Homework 1

Due: Friday, Jan 26, 11:59pm

Preface

Discussing high-level approaches to homework problems with your peers is encouraged. You must include at the top of your assignment a Collaboration Statement which declares any other people with whom you discussed homework problems. For example:

> Collaboration Statement: I discussed problems 1 and 3 with Jamie Smith. I discussed problem 2 with one of the TAs. I discussed problem 4 with a personal tutor.

If you did not discuss the assignment with anyone, you still must declare:

> Collaboration Statement: I did not discuss homework problems with anyone.

Copying answers or doing the work for another student is not allowed.

Assignment problems which refer to "Exercise X" or "Figure Y" are referring to those found in the Types and Programming Languages textbook.

Submitting

Prepare your assignment as either handwritten or using LaTeX. **I will not accept homework assignments written in Word, Google Docs, or using any other text processing software.** Handwritten assignments must be written neatly or they will receive a 0 grade. Submit the assignment either (1) via scanned pdf email to me: David.Darais@uvm.edu with "CS 225 HW1" in the subject line, or (2) placed under my office door (Votey 319) at any hour before the deadline.

Problem 1 (15 points)

Recall the definition for the divides relation:

$$\text{divides} \coloneqq \{\langle n, m\rangle \mid \exists\, o \; s.t.\; n \times o = m\}$$

Prove formally—and in as much detail as possible—that the divides relation is transitive, that is:

forall n, m and o, if n divides m and m divides o then n divides o

You may assume basic algebraic arithmetic facts like $n + n = 2n$ and $2(n + n) = 2n + 2n$. Use the example proof of reflexivity given in class as a guide for the level of detail you should strive for.

Solution 1 (informal)

Proof. Assume some arbitrary n, m, and o, and assume that n divides m and m divides o.

Because n divides m we know that there exists p_1 such that $n \times p_1 = m$.

Because m divides o we know that there exists p_2 such that $m \times p_2 = o$.

The goal is to prove n divides o, which can be shown by the existence of a p_3 such that $n \times p_3 = o$.

Let $p_3 = p_1 \times p_2$ satisfy the existential. It now must be shown that $n \times (p_1 \times p_2) = o$. This is true via calculation:

$$\begin{aligned} & n \times (p_1 \times p_2) \\ = {} & (n \times p_1) \times p_2 && \wr \text{ algebra (associativity) } \int \\ = {} & m \times p_2 && \wr \text{ assumption: } n \text{ divides } m \int \\ = {} & o && \wr \text{ assumption: } m \text{ divides } o \int \end{aligned}$$

□

Solution 2 (formal)

Proof. Assume some arbitrary n, m, and o, and assume that n divides m and m divides o.

The following facts are implied by the assumptions:

$$\begin{aligned} & n \text{ divides } m \\ \iff & \langle n, m\rangle \in \text{divides} && \wr \text{ notation } \int \\ \iff & \exists p_1.\, n \times p_1 = m && \wr \text{ def. of divides } \int \\ \implies & n \times p_1 = m && \wr \text{ assume arbitrary } p_1 \int \end{aligned}$$

and

$$\begin{aligned} & m \text{ divides } o \\ \iff & \langle m, o\rangle \in \text{divides} && \wr \text{ notation } \int \\ \iff & \exists p_2.\, m \times p_2 = o && \wr \text{ def. of divides } \int \\ \implies & m \times p_2 = o && \wr \text{ assume arbitrary } p_2 \int \end{aligned}$$

n divides m we know that there exists p_1 such that $n \times p_1 = m$.

The goal is to show n divides o which follows via the following chain of implications:

$$\begin{aligned} & n \text{ divides } o \\ \iff & \langle n, o\rangle \in \text{divides} && \wr \text{ notation } \int \\ \iff & \exists p_3.\, n \times p_3 = o && \wr \text{ def. of divides } \int \\ \impliedby & n \times (p_1 \times p_2) = o && \wr \text{ witness } p_3 \text{ with } p_1 \times p_2 \int \\ \iff & (n \times p_1) \times p_2 = o && \wr \text{ algebra (associativity) } \int \\ \iff & m \times p_2 = o && \wr \text{ by } n \times p_1 = m \int \\ \iff & o = o && \wr \text{ by } m \times p_2 = o \int \\ \iff & \mathit{true} && \wr \text{ reflexivity } \int \end{aligned}$$

□

Problem 2 (10 points)

Consider the set of boolean arithmetic terms $\mathcal{T}$ and metafunctions leaves (new) and depth (from Definition 3.3.2):

$$t \in \mathcal{T} ::= \texttt{T} \mid \texttt{F} \mid \texttt{if}\ t\ \texttt{then}\ t\ \texttt{else}\ t$$

$$\begin{aligned}
&\text{leaves}(\texttt{T}) \coloneqq 1\\
&\text{leaves}(\texttt{F}) \coloneqq 1\\
&\text{leaves}(\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3) \coloneqq \text{leaves}(t_1) + \text{leaves}(t_2) + \text{leaves}(t_3)
\end{aligned}$$

$$\begin{aligned}
&\text{depth}(\texttt{T}) \coloneqq 1\\
&\text{depth}(\texttt{F}) \coloneqq 1\\
&\text{depth}(\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3) \coloneqq \max(\text{depth}(t_1), \text{depth}(t_2), \text{depth}(t_3))
\end{aligned}$$

Define some term $t \in \mathcal{T}$ such that $\text{leaves}(t) = 7$ and $\text{depth}(t) = 3$.

Solution *...many answers possible...*

$$t = \texttt{if}\ (\texttt{if T then T else T})\ \texttt{then}\ (\texttt{if T then T else T})\ \texttt{else T}$$

$$\begin{aligned}
&\text{leaves}(t)\\
&= \text{leaves}(\texttt{if}\ (\texttt{if T then T else T})\ \texttt{then}\ (\texttt{if T then T else T})\ \texttt{else T})\\
&= \text{leaves}(\texttt{if T then T else T}) + \text{leaves}(\texttt{if T then T else T}) + \text{leaves}(\texttt{T})\\
&= 7\text{leaves}(\texttt{T})\\
&= 7
\end{aligned}$$

$$\begin{aligned}
&\text{size}(t)\\
&= \text{size}(\texttt{if}\ (\texttt{if T then T else T})\ \texttt{then}\ (\texttt{if T then T else T})\ \texttt{else T})\\
&= \max(\text{size}(\texttt{if T then T else T}), \text{size}(\texttt{if T then T else T}), \text{size}(\texttt{T})) + 1\\
&= \max(\max(\text{size}(\texttt{T}), \text{size}(\texttt{T}), \text{size}(\texttt{T})) + 1, \max(\text{size}(\texttt{T}), \text{size}(\texttt{T}), \text{size}(\texttt{T})) + 1, 1) + 1\\
&= \max(\max(1,1,1) + 1, \max(1,1,1) + 1, 1) + 1\\
&= \max(2,2,1) + 1\\
&= 3
\end{aligned}$$

Problem 3 (25 points)

Either prove by structural induction that $\text{leaves}(t)$ always produces an odd number, or give a counter-example which shows $\text{leaves}(t)$ can produce an even number.

Solution

Proof. Assume some arbitrary $t \in \mathcal{T}$. Goal is to show $\text{odd}(\text{leaves}(t))$.

Proof by structual induction on the syntax of t.

- Base cases $t = \texttt{T}$ and $t = \texttt{F}$:
  Goal is to show $\text{leaves}(\texttt{T})$ is odd.
  By the definition of leaves, $\text{leaves}(\texttt{T}) = 1$, and 1 is odd.

- Inductive case: $t = \texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3$ for some structurally smaller terms t_1, t_2 and t_3:
  Inductive hypotheses:
  1. $\text{leaves}(t_1)$ is odd
  2. $\text{leaves}(t_2)$ is odd
  3. $\text{leaves}(t_3)$ is odd

  Goal is to show $\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3$ is odd.
  By the definition of leaves, $\text{leaves}(\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3) = \text{leaves}(t_1) + \text{leaves}(t_2) + \text{leaves}(t_3)$. By inductive hypotheses, each of $\text{leaves}(t_i)$ is odd. The sum of three odd numbers is odd.

$\square$

Problem 4 (15 points)

Draw a derivation tree which justifies the following relationship:

$$\texttt{if}\ (\texttt{if}\ (\texttt{if}\ \texttt{F}\ \texttt{then}\ \texttt{F}\ \texttt{else}\ \texttt{T})\ \texttt{then}\ \texttt{T}\ \texttt{else}\ \texttt{F})\ \texttt{then}\ \texttt{T}\ \texttt{else}\ \texttt{F} \longrightarrow \texttt{if}\ (\texttt{if}\ \texttt{T}\ \texttt{then}\ \texttt{T}\ \texttt{else}\ \texttt{F})\ \texttt{then}\ \texttt{T}\ \texttt{else}\ \texttt{F}$$

Solution

$$\dfrac{\dfrac{}{\texttt{if}\ \texttt{F}\ \texttt{then}\ \texttt{F}\ \texttt{else}\ \texttt{T} \longrightarrow \texttt{T}}\ \textsc{E-IfFalse}}{\texttt{if}\ (\texttt{if}\ (\texttt{if}\ \texttt{F}\ \texttt{then}\ \texttt{F}\ \texttt{else}\ \texttt{T})\ \texttt{then}\ \texttt{T}\ \texttt{else}\ \texttt{F})\ \texttt{then}\ \texttt{T}\ \texttt{else}\ \texttt{F} \longrightarrow \texttt{if}\ (\texttt{if}\ \texttt{T}\ \texttt{then}\ \texttt{T}\ \texttt{else}\ \texttt{F})\ \texttt{then}\ \texttt{T}\ \texttt{else}\ \texttt{F}}\ \textsc{E-If}$$

Problem 5 (30 points)

Consider the extended small-step semantics described in Exercise 3.5.16 which explicitly generates the value `wrong` in place of where the semantics from Figure 3-2 gets stuck.

1. Design a big-step semantics $t \Downarrow a$ (similar to 3.5.17) which is equivalent to this small-step semantics.
2. Prove that your new big-step semantics implies the small-step semantics which generates `wrong`, that is, prove:

   $$\text{forall } t \text{ and } a, \text{ if } t \Downarrow a \text{ then } t \longrightarrow^* a.$$

   This proof need not be as detailed as your answer to Problem 1, but still must be a convincing formal proof.

You should use the following syntactic categories for terms t, numeric values nv, values v, and answers a:

$$\begin{aligned} t \in \quad \mathcal{T} &::= \texttt{T} \mid \texttt{F} \mid \texttt{if}\ t\ \texttt{then}\ t\ \texttt{else}\ t \\ &\mid\ \ \texttt{0} \mid \texttt{succ}\ t \mid \texttt{pred}\ t \mid \texttt{iszero}\ t \\ &\mid\ \ \texttt{wrong} \end{aligned}$$

$$\begin{aligned} nv \in \mathcal{NV} &::= \texttt{0} \mid \texttt{succ}\ nv \\ v \in \quad \mathcal{V} &::= \texttt{T} \mid \texttt{F} \mid nv \\ a \in \quad \mathcal{A} &::= v \mid \texttt{wrong} \end{aligned}$$

Solution *. . . many answers possible. . .* We will use the *badbool* and *badnat* syntactic categories from the book:

$$
\begin{aligned}
\mathit{badbool} &::= \mathit{nv} \mid \mathtt{wrong} \\
\mathit{badnat} &:= \mathtt{T} \mid \mathtt{F} \mid \mathtt{wrong}
\end{aligned}
$$

Specifically which name is given to each rule (e.g., Ans) isn't particularly important.

1.

$$
\frac{}{a \Downarrow a}\,\textsc{Ans} \qquad
\frac{t_1 \Downarrow \mathtt{T} \qquad t_2 \Downarrow a}{\mathtt{if}\ t_1\ \mathtt{then}\ t_2\ \mathtt{else}\ t_3 \Downarrow a}\,\textsc{If-True} \qquad
\frac{t_1 \Downarrow \mathtt{F} \qquad t_3 \Downarrow a}{\mathtt{if}\ t_1\ \mathtt{then}\ t_2\ \mathtt{else}\ t_3 \Downarrow a}\,\textsc{If-False}
$$

$$
\frac{t_1 \Downarrow \mathit{badbool}}{\mathtt{if}\ t_1\ \mathtt{then}\ t_2\ \mathtt{else}\ t_3 \Downarrow \mathtt{wrong}}\,\textsc{If-Bad} \qquad
\frac{t \Downarrow \mathit{nv}}{\mathtt{succ}\ t \Downarrow \mathtt{succ}\ \mathit{nv}}\,\textsc{Succ} \qquad
\frac{t \Downarrow \mathit{badnat}}{\mathtt{succ}\ t \Downarrow \mathtt{wrong}}\,\textsc{Succ-Bad}
$$

$$
\frac{t \Downarrow \mathtt{0}}{\mathtt{pred}\ t \Downarrow \mathtt{0}}\,\textsc{Pred-Zero} \qquad
\frac{t \Downarrow \mathtt{succ}\ \mathit{nv}}{\mathtt{pred}\ t \Downarrow \mathit{nv}}\,\textsc{Pred-Succ} \qquad
\frac{t \Downarrow \mathit{badnat}}{\mathtt{pred}\ t \Downarrow \mathtt{wrong}}\,\textsc{Pred-Bad}
$$

$$
\frac{t \Downarrow \mathtt{0}}{\mathtt{iszero}\ t \Downarrow \mathtt{T}}\,\textsc{Iszero-Zero} \qquad
\frac{t \Downarrow \mathtt{succ}\ \mathit{nv}}{\mathtt{iszero}\ t \Downarrow \mathtt{F}}\,\textsc{Iszero-Succ} \qquad
\frac{t \Downarrow \mathit{badnat}}{\mathtt{iszero}\ t \Downarrow \mathtt{wrong}}\,\textsc{Iszero-Bad}
$$

2. *Proof.* Assume some arbitrary t, a, and that $t \Downarrow a$. Goal is to show $t \longrightarrow^* a$. Proof by structural induction on the syntax of t. (Induction on the derivation is another good choice, and leads to a slightly simpler proof.)

   - Base cases $t = \mathtt{T}$, $t = \mathtt{F}$, $t = \mathtt{0}$ and $t = \mathtt{wrong}$:

     Each base case is of the form $t = a$ for some answer a'.

     Because we know $t \Downarrow a$ and t must be an answer, i.e., $t = a'$, then the only derivation rule which could have applied to construct $t \Downarrow a$ is Ans, which says $a \Downarrow a$. Because this is the only rule that applies, and it is the same a on the left and right side of $\Downarrow$, we know $t = a$.

     The goal is then to show $t \longrightarrow^* a$. Because $t = a$, it suffices to show $a \longrightarrow^* a$, which is true by reflexivity, i.e., zero iterations of the small step relation $\longrightarrow$.

   - Inductive case $t = \mathtt{if}\ t_1\ \mathtt{then}\ t_2\ \mathtt{else}\ t_3$:

     Inductive Hypotheses:

     - forall a', if $t_1 \Downarrow a'$ then $t_1 \longrightarrow^* a'$
     - forall a', if $t_2 \Downarrow a'$ then $t_2 \longrightarrow^* a'$
     - forall a', if $t_3 \Downarrow a'$ then $t_3 \longrightarrow^* a'$

     Because we know $t \Downarrow a$, and t is syntactically an if-statement, we can look at the set of derivation rules and see that $t \Downarrow a$ could have only been formed using either If-True, If-False, or If-Bad.

     - Case $t \Downarrow a$ formed via If-True.

       We know $t_1 \Downarrow \mathtt{T}$ (a hypothesis of If-True)

       We know $t_2 \Downarrow a'$ (a hypothesis of If-True)

       The goal is to show $t \longrightarrow^* a'$

       By the inductive hypotheses, and because we know $t_1 \Downarrow \mathtt{T}$ and $t_2 \Downarrow a'$, we can conclude that $t_1 \longrightarrow^* \mathtt{T}$ and $t_2 \longrightarrow^* a'$.

       It follows from $t_1 \longrightarrow^* \mathtt{T}$ and $t_2 \longrightarrow^* a'$ that $\mathtt{if}\ t_1\ \mathtt{then}\ t_2\ \mathtt{else}\ t_3 \longrightarrow^* a'$ by transitive composition of Lemma 1 (defined later) followed by E-IfTrue.

- Case $t \Downarrow a$ formed via IF-TRUE
  We know $t_1 \Downarrow \texttt{F}$ (a hypothesis of IF-TRUE)
  We know $t_3 \Downarrow a'$ (a hypothesis of IF-TRUE)
  The goal is to show $t \longrightarrow^* a'$
  Same argument as previous, except uses E-IFFALSE.
- Case $t \Downarrow a$ formed via IF-TRUE
  We know $t_1 \Downarrow \mathit{badbool}$ (a hypothesis of IF-TRUE)
  The goal is to show $t \longrightarrow^* a'$
  Same argument as previous, except uses E-IF-WRONG (from book Exercise 3.5.16).

- Inductive cases $t = \texttt{succ}\, t$, $t = \texttt{pred}\, t$, $t = \texttt{iszero}\, t$:
  Similar reasoning to the case $t = \texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3$, using lemmas similar to Lemma 1, each of which are proven using the same method: induction on the length of chain of small step derivations.

□

Lemma. *Forall t_1, t_1', t_2 and t_3, if:*

$$t_1 \longrightarrow^* t_1'$$

then:

$$\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 \longrightarrow^* \texttt{if}\ t_1'\ \texttt{then}\ t_2\ \texttt{else}\ t_3$$

Proof. Induction on the number of small steps in $t_1 \longrightarrow^* t_1'$.

- Base case $\mathit{steps} = 0$, i.e., $t_1 \longrightarrow^0 t_1'$:
  Then $t_1 = t_1'$, and

  $$\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 \longrightarrow^0 \texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3$$

  (Where $\longrightarrow^0$ is notation for a small-step derivation sequence with 0 steps.)
- Inductive case $\mathit{steps} = n + 1$, i.e., $t_1 \longrightarrow^n t_1'' \longrightarrow t_1'$ for some t_1'':
  Inductive Hypothesis:

  $$\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 \longrightarrow^* \texttt{if}\ t_1''\ \texttt{then}\ t_2\ \texttt{else}\ t_3$$

  By E-IF and $t_1'' \longrightarrow t_1'$, we know:

  $$\texttt{if}\ t_1''\ \texttt{then}\ t_2\ \texttt{else}\ t_3 \longrightarrow^* \texttt{if}\ t_1'\ \texttt{then}\ t_2\ \texttt{else}\ t_3$$

  By transitivity of the inductive hypothesis and the previous fact, we have

  $$\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 \longrightarrow^* \texttt{if}\ t_1'\ \texttt{then}\ t_2\ \texttt{else}\ t_3$$

□

Extra Credit (15 points)

Prove the other direction of Problem 5, that is:

$$\textit{forall}\ t\ a,\ \ \textit{if}\ \ t \longrightarrow^* a\ \ \textit{then}\ \ t \Downarrow a$$

Problem 6 (5 points)

Approximately how many hours did you spend working on this assignment?