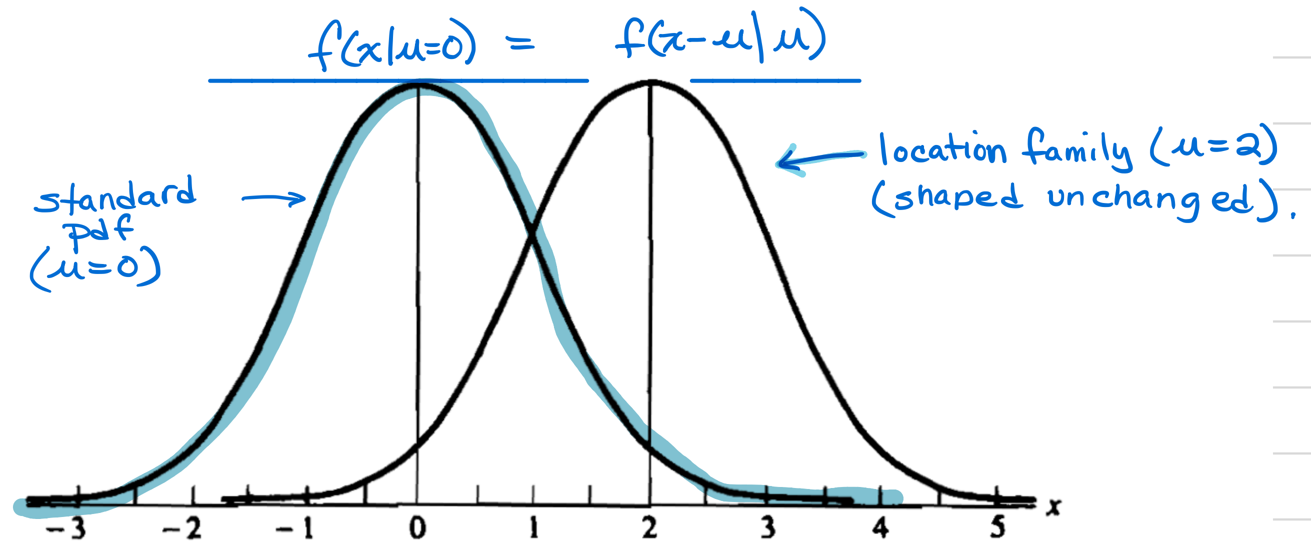


Figure 3.5.1. *Two members of the same location family: means at 0 and 2*

(same shape – shifted right by $\mu=2$)

$f(x-\mu) = f(x)$

Scale family $\frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$

Definition 3.5.4 Let $f(x)$ be any pdf. Then for any $\sigma > 0$, the family of pdfs $(1/\sigma)f(x/\sigma)$, indexed by the parameter σ, is called the *scale family with standard pdf* $f(x)$ and σ is called the *scale parameter* of the family.

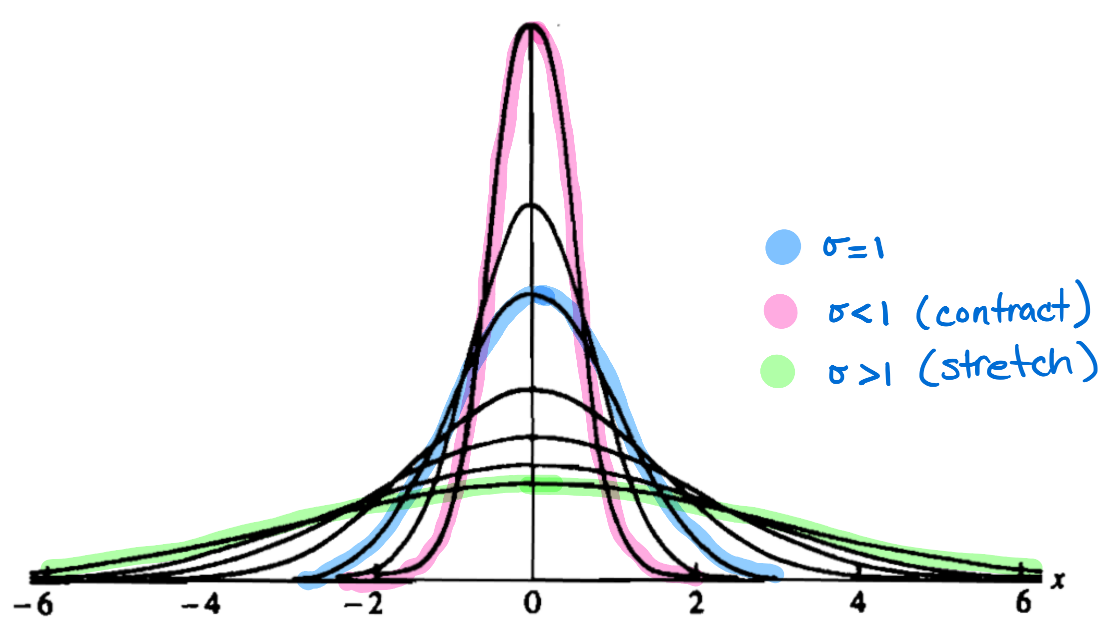


Figure 3.5.3. *Members of the same scale family*

Location-scale family $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$

Definition 3.5.5 Let $f(x)$ be any pdf. Then for any μ, $-\infty < \mu < \infty$, and any $\sigma > 0$, the family of pdfs $(1/\sigma)f((x-\mu)/\sigma)$, indexed by the parameter (μ, σ), is called the *location–scale family with standard pdf* $f(x)$; μ is called the *location parameter* and σ is called the *scale parameter*.

Location - Scale cont.

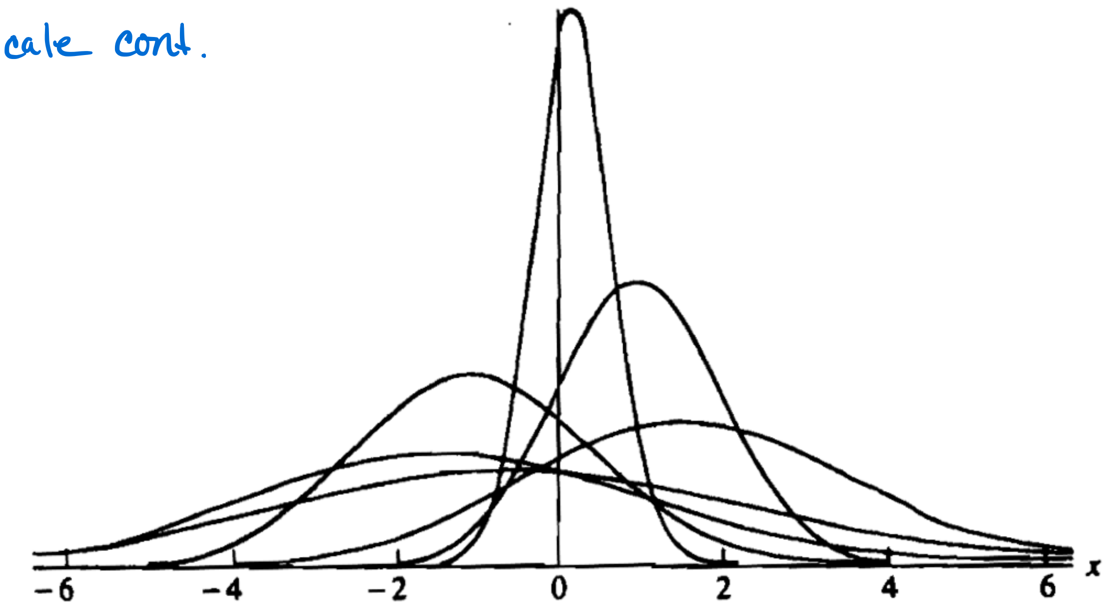


Figure 3.5.4. *Members of the same location–scale family*

Theorem 3.5.6 *Let $f(\cdot)$ be any pdf. Let μ be any real number, and let σ be any positive real number. Then X is a random variable with pdf $(1/\sigma)f((x-\mu)/\sigma)$ if and only if there exists a random variable Z with pdf $f(z)$ and $X = \sigma Z + \mu$.*

$$f_X(x) = \frac{1}{\sigma} f\left(\left(\frac{x-\mu}{\sigma}\right)\right) \overset{\text{iff}}{\Longleftrightarrow} f(z) \text{ and } X = \sigma Z + \mu$$

Z standard dist'n : $\mu = 0$, $\sigma = 1$

Theorem 3.5.7 *Let Z be a random variable with pdf $f(z)$. Suppose $\mathrm{E}Z$ and $\mathrm{Var}\,Z$ exist. If X is a random variable with pdf $(1/\sigma)f((x-\mu)/\sigma)$, then*

$$\mathrm{E}X = \sigma \mathrm{E}Z + \mu \quad \text{and} \quad \mathrm{Var}\,X = \sigma^2 \mathrm{Var}\,Z.$$

In particular, if $\mathrm{E}Z = 0$ and $\mathrm{Var}\,Z = 1$, then $\mathrm{E}X = \mu$ and $\mathrm{Var}\,X = \sigma^2$.

convient
Z standard pdf s.t. $E[Z] = 0$, $Var[Z] = 1$ — works for Normal but not Double Exponential.

Review

§ 3.6 Inequalities and Identities

Theorem 3.6.1 (Chebychev's Inequality) *Let X be a random variable and let $g(x)$ be a nonnegative function. Then, for any $r > 0$,*

$$P(g(X) \geq r) \leq \frac{\mathrm{E}g(X)}{r}.$$

Example: Cheychev's (mean & variance)

let $g(x) = \frac{(x-\mu)^2}{\sigma^2}$ (≥ 0) where $\mu = E[x]$, $\sigma^2 = \mathrm{Var}[x]$ let $r = t^2 (>0)$

$$P\left(\frac{(x-\mu)^2}{\sigma^2} \geq t^2\right) \leq \frac{1}{t^2} E\left[\left(\frac{x-\mu}{\sigma}\right)^2\right] = \frac{1}{t^2}$$

$$\rightarrow P(|x-\mu| \geq t\sigma) \leq \frac{1}{t^2}$$

$$\rightarrow P(|x-\mu| < t\sigma) \geq 1 - \frac{1}{t^2}$$

For dist'n with finite mean $\rightarrow$ bound on $|x-\mu|$ in terms of σ.

$t=2$

$$P(|x-\mu| \geq 2\sigma) \leq \frac{1}{2^2} = .25$$

$\rightarrow$ Probability random variable with 2σ of mean $= .75$
- No matter dist'n of X.

Inequality based on Standard Normal $(\mu = 0, \sigma^2 = 1)$

Example (Normal dist'n):

integrating over $x > t$, so $\frac{x}{t} > 1$

$$P(z \geq t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} dx \leq \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{x}{t} e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t}$$

$$P(|z| \geq t) \leq 2\left(\frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t}\right) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{e^{-t^2/2}}{t}$$

$$P(|z| \geq 2) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-2^2/2}}{2} = \frac{1}{\sqrt{2\pi}} e^{-2} \cong .054.$$

Prob R.V. with 2σ of mean $\cong (1 - .054) = .946$

Chapter 4

Course Overview

C&B Chapters:

1: Probability Theory
2: Transformations / Expectations
3: Common Families of Distributions
4: Multiple Random Variables
5: Properties of a Random Sample
6: Principles of Data Reduction
7: Point Estimation
8: Hypothesis Testing
9: Interval Estimation
10: Asymptotic Evaluations

Chapters 1–6: Foundation, Gaining Tools
Chapters 7–10: Applying Tools

Chapters 1–4: 1st Semester
Chapters 5–10: 2nd Semester

Chapter 4 (Multiple Random Variables) Outline

- Joint and Marginal Dist'ns — $\int_y f_{X,Y}(x,y) = f_X(x)$
- Conditional Dist'ns and independence — $f_{X|Y}(x|y) = f_{XY}(x,y)/f_Y(y)$
- Bivariate transformation — $u = g(x,y)$ $v = h(x,y)$
- Hierarchial Models + Mixture dist'n — $E[X] = E[E[X|Y]]$
- Covariance + Correlation — $\rho_{XY} = Cov(X,Y)/(\sigma_X \sigma_Y)$
- Multivariate Dist'ns — $f_{X_1,X_2 \ldots X_n}(x_1, x_2 \ldots x_n)$
- Inequalities
  - + Numerical — $|E[XY]| \le E[|XY|] \le (E[|X|^2])^{1/2}(E[|Y|^2])^{1/2}$
  - + Functional — Convex $g(x) \rightarrow E[g(X)] \ge g(E[X])$

§4.1 Joint and Marginal Dist'ns

- Univariate models → Probs involving 1 R.V. (X)
- Bivariate models → " " 2 R.V.s (X, Y)
- Multivariate models → " " n R.V.s $(X_1, \ldots X_n)$

Biostatisticians work with datasets...

- Collect an outcome on many subjects
- Collect many variables on many subjects...

Recall

Definition 1.4.1 A *random variable* is a function from a sample space S into the real numbers.

univariate : $X(s) \in \mathbb{R}^1$ or $X \in \mathbb{R}^1$

Definition 4.1.1 An *n-dimensional random vector* is a function from a sample space S into $\Re^n$, n-dimensional Euclidean space.

bivariate : $(X(s), Y(s)) \in \mathbb{R}^2$ or $(X,Y) \in \mathbb{R}^2$

multivariate $(X_1(s), X_2(s), \dots X_n(s)) \in \mathbb{R}^n$

or $(X_1, X_2, \dots X_n) \in \mathbb{R}^n$

"Classic" Example:

Roll 2 fair 6-sided dice (36 possible outcomes, all equally likely)

$$S = \begin{bmatrix} 1,1 & 1,2 & 1,3 & \boxed{1,4} & 1,5 & 1,6 \\ 2,1 & 2,2 & 2,3 & 2,4 & \boxed{\underline{2,5}} & 2,6 \\ 3,1 & 3,2 & 3,3 & \underline{3,4} & 3,5 & 3,6 \\ 4,1 & 4,2 & \underline{4,3} & 4,4 & 4,5 & 4,6 \\ 5,1 & \underline{5,2} & 5,3 & 5,4 & 5,5 & 5,6 \\ 6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6 \end{bmatrix}$$

X = sum of dice $\qquad \mathcal{X} = (2, 3, \dots 12)$

Y = | difference of 2 dice |. $\qquad \mathcal{Y} = (0, 1, \dots 5)$

example: Roll (3,3) $\quad X = 6$ & $Y = 0 \quad Pr(\text{Roll } (3,3)) = 1/36$

(4,1) $\quad X = 5 \quad Y = 3 \quad Pr(\text{Roll } (4,1)) = 1/36$

(5,2) $\quad X = 7 \quad Y = 3 \quad Pr(\text{Roll } (5,2)) = 1/36$

What is $\underline{Pr(X=6 \;\&\; Y=0)}$? $= P(\text{Roll } \underline{(3,3)}) = 1/36$

$\underline{Pr(X=5 \;\&\; Y=3)}$? $= P(\underline{(4,1)} \text{ or } \underline{(1,4)}) = \underline{2/36 = 1/18}$

$\underline{P(X=7 \;\&\; Y=3)}$? $= P(\underline{(5,2)} \text{ or } \underline{(2,5)}) = 2/36 = 1/18$

$\underline{P(X=7 \;\&\; Y=0)}$? $= P(\emptyset) = 0$

$\underline{P(X=7 \;\&\; Y \leq 4)}$? $= P(\underline{(4,3)} \text{ or } \underline{(3,4)} \text{ or } \underline{(5,2)} \text{ or } \underline{(2,5)}) = \frac{4}{36} = \frac{1}{9}$

In dice example : Are X and Y independent ?

Notation $P(X=x \text{ and } Y=y) = P(X=x, Y=y)$

Clearly (X,Y) is a discrete random vector – (has a countable (finite) number of values.)

joint pmf of X and Y : $f(x,y) = f_{XY}(x,y) = P(X=x, Y=y)$

$X=7, Y=0$

y \ x	2	3	4	5	6	7	8	9	10	11	12
0	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$	$\boxed{0}$	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$
1		$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$	
2			$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$		
3				$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$			
4					$\frac{1}{18}$		$\frac{1}{18}$				
5						$\frac{1}{18}$					

Table 4.1.1. *Values of the joint pmf $f(x,y)$*

21 possible values of (X,Y).

For other (x,y)
$P(X=x, Y=y) = 0$

(i.e. $P(X=7, Y=0) = 0$)

match earlier results using S.

What is $Pr(X=6 + Y=0)$? $= P(\text{Roll } (3,3)) = 1/36$

$Pr(X=5 + Y=3)$? $= P((4,1) \text{ or } (1,4)) = 2/36 = 1/18$

$P(X=7 + Y=3)$? $= P((5,2) \text{ or } (2,5)) = 2/36 = 1/18$

$P(X=7 + Y=0)$? $= P(\emptyset) = 0$

$P(X=7 + Y \leq 4)$? $= P((4,3) \text{ or } (3,4) \text{ or } (5,2) \text{ or } (2,5)) = \frac{4}{36} = \frac{1}{9}$

Joint pmf :

Definition 4.1.3 Let (X,Y) be a discrete bivariate random vector. Then the function $f(x,y)$ from $\Re^2$ into $\Re$ defined by $f(x,y) = P(X=x, Y=y)$ is called the *joint probability mass function* or *joint pmf* of (X,Y). If it is necessary to stress the fact that f is the joint pmf of the vector (X,Y) rather than some other vector, the notation $f_{X,Y}(x,y)$ will be used.

Joint pmf used to compute $P((X,Y)\in A)$

$$\boxed{P((X,Y)\in A) = \sum_{(x,y)\in A} f(x,y)}$$

$P(X=7, Y\le 4) = P((X,Y)\in A) = f(7,1) + f(7,3) = \frac{1}{18} + \frac{1}{18} = \frac{1}{9}$

Joint pmf used to compute $E[g(X,Y)]$

$$\boxed{E[g(X,Y)] = \sum_{(x,y)\in\mathbb{R}^2} g(x,y) f(x,y)}$$

Dice Example: Let $g(x,y) = X*Y$ Find $E[X*Y]$

$(2)(0)\frac{1}{36} + (4)(0)\frac{1}{36} + \cdots (8)(4)\frac{1}{18} + (7)(5)\frac{1}{18} = (13)(\frac{11}{18})$

Properties of Thm 2.2.5 Still apply:

Theorem 2.2.5 *Let X be a random variable and let $a, b,$ and c be constants. Then for any functions $g_1(x)$ and $g_2(x)$ whose expectations exist,*

a. $\mathrm{E}(ag_1(X) + bg_2(X) + c) = a\mathrm{E}\,g_1(X) + b\mathrm{E}\,g_2(X) + c.$

b. *If $g_1(x) \ge 0$ for all x, then $\mathrm{E}\,g_1(X) \ge 0$.*

c. *If $g_1(x) \ge g_2(x)$ for all x, then $\mathrm{E}\,g_1(X) \ge \mathrm{E}\,g_2(X)$.*

d. *If $a \le g_1(x) \le b$ for all x, then $a \le \mathrm{E}\,g_1(X) \le b$.*

i.e. $E[a\,g_1(X,Y) + b\,g_2(X,Y) + c] = a\,E[g_1(X,Y)] + b\,E[g_2(X,Y)] + c$ ...

Also, since pmf : $\sum_{(x,y)\in\mathbb{R}^2} f(x,y) = P((X,Y)\in\mathbb{R}^2) = 1$

Dice example (Table 4.1.1) $15 * (\frac{1}{18}) + 6 * \frac{1}{36} = \frac{30+6}{36} = 1$

Dice Question: What is $P(Y=0)$

Based on Rolling die $= Pr(\text{doubles}) =$

$P((1,1)\cap(2,2)\cap\ldots(6,6)) = \frac{6}{36} = \frac{1}{6}$

Based on joint pmf $= P((2,0)\cap(4,0)\cap\ldots(12,0)) = 6*(\frac{1}{36}) = \frac{1}{6}$

(Table 4.1.1) $= f_{X,Y}(2,0) + f_{X,Y}(4,0) + f_{X,Y}(6,0)$

$+ f_{X,Y}(8,0) + f_{X,Y}(10,0) + f_{X,Y}(12,0) = \frac{1}{6}$

- Usually simpler to work with joint pdf.

May be interested in only dist'n of X or Y ($f_X(x)$ or $f_Y(y)$)
'marginal' of X   'marginal' of y

Theorem 4.1.6 *Let (X,Y) be a discrete bivariate random vector with joint pmf $f_{X,Y}(x,y)$. Then the marginal pmfs of X and Y, $f_X(x) = P(X=x)$ and $f_Y(y) = P(Y=y)$, are given by*

$$f_X(x) = \sum_{y\in\Re} f_{X,Y}(x,y) \quad \text{and} \quad f_Y(y) = \sum_{x\in\Re} f_{X,Y}(x,y).$$

Example dice marginals:

	x: 2	3	4	5	6	7	8	9	10	11	12	$f_Y(y)$
y = 0	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$	$6/18$
1		$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$		$5/18$
2			$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$			$4/18$
3				$\frac{1}{18}$		$\frac{1}{18}$		$\frac{1}{18}$				$3/18$
4					$\frac{1}{18}$		$\frac{1}{18}$					$2/18$
5						$\frac{1}{18}$						$1/18$
$f_X(x)$:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1

Table 4.1.1. *Values of the joint pmf $f(x,y)$*

$f_Y(0) = f_{X,Y}(2,0) + f_{X,Y}(4,0) + f_{X,Y}(6,0) + f_{X,Y}(8,0) + f_{X,Y}(10,0) + f_{X,Y}(12,0) = 1/6$

$f_Y(1) = f_{X,Y}(3,1) + f_{X,Y}(5,1) + f_{X,Y}(7,1) + f_{X,Y}(9,1) + f_{X,Y}(11,1) = 5/18$

$f_Y(2) = f_{X,Y}(4,2) + f_{X,Y}(6,2) + f_{X,Y}(8,2) + f_{X,Y}(10,2) = 4/18 = 2/9$

$f_Y(3) = f_{X,Y}(5,3) + f_{X,Y}(7,3) + f_{X,Y}(9,3) = 3/18 = 1/6$

$f_Y(4) = f_{X,Y}(6,4) + f_{X,Y}(8,4) = 2/18 = 1/9$

$f_Y(5) = f_{X,Y}(7,5) = 1/18$

$$\sum_{y=0}^{5} f_Y(y) = \frac{1}{6} + \frac{5}{18} + \frac{2}{9} + \frac{1}{6} + \frac{1}{9} + \frac{1}{18} = \frac{3+5+4+3+2+1}{18} = \frac{18}{18} = 1$$

$$\text{Sample Space} \longrightarrow f_{XY}(x,y) \longrightarrow f_X(x),\ f_Y(y)$$
$$\cancel{\longleftarrow} \qquad\qquad \cancel{\longleftarrow}$$

$$\text{Rolling fair dice} \longrightarrow f_X(x,y) \longrightarrow f_X(x),\ f_X(y)$$
$$\cancel{\longleftarrow} \qquad\qquad \cancel{\longleftarrow}$$

<u>Example:</u> Assume Joint pmf for some bivariate vector (X,Y).

$$\left.\begin{array}{l} f(0,0) = f(0,1) = \frac{1}{6} \\ f(1,0) = f(1,1) = \frac{1}{3} \\ f(x,y) = 0 \ \text{else} \end{array}\right\rangle \left.\begin{array}{l} f(x,y) > 0 \\ \sum_{(x,y)\in\mathbb{R}^2} f(x,y) = 1 \end{array}\right\rangle \text{pmf}$$

Can Compute Prob's such as:

$P(X=Y) = f(0,0) + f(1,1) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$

$P(X>Y) = f(1,0) = 1/3$

$P(Y \geq X) = f(0,0) + f(0,1) + f(1,1) = \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \frac{2}{3}$

⋮

Given $f(x,y)$:
Don't need original sample space that defined $f(x,y)$;
X & Y (or $X(s)$ & $Y(s)$) to compute these probs.

In fact <u>many</u> sample spaces & functions that could lead to this $f_{XY}(x,y)$.

* Such as: S = 36 outcomes from rolling 2 six-sided dice

$X=0$ 1st die ≤ 2 $\qquad$ $Y=0$ 2nd die odd number
$X=1$ 1st die >2 $\qquad$ $Y=1$ 2nd die even number

(details homework.)

* Could also do with 2 3-sided dice. . . . other

⋮

<u>Different</u> Joint Dist'ns:

$f(0,0) = f(0,1) = \frac{1}{6}$
$f(1,0) = f(1,1) = \frac{1}{3}$
$f(x,y) = 0$ else

$f(0,0) = \frac{1}{12}$ $\quad f(1,0) = \frac{5}{12}$
$f(0,1) = f(1,1) = \frac{3}{12}$
$f(x,y) = 0$ else

<u>Same</u> Marginal Dist'ns

$f_X(0) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$
$f_X(1) = \frac{1}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$
$f_Y(0) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$
$f_Y(1) = \frac{1}{6} + \frac{1}{3} = \boxed{\frac{1}{2}}$

$f_X(0) = \frac{1}{12} + \frac{3}{12} = \frac{4}{12} = \frac{1}{3}$
$f_X(1) = \frac{5}{12} + \frac{3}{12} = \frac{8}{12} = \boxed{\frac{2}{3}}$
$f_Y(0) = \frac{1}{12} + \frac{5}{12} = \frac{1}{2}$
$f_Y(1) = \frac{3}{12} + \frac{3}{12} = \boxed{\frac{1}{2}}$

Continuous Random Vectors

Definition 4.1.10 A function $f(x,y)$ from $\Re^2$ into $\Re$ is called a *joint probability density function* or *joint pdf of the continuous bivariate random vector* (X,Y) if, for every $A \subset \Re^2$,

$$P((X,Y) \in A) = \underbrace{\int_A \int}_{\text{double integrals, integrate over all } (x,y)\in A.} f(x,y)\,dx\,dy.$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\,dx\,dy$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Function satisfying $f(x,y) \geq 0 \ \forall\ (x,y) \in \mathbb{R}^2$

& $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

is joint pdf of continuous random vector (X,Y).

Example: Calculating joint probabilities - I

$$f(x,y) = 6xy^2 I_{(0,1)}^{(x)} I_{(0,1)}^{(y)}$$

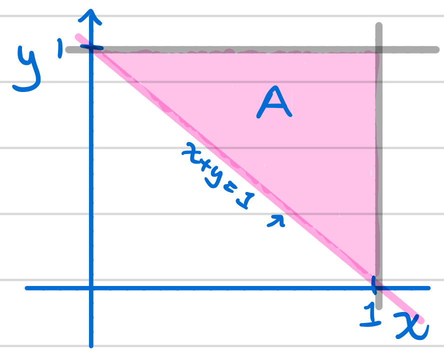
- Check pdf: Clearly $f(x,y) \geq 0$ for $0<x<1$ & $0<y<1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 6xy^2\,dx\,dy = \int_0^1 3x^2y^2\Big|_0^1 dy$$

$$= \int_0^1 3y^2\,dy = y^3\Big|_0^1 = 1$$

Example cont. $f(x,y) = 6xy^2 I_{(0,1)}^{(x)} I_{(0,1)}^{(y)}$

Calculate: $P(X+Y \geq 1)$ let $A = \{(x,y): x+y \geq 1\}$
$= P((X,Y) \in A)$



$A = \{(x,y): x+y \geq 1, 0<x<1, 0<y<1\}$
$= \{(x,y): x \geq 1-y, 0<x<1, 0<y<1\}$
$= \{(x,y): 1-y \leq x < 1, 0<y<1\}$

$$P(X+Y \geq 1) = \int_A\int f(x,y)\,dx\,dy = \int_0^1 \int_{1-y}^1 6xy^2\,dx\,dy = \int_0^1 3x^2y^2\Big|_{1-y}^1 dy$$

$$= \int_0^1 3y^2[1^2 - (1-y)^2]\,dy = \int_0^1 3y^2[\cancel{1} - (\cancel{1} - 2y + y^2)]dy = \int_0^1 6y^3 - 3y^4\,dy$$

$$= \frac{6y^4}{4} - \frac{3y^5}{5}\Big|_0^1 = \frac{3}{2}y^4 - \frac{3}{5}y^5\Big|_0^1 = -\frac{3y^4(2y-5)}{10}\Big|_0^1 = \frac{-3(2-5)}{10} = \frac{9}{10}$$

Calculate $f_X(x)$ $\quad f_X(x) = \int_y f(x,y)\,dy = \int_0^1 6xy^2\,dy$

$$= 2xy^3\Big|_0^1 = 2x \quad 0<x<1$$

$$f_X(x) = 2x\, I_{(0,1)}^{(x)}$$

Check $f_X(x)$ is a pdf

$f_X(x) \geq 0$ for $0<x<1$

$\int_0^1 2x\,dx = x^2\Big|_0^1 = 1$

Calculate $P(\frac{1}{2} < X < \frac{3}{4}) = \int_{1/2}^{3/4} 2x\,dx = x^2\Big|_{1/2}^{3/4}$

$$= \frac{9}{16} - \frac{1}{4} = \frac{9}{16} - \frac{4}{16} = \frac{5}{16}$$