What is the relationship between the mass of a damped spring-block oscillator {52.48g, 60.94g, 69.72g, 86.51g, 93.47g} and the damping ratio?

Word count: TBD

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1 Introduction

This essay extends the investigation of simple harmonic motion by studying the damping force of a damped oscillator submerged in water, aiming to scrutinize the relationship between the mass of the block and the damping ratio. By scrutinizing this relationship, the study aims to offer valuable insights that can inform the design of systems seeking to optimize damping levels for safety-related objectives.

1.1 The Research Question

What is the relationship between the mass of a damped spring-block oscillator {52.48g, 60.94g, 69.72g, 86.51g, 93.47g} and the damping ratio?

1.2 Background Information

An ideal and rather theoretical spring-mass system oscillates indefinitely, producing an ongoing sine or cosine curve. In reality, there will be a damping force that can be as minimally consequential as air resistance or as observable as viscous drag in a liquid. Anyway, energy is dissipated to the surroundings and hence the amplitude will gradually decrease until the oscillation stops. The extent to which the viscous drag force, modeled by Stoke's law, diminishes the oscillatory motion submerged in water depends on the mass of the oscillator. This investigation delves into the relationship between the independent variable of mass m, and the dependent variable of the damping ratio ζ. The mass set used in the system consists of spheres of equal radii but different densities.

Stoke's Law determines the drag force acting upon an object traveling through a fluid – proportional to the object's velocity and is the following per the Data Booklet, in Newtons

$$F_d = 6\pi r\eta v$$

This holds iff. the object's speed is low such that the flow is *laminar*, and the object is *spherical*.

Assumption 1. *The damping force is the viscous drag force.*

Moreover, the motion of the oscillator is modeled by the following differential equation, with the damping force proportional to velocity (Miller, 2004):

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0 \tag{1}$$

The damping ratio ζ is defined as the ratio of the damping coefficient to the critical damping coefficient — the damping coefficient when the system returns to equilibrium without completing an oscillation. That is

$$\zeta = \frac{b}{c_c} = \frac{b}{2mw_0}$$

where w_0 is the natural frequency of the system

$$w_0 = \sqrt{\frac{k}{m}} \implies \zeta = \frac{b}{2\sqrt{mk}}$$

1.3 Hypothesis

Suppose that the constant of proportionality of the damping force with speed is given by $b = 6\pi r\eta$, per Stoke's law. Then

$$\begin{aligned} 6\pi r\eta &= 2\zeta\sqrt{mk} \\ 9\pi^2 r^2\eta^2 &= \zeta^2 mk \\ \frac{\zeta^2}{m^{-1}} &= \frac{9\pi^2 r^2\eta^2}{k} \end{aligned}$$

Proposition 1. *For a fixed sphere radius r and viscosity η,*

$$\zeta^2 \propto m^{-1} \tag{2}$$

Equivalently, the graph ζ^2/m^{-1} is ideally a straight line, although not necessarily passing through the origin, as some form of systematic errors are expected.

Proposition 1 has a few dependencies that may affect its accuracy. It relies on Assumption 1, which, in turn, requires that the flow is laminar. This is a reasonable assumption given that the spheres are relatively small, and the speed is low; the potential for turbulence is insignificant and even if present, the impact is minimal. Furthermore, the buoyancy force on the object

$$F_b = \rho g V$$

is always present but is not modeled in the differential equation at eq. (1). Notwithstanding, the absence of considerations of buoyancy has negligible impact on precision. The setup involves spheres of radius 2.5cm and water of density $997\mathrm{kg\,m^{-3}}$, then

$$F_b \approx 10 \times 1000 \times \frac{4}{3}\pi(2.5 \times 10^{-2})^3 < 1\mathrm{N}$$

Null hypothesis (H_0): There is no relationship between the mass and the damping ratio.
Alternative hypothesis (H_1): There is a relationship between the mass and the damping ratio.

2 Research Design

2.1 Variables

Variables		Explanation	Measurement
Independent	Mass m	The masses of the spherical metal bobs	Measured using an electric balance, to two d.p. with uncertainty ± 0.01g
Dependent	Peak Amplitude	3 peaks will be measured. These data points are used to compute the damping ratio.	Ruler, with uncertainty ± 0.1cm

Table 1: Variables of investigation

Control Variables		
Variable	Rationale	Means of Control
Liquid viscosity 0.85	To keep the drag coefficient per Stoke's Law constant	Using **water from the same source** throughout the experiment, and keep the **room temperature** constant by turning off the air conditioning. In fact, during the experiment, the room temperature was 27°C
Radius of the sphere, r = 2.5cm	This is so that the drag force, per Stoke's law, is constant	Using spherical masses of the same diameter
Spring constant, k = $100\mathrm{N\,m^{-1}}$	This is so that the natural frequency of the system is constant	Using the same spring throughout the experiment. Read from the label on the box.
Initial amplitude, $A_0 = 5$cm	So that the damping ratio is not affected by the initial conditions	Pulling the spring down by equal distances for every mass. Measured by a ruler with ±0.1cm
Damping coefficient	Linearizes relationship between damping ratio and mass	Previous means of control keep fluid viscosity and object dimensions constant, so the damping coefficient will be constant.

Table 2: Control variables

2.2 Apparatus and Materials

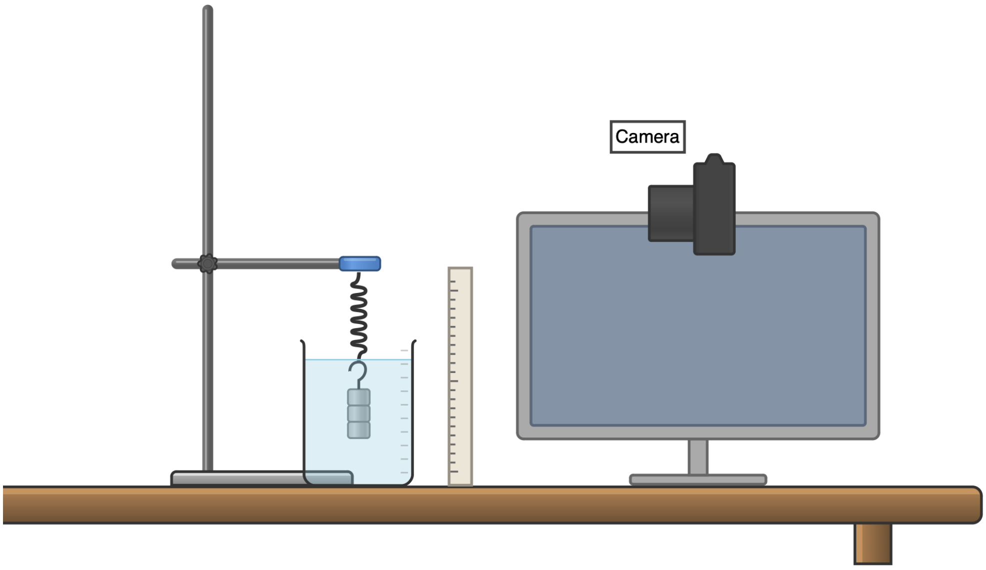

Figure 1: The experimental setup

1. Ruler (±0.1cm)
2. Electric balance (±0.01g)
3. Spring with spring constant $k = 100\,\mathrm{N\,m^{-1}}$
4. Mass set with five spherical masses of diameter 2.5cm
5. Beaker
6. Tap water ([volume here])
7. Camera — keeps a record for future reference
8. Thermometer — to ensure that the liquid stays at a constant temperature
9. Retort stand
10. Clamp

2.3 Methodology

1. Take a scrotch tape, and cut into segments of 8cm in length.
2. Arrange the masses in increasing order, i.e. $m_1 < m_2 < ... < m_n$, where n is the number of masses in the mass set.
3. Fill in the beaker with AmL of tap water
4. Measure the temperature of the water, and record as T_0
5. Attach the clamp to the retort stand.
6. Lean the ruler vertically against the retort stand.
7. Place the beaker directly under the clamp such that the clamp lies at the center of the beaker in bird's eye view.
8. For i from 1 to n inclusive, repeat the following
   (a) Take the ith mass, m_i, and attach its hook to the spring.
   (b) Take a piece of tape, wrap it as many times as possible across the point of connection.
   (c) Attach the spring-mass pair to the clamp and let the mass hang freely.
   (d) Adjust the height of the clamp so that the mass is submerged in the water.
   (e) Measure the height of the mass from the surface of the desk, and record as h_0
   (f) Adjust the vertical camera position such that it levels with the reading $(h_0 - \frac{A_0}{2})$cm to minimize parallax error of the camera.
   (g) For j from 1 to 3 inclusive, repeat the following
      i. Pull the mass down by A_0 cm by reading $h_0 - A_0$ from the ruler at eye level. The center of the mass should be the height a which readings are taken.
      ii. Start the camera recording.
      iii. Release the mass.

iv. Record the clip until the 5th negative peak is reached.

v. Save the clip as `Mass_i_Trial_j.mp4`

vi. Open the clip in a video viewer.

vii. Identify the first 3 negative peaks, one in each cycle.

viii. Record the respective heights above the desk as $\{h_k\}_{k=1}^{5}$.

ix. Then the corresponding peak amplitudes are $A_k = h_0 - h_k$

(h) Take the average values across the three trials for each peak amplitude.

(i) Measure the water temperature and wait until it stays constant at 27°C.

2.4 Preliminary Trials

Preliminary trials were conducted to identify and resolve potential challenges, as well as to refine the experimental conditions. Initially, segments of metal wire were used to attach the masses to the spring, but this approach led to inconsistent wrapping and slipping. To improve the attachment's security and uniformity, Scotch tape was tested and found to be more effective, significantly reducing variability and enhancing setup repeatability. Precise measurement of the initial height (h_0) and consistent displacement of the mass by A_0, cm were also challenging. Aligning the ruler vertically against the retort stand and centering the mass over the beaker proved crucial for accuracy. Adjustments to the clamp and careful ruler leveling minimized parallax error, further reduced by positioning the camera at eye level.

2.5 Risk Assessment

Consideration	Relevance and Mitigation
Safety	All valuable assets are kept away from the experimental setup to prevent accidental detachments of the spring-mass system leading to the mass being thrown out and causing damage. Moreover, the amplitude was kept low to prevent fierce movements that could break the beaker or computer camera.
Ethical	No use of animals or human bodies involved. However, the recording selects the location and angle that avoids the storage and exposure of any personal information or objects of the lab owner, ensuring maximum privacy protection.
Environmental	The entire experiment aims to reuse the same water throughout to minimize water wastage. Furthermore, the spring was carefully pulled to prevent it from passing its elastic limit and becoming unusable for future experiments.

Table 3: Risk Assessment

3 Results

3.1 Raw Data

3.1.1 Qualitative Data

During the entire experiment, there is no spillage of water, which means that the volume is kept constant. Moreover, the spring is always able to return to its original length without any permanent deformation, indicating that the spring is not stretched beyond its elastic limit and that Hooke's Law applies throughout.

3.1.2 Quantitative Data

Let m denote the mass of a bob, and A_n denote the n-th peak height.

$m \pm 0.01$ (g)	$A_1 \pm 0.1$ (cm)					$A_2 \pm 0.1$ (cm)					$A_3 \pm 0.1$ (cm)				
52.48	2.9	2.9	2.8	2.8	2.9	1.7	1.8	1.7	1.7	1.7	1.0	1.0	1.0	1.1	1.0
60.94	2.9	3.0	3.0	2.9	3.0	1.8	1.9	1.8	1.8	1.8	1.1	1.1	1.1	1.2	1.1
69.72	3.1	3.1	3.1	3.1	3.1	1.9	2.0	1.8	1.9	1.9	1.2	1.2	1.2	1.3	1.2
86.51	3.3	3.3	3.3	3.3	3.3	2.1	2.1	2.2	2.1	2.1	1.4	1.4	1.4	1.4	1.3
93.47	3.5	3.3	3.4	3.3	3.5	2.4	2.2	2.4	2.2	2.2	1.5	1.5	1.5	1.5	1.5

Table 4: Raw data of peak amplitudes

3.2 Data Processing

3.2.1 Transformation

Using the *logarithmic decrement* method per Inman (2008)

$$\delta = \ln\left(\frac{A_n}{A_{n+1}}\right) \text{ and } \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

where A_n and A_{n+1} denote any pair of consecutive peaks' heights.

It then follows that

$$\begin{aligned}
\delta^2 &= \frac{4\pi^2\zeta^2}{1-\zeta^2} \\
\delta^2 - \delta^2\zeta^2 &= 4\pi^2\zeta^2 \\
\delta^2 &= \zeta^2\left(4\pi^2+\delta^2\right) \\
\zeta &= \frac{\delta}{\sqrt{4\pi^2+\delta^2}}
\end{aligned}$$

This step is encapsulated in the following Python snippet that belongs to an overall script for processing the data and generating the graph.

```python
# Calculate the average damping ratio for each row in the dataframe
def avgDampingRatio(row: pd.Series):
    ratios = np.array([])  # Create an empty array to store the damping ratios

    # Iterate over the range of peak indices
    for i in range(PEAK_IND[0], PEAK_IND[1] - 1):
        # Iterate over the range of peak indices starting from i+1
        # the purpose of this double loop is to calculate the damping
        # ratio between every possible pair of peaks
        for j in range(i + 1, PEAK_IND[1]):
            n = j - i  # Calculate the number of cycles between i and j
            delta = (1 / n) * math.log(
                (row.iloc[i]) / row.iloc[j]
            )  # Calculate the logarithmic decrement between the peaks
            zeta = delta / math.sqrt(
                4 * (math.pi) ** 2 + delta**2
            )  # Calculate the damping ratio using the logarithmic difference
            ratios = np.append(
                ratios, zeta
            )  # Append the damping ratio to the ratios array

    return np.average(ratios)  # Return the average damping ratio

def processData():
    # reads the data tables and find the average across 5 trials
    df = averageData()
    # ...

    # populate the column of damping ratio in the table for each mass
    df["Damping Ratio"] = df.apply(
        avgDampingRatio,
        axis=1,
    )
```

Figure 2: Calculating the damping ratio with pandas in Python

Note that $\overline{A_i}$ denotes the average value of the i-th peak across the five trials.

$m \pm 0.01$ (10^{-3}kg)	$\overline{A_1}$ (10^{-2}m)	$\overline{A_2}$ (10^{-2}m)	$\overline{A_3}$ (10^{-2}m)	$\zeta^2/10^{-3}$ (unitless)
52.48	2.9	1.7	1.0	7.13
60.94	3.0	1.8	1.1	6.33
69.72	3.1	1.9	1.2	5.67
86.51	3.3	2.1	1.4	4.63
93.47	3.4	2.3	1.5	4.22

Table 5: Processed data

3.2.2 Uncertainty Analysis

Let Δx denote the absolute uncertainty of x. For $\Delta x \ll x$, the uncertainty of $f(x)$ is given by the following (Vacher, 2001)

$$\Delta f(x) \approx \frac{\mathrm{d}f(x)}{\mathrm{d}x} \cdot \Delta x$$

Let i, j such that $i < j$ denote the indices of any two peaks, and also let $\gamma = \dfrac{A_i}{A_{i+1}}$, the following table summarizes the uncertainties involved in the calculation of the damping ratio from a pair of peaks indexed with (i, j).

Variable	Calculation	Absolute Uncertainty
m		0.01g
m^{-1}		$(m)^{-2} \cdot 0.01\text{g}^{-1}$
A_k, $\forall k$		0.1cm
$\gamma_{i,j}$	$\frac{A_i}{A_j}$	$\Delta\lambda_{i,j} = \frac{A_i}{A_j}(\frac{\Delta A_i}{A_i} + \frac{\Delta A_j}{A_j})$
$\delta_{i,j}$	$\frac{1}{j-i} \ln \frac{A_i}{A_j}$	$\Delta\delta = \frac{\Delta\lambda}{\lambda}$
$\zeta_{i,j}$	$\frac{\delta}{\sqrt{4\pi^2+\delta^2}}$	$\Delta\zeta_{i,j} = \dfrac{\mathrm{d}\zeta}{\mathrm{d}\delta} \cdot \Delta\delta$
$\zeta_{i,j}^2$		$2(\zeta_{i,j}) \cdot (\Delta\zeta_{i,j})$

Table 6: Uncertainty Propagation

To implement the above logic in Python, the previous code snippet in fig. 2 is extended to include the uncertainty calculations.

```python
# Calculate the average damping ratio for each row in the dataframe
def avgDampingRatio(row: pd.Series, mode: Union["uncertainties", "vals"]):
    if type(row.iloc[-1]) == str:
        return 0
    ratios = np.array([])  # Create an empty array to store the damping ratios
    zetaSqUncertainties = np.array(
        []
    )  # Create an empty array to store the uncertainties of the damping ratios

    # Iterate over the range of peak indices
    for i in range(PEAK_IND[0], PEAK_IND[1] - 1):
        # Iterate over the range of peak indices starting from i+1
        # the purpose of this double loop is to calculate the damping
        # ratio between every possible pair of peaks
        for j in range(i + 1, PEAK_IND[1]):
            n = j - i  # Calculate the number of cycles between i and j
            A_i, A_j = row.iloc[i], row.iloc[j]  # Get the amplitudes of the peaks

            if type(A_i) != float:
                break

            gamma = A_i / A_j  # Calculate the ratio of the amplitudes of the peaks

            delta = (1 / n) * math.log(
                gamma
            )  # Calculate the logarithmic decrement between the peaks

            errAbsGamma = (A_i / A_j) * (
                (peakError / A_i) + (peakError / A_j)
            )  # Calculate the absolute error of gamma

            # Calculate the absolute error of delta
            errAbsDelta = errAbsGamma / gamma

            zeta = delta / math.sqrt(
                4 * (math.pi) ** 2 + delta**2
            )  # Calculate the damping ratio using the logarithmic decrement

            dzeta_ddelta = (
                4 * math.pi**2 / math.pow(delta**2 + 4 * math.pi**2, 1.5)
            )  # Calculate the derivative of zeta wrt delta

            errAbsZeta = (
                dzeta_ddelta * errAbsDelta
            )  # Calculate the absolute error of zeta

            errAbsZetaSq = (
                2 * zeta * errAbsZeta
            )  # Calculate the absolute error of the squared damping ratio

            ratios = np.append(
                ratios, zeta
            )  # Append the damping ratio to the ratios array

            zetaSqUncertainties = np.append(
                zetaSqUncertainties, errAbsZetaSq
            )  # Append the uncertainty of the damping ratio

    return (
        np.average(ratios) if mode == "vals" else np.average(zetaSqUncertainties)
    )  # Return the average damping ratio or the average
    #    uncertainties of the damping ratios based on the mode
```

Figure 3: Calculating the damping ratio with uncertainties in Python

Running the above script on the dataset produces the following table of uncertainties.

$\Delta m/10^3$ (kg)	$m^{-1}(\text{kg}^{-1})$	$\Delta m^{-1}(\text{kg}^{-1})$	$\Delta\zeta^2/10^{-3}$ (unitless)
52.48	19.230	$0.003698 \approx 0.004$	3.35
60.94	16.393	$0.002687 \approx 0.003$	2.53
69.72	14.285	$0.002041 \approx 0.002$	2.71
86.51	11.494	$0.001321 \approx 0.001$	2.42
93.47	10.753	$0.001156 \approx 0.001$	1.92

Table 7: Absolute uncertainties

3.3 Graphical Interpretation

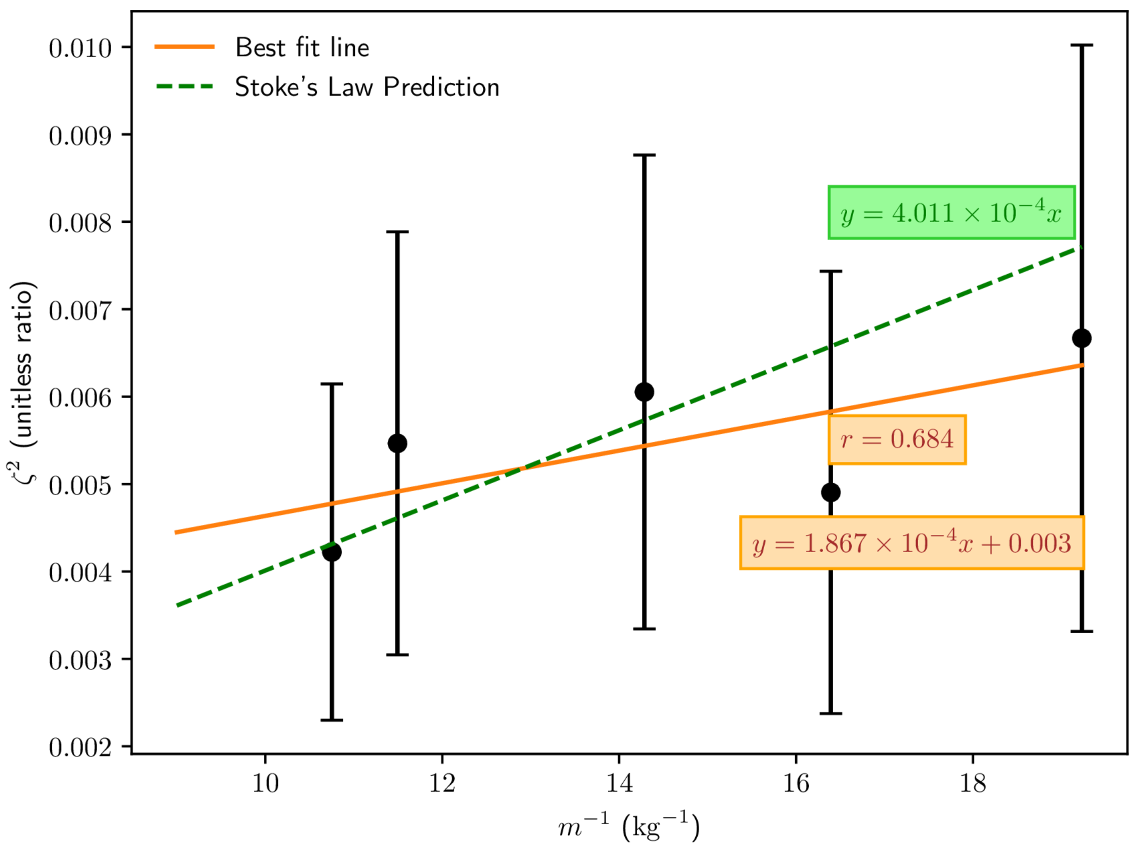

Figure 4: Graph of ζ^2 against m^{-1}, with best-fit line and predicted line

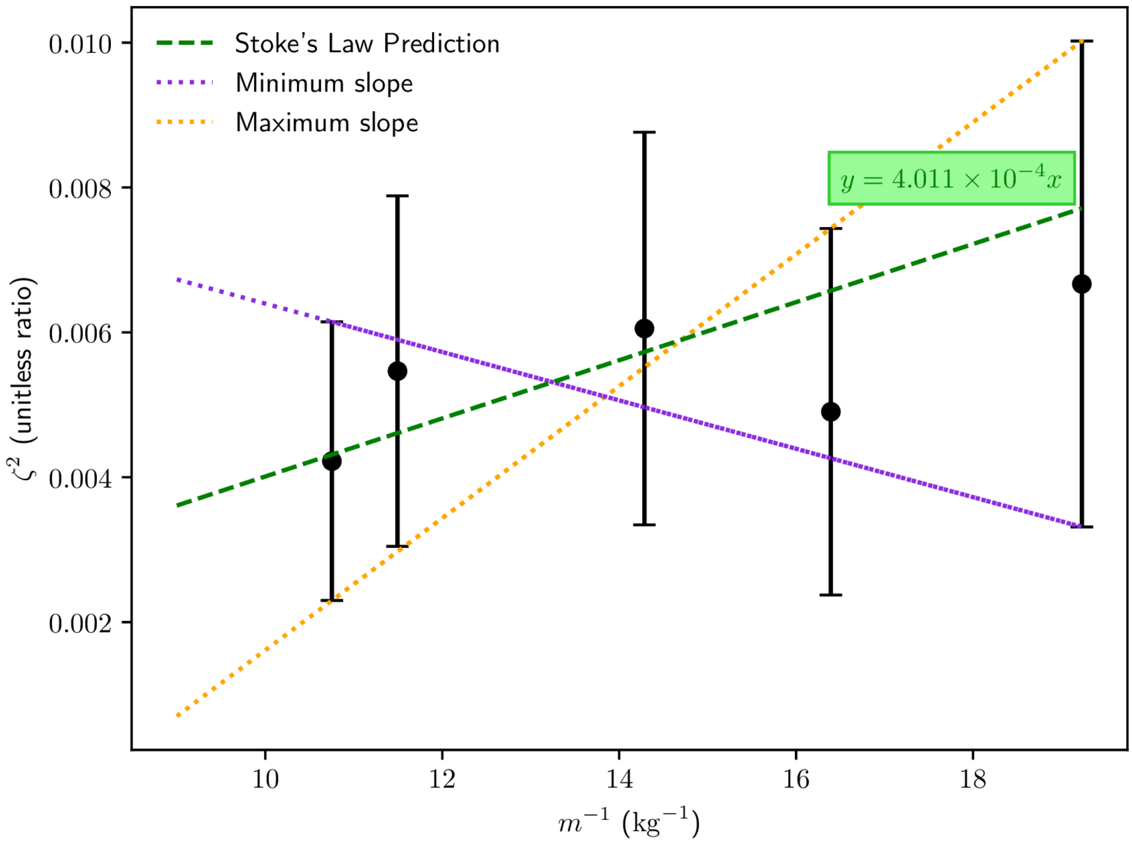

Figure 5: Graph of ζ^2 against m^{-1} — literature review

The two graphs above show the relationship between the damping ratio squared and the inverse of the mass from the resultant dataset, with error bars plotted. The first graph, fig. 4, shows a scatter plot of the data points from section 3.2.1, with the best-fit line and the line predicted using Stoke's Law stated in the hypothesis proposition section. The data points suggest a linear relationship between ζ^2 and m^{-1}, with the Pearson correlation coefficient $r = 0.684$. The second graph, fig. 5, shows the prediction line in comparison to the maximum and minimum-slope lines resulting from the propagation of uncertainty — indeed, the theoretical prediction does lie within the error bars of the data points.

4 Conclusion

This essay delved into the effect of mass m on the damping ratio ζ of a damped spring-block oscillator submerged in water, with the hypothesis that

$$\zeta^2 \propto m^{-1}$$

The results from the experiment suggest a linear relationship between the damping ratio squared and the inverse of the mass, with a Pearson correlation coefficient of 0.684, indicating a relatively strong linear trend. The theoretical prediction line falls within the interval of the maximum and minimum slopes due to the uncertainties in measurements, supporting the hypothesis that the damping ratio squared is inversely proportional to the mass, within the experimented range of 52.48 to 93.47 grams.

The observation that the relationship is not perfectly linear is a consequence of the uncertainties that arise from the limitations of measuring devices and human errors. One primary source of random error is the unintentional but inevitable extra force applied when releasing the spring in the liquid. Nonetheless, the mass set did not produce anomolies and massive systematic errors, suggesting that the environment and the experimental setup were controlled effectively. Proposition 1 suggested that the expected gradient, m_0 would be

$$\begin{aligned} m_0 &= \frac{9\pi^2 r^2 \eta^2}{k} \\ &= \frac{9\pi^2 (0.025)^2 (0.85)^2}{100} \\ &\approx 4.011 \times 10^{-4} \end{aligned}$$

The minimum and maximum slopes are -3.339×10^{-4} and 9.109×10^{-4} respectively, with the experimental gradient in between. This shows that the drag force on the oscillator in water can be estimated using Stoke's Law. The error propagation does create rather significant error bars due to the already small values of ζ^2, particularly for smaller masses. However, this does accept the theoretical value and is expected since ζ^2 is very sensitive to a change in mass.

4.1 Evaluation

4.1.1 Strengths

Strength	Observations

4.1.2 Weaknesses — Random Errors

4.1.3 Weaknesses — Systematic Errors

4.1.4 Extensions

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