



Universidad  
del País Vasco



Euskal Herriko  
Unibertsitatea

**ikerbasque**  
Basque Foundation for Science



Statistics  
Korea



## KOSTAT-UNFPA Summer Seminar on Population

### *Workshop 1. Demography in R*

## Day 2: Mortality and Fertility

Instructor:

Tim Riffe [tim.riffe@ehu.eus](mailto:tim.riffe@ehu.eus)

Assistant:

Inchan Hwang [inchan@yonsei.ac.kr](mailto:inchan@yonsei.ac.kr)

2 July 2024

## Contents

<b>1</b>	<b>Mortality</b>	<b>2</b>
1.1	Lifetable transformations as functions . . . . .	2
1.2	Death probabilities between age $x$ and $x + n$ , ${}_nq_x$ . . . . .	2
1.3	Survival probabilities between age $x$ and $x + n$ , ${}_np_x$ . . . . .	4
1.4	Survival probabilities to age $x$ , $l_x$ . . . . .	4
1.5	Death distribution, ${}_nd_x$ . . . . .	6
1.6	Person-years lived between age $x$ and $x + n$ , ${}_nL_x$ . . . . .	6
1.7	Person-years lived above age $x$ , $T_x$ . . . . .	7
1.8	Life expectancy $e_x$ . . . . .	7
1.9	Calculating a lifetable for grouped data . . . . .	8

<b>2</b>	<b>A Lifetable function</b>	<b>8</b>
2.1	reformulate function to use in a tidy pipeline . . . . .	9
<b>3</b>	<b>Fertility</b>	<b>10</b>
3.1	Crude birth rate . . . . .	10
3.2	General fertility rate . . . . .	10
3.3	Age-specific fertility rates . . . . .	11
3.4	Total fertility rate . . . . .	12
3.5	Mean age at childbearing . . . . .	13
<b>4</b>	<b>Exercises</b>	<b>13</b>
	<b>References</b>	<b>13</b>

# 1 Mortality

Mortality sets a fundamental constraint on population well-being by defining a longevity envelope within which all life happens. Mortality levels vary over age by orders of magnitude, and can also vary between populations. Demography delivers tools to understand mortality levels in terms of metrics in different units, and to adjust these metrics to be able to make valid comparisons between populations. The lifetable is the basic analytic tool to allow for valid and comparable summary metrics at the population level.

## 1.1 Lifetable transformations as functions

I describe the lifetable together with function-writing because lifetable transformations give simple practice in functionalizing mathematical formulas. When function-writing, it is desirable to work with test data handy. For this, go ahead and load theKOR2014 file from yesterday like so:

```
library(tidyverse)
KOR2014 <- read_csv("Data/KOR2014.csv",
                    show_col_types = FALSE)
```

We will take what we need from this file as we go.

## 1.2 Death probabilities between age $x$ and $x + n$ ${}_nq_x$

The first and key step is to transform a set of age-specific death rates into a set of age-specific probabilities of dying ( ${}_nq_x$ ). The relationship between  ${}_nM_x$  and  ${}_nq_x$  has been established based on analyses of actual cohorts (for mathematical proof, see Preston, Heuveline, and Guillot (2001), p. 42-43).

$${}_nq_x = \frac{n \cdot {}_nM_x}{1 + (n - {}_nA_x) \cdot {}_nM_x}$$

where  ${}_nA_x$  is the average number of person-years lived in the interval by those dying in the interval and  $n$  is the width of the age-interval.

For single ages or when we're pragmatic, we define  ${}_nA_x = n/2$  with the exceptions of the first and the last age group. Other approximations are also available, but these only matter when age groups are wider than a year. In our case, we're working with abridged lifetables, some of which represent high mortality settings, and the  ${}_nA_x$  assumptions are consequential. In our case, I provide this value so that we don't need to work so hard at deriving it in class. You could find several popular  ${}_nA_x$  approximations in the `DemoTools` package Riffe et al. (2021).

Cutting to the chase, we can rather directly convert the  ${}_nq_x$  formula to an R function:

```
calc_nqx <- function(nMx, nAx, n){
  qx      <- (n * nMx) / (1 - (n - nAx) * nMx)

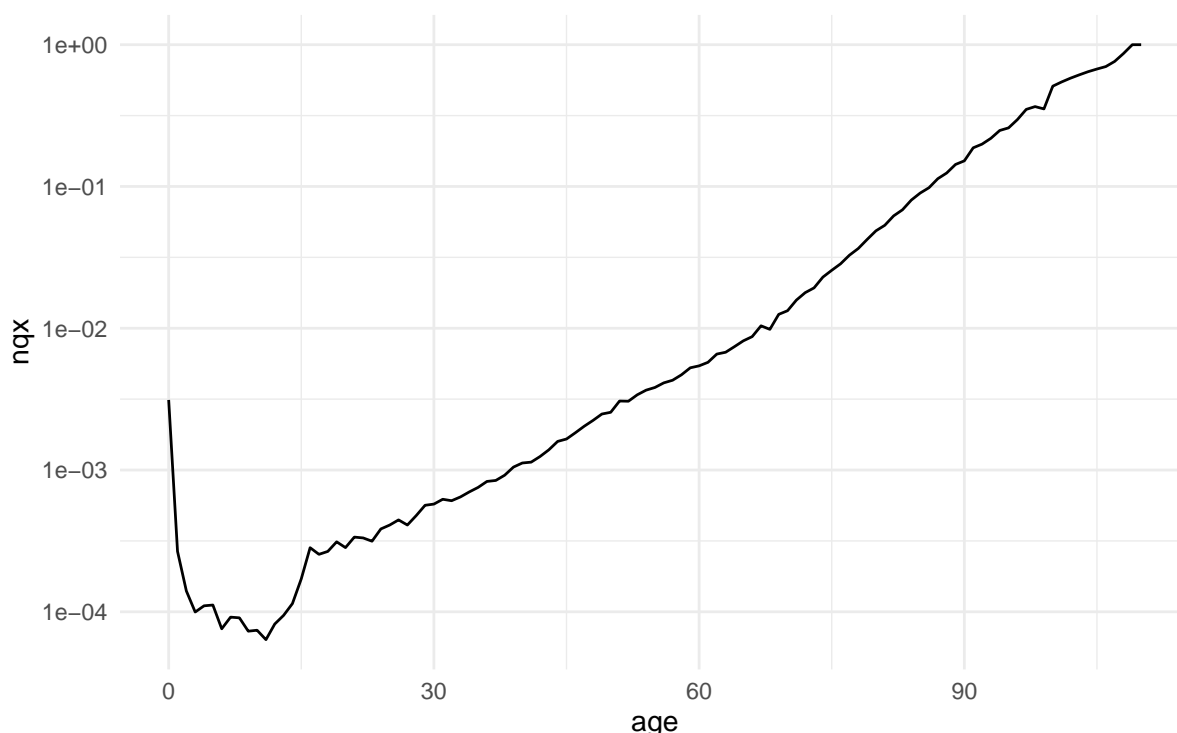
  # these are kludges, necessary to ensure nqx results as a probability
  qx[qx > 1] <- 1
  qx[qx < 0] <- 0
  qx
}
```

Here's how we can use this function in tidyverse syntax and plot the result:

```
# Example of what nqx looks like over age
KOR2014 |>
  filter(sex == "total") |>
  mutate(nMx = deaths / exposure,
         nAx = if_else(age == 0, .1, .5),
         n = 1,
         nqx = calc_nqx(nMx = nMx, nAx = nAx, n = n)) |>
  ggplot(aes(x = age, y = nqx)) +
  geom_line() +
  scale_y_log10() +
  labs(title = "nqx Korea, 2014",
       subtitle = "Data: HMD") +
  theme_minimal()
```

## nqx Korea, 2014

Data: HMD



Often we're sure to *close out* the lifetable by making the *final*  ${}_nq_x$  value equal to 1. You could optionally modify the function to impute 1 like so `nqx[length(nqx)] <- 1`.

Note, this  $q(x)$  formula needs to be committed to memory. This is something we either derive or look up as needed. For our needs we are satisfied with our values of  $q(x)$ ,  $m(x)$  and  $a(x)$ , and the remainder of the lifetable is now determined.

### 1.3 Survival probabilities between age $x$ and $x + n$ , ${}_np_x$

The survival probabilities between age  $x$  and  $x + n$  ( ${}_np_x$ ) is simply one minus  ${}_nq_x$ . It is interpreted as the chance of surviving from age  $x$  to age  $x + n$ .

$${}_np_x = 1 - {}_nq_x$$

Really there's no need to program a function for this column, as we can just use  ${}_nq_x$  as the function argument and take its complement as needed.

### 1.4 Survival probabilities to age $x$ , $l_x$

This indicator indicates the chance of surviving from birth to age  $x$  ( $l_x$ ) OR the number of survivors at age  $x$  relative to the radix  $r$  of the life table. The  $l_0$  is interpreted as the initial size (radix) of the population, generally set to 1 or 100,000. Here's one of several ways to calculate it given what we have so far:

$$l_{x+n} = r \prod_{y=0}^x {}_np_y$$

where  $r = {}_nl_0$  is the radix. To program this, our arguments should be  ${}_nq_x$  (**nqx**) or  ${}_np_x$  (**npx**) and **radix**. In this case, we can assign a default value for the radix when defining the function using **radix = 1**. Whenever the argument isn't specified by the user, 1 will be assumed.

```

# nqx is a full vector over age.
calc_lx <- function(nqx, radix = 1){
  npx <- 1 - nqx
  n <- length(nqx)
  lx <- cumprod(npx)
  # shift it back 1, as we start with 100%!
  # also ensure outgoing vector is the same length.
  lx <- radix * c(1, lx[-n])
  lx
}

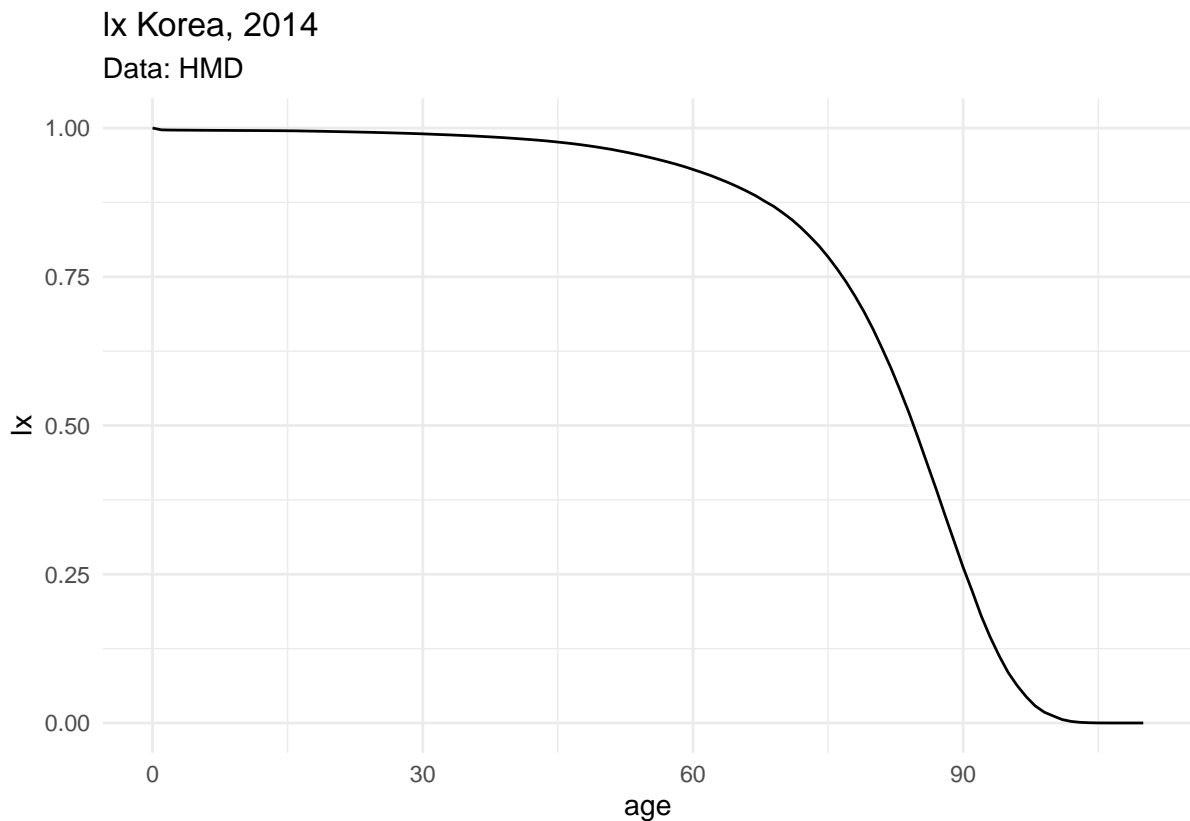
```

And here is an application of the new function to data, as per before:

```

KOR2014 |>
  filter(sex == "total") |>
  mutate(nMx = deaths / exposure,
         nAx = if_else(age == 0, .1, .5),
         n = 1,
         nqx = calc_nqx(nMx = nMx, nAx = nAx, n = n),
         lx = calc_lx(nqx = nqx)) |>
  ggplot(aes(x = age, y = lx)) +
  geom_line() +
  ylim(0,1) +
  labs(title = "lx Korea, 2014",
       subtitle = "Data: HMD") +
  theme_minimal()

```



## 1.5 Death distribution, ${}_nd_x$

The life table deaths ( ${}_nd_x$ ) is the number of (synthetic) persons dying between age  $x$  and  $x + n$ , relative to the radix, and represents the distribution of deaths over age. There are two ways of calculating  ${}_nd_x$ . When programming, this is the most pragmatic way of calculating it:

$${}_nd_x = {}_nq_x * l_x$$

One could ask, do we really need this function? This is something we can remember, right?

```
calc_ndx <- function(nqx, lx){
  nx * lx
}
```

## 1.6 Person-years lived between age $x$ and $x + n$ , ${}_nL_x$

The number of person-years between age  $x$  and  $x + n$  ( ${}_nL_x$ ) is calculated as:

$$\begin{aligned} {}_nL_x &= n(l_x - {}_nd_x) + {}_na_x \cdot {}_nd_x \\ &= n \cdot l_x - (n - {}_na_x){}_nd_x \end{aligned}$$

**Note**

$${}_nm_x = {}_nd_x / {}_nL_x$$

and

$${}_nq_x = {}_nd_x / l_x$$

```
calc_nLx <- function(lx, ndx, nAx, n){
  N <- length(lx)
  nLx <- n[-N] * lx[-1] + nAx[-N] * ndx[-N]
  # special treatment for open age
  nLx[N] <- lx[N] * nAx[N]
  nLx
}

# and application with no plot:
KOR2014 |>
  filter(sex == "total") |>
  mutate(nMx = deaths / exposure,
         nAx = if_else(age == 0, .1, .5),
         n = 1,
         nx = calc_nqx(nMx = nMx, nAx = nAx, n = n),
         lx = calc_lx(nx = nx),
         ndx = lx * nx,
         nLx = calc_nLx(lx = lx, ndx = ndx, nAx = nAx, n = n))
```

```
## # A tibble: 111 x 13
##   year age sex exposure deaths births      nMx      nAx      n      nx      lx
##   <dbl> <dbl> <chr>      <dbl> <dbl> <dbl>      <dbl> <dbl> <dbl>      <dbl> <dbl>
## 1 2014     0 total 419382. 1305.      0 0.00311      0.1      1 3.12e-3 1
```

```
## 2 2014 1 total 464824. 124. 0 0.000267 0.5 1 2.67e-4 0.997
## 3 2014 2 total 477676. 67.0 0 0.000140 0.5 1 1.40e-4 0.997
## 4 2014 3 total 481716. 48 0 0.0000996 0.5 1 9.96e-5 0.996
## 5 2014 4 total 453779. 50.0 0 0.000110 0.5 1 1.10e-4 0.996
## 6 2014 5 total 457061 51.0 0 0.000112 0.5 1 1.12e-4 0.996
## 7 2014 6 total 487581. 37 0 0.0000759 0.5 1 7.59e-5 0.996
## 8 2014 7 total 468850. 43 0 0.0000917 0.5 1 9.17e-5 0.996
## 9 2014 8 total 441220. 40 0 0.0000907 0.5 1 9.07e-5 0.996
## 10 2014 9 total 451977. 33 0 0.0000730 0.5 1 7.30e-5 0.996
## # i 101 more rows
## # i 2 more variables: ndx <dbl>, nLx <dbl>
```

## 1.7 Person-years lived above age $x$ $T_x$

Calculating the number person-years lived above age  $x$  ( $T_x$ ) is a key step to calculate life expectancy. It consists in finding the sum of  ${}_nL_x$  from age  $x$ :

$$T_x = \sum_{y=x}^{\infty} {}_nL_y$$

```
calc_Tx <- function(nLx){
  # to understand this, look at the nLx curve,
  # then imagine integrating from the right
  # to the left. Then compare with the formula!
  nLx |>
    rev() |>
    cumsum() |>
    rev()
}
```

## 1.8 Life expectancy $e_x$

The last indicator in the life table is probably one of the most used in demographic analysis. The life expectancy is the average number of years lived by a (synthetic) cohort reaching age  $x$ . It consists in dividing the number of person-years lived above age  $x$  by the number of people alive at age  $x$ :

$$e_x = \frac{T_x}{l_x}$$

Since `mutate()` let's you make columns in a sequentially dependent way, we can actually do this whole lifetable inside a single `mutate()` statement. However, each combination of **Year** and **Sex** is an independent lifetable, so we need to declare groups beforehand using `group_by()`:

```
calc_ex <- function(Tx, lx){
  Tx / lx
}

KOR2014 |>
  filter(sex == "total") |>
  mutate(nMx = deaths / exposure,
```

```

nAx = if_else(age == 0, .1, .5),
n = 1,
nqx = calc_nqx(nMx = nMx, nAx = nAx, n = n),
lx = calc_lx(nqx = nqx),
ndx = lx * nqx,
nLx = calc_nLx(lx = lx, ndx = ndx, nAx = nAx, n = n),
Tx = calc_Tx(nLx = nLx),
ex = calc_ex(Tx = Tx, lx = lx)) |>
filter(age == 0) |>
pull(ex)

```

```
## [1] 81.49283
```

## 1.9 Calculating a lifetable for grouped data

If your data has many subgroups, we can calculate a lifetable for each of them by declaring groups on the data object with `group_by()`:

```

KOR2014 |>
group_by(sex) |>
mutate(nMx = deaths / exposure,
      nAx = if_else(age == 0, .1, .5),
      n = 1,
      nqx = calc_nqx(nMx = nMx, nAx = nAx, n = n),
      lx = calc_lx(nqx = nqx),
      ndx = lx * nqx,
      nLx = calc_nLx(lx = lx, ndx = ndx, nAx = nAx, n = n),
      Tx = calc_Tx(nLx = nLx),
      ex = calc_ex(Tx = Tx, lx = lx)) |>
filter(age == 0) |>
select(sex, ex)

```

```

## # A tibble: 3 x 2
## # Groups:   sex [3]
##   sex      ex
##   <chr> <dbl>
## 1 female 84.5
## 2 male  78.2
## 3 total 81.5

```

## 2 A Lifetable function

You probably noticed that a whole lifetable operation can fit in a single `mutate()` call! Note, you could calculate all lifetables for each subset by simply including a `group_by()`. Well, that may be so, but there's still value in creating a wrapper function that does the whole thing:

```

calc_LT <- function(nMx, nAx, n, radix){
  N <- length(nMx)
  nqx <- calc_nqx(nMx, nAx, n)

```



```

lx    <- calc_lx(nqx = nqx, radix = 1)
ndx   <- calc_ndx(nqx = nqx, lx = lx)
nLx   <- calc_nLx(lx = lx, ndx = ndx, nAx = nAx, n = n)
Tx    <- calc_Tx(nLx = nLx)
ex    <- calc_ex(Tx = Tx, lx = lx)
Age   <- cumsum(c(0,n))[1:N]

tibble(Age = Age,
        nMx = nMx,
        nAx = nAx,
        nqx = nqx,
        lx = lx,
        ndx = ndx,
        nLx = nLx,
        Tx = Tx,
        ex = ex)
}

```

## 2.1 reformulate function to use in a tidy pipeline

To use easily in a tidy pipeline, we'll just want to be sure that the input consists in a whole group (or chunk) of data:

```

# data.frame in, data.frame out!
calc_LT_tidy <- function(data, radix){
  # this is hacky, but works.
  # just pick out the needed vectors from the group of data
  calc_LT(nMx = data$nMx,
          nAx = data$nAx,
          n = data$n,
          radix = radix)
}

```

Now, this is something you can easily apply in bulk using `group_modify()`. You could design this in many different ways actually.

```

KOR2014 |>
  mutate(nMx = deaths / exposure,
         nAx = if_else(age == 0, .1, .5),
         n = 1) |>
  group_by(sex) |>
  group_modify(~calc_LT_tidy(data = .x, radix = 1e5)) |>
  ungroup()

```

```

## # A tibble: 333 x 10
##   sex      Age      nMx  nAx      nqx  lx      ndx  nLx    Tx    ex
##   <chr> <dbl>    <dbl> <dbl>    <dbl> <dbl>    <dbl> <dbl> <dbl> <dbl>
## 1 female  0 0.00289    0.1 0.00290  1    0.00290  0.997  84.5  84.5
## 2 female  1 0.000296    0.5 0.000296 0.997 0.000295 0.997  83.5  83.8

```

```
## 3 female      2 0.000108      0.5 0.000108      0.997 0.000107      0.997 82.5 82.8
## 4 female      3 0.0000941     0.5 0.0000941     0.997 0.0000938     0.997 81.5 81.8
## 5 female      4 0.000100      0.5 0.000100      0.997 0.0000997     0.997 80.5 80.8
## 6 female      5 0.000108      0.5 0.000108      0.997 0.000108      0.996 79.6 79.8
## 7 female      6 0.0000634     0.5 0.0000634     0.996 0.0000631     0.996 78.6 78.8
## 8 female      7 0.0000881     0.5 0.0000881     0.996 0.0000878     0.996 77.6 77.8
## 9 female      8 0.0000846     0.5 0.0000846     0.996 0.0000842     0.996 76.6 76.9
## 10 female     9 0.0000643     0.5 0.0000643     0.996 0.0000641     0.996 75.6 75.9
## # i 323 more rows
```

### 3 Fertility

In English, fertility refers to observed births, whereas fecundity refers to the capacity to give birth. The basic elements of fertility data include

- Events: births
- Exposure: every women alive in their reproductive age (  $\approx$  15 to 50 years old)

Births information most often comes from vital registration systems. This is the case for Korea. Countries without vital registration systems, or with incomplete vital registration rely on survey data to estimate fertility indicators.

#### 3.1 Crude birth rate

The crude birth rate (CBR) is a rough measure of the occurrence/exposure of fertility.

$$CBR_{[0,T]} = \frac{\text{Number of births in the population between times } T \text{ and } T + t}{\text{Number of person - years lived in the population between times } T \text{ and } T + t}$$

Does this look familiar? It's just like the crude death rate:

```
# Person-years
KOR2014 |>
  filter(sex == "total") |>
  summarize(exposure = sum(exposure),
            births = sum(births, na.rm = TRUE)) |>
  mutate(CBR = births / exposure)
```

```
## # A tibble: 1 x 3
##   exposure births    CBR
##   <dbl>   <dbl> <dbl>
## 1 50765887. 435435. 0.00858
```

#### 3.2 General fertility rate

The general fertility rate (GFR) is generally considered a better measure of fertility, as only women in their reproductive ages can give birth, and are thus at risk of experiencing the event.

$$GFR[0, T] = \frac{\text{Number of births in the population between times } T \text{ and } T + t}{\text{Number of person - years lived by women aged 15 to 50 between times } T \text{ and } T + t}$$

This solution looks tricky because births are associated with total sex in our data, but here we need to relate them to women. For this, we use `pivot_wider()` to get all combinations of sex and measure side-by-side, then we shift births to females in `mutate()`, then we pivot the data back to its original form, filter to the desired subset, and calculate the GFR:

```
# Female population
KOR2014_2 <-
  KOR2014 |>
  select(-deaths) |>
  pivot_wider(names_from = sex, values_from = c(exposure, births)) |>
  mutate(births_female = births_total) |>
  pivot_longer(-c(year, age), names_to = c("measure", "sex"), values_to = "value", names_sep = "_") |>
  pivot_wider(names_from = "measure", values_from = "value") |>
  filter(sex == "female")

KOR2014_2 |>
  filter(between(age, 15, 50)) |>
  summarize(exposure = sum(exposure),
            births = sum(births)) |>
  mutate(GFR = births / exposure)
```

```
## # A tibble: 1 x 3
##   exposure births    GFR
##   <dbl>   <dbl> <dbl>
## 1 13308014. 435396. 0.0327
```

The GFR does not give unbiased information, however, as the age structure between ages 15 and 50 can vary wildly between populations, as can the shape and location of fertility rates within this range. Ideally, we would like a measure purged of the effects of observed population structure.

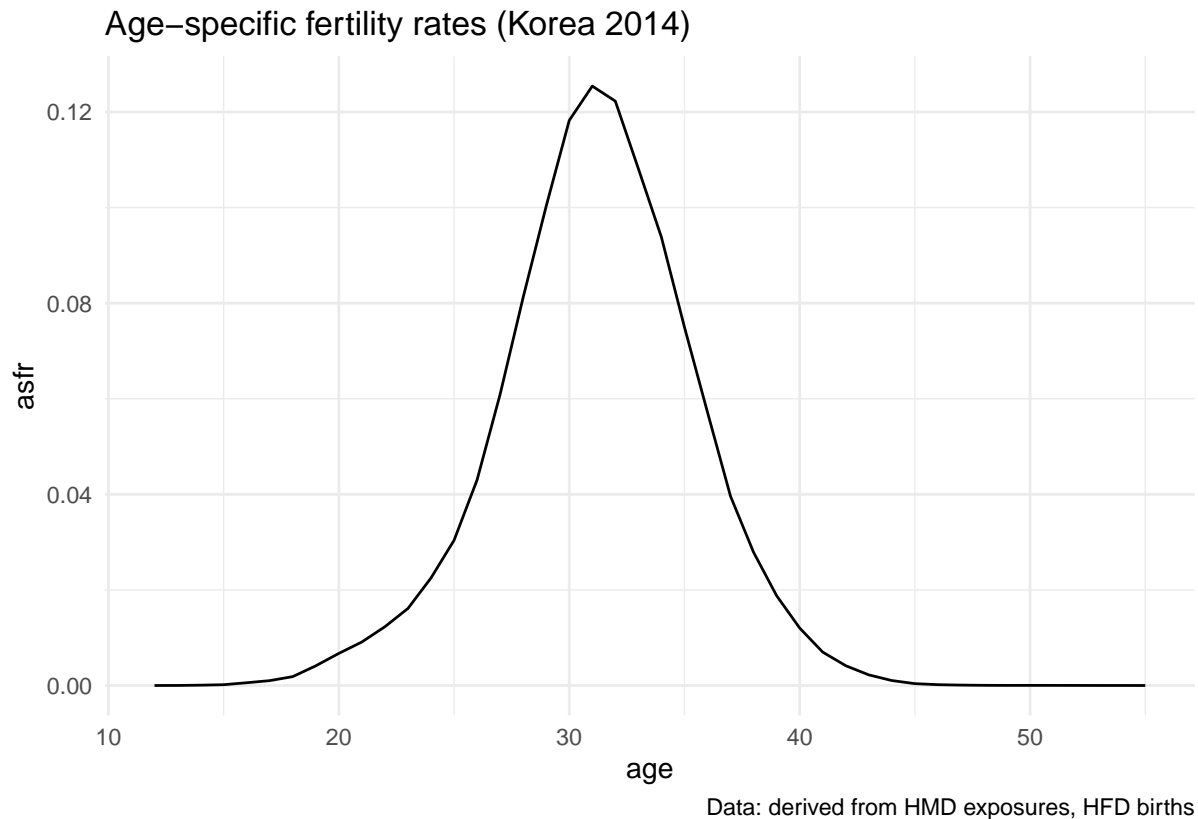
### 3.3 Age-specific fertility rates

As with age-specific death rates, age-specific fertility rates ( $F$ ) are less sensitive to the age structure of the population. This measure provides the risk of given birth from women age  $x$  to  $x + n$ :

$${}_nF_x[0, T] = \frac{\text{Number of births between times } T \text{ and } T + t \text{ to women aged } x \text{ to } x + n}{\text{Number of person - years lived by women aged } x \text{ to } x + n \text{ between times } T \text{ and } T + t}$$

```
KOR2014_2 |>
  mutate(asfr = births / exposure) |>
  ggplot(aes(x = age, y = asfr)) +
  geom_line() +
  labs(title = "Age-specific fertility rates (Korea 2014)",
```

```
caption = "Data: derived from HMD exposures, HFD births") +
xlim(12,55) +
theme_minimal()
```



There can still be unobserved heterogeneity hiding in this population: age is not the only structural determinant of fertility. For instance, fertility may be considered separately by parity, or for other subpopulations.

### 3.4 Total fertility rate

The total fertility rate (TFR) is the average number of children a woman would have if she experienced the a particular set of age-specific fertility rates and survived until the end of her reproductive age. “*The TFR is the single most important indicator of fertility*” (Preston, Heuveline, and Guillot 2001). It is also the area under the ASFR curve.

$$TFR[T, T + t] = n \sum_{x=a}^{B-n} {}_nF_x[T, T + t]$$

where  $a$  and  $B$  are the minimum and maximum age at childbearing.

```
# TFR
KOR2014_2 |>
  mutate(asfr = births / exposure,
         n = 1) |> # make this 5 if you have 5-year age groups!!
  summarize(tfr = sum(asfr * n))
```

```
## # A tibble: 1 x 1
```

```
##      tfr
##    <dbl>
## 1    1.20
```

This the most commonly calculated and cited fertility metric, but it is not without criticism. For instance, (i) we should not discount the leverage mortality can have on population reproductivity, (ii) if fertility patterns are changing over time a period TFR may not give the best signal of fertility levels, and (iii) TFR is unfortunately sometimes presented as a target.

### 3.5 Mean age at childbearing

The mean age at childbearing is not a rate, but is based on the age-specific fertility rates. The mean age at childbearing (MA) is the average age of mothers at childbearing, standardized for the age-structure of the female population at reproductive age (Human Fertility Database 2018).

$$MA[T, T + t] = \frac{\sum_{x=a}^{B-n} \bar{x} * {}_nF_x[T, T + t]}{\sum_{x=a}^{B-n} {}_nF_x[T, T + t]}$$

where  $\bar{x}$  is the mid-age of interval  $x : x + n$ , i.e.  $\bar{x} = x + n/2$ .

```
# MA
KOR2014_2 |>
  mutate(asfr = births / exposure,
         n = 1,
         xbar = age + n / 2) |>
  summarize(MA = sum(xbar * asfr) / sum(asfr))
```

```
## # A tibble: 1 x 1
##      MA
##    <dbl>
## 1    31.7
```

## 4 Exercises

In practical exercises, we will calculate trends for different populations based on different data.

## References

- Human Fertility Database. 2018. “Max Planck Institute for Demographic Research (Germany) and Vienna Institute of Demography (Austria).”
- Preston, S, Patrick Heuveline, and Michael Guillot. 2001. “Demography: Measuring and Modeling Population Processes.” *Malden, MA: Blackwell Publishers*.
- Riffe, Tim, José Manuel Aburto, Ilya Kashnitsky, Monica Alexander, Marius D. Pascariu, Sara Hertog, and Sean Fennell. 2021. *DemoTools: Standardize, Evaluate, and Adjust Demographic Data*.