The Division Algorithm

Theorem 1. *For any positive integers a and b there exists a unique pair of (q, r) of nonnegative integers such that*

$$b = aq + r, \; r < a$$

Example 1. *Prove that for all positive integers n, the fraction*

$$\frac{21n + 4}{14n + 3}$$

is irreducible.

Let $k = gcd(21n + 4, 14n + 3)$, then

$$\begin{aligned} 21n + 4 &\equiv 0 \pmod{k} \\ 14n + 3 &\equiv 0 \pmod{k} \end{aligned}$$

Multiplying the first equation by 2 an the second by 3 to match the coefficients of n

$$\begin{aligned} 42n + 8 &\equiv 0 \pmod{k} \\ 42n + 9 &\equiv 0 \pmod{k} \end{aligned}$$

$42n + 8$ and $42n + 9$ are consecutive integers and hence coprime, it then follows that $k = 1$.