Math AA HL at KCA - Chapter 13 Notes

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Contents

1 Continuity

A function is continuous if it can be drawn entirely over its domain without "lifting the pen".

The points of discontinuity are a set of points in the domain of a function at which the function is discontinuous.

A function $f(x)$ is continuous at a point $\iff \lim_{x \to a} f(x)$ exists $\iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

1.1 Removable and Essential Discontinuities

There is a *removable discontinuity* at $x = a$ if $f(a)$ does not exist and $\lim_{x \to a} f(x)$ exists such that a newly modified function $g(x)$, satisfies $g(a) = \lim_{x \to a} f(x)$. I.e. a new function can be defined such that the single "hole" would be "filled in" and the function is continuous at that point.

If such a modification cannot be made, then it is an *essential discontinuity*. This is when the two pieces are completely disjointed and not connected by a "hole".

2 The Laws of Limit

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

- Additive: $\lim_{x\to a} f(x) \pm g(x) = L \pm M$
- Multiplicative: $\lim_{x\to a} f(x) \cdot g(x) = LM$
- Reciprocal: $\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$ if $M \neq 0$
- L'Hôpital's: $\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a} \left(\frac{f'(x)}{g'(x)}\right)$

Indeterminate forms and the corresponding ways of evaluation

- $\frac{0}{0} \longrightarrow$ factorizing, lateral limits, or L'Hôpital's rule.
- $\frac{\infty}{\infty} \longrightarrow$ comparison or L'Hôpital's rule
- $\frac{1}{0} \longrightarrow$ lateral limits

3 Existence of Limits

The limit $\lim_{x\to a} f(x) = L$ exists $\iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$. I.e. approaching $x = a$ from both positive and negative directions, the function converges to the *same limit* $y = L$.

The function diverges when there is not a limit or the limit is ∞.

4 Limits at Infinity

Three cases for $\dfrac{f(x)}{g(x)}$

1. $\deg(f) > \deg(g) \implies \lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$
2. $\deg(f) = \deg(g) \implies \lim_{x\to\infty} \dfrac{f(x)}{g(x)}$ is the quotient of the coefficients of the highest power of x of both polynomials respectively
3. $\deg(f) < \deg(g) \implies \lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$

For the second case, if both $f(x)$ and $g(x)$ are cubics, then, the limit is simply the coefficient of x^3 in $f(x)$ divided by the coefficient of x^3 in $g(x)$

Limit at infinity of exponentials

$$\lim_{x\to\pm\infty} e^{\mp x} = 0$$