MS Theory I

Lecture 4

Review

§ 1.5 Distribution Functions

Cumulative distribution function (cdf)

Definition 1.5.1 The *cumulative distribution function* or *cdf* of a random variable X, denoted by $F_X(x)$, is defined by

$$F_X(x) = P_X(X \le x), \quad \text{for all } x.$$

Note: for all x, not just those in sample space $-\infty < X < \infty$.

Example: Toss coin 3 times

$X = \#$ heads $\quad S = \{0, 1, 2, 3\}$

$$F_X(x) = P_X(X \le x) = P(X \le x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ 1/8 & \text{if } 0 \le x < 1 \\ 1/2 & \text{if } 1 \le x < 2 \\ 7/8 & \text{if } 2 \le x < 3 \\ 1 & \text{if } 3 \le x < \infty \end{cases}$$

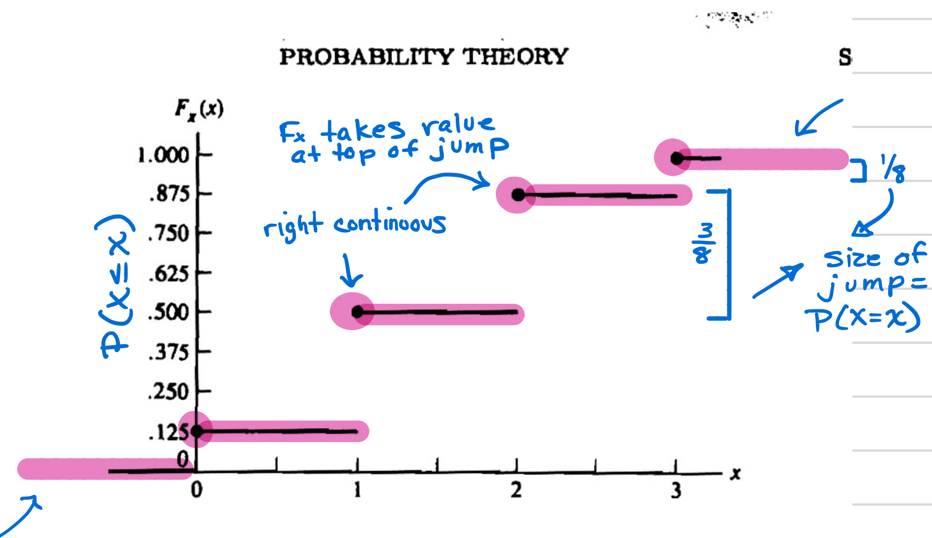

Figure 1.5.1. *Cdf of Example 1.5.2*

defined for all values of x, not just

$P(X \le 3) = 1 \qquad P(X \le 2) = 7/8 \qquad P(X < 2) = 1/2$

$P(X \le 25.2) = 1 \qquad P(X \le 2.6) = 7/8 \qquad P(X < 1.2) = 1/2$

Note:
- cdf, F_X, defined for all x $(-\infty < x < \infty)$, not just those in Sample Space.
- $P(X \le x)$ $\Rightarrow$ defined (equal sign) to be right continuous
  - equal sign
  - right continuous: ft'n is continuous when approached from right
  - right continuous: F_X takes value at top of jump
- Size of jump at any point $X = x$ is equal to $P(X = x)$
- as $x \to -\infty$ F_X goes to 0.
- as $x \to \infty$ F_X goes to 1.
- $F_X = P(X \le x)$ is nondecreasing ft'n.

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Review cont.

Theorem 1.5.3 *The function $F(x)$ is a cdf if and only if the following three conditions hold:*

a. $\lim_{x\to-\infty} F(x) = 0$ *and* $\lim_{x\to\infty} F(x) = 1$.

b. *$F(x)$ is a nondecreasing function of x.*

c. *$F(x)$ is right-continuous; that is, for every number x_0, $\lim_{x\downarrow x_0} F(x) = F(x_0)$.*

Geometric(p) C&B p. 621

pmf	$P(X = x\|p) = p(1-p)^{x-1}; \quad x = 1, 2, \ldots; \quad 0 \le p \le 1$
mean and variance	$\mathrm{E}X = \frac{1}{p}, \quad \mathrm{Var}\, X = \frac{1-p}{p^2}$

cdf of geometric ($p=0.3$) distn.

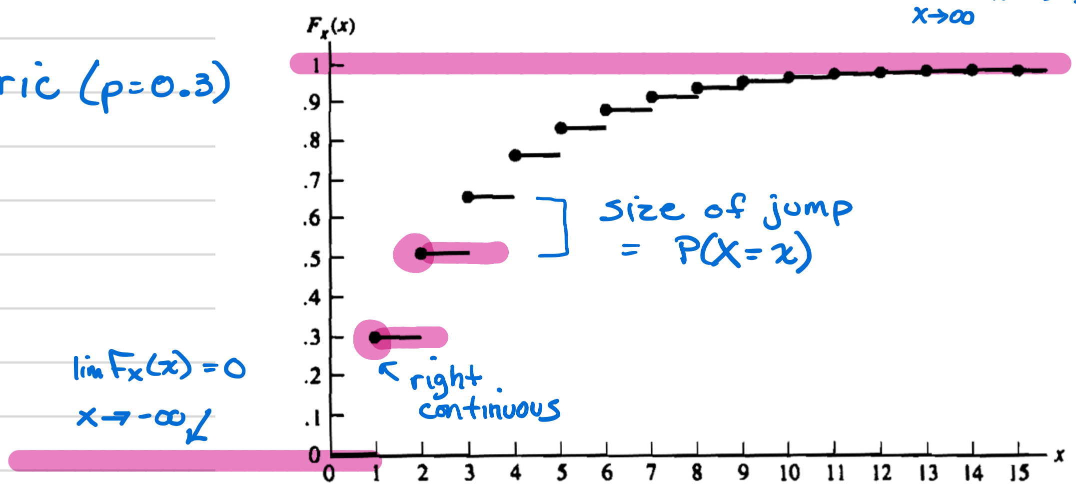
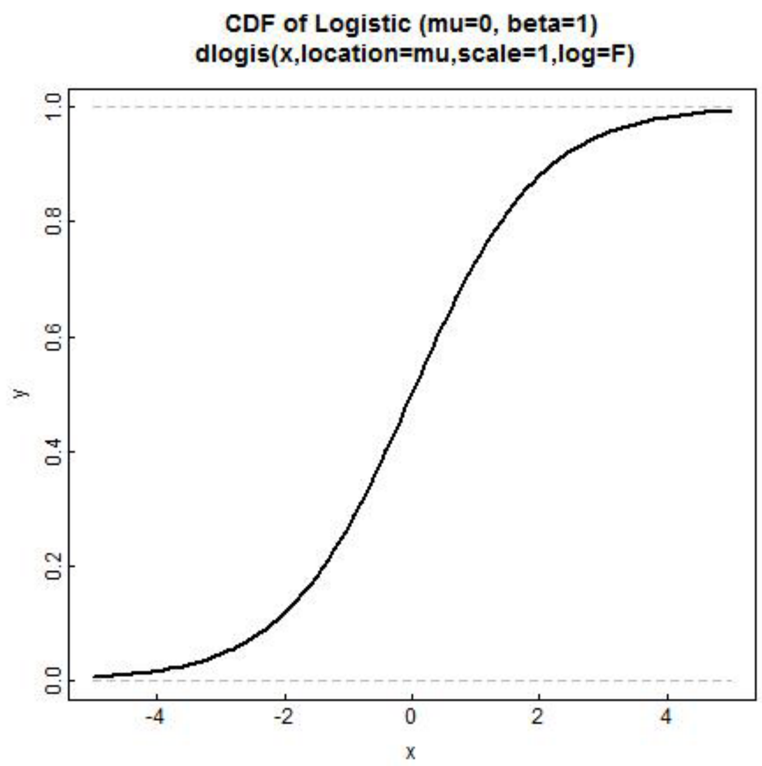


Figure 1.5.2. *Geometric cdf,* $p = .3$

Note cdf defined for $-\infty < x < \infty$

Logistic(μ, β)

pdf	$f(x\|\mu,\beta) = \frac{1}{\beta}\frac{e^{-(x-\mu)/\beta}}{[1+e^{-(x-\mu)/\beta}]^2}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \mu, \quad \mathrm{Var}\, X = \frac{\pi^2\beta^2}{3}$
mgf	$M_X(t) = e^{\mu t}\Gamma(1-\beta t)\Gamma(1+\beta t), \quad \|t\| < \frac{1}{\beta}$
notes	The cdf is given by $F(x\|\mu,\beta) = \frac{1}{1+e^{-(x-\mu)/\beta}}$.



Review cont.

Definition 1.5.7 A random variable X is *continuous* if $F_X(x)$ is a continuous function of x. A random variable X is *discrete* if $F_X(x)$ is a step function of x.

Definition 1.5.8 The random variables X and Y are *identically distributed* if, for every set $A \in \mathcal{B}^1$, $P(X \in A) = P(Y \in A)$.

§1.6 Density and Mass Functions

cdf = cumulative dist'n ft'n
$F_X(x) = P_X(X \le x) \quad \forall x$
denote cdfs by uppercase letters $F(x)$ or $F_X(x)$

Also define
pmf = probability mass function (discrete R.V. X)
and
pdf = probability dist'n function (continuous R.V. X)
denote by lowercase letters $f_X(x)$ or $f(x)$

Discrete R.V.s

Definition 1.6.1 The *probability mass function* (*pmf*) of a discrete random variable X is given by

$$f_X(x) = P(X = x) \quad \text{for all } x.$$

Example (Geometric Dist'n): (# trials to 1st success)

$$F_X(x) = \sum_{i=1}^{x} p(1-p)^{i-1} = 1-(1-p)^x$$

$$f_X(x) = P(X=x) = \underbrace{p}_{\text{prob 1 success}}\underbrace{(1-p)^{x-1}}_{\text{prob. } x-1 \text{ failures before 1st success}} \quad x = 1,2,3,\dots$$

$$= p(1-p)^{x-1} I_{[1,2,3,\dots]}(x)$$

Discrete case

pdf lowercase: $f_X(x) = P(X=x)$

$F_X(x) = P(X \le x) = \sum_{k=1}^{x} f_X(k)$

cdf uppercase: $P(a \le X \le b) = \sum_{k=a}^{b} f_X(k)$

Review cont.

Definition 1.6.3 The *probability density function* or *pdf*, $f_X(x)$, of a continuous random variable X is the function that satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \quad \text{for all } x. \tag{1.6.3}$$

We say $X \sim F_X(x)$ or $X \sim f_X(x)$
↳ is distributed as

Similarly
$$P(a \le X \le b) = \int_a^b f_X(t)\,dt$$

Note t is just a placeholder
$$\int_a^b f_X(t)\,dt = \int_a^b f_X(z)\,dz = \int_a^b f_X(y)\,dy$$
(see appendix)

and
$$P(a < X < b) = \int_a^b f_X(t)\,dt$$

and $a = b = x$
$$P(x \le X \le x) = P(X = x) = \int_x^x f_X(t)\,dt = 0$$

CDF & pdf/pmf

Since we can get the cdf from the pdf or pmf

$F_X = P_X(X \le x)$ for all x

discrete sum $\quad F_X = \sum_k^x f_X(k)$ ← sum over all values in $S \le x$.

continuous integrate $\quad F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

"add up" point probs" to get $F_X(x) = $ cdf

Or get pdf from $F_X(x)$

$$\frac{d}{dx} F_X(x) = f_X(x)$$

pdf (or pmf) and cdf give same info.
use whichever simpler.

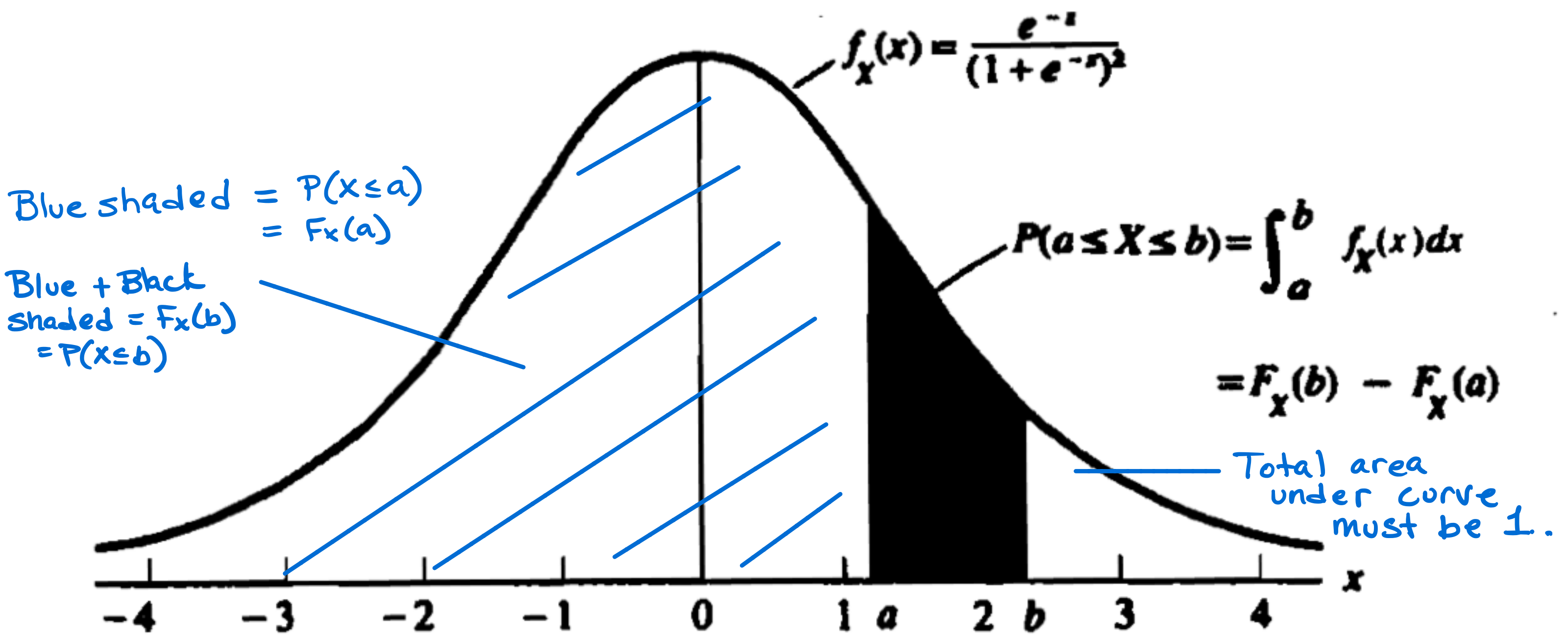


Figure 1.6.1. *Area under logistic curve*

Review Cont.

Theorem 1.6.5 *A function $f_X(x)$ is a pdf (or pmf) of a random variable X if and only if*

a. $f_X(x) \geq 0$ *for all* x.

b. $\sum_x f_X(x) = 1$ *(pmf)* *or* $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ *(pdf)*.

know $\lim_{x\to\infty} F_X(x) = 1$ $\lim_{x\to-\infty} F_X(x) = 0$

and continuous: $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

for this class we will focus on well behaved functions
(i.e. we will ignore "pathological" cases.

$$1 = \lim_{x\to\infty} F_X(x) = \int_{-\infty}^{\infty} f_X(t)\,dt$$

Distributions

Bernoulli and Binomial Distributions (Discrete)

Bernoulli(p)		
pmf	$P(X=x\mid p) = p^x(1-p)^{1-x}$;	$x = 0, 1$; $0 \leq p \leq 1$
mean and variance	$\mathrm{E}X = p$, $\mathrm{Var}\,X = p(1-p)$	
mgf	$M_X(t) = (1-p) + pe^t$	

Sample space

parameter space

Flip coin $x=1$ if head
$x=0$ if tail

$P(\text{head}) = p$ $P(\text{tail}) = 1-p$

$P(x=1) = p^1(1-p)^0 = p$

$P(x=0) = p^0(1-p)^1 = 1-p$

pmf:
$P(X=x|p) = p^x(1-p)^{1-x} I_{[0,1]}(x)$

indicator

Binomial(n,p)		
pmf	$P(X=x\mid n,p) = \binom{n}{x}p^x(1-p)^{n-x}$;	$x = 0, 1, 2, \ldots, n$; $0 \leq p \leq 1$
mean and variance	$\mathrm{E}X = np$, $\mathrm{Var}\,X = np(1-p)$	
mgf	$M_X(t) = [pe^t + (1-p)]^n$	
notes	Related to Binomial Theorem (Theorem 3.2.2). The *multinomial* distribution (Definition 4.6.2) is a multivariate version of the binomial distribution.	

sample space

parameter space (n known)

$p =$ probability success
$1-p =$ probability failure
n bernoulli trials (often n known)
$X \sim \text{bin}(n,p)$

Binomial: Flip coin 3 times; $X = \#$ heads

$X \sim \text{bin}(3,p)$

$n=3$ $p=$ prob head

pmf:

$P(X=x|n,p) = \binom{3}{x} p^x (1-p)^{3-x} \cdot I_{[0,1,2,3]}(x)$

sample space indicator

$P(X=0) = \binom{3}{0} p^0(1-p)^3 = \frac{3!}{3!\,0!}(1-p)^3 = (1-p)^3$

$P(X=1) = \binom{3}{1} p^1(1-p)^2 = \frac{3!}{2!\,1!}\,p(1-p)^2 = 3p(1-p)^2$

$P(X=2) = \binom{3}{2} p^2(1-p) = \frac{3!}{1!\,2!}\,p^2(1-p) = 3p^2(1-p)$

$P(X=3) = \binom{3}{3} p^3(1-p)^0 = \frac{3!}{0!\,3!}\,p^3(1-p)^0 = p^3$

Chapter 2: Transformations and Expectations

Random Variable: X

dist'n of X: $f_X(x)$ or $F_X(x)$ $\longrightarrow$ knowing pdf (pmf) or cdf tells how X behaves

§ 2.1 Distributions of functions of RV.

$X \sim F_X(x)$ (X is distributed as $F_X(x)$)

May be interested in behavior of function of X, say $g(X)$.

$g(X)$ is also a random variable

$Y = g(X)$ $\leftarrow$ may be interested in dist'n of Y.

Describe behavior of Y in terms of X:

for any set A

$$P(Y \in A) = P(g(X) \in A)$$

such that the dist'n of Y, depends on F_X and function, g.

Goal: Find the distribution of $Y = g(X)$.

$y = g(x)$ then $g(x)$ maps from sample space of x, $\mathcal{X}$, to new sample space of y, $\mathcal{Y}$.

$$g(x): \mathcal{X} \to \mathcal{Y}$$
$$\leftarrow : g^{-1}(y)$$

Assuming y is a point: $g^{-1}(y) = \{x \in \mathcal{X}: g(x) = y\}$

If only one x for which $g(x) = y$ then $g^{-1}(y) = x$.

For a set $A \subset \mathcal{Y}$:
$$\begin{aligned} P(Y \in A) &= P(g(X) \in A) \\ &= P(\{x \in \mathcal{X}: g(x) \in A\}) \\ &= P(X \in g^{-1}(A)) \end{aligned}$$

"Sum up": X is R.V. with know pdf

Interested in behavior (dist'n) of $Y = g(X) \leftarrow$ ft'n of X

$g(x)$: maps $\mathcal{X} \to \mathcal{Y}$

$g^{-1}(y)$: maps $\mathcal{Y} \to \mathcal{X}$

if y is a point $y = g(x)$ and $g^{-1}(y) = x$ } for now assume each x corresponds to 1 y; and each y corresponds to 1 x.

We could also be interested in a subset of $\mathcal{Y}$, say A (not necessarily a point).

Quick Simple Example

X is R.V. w/ known pdf	X = Result from rolling a fair 3-sided die $P(X=1) = 1/3$ $P(X=2) = 1/3$ $P(X=3) = 1/3$ Sample space of X: $\mathcal{X} = \{1,2,3\}$
Interested in behavior of $Y = X^2$ $g(x)$: maps $\mathcal{X} \to \mathcal{Y}$ $g^{-1}(y)$: maps $\mathcal{Y} \to \mathcal{X}$	Sample space of $\mathcal{Y} = \{1,4,9\}$ $g(x) = x^2$ maps $\{1,2,3\} \to \{1,4,9\}$ $g^{-1}(y) = \sqrt{y}$ maps $\{1,4,9\} \to \{1,2,3\}$
Random Variable $Y = g(X)$ Interested in set $A \subset \mathcal{Y}$	$Y = g(X) = X^2$ Interested in set $\underbrace{\{4,9\}}_{A} \subset \underbrace{\{1,4,9\}}_{\mathcal{Y}}$
$P(Y \in A) = P(g(X) \in A)$ $= P(\{x \in \mathcal{X} : g(x) \in A\})$ $= P(X \in g^{-1}(A))$	$P(Y \in \{4,9\}) = P(X^2 \in \{4,9\})$ $= P(\{x \in \{1,2,3\} : x^2 \in \{4,9\}\})$ $= P(X \in \{\sqrt{4}, \sqrt{9}\}) = P(X \in \{2,3\})$.
Defines probability dist'n of Y.	Defines probability dist'n of $Y = X^2$
If X is discrete R.V. $f_Y(y) = P(Y=y)$ $= \sum_{x \in g^{-1}(y)} P(X=x)$ $= \sum_{x \in g^{-1}(y)} f_X(x)$ for $y \in \mathcal{Y}$	X is discrete R.V. $f_Y(y) = P(Y=y)$ $= \sum_{x \in \{2,3\}} P(X=x)$ $= \sum_{x \in \{2,3\}} f_X(x)$ $y \in \{1,4,9\}$

Defined the dist'n of Y.

"Straightforward" to show pdf of Y satisfies Kolmogorov Axioms:

Definition 1.2.4 Given a sample space S and an associated sigma algebra $\mathcal{B}$, a *probability function* is a function P with domain $\mathcal{B}$ that satisfies

1. $P(A) \geq 0$ for all $A \in \mathcal{B}$.
2. $P(S) = 1$.
3. If $A_1, A_2, \ldots \in \mathcal{B}$ are pairwise disjoint, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Example 2.1.1 (Binomial Transform)

pmf: $f_X(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, 1, 2 \ldots n$

$= \binom{n}{x} p^x (1-p) I_{[0,1,\ldots,n]}(x)$ ← indicator ft'n

$\begin{cases} n \text{ positive integer} \\ 0 \leq p \leq 1 \end{cases}$ parameters

know, n, p

know X behavior

$Y = g(X)$, where $g(x) = n - x$

$y = n - x$

$x = n - y$

$\mathcal{X} = \{0, 1, 2, \ldots, n\}$

$\mathcal{Y} = \{y : y = g(x), x \in \mathcal{X}\}$

$= \{0, 1, \ldots, n\}$

for each x there is only one y.

each y at most one x.

$$\begin{pmatrix} x & y \\ 0 & n \\ 1 & n-1 \\ & \vdots \\ n & 0 \end{pmatrix}$$

← one to one and onto

$$f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x) = f_X(n-y) = \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}$$

one-to-one onto

$g^{-1}(y)$ is a point: $(n-y)$

$$= \binom{n}{y} (1-p)^y p^{n-y} * I_{[0,1,\ldots,n]}(y)$$

sample space indicator

$\binom{n}{y} = \binom{n}{n-y}$

Definition 1.2.17 For nonnegative integers n and r, where $n \geq r$, we define the symbol $\binom{n}{r}$, read n *choose* r, as

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

$$f_Y(y) = \binom{n}{y} (1-p)^y p^{n-y} * I_{[0,1,\ldots,n]}(y)$$

$Y \sim \text{binomial}(n, (1-p))$

Dist'ns: pdf and/or cdf and sample space (continuous)

Transformation: keep track of sample space.

If transformation is from X to $Y = g(X)$

$\mathcal{X} = \{x : f_X(x) > 0\}$ and $\mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}$.

pdf of random variable X is >0 only on $\mathcal{X}$ and zero else.

support set or support

Easiest deal with ft'ns $g(x)$ that are monotone — increasing / decreasing

$x \to g(x)$ is monotone $\to$ one to one and onto

from $\mathcal{X} \to \mathcal{Y}$

each $x \to$ one y

at most one $x \leftarrow$ each y

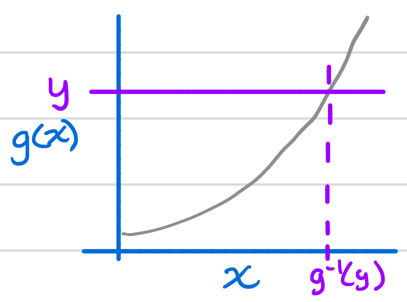
one-to-one

for each $y \in \mathcal{Y}$ there is an $x \in \mathcal{X}$ s.t. $g(x) = y$

onto

uniquely pairs x & y.

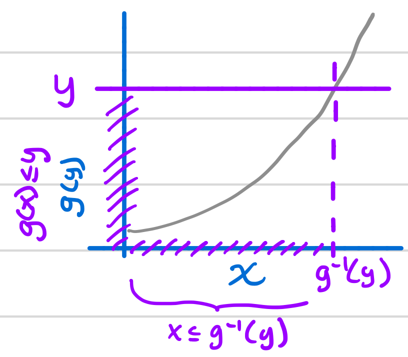
$\underline{g(x) \text{ is increasing}}$ Find $F_Y = P(Y \le y)$



$\{x \in \mathcal{X} : g(x) \le y\}$

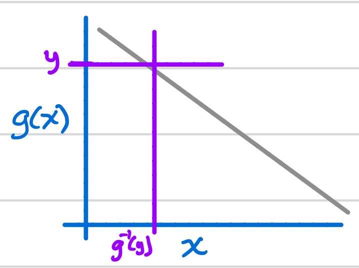
$= \{x \in \mathcal{X} : g^{-1}(g(x)) \le g^{-1}(y)\}$

$= \{x \in \mathcal{X} : x \le g^{-1}(y)\}$



$$F_Y(y) = \int_{\{x \in \mathcal{X} : x \le g^{-1}(y)\}} f_X(x)\,dx = \int_{-\infty}^{g^{-1}(y)} f_X(x)\,dx = F_X(g^{-1}(y)).$$

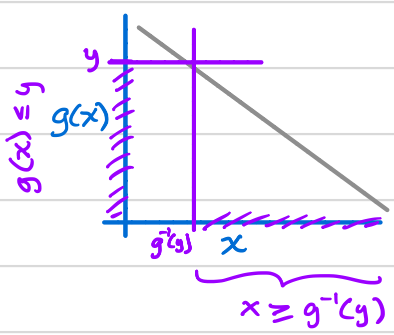
$\underline{g(x) \text{ is decreasing}}$ Find $F_Y = P(Y \le y)$



$\{x \in \mathcal{X} : g(x) \le y\}$

$= \{x \in \mathcal{X} : g^{-1}(g(x)) \ge g^{-1}(y)\}$

$= \{x \in \mathcal{X} : x \ge g^{-1}(y)\}$



$$F_Y(y) = \int_{g^{-1}(y)}^{\infty} f_X(x)\,dx = 1 - F_X(g^{-1}(y))$$

Theorem 2.1.3 *Let X have cdf $F_X(x)$, let $Y = g(X)$, and let $\mathcal{X}$ and $\mathcal{Y}$ be defined as in (2.1.7).*

a. *If g is an increasing function on $\mathcal{X}$, $F_Y(y) = F_X\left(g^{-1}(y)\right)$ for $y \in \mathcal{Y}$.*

b. *If g is a decreasing function on $\mathcal{X}$ and X is a continuous random variable, $F_Y(y) = 1 - F_X\left(g^{-1}(y)\right)$ for $y \in \mathcal{Y}$.*

2.1.7 : $\mathcal{X} = \{x : f_X(x) > 0\}$ and $\mathcal{Y} = \{y : y = g(x)$ for some $x \in \mathcal{X}\}$

A few new dist'ns.

Uniform Distribution (continuous)

Uniform(a, b)

pdf	$f(x\mid a,b) = \frac{1}{b-a}$, $\quad \underbrace{\overbrace{a \le x \le b}^{\text{parameters}}}_{\text{Sample Space}}$
mean and variance	$\mathrm{E}X = \frac{b+a}{2}$, $\quad \mathrm{Var}\, X = \frac{(b-a)^2}{12}$
mgf	$M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$
notes	If $a = 0$ and $b = 1$, this is a special case of the beta $(\alpha = \beta = 1)$.

Uniform (0,1) $\quad a = 0,\ b = 1$

$f(x \mid a=0, b=1) = \frac{1}{1-0} = 1 \quad 0 \le x \le 1$

$= 1 * I_{[0,1]}(x)$

where $I_{[0,1]}(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

$X \sim$ Uniform (0,1)

$$\underset{\text{cdf}}{P(X \le x)} = F_X(x) = \int_0^x 1\,dt = t\Big|_0^x = x$$

Exponential distribution (continuous)

Exponential(β)

pdf	$f(x\mid\beta) = \frac{1}{\beta}e^{-x/\beta}$, $\quad \underbrace{0 \le x < \infty}_{\text{sample space}}$, $\quad \underbrace{\beta > 0}_{\text{parameter space}}$
mean and variance	$\mathrm{E}X = \beta$, $\quad \mathrm{Var}\, X = \beta^2$
mgf	$M_X(t) = \frac{1}{1-\beta t}$, $\quad t < \frac{1}{\beta}$
notes	Special case of the gamma distribution. Has the *memoryless* property. Has many special cases: $Y = X^{1/\gamma}$ is *Weibull*, $Y = \sqrt{2X/\beta}$ is *Rayleigh*, $Y = \alpha - \gamma \log(X/\beta)$ is *Gumbel*.

$X \sim$ exponential (β)

$$\underset{\text{cdf}}{F_X(x)} = P(X \le x) = \int_0^x \frac{1}{\beta} e^{-t/\beta}\,dt = -\frac{\beta}{\beta} e^{-t/\beta}\Big|_0^x = -e^{-x/\beta} + 1 = 1 - e^{-x/\beta}$$

Continuous Example (Uniform - exponential) also one-to-one and onto.

$X \sim f_X(x) = 1 \quad 0<x<1 = U(0,1)$ dist'n. $\qquad f_X(x) = 1 * I_{[0,1]}(x)$

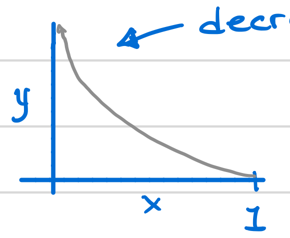
$F_X(x) = \int_0^x 1\,dt = t\big|_0^x = x$

Interested in dist'n of $Y = g(X) = -\log_e(X)$

Sample space of X: $\mathcal{X} = \{x : 0 \le x \le 1\}$ $\qquad -\log(1) = 0$; $\lim_{B \to 0} (-\log(B)) = \infty$

$\rightarrow$ Y: $\mathcal{Y} = \{y : 0 \le y < \infty\}$

decreasing



$y = -\log(x) = g(x)$

$g^{-1}(y) = e^{-y}$ (solve for x in terms of y).

$F_Y(y) = 1 - F_X(g^{-1}(y)) = 1 - F_X(e^{-y})$

$= 1 - e^{-y}$ (since $F_X = x$)

$F_Y(y) = 1 - e^{-y} \quad y > 0.$

$\hookrightarrow$ cdf of exponential $(\beta = 1)$.

If pdf of Y is continuous obtain by differentiating cdf

Theorem 2.1.5 *Let X have pdf $f_X(x)$ and let $Y = g(X)$, where g is a monotone function. Let $\mathcal{X}$ and $\mathcal{Y}$ be defined by (2.1.7). Suppose that $f_X(x)$ is continuous on $\mathcal{X}$ and that $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$. Then the pdf of Y is given by*

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \dfrac{d}{dy} g^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.10}$$

Proof: (chain rule)

$$\underset{\text{pdf}}{f_Y(y)} = \frac{d}{dy} F_Y(y) = \begin{cases} f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) & g \text{ increasing} \\ -f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) & g \text{ decreasing.} \end{cases}$$

Uniform Distribution (continuous)

Uniform(a, b)

pdf $f(x|a,b) = \frac{1}{b-a}$, $\underbrace{a \le x \le b}_{\text{Sample Space}}$ (parameters)

mean and variance $\mathrm{E}X = \frac{b+a}{2}$, $\mathrm{Var}\, X = \frac{(b-a)^2}{12}$

mgf $M_X(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$

notes If $a=0$ and $b=1$, this is a special case of the beta ($\alpha = \beta = 1$).

Uniform $(0,1)$ $a=0$, $b=1$

$f(x|a=0, b=1) = \frac{1}{1-0} = 1 \quad 0 \le x \le 1$

$= 1 * I_{[0,1]}^{(x)}$

where $I_{[0,1]}^{(x)} = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

Uniform $(0, \theta)$

pdf: $f_X(x|\theta) = \frac{1}{\theta} \quad 0 \le x \le \theta \quad = \frac{1}{\theta} I_{[0,\theta]}^{(x)}$ where $I_{[0,\theta]}^{(x)} = \begin{cases} 1 & 0 \le x \le \theta \\ 0 & \text{else} \end{cases}$

← red flag
sample space is function of parameter

Normal Distribution (continuous)

Normal(μ, σ^2)

pdf $f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$, $\underbrace{-\infty < x < \infty}_{\text{sample space}}$, $\underbrace{-\infty < \mu < \infty}_{\text{parameter space}}$,
$\sigma > 0$ ← parameter space

mean and variance $\mathrm{E}X = \mu$, $\mathrm{Var}\, X = \sigma^2$

mgf $M_X(t) = e^{\mu t + \sigma^2 t^2/2}$

notes Sometimes called the *Gaussian* distribution.

Normal $(0,1)$ $f(x|\mu=0, \sigma=1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2} \cdot I_{(-\infty,\infty)}^{(x)}$

$I_{(-\infty,\infty)}^{(x)} = \begin{cases} 1 & -\infty < x < \infty \\ 0 & \text{else} \end{cases}$

$X \sim N(\mu, \sigma^2)$ $f(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \cdot I_{(-\infty,\infty)}^{(x)}$ $-\infty < \mu < \infty$, $\sigma > 0$

Chi-Squared Distribution (continuous)

Chi squared(p)

pdf $f(x|p) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2}$; $\underbrace{0 \le x < \infty}_{\text{sample space}}$; $\underbrace{p = 1, 2, \ldots}_{\text{parameters (degree of freedom)}}$

mean and variance $\mathrm{E}X = p$, $\mathrm{Var}\, X = 2p$

$\Gamma(n) = (n-1)!$ if n is integer

$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ $\alpha > 0$

mgf $M_X(t) = \left(\frac{1}{1-2t}\right)^{p/2}$, $t < \frac{1}{2}$

notes Special case of the gamma distribution.

$X \sim \chi_p^2$ $f(x|p) = \frac{1}{\Gamma(\frac{p}{2}) 2^{p/2}} x^{(p/2)-1} e^{-x/2} I_{(0,\infty)}^{(x)}$, $I_{(0,\infty)}^{(x)} = \begin{cases} 1 & 0 \le x < \infty \\ 0 & \text{else} \end{cases}$

greek letter
χ = Chi

Gamma distribution (continuous)

***Gamma*(α, β)**

pdf	$f(x\|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$,	sample space: $0 \le x < \infty$,	parameter space: $\alpha, \beta > 0$
	$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$		$\alpha > 0$
mean and variance	$\mathrm{E}X = \alpha\beta$, $\quad \mathrm{Var}\,X = \alpha\beta^2$		
mgf	$M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha$, $\quad t < \frac{1}{\beta}$		
notes	Some special cases are exponential ($\alpha = 1$) and chi squared ($\alpha = p/2$, $\beta = 2$). If $\alpha = \frac{3}{2}$, $Y = \sqrt{X/\beta}$ is *Maxwell*. $Y = 1/X$ has the *inverted gamma distribution*. Can also be related to the Poisson (Example 3.2.1).		

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} * I_{(0,\infty)}(x) \; ; \quad \alpha,\beta > 0$$