Example 4.6.8

Theorem 4.6.7 (Generalization of Theorem 4.2.12) *Let $X_1, \ldots, X_n$ be mutually independent random variables with mgfs $M_{X_1}(t), \ldots, M_{X_n}(t)$. Let $Z = X_1 + \cdots + X_n$. Then the mgf of Z is*

$$M_Z(t) = M_{X_1}(t) \cdot \cdots \cdot M_{X_n}(t).$$

In particular, if $X_1, \ldots, X_n$ all have the same distribution with mgf $M_X(t)$, then

$$M_Z(t) = (M_X(t))^n.$$

Example (Mgf of Sum of gamma R.V.s).

Mgf of gamma $(\alpha, \beta) = \frac{1}{(1-\beta t)^{\alpha}}$

$X_1, \ldots X_n$ mutually independent random variables $X_i \sim$ gamma (α_i, β)

Find mgf of $Z = X_1 + \ldots + X_n$

$$M_Z(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \ldots \cdot M_{X_n}(t)$$

$$= (1-\beta t)^{-\alpha_1} (1-\beta t)^{-\alpha_2} \cdot \ldots \cdot (1-\beta t)^{-\alpha_n} = \underbrace{(1-\beta t)^{-(\alpha_1+\alpha_2+\ldots+\alpha_n)}}_{\text{mgf of gamma } (\sum_{i=1}^{n} \alpha_i, \beta)}$$

Example 4.6.13 Multivariate Change of Variables

X_1, X_2, X_3, X_4 have joint pdf:

order statistics $x_1 = \min(x) \ldots x_4 = \max(x)$

$$f_{\underline{X}}(\underline{x}) = f_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4) = 24\, e^{-x_1-x_2-x_3-x_4} \quad 0 < x_1 < x_2 < x_3 < x_4 < \infty$$

Let $U_1 = X_1$, $\underbrace{U_2 = X_2 - X_1}_{>0}$, $\underbrace{U_3 = X_3 - X_2}_{>0}$, $\underbrace{U_4 = X_4 - X_3}_{>0}$

Maps $\mathcal{A}$ onto $\mathcal{B} = \{\underline{u} : 0 < u_i < \infty,\ i = 1,2,3,4\}$; one-to-one

Solve for old in terms of new:

$X_1 = U_1$, $X_2 = U_1 + U_2$, $X_3 = U_1 + U_2 + U_3$, $X_4 = U_1 + U_2 + U_3 + U_4$

determinant of triangular matrix is product of diagonal

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} & \frac{\partial x_1}{\partial u_4} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_2}{\partial u_4} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3} & \frac{\partial x_3}{\partial u_4} \\ \frac{\partial x_4}{\partial u_1} & \frac{\partial x_4}{\partial u_2} & \frac{\partial x_4}{\partial u_3} & \frac{\partial x_4}{\partial u_4} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1$$

$$f_{\underline{u}}(u_1,u_2,u_3,u_4) = 24\, e^{-u_1-(u_1+u_2)-(u_1+u_2+u_3)-(u_1+u_2+u_3+u_4)} \prod_{i=1}^{4} I_{(0,\infty)}(u_i)$$

$$= 24\, e^{-4u_1-3u_2-2u_3-1u_4} \prod_{i=1}^{4} I_{(0,\infty)}(u_i)$$

$$= \underbrace{24\, e^{-4u_1} I_{(0,\infty)}(u_1) \cdot e^{-3u_2} I_{(0,\infty)}(u_2)\, e^{-2u_3} I_{(0,\infty)}(u_3)\, e^{-u_4} I_{(0,\infty)}(u_4)}$$

factor into $g_1(u_1)\, g_2(u_2)\, g_3(u_3)\, g_4(u_4)$

$\rightarrow \underbrace{u_1 \perp u_2 \perp u_3 \perp u_4}_{\text{mutually } \perp}$

$\underbrace{\text{Finding marginals (PYOR)}}$: $f_u(u_i) = \underbrace{(5-i)\, e^{-(5-i)u_i} I_{(0,\infty)}(u_i)}_{\text{exponential}\left(\frac{1}{5-i}\right)}$

note e^{-4u_1} is kernel of exponential $\left(\frac{1}{4}\right)$ dist'n

$f(u_1) = 4e^{-4u_1}$

Similarly for u_2, u_3, u_4 & $24 = 4 \cdot 3 \cdot 2 \cdot 1$

C&B Example 5.2.8

Example $X_1, \dots, X_n$ iid (random sample) $\sim N(\mu, \sigma^2)$

$$\text{mgf of normal}(\mu, \sigma^2) = \exp\left[\mu t + \sigma^2 t^2/2\right]$$

$$M_{\bar{X}}(t) = \left[e^{\mu t/n + \frac{\sigma^2 (t/n)^2}{2}}\right]^n = \exp\left[n\left[\frac{\mu t}{n} + \frac{\sigma^2 t^2/n^2}{2}\right]\right]$$

$$= \exp\left[\mu t + \left(\frac{\sigma^2}{n}\right) t^2/2\right] \leftarrow \text{mgf of } N(\mu, \sigma^2/n)$$

Theorem 5.2.7 *Let $X_1, \dots, X_n$ be a random sample from a population with mgf $M_X(t)$. Then the mgf of the sample mean is*

$$M_{\bar{X}}(t) = [M_X(t/n)]^n.$$

↑ useful if we recognize this mgf.

C&B Example 5.2.10 - Big Picture

Don't need to do "somewhat involved" integral
that is solved by "partial fraction decomposition
and some careful antidifferentiation..."

What to do if mgf doesn't exist (Cauchy).

Theorem 5.2.9 *If X and Y are independent continuous random variables with pdfs $f_X(x)$ and $f_Y(y)$, then the pdf of $Z = X + Y$ is*

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)\, dw. \tag{5.2.3}$$

Proof Change of Variables, $|J| = 1$ ...

Example

Cauchy$(0,\sigma)$ Cauchy$(0,\tau)$

$U \sim \frac{1}{\pi\sigma} \frac{1}{1+(u/\sigma)^2} I_{(-\infty,\infty)}(u)$ $V \sim \frac{1}{\pi\tau} \frac{1}{1+(v/\tau)^2} I_{(-\infty,\infty)}(v)$

$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} \frac{1}{1+(w/\sigma)^2} \frac{1}{\pi\tau} \frac{1}{1+((z-w)/\tau)^2}\, dw$

$\left\{\begin{array}{l} w = u \\ z = u+v \quad v = z-w \\ |J| = 1 \text{ (Thm 5.2.9)} \\ -\infty < z < \infty \end{array}\right.$

$\vdots$

$Z \sim$ Cauchy$(0, \sigma+\tau)$

If $Z_1, \dots Z_n$ iid Cauchy$(0,1)$

$\sum Z_i \sim$ Cauchy$(0,n)$

$\bar{Z} \sim$ Cauchy$(0,1)$ $\leftarrow$ dist'n of mean same as marginal Z_i !!

Prove Corollary 4.6.9

Corollary 4.6.9 *Let $X_1, \dots, X_n$ be mutually independent random variables with mgfs $M_{X_1}(t), \dots, M_{X_n}(t)$. Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be fixed constants. Let $Z = (a_1X_1 + b_1) + \cdots + (a_nX_n + b_n)$. Then the mgf of Z is*

$$M_Z(t) = (e^{t(\Sigma b_i)})M_{X_1}(a_1t)\cdot \cdots \cdot M_{X_n}(a_nt).$$

Proof: $M_Z(t) = E[e^{tZ}] = E[e^{t(\Sigma(a_iX_i + b_i))}] = e^{t\Sigma b_i} E[e^{t\Sigma a_iX_i}]$

$= e^{t\Sigma b_i} E[e^{ta_1} e^{ta_2} \cdot \dots \cdot e^{ta_n}]$ ← properties exponentials

$= e^{t\Sigma b_i} E[e^{ta_1}] E[e^{ta_2}] \cdot \dots \cdot E[e^{ta_n}]$ ← independence

$= e^{t\Sigma b_i} M_x(a_1t) M_x(a_2t) \cdot \dots \cdot M_x(a_nt_2)$

Prove C&B Thm 5.2.4

Theorem 5.2.4 *Let $x_1, \dots, x_n$ be any numbers and $\bar{x} = (x_1 + \cdots + x_n)/n$. Then*
a. $\min_a \sum_{i=1}^n (x_i - a)^2 = \sum_{i=1}^n (x_i - \bar{x})^2$, ← least squares
b. $(n-1)s^2 = \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n x_i^2 - n\bar{x}^2$. ← simplifies calculation

Proof: homework.

Proof: To prove part (a), add and subtract $\bar{x}$ to get

$$\begin{aligned}\sum_{i=1}^n (x_i - a)^2 &= \sum_{i=1}^n (x_i - \bar{x} + \bar{x} - a)^2 \\ &= \sum_{i=1}^n (x_i - \bar{x})^2 + 2\sum_{i=1}^n (x_i - \bar{x})(\bar{x} - a) + \sum_{i=1}^n (\bar{x} - a)^2 \\ &= \sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^n (\bar{x} - a)^2. \qquad \text{(cross term is 0)}\end{aligned}$$

It is now clear that the right-hand side is minimized at $a = \bar{x}$. (Notice the similarity to Example 2.2.6 and Exercise 4.13.)

To prove part (b), take $a = 0$ in the above. □

The expression in Theorem 5.2.4(b) is useful both computationally and theoretically because it allows us to express s^2 in terms of sums that are easy to handle.

Part b)

$$\begin{aligned}\sum_{i=1}^n (x_i - 0)^2 &= \sum (x_i - \bar{x} + \bar{x})^2 \\ &= \sum (x_i - \bar{x})^2 + 2\sum (x_i - \bar{x})(\bar{x} - 0) + \sum (\bar{x} - 0)^2 \\ &= \sum (x_i - \bar{x})^2 + 2\bar{x}\overbrace{\left(\sum x_i - n\bar{x}\right)}^{0} + \sum \bar{x}^2\end{aligned}$$

$$\sum x_i^2 = \sum (x_i - \bar{x})^2 + n\bar{x}^2$$

$$\sum (x_i - \bar{x})^2 = \sum x_i^2 - n\bar{x}^2$$

since $S^2 = \dfrac{\sum (x_i - \bar{x})^2}{(n-1)}$ $\quad (n-1)S^2 = \sum x_i^2 - n\bar{x}^2$

Easier Way! $\sum (x_i - \bar{x})^2 = \sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2) = \sum x_i^2 - 2\bar{x}\sum x_i + n\bar{x}^2$

$$= \sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 = \sum x_i^2 - n\bar{x}^2 \,//$$

Prove C&B Lemma 5.2.5

Lemma 5.2.5 *Let $X_1, \ldots, X_n$ be a random sample from a population and let $g(x)$ be a function such that* $\mathrm{E}g(X_1)$ *and* $\mathrm{Var}\, g(X_1)$ *exist. Then*

$$\mathrm{E}\left(\sum_{i=1}^{n} g(X_i)\right) = n\left(\mathrm{E}g(X_1)\right) \tag{5.2.1}$$

and

$$\mathrm{Var}\left(\sum_{i=1}^{n} g(X_i)\right) = n\left(\mathrm{Var}\, g(X_1)\right). \tag{5.2.2}$$

Proof: To prove (5.2.1), note that

$$\mathrm{E}\left(\sum_{i=1}^{n} g(X_i)\right) = \sum_{i=1}^{n} \mathrm{E}g(X_i) = n\left(\mathrm{E}g(X_1)\right).$$

Since the X_is are identically distributed, the second equality is true because $\mathrm{E}g(X_i)$ is the same for all i. Note that the independence of $X_1, \ldots, X_n$ is not needed for (5.2.1) to hold. Indeed, (5.2.1) is true for any collection of n identically distributed random variables.

To prove (5.2.2), note that

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{n} g(X_i)\right) &= \mathrm{E}\left[\sum_{i=1}^{n} g(X_i) - \mathrm{E}\left(\sum_{i=1}^{n} g(X_i)\right)\right]^2 && \text{(definition of variance)} \\
&= \mathrm{E}\left[\sum_{i=1}^{n} \left(g(X_i) - \mathrm{E}g(X_i)\right)\right]^2. && \left(\begin{array}{c}\text{expectation property and} \\ \text{rearrangement of terms}\end{array}\right)
\end{aligned}$$

In this last expression there are n^2 terms. First, there are n terms $(g(X_i) - \mathrm{E}g(X_i))^2$, $i = 1, \ldots, n$, and for each, we have

$$\begin{aligned}
\mathrm{E}\left(g(X_i) - \mathrm{E}g(X_i)\right)^2 &= \mathrm{Var}\, g(X_i) && \text{(definition of variance)} \\
&= \mathrm{Var}\, g(X_1). && \text{(identically distributed)}
\end{aligned}$$

The remaining $n(n-1)$ terms are all of the form $(g(X_i) - \mathrm{E}g(X_i))(g(X_j) - \mathrm{E}g(X_j))$, with $i \neq j$. For each term,

$$\begin{aligned}
\mathrm{E}\left[\left(g(X_i) - \mathrm{E}g(X_i)\right)\left(g(X_j) - \mathrm{E}g(X_j)\right)\right] &= \mathrm{Cov}\left(g(X_i), g(X_j)\right) && \left(\begin{array}{c}\text{definition of} \\ \text{covariance}\end{array}\right) \\
&= 0. && \left(\begin{array}{c}\text{independence} \\ \text{Theorem 4.5.5}\end{array}\right)
\end{aligned}$$

Thus, we obtain equation (5.2.2). □

$g(x_i) \perp g(x_j)$ $\mathrm{Var}(\Sigma g(x_i)) = \Sigma \mathrm{Var}(g(x_i)) = n \mathrm{Var}(g(x_1))$

Example $n=3$ $g(x_i) = x_i$ $E[x_i] = \mu_i$

$E[(\Sigma(x_i - \mu_i))^2] = E[(x_1-\mu_1)^2 + (x_2-\mu_2)^2 + (x_3-\mu_3)^2]$

$+2E[(x_1-\mu_1)(x_2-\mu_2)] + 2E[(x_1-\mu_1)(x_3-\mu_3)] + 2E[(x_2-\mu_2)(x_3-\mu_3)]$

$= 0$ since $\perp$. $\mathrm{Cov} = 0$

Prove C+B Thm 5.2.6

Theorem 5.2.6 *Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Then*

a. $\mathrm{E}\bar{X} = \mu$,

b. $\mathrm{Var}\, \bar{X} = \dfrac{\sigma^2}{n}$,

c. $\mathrm{E}S^2 = \sigma^2$.

Proof: To prove (a), let $g(X_i) = X_i/n$, so $\mathrm{E}g(X_i) = \mu/n$. Then, by Lemma 5.2.5,

$$\mathrm{E}\bar{X} = \mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n}n\mathrm{E}X_1 = \mu.$$

Similarly for (b), we have

$$\mathrm{Var}\, \bar{X} = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}n\,\mathrm{Var}\, X_1 = \frac{\sigma^2}{n}.$$

For the sample variance, using Theorem 5.2.4, we have

$$\begin{aligned}
\mathrm{E}S^2 &= \mathrm{E}\left(\frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]\right) \\
&= \frac{1}{n-1}\left(n\mathrm{E}X_1^2 - n\mathrm{E}\bar{X}^2\right) \\
&= \frac{1}{n-1}\left(n(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right) = \sigma^2,
\end{aligned}$$

establishing part (c) and proving the theorem. $\square$

Prove C&B Thm 5.2.7

Dist'n of $\bar{X}$

$$M_{\bar{X}}(t) = E[e^{t\bar{X}}] = E[e^{t/n(X_1+X_2+\dots+X_n)}] = E[e^{t/n \sum X_i}]$$

if $Y = \sum_{i=1}^{n} X_i$ $\quad M_{\bar{X}}(t) = E[e^{t/n\, Y}]$

Recall:

Theorem 4.6.7 (Generalization of Theorem 4.2.12) *Let $X_1, \dots, X_n$ be mutually independent random variables with mgfs $M_{X_1}(t), \dots, M_{X_n}(t)$. Let $Z = X_1 + \cdots + X_n$. Then the mgf of Z is*

$$M_Z(t) = M_{X_1}(t) \cdot \cdots \cdot M_{X_n}(t).$$

In particular, if $X_1, \dots, X_n$ all have the same distribution with mgf $M_X(t)$, then

$$M_Z(t) = (M_X(t))^n.$$

$$\therefore \quad M_{\bar{X}}(t) = [M_X(t/n)]^n$$

Theorem 5.2.7 *Let $X_1, \dots, X_n$ be a random sample from a population with mgf $M_X(t)$. Then the mgf of the sample mean is*

$$M_{\bar{X}}(t) = [M_X(t/n)]^n.$$

useful if we recognize this mgf.

C+B 4.43

Let $X_1, X_2, + X_3$ be uncorrelated RVs, each with mean μ + variance σ^2.
Find in terms of $\mu + \sigma^2$

$Cov(X_1+X_2, X_2+X_3)$ and $Cov(X_1+X_2, X_1-X_2)$

4.43

$$
\begin{aligned}
\mathrm{Cov}(X_1+X_2, X_2+X_3) &= \mathrm{E}(X_1+X_2)(X_2+X_3) - \mathrm{E}(X_1+X_2)\mathrm{E}(X_2+X_3) \\
&= (4\mu^2+\sigma^2) - 4\mu^2 \quad = \quad \sigma^2 \\
\mathrm{Cov}(X_1+X_2)(X_1-X_2) &= \mathrm{E}(X_1+X_2)(X_1-X_2) \quad = \quad \mathrm{E}X_1^2 - X_2^2 \quad = \quad 0.
\end{aligned}
$$

$Cov(X_1+X_2, X_2+X_3)$

$= E[(X_1+X_2)(X_2+X_3)] - E[(X_1+X_2)]\, E[(X_2+X_3)]$

$= E[X_1X_2 + X_1X_3 + X_2^2 + X_2X_3] - (2\mu)(2\mu)$

$= \mu^2 + \mu^2 + Var(X_2) + (E[X_2])^2 + \mu^2 - 4\mu^2$

$4\mu^2 + \sigma^2 - 4\mu^2 = \sigma^2$

$Cov(X_1+X_2, X_1-X_2)$

$= E[(X_1+X_2)(X_1-X_2)] - E[(X_1+X_2)]\, E[(X_1-X_2)]$

$= E[X_1^2 - X_2^2] - (2\mu)(0)$

$= (\sigma^2+\mu^2) - (\sigma^2+\mu^2) = 0$

C+B 4.45) Show that if $(X,Y) \sim$ bivariate normal $\left[\begin{pmatrix}\mu_X\\ \mu_Y\end{pmatrix}, \begin{pmatrix}\sigma_X^2 & \rho\\ \rho & \sigma_Y^2\end{pmatrix}\right]$, then

a) $f_X(x) \sim N(\mu_X, \sigma_X^2)$; Similarly $f_Y(y) \sim N(\mu_Y, \sigma_Y^2)$

b) $f_{Y|X}(y|X=x) \sim N(\mu_Y + \rho(\sigma_Y/\sigma_X)(x-\mu_X),\ \sigma_Y^2(1-\rho^2))$.

c) For any constants a + b, the dist'n of $aX+bY \sim N(a\mu_X + b\mu_Y,\ a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y)$

4.45 a. We will compute the marginal of X. The calculation for Y is similar. Start with

$$
\begin{aligned}
f_{XY}(x,y) &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \\
&\quad\times \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\}\right]
\end{aligned}
$$

and compute

$$
f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\cancel{\sigma_Y}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(\omega^2 - 2\rho\omega z + z^2)}\cancel{\sigma_Y}dz,
$$

$Y = \sigma_Y z + \mu_Y$ $\quad \frac{dy}{dz} = \sigma_Y$

change of variables

where we make the substitution $z = \frac{y-\mu_Y}{\sigma_Y}$, $dy = \sigma_Y dz$, $\omega = \frac{x-\mu_X}{\sigma_X}$. Now the part of the exponent involving ω^2 can be removed from the integral, and we complete the square in z to get

$$
\begin{aligned}
f_X(x) &= \frac{e^{-\frac{\omega^2}{2(1-\rho^2)}}}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}[(z^2 - 2\rho\omega z + \rho^2\omega^2) - \rho^2\omega^2]}dz \\
&= \frac{e^{-\omega^2/2(1-\rho^2)}e^{\rho^2\omega^2/2(1-\rho^2)}}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}(z-\rho\omega)^2}dz.
\end{aligned}
$$

The integrand is the kernel of normal pdf with $\sigma^2 = (1-\rho^2)$, and $\mu = \rho\omega$, so it integrates to $\sqrt{2\pi}\sqrt{1-\rho^2}$. Also note that $e^{-\omega^2/2(1-\rho^2)}e^{\rho^2\omega^2/2(1-\rho^2)} = e^{-\omega^2/2}$. Thus,

$$
f_X(x) = \frac{e^{-\omega^2/2}}{2\pi\sigma_X\sqrt{1-\rho^2}}\sqrt{2\pi}\sqrt{1-\rho^2} = \frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2},
$$

the pdf of $\mathrm{n}(\mu_X, \sigma_X^2)$.

b.

$$
\begin{aligned}
& f_{Y|X}(y|x) \\
&= \frac{\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}}{\frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{1}{2\sigma_X^2}(x-\mu_X)^2}} \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - (1-\rho^2)\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]} \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\rho^2\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]} \\
&\qquad\qquad * \sigma_Y^2/\sigma_Y^2 \qquad\qquad * \sigma_Y/\sigma_Y \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2\sigma_Y^2\sqrt{(1-\rho^2}}\left[(y-\mu_Y) - \left(\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right)\right]^2},
\end{aligned}
$$

which is the pdf of $\mathrm{n}\Big((\mu_Y - \rho(\sigma_Y/\sigma_X)(x-\mu_X), \sigma_Y\sqrt{1-\rho^2}\Big)$.

c. The mean is easy to check,

$$
\mathrm{E}(aX + bY) = a\mathrm{E}X + b\mathrm{E}Y = a\mu_X + b\mu_Y,
$$

4.45 cont.

as is the variance,

$$\mathrm{Var}(aX+bY) = a^2\mathrm{Var}X + b^2\mathrm{Var}Y + 2ab\mathrm{Cov}(X,Y) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y.$$

To show that $aX+bY$ is normal we have to do a bivariate transform. One possibility is $U = aX+bY$, $V = Y$, then get $f_{U,V}(u,v)$ and show that $f_U(u)$ is normal. We will do this in the standard case. Make the indicated transformation and write $x = \frac{1}{a}(u-bv)$, $y = v$ and obtain

$$|J| = \begin{vmatrix} 1/a & -b/a \\ 0 & 1 \end{vmatrix} = \frac{1}{a}.$$

Then

$$f_{UV}(u,v) = \frac{1}{2\pi a\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left[\frac{1}{a}(u-bv)\right]^2 - 2\frac{\rho}{a}(u-bv)+v^2\right]}.$$

Now factor the exponent to get a square in u. The result is

$$-\frac{1}{2(1-\rho^2)}\left[\frac{b^2+2\rho ab+a^2}{a^2}\right]\left[\frac{u^2}{b^2+2\rho ab+a^2} - 2\left(\frac{b+a\rho}{b^2+2\rho ab+a^2}\right)uv + v^2\right].$$

Note that this is joint bivariate normal form since $\mu_U = \mu_V = 0$, $\sigma_v^2 = 1$, $\sigma_u^2 = a^2+b^2+2ab\rho$ and

$$\rho^* = \frac{\mathrm{Cov}(U,V)}{\sigma_U\sigma_V} = \frac{\mathrm{E}(aXY+bY^2)}{\sigma_U\sigma_V} = \frac{a\rho+b}{\sqrt{a^2+b^2+ab\rho}},$$

thus

$$(1-\rho^{*2}) = 1 - \frac{a^2\rho^2+ab\rho+b^2}{a^2+b^2+2ab\rho} = \frac{(1-\rho^2)a^2}{a^2+b^2+2ab\rho} = \frac{(1-\rho^2)a^2}{\sigma_u^2}$$

where $a\sqrt{1-\rho^2} = \sigma_U\sqrt{1-\rho^{*2}}$. We can then write

$$f_{UV}(u,v) = \frac{1}{2\pi\sigma_U\sigma_V\sqrt{1-\rho^{*2}}}\exp\left[-\frac{1}{2\sqrt{1-\rho^{*2}}}\left(\frac{u^2}{\sigma_U^2} - 2\rho\frac{uv}{\sigma_U\sigma_V} + \frac{v^2}{\sigma_V^2}\right)\right],$$

which is in the exact form of a bivariate normal distribution. Thus, by part a), U is normal.

Corollary 4.6.9 *Let $X_1,\ldots,X_n$ be mutually independent random variables with mgfs $M_{X_1}(t),\ldots,M_{X_n}(t)$. Let $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ be fixed constants. Let $Z = (a_1X_1+b_1)+\cdots+(a_nX_n+b_n)$. Then the mgf of Z is*

$$M_Z(t) = (e^{t(\Sigma b_i)})M_{X_1}(a_1t)\cdot\cdots\cdot M_{X_n}(a_nt).$$

Corollary 4.6.10 *Let $X_1,\ldots,X_n$ be mutually independent random variables with $X_i \sim \mathrm{n}(\mu_i,\sigma_i^2)$. Let $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ be fixed constants. Then*

$$Z = \sum_{i=1}^n (a_iX_i+b_i) \sim \mathrm{n}\left(\sum_{i=1}^n (a_i\mu_i+b_i), \sum_{i=1}^n a_i^2\sigma_i^2\right).$$

C+B 5.3

1. C+B 5.3: Let $X_1, \ldots, X_n$ be iid random variables with continuous cdf F_X, and suppose $E(X_i) = \mu$. Define the random variables $Y_1, \ldots, Y_n$ by

$$Y_i = \begin{cases} 1, & \textbf{if } X_i > \mu \\ 0, & \textbf{if } X_i \leq \mu \end{cases}$$

Find the distribution of $\sum_{i=1}^n Y_i$

Note that Y_i is a Bernoulli random variable for all i: $p_i = \Pr\{X_i \geq \mu\} = 1 - \Pr\{X_i < \mu\} = 1 - F_X(\mu)\ \forall i$. Then since all Y_i's are iid Bernoulli, $\sum_{i=1}^n Y_i \sim \text{Binomial}(n, p = 1 - F_X(\mu))$

Review 4.39

1st approach

1. C+B 4.39: Let $(X_1, \dots, X_n)$ have a multinomial distribution with m trials and cell probabilities $p_1, \dots, p_n$ (see Definition 4.6.2). Show that for every i and j

$$X_i | X_j = x_j \sim \textbf{binomial}(m - x_j, \frac{p_1}{1 - p_j})$$

$$X_j \sim \textbf{binomial}(m, p_j)$$

and that $\text{Cov}(X_i, X_j) = -mp_ip_j$.

Definition 4.6.2 Let n and m be positive integers and let $p_1, \dots, p_n$ be numbers satisfying $0 \le p_i \le 1$, $i = 1, \dots, n$, and $\sum_{i=1}^n p_i = 1$. Then the random vector $(X_1, \dots, X_n)$ has a *multinomial distribution with m trials and cell probabilities* $p_1, \dots, p_n$ if the joint pmf of $(X_1, \dots, X_n)$ is

$$f(x_1, \dots, x_n) = \frac{m!}{x_1! \cdot \dots \cdot x_n!} p_1^{x_1} \cdot \dots \cdot p_n^{x_n} = m! \prod_{i=1}^n \frac{p_i^{x_i}}{x_i!}$$

on the set of $(x_1, \dots, x_n)$ such that each x_i is a nonnegative integer and $\sum_{i=1}^n x_i = m$.

Categorize the multinomial into three categories X_i, X_j, X_k such that $M = X_i + X_j + X_k$, with probabilities: $p_i, p_j, p_k = 1 - p_i - p_j$, respectively. Thus we have a trinomial, so:

$$f_{X_i, X_j}(x_i, x_j) = \frac{M!}{x_i! x_j! (M - x_i - x_j)!} p_i^{x_i} p_j^{x_j} [1 - p_i - p_j]^{M - x_i - x_j}$$

Now we find $f_{X_j}(x_j)$, which we know is binomial from the fact that the marginal pmfs are binomial: } See C&B pg 181-182.

$$f_{X_j}(x_j) = \frac{M!}{x_j!(M - x_j)!} p_j^{x_j} (1 - p_j)^{M - x_j}$$

Then we have for $x_i \in [0, M - x_j]$:

$$\begin{aligned}
f_{X_i|X_j}(x_i, x_j) &= \frac{f_{X_i, X_j}(x_i, x_j)}{f_{X_j}(x_j)} \\
&= \frac{\frac{M!}{x_i! x_j! (M - x_i - x_j)!} p_i^{x_i} p_j^{x_j} [1 - p_i - p_j]^{M - x_i - x_j}}{\frac{M!}{x_j!(M - x_j)!} p_j^{x_j} [1 - p_j]^{M - x_j}} \\
&= \frac{(M - x_j)! p_i^{x_i} [1 - p_i - p_j]^{M - x_i - x_j}}{x_i!(M - x_i - x_j)! [1 - p]^{M - x_j}} \\
&= \frac{(M - x_j)! p_i^{x_i} [(1 - p_j) - p_i]^{(M - x_j) - x_i}}{x_i!(M - x_j - x_i)! (1 - p)^{x_i} [1 - p]^{(M - x_j) - x_i}} \\
&= \frac{(M - x_j)!}{x_i! [(M - x_j) - x_i]!} \left[\frac{p_i}{(1 - p_j)}\right]^{x_i} \left[1 - \frac{p_i}{1 - p_j}\right]^{M - x_j - x_i}
\end{aligned}$$

Which is binomial with parameters $M - x_j$ and $\frac{p_i}{1 - p_j}$

$\text{Cov}(X_i, X_j) = E(X_i X_j) - E(X_i) E(X_j)$

$$\begin{aligned}
E(X_i X_j) &= E_{X_j}[E_{X_i}(X_i X_j | X_j = x_j)] \\
&= E_{X_j}[E_{X_i}(X_i x_j | X_j = x_j)] \\
&= E_{X_j}[x_j E_{X_i}(X_i | X_j = x_j)] \\
&= E_{X_j}[X_j (M - X_j) [\frac{p_i}{1 - p_j}]] \\
&= [\frac{p_i}{1 - p_j}] E_{X_j}[X_j M - X_j^2] \\
&= [\frac{p_i}{1 - p_j}][M^2 p_j - E_{X_j}(X_j^2)] \\
&= [\frac{p_i}{1 - p_j}][M^2 p_j - V_{X_j}(X_j) - [E_{X_j}(X_j)]^2] \\
&= [\frac{p_i}{1 - p_j}][M^2 p_j - M p_j (1 - p_j) - M^2 p_j^2] \\
&= [\frac{p_i}{1 - p_j}][M^2 p_j - M p_j + M p_j^2 - M^2 p_j^2] \\
&= [\frac{M p_j p_i}{1 - p_j}][M - 1 + p_j - M p_j] \\
&= [\frac{M p_j p_i}{1 - p_j}][M(1 - p_j) - (1 - p_j)] \\
&= M p_i p_j (M - 1)
\end{aligned}$$

$\text{Cov}(X_i, X_j) = E(X_i X_j) - E(X_i) E(X_j) = M p_i p_j (M - 1) - (M p_i)(M p_j) = M^2 p_i p_j - M p_i p_j - M^2 p_i p_j = -M p_i p_j$

C&B 4.39 another way: (Camille Moore)

1. Casella and Berger 4.39

Let $(X_1, X_2, ..., X_n)$ have a multinomial distribution with m trials and cell probabilities $p_1, ..., p_n$. Show that for every i and j, $X_i|X_j = x_j \sim$ binomial$\left(m - x_j, \frac{p_i}{1-p_j}\right)$, $X_j \sim$ binomial(m, p_j) and that $Cov(X_i, X_j) = -mp_ip_j$.

See C&B page 182

$$f(x_1, ...x_{j-1}, x_{j+1}...x_n|x_j) = \frac{(m-x_j)!}{x_1!...x_{j-1}!x_{j+1}!...x_n!}\left(\frac{p_1}{1-p_j}\right)^{x_1}...\left(\frac{p_{j-1}}{1-p_j}\right)^{x_{j-1}}\left(\frac{p_{j+1}}{1-p_j}\right)^{x_{j+1}}...\left(\frac{p_n}{1-p_j}\right)^{x_n}$$

Get conditional

$$\begin{aligned}
f(x_i|x_j) &= \sum_{x\neq x_i,x_j} f(x_1, ...x_{j-1}, x_{j+1}...x_n|x_j) \\
&= \sum_{x\neq x_i,x_j} \frac{(m-x_j)!}{x_1!...x_{j-1}!x_{j+1}!...x_n!}\left(\frac{p_1}{1-p_j}\right)^{x_1}...\left(\frac{p_{j-1}}{1-p_j}\right)^{x_{j-1}}\left(\frac{p_{j+1}}{1-p_j}\right)^{x_{j+1}}...\left(\frac{p_n}{1-p_j}\right)^{x_n} \\
&\quad \frac{(m-x_j-x_i)!\left(1-\frac{p_i}{1-p_j}\right)^{m-x_j-x_i}}{(m-x_j-x_i)!\left(1-\frac{p_i}{1-p_j}\right)^{m-x_j-x_i}} \\
&= \frac{(m-x_j)!}{x_i!(m-x_j-x_i)!}\left(\frac{p_i}{1-p_j}\right)^{x_i}\left(1-\frac{p_i}{1-p_j}\right)^{m-x_j-x_i} \\
&\quad \sum_{x\neq x_i,x_j} \frac{(m-x_j-x_i)!}{x_1!...x_{j-1}!x_{j+1}!...x_n!}\left(\frac{p_1}{1-p_j-p_i}\right)^{x_1}...\left(\frac{p_{j-1}}{1-p_j-p_i}\right)^{x_{j-1}}\left(\frac{p_{j+1}}{1-p_j-p_i}\right)^{x_{j+1}} \\
&\quad ...\left(\frac{p_n}{1-p_j-p_i}\right)^{x_n} \\
&= \frac{(m-x_j)!}{x_i!(m-x_j-x_i)!}\left(\frac{p_i}{1-p_j}\right)^{x_i}\left(1-\frac{p_i}{1-p_j}\right)^{m-x_j-x_i}
\end{aligned}$$

Review C&B 4.39 cont

Get marginal - See C&B page 181

$$
\begin{aligned}
f(x_j) &= \sum_{x \neq x_j} \frac{m!}{x_1! \ldots x_n!} p_1^{x_1} \ldots p_n^{x_n} \\
&= \frac{m!}{x_j!(m-x_j)!} p_j^{x_j} (1-p_j)^{m-x_j} \sum_{x \neq x_j} \frac{(m-x_j)!}{x_1! \ldots x_{j-1}! x_{j+1}! \ldots x_n!} p_1^{x_1} \ldots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \ldots p_n^{x_n} \frac{1}{(1-p_j)^{m-x_j}} \\
&= \frac{m!}{x_j!(m-x_j)!} p_j^{x_j} (1-p_j)^{m-x_j}
\end{aligned}
$$

$$Cov(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)$$

Get Joint

$$\left(\frac{1-p_j-p_i}{1-p_j}\right)^{m-x_j-x_i}$$

$$
\begin{aligned}
f_{x_i, x_j} &= f(x_i | x_j) f(x_j) \\
&= \frac{(m-x_j)!}{x_i!(m-x_j-x_i)!} \left(\frac{p_i}{1-p_j}\right)^{x_i} \left(1 - \frac{p_i}{1-p_j}\right)^{m-x_j-x_i} \frac{m!}{x_j!(m-x_j)!} p_j^{x_j} (1-p_j)^{m-x_j} \\
&= \frac{m!}{x_i! x_j! (m-x_j-x_i)!} p_i^{x_i} (1-p_i-p_j)^{m-x_j-x_i} p_j^{x_j}
\end{aligned}
$$

4.39 cont.

$$
\begin{aligned}
E(X_iX_j) &= \sum_{x_i=0}^{m}\sum_{x_j=0}^{m-x_i} x_ix_j\frac{m!}{x_i!x_j!(m-x_j-x_i)!}p_i^{x_i}(1-p_i-p_j)^{m-x_j-x_i}p_j^{x_j} \\
&= \sum_{x_i=0}^{m} x_i\frac{m!}{x_i!(m-x_i)!}p_i^{x_i}\sum_{x_j=0}^{m-x_i} x_j\frac{(m-x_i)!}{x_j!(m-x_j-x_i)!}(1-p_i-p_j)^{m-x_j-x_i}p_j^{x_j} \\
&= \sum_{x_i=1}^{m} x_i\frac{m!}{x_i!(m-x_i)!}p_i^{x_i}\sum_{x_j=1}^{m-x_i} \frac{(m-x_i)!}{(x_j-1)!(m-x_j-x_i)!}(1-p_i-p_j)^{m-x_j-x_i}p_j^{x_j} \\
&= \sum_{x_i=1}^{m} x_i\frac{m!}{x_i!(m-x_i)!}p_i^{x_i}(m-x_i)p_j\sum_{z=0}^{m-x_i-1} \frac{(m-x_i-1)!}{z!(m-x_i-z-1)!}(1-p_i-p_j)^{m-x_i-z-1}p_j^{z} \\
&= \sum_{x_i=1}^{m} x_i\frac{m!}{x_i!(m-x_i)!}p_i^{x_i}(m-x_i)p_j(p_j+1-p_i-p_j)^{m-x_i-1} \\
&= p_j\sum_{x_i=1}^{m-1} \frac{m!}{(x_i-1)!(m-x_i-1)!}p_i^{x_i}(1-p_i)^{m-x_i-1} \\
&= m(m-1)p_jp_i\sum_{y=0}^{m-2} \frac{m-2!}{y!(m-2-y)!}p_i^{y}(1-p_i)^{m-2-y} \\
&= m(m-1)p_jp_i
\end{aligned}
$$

$$
\begin{aligned}
Cov(X_i,X_j) &= E(X_iX_j) - E(X_i)E(X_j) \\
&= m(m-1)p_jp_i - mp_imp_j \\
&= (m^2-m)p_jp_i - m^2p_ip_j \\
&= -mp_jp_i
\end{aligned}
$$

Review C+B 4.46

2. C+B 4.46: A derivation of the bivariate normal distribution: Let Z_1 and Z_2 be independent $n(0,1)$ random variables, and define new random variables X and Y by,

$$X = a_X Z_1 + b_X Z_2 + c_X \text{ and } Y = a_Y Z_1 + b_Y Z_2 + c_Y$$

where $a_X, b_X, c_X, A_Y, B_Y, C_Y$ are constants.

$E(Z_1) = E(Z_2) = 0$ and $V(Z_1) = V(Z_2) = 1 = E(Z^2) - [E(Z)]^2 \implies E(Z^2) = V(Z) + [E(Z)]^2 = 1 + 0 = 1$

(a) Show that

$$E(X) = c_X, \; V(X) = a_X^2 + b_X^2$$
$$E(Y) = c_Y, \; V(Y) = a_Y^2 + b_Y^2$$
$$\mathrm{Cov}(X,Y) = a_X a_Y + b_X b_Y$$

$$E(X) = E(a_X Z_1 + b_X Z_2 + c_X) = a_X E(Z_1) + b_X E(Z_2) + c_X = a_X(0) + b_X(0) + c_X = c_X$$
$$V(X) = V(a_X Z_1 + b_X Z_2 + c_X) = a_X^2 V(Z_1) + b_X^2 V(Z_2) + 0 = a_X^2 + b_X^2$$
$$E(Y) = E(a_Y Z_1 + b_Y Z_2 + c_Y) = a_Y E(Z_1) + b_Y E(Z_2) + c_Y = a_Y(0) + b_Y(0) + c_Y = c_Y$$
$$V(Y) = V(a_Y Z_1 + b_Y Z_2 + c_Y) = a_Y^2 V(Z_1) + b_Y^2 V(Z_2) + 0 = a_Y^2 + b_Y^2$$

4.46 cont.

$$\begin{aligned}
E(XY) &= E[(a_X Z_1 + b_X Z_2 + c_X)(a_Y Z_1 + b_Y Z_2 + c_Y)] \\
&= E[a_X a_Y Z_1^2 + a_X b_Y Z_1 Z_2 + a_X c_Y Z_1 + b_X a_Y Z_1 Z_2 + b_X b_Y Z_2^2 + b_X c_Y Z_2 + c_X a_Y Z_1 + c_X b_Y Z_2 + c_X c_Y \\
&= E(a_X a_Y Z_1^2) + E(a_X b_Y Z_1 Z_2) + E(a_X c_Y Z_1) + E(b_X a_Y Z_1 Z_2) \\
&+ E(b_X b_Y Z_2^2) + E(b_X c_Y Z_2) + E(c_X a_Y Z_1) + E(c_X b_Y Z_2) + E(c_X c_Y) \\
&= a_X a_Y E(Z_1^2) + 0 + 0 + 0 + a_Y b_Y E(Z_2^2) + 0 + 0 + 0 + c_X c_Y \\
&= a_X a_Y + b_X b_Y + c_X c_Y
\end{aligned}$$

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = a_X a_Y + b_X b_Y + c_X c_Y - c_X c_Y = a_X a_Y + b_X b_Y$$

(b) If we define the constants $a_X, b_X, c_X, a_Y, b_Y, c_Y$ by

$$a_X = \sigma_X \sqrt{\frac{1+\rho}{2}},\ b_X = \sigma_X \sqrt{\frac{1-\rho}{2}},\ c_X = \mu_X$$

$$a_Y = \sigma_Y \sqrt{\frac{1+\rho}{2}},\ b_Y = -\sigma_Y \sqrt{\frac{1-\rho}{2}},\ c_Y = \mu_Y$$

where $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$ are constants, $-1 \le \rho \le 1$, then show that

$$E(X) = \mu_X,\ V(X) = \sigma_X^2$$

$$E(Y) = \mu_Y,\ V(Y) = \sigma_Y^2$$

$$\rho_{XY} = \rho$$

$$E(X) = c_X = \mu_X$$

$$V(X) = a_X^2 + b_X^2 = \sigma_X^2(\frac{1+\rho}{2}) + \sigma_X^2(\frac{1-\rho}{2}) = \sigma_X^2(\frac{1+\rho+1-\rho}{2}) = \sigma_X^2(\frac{2}{2}) = \sigma_X^2$$

$$E(Y) = c_Y = \mu_Y$$

$$V(Y) = a_Y^2 + b_Y^2 = \sigma_Y^2(\frac{1+\rho}{2}) + \sigma_Y^2(\frac{1-\rho}{2}) = \sigma_Y^2(\frac{1+\rho+1-\rho}{2}) = \sigma_Y^2(\frac{2}{2}) = \sigma_Y^2$$

$$\rho_{XY} = \frac{\text{Cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{\sigma_X\sigma_Y(\frac{1+\rho}{2}) - \sigma_X\sigma_Y(\frac{1-\rho}{2})}{\sigma_X\sigma_Y} = \frac{1+\rho-1+\rho}{2} = \frac{2\rho}{2} = \rho$$

(c) Show that (X,Y) has the bivariate normal pdf with parameters $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$ Solve for Z_1 and Z_2 in terms of X and Y:

$$\begin{cases} X = a_X Z_1 + b_X Z_2 + c_X\ (\cdot a_Y) \\ Y = a_Y Z_1 + b_Y Z_2 + c_Y\ (\cdot a_X) \end{cases}$$

$$-\begin{cases} a_Y X = a_X a_Y Z_1 + a_Y b_X Z_2 + a_Y c_X \\ a_X Y = a_X a_Y Z_1 + a_X b_Y Z_2 + a_X c_Y \end{cases}$$

$$a_Y X - a_X Y = (a_Y b_X - a_X b_Y) Z_2 + a_Y c_X - a_X c_Y$$

$$\begin{aligned}
\implies Z_2 &= \frac{a_Y X - a_X Y - a_Y c_X + a_X c_Y}{a_Y b_X - a_X b_Y} = \frac{a_Y(X - c_X) + a_X(c_Y - Y)}{a_Y b_X - a_X b_Y} \\
&= \frac{\sigma_Y(X-\mu_X)\sqrt{1+\rho} + \sigma_X(\mu_Y - Y)\sqrt{1+\rho}}{\sigma_X\sigma_Y\sqrt{2(1-\rho^2)}} = \frac{\sigma_Y(X-\mu_X) + \sigma_X(\mu_Y - Y)}{\sigma_X\sigma_Y\sqrt{2(1-\rho)}}
\end{aligned}$$

$$\begin{cases} X = a_X Z_1 + b_X Z_2 + c_X \ (\cdot b_Y) \\ Y = a_Y Z_1 + b_Y Z_2 + c_Y \ (\cdot b_X) \end{cases}$$

$$-\begin{cases} b_Y X = a_X b_Y Z_1 + b_Y b_X Z_2 + b_Y c_X \\ b_X Y = b_X a_Y Z_1 + b_X b_Y Z_2 + b_X c_Y \end{cases}$$

$$b_Y X - b_X Y = (a_X b_Y - b_X a_Y) Z_1 + b_Y c_X - b_X c_Y$$

$$\begin{aligned} \implies Z_1 &= \frac{b_Y X - b_X Y - b_Y c_X + b_X c_Y}{a_X b_Y - a_Y b_X} = \frac{b_Y (X - c_X) + b_X (c_Y - Y)}{a_X b_Y - a_Y b_X} \\ &= \frac{-\sigma_Y (X - \mu_X)\sqrt{1-\rho} + \sigma_X(\mu_Y - Y)\sqrt{1-\rho}}{-\sigma_X \sigma_Y \sqrt{2(1-\rho^2)}} = \frac{\sigma_Y(X-\mu_X) + \sigma_X(Y-\mu_Y)}{\sigma_X\sigma_Y\sqrt{2(1+\rho)}} \end{aligned}$$

Then the Jacobian is:

$$\begin{aligned} J = \begin{vmatrix} \frac{\partial Z_1}{\partial X} & \frac{\partial Z_1}{\partial Y} \\ \frac{\partial Z_1}{\partial X} & \frac{\partial Z_1}{\partial Y} \end{vmatrix} &= \begin{vmatrix} \frac{b_Y}{a_X b_Y - a_Y b_X} & \frac{-b_X}{a_X b_Y - a_Y b_X} \\ \frac{a_Y}{b_X a_Y - a_X b_Y} & \frac{-a_X}{b_X a_Y - a_X b_Y} \end{vmatrix} = \begin{vmatrix} \frac{b_Y}{a_X b_Y - a_Y b_X} & \frac{-b_X}{a_X b_Y - a_Y b_X} \\ \frac{-a_Y}{a_X b_Y - a_Y b_X} & \frac{a_X}{a_X b_Y - a_Y b_X} \end{vmatrix} \\ &= \frac{a_X b_Y}{(a_X b_Y - a_Y b_X)^2} - \frac{a_Y b_X}{(a_X b_Y - a_Y b_X)^2} = \frac{a_X b_Y - a_Y b_X}{(a_X b_Y - a_Y b_X)^2} = \frac{1}{a_X b_Y - a_Y b_X} \\ &= \frac{1}{-\sigma_X \sigma_Y \sqrt{1-\rho^2}} \end{aligned}$$

Then we apply the transformation formula:

$$\begin{aligned} f_{X,Y} &= f_{Z_1,Z_2}(Z_1^{-1}, Z_2^{-1}) |J(Z_1^{-1}, Z_2^{-1})| \\ &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\frac{[\sigma_Y(X-\mu_X) + \sigma_X(Y-\mu_Y)]^2}{2(1+\rho)\sigma_X^2\sigma_Y^2}\right] \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\frac{[\sigma_Y(X-\mu_X) + \sigma_X(\mu_Y - Y)]^2}{2(1-\rho)\sigma_X^2\sigma_Y^2}\right] \\ &\times \frac{1}{\sigma_X\sigma_Y\sqrt{1-\rho^2}} \\ &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\frac{X-\mu_X}{\sigma_X}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] \ -\infty < x < \infty, \ -\infty < \end{aligned}$$

a bivariate normal pdf.

(d) If we start with bivariate normal parameters $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$, we can define constants $a_X, b_X, c_X, a_Y, b_Y, c_Y$ as the solutions to the equations

$$\mu_X = c_X, \ \sigma_X^2 = a_X^2 + b_X^2$$

$$\mu_Y = c_Y, \ \sigma_Y^2 = a_Y^2 + b_Y^2$$

$$\rho\sigma_X\sigma_Y = a_X a_Y + b_X b_Y$$

Show that the solution given in part (b) is not unique by exhibiting another solution to these equations. How many solutions are there?

We must have $c_X = \mu_X$ and $c_Y = \mu_Y$.

For the variance, we can find more solutions because we have 2 variables a_X, b_X or a_Y, b_Y to define. Further, the equation is an ellipse:

$$1 = \frac{a_X^2}{\sigma_X^2} + \frac{b_X^2}{\sigma_X^2}$$

C&B 4.46 cont.

Thus there are an infinite number of solutions along the ellipse. Thus another solution to this problem is:

$$a_X = \rho\sigma_X b_X = \sigma_X\sqrt{1-\rho^2}$$
$$a_Y = \sigma_Y b_Y = 0$$
$$b_X = \pm\sqrt{\sigma_X^2 - a_X^2}$$
$$b_Y = \pm\sqrt{\sigma_Y^2 - a_Y^2}$$

Thus there are an infinite number of solutions along the ellipse. Thus another solution to this problem is:

$$a_X = \rho\sigma_X b_X = \sigma_X\sqrt{1-\rho^2}$$
$$a_Y = \sigma_Y b_Y = 0$$
$$b_X = \pm\sqrt{\sigma_X^2 - a_X^2}$$
$$b_Y = \pm\sqrt{\sigma_Y^2 - a_Y^2}$$

4.51 Approach -I

3. C+B 4.51: Let X, Y be independent uniform(0,1) random variables.

(a) Find $\Pr\{\frac{X}{Y} \le t\}$ and $\Pr\{XY \le t\}$

$\Pr\{XY \le t\}$: we need to define the distribution of $Z = XY$ and let $W = Y$. Then the inverses are: $X = \frac{Z}{W}$ and $Y = W$. The domain of these spaces is $X \in [0,1],\ Y \in [0,1]$. Then the Jacobian is:

$$J = \begin{vmatrix} \frac{\partial Y}{\partial W} & \frac{\partial Y}{\partial Z} \\ \frac{\partial X}{\partial W} & \frac{\partial X}{\partial Z} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{-z}{w^2} & \frac{1}{w} \end{vmatrix} = \frac{1}{w}$$

The joint distribution of z, w is:

$$f_{Z,W}(z,w) = \begin{cases} \frac{1}{w}, & \text{if } 0 < z < w < 1 \\ 0, & \text{otherwise} \end{cases}$$

The marginal distribution for Z:

$$f_Z(z) = \begin{cases} \int_z^1 \frac{1}{w}\,dw = \log w\Big|_{w=z}^{w=1} = -\log z, & \text{if } 0 < z < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then the cumulative distribution is:

$$F_Z(z) = \begin{cases} \int_0^z -\log z\,dz = z - z\log z\Big|_{z=0}^{z=1} = z - z\log z, & \text{if } 0 < z < 1 \\ 0, & \text{if } z < 0 \\ 1, & \text{if } z > 1 \end{cases}$$

$$\implies F_Z(t) = \begin{cases} t - t\log t, & \text{if } 0 < t < 1 \\ 0, & \text{if } t < 0 \\ 1, & \text{if } t > 1 \end{cases}$$

$\Pr\{\frac{X}{Y} \le t\}$: define $Z = \frac{X}{Y}, W = XY$. Then the inverse functions are: $Y = \sqrt{\frac{W}{Z}}, X = \sqrt{WZ}$. Then the Jacobian is:

$$J = \begin{vmatrix} \frac{\partial Y}{\partial W} & \frac{\partial Y}{\partial Z} \\ \frac{\partial X}{\partial W} & \frac{\partial X}{\partial Z} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{wz}} & -\frac{\sqrt{w}}{2z\sqrt{z}} \\ \frac{\sqrt{z}}{2\sqrt{w}} & \frac{\sqrt{w}}{2\sqrt{z}} \end{vmatrix} = \frac{1}{4z} + \frac{1}{4z} = \frac{1}{2z}$$

Then the joint distribution of z, w is:

$$f_{Z,W}(z,w) = \begin{cases} \frac{1}{2z}, & \text{if } w \leq \frac{1}{z}, 0 < w < 1, z > w \\ 0, & \text{otherwise} \end{cases}$$

The marginal distribution of Z:

$$f_Z(z) = \begin{cases} \int_0^z \frac{1}{2z}\ dw = \frac{w}{2z}\Big|_{w=0}^{w=z} = \frac{z}{2z} = \frac{1}{2}, & \text{if } 0 < z < 1 \\ \int_0^{\frac{1}{z}} \frac{1}{2z}\ dw = \frac{w}{2z}\Big|_{w=0}^{w=\frac{1}{z}} = \frac{\frac{1}{z}}{2z} = \frac{1}{2z^2}, & \text{if } z > 1 \\ 0, & \text{otherwise} \end{cases}$$

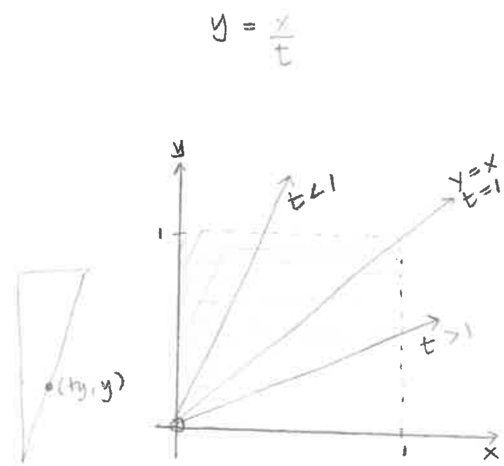
Then the cumulative distribution is:

$$F_Z(z) = \begin{cases} \int_0^z \frac{1}{2}\ dz = \frac{z}{2}\Big|_{z=0}^{z=z} = \frac{z}{2}, & \text{if } 0 < z < 1 \\ 1 - \int_1^z \frac{1}{2z^2}\ dz = 1 - \frac{-1}{2z}\Big|_{z=1}^{z=z} = 1 + \frac{1}{2z} - \frac{1}{2} = \frac{1}{2} - \frac{1}{2z}, & \text{if } z > 1 \\ 0, & \text{if } z < 0 \end{cases}$$

$$\implies F_Z(t) = \begin{cases} \frac{t}{2}, & \text{if } 0 < t < 1 \\ \frac{1}{2} - \frac{1}{2t}, & \text{if } t > 1 \\ 0, & \text{if } t < 0 \end{cases}$$

4.51 Approach-2

3. Casella and Berger 4.51



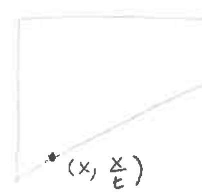
a)

For $t \leq 1$:

$$
\begin{aligned}
P(X/Y \leq t) &= \int_0^1 \int_0^{ty} 1 dx\, dy \\
&= \int_0^1 ty\, dy \\
&= ty^2/2 \Big|_0^1 \\
&= t/2
\end{aligned}
$$

For $t > 1$:

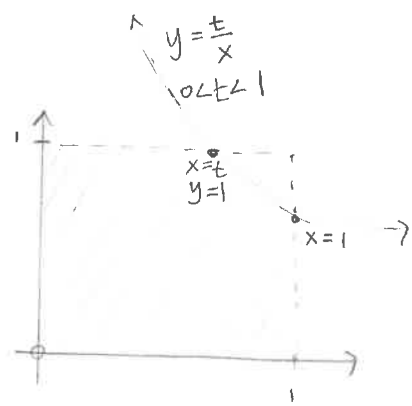
$$
\begin{aligned}
P(X/Y \leq t) &= \int_0^1 \int_{x/t}^1 1 dy\, dx \\
&= \int_0^1 \int_{x/t}^1 1 - x/t\, dx \\
&= x - x^2/2t \Big|_0^1 \\
&= 1 - 1/2t
\end{aligned}
$$



b)

For $0 < t < 1$:

$$
\begin{aligned}
P(XY \leq t) &= 1 - \int_t^1 \int_{t/x}^1 1 dy\, dx \\
&= 1 - \int_t^1 1 - t/x\, dx \\
&= 1 - [x - t \ln x] \Big|_t^1 \\
&= 1 - [1 - t \ln 1 - 1 + t \ln t] \\
&= t - t \ln t
\end{aligned}
$$



c)

For $t > 1$, $t/x > 1$:

$$\begin{aligned}
P(XY/Z \leq t) &= \int_0^1 P(Y/Z \leq t/x)dx \\
&= \int_0^1 1 - (1/2)(x/t)dx \\
&= x - (1/2)(x^2/2t)\Big|_0^1 \\
&= 1 - (1/4t)
\end{aligned}$$

For $t < 1$, $t/x > 1$ for $x \in (0,t)$ and $t/x < 1$ for $x \in (t,1)$:

$$\begin{aligned}
P(XY/Z \leq t) &= \int_0^1 P(Y/Z \leq t/x)dx \\
&= \int_0^t 1 - (1/2)(x/t)dx + \int_t^1 (1/2)(t/x)dx \\
&= x - (1/2)(x^2/2t)\Big|_0^t + (1/2)t\ln x\Big|_t^1 \\
&= t - (t^2/4t) - (1/2)t\ln t \\
&= 3t/4 - (t/2)\ln t
\end{aligned}$$

Review C+B 5.5

2. C+B 5.5: Let $X_1, \ldots, X_n$ be iid with pdf $f_X(x)$ and let $\bar{X}$ denote the sample mean. Show that

$$f_{\bar{X}}(x) = n f_{x_1+\cdots+x_n}(nx)$$

even if the mgf of X does not exist.

Proof.

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Now let

$$Y = \sum_{i=1}^{n} X_i \implies \bar{X} = \frac{1}{n}Y$$

which is a scale transformation: $\frac{1}{w}f(\frac{x}{w})$, where $w = \frac{1}{n}$

$$\implies f_{\bar{X}}(x) = \frac{1}{\frac{1}{n}} f_Y\left(\frac{x}{\frac{1}{n}}\right) = n f_Y(nx) = n f_{x_1+\cdots+x_n}(nx)$$

$\square$