Flow decompositions in multistate Markov models

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Abstract

We demonstrate the application of standard decomposition techniques to decompose differences between synthetic indices derived from age-stage Markov matrix models into differences due to each stage transition. An example is given on the basis of transition matrices from an analysis of working life expectancy in the United States.

1 Introduction

I describe the application of a generic pseudo-continuous time decomposition (Horiuchi et al. 2008) of differences between synthetic indices derived from two sets of transition probabilities into differences from each each age-stage transition. Intuitively this means we can assign how much of a difference is due to differences in each arrow in the state-space diagram of the model in question. We demonstrate this decomposition technique using published transition matrices from a recent study of working life expectancy in the United States (Dudel and Myrskylä 2017).

2 Method

2.1 Calculate the index, θ from exit probabilities

Say we have a model with s states and ω age classes, and manage to produce matrix of transition probabilities, $\mathbf{U}$, with ages nested within block matrices of state transitions. This matrix will be square of dimension $s\omega \times s\omega$, and each $\omega \times \omega$ block in $\mathbf{U}$ contains the transition probabilities from state j to state i in its subdiagonal. We'll follow the convention of *from* states j in columns and *to* states i in rows. Now, the main subdiagonal of $\mathbf{U}$ contains the probabilities of staying in one's state and advancing to the next age, often represented as *self* arrows in reduced state-space diagrams.

Now say we calculate some quantity θ from $\mathbf{U}$, that depends on all its transition probabilities, such as an expected state occpancy (e.g. health or employment). Define the function summarizing all the steps involved in calculating θ from $\mathbf{U}$:

$$\theta = f^{\theta}(\mathbf{U}) \tag{1}$$

Although expectancies and such are calculated on the basis of *living* transition probabilities, for purposes of decomposition they are best treated as a function of *exits* in general. By analogy, when decomposing differences in life epectancy from the lifetable, we always decompose with respect to death transitions and not age-specific survival probabilities. The same rule applies to state occupancies with many potential exits. If we wish to partition an expected state occupancy into contributions from each of the various transitions, we therefore need to include death transitions and exclude self-transitions from the decomposition. We therefore require a function to convert a vector of all transfer probabilities implying state exit into the matrix $\mathbf{U}$. This is more a matter of program organization that it is of mathematical notation. We know from the lifetable that $p_x = 1 - q_x$. Translating this to the present situation, given a set of exit probabilities, we can derive self-transition probabilities, $p_{s,s}$ in much the same way:

$$p_{s,s} = \left[1 - \sum p_{i,s}\right] \quad \text{for } i \neq s \tag{2}$$

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Assume we have a vector of exit probabilities, $\mathbf{p}^{(i \neq s)}$ then it is possible to convert this into the required matrix $\mathbf{U}$ via some steps in a function g^U:

$$\mathbf{U} = g^U \left(\mathbf{p}^{(i \neq s)} \right) \tag{3}$$

Then we can calculate our indicator from exit probabilities like so:

$$\theta = f^\theta \left(g^U \left(\mathbf{p}^{(i \neq s)} \right) \right) \tag{4}$$

Let's make notation easier and wrap that all in a function ζ:

$$\theta = \zeta \left(\mathbf{p}^{(i \neq s)} \right) \tag{5}$$

Now ζ is a function that is decomposable using generic decomposition methods, such as the pseudo continuous method of Horiuchi et al. (2008), which merits a brief summary.

Given two vectors of equal structure and length $\mathbf{p}^{1(i \neq s)}$ and $\mathbf{p}^{2(i \neq s)}$ (henceforth $\mathbf{p}^1$ and $\mathbf{p}^2$), which may refer to different populations or time points we may calculate θ^1 and θ^2 via (5). The difference between θ^2 and θ^1 can be decomposed into contributions due to elementwise differences in $\mathbf{p}^1$ and $\mathbf{p}^2$, Δ (also a vector). The method of Horiuchi et al. (2008) works by assuming a regular linear (though this is not necessary) change transforming $\mathbf{p}^1$ into $\mathbf{p}^2$. Say we move from $\mathbf{p}^1$ to $\mathbf{p}^2$ in n steps. In each step, each element moves $\Delta\frac{1}{n}$ of the respective difference. Within each step, there is therefore a new vector composed of an intermediate set of values, $\mathbf{p}^n$ from which θ may be recalculated. In each of the n steps, we perturb the elements of $\mathbf{p}^n$ one at a time, pushing up the i^{th} element $\frac{\Delta^i}{2n}$, calculating θ^{+i}, then pushing the same element down by the same amount and calculating θ^{-i}. The difference between θ^{+i} and θ^{-i} is taken as the contribution to $\theta^2 - \theta^1$ from element i in step n. If there are $s \times \omega$ elements of $\mathbf{p}^{(n)}$, we recalculate θ a total of $n \times 2 \times s \times \omega$ times. When complete, these contributions sum $\theta^2 - \theta^1$. Typically, the contributions for each age and stage are summed over the n "time" steps used in the procedure, producing a total of $s \times \omega$ contributions that are the output that the researcher is actually interested in. This vector of contributions can be aggregated in any way to suit the needs of the researcher and facilitate interpretation.

Even if ζ is simple, this procedure is computationally intensive. The only parameter that can easily be changed is n, the number of "time" steps. Horiuchi et al. (2008) recommend 20 as a practical number of steps, and in my own experience it make little difference whether one chooses 10 or 100 decomposition steps. Depending on the size of the exercise at hand, it may be worth the researcher's time to efficiently program ζ.

3 Application

I apply this approach to a few of the employment transition matrices published by Dudel and Myrskylä (2017) for ages 50+. The reduced (not showing age) state space graph of this model is as drawn in Figure 1.

Note that the elements of $\mathbf{p}$ are all arrows leaving a given state not including self-arrows. This is transformed to $\mathbf{U}$, which contains all arrows including self-arrows and excluding arrows pointing to death.

We haphazardly select four matrices from which to make two comparisons: black females in 2004 vs 2009, and high versus low-educated black females in 2009.[1] Black females at age 50 had a WLE of 10.1 in 2004, which dropped to 9.2 in 2009, for a difference of .9 years. This difference breaks down per Table 1

[1] The requisite matrices come from the files `Pmat_b_f_2004.csv`, `Pmat_b_f_2009.csv`, `Pmat_b_f_edu0_2009.csv`, and `Pmat_b_f_edu2_2009.csv`, respectively.

Figure 1: State space graph of Dudel and Myrskylä (2017)

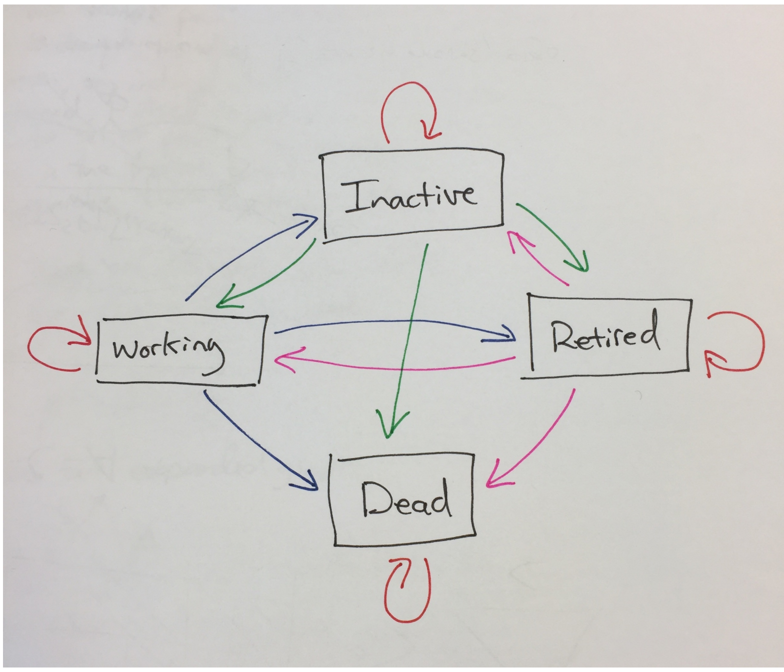


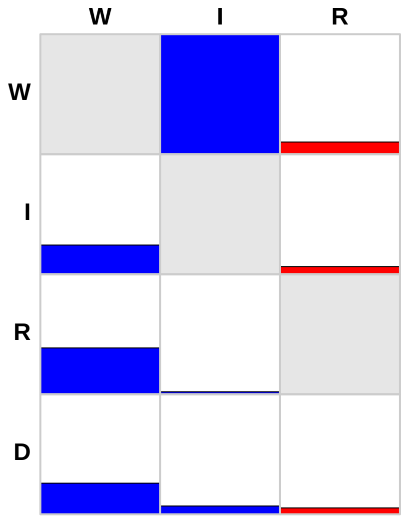
Table 1: Decomposition of .9 decrease in WLE from 2004-2009, black females. “From” states in columns, “to” states in rows.

	W	I	R
W		-0.51	0.05
I	-0.12		0.03
R	-0.20	-0.01	
D	-0.13	-0.04	0.03

From this it is clear enough that the greatest driver of the drop in WLE was due to decreased transitions from inactive to working states, as well as increased exits from working into all other states. Further, it would appear that retirement exits acted in small part to increase time spent working, either directly by transitions to work, or indirectly by decreased mortality or increasing the pool of inactive workers that then might transition into work. There are only nine numbers in this table, so it can be interpreted directly. However, for models with more states, or for many comparisons, it is better to move to visualization to detect patterns.

Table 1 may be redrawn as Figure 2, where each cell represents a table element following the same ordering. Colored bars are proportional to the values, but rescaled such that the maximum absolute contribution is a 100% filled box. Blue indicates negative contributions and red indicates positive contributions. From this view it is easier to surmise that the drop in transitions from inactive to working is roughly equal to all increased flows out of employment combined. Such tables could in principle also be composed in small multiples, in which case it would be advisable to have equal y ranges between individual plots.

Figure 2: Visualization of Table 1. Red and blue indicate positive and negative contributions, respectively.

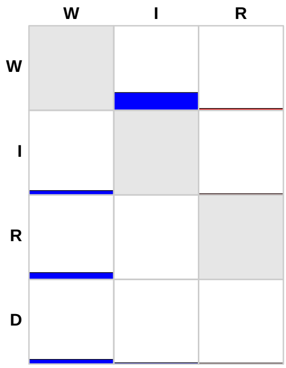


For example, highly educated black females in 2009 had a WLE 2.26 years greater than low educated bloack females. The largest single contribution to this difference was due to mortality differentials of employed persons: 2.4 years. Let's compare these decomposition results side-by-side with those of Figure 2, each proportionally scaled by the same amount.

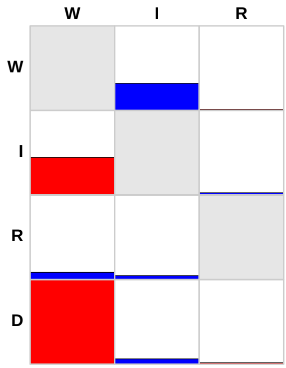
From Figure 3a it is clear that the educational gradient in WLE was greater in 2009 than the total loss in WLE from 2004 to 2009 among US black females. Most of this differential was due to lower mortality and lower transition rates from work to inactivity for highly educated black females. It's worth mentioning that this positive gradient in 2009 was offset by greater transitions from inactivity to employment for low-educated black females, and that this contribution itself was roughly equal in magnitude to the total loss in WLE from 2004 to 2009.

Figure 3: Education differentials in WLE were greater in 2009 than the downward change from 2004 to 2009 among US black females.

(a) Black females 2004 versus 2009
(0.9 year total gap)



(b) High versus low educated, black females, 2009
(2.26 year total gap)



References

Christian Dudel and Mikko Myrskylä. Working life expectancy at age 50 in the united states and the impact of the great recession. *Demography*, Oct 2017. ISSN 1533-7790. doi: 10.1007/s13524-017-0619-6. URL `https://doi.org/10.1007/s13524-017-0619-6`.

Shiro Horiuchi, John R Wilmoth, and Scott D Pletcher. A decomposition method based on a model of continuous change. *Demography*, 45(4):785–801, 2008.