Consumer and Speculator Market Dynamics - Model

Tim Schwuchow

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Contents

1 Theoretical Model

1.1 Notation

Table 1 describes the notation that will be used in the remainder of the paper:

Symbol	Value
	General Terms
i	Consumer subscript
t	Time subscript
H	Measure of the housing stock
N	Measure of consumers
θ	Vector of structural parameters
	Valuation Terms
β	Discount rate
x_i	Fixed value of housing for consumer i
$\bar{x}$	Mean of x_i over the consumer population
σ_x^2	Variance of x_i over the consumer population
η_t	"Fundamental" stochastic flow value component
ρ_η	Persistence of η_t
μ_t	Random innovation to η_t
σ_μ^2	Variance of μ_t
γ_{it}	Idiosyncratic stochastic flow value compoent
ρ_γ	Persistence of ξ_{it}
ξ_{it}	Random innovation to γ_{it}
σ_ξ^2	Variance of ξ_{it}
v_{it}	Total flow value of housing ownership for consumer i in t
	Housing Market Terms
b_{it}	Bid of consumer i in period t
p_t	Market clearing price at time t
h_t	Housing purchased by speculators at time t
$b^*(h_t)$	Marginal idiosyncratic equilibrium bid value with speculator holdings h_t
	Belief Terms
$\mathbf{\Omega}_{it}$	Information set of consumer i at the start of period t
δ_{it}	Difference between the expected and actual value of η_t by consumer i in period t
$\bar{\delta}_t$	Portion of δ_{it} common to all consumers
ι_{it}	Portion of δ_{it} idiosyncratic to each consumer
$\boldsymbol{\phi}_{it}$	Vector of bid-relevant random variables for consumer i in period t

Table 1: Notation

1.2 Model Overview

We first characterize a housing market using a discrete time, infinite horizon framework in which consumers face persistent uncertainty about the "fundamental" value of owning a home, which is characterized by the willingness to pay of some marginal buyer for housing services in each period in a sense that will become clear as we develop the model. In each period, a fixed stock of identical housing units of measure H are allocated to a set risk-neutral consumers of measure $N < H$. Housing is allocated in each period via a simple auction mechanism in which every consumer submits a bid to purchase a single unit of housing. Under this mechanism, all H housing units are allocated to the continuum of bidders who submitted the highest bids, with each winning bidder paying the bid of the marginal auction winner[1].

Consumers differ in the flow utility they receive from owning a home, which evolves stochastically over time. The flow value for consumer i at period t, which we denote v_{it}, is the sum of three components:

$$v_{it} = x_i + \eta_t + \gamma_{it} \tag{1}$$

The first term of (1), x_i, is a fixed, individual-specific flow component that varies across individuals and captures persistent differences in the value of ownership between individuals. We assume that x_i is normally distributed over the population with mean $\bar{x}$ and variance σ_x^2. The second term, η_t, is a stochastic component that varies over time but is common to all consumers, while the third, γ_{it}, varies both over individuals and time. We assume that the stochastic stochastic components both follow AR(1) processes given by

$$\eta_t = \rho_\eta \eta_{t-1} + \mu_t \tag{2}$$

$$\gamma_{it} = \rho_\gamma \gamma_{it-1} + \xi_{it} \tag{3}$$

where μ_t and ξ_{it} are independent normal random variables with mean zero and variances σ_μ^2 and σ_ξ^2, respectively.

We introduce imperfect information among consumers in this model by assuming that, while consumers privately observe their own v_{it} and x_i perfectly, they cannot decompose *ex ante* the stochastic η_t and γ_{it} terms in (1). Thus, while each consumer knows how much utility he gets from owning a home in each period, he is incapable of fully discerning how much of this flow value is due to the fundamental η_t and how is due to his own idiosyncratic valuation γ_{it}. Given that the entire stock of housing is auctioned in each period, consumers care about the resale value of housing tomorrow as well as the flow value they accrue today. As a result, consumers' bids for housing in each period will reflect both their flow utility today and their *beliefs* about prices tomorrow. Given the persistence of η_t, errors in beliefs about this component of v_{it} will cause consumers to misforecast prices tomorrow and thus bid imperfectly today. An interesting feature of this model is that belief errors can be systematic as well as idiosyncratic, such that consumers can remain misinformed even after market-clearing prices are observed.

Apart from consumers, risk-neutral, profit-maximizing housing speculators also buy and sell homes in each period. Speculators differ from regular consumers along several dimensions. First, unlike consumers, speculators may own multiple units of housing. However, they receive no flow utility from any of the housing they hold and only derive utility from the (discounted) cash flows they accrue from buying and selling homes. Because the value of housing is intrinsically higher for consumers, the profitability of speculation in this model requires that speculators have superior information. Specifically, we assume that speculators perfectly observe η_t in each period prior

[1]We eliminate transaction costs and other trade frictions from the model to simplify the model and focus on the role of information in the market. That said, we expect that introducing market frictions will not change any of the salient features of our model.

to bidding for housing, which allows them to profit from the information problem that ordinary consumers face.

Apart from this, we allow for varying degrees of intensity of speculator competition ranging from a single monopolist to perfect competition[2]. Additionally, we allow for differences in consumer awareness about the effect of speculation on prices, and show that consumer naivete about speculation can lead to situations where positive price pressure from speculative buying is incorrectly interpreted as positive innovation in the fundamental, which can lead to bubble-like behavior in the model equilibrium.

1.3 Timing and Learning

In each period, each consumer i enters with an initial information set $\boldsymbol{\Omega}_{it} = \{v_{it-s}, p_{t-s}\}_{s=1}^{\infty}$ comprising his entire history of flow values v_{it-s} as well as past market prices p_{t-s}. We denote each consumer's initial belief bias δ_{it-1} in period t as the error in his beliefs about yesterday's common component given $\boldsymbol{\Omega}_{it}$:

$$\delta_{it-1} \equiv E[\eta_{t-1}|\boldsymbol{\Omega}_{it}] - \eta_{t-1} \tag{4}$$

We assume that this bias term can be decomposed into two components, one of which is idiosyncratic and another of which is common to all consumers[3]

$$\delta_{it-1} = \bar{\delta}_{t-1} + \iota_{it-1} \tag{5}$$

As we develop the model we will derive expressions for these bias terms as functions of past values of the exogenous shocks μ_t and ξ_{it}. For now, suffice it to say that at the beginning of each period consumers believe that each term is normally distributed with mean zero and some variance that we will derive.

In the model we've developed up to this point, the uncertainty that consumers face at the start of each period can be reduced to a vector $\boldsymbol{\phi}_{it} = (\mu_t, \xi_{it}, \bar{\delta}_{t-1}, \iota_{it-1})'$. That is, consumers begin each period with uncertainty about the location of their flow components yesterday in addition to uncertainty about how component has since evolved.

Throughout each period, consumers incorporate new information about $\boldsymbol{\phi}_{it}$ as it is received and update their beliefs according to Bayes' rule. The timing of learning in each period proceeds as follows:

1. Consumers enter period t with information $\boldsymbol{\Omega}_{it}$.

2. Random innovations μ_t and ξ_{it} are realized and η_t and γ_{it} transition according to equations (2) and (3)

3. Consumers privately observe their flow values v_{it} and update their beliefs about $\boldsymbol{\phi}_{it}$ conditional on $\boldsymbol{\Omega}_{it}, v_{it})$ using Bayes Rule.

4. Consumers bid for housing using $E[\boldsymbol{\phi}_{it}|\boldsymbol{\Omega}_{it}, v_{it}]$ formed from the previous step.

5. Consumers observe the market-clearing price p_t and update their beliefs to $\boldsymbol{\phi}_{it}$ conditional on $\boldsymbol{\Omega}_{it+1})$

[2]In the perfectly competitive outcome, aggregate speculative holdings yield zero expected profit.

[3]We will show that this assumption holds in the equilibrium we derive in the next section

6. The next period begins.

For the sake of illustration and to make sure these steps are clear, I will quickly proceed though these steps. Given the initial beliefs generated from $\mathbf{\Omega}_{it}$, consumers first have an opportunity to update their beliefs prior to bidding after they learn their flow value v_{it}. In order to apply Bayes's Rule, we first begin with the following expression, which represents various elements of consumers' information set after v_{it} is observed in terms of the unobserved vector $\boldsymbol{\phi}_{it}$:

$$\begin{aligned} v_{it} - \rho_\gamma v_{it-1} + (\rho_\gamma - \rho_\eta) E[\eta_{it-1} | \Omega_{it}] &= \mu_t + \xi_{it} + (\rho_\gamma - \rho_\eta)(\bar{\delta}_{t-1} + \iota_{it-1}) \\ &= \mathbf{c}_1 \boldsymbol{\phi}_{it} \end{aligned}$$

where and

$$\mathbf{c}_1 = \begin{pmatrix} 1 & 1 & \rho_\gamma - \rho_\eta & \rho_\gamma - \rho_\eta \end{pmatrix} \tag{6}$$

Letting

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_\mu^2 & 0 & 0 & 0 \\ 0 & \sigma_\xi^2 & 0 & 0 \\ 0 & 0 & \sigma_\delta^2 & 0 \\ 0 & 0 & 0 & \sigma_\iota^2 \end{pmatrix}$$

denote the variance-covariance matrix of $\boldsymbol{\phi}_{it}$, we next define a projection matrix $\mathbf{\Gamma}_1$ that projects the $\boldsymbol{\phi}_{it}$ onto the space of the updated information set:

$$\mathbf{\Gamma}_1 = \mathbf{\Sigma}\mathbf{c}_1'(\mathbf{c}_1\mathbf{\Sigma}\mathbf{c}_1')^{-1}\mathbf{c}_1 - \mathbf{I_4}$$

With this projection matrix, the distribution of the bias of the updated beliefs implied by Bayes' Rule is

$$E[\boldsymbol{\phi}_{it} | \mathbf{\Omega}_{it}, v_{it}] - \boldsymbol{\phi}_{it} \sim N\left(\mathbf{\Gamma}_1 \boldsymbol{\phi}_{it}, \mathbf{\Gamma}_1 \mathbf{\Sigma} \mathbf{\Gamma}_1'\right) \tag{7}$$

After each consumer updates his beliefs conditional on his private flow value signalv_{it}, consumers bid simultaneously for a single unit of housing. As with second-price auctions in other settings, bidding one's true expected value of owning a home is a weakly dominant strategy for each consumer, such that each bids according to

$$b_{it} = v_{it} + \beta E[p_{t_1} | \mathbf{\Omega}_{it}] \tag{8}$$

Given that v_{it} is known to each consumer, equation (8) suggests that any "errors" in consumers' bids are due to biased beliefs about tomorrow's price, which we'll derive as a function of θ_{it} when we construct a price equilibrium in this model.

After the consumers bid, the market clearing price p_t is commonly observed, and consumers' again update their beliefs about $\boldsymbol{\phi}_{it}$ based on the new information conveyed by p_t. While consumers cannot observe the full distribution of bids, we will establish that the bid distribution is normal with variance that does not vary over time, such that prices reveal to consumers their relative position in this distribution. Although this information provides each consumer with a signal of $\boldsymbol{\phi}_{it}$ that does not depend on his own idiosyncratic noise, he still faces uncertainty about whether the clearing price this period reflects the true common fundamental η_t or the common bias $\bar{\delta}_{t-1}$ with which consumers enter each period, such that consumers remain incompletely informed even after prices are revealed.

1.4 Consumer-Only Equilibrium

Before we characterize the full model with speculation, it is useful to derive equilibrium prices in a market with only ordinary consumers. The following proposition does this:

Proposition 1. *Equilibrium prices are given by*

$$p_t = \frac{\bar{x} + b^*(0)}{1-\beta} + \frac{\eta_t}{1-\beta\rho_\eta} + e_t \tag{9}$$

where error term $e_t = c_{1\delta}\bar{\delta}_{t-1} + c_{1\mu}\mu_t$ captures deviations of market prices from the fundamental due to the common error terms $\bar{\delta}_{t-1}$ and μ_t and $b^*(0)$ is a term that collects the idiosyncratic components of the marginal bidder's bid.

Proof. Our proof of the equilibrium in (9) is constructive and proceeds first by deriving consumer bids under the assumption that equilibrium prices are as in (9) is true and then showing that such bidding by consumers indeed gives rise to market-clearing prices that are consistent with this assumption.

To begin, given the structure of the pricing game, truthful bidding is a weakly dominant strategy for each bidder, such that each consumer bids his expected value of owning a unit of housing that period, which is the sum of the flow value today plus the discounted resale value next period:

$$b_{it} = v_{it} + \beta E[p_{t+1}|\mathbf{\Omega}_{it}, v_{it}] \tag{10}$$

After some manipulation, plugging (9) into (10) yields

$$\begin{aligned} b_{it} &= \frac{\bar{x} + \beta b^*(0)}{1-\beta} + \frac{\eta_t}{1-\beta\rho_\eta} + (x_i - \bar{x}) + \gamma_{it} + \frac{\beta\rho_\eta}{1-\beta\rho_\eta}\left(E\left[\rho_\eta\eta_{t-1} + \mu_t|\mathbf{\Omega}_{it}, v_{it}\right] - \rho_\eta\eta_{t-1} - \mu_t\right) \\ &= \frac{x + \beta b^*(0)}{1-\beta} + \frac{\eta_t}{1-\beta\rho_\eta} + \gamma_{it} + (x_i - \bar{x}) \\ &+ \frac{\beta\rho_\eta}{1-\beta\rho_\eta}\left(E\left[\mu_t - \rho_\eta(\bar{\delta}_{t-1} + \iota_{it-1})|\mathbf{\Omega}_{it}, v_{it}\right] - (\mu_t - \rho_\eta(\bar{\delta}_{t-1} + \iota_{it-1}))\right) \end{aligned} \tag{11}$$

We next observe that the last set of terms in equation (11) can be represented with the matrix representation of Bayes' rule expressed in (7). First we define a vector $\mathbf{c}_p$ of coefficients as

$$\mathbf{c}_p \equiv \begin{pmatrix} 1 & 0 & \text{-}\rho_\eta & -\rho_\eta \end{pmatrix}$$

It then follows that consumer bids can be expressed as

$$b_{it} = \frac{\bar{x} + \beta b^*(0)}{1-\beta} + \frac{\eta_t}{1-\beta\rho_\eta} + (x_i - \bar{x}) + \gamma_{it} + \frac{\beta\rho_\eta}{1-\beta\rho_\eta}(\mathbf{c}_p\mathbf{\Gamma}_1\boldsymbol{\phi}_{it}) \tag{12}$$

Partitioning this into common and idiosyncratic terms then yields

$$b_{it} = \left[\frac{\bar{x} + \beta b^*(0)}{1-\beta} + \frac{\eta_t}{1-\beta\rho_\eta}\right] + \left[(x_i - \bar{x}) + \rho_\gamma\gamma_{it-1} + (c_{1\xi}+1)\xi_{it} + c_{1\iota}\iota_{it-1}\right] + \left[c_{1\mu}\mu_t + c_{1\delta}\bar{\delta}_{t-1}\right] \tag{13}$$

where c_{1u}, $c_{1\xi}$, $c_{1\delta}$, and $c_{1\iota}$ are constants resulting from the expansion of the last term in (12).

The first bracketed term, which is almost the same as equation (9), is portion of each consumer's bid that reflects the fundamental value of housing. The second bracketed term characterizes the idiosyncratic bid component, with $b^*(0)$ being defined as the value of this term for the marginal bidder. This distribution of this term is stationary over the population with mean zero and variance given by:

$$\sigma_x^2 + \sigma_\gamma^2 \left((1 + c_{1\xi})^2 + \frac{\rho_\gamma^2}{1 - \rho_\gamma^2} + \frac{2\rho_\gamma c_{1\iota} c_{2\xi}}{1 - \rho_\gamma c_{2\iota}} + \frac{(c_{1\iota} c_{2\xi})^2}{1 - c_{2\iota}^2} \right)$$

Though some of this term represents each consumer's true value for housing, it also contains terms related to idiosyncratic bias. Because each of these terms is stationary, however, none of these will affect prices. Further, given that this term is the only one that varies by consumer, the $b^*(0)$ term from (9) is implicitly defined by cumulative joint distribution of this term for consumers whose bids are equal to the clearing price.

The final bracketed term is the functional form of the error term e_t from (9). Given that $\bar{\delta}_{t-1}$ is determined before μ_t is realized, they must be independent. Furthermore, it must be the case that $E[e_t|\mathbf{\Omega}_{it}, v_{it}] = 0$ since consumers form beliefs rationally.

□

1.5 State Transition

To close the model, we last need to derive consumers' posterior beliefs after observing p_t and use this to derive a transition equation for the bias term δ_{it}. As before, we can express p_t and other terms in the consumer's information set in terms of the underlying errors $\boldsymbol{\phi}_{it}$. To do this, note that because consumers know $b^*(0)$, it follows that[4]:

$$\begin{aligned}
p_t - \frac{\bar{x} + b^*(0)}{1-\beta} - \frac{\rho_\eta E[\eta_{t-1}|\mathbf{\Omega}_{it}]}{1 - \beta\rho_\eta} &= \frac{\eta_t - \rho_\eta E[\eta_{t-1}|\mathbf{\Omega}_{it}]}{1 - \beta\rho_\eta} + c_{1\delta}\delta_{t-1} + c_{1\mu}\mu_t \qquad (14) \\
&= \frac{\rho_\eta(\eta_{t-1} - E[\eta_{t-1}|\Omega_{it}]) + \mu_t}{1 - \beta\rho_\eta} + c_{1\delta}\delta_{t-1} + c_{1\mu}\mu_t \\
&= \left(\frac{1}{1 - \beta\rho_\eta} + c_{1\mu} \right) \mu_t + \left(c_{1\delta} - \frac{\rho_\eta}{1 - \beta\rho_\eta} \right) \bar{\delta}_{t-1} - \frac{\rho_\eta}{1 - \beta\rho_\eta} \iota_{it-1} \\
&= \mathbf{c}_2 \boldsymbol{\phi}_{it} \qquad (15)
\end{aligned}$$

where $\mathbf{c}_2 \equiv \begin{pmatrix} \left(\frac{1}{1-\beta\rho_\eta} + c_{1\mu}\right) & 0 & \left(c_{1\delta} - \frac{\rho_\eta}{1-\beta\rho_\eta}\right) & -\frac{\rho_\eta}{1-\beta\rho_\eta} \end{pmatrix}$ Let $\mathbf{C}_{12}$ be a stacked matrix made up of the vector $\mathbf{c}_1$ from equation (6) along with $\mathbf{c}_2$. Proceeding as before, we define a projection matrix that incorporates the new information conveyed by p_t:

$$\mathbf{\Gamma}_2 = \mathbf{\Sigma}\mathbf{C}_{12}'(\mathbf{C}_{12}\mathbf{\Sigma}\mathbf{C}_{12}')^{-1}\mathbf{C}_{12} - \mathbf{I}_4$$

Bayes' Rule then implies that bias in beliefs after observing p_t are distributed for each consumer according to:

$$E[\boldsymbol{\phi}_{it}|\mathbf{\Omega}_{it+1}] - \boldsymbol{\phi}_{it} \sim N\left(\mathbf{\Gamma}_2\boldsymbol{\phi}_{it}, \mathbf{\Gamma}_2\mathbf{\Sigma}\mathbf{\Gamma}_2'\right) \qquad (16)$$

[4] We could also have constructed an equivalent signal out of individual's private signals using the information that p_t conveys about the location of each consumers' own idiosyncratic bid relative to the population. As a practical matter, these pieces of information are the same and using either yields the same results.

Now that we have a function for posterior beliefs biases in terms of $\boldsymbol{\phi}_{it}$, we can construct the law of motion for δ_{it} by solving the price equation (9) for η_t and taking expectations:

$$
\begin{aligned}
\eta_t &= (1-\beta\rho_\eta)\left(p_t - \frac{\bar{x}+b^*(0)}{1-\beta} - c_{1\delta}\delta_{t-1} - c_{1\mu}\mu_t\right) \\
\delta_{it} &= E[\eta_t|\boldsymbol{\Omega}_{it+1}] - \eta_t \\
&= (1-\beta\rho_\eta)c_{1\delta}\left(\bar{\delta}_{t-1} - E[\bar{\delta}_{t-1}|\boldsymbol{\Omega}_{it+1}]\right) + (1-\beta\rho_\eta)c_{1\delta}\left(\iota_{it-1} - E[\iota_{it-1}|\boldsymbol{\Omega}_{it+1}]\right) \\
&+ (1-\beta\rho_\eta)c_{1\mu}\left(\mu_t - E[\mu_t|\Omega_{it+1}]\right) \\
&= \mathbf{c}_\delta \boldsymbol{\Gamma}_2 \boldsymbol{\phi}_{it} \qquad (17)
\end{aligned}
$$

where

$$
\mathbf{c}_\delta \equiv \begin{pmatrix} (1-\beta\rho_\eta)c_{1\mu} & 0 & (1-\beta\rho_\eta)c_{1\delta} & 0 \end{pmatrix}'
$$

Breaking this apart into common and idiosyncratic components then yields:

$$
\bar{\delta}_t = c_{2\delta}\bar{\delta}_{t-1} + c_{2\mu}\mu_t \qquad (18)
$$

$$
\iota_{it} = c_{2\iota}\iota_{it-1} + c_{2\xi}\xi_{it} \qquad (19)
$$

where the constants come from multiplying out the matrices in equation (17)

The expected value of both these terms is zero conditional on $\boldsymbol{\Omega}_{it+1}$, as we assumed. To find the stationary variances σ_δ^2 and σ_ι^2 that appear in Σ, we must search for a fixed point. The procedure works taking an initial guess at these variances, using them to get the constants in (18) and (19) and computing the implied variance of these equations, which is used as the guess for the next iteration until convergence is achieved.

1.6 Models with Speculators

We next bring speculators into the model, which requires some adjustments to the equilibrium derived above but, particularly when consumers are unaware of speculator activity in the market, leave most of the properties of the equilibrium above unchanged. The primary difference between the model in the previous section and one with speculators regards the beliefs about $b^*(\cdot)$ - that is, how the marginal bidder changes with speculator inventory. With speculation, equilibrium prices are given by

$$
p_t(h_t, h_{t-1}) = \frac{\bar{x}}{1-\beta} + b^*(h_t) + b^s(h_{t-1}, \mu_t, \bar{\delta}_{t-1}) + \frac{\eta_t}{1-\beta\rho_\eta} + c_{1\delta}\bar{\delta}_{t-1} + c_{1\mu}\mu_t \qquad (20)
$$

where $b^s(h_{t-1}, \mu_t, \bar{\delta}_{t-1})$ is a term that reflects consumers' expectations about the location of tomorrow's marginal bidder in the bid distribution given that they observe speculator holdings of h_{t-1} prior to bidding. It is through this term that the degree of consumer naivete is expressed. The case of total consumer naivete corresponds to $b^s(h_{t-1}, \mu_t, \bar{\delta}_{t-1}) = \frac{\beta b^*(0)}{1-\beta}$ - that is, consumers believe that speculator holdings will be zero forever as of period t. In contrast, rational consumers will use the information they have to predict future speculator behavior, with the result that $b^s(h_{t-1}, \mu_t, \bar{\delta}_{t-1})$ will depend on μ_t and $\bar{\delta}_{t-1}$ as well as h_t.

Except when consumers are fully naive, the specific form of $b^s(h_{t-1}, \mu_t, \bar{\delta}_{t-1})$ will also depend on the degree of competition among speculators. Let J denote the number of speculators with

$J \in \mathbb{Z}_+ \cup \infty$. Then in equilibrium each speculator j solves

$$\pi_t\left(h_{j,t-1}, \sum_{k\neq j} h_{k,t}\right) = \max_{\{h_{j,t+s}\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s E_t\left[(h_{j,t+s-1} - h^*_{j,t+s})p_{t+s}\left(h_{j,t+s} + \sum_{k\neq j} h^*_{k,t+s}, \sum_{j=1}^{J} h^*_{j,t+s-1}\right)\right] \tag{21}$$

Which gives rise to the following dynamic programming problem:

$$V_t\left(h_{j,t-1}, \sum_{k\neq j} h_{k,t-1}\right) = \max_h \left(h_{j,t-1} - h\right) p_t\left(h + \sum_{k\neq j} h^*_{k,t}, \sum_{j=1}^{J} h_{j,t-1}\right) + \beta E_t\left[V_{t+1}\left(h, \sum_{k\neq j} h^*_{k,t}\right)\right] \tag{22}$$

where the expectation is taken to the state vector[5].

For any finite J, we can solve for a symmetric equilibrium using standard solution methods. In particular:

$$V_t\left(h^{*J}_{t-1}, (J-1)h^{*J}_{t-1}\right) = \max_h \left(h^{*J}_{t-1} - h\right) p_t\left(h + (J-1)h^{*J}_t, Jh^{*J}_{t-1}\right) + \beta E_t\left[V\left(h, (J-1)h^{*J}_t\right)\right]$$

Taking first order conditions and applying symmetry then gives

$$\begin{aligned} p_t\left(Jh^{*J}_t, Jh^{*J}_{t-1}\right) &= \left(h^{*J}_{t-1} - h^{*J}_t\right) p_{t\,1}(Jh^{*J}_t, Jh^{*J}_{t-1}) + \beta E_t\left[V_{t1}\left(h^{*J}_t, (J-1)h^{*J}_t\right)\right] \\ V_{t1}\left(h^{*J}_t, (J-1)h^{*J}_t\right) &= p_{t+1}(Jh^{*J}_{t+1}, Jh^{*J}_t) + \left(\left(h^{*J}_t - h^{*J}_{t+1}\right)\right) p_{t+1\,2}(Jh^{*J}_{t+1}, Jh^{*J}_t) \end{aligned}$$

Substituting these together then yields:

$$\begin{aligned} p_t\left(Jh^{*J}_t, Jh^{*J}_{t-1}\right) = &\left(h^{*J}_{t-1} - h^{*J}_t\right) p_{t\,1}(Jh^{*J}_t, Jh^{*J}_{t-1}) + \beta E_t\left[p_{t+1}(Jh^{*J}_{t+1}, Jh^{*J}_t)\right] \\ &+ \beta E_t\left[\left(\left(h^{*J}_t - h^{*J}_{t+1}\right)\right) p_{t+1\,2}(Jh^{*J}_{t+1}, Jh^{*J}_t)\right] \end{aligned} \tag{23}$$

which says that speculative holdings equate the cost of purchasing a marginal unit of housing today with the expected price received tomorrow plus the effect of purchasing on prices both today and tomorrow, which affects the value of each speculator's total stock of housing inventory. As the market becomes increasingly competitive, intuition suggests that these marginal price effects should vanish. Assuming that $b^s(\cdot)$ is another Gaussian inverse CDF, after a lengthy derivative, we can use the implicit function theorem to show that Jh^{*J}_t is increasing h^{*J}_t is decreasing in J. In the limit as $J \to \infty$, the two derivative terms in the (23) will vanish, yielding:

$$p_t(h^{*\infty}_t, h^{*\infty}_{t-1}) = \beta E_t\left[p_{t+1}(h^{*\infty}_{t+1}, h^{*\infty}_t)\right] \tag{24}$$

In the case where consumers are completely naive, we can explicitly solve for $h^{*\infty}_t$ as

$$h^{*\infty}_t = b^{*-1}\left(-\frac{x + \beta b^*(0)}{1-\beta} - \frac{\eta_t}{1-\beta\rho_\eta} - c_{1\mu}\mu_t - c_{1\delta}\bar{\delta}_{t-1}\right) \tag{25}$$

Somewhat unexpectedly, this implies that competitive speculators drive the price of housing to a fixed level equal to zero. The mathematical justification from this is easiest to see once one notes that, given that the expected price tomorrow is zero due to speculative activity, the zero-profit

[5] Out of sloppiness, the state is implied by the time subscript at the moment.

condition requires that prices be zero today as well[6]. If there are short-selling constraints, prices can be strictly positive but will be zero in any case where speculators hold positive inventory.

There are three factors that drive speculator holdings vary over time. The first (trivial) factor that induces speculators to purchase inventory is a highly negative fundamental η_t, which can induce speculators to purchase inventory by making the implicit holding cost of speculation (the foregone flow utility) negative. Intuitively, this might cover cities like Detroit, where speculators are buyers of last resort when homeowners would rather abandon their houses than live in them[7].

The final two factors are positive realizations of the current fundamental shock μ_t, which are not fully incorporated into bids by consumers in the short run, and positive realizations of yesterday's *ex post* belief bias $\bar{\delta}_{t-1}$, which accumulates from positive *past* realizations of the fundamental shock. In both cases, profit opportunities arise as these sources of mispricing are corrected over time and, under perfectly competitive speculation, such opportunities are completely eroded.

In all three cases, the intuitive reason for the zero-price result is that a clearing price of zero is potentially efficient in the sense that (a) only consumers holding housing receive non-negative flow utility from holding it and (b) the collective net transfer of surplus from consumers to speculators is zero, implying that speculators earn zero trading profits collectively. Regardless of the factor driving it, speculation here is a fringe activity, however, and only takes place when prices would otherwise be negative. In any case when this is true marginal consumers would be better off not holding housing since they are either incurring negative flow utility or trading losses from incorrect expectations. Given the empirical fact that housing is almost never priced in this manner, this model is not likely to be successful in matching actual prices.

For a monopolist speculator, substituting the equilibrium price equation and rearranging a bit gives

$$\begin{aligned} b^*(h_t^*) - \beta b^s(h_t^*) + \beta(E[h_{t+1}] - h_t^*)b^{s'}(h_t^*) + (h_t^* - h_{t-1})b^{*'}(h_t^*) = \beta E[b^*(h_{t+1})] - b^s(h_{t-1}) \\ - x - \eta_t + c_{1\delta}(\beta E[\bar{\delta}_t] - \bar{\delta}_{t-1}) - c_{1\mu}\mu_t \end{aligned} \tag{26}$$

Under complete consumer naivete, we can telescope this equation forward, yielding

$$\begin{aligned} b^*(h_t^*) = h_{t-1}b^{*'}(h_t^*) + h_t(\beta E[b^{*'}(h_{t+1})] - b^{*'}(h_t^*)) + \sum_{s=1}^{\infty} \beta^s E[h_{t+s}(\beta E[b^{*'}(h_{t+1+s})] - b^{*'}(h_{t+s}))] \\ - (\frac{\bar{x} + \beta b^*(0)}{1-\beta} + \frac{\eta_t}{1-\beta\rho_\eta} + c_{1\mu}\mu_t + c_{1\delta}\delta_{t-1}) \end{aligned} \tag{27}$$

which shows that the basic strategic mechanics still hold from the perfectly competitive model - the only difference is the existence of forward price effects that arise from strategic dependence of inventory choices yesterday on the choice made today.

1.6.1 Learning with Speculation

For any set of consumer beliefs, the consumer belief formation process needs to be modified to allow speculative activity to potentially influence consumer behavior. A good starting point is

[6] One might also note that, for $\beta < 1$, $x = 0$ is the only solution of $x = \beta x$.

[7] This assumes, of course, that consumers themselves cannot hold housing they do not occupy. However, such consumers are *de facto* speculators regardless.

to consider a case in which consumers are completely unaware of the effect that speculators have on prices. Specifically, consumers mistakenly believe that $E[b^*(h_t)|\mathbf{\Omega}_{it}] = b^*(0)$; that is, that speculators are absent from the market. To construct beliefs in this case, we proceed analogously to the consumer-only model, defining:

$$
\begin{aligned}
\boldsymbol{\phi}_{it} &= \begin{pmatrix} \mu_t & \xi_{it} & \bar{\delta}_{t-1} & \iota_{it-1} & b^*(0) - b^*(h_t) \end{pmatrix}' \\
\mathbf{\Sigma}^s &= \begin{pmatrix} \sigma_\mu^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_\xi^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_\delta^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_\iota^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_b^2 \end{pmatrix} \\
\mathbf{c}_1^s &= \begin{pmatrix} 1 & 1 & \rho_\gamma - \rho_\eta & \rho_\gamma - \rho_\eta & 0 \end{pmatrix}' \\
\mathbf{c}_2^s &= \begin{pmatrix} \left(\frac{1}{1-\beta\rho_\eta} + c_{1\mu}\right) & 0 & \left(c_{1\delta} - \frac{\rho_\eta}{1-\beta\rho_\eta}\right) & -\frac{\rho_\eta}{1-\beta\rho_\eta} & 1 \end{pmatrix}' \\
\mathbf{\Gamma}_1^s &= \mathbf{\Sigma}^s \mathbf{c}_1^{s'} (\mathbf{c}_1^s \mathbf{\Sigma}^s \mathbf{c}_1^{s'})^{-1} \mathbf{c}_1 - \mathbf{I_5} \\
\mathbf{c}_p^s &= \begin{pmatrix} 1 & 0 & \text{-}\rho_\eta & -\rho_\eta \; 0 \end{pmatrix}' \\
\mathbf{\Gamma}_2^s &= \mathbf{\Sigma^s} \mathbf{c}_2^{s'} (\mathbf{c}_2^s \mathbf{\Sigma}^s \mathbf{c}_2^{s'})^{-1} \mathbf{c}_2^s - \mathbf{I}_5 \\
\mathbf{c}_\delta^s &= \begin{pmatrix} (1-\beta\rho_\eta)c_{1\mu} & 0 & (1-\beta\rho_\eta)c_{1\delta} & 0 \end{pmatrix}' \\
\Rightarrow \delta_{it} &= \mathbf{c}_\delta^s \mathbf{\Gamma}_2^s \boldsymbol{\phi}_{it} \qquad (28)
\end{aligned}
$$

Beliefs can be computed by using these matrices in place of their counterparts in the consumer-only model. The only material difference in this model is in the transition equation of the mean bias term:

$$\bar{\delta}_t = c_{2\delta}\bar{\delta}_{t-1} + c_{2\mu}\mu_t + c_{2b}(b^*(0) - b^*(h_t)) \qquad (29)$$

$$\iota_{it} = c_{2\iota}\iota_{it-1} + c_{2\xi}\xi_{it} \qquad (30)$$

which now includes an additional component that reflects consumers' misattribution of high prices from speculative buying as high common valuation.

We can relax the strength of the naivete assumption somewhat by allowing consumers to form (possibly noisy) beliefs both before and after bidding, which amounts to replacing the $b^*(0)$ term in (29) with some forecast $E_t[b^*(h_t)]$ of the actual marginal bid value.

Expectations of this class are still not fully rational, however, as they only prevent speculator behavior from contributing to consumer bias. In a fully rational model, consumers could reverse-engineer speculators' optimization problem and use this information to reduce or eliminate their belief biases regarding the true state. This could potentially make it much harder to solve the speculative policy function since the policy function would appear non-linearly in equilibrium prices, though I haven't tried to solve this problem yet.