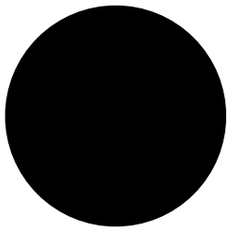
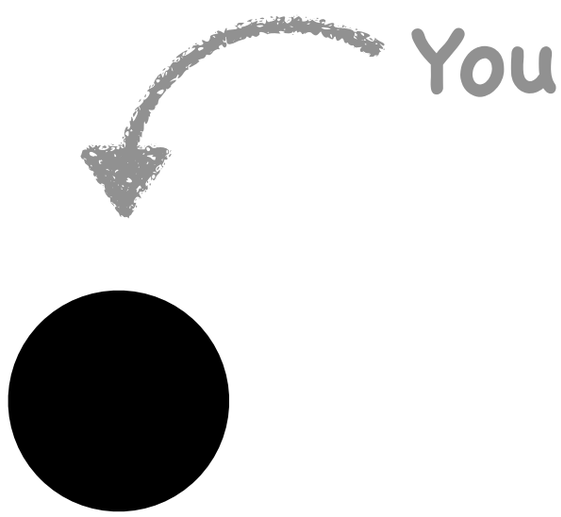
Constructive Galois Connections

David Darais
University of Maryland

David Van Horn
University of Maryland

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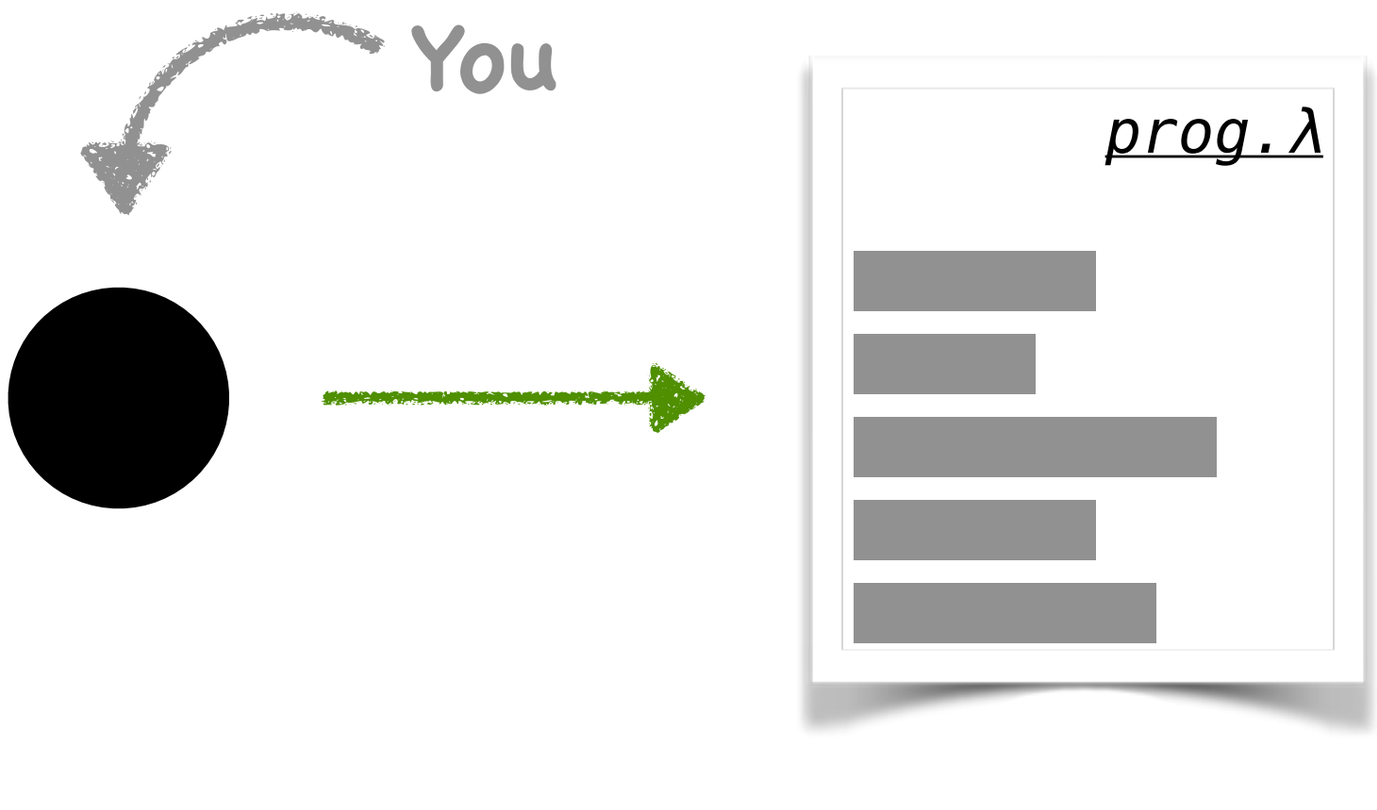
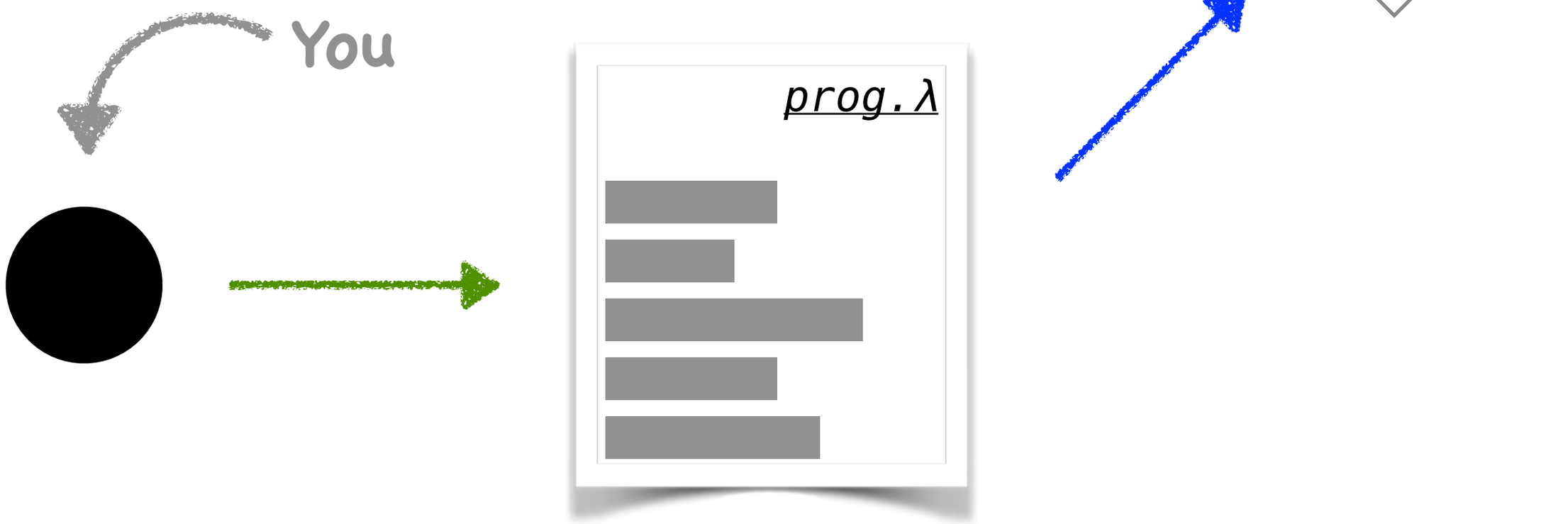
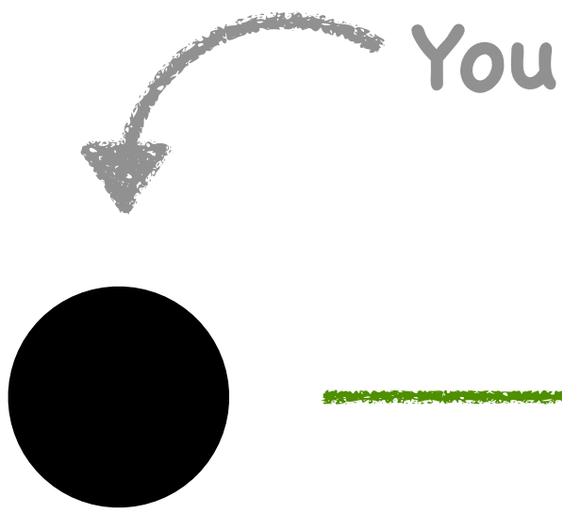

You
prog.λ


Static
Analysis
You
prog.λ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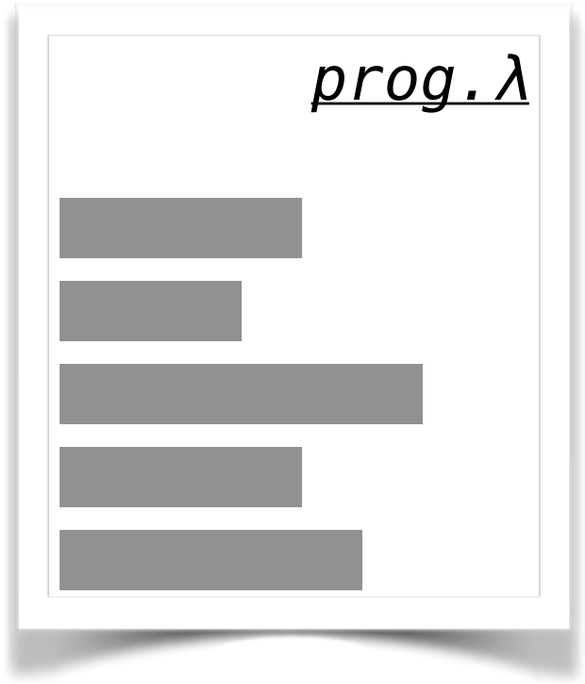
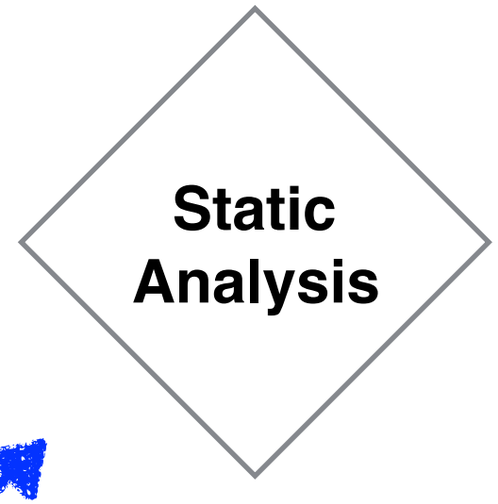
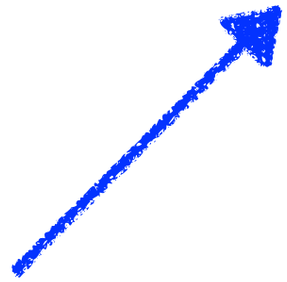
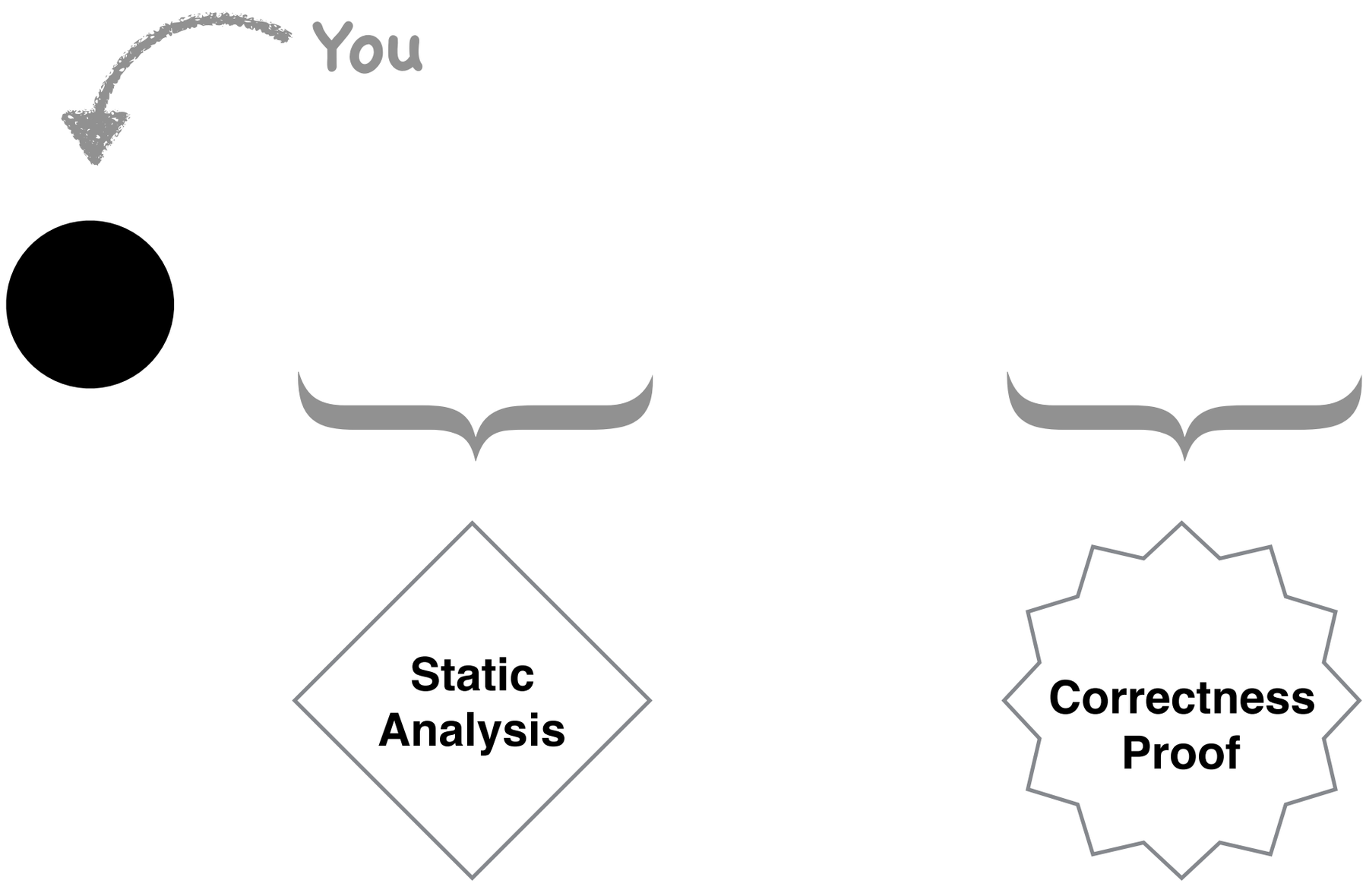
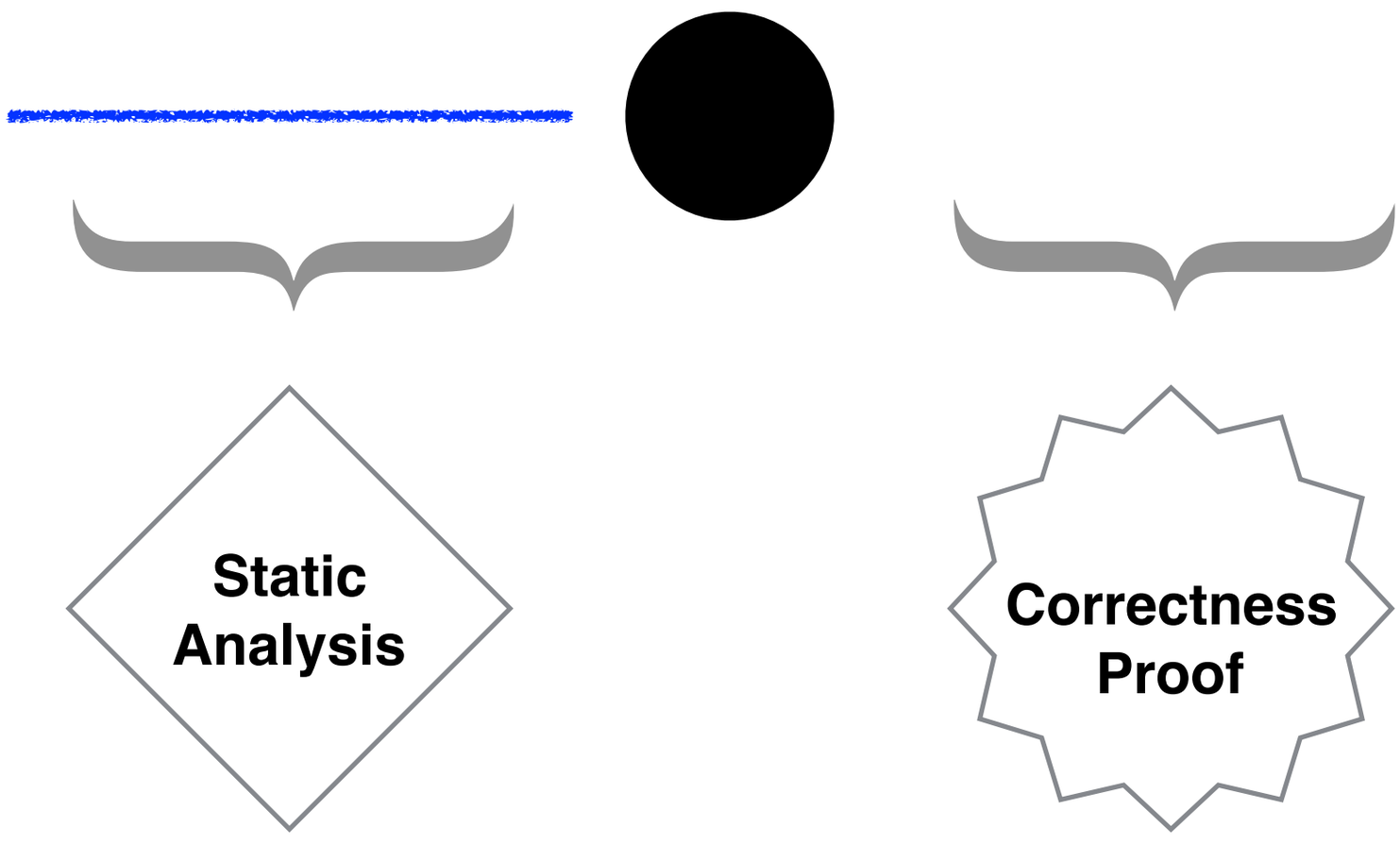

prog.λ

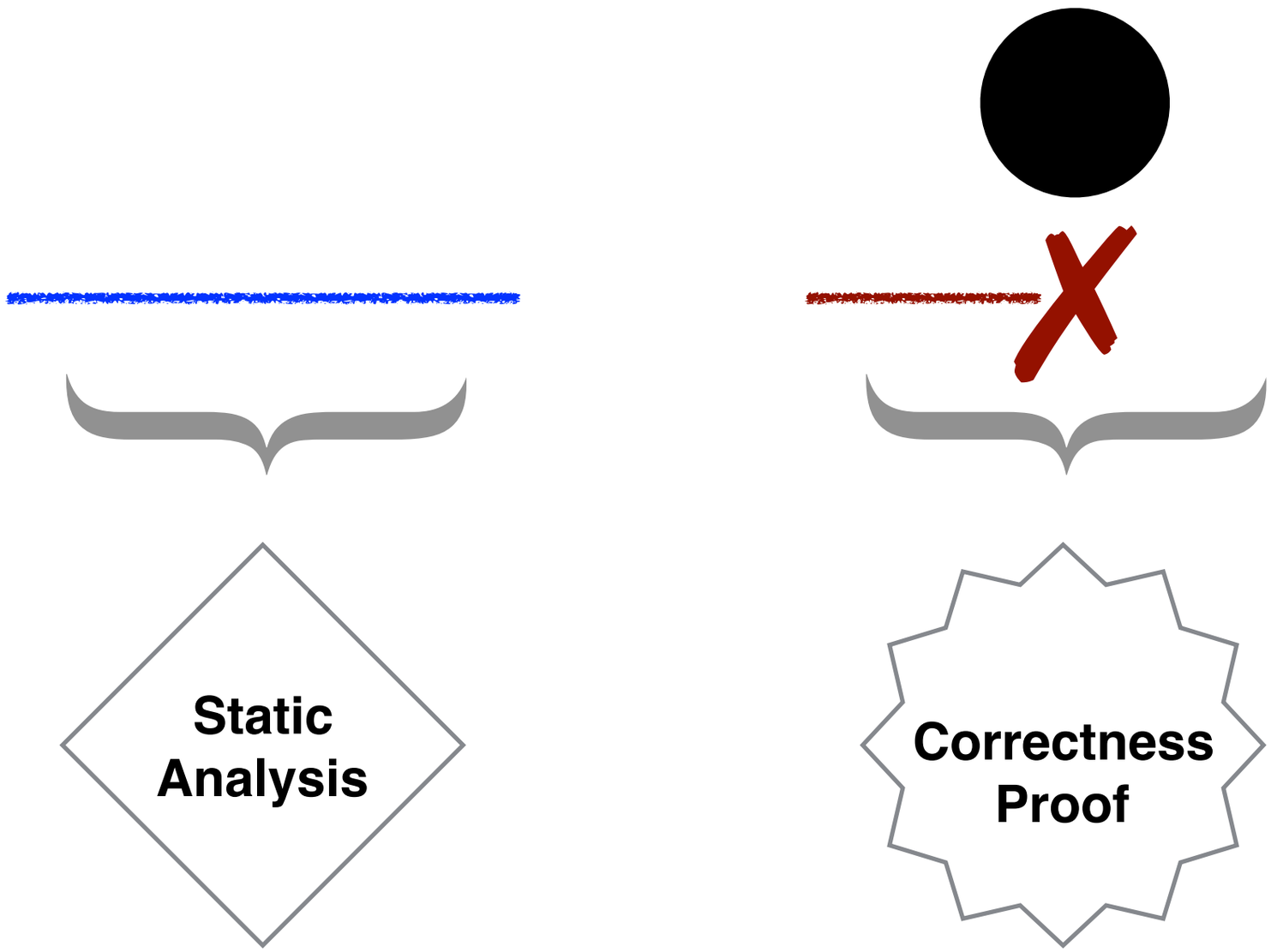

Static
Analysis

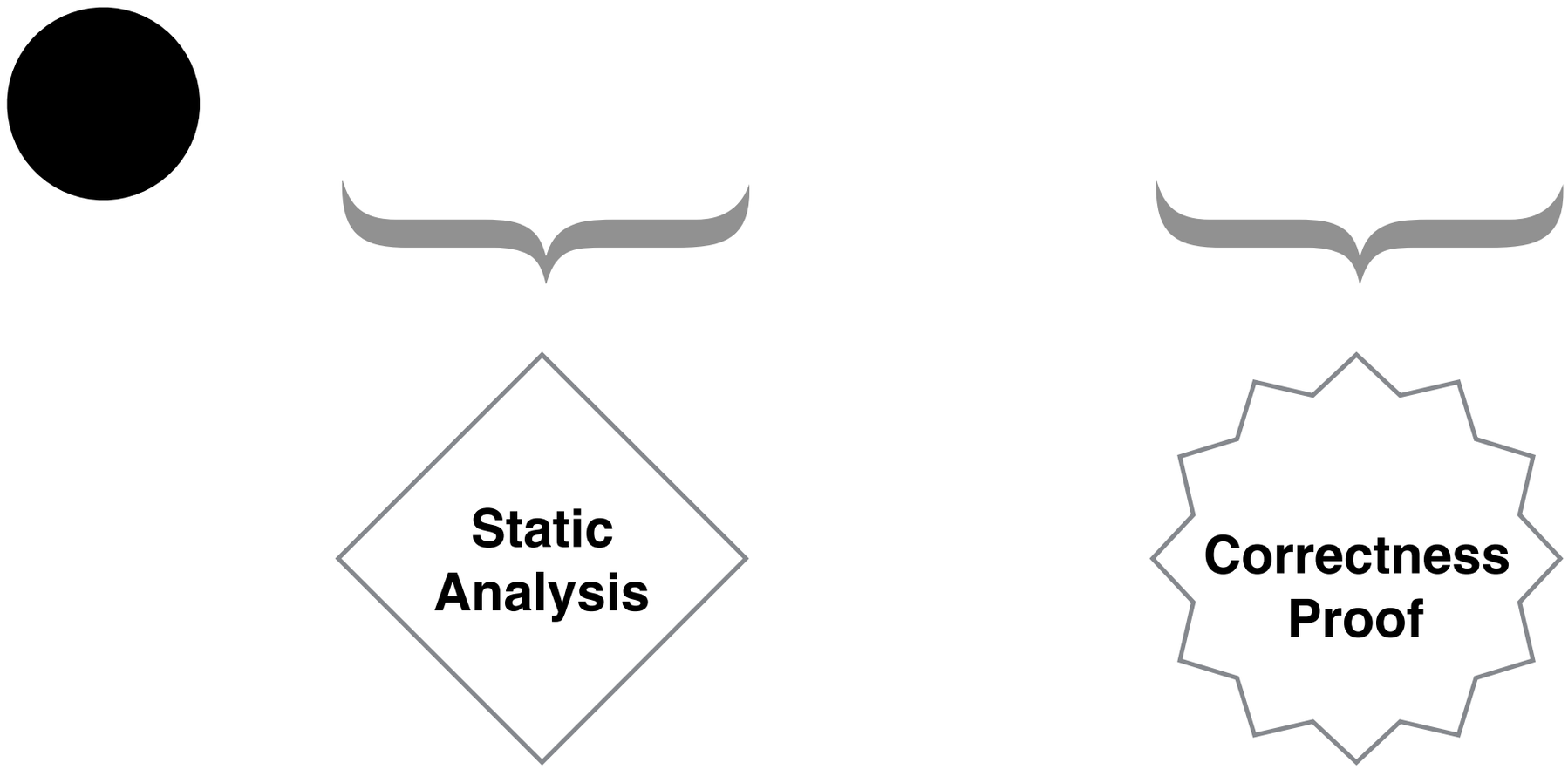



Correctness
Proof

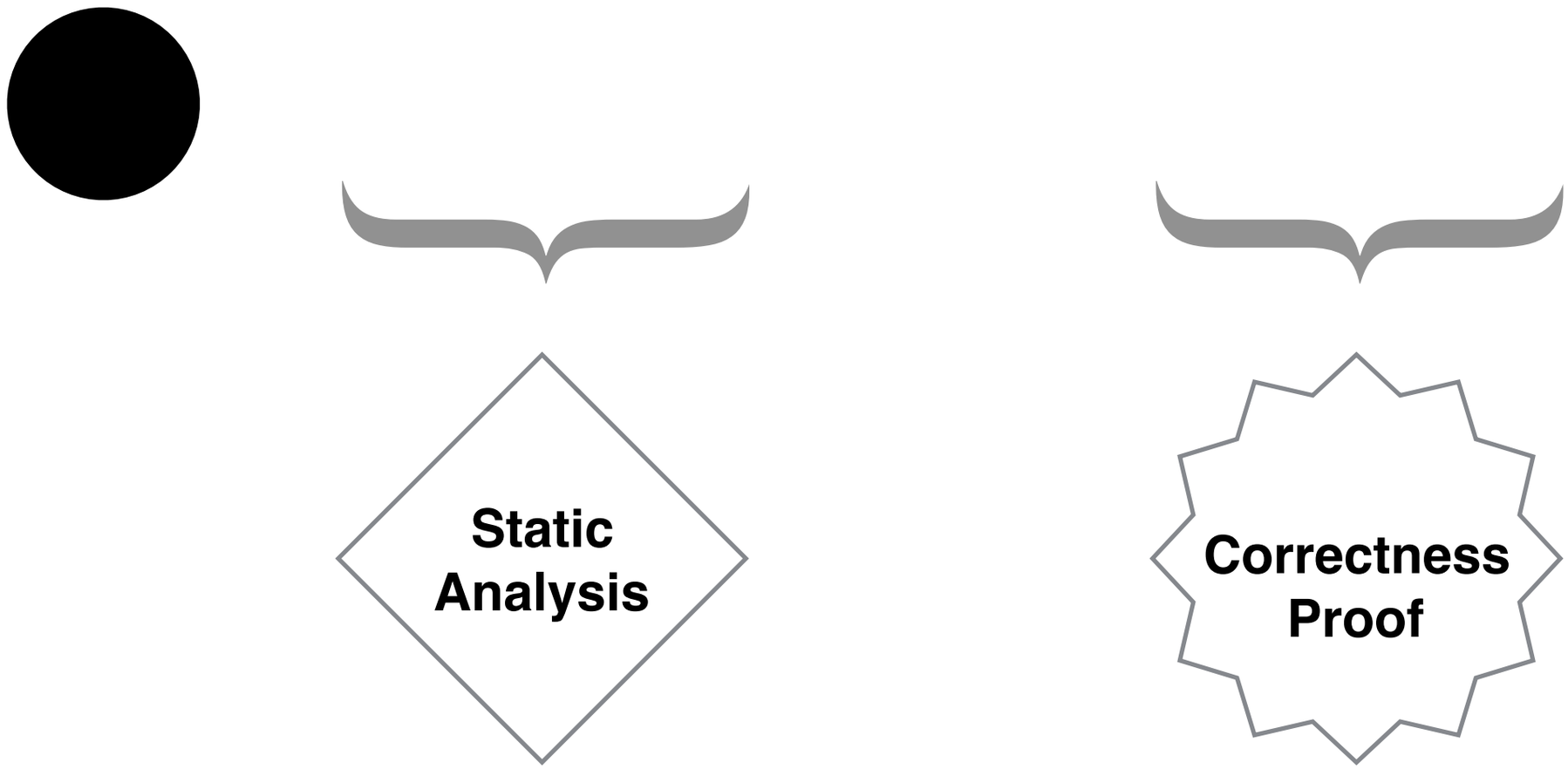

You
Static Analysis
Correctness Proof


Static
Analysis
Correctness
Proof

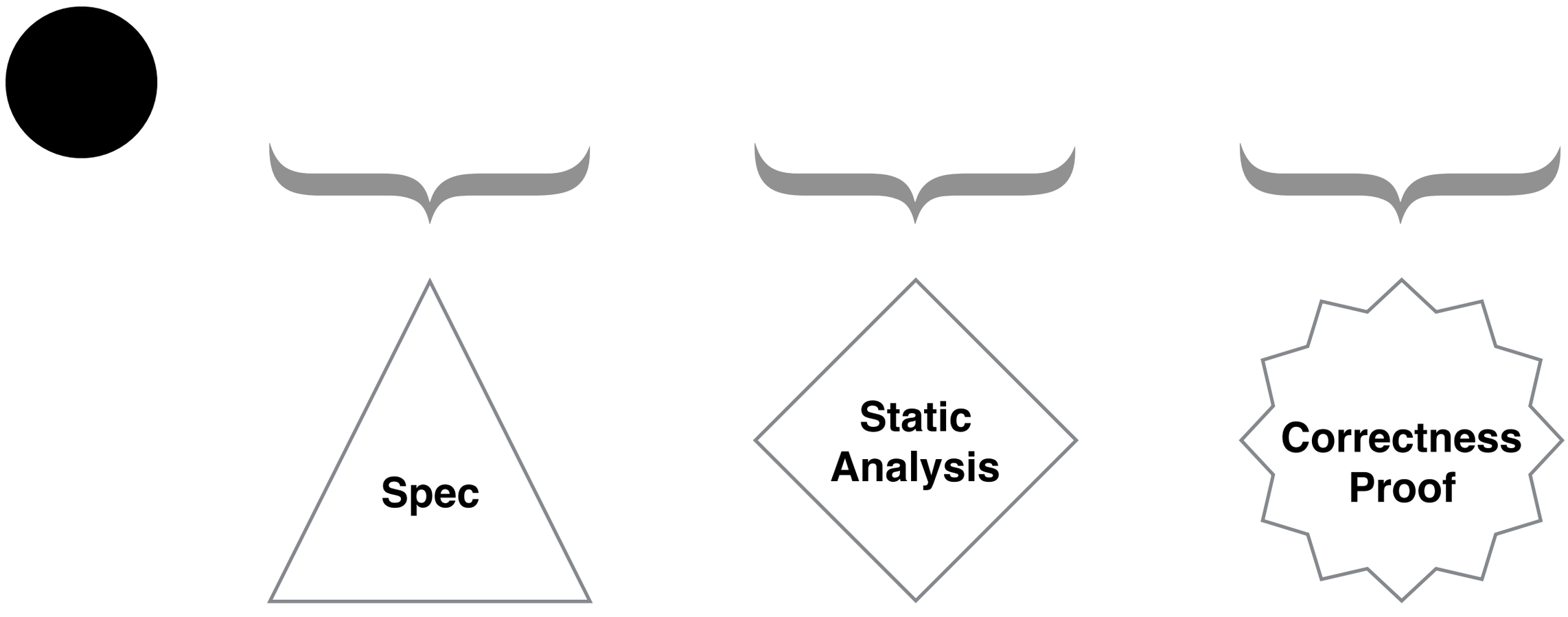

Static
Analysis
Correctness
Proof


Static
Analysis
Correctness
Proof

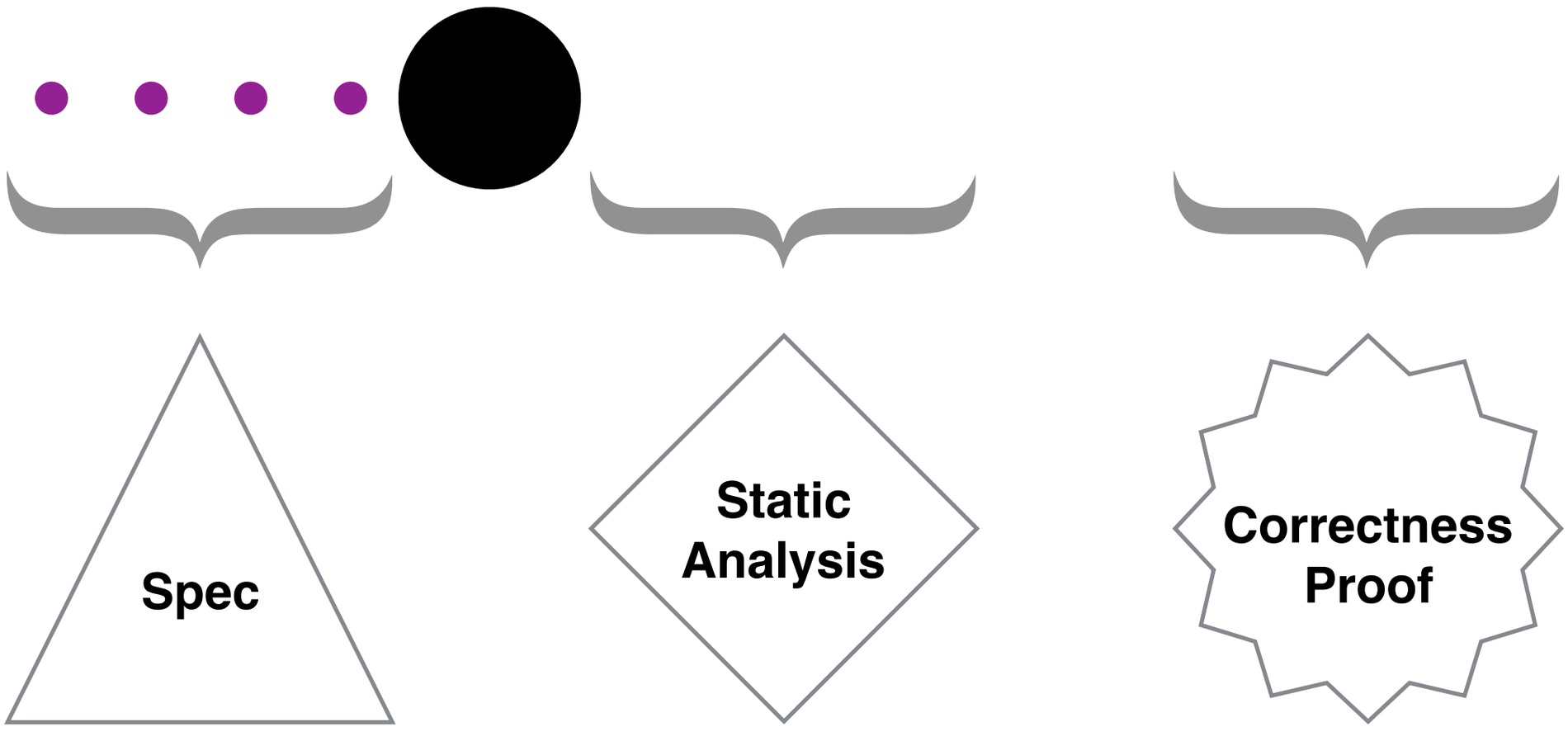
Abstract Interpretation



Abstract Interpretation

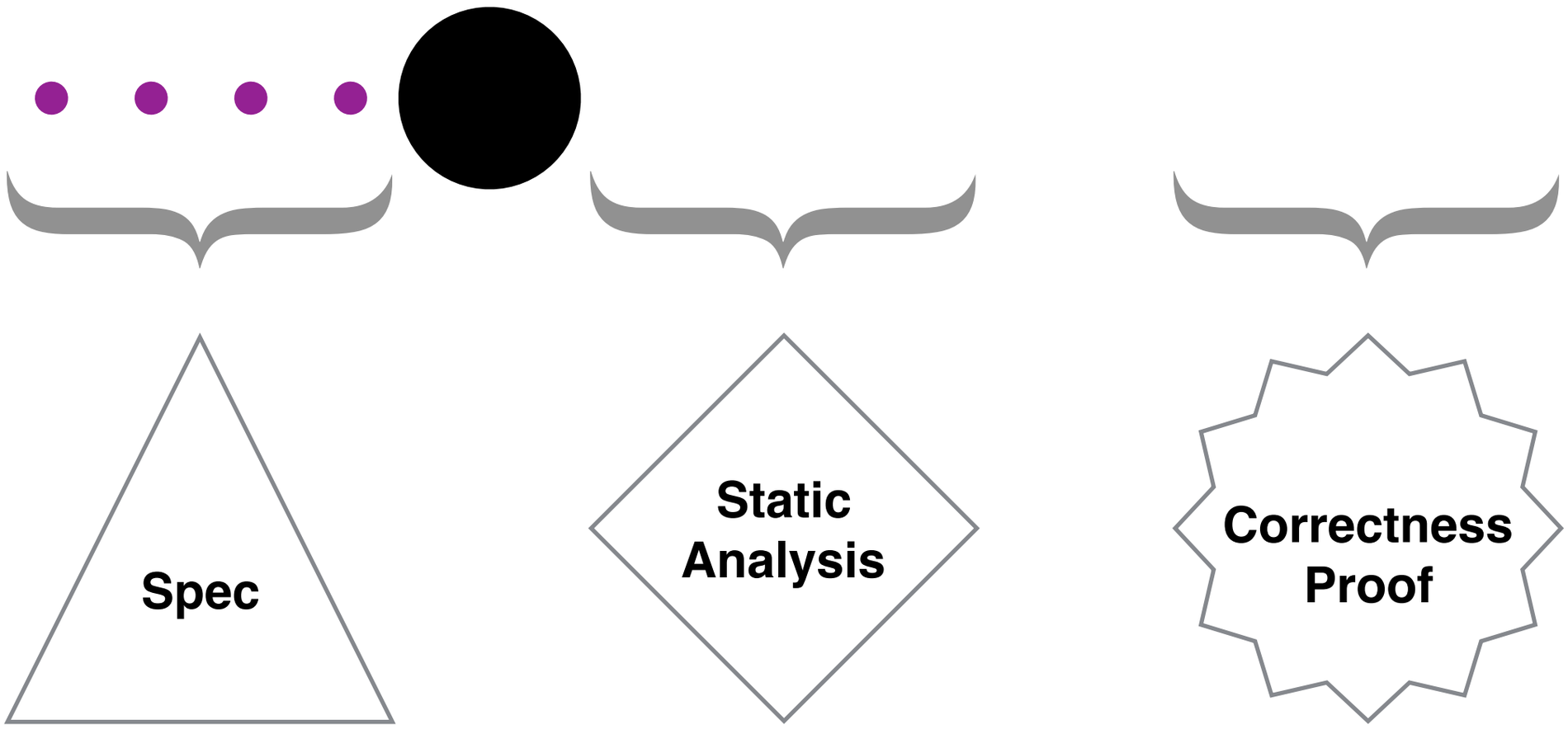


Abstract Interpretation



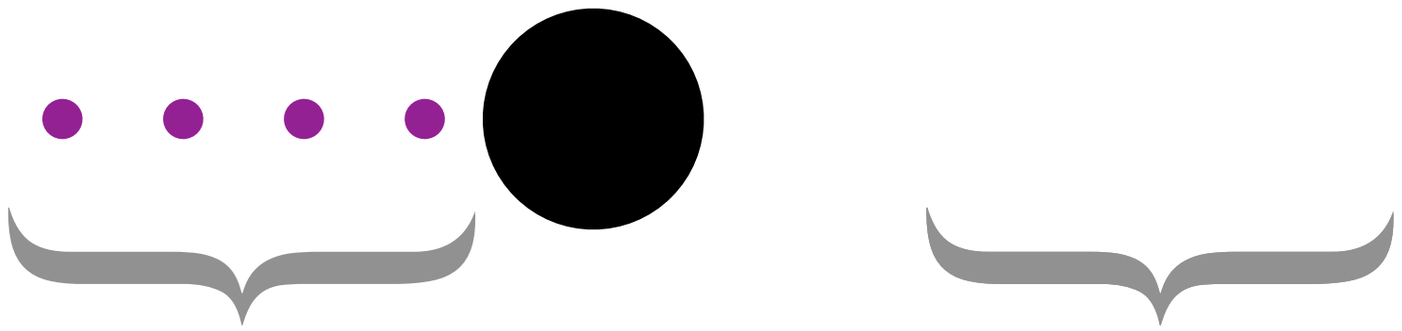
Abstract Interpretation

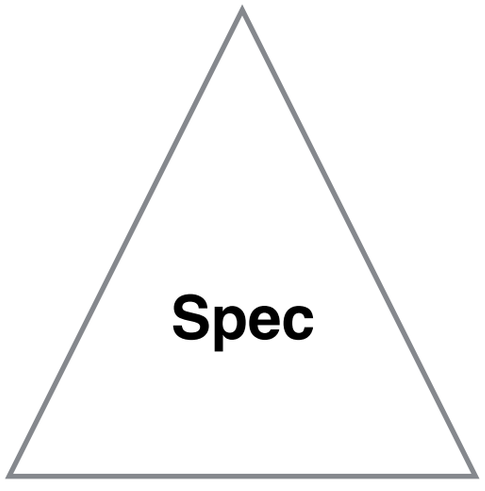
Calculational Design

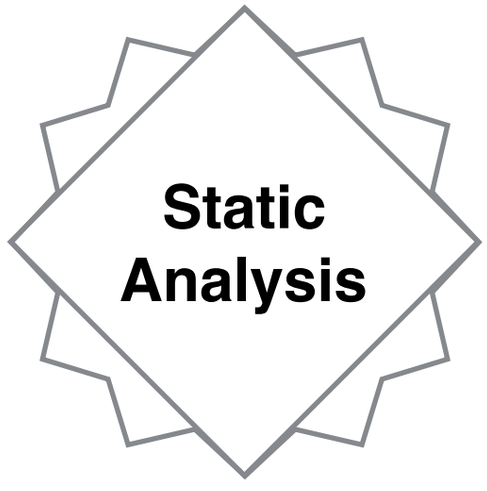


Abstract Interpretation

Calculational Design

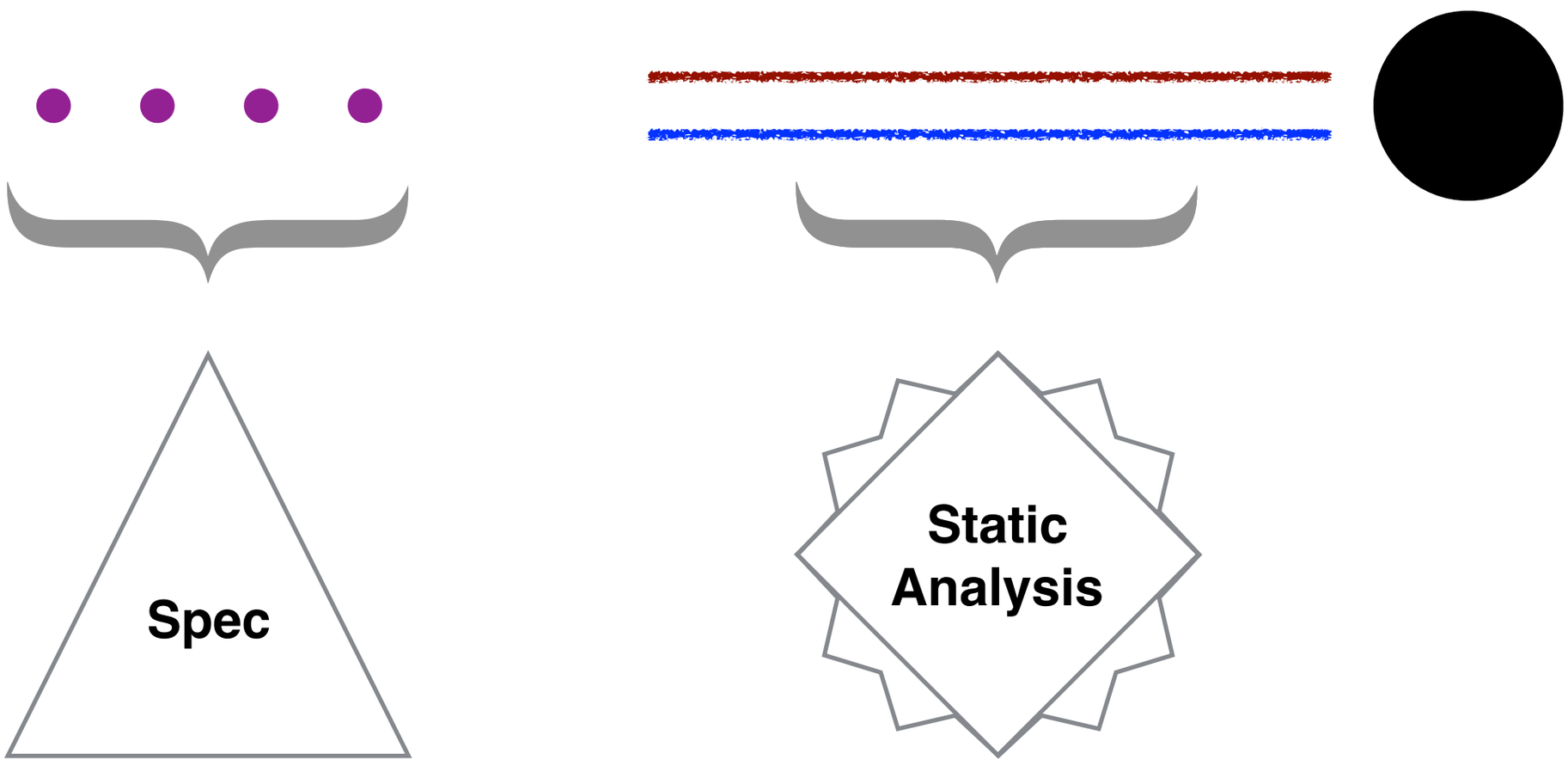






Abstract Interpretation

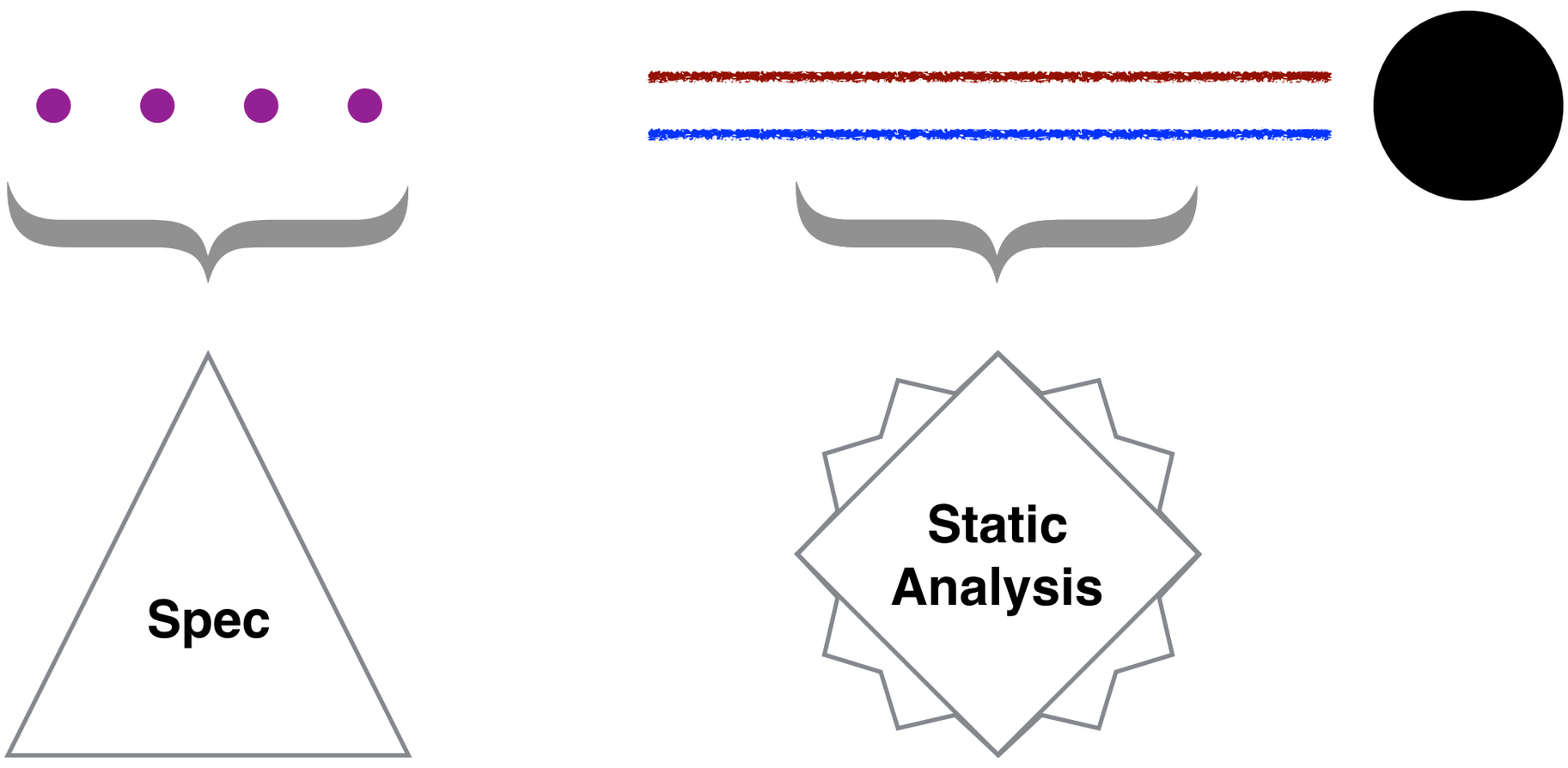
Calculational Design



Abstract Interpretation

Calculational Design

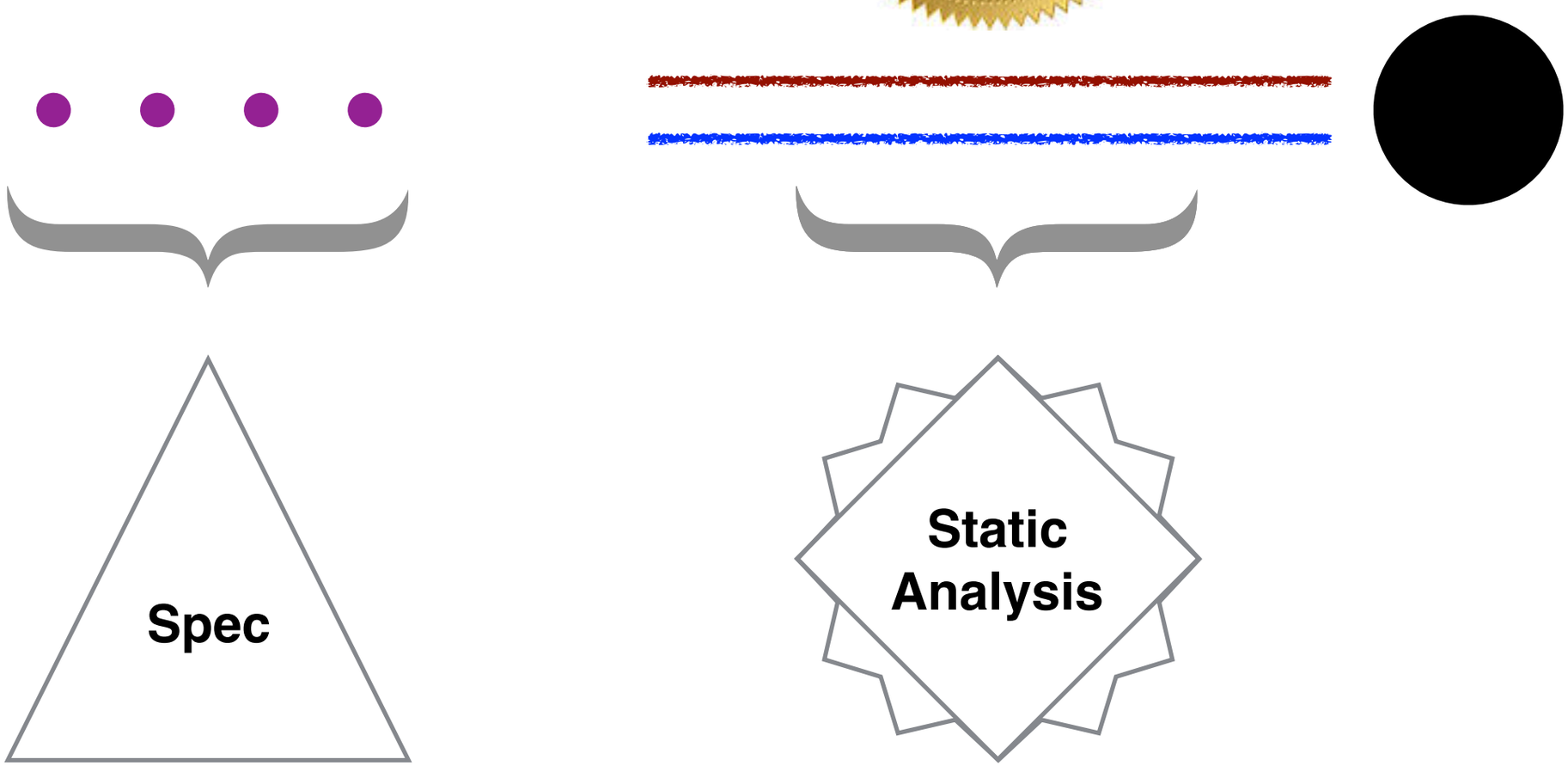
Proof Assistants



Abstract Interpretation

Calculational Design

Proof Assistants



The Dream

Abstract Interpreters

{

Synthesized from spec

Correct by construction

Certified Implementation

The Calculational Design of a Generic Abstract Interpreter

Patrick COUSOT
LIENS, Département de Mathématiques et Informatique
École Normale Supérieure, 45 rue d'Ulm, 75230 Paris cedex 05, France

Abstract. We present in extenso the calculation-based development of a generic compositional reachability static analyzer for a simple imperative programming language by abstract interpretation of its formal rule-based/structured small-step operational semantics.

Contents

The emphasis in these notes [has been the] correctness of the design **by calculus**. The **mechanized verification** [of this technique] can be foreseen with **automatic extraction** of a **correct program** from its **correctness proof**.

–Patrick Cousot [Monograph 1999]

N° d'ordre: 3262

THÈSE

présentée

devant l'Université de Rennes 1

pour obtenir

le grade de : DOCTEUR DE L'UNIVERSITÉ DE RENNES 1
Mention INFORMATIQUE

par

David PICHARDIE

Équipe d'accueil : Lande (Irisa,Rennes)
École Doctorale : Matisse
Composante universitaire : IFSIC

Titre de la thèse :

Interprétation abstraite en logique intuitionniste :
extraction d'analyseurs Java certifiés

Soutenue le 6 décembre 2005 devant la commission d'examen

M. :	Jean-Pierre	Banâtre	Président
M. :	Patrick	Cousot	Rapporteurs
M. :	Xavier	Leroy	
Mme. :	Christine	Paulin-Mohring	Examinateurs
M. :	David	Schmidt	
M. :	Thomas	Jensen	Directeurs
M. :	David	Cachera	

[Our] framework [loses] an important property of the standard framework: **the ability to derive a correct approximation from** [**its specification**]. Several examples of such derivations are given by Cousot. **It seems interesting to find a framework for** [**deriving approximations**]**, while remaining easily formalizable in Coq**.

–David Pichardie [PhD Thesis 2005]

```
…
if (b) {x ≔ 10} else {x ≔ 20}
…
```

```
…
if (b) {x ≔ 10} else {x ≔ 20}
…
```

$$x \in [10,20]$$

```
…
if (b) {x ≔ 10} else {x ≔ 20}
…
```

$\wp(\mathbb{Z})$

$\mathbb{Z} \times \mathbb{Z}$

x ∈ [10,20]

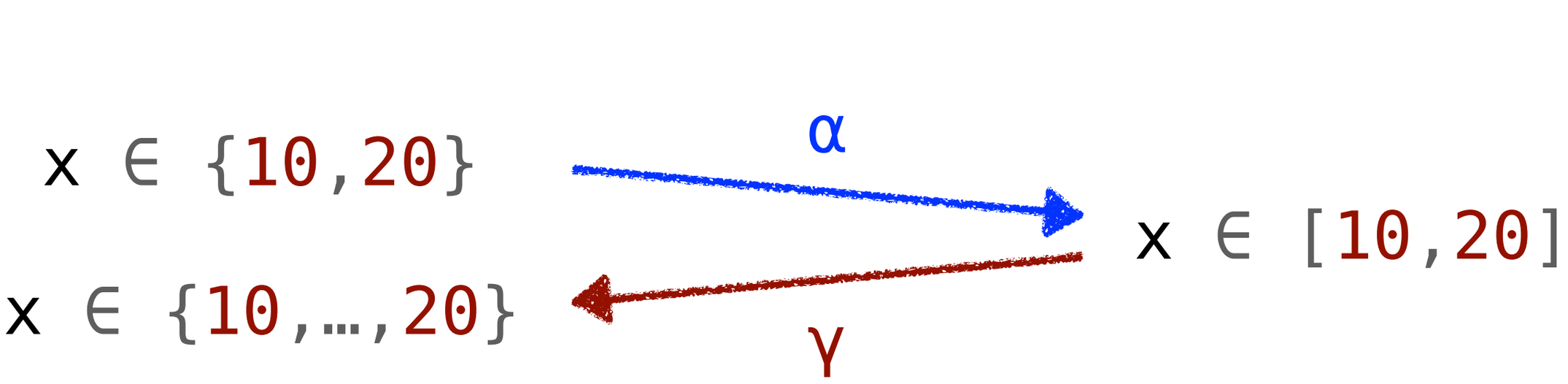
```
…
if (b) {x ≔ 10} else {x ≔ 20}
…
```

$\wp(\mathbb{Z})$ $\qquad\qquad$ $\mathbb{Z} \times \mathbb{Z}$

$x \in \{10,20\}$ $\xrightarrow{\alpha}$ $x \in [10,20]$

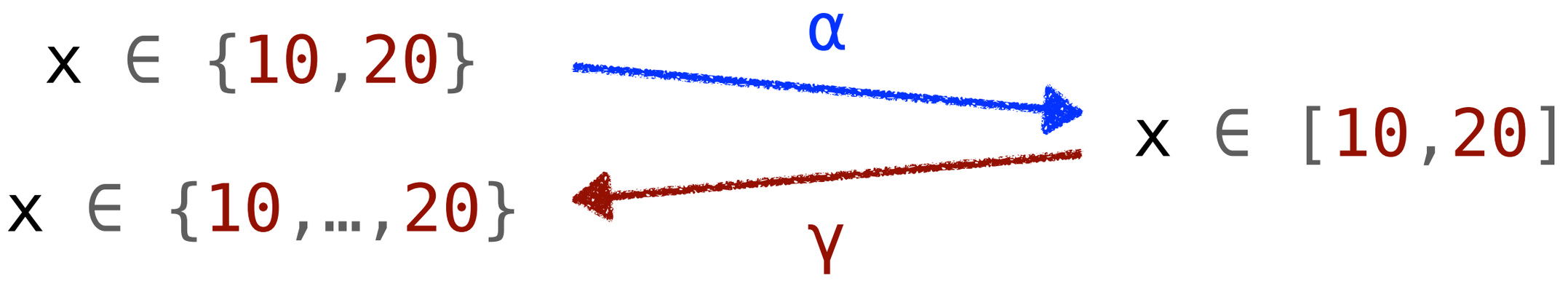
```
…
if (b) {x ≔ 10} else {x ≔ 20}
…
```

$\wp(\mathbb{Z})$ $\mathbb{Z} \times \mathbb{Z}$



```
…
if (b) {x ≔ 10} else {x ≔ 20}
…
```

$\wp(\mathbb{Z})$ $\qquad\qquad$ $\mathbb{Z} \times \mathbb{Z}$



Uncomputable $\qquad\qquad$ *Computable*

$\wp(\mathbb{Z})$ $\mathbb{Z} \times \mathbb{Z}$

Classical Reasoning ↯ *Program Extraction*

calculate.cousot

```
α(eval[n])(ρ♯)
⟅ defn of α ⟆
= αᴵ(eval[n](γᴿ(ρ♯)))
⟅ defn of eval[n] ⟆
= αᴵ({i | ρ ⊢ n ↦ i})
⟅ defn of _⊢_↦_ ⟆
= αᴵ({i})
⟅ defn of eval♯[n] ⟆
≜ eval♯[n](ρ♯)
```

calculate.cousot

```
α(eval[n])(ρ♯)
≀ defn of α ≀
= αᴵ(eval[n](γᴿ(ρ♯)))
≀ defn of eval[n] ≀
= αᴵ({i | ρ ⊢ n ↦ i})
≀ defn of _⊢_↦_ ≀
= αᴵ({i})
≀ defn of eval♯[n] ≀
≜ eval♯[n](ρ♯)
```

calculate.agda

```
▸ ⦇ α[ ⇄ᴿ ⇗ ⇄ᴵ ] · eval[ Num n ] · ρ♯ ⦈
▸ ⦇ (αᴵ * · (eval[ Num n ] * · (γᴿ · ρ♯)) ⦈
▸ [focus-right [·] of αᴵ * ]
    ≀ defn[eval[ Num n ]] ≀
▸ ⦇ αᴵ * · (return · n) ⦈
▸ ≀ right-unit[*] ≀
▸ ⦇ pure · eval♯[ Num n ] · ρ♯ ⦈
```

Four Stories

Direct Verification ✗ `calculate`

Abstract Interpretation ✗ `mechanize`

Four Stories

Direct Verification	✗ `calculate`
Abstract Interpretation	✗ `mechanize`
Kleisli GCs	✓ `calculate` ½ `mechanize`
Constructive GCs	✓ `calculate` ✓ `mechanize`

Direct Verification

Direct Verification

$$\mathtt{succ} \;:\; \mathbb{N} \to \mathbb{N}$$

Direct Verification

```
succ : ℕ → ℕ
```

```
ℙ ≔ {E,O}
```

Direct Verification

```
succ  : ℕ → ℕ                ℙ ≔ {E,O}
succ♯ : ℙ → ℙ
succ♯(E) ≔ O
succ♯(O) ≔ E
```

Direct Verification

```
succ  : ℕ → ℕ
succ♯ : ℙ → ℙ
succ♯(E) ≔ O
succ♯(O) ≔ E
```

```
ℙ ≔ {E,O}
⟦_⟧ : ℙ → ℘(ℕ)
⟦E⟧ ≔ { n | even(n) }
⟦O⟧ ≔ { n |  odd(n) }
```

Direct Verification

$succ : \mathbb{N} \to \mathbb{N}$

$succ^\sharp : \mathbb{P} \to \mathbb{P}$

$succ^\sharp(E) \coloneqq O$

$succ^\sharp(O) \coloneqq E$

$\mathbb{P} \coloneqq \{E, O\}$

$[\![\_]\!] : \mathbb{P} \to \wp(\mathbb{N})$

$[\![E]\!] \coloneqq \{\ n \mid even(n)\ \}$

$[\![O]\!] \coloneqq \{\ n \mid odd(n)\ \}$

$sound : n \in [\![p]\!] \implies succ(n) \in [\![succ^\sharp(p)]\!]$

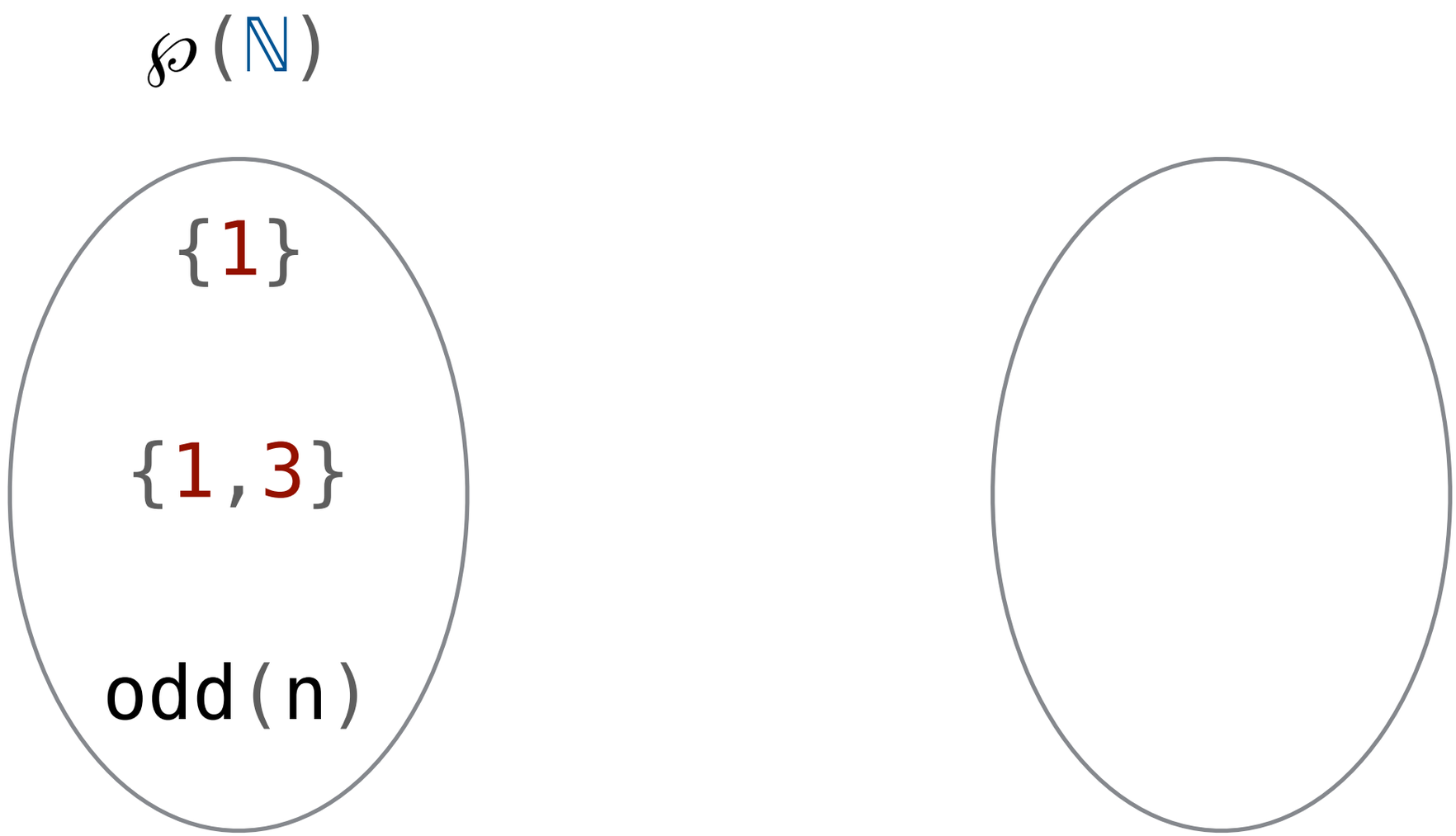
Four Stories

Direct Verification	✗ `calculate`
Abstract Interpretation	✗ `mechanize`
Kleisli GCs	✓ `calculate` ½ `mechanize`
Constructive GCs	✓ `calculate` ✓ `mechanize`

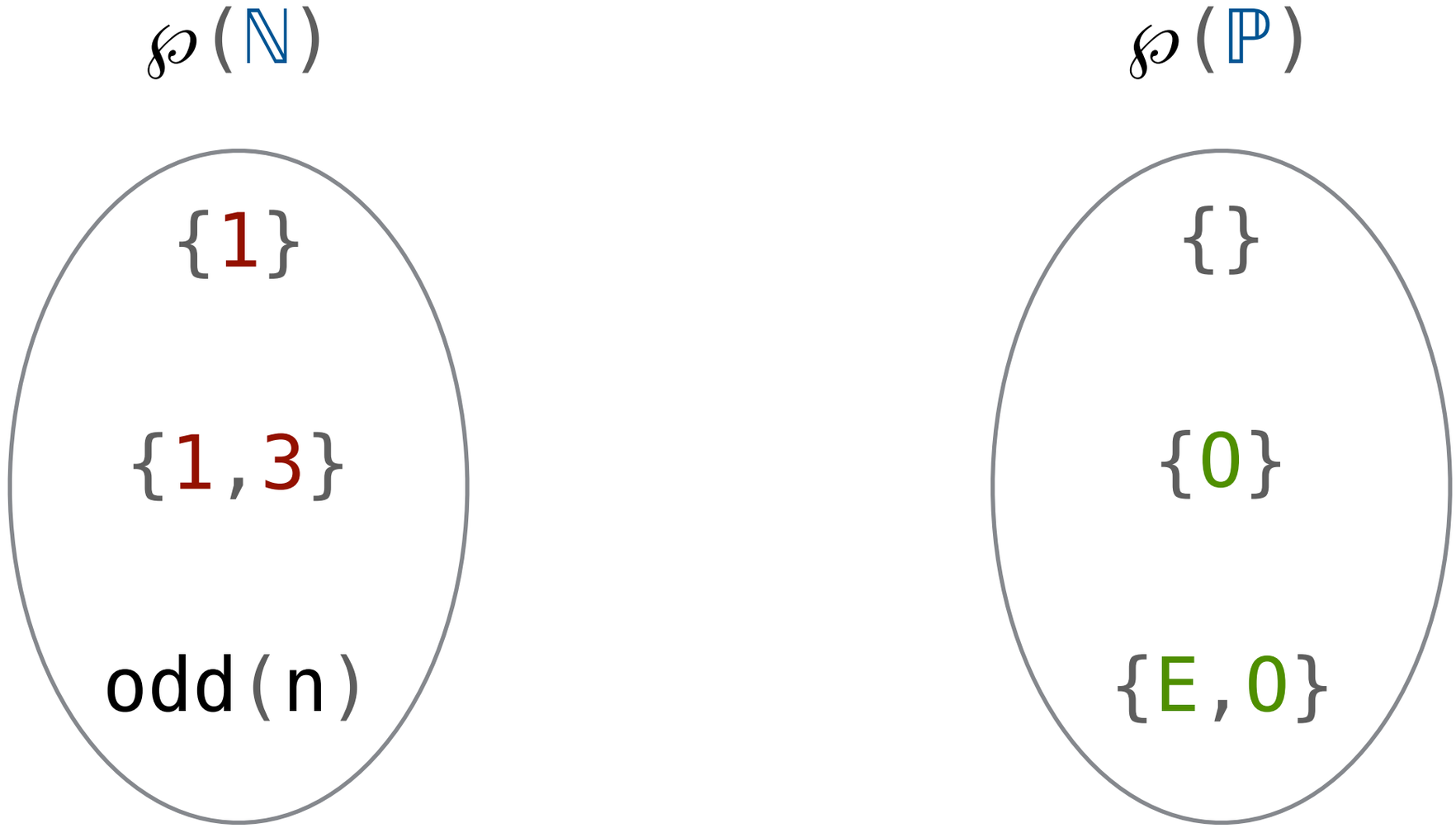
Abstract Interpretation

Abstract Interpretation

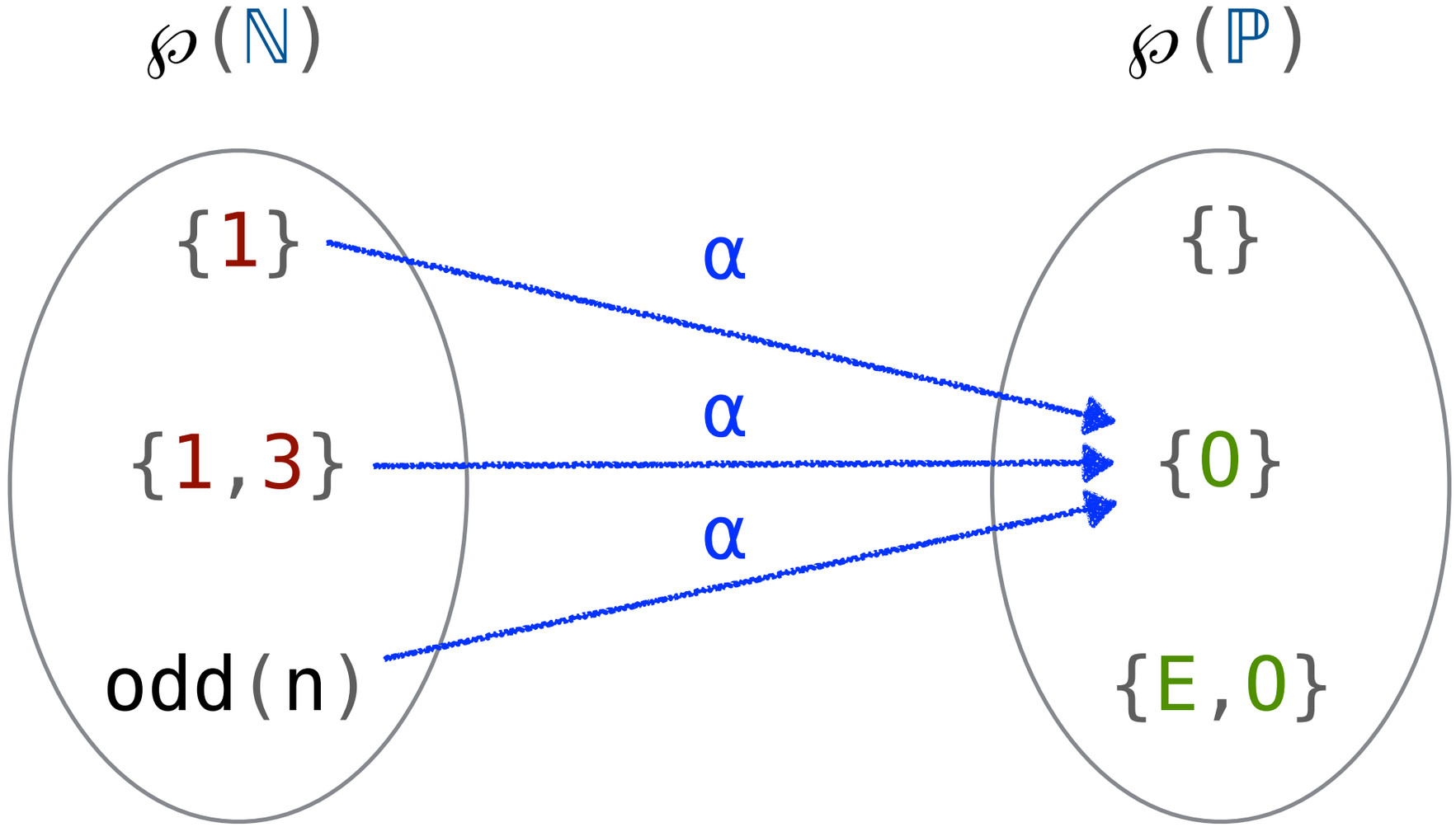
Abstract Interpretation



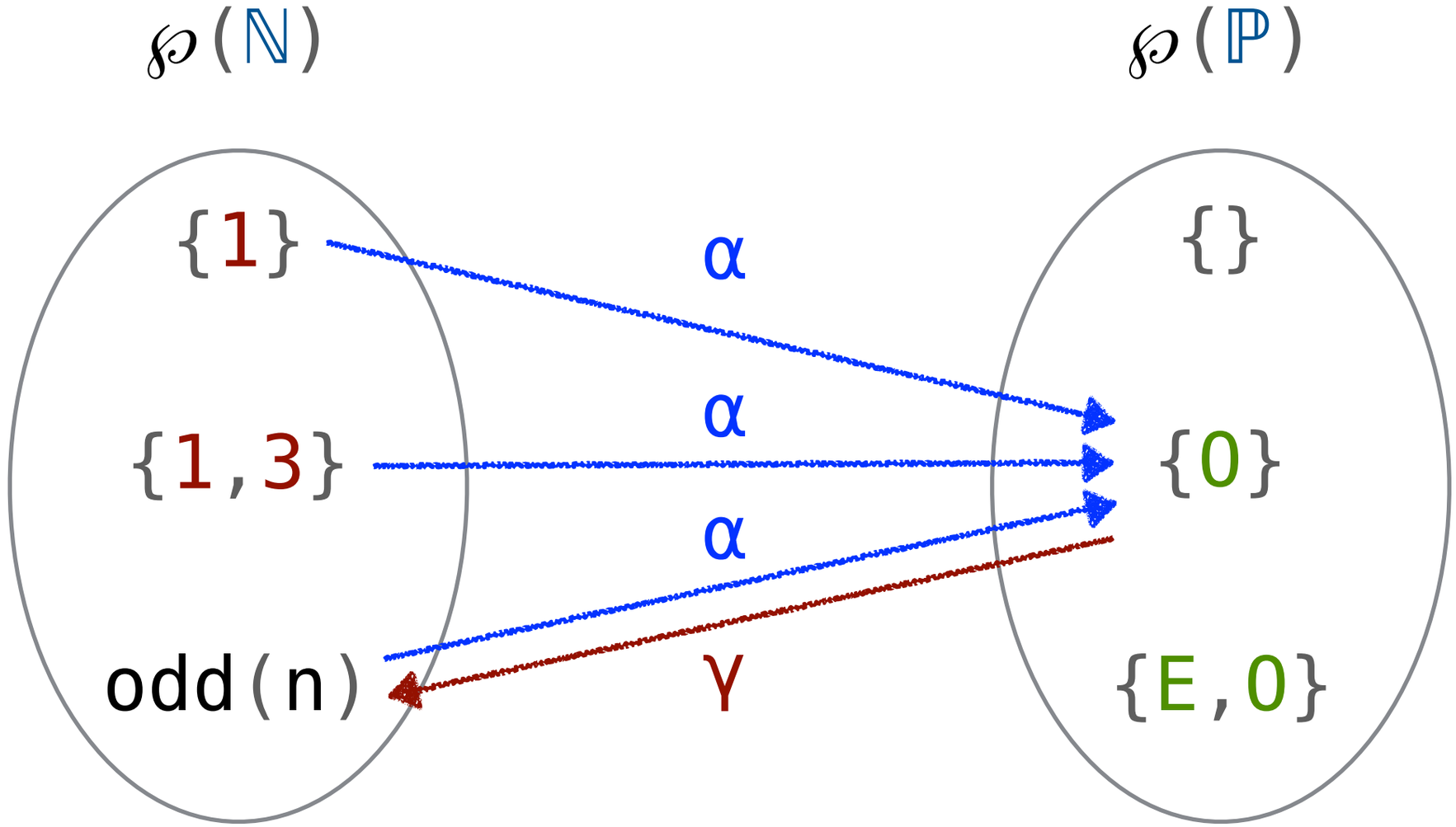
Abstract Interpretation



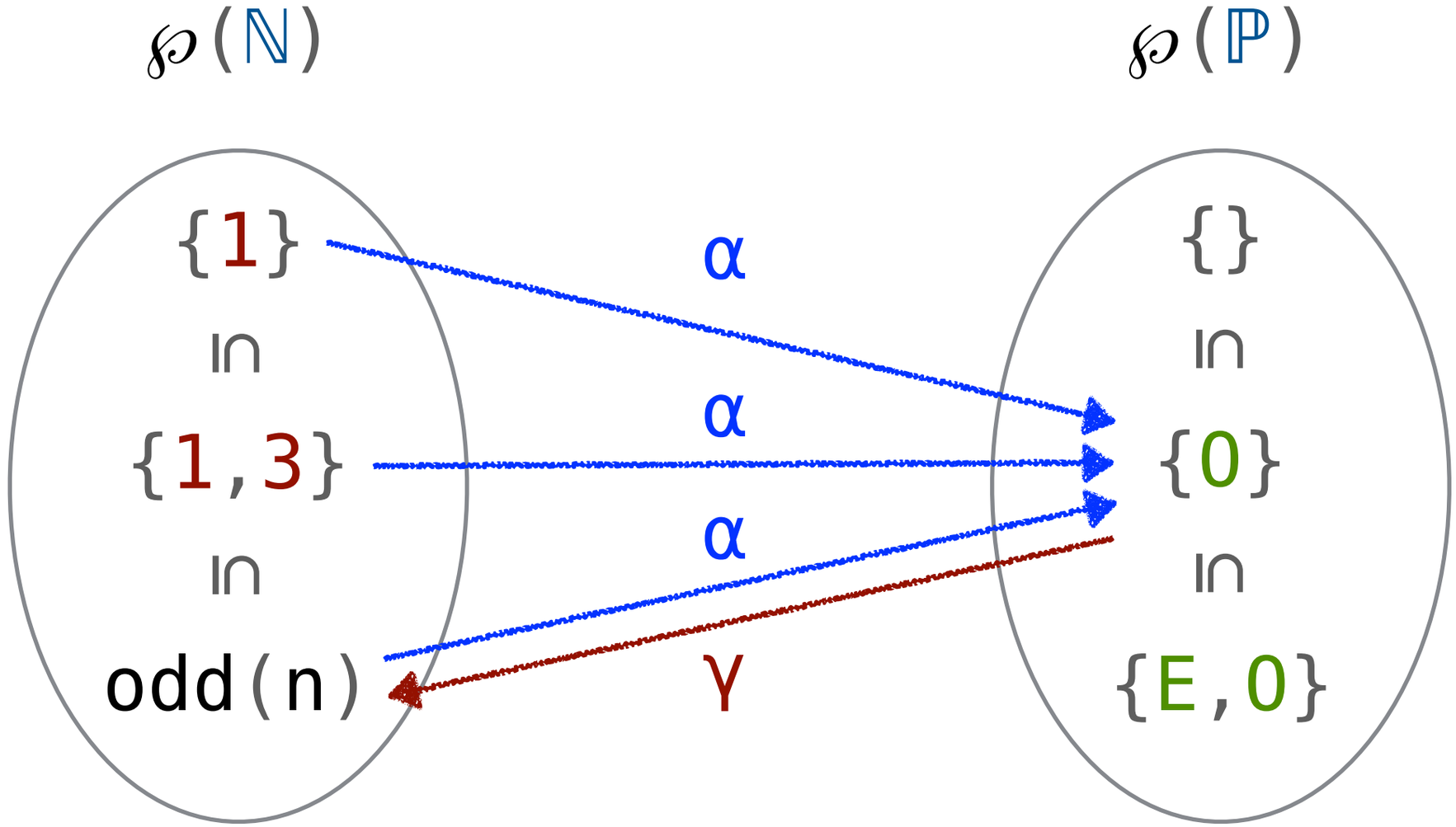
Abstract Interpretation



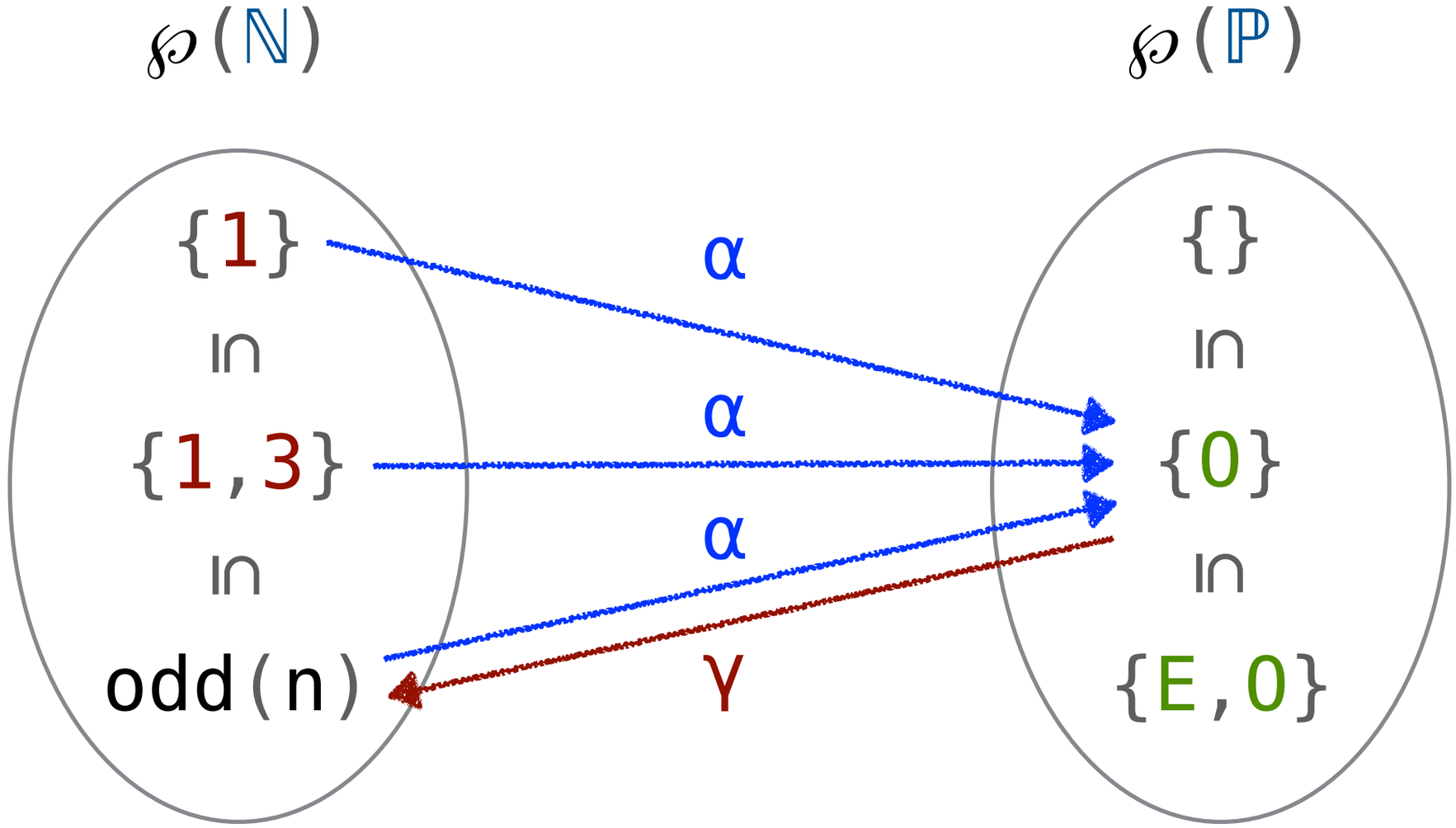
Abstract Interpretation



Abstract Interpretation

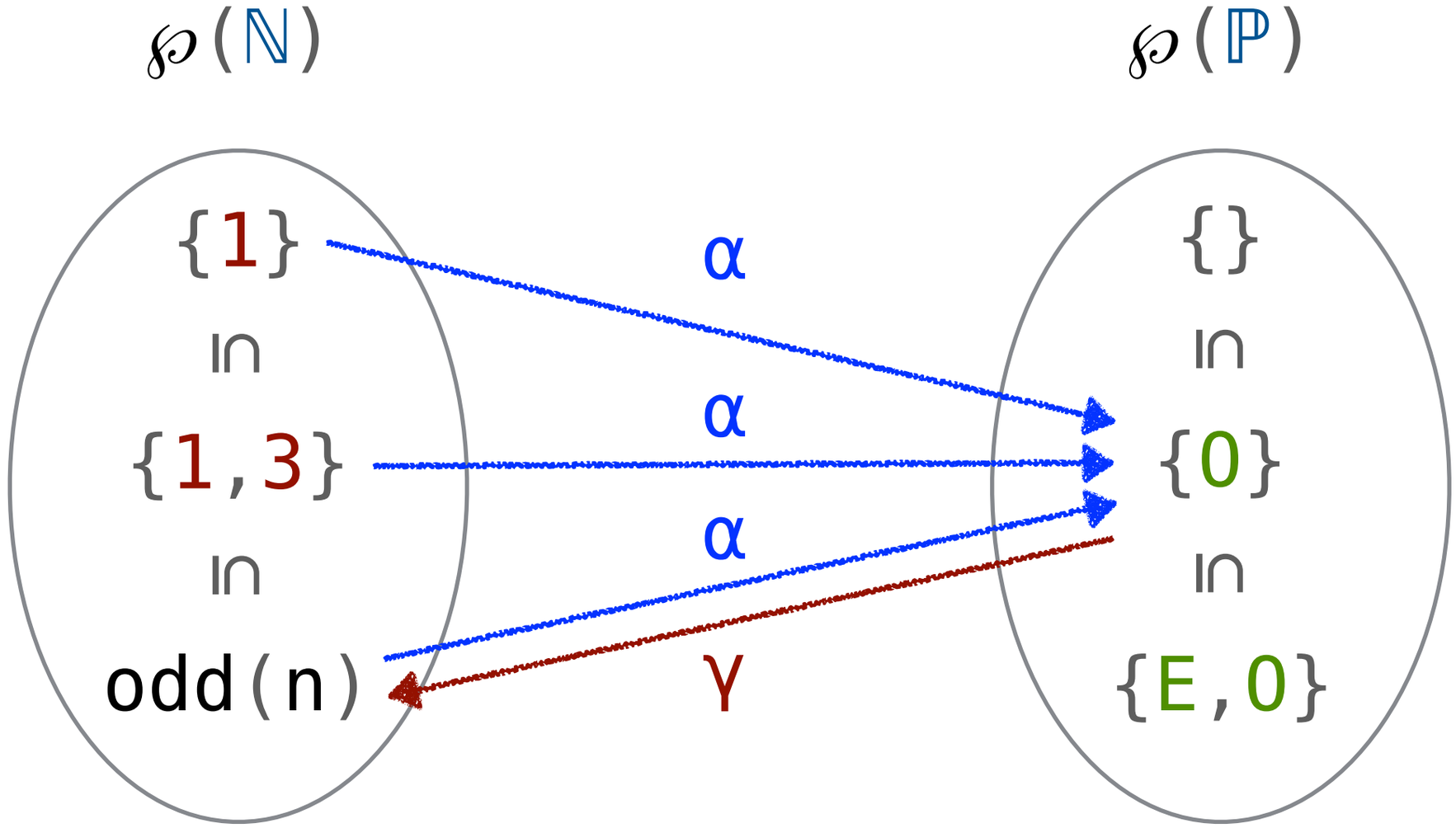


Abstract Interpretation



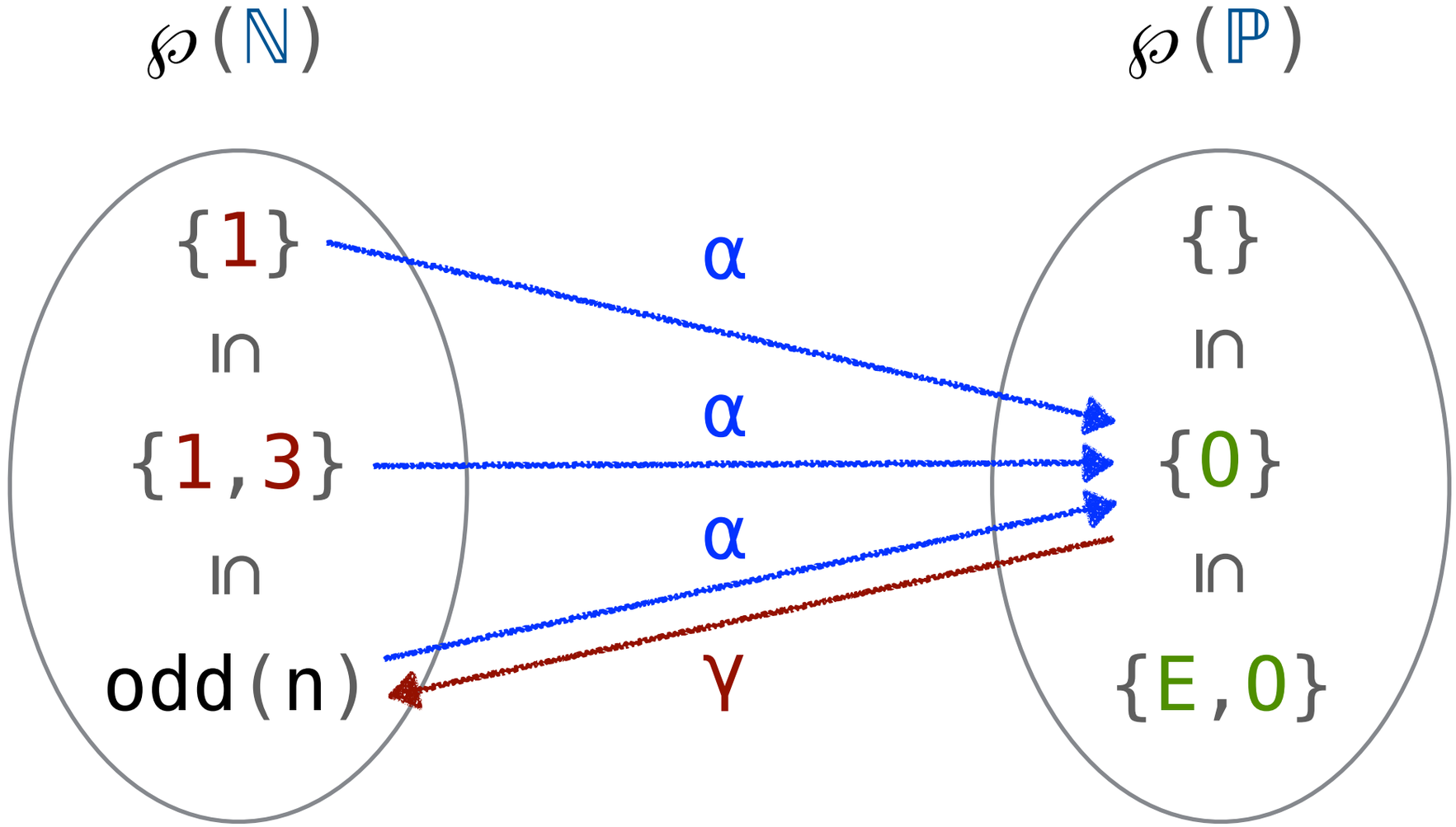
N ⊆ γ(α(N))

Abstract Interpretation



N ⊆ γ(α(N)) α(γ(P)) ⊆ P

Abstract Interpretation



$$N \subseteq \gamma(\alpha(N)) \quad \wedge \quad \alpha(\gamma(P)) \subseteq P$$

$$N \subseteq \gamma(P) \iff \alpha(N) \subseteq P$$

Abstract Interpretation

$$\mathtt{N} \in \wp(\mathbb{N})$$

$$\mathtt{P} \in \wp(\mathbb{N})$$

"P is sound for N"

$$\alpha(\mathtt{N}) \subseteq \mathtt{P}$$

Abstract Interpretation

$\mathtt{N} \in \wp(\mathbb{N})$

$\mathtt{P} \in \wp(\mathbb{N})$

"$\mathtt{P}$ is sound for $\mathtt{N}$"

$\alpha(\mathtt{N}) \subseteq \mathtt{P}$

$\mathtt{f}^{\mathtt{N}} \in \wp(\mathbb{N}) \to \wp(\mathbb{N})$

$\mathtt{f}^{\mathtt{P}} \in \wp(\mathbb{P}) \to \wp(\mathbb{P})$

"$\mathtt{f}^{\mathtt{P}}$ is sound for $\mathtt{f}^{\mathtt{N}}$"

Abstract Interpretation

$\mathtt{N} \in \wp(\mathbb{N})$

$\mathtt{P} \in \wp(\mathbb{N})$

"$\mathtt{P}$ is sound for $\mathtt{N}$"

$\alpha(\mathtt{N}) \subseteq \mathtt{P}$

$\mathtt{f}^{\mathtt{N}} \in \wp(\mathbb{N}) \to \wp(\mathbb{N})$

$\mathtt{f}^{\mathtt{P}} \in \wp(\mathbb{P}) \to \wp(\mathbb{P})$

"$\mathtt{f}^{\mathtt{P}}$ is sound for $\mathtt{f}^{\mathtt{N}}$"

$\vec{\alpha}(\mathtt{f}^{\mathtt{N}}) \subseteq \mathtt{f}^{\mathtt{P}}$

Abstract Interpretation

$N \in \wp(\mathbb{N})$

$P \in \wp(\mathbb{N})$

"P is sound for N"

$\alpha(N) \subseteq P$

$f^N \in \wp(\mathbb{N}) \to \wp(\mathbb{N})$

$f^P \in \wp(\mathbb{P}) \to \wp(\mathbb{P})$

"f^P is sound for f^N"

$\alpha \circ f^N \circ \gamma \subseteq f^P$

Abstract Interpretation

$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$
$$\alpha(N) \coloneqq \{\text{parity}(n) \mid n \in N\}$$

Abstract Interpretation

$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$
$$\alpha(N) \coloneqq \{parity(n) \mid n \in N\}$$

$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$
$$\gamma(P) \coloneqq \{n \mid p \in P \wedge n \in [\![p]\!]\}$$

Abstract Interpretation

$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$
$$\alpha(N) \coloneqq \{parity(n) \mid n \in N\}$$

$$\alpha(N) \approx map(parity)(N)$$
$$map : (\mathbb{N} \to \mathbb{P}) \to \wp(\mathbb{N}) \to \wp(\mathbb{P})$$

$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$
$$\gamma(P) \coloneqq \{n \mid p \in P \wedge n \in [\![p]\!]\}$$

Abstract Interpretation

$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$
$$\alpha(N) \coloneqq \{parity(n) \mid n \in N\}$$

$$\alpha(N) \approx map(parity)(N)$$
$$map : (\mathbb{N} \to \mathbb{P}) \to \wp(\mathbb{N}) \to \wp(\mathbb{P})$$

$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$
$$\gamma(P) \coloneqq \{n \mid p \in P \wedge n \in [\![p]\!]\}$$

$$\gamma(P) \approx P^*$$
$$\_^* : (\mathbb{P} \to \wp(\mathbb{N})) \to \wp(\mathbb{P}) \to \wp(\mathbb{N})$$

Abstract Interpretation

$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$

$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$

$\mathtt{succ} : \mathbb{N} \to \mathbb{N}$

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Abstract Interpretation

$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$

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$\mathtt{succ} : \mathbb{N} \to \mathbb{N}$

$\mathtt{succ}^\sharp : \mathbb{P} \to \mathbb{P}$

$\mathtt{f}^{\mathtt{N}} : \wp(\mathbb{N}) \to \wp(\mathbb{N})$

$\mathtt{f}^{\mathtt{N}}(N) \coloneqq ?$

$\mathtt{f}^{\mathtt{P}} : \wp(\mathbb{P}) \to \wp(\mathbb{P})$

$\mathtt{f}^{\mathtt{P}}(P) \coloneqq ?$

Abstract Interpretation

$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$

$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$

$\mathsf{succ} : \mathbb{N} \to \mathbb{N}$

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$\uparrow\mathsf{succ} : \wp(\mathbb{N}) \to \wp(\mathbb{N})$

$\uparrow\mathsf{succ}(N) \coloneqq \{\mathsf{succ}(n) \mid n \in N\}$

$\uparrow\mathsf{succ}^\sharp : \wp(\mathbb{P}) \to \wp(\mathbb{P})$

$\uparrow\mathsf{succ}^\sharp(P) \coloneqq \{\mathsf{succ}^\sharp(p) \mid p \in P\}$

Abstract Interpretation

$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$

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$\mathsf{sound} : \alpha(\uparrow\mathsf{succ}(\gamma(P))) \subseteq \uparrow\mathsf{succ}^\sharp(P)$

Abstract Interpretation

$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$
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$\mathsf{sound} : \alpha(\uparrow\mathsf{succ}(\gamma(P))) \subseteq \uparrow\mathsf{succ}^\sharp(P)$
$\mathsf{optimal} : \alpha(\uparrow\mathsf{succ}(\gamma(P))) = \uparrow\mathsf{succ}^\sharp(P)$

Abstract Interpretation

$$\text{optimal} : \alpha(\uparrow\text{succ}(\gamma(P))) = \uparrow\text{succ}^{\sharp}(P)$$

Abstract Interpretation

$$\mathtt{calc} : \alpha(\uparrow \mathtt{succ}(\gamma(P))) = \ldots \triangleq \uparrow \mathtt{succ}^{\sharp}(P)$$

Abstract Interpretation

$$\mathtt{calc} : \alpha(\uparrow\mathtt{succ}(\gamma(P))) = \ldots \triangleq \uparrow\mathtt{succ}^{\sharp}(P)$$

$$\alpha(\uparrow\mathtt{succ}(\gamma(\{E\})))$$

Abstract Interpretation

$$\mathtt{calc} : \alpha(\uparrow \mathtt{succ}(\gamma(P))) = \ldots \triangleq \uparrow \mathtt{succ}^{\sharp}(P)$$

$$\begin{aligned}
&\alpha(\uparrow \mathtt{succ}(\gamma(\{E\}))) \\
&= \alpha(\uparrow \mathtt{succ}(\{n \mid \mathtt{even}(n)\}))
\end{aligned}$$

Abstract Interpretation

$$\mathrm{calc} : \alpha(\uparrow\mathrm{succ}(\gamma(P))) = \ldots \triangleq \uparrow\mathrm{succ}^{\sharp}(P)$$

$$\begin{aligned}
&\alpha(\uparrow\mathrm{succ}(\gamma(\{E\})))\\
&= \alpha(\uparrow\mathrm{succ}(\{n \mid \mathrm{even}(n)\}))\\
&= \alpha(\{\mathrm{succ}(n) \mid \mathrm{even}(n)\})
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&\alpha(\uparrow\mathrm{succ}(\gamma(\{E\})))\\
&= \alpha(\uparrow\mathrm{succ}(\{n \mid \mathrm{even}(n)\}))\\
&= \alpha(\{\mathrm{succ}(n) \mid \mathrm{even}(n)\})\\
&= \alpha(\{n \mid \mathrm{odd}(n)\})
\end{aligned}$$

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$$\mathrm{calc} : \alpha(\uparrow\mathrm{succ}(\gamma(P))) = \ldots \triangleq \uparrow\mathrm{succ}^{\sharp}(P)$$

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&= \{0\}
\end{aligned}$$

Abstract Interpretation

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&\alpha(\uparrow\mathrm{succ}(\gamma(\{E\})))\\
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&= \alpha(\{\mathrm{succ}(n) \mid \mathrm{even}(n)\})\\
&= \alpha(\{n \mid \mathrm{odd}(n)\})\\
&= \{0\}\\
&\triangleq \uparrow\mathrm{succ}^\sharp(\{E\})
\end{aligned}$$

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$$\mathrm{calc} : \alpha(\uparrow\mathrm{succ}(\gamma(P))) = \ldots \triangleq \uparrow\mathrm{succ}^\sharp(P)$$

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Abstract Interpretation

$$\wp(\mathbb{P}) \quad \mapsto \quad \ldots \quad \mapsto \quad \wp(\mathbb{P})$$

```
calc : α(↑succ(γ(P))) = … ≜ ↑succ♯(P)
```

```
↑succ♯({E}) ≜ {0}
```

Abstract Interpretation

$$\wp(\mathbb{P}) \quad \mapsto \quad \ldots \quad \mapsto \quad \wp(\mathbb{P})$$

$$\mathtt{calc} : \alpha(\uparrow \mathtt{succ}(\gamma(P))) = \ldots \triangleq \uparrow \mathtt{succ}^\sharp(P)$$

$$\uparrow \mathtt{succ}^\sharp(\{E\}) \triangleq \{0\}$$

$$\wp(\mathbb{P}) \coloneqq (\mathbb{P} \to \mathtt{prop}) \approx \text{“specification”}$$
$$\wp(\mathbb{P}) \coloneqq [\mathbb{P}] \approx \text{“constructed”}$$

Abstract Interpretation

$$\wp(\mathbb{P}) \quad \mapsto \quad \ldots \quad \mapsto \quad \wp(\mathbb{P})$$

$$\mathtt{calc} : \alpha(\uparrow \mathtt{succ}(\gamma(P))) = \ldots \triangleq \uparrow \mathtt{succ}^\sharp(P)$$

$$\uparrow \mathtt{succ}^\sharp(\{E\}) \triangleq \{O\}$$

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no extraction

Abstract Interpretation

$\wp(\mathbb{N}) \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} \wp(\mathbb{P}) \quad \mapsto \quad \ldots \quad \mapsto \quad \wp(\mathbb{P})$

$\mathtt{calc} : \alpha(\uparrow \mathtt{succ}(\gamma(P))) = \ldots \triangleq \uparrow \mathtt{succ}^\sharp(P)$

$\uparrow \mathtt{succ}^\sharp(\{E\}) \triangleq \{O\}$

$\wp(\mathbb{P}) \coloneqq (\mathbb{P} \to \mathtt{prop}) \approx$ “specification”

$\wp(\mathbb{P}) \coloneqq [\mathbb{P}] \approx$ “constructed”

Abstract Interpretation

$\wp(\mathbb{N}) \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} \wp(\mathbb{P}) \quad \mapsto \quad \ldots \quad \mapsto \quad \wp(\mathbb{P})$

```
calc : α(↑succ(γ(P))) = … ≜ ↑succ♯(P)
```

```
↑succ♯({E}) ≜ {0}
```

```
℘(ℙ) ≔ (ℙ → prop) ≈ “specification”
℘(ℙ) ≔ [ℙ]        ≈ “constructed”
```

no extraction

Four Stories

Direct Verification	✗ `calculate`
Abstract Interpretation	✗ `mechanize`
Kleisli GCs	✓ `calculate` ½ `mechanize`
Constructive GCs	✓ `calculate` ✓ `mechanize`

Kleisli GCs

$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P}) \qquad \uparrow\mathsf{succ} : \wp(\mathbb{N}) \to \wp(\mathbb{N})$$
$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N}) \qquad \uparrow\mathsf{succ}^\sharp : \wp(\mathbb{P}) \to \wp(\mathbb{P})$$

Kleisli GCs

$$\alpha : \mathbb{N} \to \mathbb{P} \to \mathtt{prop} \qquad \uparrow\mathtt{succ} : \mathbb{N} \to \mathbb{N} \to \mathtt{prop}$$
$$\gamma : \mathbb{P} \to \mathbb{N} \to \mathtt{prop} \qquad \uparrow\mathtt{succ}^\sharp : \mathbb{P} \to \mathbb{P} \to \mathtt{prop}$$

$$\wp(X) \coloneqq X \to \mathtt{prop}$$

Kleisli GCs

$$\alpha : \mathbb{N} \to \wp(\mathbb{P}) \qquad \uparrow\mathsf{succ} : \mathbb{N} \to \wp(\mathbb{N})$$

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Kleisli GCs

$$\alpha : \mathbb{N} \to \wp(\mathbb{P}) \qquad \uparrow\mathsf{succ} : \mathbb{N} \to \wp(\mathbb{N})$$
$$\gamma : \mathbb{P} \to \wp(\mathbb{N}) \qquad \uparrow\mathsf{succ}^\sharp : \mathbb{P} \to \wp(\mathbb{P})$$

$$N \subseteq \gamma(\alpha(N)) \quad \wedge \quad \alpha(\gamma(P)) \subseteq P$$
$$\Longleftrightarrow$$
$$N \subseteq \gamma(P) \iff \alpha(N) \subseteq P$$

Kleisli GCs

$$\alpha : \mathbb{N} \to \wp(\mathbb{P}) \qquad \uparrow\mathsf{succ} : \mathbb{N} \to \wp(\mathbb{N})$$
$$\gamma : \mathbb{P} \to \wp(\mathbb{N}) \qquad \uparrow\mathsf{succ}^\sharp : \mathbb{P} \to \wp(\mathbb{P})$$

$$\mathsf{id} \sqsubseteq \gamma \circ \alpha \quad \wedge \quad \alpha \circ \gamma \sqsubseteq \mathsf{id}$$
$$\Longleftrightarrow$$
$$\mathsf{id}(N) \subseteq \gamma(P) \iff \alpha(N) \subseteq \mathsf{id}(P)$$

Kleisli GCs

$$\alpha : \mathbb{N} \to \wp(\mathbb{P}) \qquad \uparrow\mathsf{succ} : \mathbb{N} \to \wp(\mathbb{N})$$
$$\gamma : \mathbb{P} \to \wp(\mathbb{N}) \qquad \uparrow\mathsf{succ}^\sharp : \mathbb{P} \to \wp(\mathbb{P})$$

$$\mathsf{ret} \sqsubseteq \gamma \circledast \alpha \quad \wedge \quad \alpha \circledast \gamma \sqsubseteq \mathsf{ret}$$
$$\Longleftrightarrow$$
$$\mathsf{ret}(n) \subseteq \gamma(p) \iff \alpha(n) \subseteq \mathsf{ret}(p)$$

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$$\alpha : \mathbb{N} \to \wp(\mathbb{P}) \qquad \uparrow\mathsf{succ} : \mathbb{N} \to \wp(\mathbb{N})$$
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$$\Longleftrightarrow$$
$$\mathsf{ret}(n) \subseteq \gamma(p) \iff \alpha(n) \subseteq \mathsf{ret}(p)$$

$$\mathsf{sound} : \alpha \circ \uparrow\mathsf{succ} \circ \gamma \sqsubseteq \uparrow\mathsf{succ}^\sharp$$

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Four Stories

Direct Verification	✗ `calculate`
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Constructive GCs	✓ `calculate` ✓ `mechanize`

Constructive GCs

$$\alpha : \mathbb{N} \to \wp(\mathbb{P}) \qquad \gamma : \mathbb{P} \to \wp(\mathbb{N})$$

$$\wedge$$

$$\mathtt{ret} \sqsubseteq \gamma \circledast \alpha \qquad \alpha \circledast \gamma \sqsubseteq \mathtt{ret}$$

Constructive GCs

$$\alpha : \mathbb{N} \to \wp(\mathbb{P}) \qquad \gamma : \mathbb{P} \to \wp(\mathbb{N}) \qquad \wedge \qquad \mathtt{ret} \sqsubseteq \gamma \circledast \alpha \qquad \alpha \circledast \gamma \sqsubseteq \mathtt{ret}$$

$$\exists(\eta : \mathbb{N} \to \mathbb{P}).\ \alpha(x) = \mathtt{ret}(\eta(x))$$

Constructive GCs

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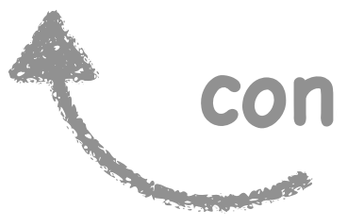

Constructive GCs

$$\alpha : \mathbb{N} \to \wp(\mathbb{P}) \qquad \gamma : \mathbb{P} \to \wp(\mathbb{N}) \qquad \wedge \qquad \mathtt{ret} \sqsubseteq \gamma \circledast \alpha \qquad \alpha \circledast \gamma \sqsubseteq \mathtt{ret}$$

$$\exists(\eta : \mathbb{N} \to \mathbb{P}).\ \alpha(x) = \mathtt{ret}(\eta(x))$$

The “specification effect” in α is always benign

Constructive GCs

$$\alpha : \mathbb{N} \to \wp(\mathbb{P})$$
$$\gamma : \mathbb{P} \to \wp(\mathbb{N})$$

Constructive GCs

$$\eta : \mathbb{N} \to \mathbb{P}$$
$$\mu : \mathbb{P} \to \wp(\mathbb{N})$$

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Constructive GCs

$$\eta : \mathbb{N} \to \mathbb{P}$$
$$\mu : \mathbb{P} \to \wp(\mathbb{N})$$

$$\mathtt{ret} \sqsubseteq \mu \circledast \uparrow(\eta) \quad \wedge \quad \uparrow(\eta) \circledast \gamma \sqsubseteq \mathtt{ret}$$

$$\mathtt{ret}(n) \subseteq \mu(p) \iff \uparrow(\eta)(n) \subseteq \mathtt{ret}(p)$$

Constructive GCs

$$\texttt{parity} : \mathbb{N} \to \mathbb{P}$$
$$[\![\_]\!] : \mathbb{P} \to \wp(\mathbb{N})$$

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$$\texttt{parity} : \mathbb{N} \to \mathbb{P}$$
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$$n \in [\![\texttt{parity}(n)]\!]$$

Constructive GCs

$$\texttt{parity} : \mathbb{N} \to \mathbb{P}$$
$$[\![\_]\!] : \mathbb{P} \to \wp(\mathbb{N})$$

$$n \in [\![p]\!] \Rightarrow \texttt{parity}(n) \sqsubseteq p$$

Constructive GCs

$$\texttt{parity} : \mathbb{N} \to \mathbb{P}$$
$$[\![\_]\!] : \mathbb{P} \to \wp(\mathbb{N})$$

$$n \in [\![p]\!] \iff \texttt{parity}(n) \sqsubseteq p$$

Constructive GCs

$$\texttt{parity} : \mathbb{N} \to \mathbb{P}$$
$$[\![\_]\!] : \mathbb{P} \to \wp(\mathbb{N})$$

$$\texttt{sound} : n \in [\![p]\!] \implies \texttt{succ}(n) \in [\![\texttt{succ}^\sharp(p)]\!]$$

Results

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- Metatheory complete w.r.t. subset of classical GC

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Results

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- Sound, optimal and *computable* AIs by construction
- Metatheory and case studies all verified in Agda

Constructive GCs

$$\eta : \mathbb{N} \to \mathbb{P}$$
$$\mu : \mathbb{P} \to \wp(\mathbb{N})$$

$$n \in \mu(p) \iff \eta(n) \sqsubseteq p$$

✓ calculate
✓ mechanize