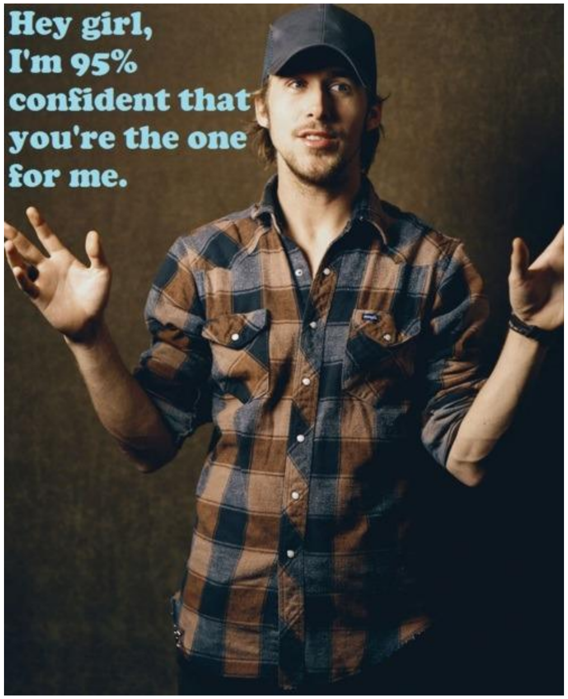

BIOS 6612
Lecture 4

Logistic Regression III
Wald, Score, & LRT

KKMN Chapters 21, 22

Agresti (2002) Catgorical Data Analysis, 2nd Edition. Section 4.2, Chapter 5 up to 5.1.3, Section 6.6

Vittinghoff. Regression Methods in Biostatistics. Chapter 6

Review (Lecture 3)/ Current (Lecture 4)/ Preview (Lecture 5)

- Lecture 3: Logistic Regression II
  - Maximum Likelihood Estimation
    - Analytic solution for intercept only model
    - No analytic solution for more complicated models
      - Newton Raphson Algorithm
- Lecture 4: Logistic Regression III
  - Hypothesis testing
    - Wald
    - Score
    - Likelihood Ratio Test (LRT)
- Lecture 5: Logistic Regression IV
  - Comparing Models
    - LRTs
  - Interactions

Likelihood Ratio Test

- Likelihood Ratio Test / Likelihood Ratio Statistic
  - The ratio of the likelihood at the hypothesized parameter values to the likelihood of the data at the MLE

    LR= -2 log([Likelihood at H_0] / [Likelihood at MLE])= -2LogL(H_0) - -2LogL(MLE)

    - For large n, the LRT statistic has a chi-square distribution with degrees of freedom based on the number of parameters being estimated.

  - The Likelihood Ratio Test Statistic can also be viewed as the change in value of the log-likelihood (Log L) between two nested models (a full model with and reduced model without the parameter(s) of interest).

    $$-2LogL(reduced) - -2LogL(full) \sim \chi^2_{p(full)-p(reduced)}$$

    p is the number of parameters

    You can use AIC or SC(BIC) for non-nested models.

  - This test is similar to the multiple partial *F* test in ideology

  - Performs best in small samples

Wald Test

- Wald (Chi-Square) Test Statistic (reported as Z or $Z^2=\chi^2$)
  - The Wald Chi-Square Statistic is a function of the difference in the MLE and the hypothesized value, normalized by an estimate of the standard error which is calculated by the inverse of the information matrix
  - This is the p-value reported in many parameter estimate tables

- Wald tests are constructed using MLE's and estimates of their variance obtained by maximizing the likelihood under the alternative hypothesis

- It has long been known that Wald tests and confidence intervals can behave in an aberrant manner (e.g., Fears et al., Am. Stat., 1996, Hauck and Donner, JASA, 1977, Meeker and Escobar, Am. Stat., 1995)

- For example, Hauck and Donner (1977) have shown that the power of Wald tests for logistic regression decreases to its Type I error rate for alternative values sufficiently far from the null value

- However, Wald tests easily come out of any software package and they relate to CI estimation

Score Test

- Score Test / Score Statistic
    - The score test statistic is based on the first derivative of the log-likelihood evaluated under the null hypothesis

    - Score functions can be interpreted as the slope at a point on the likelihood function
        - The variance of the score function is equal to the expected information

        - The information is a function of the curvature at a point on the likelihood function, and hence of the speed with which the slope changes

- Consider testing H_0: $\theta = \theta^*$
    - The first derivative of the log-likelihood denoted by

$$U(\theta) = \frac{\partial \log L(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial \log L_i(\theta)}{\partial \theta}$$

    where $L_i(\theta)$ is the likelihood from the ith observation

    - Since the score can be written as a sum of independent observations, we can apply the Central Limit Theorem

$$U(\theta^*) \sim N(0, \mathrm{Var}[U(\theta^*)]) \;\; b/c \;\; \mathrm{E}[U(\theta^*)] = 0 \;\; under \;\; H_0$$

- The score test statistic for H_0: $\theta = \theta^*$ is

$$\chi_S^2 = \frac{U(\theta^*)^2}{Var[U(\theta^*)]}$$

Notes

- The sampling distributions for Wald, score, and likelihood ratio test statistics are asymptotically chi-square when the null hypothesis is true, and subject to the usual assumptions
- Degrees of freedom for these test statistics are equal to the numbers of parameters being tested
- Degrees of freedom can also be determined as the difference in the number of unknown parameters under the alternative and the null hypothesis
- Wald, score, and likelihood ratio test statistics are asymptotically equivalent when the null hypothesis is true
- They differ in small samples or when the alternative hypothesis is true
    - Score tests tend to give Type I error rates closest to the nominal levels when the sample size is small
    - Likelihood ratio tests tend to have the greatest power

Confidence Interval Construction

- Each of the three test statistics (Wald, Score, and Likelihood Ratio) can be inverted to construct confidence intervals (Alho, Statistics in Medicine, 1992) for $H_0 : \beta_1 = \beta_1^*$

- The 95% confidence interval for an odds ratio obtained by inverting a Wald test constructed using logistic regression is given by

$$\left\{\exp\left(\hat{\beta}_1 - 1.96\hat{\mathrm{S}}\mathrm{E}[\hat{\beta}_1]\right), \exp\left(\hat{\beta}_1 + 1.96\hat{\mathrm{S}}\mathrm{E}[\hat{\beta}_1]\right)\right\}$$

- The 95% confidence interval for an odds ratio obtained by inverting a likelihood ratio test is found by solving the equation

$$2\log\left(\frac{L(\hat{\beta}_0, \hat{\beta}_1)}{L(\tilde{\beta}_0 \mid \beta_1 = \beta_1^*)}\right) \le \chi^2_{1,1-\alpha}$$

  - where $\chi^2_{1,1-\alpha}$ is the (1 − α)th quantile from a chi- square distribution with 1 degree of freedom. Accordingly $\chi^2_{1,1-0.05} = 3.841$
  - These CI's are called profile likelihood-based confidence intervals

- Confidence intervals obtained by inverting likelihood ratio and score tests require iterative solutions for even quite simple problems

Bernoulli Example: LRT

- Recall:
  - Pr(Y=1)=p and the likelihood of the data $L = \prod_{i=1}^{n} p^{Yi}(1-p)^{1-Yi}$
  - Log Likelihood $LL = \ln(L) = \ln\left(\prod_{i=1}^{n} p^{Yi}(1-p)^{1-Yi}\right) = \left(\sum_{i=1}^{n} Y_i\right)\ln(p) + \left(n - \sum_{i=1}^{n} Y_i\right)\ln(1-p)$
  - The MLE, which maximizes the log likelihood is $\hat{p} = \dfrac{\sum_{i=1}^{n} Y_i}{n}$
- For the null hypothesis: H0:p=p_0 vs HA: p≠ p_0
  - Log L(H_0) $= \left(\sum_{i=1}^{n} Y_i\right)\ln(p_0) + \left(n - \sum_{i=1}^{n} Y_i\right)\ln(1-p_0)$
  - Log L(MLE) $= \left(\sum_{i=1}^{n} Y_i\right)\ln(\hat{p}) + \left(n - \sum_{i=1}^{n} Y_i\right)\ln(1-\hat{p})$

- The LRT statistic $= -2\left[\left(\sum_{i=1}^{n} Y_i\right)\ln\left(\dfrac{p_0}{\hat{p}}\right) + \left(n - \sum_{i=1}^{n} Y_i\right)\ln\left(\dfrac{1-p_0}{1-\hat{p}}\right)\right] \sim \chi_1^2$

Bernoulli Example: Wald Test

- Fisher information

$$I(p) = E\left[-\frac{\partial^2 LL}{\partial p^2}\right] = E\left[\frac{\sum_{i=1}^{n} Y_i}{p^2} + \frac{n - \sum_{i=1}^{n} Y_i}{\left(1-p\right)^2}\right] = \frac{n}{p(1-p)}$$

- Substituting the MLE of p into $I(p)$, yielding: $I(\hat{p}) = \dfrac{n}{\hat{p}(1-\hat{p})}$

- The inverse of the information evaluated at the MLE: $\text{var}(\hat{p}) = I(\hat{p})^{-1} = \dfrac{\hat{p}(1-\hat{p})}{n}$

- For the null hypothesis: H0:p=p_0 vs HA: p$\neq$ p_0

  - The Wald statistic $\dfrac{\left(\hat{p}-p_0\right)^2}{I(\hat{p})^{-1}} = \dfrac{\left(\hat{p}-p_0\right)^2}{\left(\dfrac{\hat{p}(1-\hat{p})}{n}\right)} \sim \chi_1^2$

    - Issues for $\hat{p}$ near 0 or 1
    - In R, it reports the square root of the above test statistic, which is distributed N(0,1)

Bernoulli Example: Score Test

- The score function $U(p) = \frac{\partial LL}{\partial p} = \frac{\sum_{i=1}^{n} Y_i}{p} - \frac{n - \sum_{i=1}^{n} Y_i}{1-p}$

- Fisher's Information $I(p) = E\left[-\frac{\partial^2 LL}{\partial p^2}\right] = E\left[\frac{\sum_{i=1}^{n} Y_i}{p^2} + \frac{n - \sum_{i=1}^{n} Y_i}{(1-p)^2}\right] = \frac{n}{p(1-p)}$

- For the null hypothesis: H0:p=p_0 vs HA: p≠ p_0

  - The score test= $\frac{U(p_0)^2}{I(p_0)} = \frac{\left(\frac{\sum_{i=1}^{n} Y_i}{p_0} - \frac{n - \sum_{i=1}^{n} Y_i}{1-p_0}\right)^2}{\left(\frac{n}{p_0(1-p_0)}\right)} \sim \chi_1^2$

Multiple Logistic Regression

$$\text{logit}(\pi_i) = \mathbf{X}_i^T\boldsymbol{\beta} = \beta_0 + \beta_1 X_{1i} + \ldots + \beta_p X_{pi} \; \textit{where} \; \pi_i = P(\mathrm{Y}_i = 1 \mid \mathbf{X}_i)$$

$$\mathrm{L}(\beta) = \prod_{i=1}^{n}\left\{\pi_i^{Y_i}(1-\pi_i)^{1-Y_i}\right\} = \prod_{i=1}^{n}\left\{\left(\frac{\exp(\mathbf{X}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i^T\boldsymbol{\beta})}\right)^{Y_i}\left(\frac{1}{1+\exp(\mathbf{X}_i^T\boldsymbol{\beta})}\right)^{1-Y_i}\right\} = \frac{\exp\left[\sum_{i=1}^{n} Y_i\left(\mathbf{X}_i^T\boldsymbol{\beta}\right)\right]}{\prod_{i=1}^{n}\left[1+\exp\left(\mathbf{X}_i^T\boldsymbol{\beta}\right)\right]}$$

$$U = \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_p \end{bmatrix} = \sum_{i=1}^{n}\begin{bmatrix} 0 \\ X_{1i} \\ \vdots \\ X_{pi} \end{bmatrix}(\mathrm{Y}_i - \hat{\pi}_i) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \sum_{i=1}^{n}\mathbf{X}_i\left(Y_i - \hat{\pi}_i\right) = 0 \;\; \textit{where} \;\; \mathbf{X}_i^T = (1 \quad X_{1i} \quad \ldots \quad X_{pi})$$

$$I(\beta) = \sum_{i=1}^{n}\pi_i\left(1-\pi_i\right)\mathbf{X}_i\mathbf{X}_i^T$$

$$\textit{In large samples } \hat{\beta} \sim MVN_{(\mathrm{p}+1)}\left(\beta, Var(\hat{\beta})\right) \;\; \textit{where } Var(\hat{\beta}) = \left[I(\beta)\right]^{-1}$$

- There is usually no closed form solution for the MLE. (Newton-Raphson)

Multiple Logistic Regression: Likelihood Ratio Tests

- Hypothesis tests for two or more parameters are usually constructed using likelihood ratio tests

- For example, for the logistic regression model

  - $H_0 : \beta_1 = \beta_2 = 0$ vs
  - H_A : At least one of β_1 or β_2 does not = 0

- Can be tested using the two degree of freedom likelihood ratio statistic given by

$$\text{logit}(\pi_i) = \mathbf{X}_i^T \boldsymbol{\beta} = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i}$$

$$\chi^2_{LR} = 2\left\{\log[\mathrm{L}(\hat{\beta} \mid \mathrm{H}_A)] - \log[\mathrm{L}(\tilde{\beta} \mid \mathrm{H}_0)]\right\} = 2\sum_{i=1}^{n}\left[Y_i \log\left(\frac{\hat{\pi}_i}{\tilde{\pi}_i}\right) + (1-Y_i)\log\left(\frac{1-\hat{\pi}_i}{1-\tilde{\pi}_i}\right)\right]$$

Multiple Logistic Regression: Wald Tests and Confidence Intervals

- In large samples

$$\hat{\beta}_j \sim N(\beta_j, \mathrm{Var}\left(\hat{\beta}_j\right)), \mathrm{j} = 0,\ldots,\mathrm{p}$$

$$\mathrm{H}_o : \beta_j = \beta_{j0}$$

$$\chi_W^2 = \frac{(\hat{\beta}_j - \beta_{j0})^2}{V\hat{a}r(\hat{\beta}_j)}$$

$$95\%CI \;\; for \;\; \exp(\beta_j) \Rightarrow \left(\exp(\hat{\beta}_j - 1.96S\hat{E}(\hat{\beta}_j)) \quad \exp(\hat{\beta}_j + 1.96S\hat{E}(\hat{\beta}_j))\right)$$

• Multiple Logistic Regression: Score Tests

- Score tests for single parameters, e.g., $H_0 : \beta_1 = 0$ can be written as

$$\chi_S^2 = \frac{\left\{\sum_{i=1}^{n} X_{i1}(Y_i - \tilde{\pi}_i)\right\}^2}{V\hat{a}r\left\{\sum_{i=1}^{n} X_{i1}(Y_i - \tilde{\pi}_i)\right\}} \; where \; \tilde{\pi}_i$$

denotes the estimate obtained under the null hypothesis

- In general iteration is required to obtain πthe MLE (i.e. Newton Raphson)

- Simple algebraic expressions are assured only when the model specified under the null is saturated
  - i.e., there are as many data points as there are parameters, e.g., logistic regression models with a single factor taking on K levels modeled using K−1 dummy variables

- Score tests and confidence intervals?
  - Conceptually straightforward, somewhat difficult to compute

Example: Logistic regression model examining factors associated with post-surgery hypothermia at PACU entry (tympanic temperature ≤ 96.8 degrees).

```
PROC LOGISTIC;
 MODEL hypotherm (EVENT= 'Yes') = age surgtime bmi blanket ;
RUN;
```

MODEL 1:
- Age
- Surgery Time
- BMI
- Use of Warmed Blanket

```
                   The LOGISTIC Procedure

                     Model Information
Data Set                        WORK.BODYCOVERS
Response Variable               hypotherm Hypothermia
Number of Response Levels       2
Number of Observations          60
Model                           binary logit
Optimization Technique          Fisher's scoring
```

Response Profile

Ordered Value	hypotherm	Total Frequency
1	Yes	14
2	No	46

Probability modeled is hypotherm='Yes'.

Model Fit Statistics

Criterion	Intercept Only	Intercept and Covariates
AIC	67.193	59.199
SC	69.287	69.670
-2 Log L	65.193	49.199

-2LL for Intercept and Covariates: See Next Page

-2LL for Intercept Only:

$\hat{p} = 14/60 = 0.23333$

LL = 14LN(0.2333)+ 46LN(1-.23333) = -32.596

-2LL = -2 ×-32.596 = 65.193

Calculation of -2LL

Parameter	DF	Estimate	Standard Error
Intercept	1	2.8881	2.6575
age	1	0.0143	0.0242
surgtime	1	1.0930	0.5332
bmi	1	-0.2413	0.0963
blanket	1	-0.9906	0.8068

```
DATA phat;
SET phat;
ll = hypotherm*LOG(phat)+(1-hypotherm)*LOG(1-phat);
RUN;

PROC PRINT DATA=phat;
RUN;

PROC MEANS DATA=phat N SUM;
VAR ll;
RUN;
```

ID #1: phat = 1/(1+e-(2.8881+0.0143*59+1.0930*1.2)-0.2431*19.2-0.9905*1

ll = LN(0.35863)

Obs	id	age	surgtime	bmi	blanket	hypotherm	phat	ll
1	1	59	1.2	19.2	Warmed	No	0.35853	-0.44400
2	2	39	1.3	26.6	Unwarmed	Yes	0.17468	-1.74482
3	3	75	1.7	23.7	Unwarmed	Yes	0.52444	-0.64543
4	4	34	0.8	24.0	Warmed	No	0.07352	-0.07636
5	5	71	1.3	18.2	Unwarmed	Yes	0.71721	-0.33239
...								
55	55	34	1.5	20.4	Warmed	Yes	0.28902	-1.24127
56	56	70	1.3	27.5	Warmed	No	0.08963	-0.09390
57	57	41	1.3	27.4	Warmed	Yes	0.06251	-2.77244
58	58	43	1.3	24.6	Warmed	No	0.11881	-0.12648
59	59	65	2.1	24.8	Unwarmed	No	0.53168	-0.75860
60	60	45	1.9	21.5	Warmed	Yes	0.36091	-1.01913

The MEANS Procedure

Analysis Variable : ll

N	Sum
60	-24.5993579

-2LL for Intercept and Covariates:
-24.599357*2 = 49.1987

Model with age, bmi, blanket, surgery time

Testing Global Null Hypothesis: BETA=0

H_0: $\beta_{age}=\beta_{surgtime}=\beta_{BMI}=\beta_{blanket} = 0$

Test	Chi-Square	DF	Pr > ChiSq
Likelihood Ratio	15.9940	4	0.0030
Score	13.9847	4	0.0073
Wald	9.5787	4	0.0482

65.193 – 49.199

= 15.9940

Analysis of Maximum Likelihood Estimates

Parameter	DF	Estimate	Standard Error	Wald Chi-Square	Pr > ChiSq
Intercept	1	2.8881	2.6575	1.1810	0.2771
age	1	0.0143	0.0242	0.3465	0.5561
surgtime	**1**	**1.0930**	**0.5332**	**4.2013**	**0.0404**
bmi	1	-0.2413	0.0963	6.2738	0.0123
blanket	1	-0.9906	0.8068	1.5075	0.2195

Odds Ratio Estimates

Effect	Point Estimate	95% Wald Confidence Limits	
age	1.014	0.967	1.064
surgtime	**2.983**	**1.049**	**8.483**
bmi	0.786	0.650	0.949
blanket	0.371	0.076	1.805

Interpretation: Length of surgery is a significant predictor of post-surgery hypothermia after adjusting for age, BMI, and use of a warmed blanket during surgery (p = 0.0404). Each one-hour increase in the length of surgery is associated with a 2.98-fold increase in the odds of post-surgery hypothermia (95% CI: 1.05,8.48).

Wald Test Statistic: $Z = 1.0930 / 0.5332 = 2.0499 \sim N(0,1)$, p = 0.0404

$Z^2 = 4.20 \sim \chi^2_1$, p = 0. 0404

Likelihood Ratio Test Statistic:

```
PROC LOGISTIC;
 MODEL hypotherm (EVENT= 'Yes') = age bmi blanket ;
RUN;
```

To perform the likelihood ratio test for the effect of surgery time, we need to compare the likelihood in the model above (-2LogL = **49.199**) to the likelihood for the model without surgery time included (-2LogL = **54.107**):

MODEL 2:
Age
BMI
Use of Warmed Blanket

```
                    The LOGISTIC Procedure
                      Model Information

Data Set                      TEACH.BODYCOVERS
Response Variable             hypotherm             Hypothermia
Number of Response Levels     2
Number of Observations        60
Model                         binary logit
Optimization Technique        Fisher's scoring

                       Response Profile
             Ordered                           Total
               Value      hypotherm        Frequency
                   1      Yes                     14
                   2      No                      46
           Probability modeled is hypotherm='Yes'.

                  Model Convergence Status
        Convergence criterion (GCONV=1E-8) satisfied.

                    Model Fit Statistics

                                            Intercept
                               Intercept       and
           Criterion             Only       Covariates
           AIC                 67.193         62.107
           SC                  69.287         70.484
           -2 Log L            65.193         54.107
```

The LOGISTIC Procedure

Testing Global Null Hypothesis: BETA=0

H_0: $\beta_{age}=\beta_{BMI}=\beta_{blanket} = 0$

Test	Chi-Square	DF	Pr > ChiSq
Likelihood Ratio	**11.0861**	**3**	**0.0113**
Score	8.8687	3	0.0311
Wald	7.4666	3	0.0584

65.193 - 54.107

= 11.0861

Analysis of Maximum Likelihood Estimates

Parameter	DF	Estimate	Standard Error	Wald Chi-Square	Pr > ChiSq
Intercept	1	3.1798	2.5322	1.5769	0.2092
age	1	0.0230	0.0233	0.9796	0.3223
bmi	1	-0.1941	0.0863	5.0588	0.0245
blanket	1	-1.1521	0.7517	2.3491	0.1254

Odds Ratio Estimates

Effect	Point Estimate	95% Wald Confidence Limits	
age	1.023	0.978	1.071
bmi	0.824	0.695	0.975
blanket	0.316	0.072	1.379

Recall (from p.17/18):
Wald Chi-Square was 4.2 ~ χ^2 , p = 0.0404

Likelihood Ratio Test Statistic:

$$LR = \underbrace{54.107}_{\text{-2LogL (reduced)}} - \underbrace{49.199}_{\text{-2LogL (full)}} = 4.908 \sim \chi_1^2, \qquad p = 0.02673$$

Model with age, bmi, blanket, surgery time and cover type

In the study, patients were randomized to one of four experimental groups to test the effect of different aluminum covers (no cover, cover A, cover B, cover C) on hypothermia.

- After adjusting for age, BMI, length of surgery, and use of a warmed blanket, is treatment group associated with hypothermia at PACU entry?

```
PROC LOGISTIC;
 CLASS covertype (REF='None') /PARAM=REF;
 MODEL hypotherm (EVENT='Yes')= age bmi blanket surgtime covertype;
RUN;
```

```
/* Request Reference
Coding with 'None' as
Reference Group */
```

```
                    The LOGISTIC Procedure

                      Model Information

Data Set                       WORK.BODYCOVERS
Response Variable              hypotherm            Hypothermia
Number of Response Levels      2
Number of Observations         60
Model                          binary logit
Optimization Technique         Fisher's scoring


                       Response Profile
              Ordered                        Total
                Value     hypotherm      Frequency

                    1     Yes                   14
                    2     No                    46

          Probability modeled is hypotherm='Yes'.
```

Model with age, bmi, blanket, surgery time and cover type

Class Level Information

Class	Value	Design Variables 1	2	3
covertype	Cover A	1	0	0
	Cover B	0	1	0
	Cover C	0	0	1
	None	0	0	0

Model Fit Statistics

Criterion	Intercept Only	Intercept and Covariates
AIC	67.193	57.335
SC	69.287	74.090
-2 Log L	**65.193**	**41.335**

Testing Global Null Hypothesis: BETA=0

Test	Chi-Square	DF	Pr > ChiSq
Likelihood Ratio	23.8572	7	0.0012
Score	19.5466	7	0.0066
Wald	12.7202	7	0.0792

Model with age, bmi, blanket, surgery time and cover type

Type III Analysis of Effects

Effect	DF	Wald Chi-Square	Pr > ChiSq
age	1	0.6687	0.4135
bmi	1	6.5588	0.0104
blanket	1	4.0341	0.0446
surgtime	1	5.1234	0.0236
covertype	**3**	**5.7673**	**0.1235**

H_0: LnOdds(coverA) = LnOdds(coverB) = LnOdds|coverC) = LnOdds(None)

H_0: $\beta_{coverA} = \beta_{coverB} = \beta_{coverC} = 0$

Analysis of Maximum Likelihood Estimates

Parameter	DF	Estimate	Standard Error	Wald Chi-Square	Pr > ChiSq
Intercept	1	4.1550	3.1010	1.7953	0.1803
age	1	0.0240	0.0293	0.6687	0.4135
bmi	1	-0.2808	0.1097	6.5588	0.0104
blanket	1	-2.0345	1.0129	4.0341	0.0446
surgtime	1	1.3908	0.6144	5.1234	0.0236
covertype Cover A	1	0.0640	1.1156	0.0033	0.9543
covertype Cover B	1	-3.1271	1.4774	4.4805	0.0343
covertype Cover C	1	-0.9523	1.2347	0.5949	0.4405

Model with age, bmi, blanket, surgery time and cover type

Odds Ratio Estimates

Effect	Point Estimate	95% Wald Confidence Limits	
age	1.024	0.967	1.085
bmi	0.755	0.609	0.936
blanket	0.131	0.018	0.952
surgtime	4.018	1.205	13.398
covertype Cover A vs None	1.066	0.120	9.493
covertype Cover B vs None	0.044	0.002	0.793
covertype Cover C vs None	0.386	0.034	4.339

<u>Likelihood Ratio Test:</u>

H_0: Log-odds|coverA = Log-odds|coverB = Log-odds|coverC = Log-odds|None

$$-2\text{LogL(reduced)} - -2\text{LogL(full)} \sim \chi^2_{p(full)-p(reduced)}$$

$$\text{LR} = 49.199 - 41.335 = 7.864 \sim \chi^2_3, \quad p = .0489$$

After adjusting for age, length of surgery, BMI, and use of a warmed blanket, treatment cover is significantly associated with hypothermia at PACU entry (p = .0489).

Practice Problem: Post-Surgery Hypothermia and Use of a Warmed Blanket.

Warmed Blanket	*Hypothermia* Yes	No	
No (0)	10	20	30
Yes (1)	4	26	30
	14	46	60

$p_1 = 10/30 = 0.3333$
$p_2 = 4/30 = 0.1333$

$\bar{p} = 14/60 = .23333$

Find β_0 and β_1 for the logistic regression equation predicting post-surgery hypothermia from use of a warmed blanket:

$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta_{blanket} blanket$$

$$\log\left(\frac{\hat{p}_{hypothermia|blanket=0}}{1-\hat{p}_{hypothermia|blanket=0}}\right) = \hat{\beta}_0 \Rightarrow \hat{\beta}_0 = \log\left(\frac{\frac{10}{30}}{1-\frac{10}{30}}\right) = -0.693$$

$$\log\left(\frac{\hat{p}_{hypothermia|blanket=1}}{1-\hat{p}_{hypothermia|blanket=1}}\right) = \hat{\beta}_0 + \hat{\beta}_{blanket} \Rightarrow \log\left(\frac{\frac{4}{30}}{1-\frac{4}{30}}\right) = \hat{\beta}_0 + \hat{\beta}_{blanket} \Rightarrow \hat{\beta}_{blanket} = \log\left(\frac{\frac{4}{30}}{1-\frac{4}{30}}\right) - \log\left(\frac{\frac{10}{30}}{1-\frac{10}{30}}\right) = -1.179$$

Warmed Blanket (E)	*Hypothermia (O)* Yes (1)	No (0)	
No (0)	10=$n_{(O=1\&E=0)}$	20=$n_{(O=0\&E=0)}$	30=$n_{(E=0)}$
Yes (1)	4=$n_{(O=1\&E=1)}$	26=$n_{(O=0\&E=1)}$	30=$n_{(E=1)}$
	14=$n_{(O=1)}$	46=$n_{(O=0)}$	60=n

p_1= 10/30 = 0.3333
p_2 = 4/30 = 0.1333

$\overline{p}$ = 14/60 = .23333

Perform a Likelihood Ratio test for the effect of a warmed blanket on post-surgery hypothermia:

$$-2\log L(\text{null}) = -2\left[\sum_{i=1}^{n} Y_i \log\left(\frac{e^{\beta_0}}{1+e^{\beta_0}}\right) + \left(n - \sum_{i=1}^{n} Y_i\right)\log\left(1 - \frac{e^{\beta_0}}{1+e^{\beta_0}}\right)\right]$$

$$= -2\left[n_{O=1}\log\left(\hat{p}_{hypothermia}\right) + n_{O=0}\log\left(\hat{p}_{NOhypothermia}\right)\right] = -2\left[n_{O=1}\log\left(\frac{n_{O=1}}{n}\right) + n_{O=0}\log\left(\frac{n_{O=0}}{n}\right)\right]$$

$$= -2\left[14\log\left(\frac{14}{60}\right) + 46\log\left(\frac{46}{60}\right)\right] = 65.19273$$

Warmed Blanket (E)	*Hypothermia (O)* Yes (1)	No (0)	
No (0)	$10=n_{(O=1\&E=0)}$	$20=n_{(O=0\&E=0)}$	$30=n_{(E=0)}$
Yes (1)	$4=n_{(O=1\&E=1)}$	$26=n_{(O=0\&E=1)}$	$30=n_{(E=1)}$
	$14=n_{(O=1)}$	$46=n_{(O=0)}$	$60=n$

Perform a Likelihood Ratio test for the effect of a warmed blanket on post-surgery hypothermia:

$$-2\log L(\text{full}) = -2\left[\sum_{i=1}^{n} Y_i \log\left(\frac{e^{\beta_0+\beta_{blanket}blanket_i}}{1+e^{\beta_0+\beta_{blanket}blanket_i}}\right) + \left(n-\sum_{i=1}^{n} Y_i\right)\log\left(1-\frac{e^{\beta_0+\beta_{blanket}blanket_i}}{1+e^{\beta_0+\beta_{blanket}blanket_i}}\right)\right]$$

$$= -2\left[n_{O=1\&E=1}\log\left(\frac{e^{\hat\beta_0+\hat\beta_{blanket}}}{1+e^{\hat\beta_0+\hat\beta_{blanket}}}\right) + n_{O=1\&E=0}\log\left(\frac{e^{\hat\beta_0}}{1+e^{\hat\beta_0}}\right) + n_{O=0\&E=1}\log\left(\frac{1}{1+e^{\hat\beta_0+\hat\beta_{blanket}}}\right) + n_{O=0\&E=0}\log\left(\frac{1}{1+e^{\hat\beta_0}}\right)\right]$$

$$= -2\left[n_{O=1\&E=1}\log\left(\hat p_{\text{O=1|E=1}}\right) + n_{O=1\&E=0}\log\left(\hat p_{\text{O=1|E=0}}\right) + n_{O=0\&E=1}\log\left(\hat p_{\text{O=0|E=1}}\right) + n_{O=0\&E=0}\log\left(\hat p_{\text{O=0|E=0}}\right)\right]$$

$$= -2\left[n_{O=1\&E=1}\log\left(\frac{n_{O=1\&E=1}}{n_{E=1}}\right) + n_{O=1\&E=0}\log\left(\frac{n_{O=1\&E=0}}{n_{E=0}}\right) + n_{O=0\&E=1}\log\left(\frac{n_{O=0\&E=1}}{n_{E=1}}\right) + n_{O=0\&E=0}\log\left(\frac{n_{O=0\&E=0}}{n_{E=0}}\right)\right]$$

$$= -2\left[4\log\left(\frac{4}{30}\right) + 10\log\left(\frac{10}{30}\right) + 26\log\left(\frac{26}{30}\right) + 20\log\left(\frac{20}{30}\right)\right] = 61.75132$$

$$LR = -2\log L(\text{null}) - -2\log L(\text{full}) = 65.19273 - 61.75132 = 3.44141 \sim \chi_1^2 \Rightarrow p = 0.0636$$

The LOGISTIC Procedure

```
PROC LOGISTIC;
 MODEL hypotherm (EVENT= 'Yes') = blanket;
RUN;
```

Model Information

Data Set	TEACH.BODYCOVERS	
Response Variable	hypotherm	Hypothermia
Number of Response Levels	2	
Model	binary logit	
Optimization Technique	Fisher's scoring	

Number of Observations Read	60
Number of Observations Used	60

Response Profile

Ordered Value	hypotherm	Total Frequency
1	No	46
2	Yes	14

Probability modeled is hypotherm='Yes'.

Model Convergence Status

Convergence criterion (GCONV=1E-8) satisfied.

Model Fit Statistics

Criterion	Intercept Only	Intercept and Covariates
AIC	67.193	65.751
SC	69.287	69.940
-2 Log L	**65.193**	**61.751**

Testing Global Null Hypothesis: BETA=0

Test	Chi-Square	DF	Pr > ChiSq
Likelihood Ratio	**3.4414**	**1**	**0.0636**
Score	3.3540	1	0.0670
Wald	3.1684	1	0.0751

Analysis of Maximum Likelihood Estimates

Parameter	DF	Estimate	Standard Error	Wald Chi-Square	Pr > ChiSq
Intercept	**1**	**-0.6931**	0.3873	3.2030	0.0735
blanket	**1**	**-1.1786**	0.6622	3.1684	0.0751

The LOGISTIC Procedure

Odds Ratio Estimates

Effect	Point Estimate	95% Wald Confidence Limits	
blanket	0.308	0.084	1.127