Solutions to Homework 2
BIOS 7731

1. **BD 1.1.1**

   (a) Let X = pebble diameter, then $Z = \log X$ has a $N(\mu, \sigma^2)$ distribution.
   Since $X = e^Z$, using a change of variables, X has density

   $$p(x, \theta) = \frac{1}{x\sigma}\ \phi(\frac{\log x - \mu}{\sigma}),\ x > 0,\ \theta = (\mu, \sigma)$$

   where ϕ is the $N(0, 1)$ density.
   Assuming that the pebbles are iid, $p(\mathbf{x}, \theta) = \prod_{i=1}^n p(x_i, \theta)$.
   Since there is no knowledge about the magnitude of μ and σ^2, the parameter space is

   $$\Theta = \{(\mu, \sigma) : -\infty < \mu < \infty, \sigma > 0\}.$$

   The model is parametric since a log normal model is specified.

   (b) The density of the n independent determinations, X_i, $i = 1, \ldots, n$ is

   $$p(x, \theta) = \prod_{i=1}^n \frac{1}{\sigma}\ \phi(\frac{x_i - \mu - 0.1}{\sigma}),$$

   where ϕ is the $N(0, 1)$ density and σ is known.
   The parameter space is

   $$\Theta = \{\mu : -\infty < \mu < \infty\}.$$

   The model is parametric since a Gaussian model is specified.

   (c) Let $\beta > 0$ stand for the unknown bias. Replace 0.1 with β above for the density.
   Since σ is known, the parameter space is

   $$\Theta = \{(\mu, \beta) : -\infty < \mu < \infty, \beta > 0\}.$$

   We can only determine the sum $\mu + \beta$ (see next problem).
   The model is parametric since a Gaussian model is specified.

   (d) Let Y = # of eggs laid, and X = # of eggs hatching.
   From $A.6.3$, $X|Y = y$ is $Binomial(y, p)$
   Since Y is $Poisson(\lambda)$, the joint frequency function for n insects is

   $$p(x, y, \theta) = \prod_{i=1}^n \binom{y_i}{x_i} p^{x_i}(1-p)^{y_i - x_i}\frac{e^{-\lambda}\lambda^{y_i}}{y_i!}, x = 0, 1, \ldots y.$$

The parameter space is

$$\Theta = \{(\lambda, p) : \lambda > 0, 0 \leq p \leq 1\}.$$

The model is parametric since a Poisson model is specified for Y and a Binomial model is specified for the conditional distribution $X|Y$.

2. **BD 1.1.2**

(a) Unidentifiable. Two different values of $\theta = (\mu, \beta)$ can yield the same values for P_θ. For example, $\theta = (1, 1)$ and $\theta' = (0, 2)$ both generate a $N(2, \sigma^2)$ distribution (with σ known). (Can show generally if we add/subtract constant c for μ and β).

(b) Identifiable. The joint distribution from 1.1.1d with $\theta = (\lambda, p)$ is

$$p(x, y, \theta) = \binom{y}{x} p^x (1-p)^{y-x} \frac{\lambda^y \exp -\lambda}{y!}.$$

Suppose $\theta' \neq \theta$, then the ratio of the joint densities should equal to 1 if the parameterization is not identifiable:

$$\frac{p(x, y, \theta')}{p(x, y, \theta)} = (\frac{p'}{p})^x (\frac{1-p'}{1-p})^{y-x} (\frac{\lambda'}{\lambda})^y \exp -(\lambda' - \lambda).$$

Each component of this ratio is $\neq 1$ if $p' \neq p$ and $\lambda' \neq \lambda$, so the ratio is $\neq 1$, and $p(x, y, \theta') \neq p(x, y, \theta)$. Therefore, the parameterization is identifiable.

(c) Unidentifiable. First, find the marginal frequency function $p(x, \theta)$,

$$p(x, \theta) = \sum_{y=x}^{\infty} p(x, y, \theta) = \sum_{y=x}^{\infty} \frac{y!}{(y-x)!x!} p^x (1-p)^{y-x} \frac{e^{-\lambda} \lambda^y}{y!}.$$

Factor out terms not depending on y, to get

$$p(x, \theta) = p^x \frac{e^{-\lambda}}{x!} \sum_{y=x}^{\infty} \lambda^y \frac{(1-p)^{y-x}}{y-x!}.$$

Let $k = y - x$ and perform a change of variables,

$$p(x, \theta) = p^x \frac{e^{-\lambda}}{x!} \sum_{k=0}^{\infty} \lambda^{k+x} \frac{(1-p)^k}{k!}.$$

Recall that $\sum_{k=0}^{\infty} \frac{(\lambda - \lambda p)^k}{k!} = e^{\lambda - \lambda p}$, to obtain

$$p(x, \theta) = \frac{e^{-\lambda p} (\lambda p)^x}{x!}.$$

Thus, X has a Poission(λp) distribution. Two different values of $\theta = (\lambda, p)$ can yield the same P_θ. For example, $\theta = (2, 1/2)$ and $\theta' = (3, 1/3)$ both generate a Poisson(1) distribution for X.

3. **BD 1.2.7**

First, note that the conditional frequency of $X = k|D = d$ can be expressed as

$$\frac{\binom{d}{k}\binom{N-d}{n-k}}{\binom{N}{n}} = \frac{\binom{n}{k}\binom{N-n}{d-k}}{\binom{N}{d}}.$$

The posterior frequency function of $D|X = k$ is

$$P(D = d'|X = k) = \frac{P(X = k|D = d')P(D = d')}{\sum_d P(X = k|D = d)P(D = d)}$$

$$= \frac{\frac{\binom{n}{k}\binom{N-n}{d'-k}}{\binom{N}{d'}}\binom{N}{d'}\pi_0^{d'}(1-\pi_0)^{N-d'}}{\sum_{d=k}^{N-n+k}\frac{\binom{n}{k}\binom{N-n}{d-k}}{\binom{N}{d}}\binom{N}{d}\pi_0^{d}(1-\pi_0)^{N-d}}.$$

The sum in the denominator is over the range of d that is possible: d must be at least k and there are $N - n + k$ objects left that can be defective.

The posterior frequency simplifies to

$$= \frac{\binom{N-n}{d'-k}\pi_0^{d'}(1-\pi_0)^{N-d'}}{\sum_{d=k}^{N-n+k}\binom{N-n}{d-k}\pi_0^{d}(1-\pi_0)^{N-d}}$$

$$= \frac{\binom{N-n}{d'-k}\pi_0^{d'}(1-\pi_0)^{N-d'}}{\sum_{d=0}^{N-n}\binom{N-n}{d}\pi_0^{d+k}(1-\pi_0)^{N-d+k}}$$

$$= \frac{\binom{N-n}{d'-k}\pi_0^{d'}(1-\pi_0)^{N-d'}}{\pi_0^{k}(1-\pi_0)^{n-k}\sum_{d=0}^{N-n}\binom{N-n}{d}\pi_0^{d}(1-\pi_0)^{N-n-d}}$$

$$= \frac{\binom{N-n}{d'-k}\pi_0^{d'}(1-\pi_0)^{N-d'}}{\pi_0^{k}(1-\pi_0)^{n-k}}$$

$$= \binom{N-n}{d'-k}\pi_0^{d'-k}(1-\pi_0)^{N-n-(d'-k)}.$$

Let $z = d' - k$, then the last term is a binomial for Z, $Bin(N - n, \pi)$ and therefore D given $X = k$ is $k + Z$ where Z is $Bin(N - n, \pi)$.

4. **BD 1.2.12**

(a) The conditional density of X given θ is

$$p(x|\theta) = \Pi_i \theta^{\frac{1}{2}} \frac{1}{\sqrt{(2\pi)}} \exp\{-\frac{1}{2}\theta(x_i - \mu_0)^2\}$$

$$= (2\pi)^{-\frac{1}{2}n}\theta^{\frac{1}{2}n} \exp\{-\frac{1}{2}\theta \sum_{i=1}^{n}(x_i - \mu_0)^2\} \propto \theta^{\frac{1}{2}n} \exp\{-\frac{1}{2}\theta t\},$$

where $t = \sum_{i=1}^{n}(x_i - \mu_0)^2$.

(b) The posterior density of θ given $X = x$ is

$$\pi(\theta|x) \propto \pi(x|\theta)p(\theta) \propto \theta^{\frac{1}{2}n} \exp\{-\frac{1}{2}\theta t\}\theta^{\frac{1}{2}(\lambda-2)} \exp\{-\frac{1}{2}\nu\theta\}$$

$$\propto \theta^{\frac{1}{2}(\lambda+n-2)} \exp\{-\frac{1}{2}(\nu + t)\theta\}.$$

Recall that for $X \sim \Gamma(\beta, \alpha)$, the density is $\frac{\alpha^\beta x^{\beta-1} \exp^{-x\alpha}}{\Gamma(\beta)}$.
Thus, the posterior density is a gamma density with parameters $\alpha = \frac{1}{2}(\nu + t), \beta = \frac{1}{2}(\lambda + n)$. Recall that the $\chi^2(r)$ density is a special case of the gamma when $\alpha = \frac{1}{2}$ and $\beta = \frac{r}{2}$. The change of variable $\theta' = (\nu + t)\theta$, gives us

$$\pi(\theta|x) \propto (\theta^{'})^{\frac{1}{2}(\lambda+n-2)} \exp -\frac{1}{2}\theta^{'},$$

where the derivative from the change of variables $(\nu + t)$ is absorbed into the constants (not shown). This is the density of a χ^2 distribution with $r = \lambda + n$ degrees of freedom.

(c) Using transformations of random variables, $\sigma = g(\theta) = \theta^{-\frac{1}{2}}$, the density of σ given $X = x$ is

$$p(\sigma|x) = p_\theta(g^{-1}(\sigma))|\frac{dg^{-1}(\sigma)}{d\sigma}|,$$

where $|\frac{dg^{-1}(\sigma)}{d\sigma}| = 2\sigma^{-3}$.
From part b), $p(\theta|x)$ is the gamma density with parameters $\alpha = \frac{1}{2}(\nu + t), \beta = \frac{1}{2}(\lambda + n)$, thus

$$p(\sigma|x) = \frac{\alpha^\beta(\sigma^{-2})^{\beta-1} \exp^{-\sigma^{-2}\alpha}}{\Gamma(\beta)} 2\sigma^{-3}$$

$$= \frac{2\alpha^\beta\sigma^{-2\beta-1} \exp^{-\sigma^{-2}\alpha}}{\Gamma(\beta)}, \quad \sigma > 0.$$

5. **BD 1.3.8**

(a) The expectation of s^2 is

$$E[s^2] = \frac{1}{n-1}E(\sum_{i=1}^{n}(X_i - \bar{X})^2)$$

$$= \frac{1}{n-1}E[\sum_{i=1}^{n}(X_i - \mu)^2 - 2(X_i - \mu)(\mu - \bar{X}) + (\mu - \bar{X})^2]$$

$$= \frac{1}{n-1}(n\sigma^2 - n2\frac{\sigma^2}{n} + n\frac{\sigma^2}{n}) = \sigma^2.$$

Since $E[s^2] = \sigma^2$, s^2 is unbiased.

(b) i. Since $Bias(s^2) = 0$ from part a), the MSE simplifies to

$$MSE(s^2) = Var(s^2) = \sigma^4(n-1)^2 Var[\sigma^{-2}\sum_{i=1}^{n}(X_i - \bar{X})^2].$$

Note that $\sigma^{-2}\sum_{i=1}^{n}(X_i - \bar{X})^2 \sim \chi^2_{n-1}$ and the variance of a χ^2_r distributed random variable is $2r$. Therefore,

$$Var(s^2) = \sigma^4(n-1)^{-2}2(n-1) = 2(n-1)^{-1}\sigma^4.$$

ii. The MSE of $\hat{\sigma}^2_c$ is

$$MSE(c(n-1)s^2) = Var(c(n-1)s^2) + Bias(c(n-1)s^2)^2,$$

where

$$Bias(c(n-1)s^2)^2 = (E[(c(n-1)s^2)] - \sigma^2)^2 = (c(n-1)\sigma^2 - \sigma^2)^2 = \sigma^4[c^2(n-1)^2 - 2c(n-1) + 1]$$

and

$$Var((c(n-1)s^2) = c^2(n-1)^2 2(n-1)^{-1}\sigma^4 = \sigma^4 c^2 2(n-1).$$

Therefore,

$$MSE(c(n-1)s^2) = \sigma^4[c^2 2(n-1) + c^2(n-1)^2 - 2c(n-1) + 1]$$

$$= \sigma^4[c^2(n^2-1) - 2c(n-1) + 1].$$

(Sketch of solution) Take the derivative of the MSE with respect to c and solve for 0. This function is minimized at $c = (n+1)^{-1}$, and the second derivative is > 0 for $n > 1$.

6. **BD 1.3.19 a,c**

(a) The risk for loss l is defined as

$$R(\theta_i, \delta_j) = \sum_{k=1}^{2} l(\theta_i, \delta_j(X_k))P(X = X_k).$$

For this example, we can calculate the risk for the four decision procedures δ for the two values of θ.

	δ_1	δ_2	δ_3	δ_4
θ_1	0	1.6	0.4	2
θ_2	3	1.8	2.2	1

The minimax value $\min_j \max_i R(\theta_i, \delta_j)$ is achieved for δ_2 with $R(\theta_2, \delta_2) = 1.8$. Therefore, the non-randomized minimax rule is δ_2.

(b) The Bayes risk is $r(\delta_j) = \sum_i \pi(\theta_i)R(\theta_i, \delta_j)$.

δ_1	δ_2	δ_3	δ_4
2.7	1.78	2.02	1.1

which is minimized for δ_4.