Calc Assignment 08

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1. Determine whether each of the following series is convergent or divergent. State which test you are using in each case. **Do not use the Ratio Test in this question**

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5^n}{n^2+8^n}$

$$\text{let } a_n = \frac{n+5^n}{n^2+8^n} < \frac{n+5^n}{8^n} < \frac{5^n+5^n}{8^n} = \frac{2}{1}\cdot\frac{5^n}{8^n} = 2\left(\frac{5}{8}\right)^n \quad \forall n > 0$$

$$\begin{aligned} a_n &< 2\left(\frac{5}{8}\right)^n && (1)\\ \sum_{n=1}^{\infty}\frac{n+5^n}{n^2+8^n} &< \sum_{n+1}^{\infty} 2\left(\frac{5}{8}\right)^n && (2)\\ &< 2\sum_{n=1}^{\infty}\frac{5}{8}^n && (3) \end{aligned}$$

   By GST, we know that $2\sum_{n=1}^{\infty}(\frac{5}{8})^n$ will converge, and therefore, by our inequality (and the comparison test), $\sum_{n=1}^{\infty}\frac{n+5^n}{n^2+8^n}$ will also converge.

   (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n e^{-n}$

   let $b_n = \left(\frac{1}{e}\right)^n$, then we know the following:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{1}{e}\right)^n = 0$$

$$\begin{aligned} b_{n+1} - b_n &= \left(\frac{1}{e}\right)^{n+1} - \left(\frac{1}{e}\right)^n && (4)\\ &= \left(\frac{1}{e}\right)^n\left(\frac{1}{e}\right) - \left(\frac{1}{e}\right)^n && (5)\\ &= \left(\frac{1}{e}\right)^n\left[\frac{1}{e}-1\right] < 0 \quad \forall n > 0 && (6) \end{aligned}$$

   Hence, we know that the limit of b_n is equal to 0 and by the principal of mathematical induction (leaving trivalties aside) that b_n is also decreasing; therefore by the alternating series test, $\sum_{n=1}^{\infty}(-1)^n e^{-n}$ is convergent.

(c) $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3-n+1}$

$$
\begin{aligned}
a_n = \frac{n^2+n+1}{n^3-n+1} &> \frac{n^2+n}{n^3-n} \quad \forall n > 2 && (7)\\
&> \frac{n(n+1)}{n(n^2-1)} = \frac{(n+1)}{(n+1)(n-1)} = \frac{1}{n-1} > \frac{1}{n} \quad \forall n > 1 && (8)\\
& && (9)
\end{aligned}
$$

Let $n := Max\{1, 2\}$ then we know that since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges then by the comparison test that $\sum_{n=2}^{\infty} \frac{n^2+n+1}{n^3-n+1}$ must also diverge; changing the index to start from $n = 1$ is trivial since all but a finite number of terms converge.

(d) $\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$

n	1	2	3	4	$\cdots$	k
$\cos n\pi$	-1	1	-1	1	$\cdots$	$(-1)^k$

Hence,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{n \ \cos n\pi}{2^n} &= \sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n} && (10)\\
\text{Let } b_n &= \frac{n}{2^n} && (11)\\
\lim_{n\to\infty} b_n &\overset{H}{=} \frac{1}{2^n \ln 2} = 0 \qquad \text{(bounded below by 0)} && (12)\\
b_{n+1} - b_n &= \frac{n+1}{2^{n+1}} - \frac{n}{2^n} = \frac{n+1}{2^n \cdot 2} - \frac{n}{2^n} && (13)\\
&= \frac{n+1-2n}{2^n \cdot 2} = \frac{1}{2}\left(\frac{-n+1}{2^n}\right) && (14)\\
&< 0 \ \ (\forall \text{n} > 1) && (15)
\end{aligned}
$$

Therefore by the alternating series test, $\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$ must be convergent.

2. Show that if $a_n > 0$ and $\lim\limits_{x\to\infty} na_n = L \neq 0$, where $L \,\epsilon\, \mathbb{N}$, then $\sum a_n$ is divergent.

3. Assume that $\sum a_n$ is convergent (with positive terms); prove whether each of the following are convergent or divergent.

   (a) $\sum\limits_{n=1}^{\infty} \sin(a_n)$

   Let us define $\sin x$ on the inteval $I := (0, 2\pi]$, then by the mean-value theorem we can guarantee that $\exists\; c \,\epsilon\, I$ such that

$$\begin{aligned} \frac{\sin x - \sin 0}{x - 0} &= \frac{d}{dx} sinx\Big|_{x=c} && (16)\\ \frac{\sin x}{x} &= \cos c && (17)\\ \sin x &= x \cos c && (18)\\ \sin x &< x && (19) \end{aligned}$$

   Now let $\sin(a_n) = b_n$, by taking the $\lim\limits_{n\to\infty} a_n = 0$, we also get that $\lim\limits_{n\to\infty} \sin(a_n) = 0$, and by our previous inequality we have also shown that $\sin(a_n) < a_n$ which means that as $n \to \infty$ a_n will approach 0, but b_n must always be less than a_n hence it is monotonically decreasing and bounded below by 0, thus by the monotone convergence theorem, $\sum\limits_{n=1}^{\infty} \sin(a_n)$ must converge.

   (b) $\sum\limits_{n=1}^{\infty} a_n^2$

$$\sum a_n^2 = \sum a_n \cdot \sum a_n < M \cdot M = M^2$$

   with this we have shown that $\sum\limits_{n\to\infty} a_n^2$ is bounded above and because we know that a_n is made of positive numbers, then $\sum a_n^2$ is also just the sum of positive numbers which implies monotonicity and hence by the monotone convergence theorem, $\sum\limits_{n\to\infty} a_n^2$ must also converge.

(c) $\sum_{n=1}^{\infty} \sqrt{a_n}$ Let $a_n = \frac{1}{n^2}$, then we evalute

$$\sum_{n=1}^{\infty} \sqrt{\frac{1}{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n}$$

(which we know to be divergent)

(d) $\sum_{n=1}^{\infty} \ln(1 + a_n)$

Applying the limit comparison test we get $\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n}$ and because $\lim_{n\to\infty} a_n = 0$ we have

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} \stackrel{H}{=} \lim_{x\to 0} \frac{1}{1+x} = 1$$

Hence, by our limit comparison test, $\sum_{n=1}^{\infty} \ln(1 + a_n)$ converges because $\sum_{n=1}^{\infty} a_n$ converges

4. Determine if each of the following is absolutely convergent, conditionally convergent or divergent. In each case, state which test you are using (e.g., Ratio Test).

   (a) $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2+7}$

   By the alternating series test we note that $\lim_{n\to\infty}\dfrac{n}{n^2+7}=0$ and that $\dfrac{n}{n^2+7}$ is positive and decreasing over the interval $[3,\infty)$, hence it will converge, however, using the absolute convergence test we see that $|(-1)^{n-1}\frac{n}{n^2+7}|$ does not converge by comparison to the harmonic series, and therefore $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+7}$ converges conditionally

   (b) $\sum_{n=1}^{\infty}\dfrac{n^5}{(-5)^{n+1}}$

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{-0.2^{(n+1)-1}(n+1)^5}{-0.2^{n-1}n^5}\right| && (20)\\
&= \left|\frac{-0.2(n+1)^5}{n^5}\right| && (21)\\
&= \left|\frac{(n+1)^5}{5n^5}\right| \overset{H}{=} \frac{1}{5}<1 \quad \text{Hence, absolutely convergent} && (22)
\end{aligned}$$

   (c) $\sum_{n=1}^{\infty}\dfrac{n!}{n^n}$

   By the ratio test we get

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^{-n}}{n}=\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^n}=\frac{1}{e}<1$$

   Hence, $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges absolutely.

   (d) $\sum_{n=1}^{\infty}(-1)^n ne^{-n}$

   Again by using the ratio test we can get

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{-1(n+1)}{(e^{n+1})}\cdot\frac{e^n}{n}=\lim_{n\to\infty}-\frac{1}{e}\cdot\frac{n+1}{n}=-\frac{1}{e}$$

   Since $|\dfrac{-1}{e}|<1$, Our series is once again absolutely convergent.

5. Consider the alternating series $\sum_{n=1}^{\infty}(-1)^n \frac{e^{\frac{1}{n}}}{n^4}$. This series converges because its coefficients satisfy the conditions of the alternating series test. How many terms are required to estimate the sum with error ($|R_n|$) less than 10^{-5}

If we wish to find the partial sum of an alternating series with an error at most less than 10^{-5} then we need to find the max value of n such that b_{n+1} (the absolute value of our series) is greater than our allowed error.

$$\frac{e^{\frac{1}{n+1}}}{(n+1)^4} < \frac{1}{10^5}$$

$$\frac{e^{\frac{1}{n+1}}}{(n+1)^4} - \frac{1}{10^5} < 0$$

n	17	18
b_{n+1}	$7.0 \cdot 10^{-8}$	$-1.9 \cdot 10^{-6}$

Thus we choose $n = 17$ to be the number of terms required for an error of at most 10^{-5}