Time spent and left of transient states in stationary populations

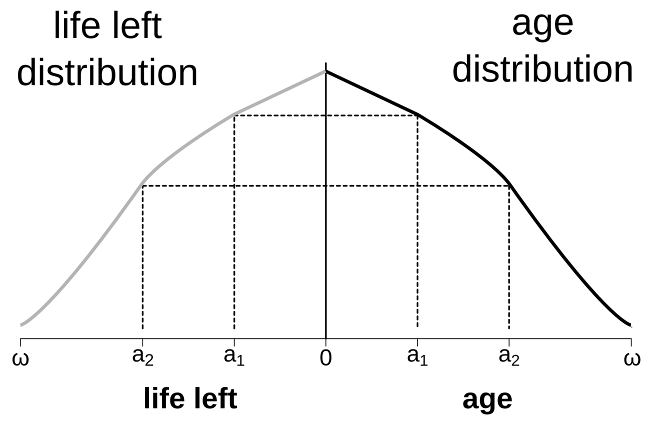
Summary

In **stationary multistate populations**, the distribution of individuals in a given state by **time spent** continuously in that state is **identical** to the distribution by **time left**. We aim to prove this symmetry.

This might be useful to **estimate one of these distributions from the other**. It might be help to estimate the unobserved age pattern of onset for health conditions.

Background

In stationary populations there is a symmetry between the age compositon on the distribution of remaining life: the **Brouard-Carey Equality**.

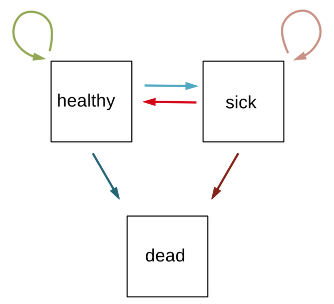


Multistate stationary populations produce the expectation of a fixed multistate trajectory composition.

Example

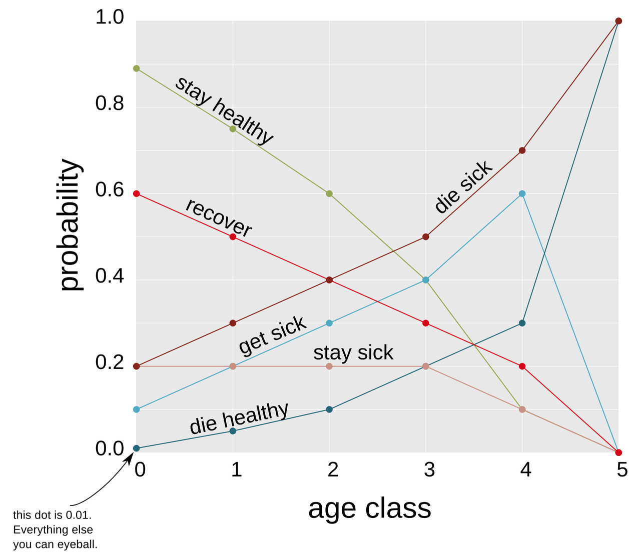
Let's generate a multistate stationary population using a Markov process.
Define a 2-state model (healthy-sick) with 6 discrete age classes.

state space diagram



This implies six transition probability schedules. The probability of being sick at birth is .1.

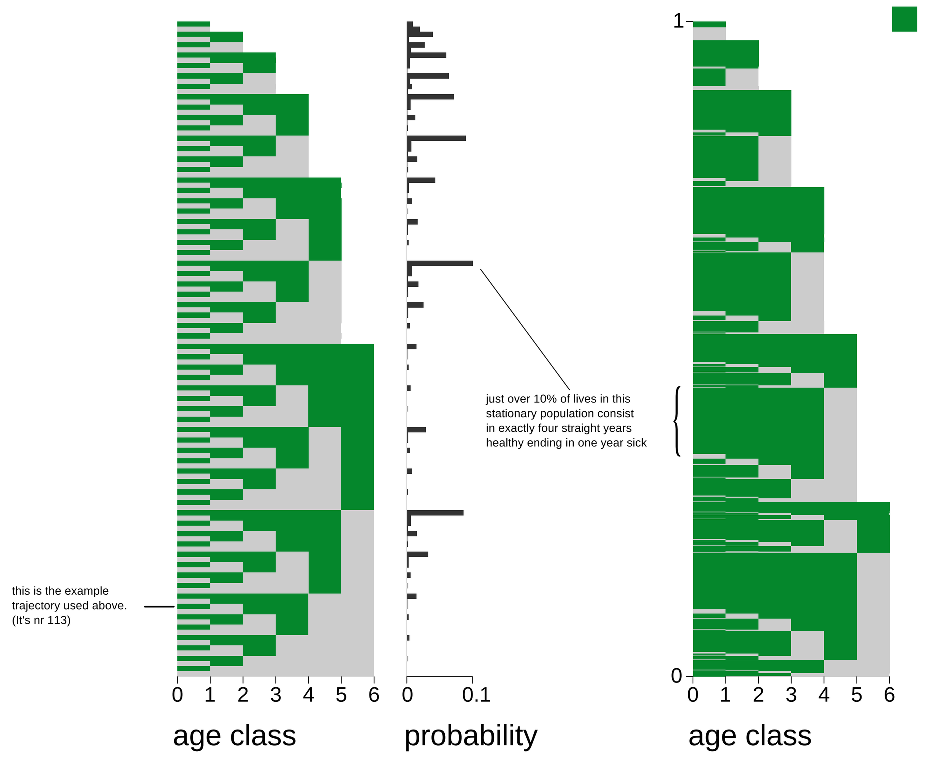
toy transition probabilities



This setup implies **126 possible life trajectories** (left panel). Each trajectory has a single probability of occurring, allowing us to weight each possible trajectory to determine the stationary population composition (right panel).

trajectory space and probability of each

stationary trajectory composition



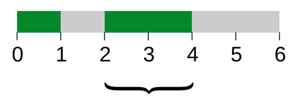
this is the point of departure

Pictoral proof

1 individual symmetry

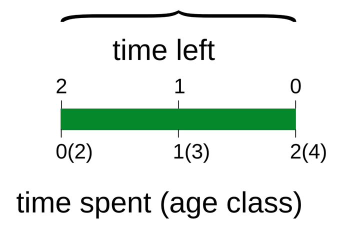
A life trajectory can be broken into distinct **state episodes**.

a discrete trajectory



The second episode of good health starts at age 2 and ends at age 4. This episode passes through age classes, but can also have its own clocks, **time left** and **time spent** in the state episode.

symmetry within episodes



Over the course of this single episode, values of time spent and left form two **identical sets**.

$A^{i,j}$: set of time spent values for the j^{th} episode of the i^{th} trajectory.

$T^{i,j}$: set of time left values

such that $A^{i,j} = T^{i,j}$

From our example trajectory, we have:

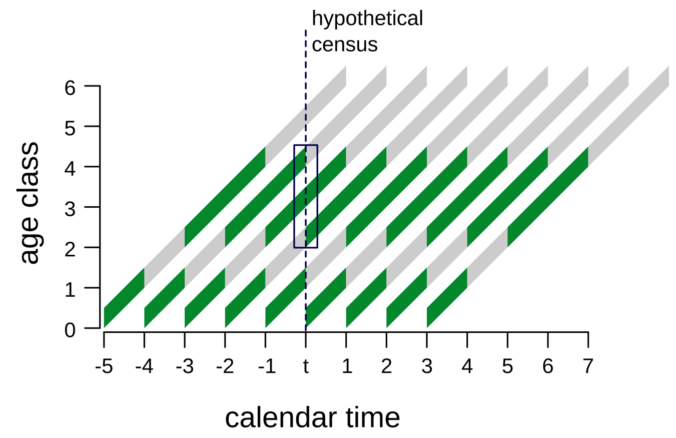
$A^{113,2} = \{0,1,2\}$

$T^{113,2} = \{2,1,0\}$

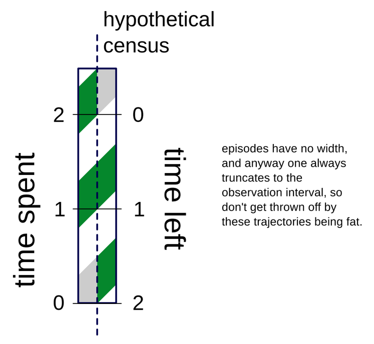
2 symmetry over a renewed trajectory

Once in **stationary renewal**, the probability of observing this very episode is fixed in each time interval. If renewal is happening in lock-step with age, then we expect to **see each time spent and time left** value of this episode **simultaneously**, but over sequentially replicated life trajectories.

stationary renewal of a single trajectory



zoom on single renewed episode



In a **census** we expect to see the **same time spent and time left values**, but drawn from trajectories generated in **different birth cohorts**.

3 identical sets concatenate identically

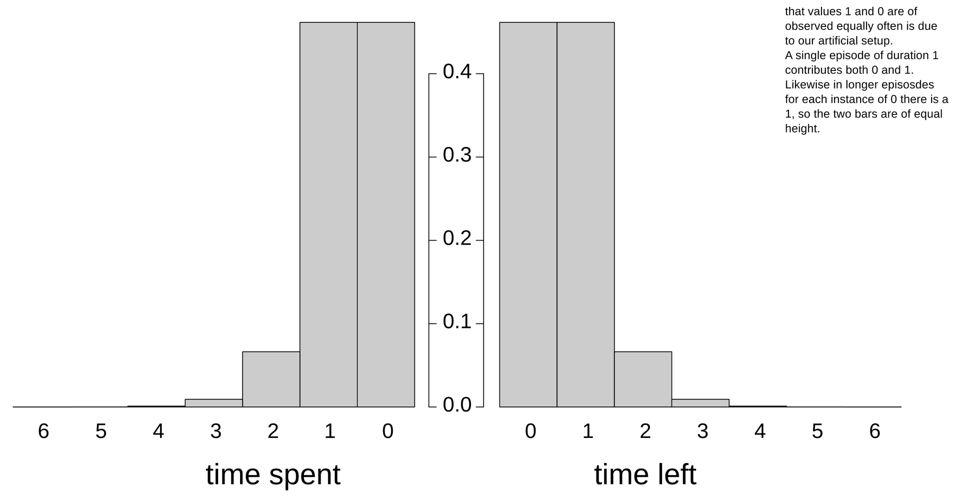
Say we collect values from both the first and second episodes over the life course of the 113th individual.
Both **sets** are **identical**:

$$T^{113} = \{T^{113,1}, T^{113,2}\} = \{1,0,2,1,0\}$$

$$A^{113} = \{A^{113,1}, A^{113,2}\} = \{0,1,0,1,2\}$$

The **same must hold** in the **period cross section** for episodes from distinct trajectories that are subject to stationary renewal. By **induction**, we can keep concatenating time spent and left values as of a **hypothetical census**. This owes to the stationary trajectory composition, itself derived from fixed transition rates. ∎

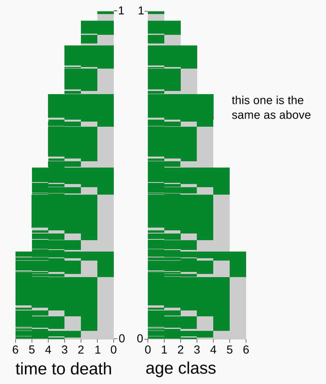
Symmetrical distributions of time spent and left sick.



this is the primary result

Aside: related things that are not symmetrical

The stationary trajectory composition is not symmetrical in the aggregate Brouard-Carey sense.



The resulting multistate occupancy patterns are also not symmetrical.

The resulting multistate prevalence patterns are also not symmetrical.

Authors

Tim Riffe
Max Planck Institute for Demographic Research
riffe@demogr.mpg.de
@timriffe1

Francisco Villavicencio
University of Southern Denmark
fvillavicencio@health.sdu.dk
@VillavicencioFG

Nicolas Brouard
Institut National d'Études Démographiques
brouard@ined.fr

Further material

F. Villavicencio and T. Riffe (2016) Symmetries between life lived and left in finite stationary populations, *Demographic Research*.

T. Riffe and N. Brouard (2018) Structure and dynamics of populations: The 'years-to-live' pyramid, national aspects and regional examples. Socarxiv. An English translation of:

N. Brouard (1986) Structure et dynamique des populations. La pyramide des années à vivre, aspects nationaux et exemples régionaux. *Espace, Populations, Sociétés.*

Code

Any R code or LaTeX related to this work is available in a public repository, including while in development:

https://github.com/timriffe/TransientSymmetry

Extensions

Other symmetries follow logically from the episode-based time-spent and left symmetry:
1) episodes considered cumulatively
2) merged states
3) episodes conditioned on order or duration.

This relationship is also tractable in **stable populations**. Future research, potentially using simulations, should aim to establish the impact of departures from stationarity and suggest bounds of usability for this relationship.

Conclusions

We give an **intuitive proof** that the **distributions of time spent and left are equal within states** in stationary populations.

This relationship may inspire new measurement approaches when **estimating unobserved onset of health conditions**.

We think that researchers working with **censored or truncated data** or any kind of multistate models should be generally aware of this equality in case it may come in handy as a heuristic.