# IB Physics Topic A4 Rigid Body Mechanics; HL

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# Table of Contents

# 1 Kinematic Equation — Rotational Equivalent

| Quantity | Linear | Angular |
|---|---|---|
| Displacement | $s$ | $\theta$ |
| Average velocity | $v = \frac{\Delta s}{\Delta t}$ | $\omega = \frac{\Delta \theta}{\Delta t}$ |
| Acceleration | $a = \frac{\Delta v}{\Delta t}$ | $\alpha = \frac{\Delta \omega}{\Delta t}$ |
| | $v = u + at$ | $\omega = \omega_0 + \alpha t$ |
| | $s = ut + \frac{1}{2}at^2$ | $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ |
| | $v^2 = u^2 + 2as$ | $\omega^2 = \omega_0^2 + 2\alpha\theta$ |
| | $s = \frac{1}{2}(u+v)t$ | $\theta = \frac{1}{2}(\omega_0 + \omega)t$ |
| Kinetic energy | $E_K = \frac{1}{2}mv^2$ | $E_K = \frac{1}{2}I\omega^2$ |

The angular speed and the linear speed are related by the equation

$$v = \omega r$$

where $r$ is the radius of the circular trajectory. Similarly,

$$a = \alpha r$$

# 2 Torque

Torque is defined as the measure of the force that leads to the rotation of an object about its axis. It is the rotational equivalent of force. Put formally, we define the torque $\tau$ of the force $F$ about the rotational axis to be the product of the force and the perpendicular distance between the line of action of the force and the axis. Mathematically,

$$\tau = Fr\sin\theta$$

where $r$ is the distance between the axis and the line of action of the force, $F$ is the force, and $\theta$ is the angle between the force and the line of action of the force. The unit of torque is the newton-meter (N·m).

# 3 Moment of Inertia

- Translational equilibrium: The net force acting on the object is zero. The center of mass of the body remains at rest or moves in a straight line at constant speed.
- Rotational equilibrium: The net torque acting on the object is zero.

The moment of inertia of a rigid body is a measure of the body's resistance to rotational motion about a given axis. It is the rotational equivalent of mass. The moment of inertia is given as

$$I = \sum m_i r_i^2$$

where $m_i$ is the mass of the $i$th particle and $r_i$ is the distance of the $i$th particle from the axis of rotation. The unit of moment of inertia is the kilogram-meter squared (kg·m$^2$).

These so-called particles refer to the small individual masses that make up a rigid body. These particles are conceptualized as point masses, each having a specific mass $(m_i)$ and a defined position relative to the axis of rotation.

To calculate the moment of inertia $I$, the rigid body is thought of as being composed of these discrete particles. Each particle's contribution to the moment of inertia is determined by its mass $m_i$ and the square of its perpendicular distance $r_i$ from the axis of rotation.

In exam questions, you will often be given the formula of moment of inertia for the object under study, and it is in the form of $I = kmr^2$, where $k$ is some given constant.

When comparing the moments of inertia of two rigid bodies, the one whose mass is distributed closer to the axis of rotation will have a smaller moment of inertia.

# 4 Newton's Laws — Rotational Equivalent

1. Newton's first law: An object moves at a constant angular velocity unless acted upon by a net external torque.

2. Newton's second law: The net torque acting on an object is equal to the product of the moment of inertia and the angular acceleration. Mathematically,

$$\tau = I\alpha$$

   where $\tau$ is the net torque, $I$ is the moment of inertia, and $\alpha$ is the angular acceleration.

3. Newton's third law: When object A applies a torque to object B, then object B will apply an equal and opposite torque to object A.

# 5 Angular Momentum

The angular momentum is denoted by $L$ and is defined as follows:

$$L = I\omega$$

where $I$ is the moment of inertia and $\omega$ is the angular velocity. The unit of angular momentum is the kilogram-meter squared per second (kg·m$^2$/s).

The **conservation of momentum** also applies to rotational motion:

> The total angular momentum of a system remains constant provided no external torque acts on the system.

Similar to how we formulated $E_K = \dfrac{p^2}{2m}$ in linear motion, we can also express the kinetic energy in rotational motion as

$$E_K = \frac{L^2}{2I}$$

## 5.1 Angular Impulse

This is the change in angular momentum of an object. It is given by

$$\Delta L = \tau \Delta t = \Delta(I\omega)$$

which is analogous to

$$\Delta p = \Delta(Ft) = \Delta(mv)$$

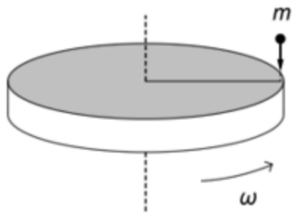
where $\tau$ is the torque, $\Delta t$ is the time interval, $I$ is the moment of inertia, and $\omega$ is the (change in) angular velocity.

Again, recall the definition of force as $F = \dfrac{\mathrm{d}p}{\mathrm{d}t} = \dfrac{\mathrm{d}(mv)}{\mathrm{d}t} = m\dfrac{\mathrm{d}v}{\mathrm{d}t} + v\dfrac{\mathrm{d}m}{\mathrm{d}t}$, we can do the same for torque:

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}(I\omega)}{\mathrm{d}t} = I\frac{\mathrm{d}\omega}{\mathrm{d}t} + \omega\frac{\mathrm{d}I}{\mathrm{d}t}$$

## 5.2 Conservation of Angular Momentum

A turntable of mass $M$ and radius $R$ spins freely about the vertical axis at an initial angular velocity $\omega$. The moment of inertia of the turntable about the axis of rotation is $\frac{1}{2}MR^2$. A small body of mass $m$ is dropped close to the edge of the turntable with a negligible initial velocity.




The body comes to rest relative to the turntable. What is the final angular velocity of the turntable?

Figure 1: Conservation of angular momentum

The initial angular momentum is given by

$$L_0 = \frac{1}{2}MR^2\omega$$

The final angular momentum is given by

$$L' = \omega'\left(\frac{1}{2}MR^2 + mR^2\right)$$

This is because the moment of inertia of the point mass $m$ is simply $mR^2$.

By the conservation of angular momentum, we have

$$\frac{1}{2}MR^2\omega = \omega'\left(\frac{1}{2}MR^2 + mR^2\right)$$

$$M\omega = \omega'(M + 2m)$$

$$\omega' = \frac{M\omega}{M + 2m}$$

# 6 Rolling and Sliding

Rolling and sliding are two distinct motions:

- Rolling: The object rotates about its axis along the surface.
- Sliding: The object moves along a surface without rotating. When an object moves on a perfectly frictionless surface, it cannot roll and must slide.

When there is friction between surface and object, the point of contact between the two is instantaneously at rest; this implies that the coefficient of static friction $\mu_s$ must be used in any calculation.

## 6.1 Rolling without Slipping

When an object rolls without slipping, the point of contact between the object and the surface is instantaneously at rest. Another way to think about this is that the center of mass of the body has moved forward a distance of $2\pi r$ in a time equal to the period of revolution $T$, where $r$ is the distance from the center of mass to the point of contact. This is much like unfolding the circumference of a circle to form a straight line.

Let us now consider a rotating wheel: Its top point has a combined velocity of $v + \omega r$, while the bottom point has a combined velocity of $v - \omega r$. The top point has a greater velocity than the bottom point, and this difference in velocity is what causes the wheel to rotate. The velocity of the center of mass is $v$, and the angular velocity is $\omega$.

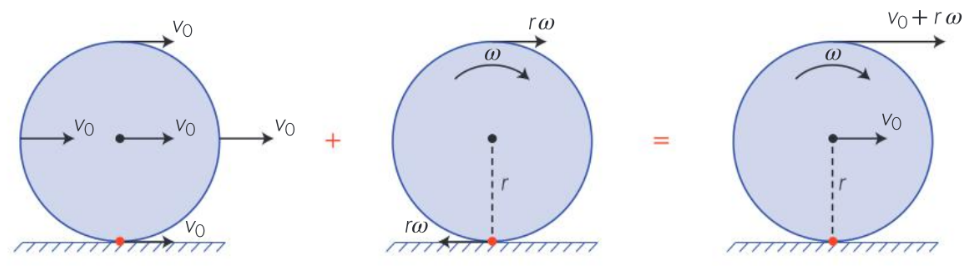



Figure 2: Rolling without slipping

However, based on the assumption that the object does not slip, $\omega r$ must be equal to the tangential velocity $v$. Hence, the velocity at the top is $2v$ and the velocity at the bottom is zero (this matches the assumption of no slipping). The acceleration of the center of mass is $a = \alpha r$.

#### 6.1.1 Energy

In this case, while the object is rotating, it is also moving forward from a translational perspective. Then, the total kinetic energy is the sum of both the linear and the rotational kinetic energies:

$$E_K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

where $m$ is the mass of the object, $v$ is the linear velocity, $I$ is the moment of inertia, and $\omega$ is the angular velocity. If the object is rolling down a slope of height $\Delta h$, then, we can say that $(-)\Delta GPE = (+)\Delta KE$

$$mg\Delta h = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Assuming that the moment of inertia is given as $I = kmr^2$, then we can derive a simplified form:

$$\begin{aligned} g\Delta h &= \frac{1}{2}v^2 + \frac{1}{2}kr^2\omega^2 \\ &= \frac{1}{2}v^2(k+1) \end{aligned}$$

# 7 Exam Questions

## 7.1 Comparing Rotational KE

A ring of mass $M$ and radius $R$ is accelerated from rest by a constant torque of $\tau$. The moment of inertia of the ring is $MR^2$. A solid disc of the same mass and radius as the ring is accelerated by the same torque. Compare, without calculation:

(a) The angular impulse delivered to the disc and to the ring during the first 5.0s.

- We know that

$$\Delta L = \tau \Delta t$$

- Since both are accelerated by the same torque in the same time, the angular impulse delivered to both the disc and the ring are the same.

(b) the final kinetic energy of the disc and the ring.

- We recall that the kinetic energy gained through an angular impulse is given by

$$\Delta E_K = \frac{L^2}{2I}$$

- From the previous part, we deduced that the angular impulse is the same, so all that is left is to compare the moment of inertia.
- The moment of inertia of the disk is lower because the mass is distributed closer to the axis of rotation. This means that the disk will gain a higher level of KE than the ring.

## 7.2 Planetary Rotation

A planet orbits around a star in an elliptical orbit as shown below.

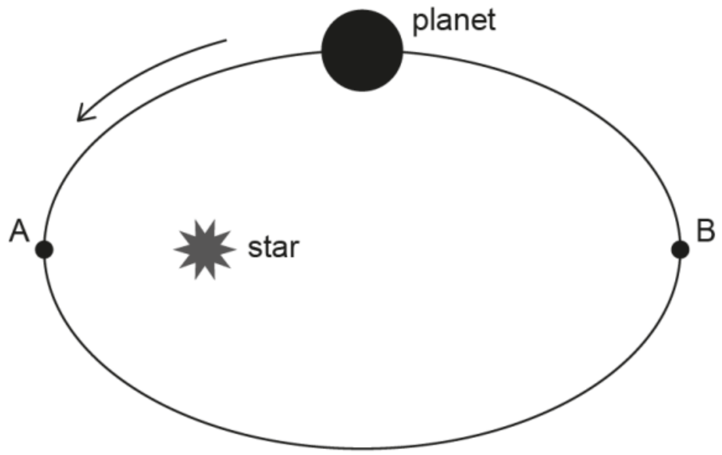



Figure 3: Diagram

At point A the planet is closest to the star and at point B it is furthest from the star. As the planet orbits the star it has a moment of inertia $I = mr^2$ where $m$ is the mass of the planet.

(a) State what represents in this situation.

- The distance between the centre of planet and the centre of the star.

(b) Show, using conservation of angular momentum, that the linear speed of the planet is greater at A than at B.

$$L_x = L_y$$

$$mv_A r_A = mv_B r_B$$

$$\frac{v_A}{v_B} = \frac{r_B}{r_A}$$

Since $r_B > r_A$, we can conclude that $v_A > v_B$.

(c) Suggest why, in this situation, the magnitude of the linear momentum of the planet is not conserved whereas the magnitude of its angular momentum is conserved.

- For the momentum of a system to be conserved, it is required that no external forces act on the system. In this case, the planet is under the influence of the star's gravitational force, which is an external force. Hence, the linear momentum is not conserved.
- In contrast, gravity acts inwards and so does not exert a torque about the center of the planet. This means that the angular momentum is conserved.

## 7.3 Misc #1

N.b. this question covers basically everything on the calculation side, in my opinion.

A uniform cylinder, of mass $M$ and length $L$, has a moment of inertia of $\frac{1}{12}ML^2$ when rotated about an axis through its centre.

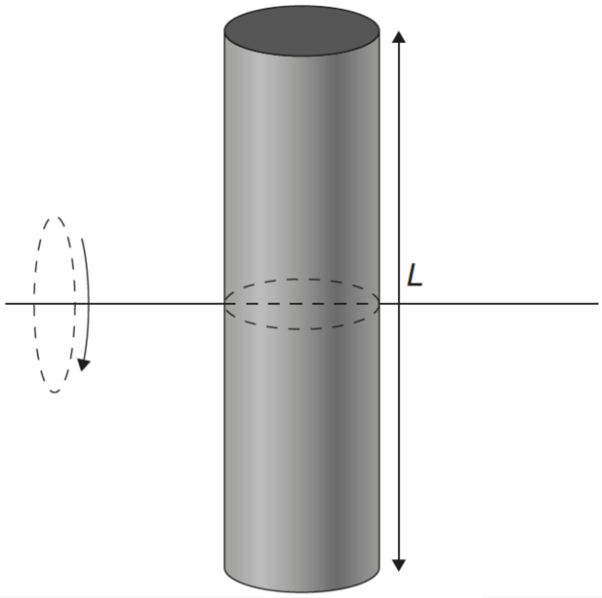



Figure 4: Diagram

(a) (i) State the condition for rotational equilibrium.

- The net torque acting on the object is zero.

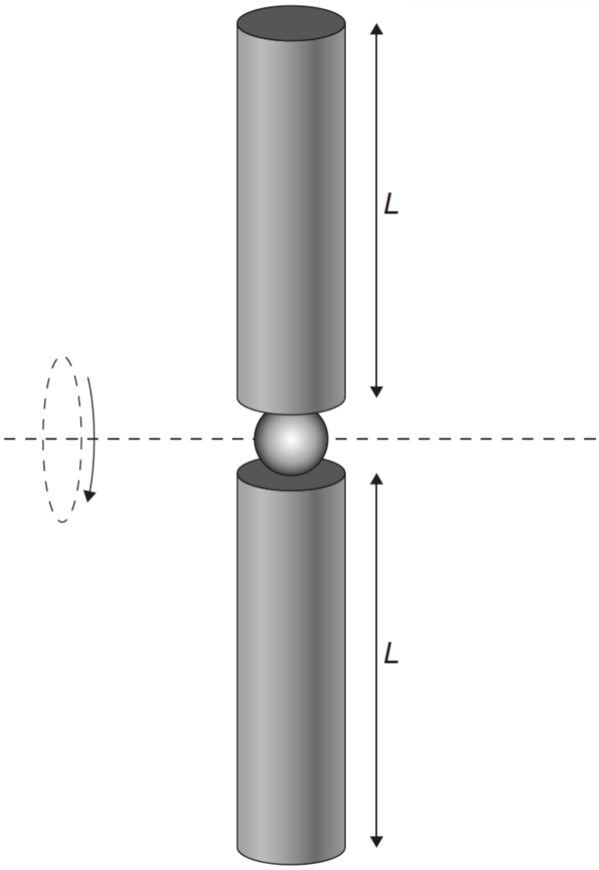



Figure 5: Diagram

(ii) Two identical cylinders, each of mass $M$ and length $L$, are connected end to end. Show that the moment of inertia when these cylinders are rotated about their combined centre is $\frac{2}{3}ML^2$.

$$\Sigma M = 2M \quad \text{and} \quad \Sigma L = 2L$$
$$I = \frac{1}{12}(2M)(2L)^2 = \frac{2}{3}ML^2$$

(b) A two-blade propeller can be modelled using the two-cylinder arrangement in (a)(iii).

The following data for the two-blade propeller are available:

- Length of each blade: 0.60 m

- Mass of each blade: 2.2 kg

Show that the moment of inertia of the two-blade propeller is about $0.5\,\mathrm{kg\,m^2}$.

$$\begin{aligned} I &= \frac{2}{3}ML^2 \\ &= \frac{2}{3}(2.2)(0.6)^2 \\ &= 0.528 \\ &\approx 0.5\,\mathrm{kg\,m^2} \end{aligned}$$

(c) The two-blade propeller is initially at rest. When a constant torque of 140 N m acts on the two-blade propeller it reaches an angular speed of $750\,\mathrm{rad\,s^{-1}}$. Ignore any frictional torque.

(i) Calculate the time taken for the two-blade propeller to reach the angular speed of $750\,\mathrm{rad\,s^{-1}}$.

$$\begin{aligned} \tau &= \frac{I(\omega - \omega_0)}{t} \\ t &= \frac{I(\omega - \omega_0)}{\tau} \\ &= \frac{0.528(750 - 0)}{140} \\ &\approx 2.8\text{ s} \end{aligned}$$

(ii) Calculate the number of revolutions of the two-blade propeller to reach the angular speed of $750\,\mathrm{rad\,s^{-1}}$.

$$
\begin{aligned}
\theta &= \frac{1}{2}(\omega + \omega_0)t \\
&= \frac{1}{2}(750 + 0)(2.8) \\
&= 1065 \text{ rad} \\
&= \frac{1065}{2\pi} \\
&\approx 167 \text{ rev}
\end{aligned}
$$

(iii) The propeller is brought to rest in 5.0 s. Determine the average value of the external torque applied.

$$
\begin{aligned}
\tau &= \frac{I(\omega - \omega_0)}{t} \\
&= \frac{0.528(0 - 750)}{5.0} \\
&= -79.2 \text{ N m} \\
&\approx 80 \text{ N m}
\end{aligned}
$$

## 7.4 Momentum Conservation #1

A net torque acts on a horizontal disk of mass 0.20 kg and radius 0.40 m that is initially at rest. The disk begins to rotate. The graph shows the variation with time $t$ of the angular speed $\omega$ of the disk.

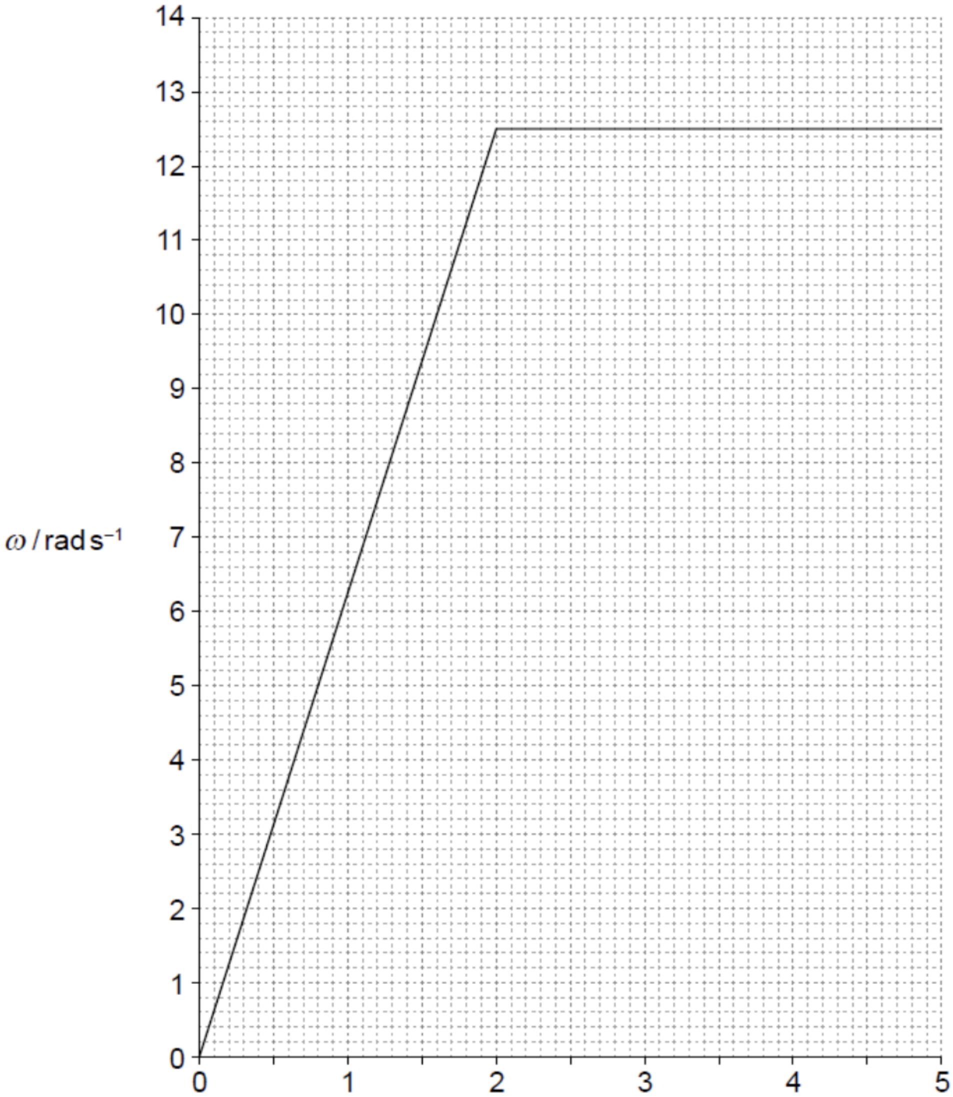



Figure 6: Graph

The moment of inertia of a disk of mass $M$ and radius $R$ about a vertical axis through its centre is $\frac{1}{2}MR^2$.

(a) Show that the angular acceleration of the disk is about $6\,\mathrm{rad\,s^{-1}}$.

$$\Delta\omega = 12.5$$
$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{12.5}{2}$$
$$= 6.25 \text{ rad/s}^2$$

(b) While the disk is rotating at its final constant angular speed, a small object of mass 0.10 kg falls on the disk and sticks to the edge of the disk.

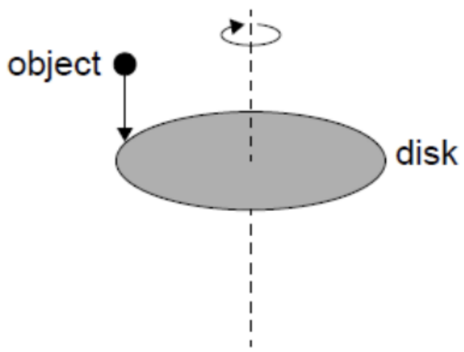



Figure 7: Diagram

(i) Calculate the new angular speed of the disk.

$$\Sigma p_{\text{before}} = \Sigma p_{\text{after}}$$
$$\omega I = (I + I_{\text{object}})\omega'$$
$$\omega' = \frac{\omega I}{I + I_{\text{object}}}$$
$$I_{\text{disk}} = \frac{1}{2}(0.2)R^2 = 0.1R^2$$
$$I_{\text{object}} = mR^2 = 0.1R^2$$
$$\implies I_{\text{object}} = I_{\text{disk}}$$
$$\omega' = \frac{12.5I}{2I}$$
$$= \frac{12.5}{2} \approx 6.3\,\mathrm{rad\,s^{-1}}$$

(ii) Determine the fraction of the total energy of the disk that was lost.

$$E_0 = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)(12.5)^2 \approx 1.25 \text{ J}$$

$$E' = \frac{1}{2}(I + I_{\text{object}})\omega'^2 = \frac{1}{2}\left(MR^2\right)(6.25)^2 \approx 0.625 \text{ J}$$

$$\Delta E = E_0 - E' = 1.25 - 0.625 \approx 0.625 \text{ J}$$

$$\text{fractional loss} = \frac{\Delta E}{E_0} = \frac{0.625}{1.25} \approx 0.5 \text{ or } 50\%$$

## 7.5 Rolling – Energy Conservation

A solid sphere of radius $r$ and mass $m$ is released from rest and rolls down a slope, without slipping. The vertical height of the slope is $h$. The moment of inertia $I$ of this sphere about an axis through its centre is $\frac{2}{5}mr^2$.

Find an expression for the linear velocity of the sphere as it leaves the slope.

$$\begin{aligned}
\Delta E_P &= \Delta E_{\text{K, linear}} + \Delta E_{\text{K, rotational}} \\
mgh &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \\
&= \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{v}{r}\right)^2 \\
2gh &= v^2 + \frac{2}{5}v^2 \\
v^2 &= \frac{2gh}{\frac{2}{5}+1} \\
v &= \sqrt{\frac{10gh}{7}}
\end{aligned}$$

## 7.6 Misc – M19 P3 TZ1 Q8

A solid cylinder of mass $M$ and radius $R$ is free to rotate about a fixed horizontal axle. A rope is tied around the cylinder and a block of mass $\frac{M}{4}$ is attached to the end of the rope.

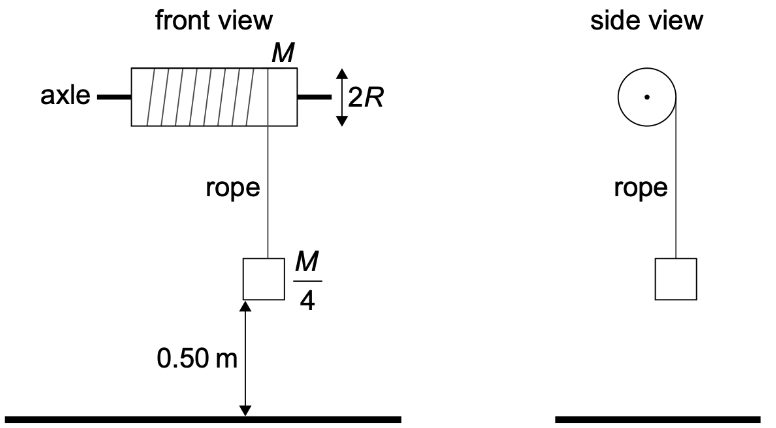



Figure 8: Diagram

The system is initially at rest and the block is released. The moment of inertia of the cylinder about the axle is $\frac{1}{2}MR^2$.

(a) Show that

(i) the angular acceleration $\alpha$ of the cylinder is $\frac{g}{3R}$

- The key is to consider the forces acting on the block.
  - The downward force is the gravitational force and is given by $F_g = \frac{M}{4}g$
  - The upward force is the tension in the string arising from the cylinder's moment of inertia. This is given by $T = I\alpha/r = \frac{1}{2}MR\alpha$
  - We form the following equation about the block's resultant force

$$\frac{M}{4}a_{\text{block}} = \frac{M}{4}g - \frac{1}{2}MR\alpha$$

– We also know that, by the nature of the movement, the linear acceleration of the block is equal to the tangential acceleration of the cylinder, which is given by $a_{\text{block}} = R\alpha$.

– We combine these two equations to form the following:

$$\begin{aligned}\frac{M}{4}R\alpha &= \frac{M}{4}g - \frac{1}{2}MR\alpha \\ R\alpha &= g - 2R\alpha \\ \alpha &= \frac{g}{3R}\end{aligned}$$

(ii) the tension $T$ in the string is $\dfrac{Mg}{6}$

$$\begin{aligned}T &= \frac{1}{2}MR\alpha \\ &= \frac{1}{2}MR\left(\frac{g}{3R}\right) \\ &= \frac{Mg}{6}\end{aligned}$$

(b) The following data are available:

- $R = 0.20$ m
- $M = 12$ kg

Calculate, for the cylinder, at the instant just before the block hits the ground

(i) the angular momentum

- Let us first find the change in angular speed

$$\Delta\omega = \alpha t = \frac{g}{3R}t$$

This is a good start because we are now given both $R$ and $t$ so we can compute $\Delta\omega$.

- Then, we have

$$
\begin{aligned}
L = I(\Delta\omega) &= \frac{1}{2}MR^2(\frac{g}{3R}t) \\
&= \frac{MRgt}{6} \\
&= \frac{12 \times 0.2 \times 9.8 \times 0.55}{6} \\
&= 2.156\,\mathrm{kg\,m^2\,s^{-1}}
\end{aligned}
$$

(ii) the kinetic energy

$$
\begin{aligned}
E_K &= \frac{L^2}{2I} \\
&= \frac{(2.156)^2}{2 \times \frac{1}{2}(12)(0.2)^2} \\
&\approx 9.7 \text{ J}
\end{aligned}
$$