

Homework #1

1. a) Let \mathcal{A} denote a class of subsets of the sample space Ω , to which we can assign probabilities. \mathcal{A} is taken to be a σ -field, so

$$A \in \mathcal{A}, A^c \in \mathcal{A}, \text{ and } \emptyset \in \mathcal{A}.$$

Also, $P(A) = P(\Omega) = 1$. A and A^c are pairwise disjoint by definition, so $P(A) + P(A^c) = 1$, which means $P(A^c) = 1 - P(A)$.

Don't forget 2nd part

b) Take the identity $A \cup B = A \cup (B \cap A^c)$ (Casella & Berger 1.2.7). A and A^c are disjoint by definition, so A and $(B \cap A^c)$ are also disjoint. Therefore,

$$P(A \cup B) = P(A) + P(B \cap A^c) = P(A) + P(B) - P(A \cap B)$$

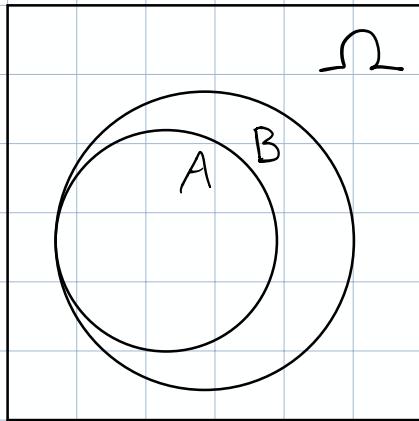
If $A \subset B$, $A \cap B = A$, so

$$P(A \cap B) = P(A) \quad \text{and}$$

$$P(B \cap A^c) = P(B) - P(A)$$

Because $P(A^c \cap B)$ must be ≥ 0 , we see

that $O \subseteq P(A^c \cap B) = P(B) - P(A)$ and
 $\therefore P(A) \subseteq P(B)$.



I know Venn diagrams aren't a formal proof, but they help me think about this stuff.

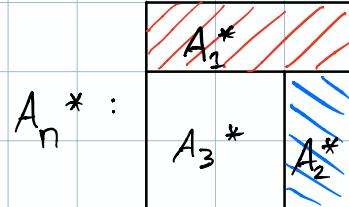
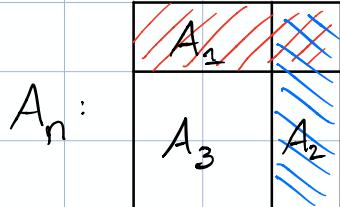
Figures are also good!

$$c) P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$$

First construct a disjoint set A_n^* such that $\bigcup_{n=1}^{\infty} A_n^* = \bigcup_{n=1}^{\infty} A_n$ with

$$A_1^* = A_1 \text{ and for } i = 2, 3, \dots, A_i^* = A_i \setminus \bigcup_{j=1}^{i-1} A_j$$

In other words, a set that covers the same space as $\bigcup_{n=1}^{\infty} A_n$ but is pairwise disjoint.



$$\text{Therefore } P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} A_n^*\right) = \sum_{n=1}^{\infty} P(A_n^*)$$

by the properties of probability models.

$$\text{By construction, } P(A_n^*) \leq P(A_n)$$

$$\therefore \sum_{n=1}^{\infty} P(A_n^*) = P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$$

d) The approach for this is essentially the same as c) above, but applied to the complement. So,

$$P\left(\bigcup_{i=1}^K A_i^c\right) \leq \sum_{i=1}^K P(A_i^c) \text{ by A.2.5.}$$

From this we can rearrange each side:

$$\sum_{i=1}^K P(A_i^c) = \sum_{i=1}^K (1 - P(A_i)) = K - \sum_{i=1}^K P(A_i)$$

and

$$P\left(\bigcup_{i=1}^K A_i^c\right)^{\text{why?}} = P\left(\bigcap_{i=1}^K A_i\right)^c = 1 - P\left(\bigcap_{i=1}^K A_i\right)$$

$$\therefore 1 - P\left(\bigcap_{i=1}^K A_i\right) \leq K - \sum_{i=1}^K P(A_i^c)$$

$$P\left(\bigcap_{i=1}^K A_i\right) \geq 1 - \sum_{i=1}^K P(A_i^c)$$

$$2. \text{ a) } f_x(x) = \frac{x^2 e^{-x}}{2} \quad \text{for } x \in (0, \infty)$$

$$= \frac{x^2}{2e^x} \quad \text{which is a positive}$$

Function because $x > 0$. Next integrate over x :

$$\int_0^\infty \frac{x^2 e^{-x}}{2} dx = \frac{1}{2} \int_0^\infty x^2 e^{-x} dx$$

$$u = x^2$$

$$v = -e^{-x}$$

$$du = 2x dx$$

$$dv = e^{-x} dx$$

$$\frac{1}{2} \int_0^\infty x^2 e^{-x} dx = \left. \frac{-x^2 e^{-x}}{2} \right|_0^\infty + \int_0^\infty x e^{-x} dx = \int_0^\infty x e^{-x} dx$$

$$\lim_{x \rightarrow \infty} \frac{-x^2}{e^{-x}} = 0$$

$$\int_0^\infty x e^{-x} dx = \left. -x e^{-x} \right|_0^\infty + \int_0^\infty e^{-x} dx = \int_0^\infty e^{-x} dx$$

$$\lim_{x \rightarrow \infty} -x e^{-x} = 0$$

$$\int_0^{\infty} e^{-x} dx = -e^{-x} \Big|_0^{\infty} = 0 - (-1) = 1$$

$\therefore \int_0^{\infty} \frac{x^2 e^{-x}}{2} dx = 1$ and $f_x(x)$ is a density function.

$$\begin{aligned} b) F(x) &= \int_0^x \frac{t^2 e^{-t}}{2} dt = \frac{-te^{-t}}{2} \Big|_0^x + \int_0^x te^{-t} dt \\ &= \frac{-x^2 e^{-x}}{2} + \int_0^x te^{-t} dt \end{aligned}$$

$$\int_0^x te^{-t} dt = -te^{-t} \Big|_0^x + \int_0^x e^{-t} dt$$

$$= -xe^{-x} + -e^{-t} \Big|_0^x$$

$$-e^{-t} \Big|_0^x = -e^{-x} + 1$$

$$\therefore F(x) = -\frac{x^2 e^{-x}}{2} - xe^{-x} - e^{-x} + 1$$

$$= \frac{-e^{-x}(x^2 + 2x + 2)}{2}$$

$$c) F(2) = \frac{-e^{-2}(4+4+2)}{2} = \frac{10}{2e^2} = \frac{5}{e^2} \approx 0.323$$

$$3. a) \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad \alpha > 0$$

$$\Gamma(\alpha+1) = \int_0^\infty x^{(\alpha-1)+1} e^{-x} dx = \int_0^\infty x^\alpha e^{-x} dx$$

$$u = x^\alpha \quad v = -e^{-x}$$

$$du = \alpha x^{\alpha-1} dx \quad dv = e^{-x} dx$$

$$\int_0^\infty x^\alpha e^{-x} dx = -x^\alpha e^{-x} \Big|_0^\infty + \int_0^\infty e^{-x} \alpha x^{\alpha-1} dx$$

$$-x^\alpha e^{-x} \Big|_0^\infty = 0$$

$$\therefore \Gamma(\alpha+1) = \int_0^\infty e^{-x} \alpha x^{\alpha-1} dx = \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx =$$

$$\alpha \Gamma(\alpha).$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx = -x^{\alpha-1} e^{-x} \Big|_0^\infty + (\alpha-1) \int_0^\infty x^{\alpha-2} e^{-x} dx$$

$$\Gamma(\alpha) = (\alpha-1) \Gamma(\alpha-1)$$

$$\therefore \Gamma(\alpha+1) = (\alpha)(\alpha-1) \dots \Gamma(1) = \alpha!$$

-2 3b, c?

4) $X \sim N(0, 1)$, find the mean and covariance of $(X, I(X > c))$.

The mean of X is 0 because it is a standard normal distribution. $I(X > c)$ is a Bernoulli with $p = 1 - \Phi(c)$.

Call this function Y , so $Y \sim \text{Bernoulli}(p)$ and $E[Y] = p = 1 - \Phi(c)$

$$\text{Mean vector} = \{0, 1 - \Phi(c)\}$$

The variance of X is 1, again because it's a standard normal-distributed RV.

$$\begin{aligned} \text{The variance of } Y \text{ is } p(1-p) &= \\ (1 - \Phi(c))(1 - (1 - \Phi(c))) &= \Phi(c)(1 - \Phi(c)) \end{aligned}$$

The covariance of X and Y is:

$$E[XY] - E[X]E[Y] = E[XY] - \text{Cov}[X, Y]$$

$$E[XY] = E[X I(X > c)] = \int_x^{\infty} x I(x > c) P_x(x)$$

$$= E[X | X > c]$$

$$= \int_c^{\infty} x P_x(x) dx = \int_c^{\infty} \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_c^{\infty} x e^{-\frac{x^2}{2}} dx = \frac{e^{-c^2/2}}{\sqrt{2\pi}}$$

Variance-covariance :

$$\begin{bmatrix} 1 & \frac{e^{-c^2/2}}{\sqrt{2\pi}} \\ \frac{e^{-c^2/2}}{\sqrt{2\pi}} & \left(\Phi(c)(1 - \Phi(c)) \right) \end{bmatrix}$$

5. $T \sim \text{exponential } (\beta)$ $f_x(x|\beta) = \frac{1}{\beta} e^{-x/\beta} \quad 0 \leq x < \infty$

$U \sim \text{uniform}(0, T)$ $F_x(x|a, b) = \frac{1}{b-a} \quad a \leq x \leq b$

By the conditional expectation theorem:

$$E[U] = E[E[U|T]]$$

$$E[U|T] = \frac{T+0}{2}$$

$$E[E[U|T]] = E[T] = \frac{\beta}{2}$$
✓

By the conditional variance identity:

$$\begin{aligned} \text{Var}[U] &= E[\text{Var}[U|T]] + \text{Var}[E[U|T]] \\ &= E[T^2/12] + \text{Var}[T/2] \\ &= \frac{E[T^2]}{12} + \frac{\text{Var}[T]}{4} \end{aligned}$$

$$\begin{aligned} E[T^2] &= \text{Var}[T] + E[X]^2 \\ &= \beta^2 + \beta^2 \end{aligned}$$

$$\text{Var}[U] = \frac{2\beta^2}{12} + \frac{\beta^2}{2} = \frac{\beta^2}{6} + \frac{3\beta^2}{6} = \frac{2\beta^2}{3}$$

$$6. \text{ a) } \text{Cov}(X, E[Y|X]) = E[X E[Y|X]] - E[X] E[E[Y|X]]$$

$$\begin{aligned} &E[E[X Y | X]] - E[X] E[Y] \\ &= E[X Y] - E[X] E[Y] = \text{Cov}(X, Y) \end{aligned}$$

$$\text{b) } \text{Cov}(X, Y - E[Y|X]) =$$

$$E[X(Y - E[Y|X])] - E[X] E[Y - E[Y|X]]$$

$$\begin{aligned}
 &= E[XY - XE[Y|X]] - E[X]E[Y - E[Y|X]] \\
 &= E[XY] - E[X]E[Y|X] - E[X](E[Y] - E[E[Y|X]]) \\
 &= E[XY] - E[XY] - E[X](E[Y] - E[Y]) \\
 &= 0 - 0 = 0
 \end{aligned}$$

✓

$$\text{Correlation}(X, Y - E[Y|X]) = \frac{\text{Cov}(X, Y - E[Y|X])}{\sigma_X \sigma_{Y - E[Y|X]}}$$

$$\therefore \text{Correlation}(X, Y - E[Y|X]) = 0$$