

Formal Relationships: Introduction and Orientation

Joshua R. Goldstein
James W. Vaupel

In addition to Research Articles, Reflexions, Descriptive Findings, Research Material, and Reviews, *Demographic Research* will publish short reports on mathematical relationships in formal demography. These reports will be published as part of a new, ongoing Special Collection, called Formal Relationships, that will be edited by the two of us, the current and founding Publishers of the journal. Note that these reports should pertain to mathematical formulas that can be proven and that concern relationships among demographic measures. The reports should not focus on statistical findings about empirical regularities and the results should not depend on behavioral assumptions.

Submissions to Formal Relationships should be sent to *Demographic Research* according to the usual procedures. These submissions will be forwarded to the two of us and we will review them. They will also be posted so that members of the journal's Scientific Review Board can review them. If we need additional help, we will ask others to assist us as referees. If, given the reviews, the two of us agree that it is appropriate to publish a submission in the Formal Relationships department of the journal, we will then send the item to the Editor of the journal, currently Nico Keilman, who will make the final decision.

If and when enough Formal Relationships have been published in *Demographic Research*, we will compile them into a book to be published by Springer Verlag as part of the series of *Demographic Research Monographs*.

Authors submitting to Formal Relationships do not have to be the discoverers of the mathematical formulas in their submission. Indeed, most of the Formal Relationships we publish will be especially beautiful equations that are known to many demographers or striking expressions that deserve more attention. Our goal is to compile demographic relationships that are, as Michel Guillot put it, "sheer poetry". We aspire to collect the pearls of formal demography, the champagne, the quintessence. Submissions to Formal Relationships should be original manuscripts that have not been previously published and they should be written in a cogent, lucid manner that conveys the elegance of a mathematical equation and its proof as well as indicating why the relationship is interesting and important for formal demography.

Articles published in Formal Relationships will generally be concise, often only a page long, rarely more than two or three. Articles will not cover a topic, such as "life expectancy" or "decomposition", but will focus on a specific relationship that might fit under such a rubric when the collection is compiled and published as a book.

Articles will typically consist of the following sections:

1. The key relationship.
2. Proof of the relationship.
3. Optionally, some other relationships that are closely linked to and can be readily derived from the key relationship.
4. History of the discovery of the relationship and related relationships, with appropriate citations.
5. A brief sketch of a possible application or perhaps two or three.

To improve readability and to enhance communication, we will try to edit Formal Relationships so that notation is as consistent as feasible and follows general practice in formal demography. We will strive for compact, elegant notation that is clear and unambiguous. Proofs and exposition should be lucid, readable and not excessively technical. We aspire to publish a series of Formal Relationships that are concise and clear—and that make a reader smile with joy about the beauty of pure demography.

We encourage readers of Formal Relationships to make corrections and suggestions, especially regarding the history of the discovery of a relationship. Particularly useful or interesting remarks will be appended to the text of Formal Relationships, with credit given to the person making the contribution.

The force of mortality by life lived is the force of increment by life left in stationary populations

Author Redacted

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Structured abstract

Background

The age distribution and remaining lifespan distribution are identical in stationary populations. The lifetable survival function is proportional to the age distribution in stationary populations.

Objective

We provide an alternative interpretation of the lifetable when viewed by remaining years of life.

Conclusions

The functions describing the mortality of birth cohorts over age are identical to the functions describing the growth of death cohorts as time to death decreases in stationary populations.

Relationship

Age can be defined as chronological, a (time since birth), or thanatological, y (time until death). In a stationary population, decrement over chronological age, as described by the lifetable, is identical to increment over thanatological age. Define the chronological force of mortality, $\mu(a)$, and the thanatological force of increment, $\mu^*(y)$. We have:

$$\mu(a) = \mu^*(y) \quad \text{for } a = y \quad . \quad (1)$$

Proof

Chronological age structure equals thanatological age structure in a stationary population (Brouard 1989, Vaupel 2009, Rao and Carey 2014):

$$c(a) = g(a) \quad , \quad (2)$$

where $c(a)$ is the stable chronological age structure expressed as a proportion, and $g(a)$ is the proportion of the population with a remaining years of life. Using a to index chronological age and y to index thanatological age (time left until death), equation (2) is the same as:

$$c(a) = c^*(y) \quad \text{for } a = y \quad , \quad (3)$$

where $c^*(y)$ refers to the stable thanatological age structure. $c(a)$ is proportional to the survival function, $l(a)$, which means that $c^*(y)$ is proportional to some thanatological function, $l^*(y)$, a stationary thanatological cumulative *increment* function. To untangle how this is so, it helps to be explicit about what $l(a)$ represents. A birth cohort is a group of individuals that are born at the same

point in time and experience attrition over age until extinction, as described by $l(a)$. $l(a)$ is also the sum of all deaths to the birth cohort at ages greater than a , and can be thought of as the de-accumulation of the stock of future deaths over age.

$$l(a) = \int_0^{\omega-a} f(a+t) dt \quad , \quad (4)$$

where $f(a)$ is the lifetable density function, often denoted by $d(a)$. $f(a)$ gives the probability that a member of the birth cohort born in year t will die in the year $t+a$. Since $t+a$ refers to the future, let us switch to index y , for thanatological age. Under stationarity, $f(y)$ is also the probability at birth that an individual born y years ago will die in present year t . The deaths that occur together in year t comprise a *death cohort*. In year $t-1$, $l(0) \cdot f(1)$ births increment to the year t death cohort in the stationary population. Death cohorts grow monotonically in this way, starting with a few members that will enjoy the maximum attainable lifespan, $y = \omega$, i.e., born ω years ago in year $t-\omega$ and then expiring in year t . New births accumulate into a given death cohort as y decreases from ω toward thanatological age 0. From the vantage point of year t , we define the year $t+y$ death cohort (y years in the future), $l^*(y)$ as the members born up to year t that will die in exactly y years.

$$l^*(y) = \int_{-(\omega-y)}^0 f(y-t) dt \quad . \quad (5)$$

Extinction of the death cohort is simultaneous upon reaching thanatological age 0, y years in the future. Equation 5 is equal to 4:

$$l^*(y) = \int_{-(\omega-y)}^0 f(y-t) dt = \int_{-\omega}^{-y} f(-t) dt = \int_y^{\omega} f(t) dt = l(y) \quad . \quad (6)$$

The lifetable deaths distribution is under this perspective a distribution of the births to a death cohort, when read from the highest to the lowest chronological ages. The rate at which new births accumulate to death cohorts over thanatological age, $\mu^*(y)$, is given by

$$\mu^*(y) = \frac{-l^{*\prime}(y)}{l^*(y)} \quad , \quad (7)$$

and since $l(a) = l^*(y)$ for $a = y$,

$$\begin{aligned} &= \frac{-l'(a)}{l(a)} \\ &= \mu(a) \quad . \end{aligned} \quad (8)$$

The rate of birth accumulation into death cohorts over thanatological age, $\mu^*(y)$, is equal to the rate of attrition of birth cohorts over chronological age, $\mu(a)$ in stationary populations. The stock of deaths in the future for chronological age a , $l(a)$, is symmetrically a description of births in the past when structured by thanatological age, $l^*(y)$. The remaining lifetable columns are subject to similar reinterpretations when viewed under thanatological age. The main contribution of this relationship is to point out the symmetry of chronological decrement and thanatological renewal processes in the case of stationary populations.

History and related results

The term “thanatological age” has not previously appeared in the literature, and was coined by Ken Wachter sometime prior to 2001. Thanatos was the Greek god of death, which is used as the reference point for age under this perspective. Explicit decompositions of chronological age groups into remain-

ing lifespan classes is to our knowledge only found in Brouard (1986), who redistributed population pyramids by remaining years of life, and Miller (2001), who examined medicare expenditure as a function of time until death using the same lifetable identities. Equivalence of chronological and thanatological age structures in stationarity, also known as Carey’s equality, is proven by Brouard (1989), again in Vaupel (2009),¹ and in more detail by Rao and Carey (2014). These latter two papers and Goldstein (2012) describe more of the lineage of the result presented here.

The estimation of remaining lifespan has motivated demography since the invention of the lifetable, but this notion is typically dealt with as a mean (expectancy). Age-specific remaining lifespan distributions are not often used to explicitly decompose demographic quantities, such as population counts. Using the above identities, one may estimate the total population with y remaining years of life, $P(y)$ directly

$$P(y) = \int_0^\infty P(a)\mu(a+y)\frac{l(a+y)}{l(a)} da \quad , \quad (9)$$

as Brouard (1986) did, and so approximate a population’s thanatological age structure according to some mortality assumptions. This result differs from the more common approach of summing the population within age classes bounded by some values interpolated along the remaining life expectancy function (e.g. Sanderson and Scherbov (2005; 2007; 2010), who follow the line of Hersch (1944) and Ryder (1975)).

In the present case of a stationary population, one decomposes back to birth cohorts in much the same way as is evident from (9). The probability that a member of death cohort y' was born a' years ago is equal to the probability

¹Vaupel (2009) added detail to the more summary result reported in Goldstein (2009), which showed that the mean (or total) remaining lifespan is equal to mean (or total) life lived in the stationary population.

that a member of birth cohort a will die y years in the future when $a = y'$ and $y = a'$.

$$f(y|a) = f(a'|y') \quad \text{for } a = y', y = a' \quad (10)$$

$$= \mu(a+y) \frac{l(a+y)}{l(a)} \quad (11)$$

$$= \mu^*(a'+y') \frac{l^*(a'+y')}{l^*(y')} \quad (12)$$

Equation (12) seems redundant, but is less obvious when put into words: The probability of dying y years in the future given survival to chronological age a is the probability of surviving to chronological age $a+y$ given survival to age a times the force of mortality at age $a+y$, $\mu(a+y)$. Viewed thanatologically, the probability of having been born a' years in the past given that one has y' remaining years of life is equal to the probability that someone in the y' death cohort was born more than a' years ago given that they have already been born times the force of increment at thanatological age $a' + y'$, $\mu^*(a' + y')$.

Similarly, taking $f(y|a)$ as the conditional density of remaining lifespans, one may calculate the variance of age-specific remaining life spans (thanatological age), where $e(a)$ is remaining life expectancy at age a :

$$e(a) = \frac{\int_{y=0}^{\infty} l(a+y) dy}{l(a)} \quad (13)$$

The variance of y given survival to age a is

$$\sigma^2(y|a) = \int_{y=0}^{\infty} (e(a) - y)^2 f(y|a) dy \quad , \quad (14)$$

and this function will have some non-monotonic pattern over age (in human

populations) that remains to be explored.² In the reliability literature, $\sigma^2(y|a)$ is called the variance residual life function (VRLF), and its properties have been described for various common distributions (see for example Gupta 2006).

The thanatological age perspective only offers the kind of profile symmetry presented in this paper for the theoretical case of stationary populations. For changing populations, the chronological and thanatological age perspectives typically offer different profiles of the same phenomena, due to changes in lifespan distributions and fluctuations in the birth flow, and therefore offer complementary information on population structure.

Applications

Thanatological age structure can be applied to stable populations (subject to a growth rate, r), though we leave the description of a thanatological renewal model for future work. Thanatological age equates individuals that share a common terminal state rather than a common origin state. In the present relationship, this is the absorbing state of death, but the method generalizes to any terminal state or lifecourse transition that can be modeled using lifetable techniques. Potential area applications that may gain insights using such remaining-time methods include morbidity, disability, late-life savings and investment behavior, or perhaps time to birth, menopause, retirement, or graduation. Glacial or open ice pack, old growth forest, and prison populations are other examples of aggregates for which remaining time structure is inherently of equal or greater interest than time passed. Populations of fixed or controlled size, or where entries are largely a function of exits are also prime candidates for analysis under some analogy to thanatological age. Examples of such populations include professional athletes in leagues, tenured professors, company directors, and vehicle

²This definition works out to be the same as Chiang (1984), Chapter 10, Equation 6.10.

fleets.

As a specific example, the question of morbidity compression has often been posed as a matter of comparing age of onset with remaining life expectancy (e.g., Fries 2002; 2003).³ In retrospect (within death cohorts), one can analytically separate between changes in lifespan and changes in morbidity as a function of time until death. Cross-tabulations of morbidity by chronological and thanatological age for two birth cohorts would allow the researcher to estimate changes in the morbidity profile over time free from distortion due to changes in the lifespan distribution, and so directly settle the question of morbidity compression.

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³In the health literature, one often sees $e(65)$ compared with average disability levels above age 65.

submitted as a relationship report to be included in the Special Collection “Formal Relationships” edited by Joshua R. Goldstein and James W. Vaupel

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