Semantic theories of truth: Lecture 2
Nixon-Dean and Fixed-point semantics

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Outline

- Discussion of Tarskian semantics.
- Interlude: Analysis of the Liar Paradox.
- The fixed-point conception of truth.
- Mathematical interlude: Order-theoretic facts.
- Fixed-point semantic theories of truth.
- References.

The formal setting

The object language

The truth language $\mathcal{L}_\mathsf{T}(N, P)$: $p, \mathsf{T}a, \mathsf{T}b \wedge \neg p, \ldots$
The base language $\mathcal{L}(P)$: $p, q, q \wedge \neg p, \ldots$

A formal semantics

A family $\mathcal{V}$ of admissible valuations of the sentences of the truth language into a set of truth values.

A formalised example

A support X: A set of sentences of the truth language.
A reference list π: A function from the set of names occurring in X into X.

A puzzle

A triple $(X, \pi, \mathcal{V})$. A solution is an assignment of truth values to the sentences in X which agrees with at least one admissible valuation.

Example 2 - The Nixon-Dean dialogue

Suppose Dean asserts
(a) All of Nixon's utterances about Watergate are false,
while Nixon in turn asserts
(b) Everything Dean says about Watergate is false.

Dean, in asserting the sweeping (a), wishes to include Nixon's assertion (b) within its scope (as one of the Nixonian assertions about Watergate which is said to be false); and Nixon, in asserting (b), wishes to do the same with Dean's (a). Now on any theory that assigns intrinsic "levels" to statements, so that a statement of a given level can speak only of the truth or falsity of statements of lower levels, it is plainly impossible for both to succeed: if the two statements are on the same level, neither can talk about the truth or falsity of the other , while otherwise the higher can talk about the lower, but not conversely. Yet intuitively, we can often assign unambiguous truth values to (a) and (b) [...] It seems difficult to accommodate these intuitions within the confines of the orthodox approach. Kripke, *Outline of a theory of truth*, 1975.

Formalised Liar Paradox

Liar sentence

$$a{:}\ \neg Ta.$$

Formal contradiction

1.	$Ta \leftrightarrow \neg Ta$	[Truth schema for a]
2.	$\neg(Ta \wedge \neg\neg Ta) \wedge \neg(\neg Ta \wedge \neg Ta)$	[Equivalence]
3.	$\neg(Ta \wedge Ta) \wedge \neg(\neg Ta \wedge \neg Ta)$	[Double negation]
4.	$\neg Ta \wedge \neg\neg Ta$	[Conjunction]
5.	$\neg Ta \wedge Ta$	[Double negation]

Liar Paradox ingredients

Key ingredients

- A language capable of self-reference.
- Classical logic and semantics.
- The truth schema for the truth language.

The Tarskian operator

Fix a ground model $\mathcal{M} = (D, I^-)$ such that $\mathcal{L}_\mathsf{T}(N, P) \subseteq D$.

The Tarskian operator is the function $\tau : \mathcal{P}(D) \to \mathcal{P}(D)$ defined by

$$\tau(Z) = \{\phi \in \mathcal{L}_\mathsf{T}(N, P) \mid \mathsf{Val}_Z(\phi) = \mathbf{t}\}.$$

Theorem (Theorem 1)

Suppose that every sentence in $\mathcal{L}_\mathsf{T}(N, P)$ has a name in the ground model $\mathcal{M}$. Hence, Z is a fixed point of τ iff $\mathcal{M} + Z$ validates the full truth schema.

Theorem (Theorem 2)

If there is a name a such that $I^-(a) = \neg Ta$, then there is no fixed point of τ.

Partial valuations and interpretations

Partial valuation

A partial valuation v is a function from $\mathcal{L}_{\mathrm{T}}(N, P)$ to the set of truth values $\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ or, equivalently, a *partial function* from $\mathcal{L}_{\mathrm{T}}(N, P)$ to the set of *classic truth values* $\{\mathbf{t}, \mathbf{f}\}$.

Partial interpretation

A partial interpretation $I(\mathrm{T})$ into a domain D is a function from D to the set of truth values $\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ or, equivalently, a *partial set* of D, namely, a pair $(Z^{\mathbf{t}}, Z^{\mathbf{f}})$, where $Z^{\mathbf{t}}, Z^{\mathbf{f}}$ are disjoint subsets of D.

Classic semantics

$v : \mathcal{L}_{\mathrm{T}}(N, P) \to \{\mathbf{t}, \mathbf{f}\}$ is admissible in classic semantics iff

- $v(\neg\phi) = -v(\phi)$
- $v(\phi \wedge \psi) = v(\phi) * v(\psi)$.

Classic negation

$v(\phi)$	$-v(\phi)$
t	**f**
f	**t**

Classic conjunction

$v(\phi) * v(\psi)$	**t**	**f**
t	**t**	**f**
f	**f**	**f**

Strong Kleene semantics

$v : \mathcal{L}_{\mathrm{T}}(N, P) \to \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ is admissible in Strong Kleene semantics iff it satisfies the negation and conjunction conditions interpreted as follows:

Strong Kleene negation

$v(\phi)$	$-v(\phi)$
t	**f**
f	**t**
n	**n**

Strong Kleene conjunction

$v(\phi) * v(\psi)$	**t**	**f**	**n**
t	**t**	**f**	**n**
f	**f**	**f**	**f**
n	**n**	**f**	**n**

The Strong Kleene operator

A partial model for $\mathcal{L}_{\mathrm{T}}(N, P)$ is denoted by $\mathcal{M} + (Z^{\mathbf{t}}, Z^{\mathbf{f}})$, where $\mathcal{M}$ is a ground model and $(Z^{\mathbf{t}}, Z^{\mathbf{f}})$ is a partial set of D.

$\mathrm{Val}^{\kappa}_{(Z^{\mathbf{t}}, Z^{\mathbf{f}})}(\phi)$ denotes the evaluation of the sentence ϕ in the partial model $\mathcal{M} + (Z^{\mathbf{t}}, Z^{\mathbf{f}})$ according to the Strong Kleene semantics.

The Strong Kleene operator is the function κ on partial sets of D defined by

$$\kappa(Z^{\mathbf{t}}, Z^{\mathbf{f}}) = (W^{\mathbf{t}}, W^{\mathbf{f}}),$$

where

- $W^{\mathbf{t}} = \{\phi \in \mathcal{L}_{\mathrm{T}}(N, P) \mid \mathrm{Val}^{\kappa}_{(Z^{\mathbf{t}}, Z^{\mathbf{f}})} = \mathbf{t}\}$.
- $W^{\mathbf{f}} = \{\phi \in \mathcal{L}_{\mathrm{T}}(N, P) \mid \mathrm{Val}^{\kappa}_{(Z^{\mathbf{t}}, Z^{\mathbf{f}})} = \mathbf{f}\}$.

The fixed-point conception of truth

Partial truth schema

Fix an interpretation I^- of the names. A partial valuation v admissible in Strong Kleene semantics has to satisfy, for every name $a \in N$,

$$v(\mathrm{T}a) = v(I^-(a)).$$

Theorem (Theorem 3)

Suppose that every sentence in $\mathcal{L}_\mathrm{T}(N, P)$ has a name in the ground model $\mathcal{M}$. Hence, $(Z^\mathbf{t}, Z^\mathbf{f})$ is a fixed point of κ iff $(Z^\mathbf{t}, Z^\mathbf{f})$ validates the partial truth schema.

Theorem (Theorem 4)

There are fixed points of κ for every ground model $\mathcal{M}$.

Coherent complete partial orders

Let $(Q, \preceq)$ be a *partially ordered set*. We say that $(Q, \preceq)$ is coherent complete (or ccpo) iff every coherent subset of Q has the least upper bound in Q (where $X \subseteq Q$ is coherent iff every two members of X are compatible in Q).

A function $f : Q \to Q$ is monotonic iff $\forall x, y \in Q\,(x \preceq y \Rightarrow f(x) \preceq f(y))$.

Fact (Fact 1)

If $(Q, \preceq)$ is a ccpo and $f : Q \to Q$ is monotonic, then the set $\mathrm{Fix}(f)$ of fixed points of f forms a sub-ccpo of Q. In particular, $\mathrm{Fix}(f)$ has the minimum, thus it is not empty.

Weak Kleene semantics

$v : \mathcal{L}_\mathrm{T}(N, P) \to \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ is admissible in Weak Kleene semantics iff it satisfies the negation and conjunction conditions interpreted as follows:

Weak Kleene negation

$v(\phi)$	$-v(\phi)$
t	**f**
f	**t**
n	**n**

Weak Kleene conjunction

$v(\phi) * v(\psi)$	**t**	**f**	**n**
t	**t**	**f**	**n**
f	**f**	**f**	**n**
n	**n**	**n**	**n**

Fixed-point semantics

Semantics		all	lfp	gifp
Strong Kleene	κ			
Weak Kleene	μ			
Supervaluation	σ			
Consistent supervaluation	σ_1			
Max consistent supervaluation	σ_2			