Math AA HL at KCA - Chapter 9 Notes

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1 Partial Fractions

When splitting a rational function into partial fractions, the denominator must have a **strictly higher degree** than the numerator. If not, the rational function has to be rewritten.
Example:

$$\frac{3x^2 + 2x - 11}{(x-1)(x+2)} \equiv 3 - \frac{x+5}{(x-1)(x+2)}$$

In general, a rational function that cannot be decomposed yet is to be rewritten in the form of

$$p(x) + \frac{q(x)}{r(x)}$$

where

- $\deg(p) = \deg(\text{numerator}) - \deg(\text{denominator})$
- $\deg(q) < \deg(r)$

1.1 Case 1 - Linear Factors

A fraction with only linear factors $Q_1(x), Q_2(x), \ldots, Q_n(x)$ in the denominator is decomposed into a sum of fractions where $A, B, \ldots$ are constants:

$$\frac{P(x)}{Q_1(x)Q_2(x)\ldots Q_n(x)} \equiv \frac{A}{Q_1(x)} + \frac{B}{Q_2(x)} + \cdots + \frac{Z}{Q_n(x)}$$

The first way of decomposition:

1. Multiply both sides by $Q_1(x)Q_2(x)\ldots Q_n(x)$
2. Expand
3. Compare coefficients to form a system of equations and solve for the values

The second way of decomposition:

1. Multiply both sides by $Q_1(x)Q_2(x)\ldots Q_n(x)$
2. Choose and substitute values of x that can eliminate one of the unknowns, i.e. $x \mid Q_i(x) = 0$ to form an equation.
3. Repeat this for other $Q(x)$ until a solvable system is formed.

1.2 Case 2 - Repeated Linear Factors

When a linear factor $Q(x)$ is raised to the nth power, there should be one decomposed fraction for each power of $Q(x)$.

Example:

$$\frac{-x^2+6x+11}{(x-1)^2(x+3)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

1.3 Case 3 - Higher Order Factors

When there is a polynomial $R(x)$ of a higher (than linear) order n in the denominator that cannot be further factored into linear factors, for the corresponding decomposed fraction, the unknown polynomial in its numerator must have a degree of $n-1$.

Example

$$\frac{5x^2-5x+12}{(x^2+5)(x-1)} \equiv \frac{Ax+B}{x^2+5} + \frac{C}{x-1}$$

2 Polynomial Long Division

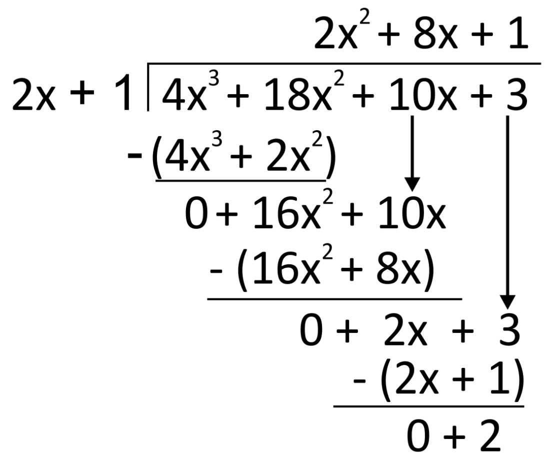

Figure 1: Long division example

3 The Factor and Remainder Theorems

1. *The remainder theorem:* The remainder of a polynomial $P(x)$ when divided by $(x - \alpha)$ is $P(\alpha)$

2. *The factor theorem:* A value α is a root of the polynomial $P(x)$ if and only if $(x - \alpha)$ is a factor of $P(x)$. In other words:

$$P(\alpha) = 0 \iff (x - \alpha) \mid P(x)$$

4 Roots of Polynomial

For any polynomial

$$P(x) = ax^n + bx^{n-1} + ...$$

The sum of the roots is $\dfrac{-b}{a}$.

Let a_0 be the constant coefficient, then the product of the roots are

1. if $P(x)$ has an *even* degree, then $\dfrac{a_0}{a}$
2. if $P(x)$ has an *odd* degree, then $-\dfrac{a_0}{a}$