Introduction to Computational Fluid Dynamics:

Coursework 1

Due date: see Blackboard

Before the submission deadline, you should submit a pdf file containing all your answers to all the questions in order, together with Matlab .m files (or source code files) that you used to answer the questions. Make your submission on Blackboard. Do not format your pdf file as a report, just answer the questions in order. Your pdf should be a self-contained document, and it should not be necessary to look at the Matlab/source files to mark the coursework (they are provided to give partial credit in case of errors). It is not necessary to provide an abstract/introduction/conclusion etc.

We shall consider two second order accurate in time and space schemes to discretise the linear advection equation $u_t + a u_x = 0$. The first scheme is the leap frog scheme

$$u_i^{n+1} = u_i^{n-1} - \sigma(u_{i+1}^n - u_{i-1}^n)$$

and the second is the angled derivative scheme

$$u_i^{n+1} = u_{i-1}^{n-1} + (1 - 2\sigma)(u_i^n - u_{i-1}^n)$$

where $\sigma = a\Delta t/\Delta x$ is the Courant number.

1. Solve the two numerical schemes under the following conditions
   - $-1 \le x \le 1$
   - $u(x, 0) = \sin(2\pi x)$
   - Periodic boundary conditions
   - Propagation velocity $a = 0.75$
   - Courant number $\sigma = 0.6$
   - Mesh spacing $\Delta x = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002, 0.001$
   - Final time $T = 1.0$

   (a) Concisely describe your implementation. Note 10% of marks to this problem will be awarded for code clarity and appropriate commenting. [30%]

   (b) To start the multi-level scheme you will need to know the initial conditions at two time levels. Calculate the exact solution at $t = \Delta t$ and use this condition to start your simulation. What alternative strategy might have been adopted which only uses information at time $t = 0$? [10%]

   (c) On a log-log axis plot the maximum error of the solution at time $T = 1$ as a function of Δx. Calculate the maximum error given by

   $\epsilon_{max}(\Delta x) = max|u_i - \bar{u}_i|$ for all i

   where N is the number of mesh points ($N = \frac{2}{\Delta x} + 1$), u_i and $\bar{u}_i$ are the numerical and exact solutions at point x_i, respectively. Does this plot demonstrate second order accuracy in time and space? [20%]

   (d) If this test demonstrates numerical convergence and we know the scheme is stable what other condition would this imply? [10%]

2. Using the initial conditions

$$u(x,0) = \begin{cases} 0 & x < -0.2 \\ 1 & -0.2 \le x \le 0.2 \\ 0 & x > 0.2 \end{cases}$$

solve the two numerical schemes with $\sigma = 0.6$, $\Delta x = 0.005$ at a final time $T = 0.5$ and plot the solutions. Comment on the dispersion and diffusion properties. [30%]