Lecture - 10

Ms Theory I

Review
Chapter - 3

Course Overview

C&B Chapters:

1: Probability Theory
2: Transformations / Expectations
3: Common Families of Distributions
4: Multiple Random Variables
5: Properties of a Random Sample
6: Principles of Data Reduction
7: Point Estimation
8: Hypothesis Testing
9: Interval Estimation
10: Asymptotic Evaluations

Chapters 1–5: Foundation, Gaining Tools
Chapters 6–10: Applying Tools

1st Semester
2nd Semester

Chapter - 3

3.1 Intro
3.2 Discrete Dist'ns
3.3 Continuous Dist'ns
3.4 Exponential families
3.5 Location and Scale families
3.6 Inequalities and Identities

- We've seen families of dist'n

  Binomial (n, p)
  Poisson (λ)
  Normal (μ, σ^2)
  Uniform (a, b)
  Beta (α, β)
  Gamma (α, β)
  Geometric (p)
  ⋮

  Note 'family' characterized by parameters.

  Know parameters, know how random variables behave
  ↳ See chapter 7 !

- For 'common' dist'ns:
  - $f_X(x \mid \theta)$ pdf "|" given parameter θ
  - $F_X(x \mid \theta)$
  - Sample space (support) of X. → $I(x)$
  - $E[X]$ = mean
  - $Var[X]$ = variance
    - ft'n of parameters
  - mgfs ← if exists / helpful

Review

Discrete Uniform Dist'n : (rolling standard die N=6)

Discrete uniform

get cdf by summing

pmf	$P(X=x\|N)=\frac{1}{N};\quad x=1,2,\ldots,N;\quad N=1,2,\ldots$
mean and variance	$\mathrm{E}X=\frac{N+1}{2},\quad \mathrm{Var}\,X=\frac{(N+1)(N-1)}{12}$
mgf	$M_X(t)=\frac{1}{N}\sum_{i=1}^{N}e^{it}$

$I_{[1,\ldots N]}(x) = \begin{cases} 1 & x\in[1,\ldots N] \\ 0 & \text{else} \end{cases}$

Bernoulli(p)

Binomial with $n=1$

pmf	$P(X=x\|p)=p^x(1-p)^{1-x};\quad x=0,1;\quad 0\le p\le 1$
mean and variance	$\mathrm{E}X=p,\quad \mathrm{Var}\,X=p(1-p)$
mgf	$M_X(t)=(1-p)+pe^t$

Binomial(n,p)

usually parameter n assumed known

pmf	$P(X=x\|n,p)=\binom{n}{x}p^x(1-p)^{n-x};\quad x=0,1,2,\ldots,n;\quad 0\le p\le 1$
mean and variance	$\mathrm{E}X=np,\quad \mathrm{Var}\,X=np(1-p)$
mgf	$M_X(t)=[pe^t+(1-p)]^n$
notes	Related to Binomial Theorem (Theorem 3.2.2). The *multinomial* distribution (Definition 4.6.2) is a multivariate version of the binomial distribution.

Theorem 3.2.2 (Binomial Theorem) *For any real numbers x and y and integer $n\ge 0$,*

$$(x+y)^n=\sum_{i=0}^{n}\binom{n}{i}x^iy^{n-i}. \tag{3.2.4}$$

If $x=y=1$

$(1+1)^n = 2^n = \sum_{i=0}^{n}\binom{n}{i}$

Review: Poisson Dist'n

Poisson(λ)	
pmf	$P(X=x\mid\lambda)=\frac{e^{-\lambda}\lambda^x}{x!};\quad x=0,1,\ldots;\quad 0\le\lambda<\infty$
mean and variance	$\mathrm{E}X=\lambda,\quad \mathrm{Var}\,X=\lambda$
mgf	$M_X(t)=e^{\lambda(e^t-1)}$

- Model wait times (time between hormone pulses).
- Number successes in given time (small time interval)
- Count data (often $E[X]=Var[X]$ not valid medical research).

$$f_X(x)=P(X=x\mid\lambda)=\frac{e^{-\lambda}\lambda^x}{x!}\cdot I_{[0,1,2,\ldots)}(x)$$

Recall:

$$\boxed{\text{Tool: } e^y=\sum_{i=0}^{\infty} y^i/i!}$$

Show pmf sums to 1.

$$\sum_{x=0}^{\infty}P(X=x\mid\lambda)=\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}=e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!}=e^{-\lambda}e^{\lambda}=1$$

$$E[X]=\sum_{x=0}^{\infty}\frac{xe^{-\lambda}\lambda^x}{x!}=\sum_{x=1}^{\infty}\frac{\cancel{x}e^{-\lambda}\lambda\cdot\lambda^{x-1}}{\cancel{x}(x-1)}=\lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

let $y=x-1$ $\quad x=1\Rightarrow y=0$, $x=\infty\Rightarrow y=\infty$

$$E[X]=\lambda e^{-\lambda}\sum_{y=0}^{\infty}\frac{\lambda^y}{y!}=\lambda\cancel{e^{-\lambda}}\cancel{e^{\lambda}}=\lambda$$

Similarly: $Var[X]=\lambda$, $M_X(t)=e^{\lambda(e^t-1)}$ homework (after exam)

Review

→ Hormone Pulse Example (waiting time)

- A hormone pulses on average 5 times in 3 hours
- Find probability there are no pulses in next hour.
- Find probability there are at least 2 pulses.

X = # pulses in hour Possible X $0, 1, 2, \dots$

$X \sim \text{Poisson}(\lambda)$ λ?

$\lambda = E[X] = 5 \text{ pulses}/3 \text{ hours} = \frac{5}{3}$

$$P(0 \text{ pulses in next hour}) = P(X=0) = \frac{e^{-5/3}\left(\frac{5}{3}\right)^0}{0!} = .189$$

$$P(\geq 2 \text{ calls in next hr}) = 1 - P(X=0) - P(X=1)$$

$$= 1 - .189 - \frac{e^{-5/3}\left(\frac{5}{3}\right)^1}{1!} = .496$$

→ Poisson Approx. to Binomial

Recall mgf of Binomial $\xrightarrow{\text{converges}}$ Mgf of Poisson] n large, np small
(Ex 2.3.13) $\Rightarrow$ cdfs converge

If $X \sim \text{Bin}(1500, \frac{1}{500})$ $P(X \leq 2) = \sum_{x=0}^{2} \binom{1500}{x} \left(\frac{1}{500}\right)^x \left(\frac{499}{500}\right)^{1500-x} = .4230$

easy in R; tough by hand...

$E[X] = np = \frac{1500}{500} = 3$

→ Poisson approx.

$$P(X \leq 2) \cong \sum_{x=0}^{2} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-3}\left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!}\right] = .4232$$

Negative binomial(r, p)

pmf $P(X = x|r,p) = \binom{r+x-1}{x} p^r (1-p)^x; \quad x = 0, 1, \dots; \quad 0 \leq p \leq 1$

mean and variance $\text{E}X = \frac{r(1-p)}{p}, \quad \text{Var}\, X = \frac{r(1-p)}{p^2}$

mgf $M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$

notes An alternate form of the pmf is given by $P(Y = y|r,p) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$, $y = r, r+1, \dots$. The random variable $Y = X + r$. The negative binomial can be derived as a gamma mixture of Poissons. (See Exercise 4.34.)

Negative Bionomial

\# trials to r^{th} success, r fixed & known.

X = trial r^{th} success

r successes ; $(x-r)$ failures [note last must be success!]

→ in $(x-1)$ trials have $(r-1)$ successes

success on r^{th} trial

$$p\left[\binom{x-1}{r-1} p^{(r-1)}(1-p)^{(x-1)-(r-1)}\right]$$

$$I_{[r,r+1,r+2,\dots)}(x) = \begin{cases} 1 & \text{if } x \in [r, r+1, \dots) \\ 0 & \text{else} \end{cases}$$

$$= \binom{x-1}{r-1} p^r (1-p)^{x-r} * I_{[r,r+1,r+2,\dots)}(x)$$

Alternative form (homework)

y = # failures to r^{th} success

r successes & y fail $\quad y = x-r \rightarrow x = y+r \quad y = x-r$

If $x=r$ $y=0$

$x=r+1$ $y=1$

$\vdots$

$$\binom{x-1}{r-1} p^r (1-p)^{x-r} I_{[r,r+1,\dots)}(x)$$

$$\binom{r+y-1}{y} p^r (1-p)^y I_{[0,1,\dots)}(y)$$

$r+y$ trial success
need $(r-1)$ fail
$(r+y-1)$ trials
$y \rightarrow$ success
$(r+y-1)-(r-1)=y$

$y = x-r \quad x = y+r \ : \ (x-1) = (y+r-1)$

$(x-r) = (y+\cancel{r}-1-\cancel{r})$

by symmetry: $\binom{r+y-1}{y} = \binom{r+y-1}{r-1}$

"Sometimes see" $\binom{r+y-1}{y} = (-1)^y\binom{-r}{y}$

$$\binom{r+y-1}{y} = \frac{(r+y-1)(r+y-2)\dots(r)\overbrace{\cancel{(r-1)(r-2)\cdots(1)}}^{(r-1)!}}{y!\,\cancel{[(r-1)!]}}$$

$$= (-1)^y \frac{\overbrace{(-r)(-r-1)(-r-2)\dots(-r-y+1)}^{y \text{ terms each } *(-1)}}{(y)(y-1)(y-2)\cdots(2)(1)} = (-1)^y\binom{-r}{y}$$

$$P(Y=y) = (-1)^y\binom{-r}{y} p^r (1-p)^y * I_{[0,1,\dots)}(y)$$

name: negative binomial
"striking resemblance" to binomial.

$$E[y] = \sum_{y=1}^{\infty} y \binom{r+y-1}{y} p^r (1-p)^y$$

$$= \sum_{y=1}^{\infty} \frac{y}{y}\frac{(r+y-1)!}{(y-1)!\,(r-1)!} p^r(1-p)^y = \sum_{y=1}^{\infty} \frac{r}{r}\frac{r\,(r+y-1)!}{(y-1)!\,(r-1)!} = \sum_{y=1}^{\infty} r\binom{r+y-1}{y-1} p^r (1-p)^y$$

$$z = y-1 \Rightarrow \sum_{y=0}^{\infty} r \binom{r+z}{z} p^{r-1+1} (1-p)^{z+1} = r\,\frac{(1-p)}{p} \underbrace{\sum_{z=0}^{\infty} \binom{(r+1)+z-1}{z} p^{r+1}(1-p)^z}_{\text{negative binomial pmf}}$$

$$E[Y] = \frac{r(1-p)}{p} \qquad \text{Similarly... } Var(y) = \frac{r(1-p)}{p^2}$$

Uses in medical research: Define $\mu = \frac{r(1-p)}{p}$

$Var(y) = \mu + \frac{1}{r}\mu^2$

Variance is quadratic ft'n of mean

Analyze count data (vs Poisson) Poisson limiting case $r \to \infty$, $p \to 1$ ...

Geometric(p)

pmf	$P(X = x\|p) = p(1-p)^{x-1}; \quad x = 1, 2, \ldots; \quad 0 \le p \le 1$
mean and variance	$EX = \frac{1}{p}, \quad \text{Var}\, X = \frac{1-p}{p^2}$
mgf	$M_X(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$
notes	$Y = X - 1$ is negative binomial$(1, p)$. The distribution is *memoryless*: $P(X > s\|X > t) = P(X > s - t)$.

Geometric Dist'n (special case of neg binom dist'n, r=1)

\# trials to 1st success $(r=1)$

$P(X=x|p) = p(1-p)^{x-1} \quad x = 1, 2, \ldots$

Show pdf sums to 1 (recall $|a|<1$: $\sum_{x=1}^{\infty} a^{x-1} = \frac{1}{1-a}$)

$$p\sum_{x=1}^{\infty}(1-p)^{x-1} = p\sum_{y=0}^{\infty}(1-p)^y = \frac{p}{1-(1-p)} = 1$$

$E[X] = \frac{1}{p} \qquad Var(X) = \frac{1-p}{p^2}$ } from neg binom.

Cool Property: 'memoryless'

$$P(X>s \mid X>t) = P(X>s-t) \qquad \text{If } s=5,\ t=2$$

$$P(X>s \mid X>t) \qquad\qquad P(X>5 \mid X>2) = P(X>3)$$

$$= \frac{P(X>s \text{ and } X>t)}{P(X>t)} \qquad\qquad \frac{P(X>5 \text{ and } X>2)}{P(X>2)}$$

$$= \frac{P(X>s)}{P(X>t)} \qquad\qquad = \frac{P(X>5)}{P(X>2)} = \frac{(1-p)^5}{(1-p)^2} = (1-p)^3$$

$$= \frac{(1-p)^s}{(1-p)^t}$$

Aside $P(X>s)$
$= P(0 \text{ success } s \text{ trials})$
$= (1-p)^s$

$$= (1-p)^{s-t}$$

$$= P(X>s-t)$$

Continuous Dist'ns

Sample space is f't'n of parameters $a \le x \le b$

Uniform(a, b)

pdf	$f(x\|a,b) = \frac{1}{b-a}, \quad a \le x \le b$
mean and variance	$\mathrm{E}X = \frac{b+a}{2}, \quad \mathrm{Var}\, X = \frac{(b-a)^2}{12}$
mgf	$M_X(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$
notes	If $a=0$ and $b=1$, this is a special case of the beta ($\alpha=\beta=1$).

Pdf: integrates to 1: $\int_a^b \frac{1}{b-a}\,dx = \frac{x}{b-a}\Big|_a^b = \frac{1}{(b-a)}(b-a) = 1$

$$E[X] = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{(b-a)}\left[\frac{x^2}{2}\right]\Big|_a^b = \frac{1}{(b-a)}\frac{(b^2-a^2)}{2} = \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$E[X^2] = \int_a^b \frac{x^2}{(b-a)}\,dx = \frac{1}{(b-a)}\left[\frac{x^3}{3}\right]\Big|_a^b = \frac{1}{3(b-a)}(b^3-a^3) = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$\mathrm{Var}[X] = \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4} = \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

$$= (b^2-2ab+a^2)/12 = (b-a)^2/12$$

Gamma Dist'n

Gamma(α, β)

pdf $f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 \le x < \infty, \quad \alpha, \beta > 0$

$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$\Gamma(n) = (n-1)!$ $n>0$, integer

$\Gamma(1/2) = \sqrt{\pi}$

mean and variance $\mathrm{E}X = \alpha\beta, \quad \mathrm{Var}\, X = \alpha\beta^2$

mgf $M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < \frac{1}{\beta}$

notes Some special cases are exponential ($\alpha = 1$) and chi squared ($\alpha = p/2$, $\beta = 2$). If $\alpha = \frac{3}{2}$, $Y = \sqrt{X/\beta}$ is *Maxwell*. $Y = 1/X$ has the *inverted gamma distribution*. Can also be related to the Poisson (Example 3.2.1).

$\beta = 1 \quad f(t|\alpha, \beta=1) = \frac{t^{\alpha-1} e^{-t}}{\Gamma(\alpha)} \quad 0 < t < \infty \qquad f(t|\alpha,\beta=1) > 0$

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I_{(0,\infty)}(x) \qquad \alpha>0, \beta>0$$

α = shape param
β = scale param

$$E[X] = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} (x)(x^{\alpha-1}) e^{-x/\beta} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{(\alpha+1)-1} e^{-x/\beta} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} * \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)} \times \frac{\beta}{\beta} \int_0^\infty x^{(\alpha+1)-1} e^{-x/\beta} dx$$

$$= \frac{\alpha\cancel{\Gamma(\alpha)}\beta}{\cancel{\Gamma(\alpha)}} * \underbrace{\int_0^\infty \frac{1}{\Gamma(\alpha+1)\beta^{\alpha+1}} x^{(\alpha+1)-1} e^{-x/\beta} dx}_{\text{gamma}(\alpha+1,\beta) \text{ integrates to } 1.}$$

$$= \boxed{\alpha\beta} \; //$$

Calculate Var(X) Use mgf to get moments

$$M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha \qquad \frac{d}{dt}(1-\beta t)^{-\alpha} = (-\alpha)(1-\beta t)^{-\alpha-1}(-\beta) = \alpha\beta(1-\beta t)^{-(\alpha+1)}$$

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = \frac{\alpha\beta}{(1-\beta t)^{\alpha+1}}\Big|_{t=0} = \boxed{\alpha\beta}$$

$$\frac{d^2}{dt^2} M_X(t) = \frac{d}{dt} \alpha\beta(1-\beta t)^{-(\alpha+1)} = \alpha\beta(-(\alpha+1))(1-\beta t)^{-(\alpha+1)-1}(-\beta) = \frac{\alpha\beta^2(\alpha+1)}{(1-\beta t)^{\alpha+2}}$$

$$\frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \boxed{\alpha\beta^2(\alpha+1)}$$

$$\mathrm{Var}(X) = \alpha\beta^2(\alpha+1) - (\alpha\beta)^2 = \boxed{\alpha\beta^2}$$

Normal Dist'n (Gaussian Dist'n)

Normal(μ, σ^2)

pdf	$f(x\|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$, $\quad -\infty < x < \infty, \quad -\infty < \mu < \infty,$ $\sigma > 0$
mean and variance	$\mathrm{E}X = \mu, \quad \mathrm{Var}\, X = \sigma^2$
mgf	$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$
notes	Sometimes called the *Gaussian* distribution.

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} I_{(-\infty,\infty)}(x)$$

Standard normal:

$$f(x \mid \mu=0, \sigma^2=1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} I_{(-\infty,\infty)}(x)$$

If $Z \sim N(0,1)$ $\quad E[Z] = \int_{-\infty}^{\infty} \frac{z e^{-z^2/2}}{\sqrt{2\pi}} dz = \int \frac{e^{-u}}{\sqrt{2\pi}} du \qquad u = z^2/2,\ du = z$

$$= \frac{(-1)}{\sqrt{2\pi}} e^{-u}\Big| = \frac{(-1)}{\sqrt{2\pi}} e^{-z^2/2}\Big|_{-\infty}^{\infty} = \frac{-1}{\sqrt{2\pi}}(0-0) = 0$$

If $X = \mu + \sigma Z \quad E[X] = \mu + \sigma E[Z] = \mu$

$$\mathrm{Var}[X] = \mathrm{Var}(\mu + \sigma Z) = \sigma^2 \mathrm{Var}(Z) = \sigma^2$$

Next Lecture
Show $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1$

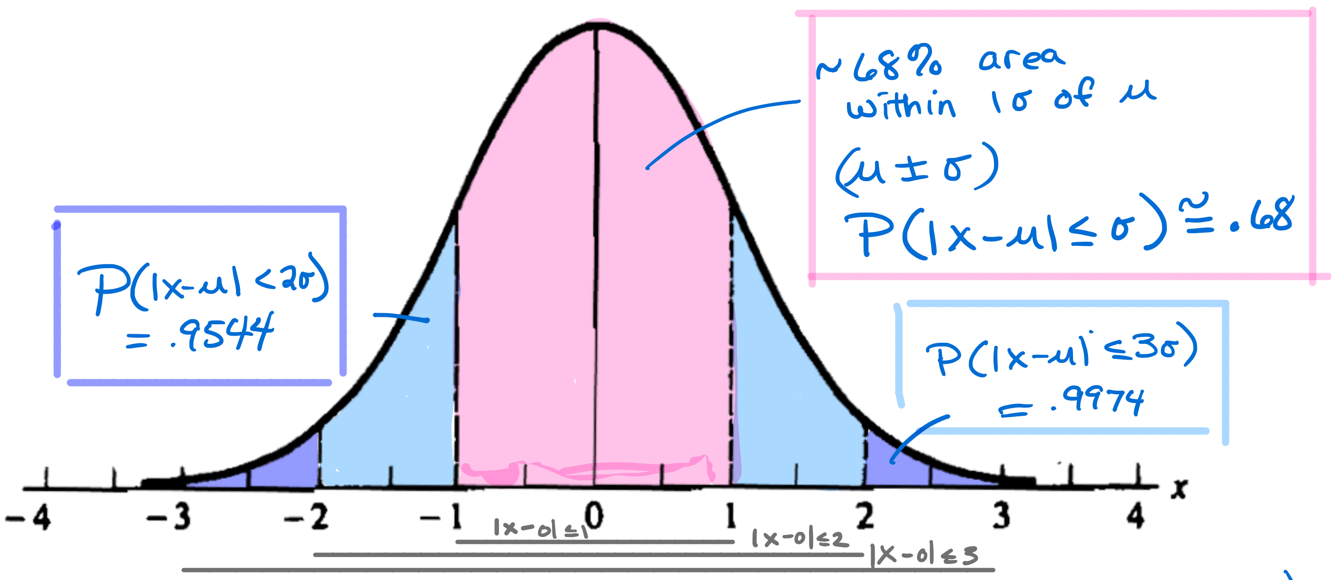

Figure 3.3.1. *Standard normal density* $(\mu = 0, \sigma = 1)$

Use mgf to get $E[X], E[X^2]$ and $Var[X]$

$$mgf = e^{\mu t + \sigma^2 t^2/2}$$

$$\frac{d}{dt} M_X(t) = (\mu + \sigma^2 t)\, e^{\mu t + \sigma^2 t^2/2} \Big|_{t=0} = \mu$$

$$\frac{d^2}{dt^2} M_X(t) = \sigma^2 e^{\mu t + \sigma^2 t^2/2} + (\mu + \sigma^2 t)^2 e^{\mu t + \sigma^2 t^2/2} \Big|_{t=0} = \sigma^2 + \mu^2$$

$$Var(X) = E[X^2] - (E[X])^2 = \sigma^2 + \cancel{\mu^2} - \cancel{\mu^2} = \sigma^2$$

Beta Dist'n

Beta(α, β)

bounded support ⟩ used to model proportions

pdf $\quad f(x|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad \underline{0 \le x \le 1}, \quad \alpha > 0, \quad \beta > 0$

mean and variance $\quad EX = \frac{\alpha}{\alpha+\beta}, \quad Var\, X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

mgf $\quad M_X(t) = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$

notes $\quad$ The constant in the beta pdf can be defined in terms of gamma functions, $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. Equation (3.2.18) gives a general expression for the moments.

$$f(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} * I_{(0,1)}(x) \qquad \alpha>0,\ \beta>0$$

$$E[X^n] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^n x^{\alpha-1}(1-x)^{\beta-1}dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{\int_0^1 x^{(n+\alpha)-1}(1-x)^{\beta-1}dx}_{\text{Kernel of Beta}(n+\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+n+\beta)}{\Gamma(\alpha+n+\beta)} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n)} \int_0^1 x^{(n+\alpha)-1}(1-x)^{\beta-1}dx$$

$$= \frac{\Gamma(\alpha+\beta)\,\Gamma(\alpha+n)}{\Gamma(\alpha)\,\Gamma(\alpha+n+\beta)} \underbrace{\int_0^1 \frac{\Gamma(\alpha+n+\beta)}{\Gamma(\beta)\Gamma(\alpha+n)} x^{(n+\alpha)-1}(1-x)^{\beta-1}dx}_{1}$$

$$= \frac{\Gamma(\alpha+\beta)\,\Gamma(\alpha+n)}{\Gamma(\alpha)\,\Gamma(\alpha+n+\beta)}$$

$$E[X] = \frac{\cancel{\Gamma(\alpha+\beta)}\,\overbrace{\Gamma(\alpha+1)}^{\alpha\cancel{\Gamma(\alpha)}}}{\cancel{\Gamma(\alpha)}\,\underbrace{\Gamma(\alpha+\beta+1)}_{(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}}} = \frac{\alpha}{\alpha+\beta}; \quad \text{Similarly} \quad Var[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

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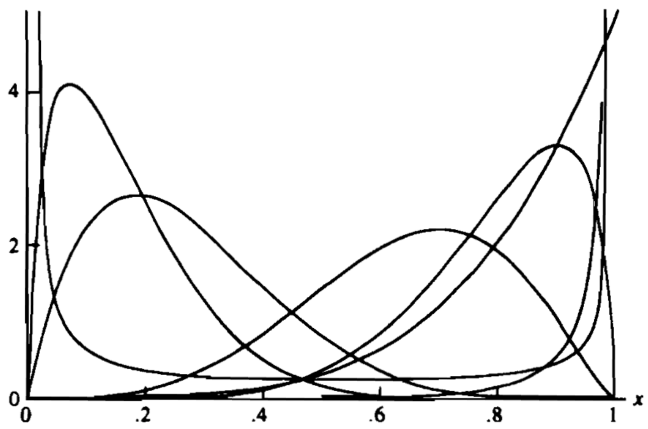


Figure 3.3.3. *Beta densities*

strictly increasing $\alpha>1, \beta=1$
Strictly decreasing $\alpha=1, \beta>1$
U shaped $(\alpha<1, \beta<1)$
Unimodal $(\alpha>1, \beta>1)$

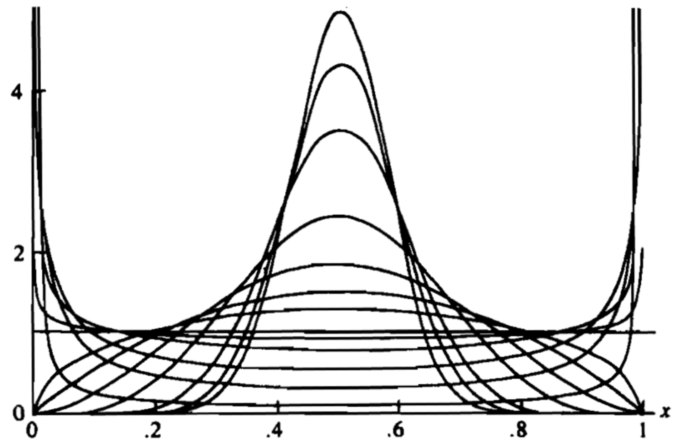
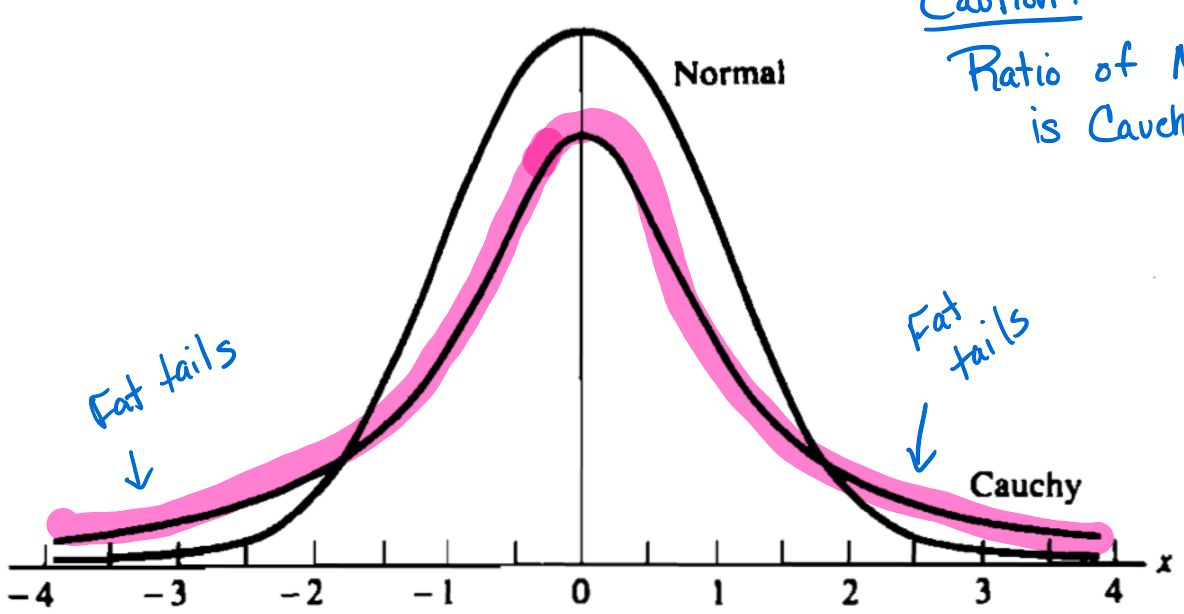


Figure 3.3.4. *Symmetric beta densities*

$\alpha=\beta$ symmetric around $1/2$.
mean $= \frac{\alpha}{\alpha+\alpha} = \frac{1}{2}$

Cauchy Dist'n

Cauchy(θ,σ)	
pdf	$f(x\|\theta,\sigma)=\frac{1}{\pi\sigma}\frac{1}{1+\left(\frac{x-\theta}{\sigma}\right)^2}$, $\quad -\infty<x<\infty; \quad -\infty<\theta<\infty, \quad \sigma>0$
mean and variance	do not exist
mgf	does not exist
notes	Special case of Student's t, when degrees of freedom $=1$. Also, if X and Y are independent $\mathrm{n}(0,1)$, X/Y is Cauchy.



Caution:
Ratio of Normals is Cauchy.

Figure 3.3.5. *Standard normal density and Cauchy density*

Lognormal(μ, σ^2)

pdf $\quad f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\frac{e^{-(\log x-\mu)^2/(2\sigma^2)}}{x}, \quad 0 \le x < \infty, \quad -\infty < \mu < \infty,$ $\sigma > 0$

mean and variance $\quad \mathrm{E}X = e^{\mu+(\sigma^2/2)}, \quad \mathrm{Var}\,X = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$

moments (mgf does not exist) $\quad \mathrm{E}X^n = e^{n\mu+n^2\sigma^2/2}$

notes $\quad$ Example 2.3.5 gives another distribution with the same moments.

Homework #4 mgf

Double exponential(μ, σ)

pdf $\quad f(x|\mu,\sigma) = \frac{1}{2\sigma}e^{-|x-\mu|/\sigma}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$

mean and variance $\quad \mathrm{E}X = \mu, \quad \mathrm{Var}\,X = 2\sigma^2$

mgf $\quad M_X(t) = \frac{e^{\mu t}}{1-(\sigma t)^2}, \quad |t| < \frac{1}{\sigma}$

notes $\quad$ Also known as the *Laplace* distribution.

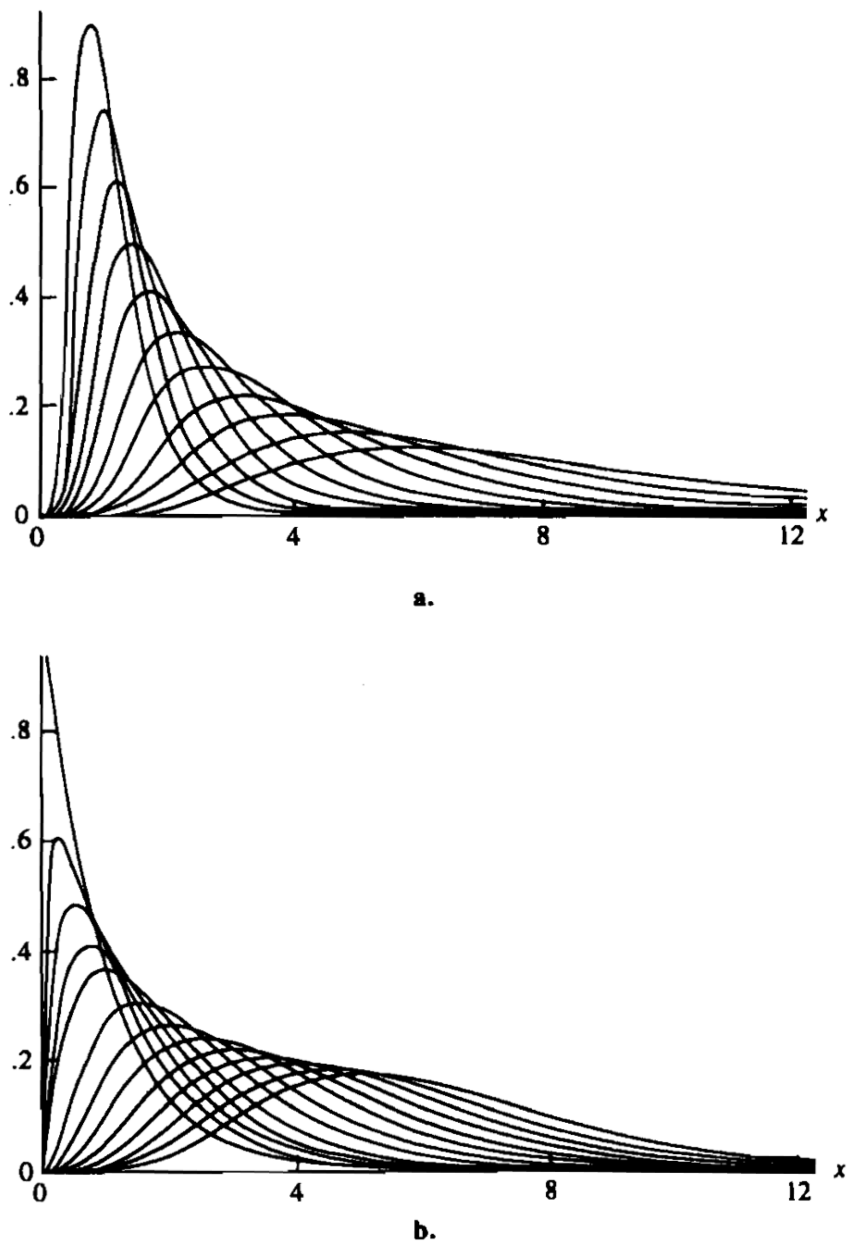


Figure 3.3.6. (a) *Some lognormal densities;* (b) *some gamma densities*