Math AA HL at KCA - Chapter 5 Notes

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Contents

1 Forms of Complex Numbers and Operations

1.1 Cartesian Form

$$z = a + bi$$

where

- $a = \mathrm{Re}(z)$
- $b = \mathrm{Im}(z)$

The complex conjugate of z will then be

$$z^* = a - bi$$

The addition and subtracting of complex numbers in the Cartesian form are done by adding/subtracting the real components and the imaginary components respectively.

Division insolves multiplying by the **complex conjugate** of the denominator throughout

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

this is so that the imaginary component will be canceled out (similar to rationalizing denominators)

1.2 Modulus-Argument Form

$$z = r(\cos\theta + i\sin\theta)$$

where

- $r = |z| =$ modulus; the straight line distance from the origin to z on the Argand diagram
- $\theta = \arg(z) =$ argument; the angle measured counter-clockwise from the positive x-axis.

Operations on moduli and arguments

Suppose z and w are two complex numbers, then

$$|zw| = |z||w|$$
$$|\frac{z}{w}| = \frac{|z|}{|w|}$$

$$\arg(zw) = \arg(z) + \arg(w)$$
$$\arg(\frac{z}{w}) = \arg(z) - \arg(w)$$

Operations on modulus-argument

Multiplication:

$$(r_1(\cos\alpha + i\sin\alpha))(r_2(\cos\beta + i\sin\beta)) = r_1 r_2(\cos(\alpha+\beta) + i\sin(\alpha+\beta))$$

Division:

$$\frac{r_1(\cos\alpha + i\sin\alpha)}{r_2(\cos\beta + i\sin\beta)} = \frac{r_1}{r_2}(\cos(\alpha-\beta) + i\sin(\alpha-\beta))$$

1.3 Exponential/Euler's Form

$$z = re^{i\theta}$$

where

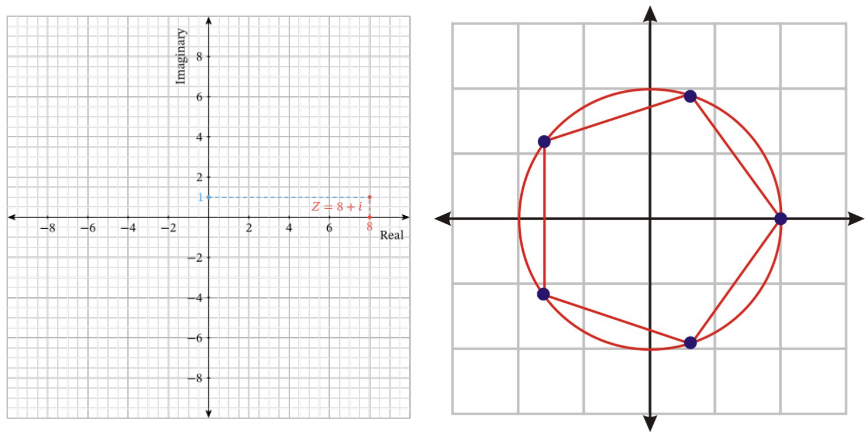
- $r = |z|$
- $\theta = \arg(z)$

Multipliation and division:

$$zw = r_1 r_2 e^{i(\alpha+\beta)}$$

$$\frac{z}{w} = \frac{r_1}{r_2} e^{i(\alpha-\beta)}$$

2 The Argand Diagram



(a) The Argand diagram

(b) 5th roots

Key points

1. For a complex number $z = a + bi$
   (a) $|z| = \sqrt{a^2 + b^2}$
   (b) $\arg(z) = \arctan(\frac{b}{a})$, but a further step of using other information such as the quadrant is required to determine the argument, since $\tan(\theta) = \tan(\pi + \theta)$.

2. The real axis is a line of symmetry for the complex solutions of a polynomial equation; this is because complex roots are **pairwise conjugate**.

3. The n-th roots of a complex number **have the same modulus** and **equally divide a full turn**. This leads to implications such as
   (a) The roots join to form a circle with a radius equal to their mutual modulus.
   (b) By joining the edges, a regular n-sided polygon is formed.
   (c) The area of the regular polygon formed by the n-th root of a complex number with modulus r is given by $\dfrac{n}{2}\sqrt[n]{r^2}\sin(\frac{2\pi}{n})$