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AN EIGENVALUE CRITERION FOR THE STUDY OF THE HAMILTONIAN ACTION'S EXTREMALITY

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Introduction

The question of the characterization of the extremum in Hamilton's Principle, although admittedly not so vital to the more utilitarian aspects of Analytical Dynamics, has nevertheless occupied the attention of several authors, among which are some of Mechanics' best (see e.g., [1, Ch. 9] [2, Arts. 646-654], [3, pp. 649-665], and [4]). The reasons for this have not always been mere mathematical curiosity and/or desire toward conceptual clarification and rigor. Aside from the recent revival of interest in the study of the structure of Jacobi's "équation aux variations" (due to its relation with the "Inverse" Problem of Mechanics, see [5], and references cited therein),the answer to the extremum question seems to furnish an Action based criterion for the study of certain types of kinetic stability (see e.g., [6, pp. 416-439], [7, Ch. VIII], and [8]); this would constitute the kinetic counterpart of the well-known (minimum) Potential Energy Theorem of static stability.

For mechanical systems whose Action functional is a single integral involving ordinary derivatives, i.e., for discrete systems, the extremality (minimality) problem has been answered by the classical Jacobi's Conjugate Points Criterion (readable accounts of the relevant mathematical theory can be found for e.g., in [9, Ch. 5], or [10, Ch. 6.2]). In applying this criterion various authors have used either the explicit solution of Jacobi's second variation or linear perturbation equation, or have invoked qualitative results from the Oscillation/Comparison (Sturmian) theory of differential equations (see [14]). The former is best suited when Jacobi's equation has known constant coefficients, whereas the latter is to be preferred in the case of variable unknown coefficients when a qualitative result can still be obtained.

The conjugate points test for the study of the Action's minimum, with or without the Oscillation/Comparison methodology, is one way to proceed. A second method is what is known in the variational calculus literature as the Accessory Jacobi's Eigenvalue Criterion. It consists of the following: The Action's second variation, a homogeneous quadratic functional in the admissible virtual displacements, is expressed in its "principal axes", or "diagonalized" form. This transformation results in an "accessory" linear variational equation with a new parameter containing term: the standard Jacobi's conjugate points equation (which is basically an initial-value problem!) has now been replaced by Jacobi's accessory equation, a linear eigenvalue problem with homogeneous boundary conditions. Now, if the smallest eigenvalue of this accessory equation is positive, then the original second variation is positive-definite, and therefore (excluding pathological cases, such as those that may appear in multiple integral extrema), the original functional is a (weak) minimum; the positivity of the smallest eigenvalue can be checked by several methods, the Oscillation/Comparison being one of them; we take the opportunity to draw attention to this simple and powerful method since its conscious and explicit understanding seems to be lacking among engineers.

This communication is utilizing this Eigenvalue Criterion first, and subsequently the Oscillation/Comparison methodology to bracket the lowest eigenvalue of the accessory Jacobi's equation; the results of [14] are thus rediscovered.

The problem is interesting not only physically, as explained earlier, but also from the analytical viewpoint since its methodology can be profitably extended to other areas of mechanics, such as that of elastostatic stability when the prebuckling state cannot be linearized, i.e., when the influence of displacements, strains, and rotations preceding the loss of stability cannot be ignored). For the Eigenvalue Criterion theory, see for e.g., [10, Ch. 6]; for the Oscillation/Comparison theory see for e.g., [10, Ch. 8], and [15]. The real power and advantages of the Eigenvalue Criterion, over the Conjugate Points one, appear in the treatment of multiple integrals; a rather complete exposition (for the analytically minded engineer) appears in [16].

Theory

Consider a system of mass m, generalized coordinate $q = q(t)$, and force $Q = Q(q)$ Then, according to Hamilton's variational principle, its equation of motion

$$m\ddot{q} = Q(q), \qquad \left((\ldots)^{\cdot} = d(\ldots)/dt\right) \tag{1}$$

is obtained from

$$\delta A(q, \delta q) = 0 , \tag{2}$$

where

$$A(q) = \int_{t_0}^{t_1} L(q,\dot{q})\, dt = \text{action functional (or "principal function")} , \tag{3}$$

$$L(q,\dot{q}) = T(\dot{q}) - V(q) = \text{Lagrangean} , \tag{4}$$

$$T(\dot{q}) = \tfrac{1}{2} m\dot{q}^2 = \text{kinetic energy} , \tag{5}$$

$$V(q) = V(q_0) + \int_{q}^{q_0} Q(x)\,dx = \text{potential energy}, \tag{6}$$

and the (weak and contemporaneous) variations $\delta q(t)$ satisfy

$$\delta q(t_0) = \delta q(t_1) = 0 , \tag{7}$$

for arbitrary time limits t_0, t_1 $(t_0 > t_1)$.

As is well known, the sufficient conditions for the extremum of $A(q)$ come from the study of $\delta[\delta A(q,\delta q)] = \delta^2 A(q,\delta q)$, i.e., from the second variation of $A(q)$. This latter is found from the Taylor-like expansion

$$\Delta A = A(q + \delta q) - A(q) \qquad (= \text{total variation})$$

$$= \delta A(q,\delta q) + \tfrac{1}{2}\delta^2 A(q,\delta q) + O_3(\delta q)$$

$$\simeq \tfrac{1}{2}\,\delta^2 A(q,\delta q) \qquad \text{(due to (2)}\ , \tag{8}$$

where

$$\delta^2 A(q,\delta q) = \int_{t_0}^{t_1} (\delta^2 L)\, dt\ ,$$

$$\delta^2 L = \left(\frac{\partial^2 L}{\partial \dot{q}^2}\right)(\delta\dot{q})^2 + 2\left(\frac{\partial^2 L}{\partial\dot{q}\partial q}\right)(\delta\dot{q})\ (\delta q) + \left(\frac{\partial^2 L}{\partial q^2}\right)(\delta q)^2\ . \tag{8a}$$

In our case this gives

$$\delta^2 A\ (q,\delta q) = \int_{t_0}^{t_1} [m(\delta\dot{q})^2 - (d^2V/dq^2)(\delta q)^2]\ dt$$

$$= \int_{t_0}^{t_1} [m(\delta\dot{q})^2 + (dQ/dq)\ (\delta q)^2]\ dt\ , \tag{9}$$

and all q-derivatives are evaluated at the solution(-s) of (1), with (7), i.e., along the actual trajectory(-ies). Now, according to standard variational calculus theory (see for e.g., [9, Ch. 5], or [10, pp. 369-401]):

i) The extremum (minimum) of (3), for solutions of (1) satisfying (7), is determined by the (positive) sign-definiteness of $\delta^2 A(q,\delta q)$ i.e., if $\delta^2 A(q,\delta q) > 0$ for every $\delta q(t)$ (not identically zero), then $q(t)$ minimizes $A(q)$; for a maximum the inequality sign should be reversed ,

ii) If $\delta^2 A(q,\delta q)$ is sign-indefinite, then $A(q)$ has a minimax (saddle-point), i.e., there is no extremum,

iii) If $\delta^2 A(q,\delta q)$ is sign-semi-definite, i.e., it does not change its sign but may turn into zero, or if it is identically zero as a $\delta q(t)$ dependent functional, one has to resort to higher variations or other means; in this case $\delta^2 A$'s behavior cannot inform us about the existence/character of the extremum.

Let us now translate these results into direct conditions for the integrand of (3). Due to (7), and since here $\delta d(\ldots) = d\delta(\ldots)$, we can transform $\delta^2 A$ as follows:

$$\delta^2 A(q,\delta q) = [(m\delta\dot{q})\,\delta q]_{t_0}^{t_1} - \int_{t_0}^{t_1} (m\delta\ddot{q})\,\delta q dt - \int_{t_0}^{t_1} [(d^2V/dq^2)\;\delta q]\;\delta q\;dt$$

$$= - \int_{t_0}^{t_1} [m\;\delta\ddot{q} + (d^2V/dq^2)\;\delta q]\;\delta q\;dt$$

$$= - \int_{t_0}^{t_1} J(\delta q)\;\delta q\,dt \quad (\text{"Jacobi's" form}) \;, \tag{10}$$

where

$$J(\delta q) \equiv \left[\left(\frac{\partial}{\partial(\delta\dot{q})}\right)^{\cdot} - \left(\frac{\partial}{\partial(\delta q)}\right)\right]\left(\frac{1}{2}\,\delta^2 L\right)$$

$$= m\;\delta\ddot{q} + (d^2V/dq^2)\;\delta q\;. \tag{11}$$

Equation $J(\delta q) = 0$ (plus appropriate <u>initial</u> conditions, in general incompatible with (7)) is known as Jacobi's equation associated with the original functional (3). For a general Lagrangean $L(q,q,t)$ Jacobi's equation is

$$J(\delta q) = \left(\frac{\partial^2 L}{\partial\dot{q}^2}\right)\delta\ddot{q} + \left(\frac{\partial^2 L}{\partial\dot{q}^2}\right)^{\cdot}\delta\dot{q} + \left[\left(\frac{\partial^2 L}{\partial\dot{q}\partial q}\right)^{\cdot} - \left(\frac{\partial^2 L}{\partial q^2}\right)\right]\delta q = 0 \;;$$

the study of the roots of the solution of this <u>initial-value problem</u> is central to the method of conjugate points; this latter's significance to our extremum problem is described in [14]. Here, however, (11) will be used in a different way: expanding $\delta q(t)$ into a series with respect to the (complete) system of orthonormal eigenfunctions $\{\delta q_k(t)\}_{k=1}^{\infty}$ of the linear and, due to (7), <u>self-adjoint</u> "Jacobi's" operator $-J$

$$\delta q(t) = \sum_{k=1}^{\infty} C_k\,\delta q_k(t) \;,$$

where

$$C_k = \int_{t_0}^{t_1} (\delta q\,\delta q_k)\;dt \qquad (k = 1,\,2,\,\ldots)\;,$$

and

$$\int_{t_0}^{t_1} (\delta q_k\,\delta q_\ell)\;dt = \delta_{k\ell} \;\;(\,= \text{Kronecker's delta};\; \ell = 1,\,2,\,\ldots)\;, \tag{12}$$

and substituting it into (10) we find

$$\delta^2 A(q,\delta q) = -\int_{t_0}^{t_1} J\left(\sum_{k=1}^{\infty} C_k \delta q_k\right)\cdot\left(\sum_{\ell=1}^{\infty} C_\ell \delta q_\ell\right) dt$$

$$= -\int_{t_0}^{t_1} \left[\sum_{k=1}^{\infty} C_k J(\delta q_k)\right]\left(\sum_{\ell=1}^{\infty} C_\ell \delta q_\ell\right) dt$$

$$= \sum_{k,\ell=1}^{\infty} C_k C_\ell \left(\int_{t_0}^{t_1} \lambda_k \delta q_k \delta q_\ell dt\right)$$

$$= \sum_{k=1}^{\infty} \lambda_k C_k^2 , \tag{13}$$

where

$$-J(\delta q_k) = \lambda_k \cdot \delta q_k \qquad (\lambda_1 \le \lambda_2 \le \lambda_3 \ldots) ,$$

and

$$\left\{\lambda_k\right\}_{k=1}^{\infty} = \text{eigenvalues of } -J \text{, with (7).} \tag{14}$$

Since $\delta q(t)$ is, apart from (7) and appropriate smoothness requirements, arbitrary the coefficients C_k are arbitrary, and accordingly:

i) If all the eigenvalues λ_k are positive, then $\delta^2 A(q,\delta q)$ is positive-definite, and $A(q)$ has a minimum,

ii) If there is at least one negative eigenvalue, then $\delta^2 A(q,\delta q)$ is sign-indefinite, and therefore $A(q)$ has a minimax (saddle-point).

In terms of the least eigenvalue, the above conclusions can be recast in the following form:

i) If $\min(\lambda_k) \equiv \lambda_1 > 0$, then $A(q)$ = minimum ,

ii) If $\min(\lambda_k) \equiv \lambda_1 < 0$, then $A(q)$ = minimax (saddle-point) ;

If $\min(\lambda_k) \equiv \lambda_1 = 0$, then $\delta^2 A$ is <u>positive semi-definite</u>, i.e., one has to examine the <u>higher</u> variations of A.

Finally, from (10)-(14), and invoking the well-known Rayleigh's Principle methodology, one can easily deduce the following minimum characterization of λ_1 as:

$$\lambda_1 = \min\left(\frac{\delta^2 A(q,\delta q)}{\int_{t_0}^{t_1} (\delta q)^2\, dt}\right)$$

$$= \min\left(\frac{-\int_{t_0}^{t_1} J(\delta q)\,\delta q\, dt}{\int_{t_0}^{t_1} (\delta q\cdot\delta q)\, dt}\right)$$

$$= \min\left(\frac{-\int_{t_0}^{t_1} [m\delta\ddot{q} + (d^2V/dq^2)\delta q]\delta q\, dt}{\int_{t_0}^{t_1} (\delta q\cdot\delta q)\, dt}\right) \tag{15}$$

where the competing (admissible) functions $\delta q(t)$ satisfy (7).

Now, (15) clearly shows that the sign of λ_1 depends not only on the structure of the integrand of $\delta^2 A$, but also on the <u>length</u> of the integration interval $\Delta t = t_1 - t_0$, i.e., $\lambda_1 = \lambda_1(\Delta t)$. Since $q(t)$, and therefore $d^2V\,[q(t)]/dq^2$, are in general unknown, a statement concerning the sign of λ_1 can be made only by utilizing some qualitative (inequality) result that furnishes eigenvalue bounds in terms of bounds for the coefficients of the -J operator. From the general Sturm-Liouville problem eigenvalue Theorems (see e.g., [10, pp. 399-401], or [17, pp. 197-201]) one readily derives the following pertinent <u>Corollary</u>: the eigenvalues of the Extended (Jacobi's) Eigenvalue Problem

$$-\,J(\delta q) = -\,[m\delta\ddot{q} + (d^2V/dq^2)\,\delta q] = \lambda\,\delta q\ ,$$

or

$$\delta\ddot{q} + m^{-1}(d^2V/dq^2 + \lambda)\ \delta q = 0\ , \tag{16}$$

with $\delta q(t_0) = \delta q(t_1) = 0$, are bracketed by the following inequalities

$$\min\,(\Lambda_k) \le \lambda_k \le \max\,(\Lambda_k) \qquad (k = 1,2,3\ \ldots)\ , \tag{17}$$

where

$$\Lambda_k \equiv m\ (k\pi/\Delta t)^2 + (-d^2V/dq^2)\ , \tag{18}$$

and the max/min are to be sought in $[t_0\,, t_1] \equiv I$.

The left side of (17), (18), with $k = 1$, shows that:

i) If $(d^2V/dq^2) \le 0$ in I, then $\lambda_1 > 0$ for arbitrary Δt, and

ii) If $(d^2V/dq^2) > 0$ in I, and since $\min(-d^2V/dq^2) = -\max(d^2V/dq^2)$, then $\lambda_1 > 0$ for $\Delta t < \tau$,

where

$$\tau \equiv \pi\,[m/\max(d^2V/dq^2)]^{\frac{1}{2}} \quad ; \tag{19}$$

for $\Delta t > \tau$ the Action A(q) may have a saddle-point.

These are precisely the conditions for the minimum of A(q) reached in [14] via the oscillation/comparison method.

Similar reasoning utilizing the right side of (17) readily shows that for $\Delta t > \tau^*$, where

$$\tau^* \equiv \pi\,[m/\min(d^2V/dq^2)]^{\frac{1}{2}} \qquad (d^2V/dq^2 > 0 \text{ on } I), \tag{20}$$

$\lambda_1 < 0$.

In summary:

i) If $\Delta t < \tau$, then A is guaranteed to have a minimum,

ii) If $\Delta t > \tau^*$, then A is guaranteed to have a minimax (saddle-point), and

iii) If $\tau \le \Delta t \le \tau^*$, then our criteria become inconclusive (i.e., A may have minimum, or minimax, or sign-semi-definiteness) and the problem needs further investigation.

The reader is referred to [14] for additional insights to the problem.

N d.o.f. case - Maupertuis' Action

Consider the three-dimensional unconstrained motion of a unit mass particle in a potential field $V(q_1, q_2, q_3)$. Lagrange's and Jacobi's equations are

$$\ddot{q}_k + \partial V/\partial q_k = 0 \quad , \tag{21}$$

and

$$\delta\ddot{q}_k + f_{k\ell}(t)\,\delta q_\ell = 0 \quad , \tag{22}$$

respectively, where $f_{k\ell}(t) = f_{\ell k}(t) \equiv \partial^2 V[q(t)]/\partial q_\ell \partial q_k$ is evaluated at the solution of (21) (plus (7)-like boundary conditions), k, ℓ = 1,2,3, and the usual summation convention is applied; in general $f_{k\ell}$ might be asymmetric. This simple example illustrates the problems associated with the extension of the above methodology to the N d.o.f. case: equations (22) constitute a coupled variable coefficient system, and for such a case Sturmian eigenvalue theorems are presently unavailable. Uncoupling them, which is the main problem, i e., developing a variable coefficient "normal mode" theory, is possible only under certain special conditions on $f_{k\ell}$; at any rate, for this, as well as for the (related) more complex problem of uncoupling (21), one needs definitions and analytical tools that far exceed the scope of this note. Once the uncoupling has been effected the Sturmian results, like (17) and (18), can be applied intact to each new "normal coordinate's" equation.

Finally, the Sturmian methodology should be extended to the study of the extremum of the Maupertuisian Action (a special case of Hamilton's characteristic function)

$$\mathfrak{A} = \int_{t_0}^{t_1} (2T)\, dt \quad . \tag{23}$$

Here not only the functional is different, but also the coordinate variations are noncontemporaneous (and noncommutative!), and the varied path obeys the same energy conservation constraint as the actual one. As a result, one is led to a variable endpoints extremum problem and, instead of the Hamiltonian conjugate kinetic foci, to the concept of the Maupertuisian focal points; see e.g., [18, Ch. 29]. Relevant results will be hopefully presented in the near future.

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