

# Math AA HL at KCA - Chapter 5 Notes

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# 1 Forms of Complex Numbers and Operations

## 1.1 Cartesian Form

$$z = a + bi$$

where

- $a = \operatorname{Re}(z)$
- $b = \operatorname{Im}(z)$

The complex conjugate of  $z$  will then be

$$z^* = a - bi$$

The addition and subtracting of complex numbers in the Cartesian form are done by adding/subtracting the real components and the imaginary components respectively.

Division involves multiplying by the **complex conjugate** of the denominator throughout

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)}$$

this is so that the imaginary component will be canceled out (similar to rationalizing denominators)

## 1.2 Modulus-Argument Form

$$z = r (\cos \theta + i \sin \theta)$$

where

- $r = |z|$  = modulus; the straight line distance from the origin to  $z$  on the Argand diagram
- $\theta = \arg(z)$  = argument; the angle measured counter-clockwise from the positive  $x$ -axis.

### Operations on moduli and arguments

Suppose  $z$  and  $w$  are two complex numbers, then

$$|zw| = |z||w|$$
$$\left| \frac{z}{w} \right| = \frac{|z|}{|w|}$$

$$\arg(zw) = \arg(z) + \arg(w)$$
$$\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$$

### Operations on modulus-argument

Multiplication:

$$(r_1(\cos \alpha + i \sin \alpha)) (r_2(\cos \alpha + i \sin \alpha)) = r_1 r_2 (\cos(\alpha + \beta) + i \sin(\alpha + \beta))$$

Division:

$$\frac{r_1 (\cos \alpha + i \sin \alpha)}{r_2 (\cos \beta + i \sin \beta)} = \frac{r_1}{r_2} (\cos(\alpha - \beta) + i \sin(\alpha - \beta))$$

### 1.3 Exponential/Euler's Form

$$z = re^{i\theta}$$

where

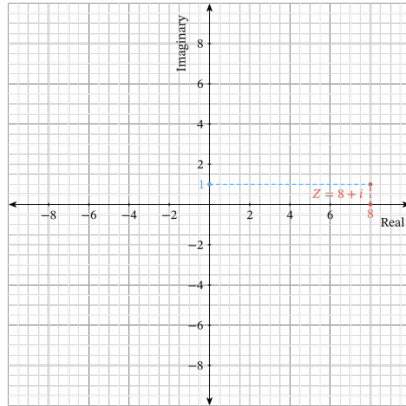
- $r = |z|$
- $\theta = \arg(z)$

Multiplication and division:

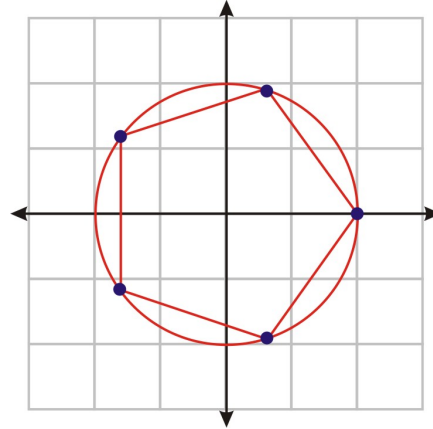
$$zw = r_1 r_2 e^{i(\alpha+\beta)}$$

$$\frac{z}{w} = \frac{r_1}{r_2} e^{i(\alpha-\beta)}$$

## 2 The Argand Diagram



(a) The Argand diagram



(b) 5th roots

Key points

1. For a complex number  $z = a + bi$ 
  - (a)  $|z| = \sqrt{a^2 + b^2}$
  - (b)  $\arg(z) = \arctan(\frac{b}{a})$ , but a further step of using other information such as the quadrant is required to determine the argument, since  $\tan(\theta) = \tan(\pi + \theta)$ .
2. The real axis is a line of symmetry for the complex solutions of a polynomial equation; this is because complex roots are **pairwise conjugate**.
3. The  $n$ -th roots of a complex number **have the same modulus** and **equally divide a full turn**. This leads to implications such as
  - (a) The roots join to form a circle with a radius equal to their mutual modulus.
  - (b) By joining the edges, a regular  $n$ -sided polygon is formed.