Math AA HL at KCA - Chapters 6 to 8 Notes

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February 25, 2024

Contents

1 Quadratics

A quadratic function can be in any of the following three forms

Form	General expression	Features
Standard	$ax^2 + bx + c$	$c = y$-intercept
Vertex	$a(x-h)^2 + k$	(h, k) vertex; $x = h$ line of symmetry
Factored	$a(x-\alpha)(x-\beta)$	α, β roots

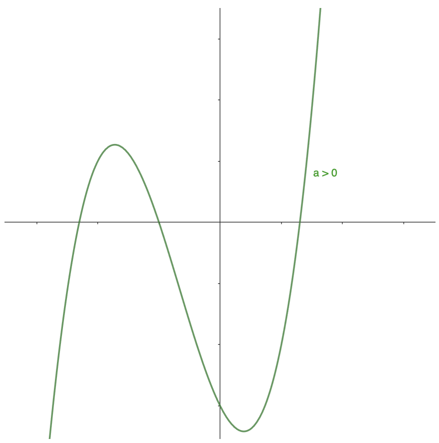
The leading coefficient a determines the concavity of the function

- $a > 0 \implies$ the function is convex (tends to ∞)
- $a < 0 \implies$ the function is concave (tends to $-\infty$)

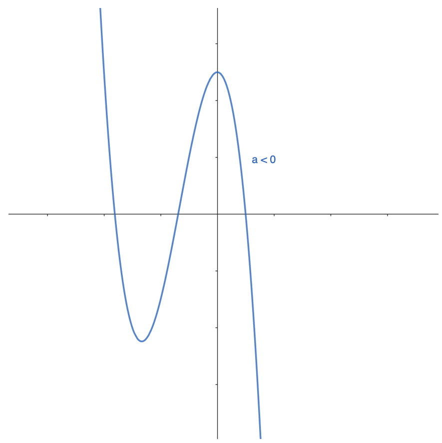
2 Sketching Cubics and Higher Degree Graphs

A polynomial of degree n can have at most n distinct roots.

The following features are to be considered when sketching a polynomial $f(x)$ with leading coefficient a



(a) $a > 0$



(b) $a < 0$

1. Behaviors as $x \to \pm\infty$; for a cubic:
   (a) $a > 0 \implies$ as $x \to \pm\infty$, $y \to \pm\infty$
   (b) $a < 0 \implies$ as $x \to \pm\infty$, $y \to \mp\infty$
2. If the roots are α, β, γ, then the x-intercepts are $(\alpha, 0)$, $(\beta, 0)$, $(\gamma, 0)$.
3. The y-intercept, this can be found by $f(0)$
4. The turning point(s), if applicable and necessary.

3 Mappings

A **relation** describes how values are mapped from one set onto another.

In the case of functions, values from the **domain** are mapped onto values in the **range**.

There are four types of mapping

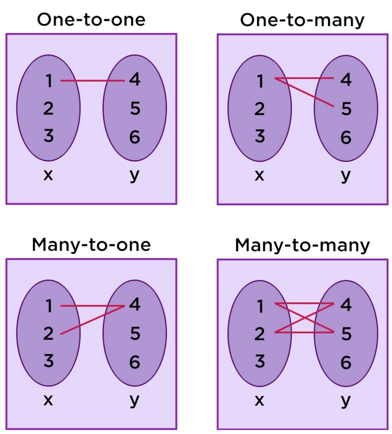


Figure 2: Mappings

For a relation to be a function, it has to be either one-to-one or many-to-one. This is because each x value inputted into the function must be associated with a single y value.

3.1 Interval Notations

An opening/closing square bracket represents an inclusive end.

An opening/closing parenthesis represents an exclusive end.

E.g.

- $x \in [a, b) \iff a \leq x < b$
- $x \in (-\infty, h] \iff x \leq h$

3.2 Domain and Range

The **range** is the set of x values that can go into a function, and the **range** is the set of y values that can come out of a function.

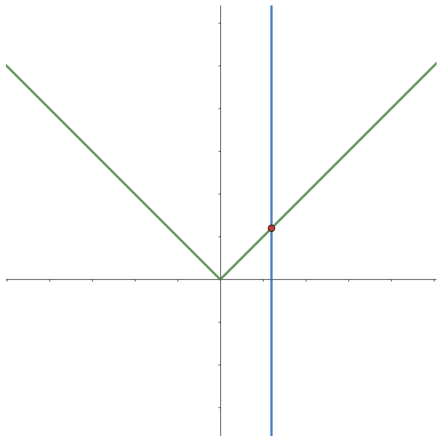
Notations for expressing a domain

1. $x \in [a, b]$
2. $a \leq x \leq b$
3. $R = \{x \mid a \leq x \leq b\}$

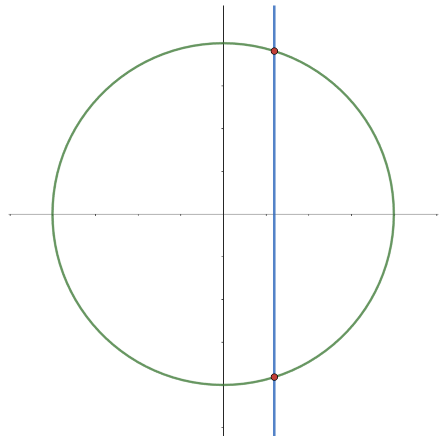
Notations for expressing a range

1. $y \in [a, b]$
2. $f(x) \in [a, b]$
3. $a \leq f(x) \leq b$
4. $D = \{y \mid a \leq y \leq b\}$

3.3 Restricting the Domain



(a) Function



(b) Not a function

A relation is not a function if it does not pass the **vertical line test**, which states that: If *at any point* in the domain of a relation, a vertical line intersects the graph at two or more points, then it is not a function.

For such a relation to be a function, the domain needs to be restricted such that the new domain passes the vertical line test.

4 Composite Functions

$$(f \circ g)(x) = f(g(x))$$

Properties

- Associativity $f \circ (g \circ h) = (f \circ g) \circ h$
- $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

5 Inverse Functions

The inverse function of a function $f(x)$ is denoted $f^{-1}(x)$, it is a reflection in the line $y = x$.

A **self-inverse** function is its own inverse, i.e. $f(x) \equiv f^{-1}(x)$. E.g. $y = \frac{1}{x}$ is a self-inverse function.

Implications:

- $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$
- The *domain* of f is the *range* of f^{-1}, and vice versa
- The intersections of $f(x)$ and $f^{-1}(x)$ lie on the line $y = x$ and can be found by $f(x) = x$ or $f^{-1}(x) = x$

To find an expression for the inverse function, exchange the x's and y's, then make the new y the subject.

A function has an inverse if and only if it is **injective** over its domain, i.e. it is a one-to-one mapping over its domain. This can be examined using the *horizontal line test*.

Sometimes to restrict the domain such that there is an inverse, the turning point is used.

6 Even and Odd Functions

1. Even functions: $f(-x) = f(x)$; symmetrical about the y-axis
   (a) $f(x)$ has only even powers of x
2. Odd functions: $f(-x) = -f(x)$; rotational symmetry of $180°$ about the origin
   (a) $f(x)$ has only odd powers of x

7 Function Transformations and Descriptions

$-f(x)$	Reflection in the x-axis
$f(-x)$	Reflection in the y-axis
$f(x+a)$	Translation to the left by a units
$f(x-a)$	Translation to the right by a units
$f(kx)$	Horizontal stretch by a factor of $\frac{1}{k}$
$f(x)+b$	Upward translation by b units
$f(x)-b$	Downward translation by b units
$kf(x)$	Vertical stretch by a factor of k

8 Special Functions and Their Graphs

Sketching these functions involves looking at

1. x and y intercepts
2. horizontal and vertical asymptotes
3. where $f(x)$ tends to as $x \to \pm\infty$
4. sometimes the turning points too

8.1 Rational/Reciprocal Functions

They take the general form of

$$f(x) = \frac{ax+b}{cx+d}$$

and have the following properties

- x-intercept $= (-\frac{b}{a}, 0)$; when $y = 0$ and $a \neq 0$
- y-intercept $= (0, \frac{b}{d})$; when $x = 0$ and $d \neq 0$
- vertical asymptote $x = -\frac{d}{c}$; when the denominator is 0
- horizontal asymptote $y = \frac{a}{c}$; cancel out the b, d, and x's, as the limit is obtained for very large values of x.

8.2 Exponential Functions

They can take the form of

$$f(x) = ka^{bx} + c, \quad a > 0$$

and have the properties

1. y-intercept $(k + c, 0)$
2. Horizontal asymptote $y = c$
3. No vertical asymptote
4. $b > 0 \implies \lim_{x \to \infty} f(x) = \infty$; growth
5. $b < 0 \implies \lim_{x \to -\infty} f(x) = \infty$; decay

8.3 Logarithmic Functions

They are the reflections of exponential functions in $y = x$. They have vertical asymptotes but not horizontal asymptotes.

8.4 Modulus Functions

Arithmetic rules

1. $|-x| = |x|$
2. $|x|^2 = x^2$
3. $|xy| = |x||y|$
4. $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$

These functions are formally defined as

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

The graph is obtained by reflecting all parts below the y-axis through the y-axis so that they are all above it now.

In general, there are three ways of solving modulus equations/inequalities; solving manually will require the results to be substituted back into the equation to validate the answers and discard any invalid ones

1. Split into cases. e.g. for $|a| = 3$, solve when $a = 3$ and also $a = -3$, then validate the results
2. If both sides are in absolute value, or only one side is and the other is a constant (e.g. $|a| = |b|$ or $|a| = 3$), then square both sides and solve from there
3. Sketch graphs and find intersections
4. Use the G.D.C

The transformation of absolute value functions can be *done step by step* and it is normally easier to visualize this way.

8.5 Graphs of $[f(x)]^2$

Feature of $f(x)$	Feature of $[f(x)]^2$
Parts where $y < 0$	$y > 0$
this i the p $(a, 0)$ is an x-intercept	$(a, 0)$ local minimum
$(0, b)$ is a y-intercept	$(0, b^2)$ y-intercept
$x = a$ is a vertical asymptote	unchanged
$y = b$ is a horizontal asymptote	$y = b^2$ becomes a horizontal asymptote
$y \to \pm\infty$	$y \to \infty$

8.6 Graphs of $\frac{1}{f(x)}$

Feature of $f(x)$	Feature of $\frac{1}{f(x)}$
$x = a$ is an x-intercept	$x = a$ is a vertical asymptote
$(0, b)$, $b \neq 0$ is a y-intercept	$(0, \frac{1}{b})$ becomes the y-intercept
$x = a$ is a vertical asymptote	x-intercept at $(a, 0)$
$y = b$ is a horizontal asymptote	$y = \frac{1}{b}$ is a horizontal asymptote
$y = 0$ is a horizontal asymptote	$y \to \pm\infty$
$y \to \pm\infty$	$y = 0$ is a horizontal asymptote
(a, b), $b \neq 0$ is a turning point	$(a, \frac{1}{b})$ is a turning point