Intuitionistic modal logic

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- Intermediate logics
- Modal logics
- **Combining logics**
- Two peculiar intuitionistic modal logics
- A minimal setting

Combining logics

Outline

- Combining modal logics
- Combining intuitionistic connectives and classical connectives
- Combining intuitionistic connectives and modal connectives

Combining logics: combining modal logics

Combining logics: combining modal logics

Introduction

- When can we say that a modal logic is a combination of others?
- Given a family $\mathbb{L}$ of modal logics and a combination method C, do certain properties of the component logics $\mathbf{L} \in \mathbb{L}$ transfer to their combination $C(\mathbb{L})$?

Most of the combination methods are such that

- C is defined only on finite families $\mathbb{L}$ of modal logics
- $C(\mathbb{L})$ is a modal logic itself
- $C(\mathbb{L})$ is an extension of each component logic $\mathbf{L} \in \mathbb{L}$

Combining logics: combining modal logics

Transfer of axiomatization/completeness

- does the combination of recursively axiomatizable (finitely axiomatizable) modal logics remain recursively axiomatizable (finitely axiomatizable)?
- does the combination of Kripke-complete modal logics remain Kripke-complete?

Transfer of decidability/complexity

- does the combination of decidable modal logics remain decidable and if so, what is the change in the complexity?
- does the combination of finitely approximable modal logics remain finitely approximable?

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Fusion of modal logics

The fusion $\mathbf{L}_1 \otimes \mathbf{L}_2$ of $\mathbf{L}_1$ formulated with $\Box_1$ and $\mathbf{L}_2$ formulated with $\Box_2$ is

- the least normal modal logic formulated with $\Box_1$ and $\Box_2$ and containing $\mathbf{L}_1$ and $\mathbf{L}_2$

Examples

- $\mathbf{K}_{\Box_1} \otimes \mathbf{K}_{\Box_2}$
- $\mathbf{S}5_{\Box_1} \otimes \mathbf{S}5_{\Box_2}$
- $\mathbf{S}5_{\Box_1} \otimes \mathbf{S}5_{\Box_2}$

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Transfer results

Theorem

If $\mathbf{L}_1$ and $\mathbf{L}_2$ are consistent then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is a conservative extension of $\mathbf{L}_1$ and $\mathbf{L}_2$.

Theorem

If $\mathbf{L}_1$ and $\mathbf{L}_2$ are characterized by classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of frames then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is characterized by the class of frames of the form (W, R_1, R_2) where (W, R_1) is in $\mathcal{C}_1$ and (W, R_2) is in $\mathcal{C}_2$.

Theorem

If $\mathbf{L}_1$ and $\mathbf{L}_2$ have the finite model property then $\mathbf{L}_1 \otimes \mathbf{L}_2$ has the finite model property.

Theorem

If $\mathbf{L}_1$ and $\mathbf{L}_2$ are decidable then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is decidable.

Theorem

If $\mathbf{L}_1$ and $\mathbf{L}_2$ are in $\{\mathbf{K}, \mathbf{KT}, \mathbf{K}4, \mathbf{S}4, \mathbf{KD}45, \mathbf{S}5\}$ then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is **PSPACE**-complete.

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Product of frames

The product frame $\mathcal{F}_1 \times \mathcal{F}_2$ of the frames $\mathcal{F}_1 = (W_1, R_1)$ and $\mathcal{F}_2 = (W_2, R_2)$ is

- the frame (W, S_1, S_2) where $W = W_1 \times W_2$ and for all $u_1, v_1 \in W_1$ and for all $u_2, v_2 \in W_2$
  - $(u_1, u_2)S_1(v_1, v_2)$ if and only if $u_1 R_1 v_1$ and $u_2 = v_2$
  - $(u_1, u_2)S_2(v_1, v_2)$ if and only if $u_1 = v_1$ and $u_2 R_2 v_2$

Examples

- Time and knowledge
- Time and space

Product classes

For all classes $\mathcal{C}_1$, $\mathcal{C}_2$ of frames, let

- $\mathcal{C}_1 \times \mathcal{C}_2 = \{\mathcal{F}_1 \times \mathcal{F}_2 : \ \mathcal{F}_1 \in \mathcal{C}_1 \ \& \ \mathcal{F}_2 \in \mathcal{C}_2\}$

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Product of frames

Proposition

For all frames $\mathcal{F}_1 = (W_1, R_1)$ and $\mathcal{F}_2 = (W_2, R_2)$, the product frame $\mathcal{F}_1 \times \mathcal{F}_2 = (W, S_1, S_2)$ is such that

- $S_1 \circ S_2 \subseteq S_2 \circ S_1$
- $S_2 \circ S_1 \subseteq S_1 \circ S_2$
- $S_1^{-1} \circ S_2 \subseteq S_2 \circ S_1^{-1}$
- $S_2^{-1} \circ S_1 \subseteq S_1 \circ S_2^{-1}$

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Product of frames

Proposition

For all frames $\mathcal{F}_1 = (W_1, R_1)$, $\mathcal{F}_1' = (W_1', R_1')$, $\mathcal{F}_2 = (W_2, R_2)$, $\mathcal{F}_2' = (W_2', R_2')$,

- If $\mathcal{F}_1'$ is a bounded morphic image of $\mathcal{F}_1$ and $\mathcal{F}_2'$ is a bounded morphic image of $\mathcal{F}_2$ then $\mathcal{F}_1' \times \mathcal{F}_2'$ is a bounded morphic image of $\mathcal{F}_1 \times \mathcal{F}_2$
- If $\mathcal{F}_1'$ is a generated subframe of $\mathcal{F}_1$ and $\mathcal{F}_2'$ is a generated subframe of $\mathcal{F}_2$ then $\mathcal{F}_1' \times \mathcal{F}_2'$ is a generated subframe of $\mathcal{F}_1 \times \mathcal{F}_2$

The product logic $\mathbf{L}_1 \times \mathbf{L}_2$ of a Kripke-complete modal logic $\mathbf{L}_1$ formulated with $\Box_1$ and a Kripke-complete modal logic $\mathbf{L}_2$ formulated with $\Box_2$ is

- the modal logic $\mathbf{Log}(\{\mathcal{F}_1 \times \mathcal{F}_2 : \ \mathcal{F}_1 \models \mathbf{L}_1 \ \& \ \mathcal{F}_2 \models \mathbf{L}_2\})$ formulated with $\Box_1$ and $\Box_2$

Examples

- $\mathbf{K}_{\Box_1} \times \mathbf{K}_{\Box_2}$
- $\mathbf{K}_{\Box_1} \times \mathbf{S}5_{\Box_2}$
- $\mathbf{S}5_{\Box_1} \times \mathbf{S}5_{\Box_2}$

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Notes about products

There exists classes $\mathcal{C}_1$, $\mathcal{C}_1'$, $\mathcal{C}_2$, $\mathcal{C}_2'$ of frames such that

- $\mathbf{Log}(\mathcal{C}_1) = \mathbf{Log}(\mathcal{C}_1')$
- $\mathbf{Log}(\mathcal{C}_2) = \mathbf{Log}(\mathcal{C}_2')$
- $\mathbf{Log}(\mathcal{C}_1 \times \mathcal{C}_2) \neq \mathbf{Log}(\mathcal{C}_1' \times \mathcal{C}_2')$

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Properties of products

Proposition

For all consistent Kripke-complete modal logics $\mathbf{L}_1$, $\mathbf{L}_2$,

- $\mathbf{L}_1 \otimes \mathbf{L}_2 \subseteq \mathbf{L}_1 \times \mathbf{L}_2$
- $\mathbf{L}_1 \times \mathbf{L}_2$ is a conservative extension of $\mathbf{L}_1$ and $\mathbf{L}_2$

Proposition

For all consistent Kripke-complete modal logics $\mathbf{L}_1$, $\mathbf{L}_2$, if $\mathbf{Fr}(\mathbf{L}_1)$ and $\mathbf{Fr}(\mathbf{L}_2)$ are first-order definable then $\mathbf{L}_1 \times \mathbf{L}_2$ is determined by the class of its countable product frames

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Axiomatizing products

Definition

For all consistent Kripke-complete modal logics $\mathbf{L}_1$, $\mathbf{L}_2$, let $[\mathbf{L}_1, \mathbf{L}_2]$ be the least modal logic formulated with $\Box_1$ and $\Box_2$ and containing

- $\mathbf{L}_1$
- $\mathbf{L}_2$
- $\Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_1 p$
- $\Diamond_2\Diamond_1 p \to \Diamond_1\Diamond_2 p$
- $\Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$
- $\Diamond_2\Box_1 p \to \Box_1\Diamond_2 p$

Proposition

For all consistent Kripke-complete modal logics $\mathbf{L}_1$, $\mathbf{L}_2$,
$\mathbf{L}_1 \otimes \mathbf{L}_2 \subseteq [\mathbf{L}_1, \mathbf{L}_2] \subseteq \mathbf{L}_1 \times \mathbf{L}_2$

Horn sentences

A Horn sentence is a universal first-order sentence of the form

- $\forall x\ \forall y\ \forall \mathbf{z}\ (\varphi(x, y, \mathbf{z})\ \rightarrow\ R(x, y))$

where $\varphi(x, y, \mathbf{z})$ is positive

Examples of Horn sentences

- $\forall x\ \forall y\ \forall z\ (R(x, z)\ \&\ R(z, y)\ \rightarrow\ R(x, y))$
- $\forall x\ \forall y\ \forall z\ (R(z, x)\ \&\ R(z, y)\ \rightarrow\ R(x, y))$
- $\forall x\ \forall y\ (R(y, x)\ \rightarrow\ R(x, y))$

Horn modal formulas

A Horn modal formula is a modal formula corresponding to a Horn sentence

Examples of Horn modal formulas

- $\Box p \to \Box\Box p$ vs. $\forall x\ \forall y\ \forall z\ (R(x,z)\ \&\ R(z,y)\ \to\ R(x,y))$
- $\Diamond p \to \Box\Diamond p$ vs. $\forall x\ \forall y\ \forall z\ (R(z,x)\ \&\ R(z,y)\ \to\ R(x,y))$
- $p \to \Box\Diamond p$ vs. $\forall x\ \forall y\ (R(y,x)\ \to\ R(x,y))$

Horn-axiomatizable modal logics

A Horn-axiomatizable modal logic is a modal logic axiomatizable by Horn modal formulas and variable-free formulas

Examples of Horn-axiomatizable modal logics

- **K**4 — axiomatizable by $\Box p \to \Box\Box p$
- **K**5 — axiomatizable by $\Diamond p \to \Box\Diamond p$
- **KB** — axiomatizable by $p \to \Box\Diamond p$
- **S**4, **S**5, ...

Proposition

For all consistent Kripke-complete modal logics $\mathbf{L}_1$, $\mathbf{L}_2$, if $\mathbf{L}_1$ and $\mathbf{L}_2$ are Horn-axiomatizable then $[\mathbf{L}_1, \mathbf{L}_2] = \mathbf{L}_1 \times \mathbf{L}_2$

Combining logics:
combining intuitionistic connectives and classical connectives

Combining logics: combining intuitionistic connectives and classical connectives

The approach of Fariñas del Cerro and Herzig (1996)

Intuitionistic syntax

- Atomic formulas: $p \in AF$
- Formulas: $\phi \in Fma_i(AF)$

$$\phi ::= p \mid \bot_i \mid \top_i \mid (\phi_1 \vee_i \phi_2) \mid (\phi_1 \wedge_i \phi_2) \mid (\phi_1 \rightarrow_i \phi_2)$$

Classical syntax

- Atomic formulas: $p \in AF$
- Formulas: $\phi \in Fma_c(AF)$

$$\phi ::= p \mid \bot_c \mid \top_c \mid (\phi_1 \vee_c \phi_2) \mid (\phi_1 \wedge_c \phi_2) \mid (\phi_1 \rightarrow_c \phi_2)$$

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Combining intuitionistic syntax and classical syntax

- Atomic formulas: $p \in AF$
- Formulas: $\phi \in Fma_{i,c}(AF)$

$$\phi ::= p \mid \bot \mid \top \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \rightarrow_i \phi_2) \mid (\phi_1 \rightarrow_c \phi_2)$$

Abbreviations

- $\neg_i \phi ::= (\phi \rightarrow_i \bot)$
- $\neg_c \phi ::= (\phi \rightarrow_c \bot)$

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Semantics

As expected

- Frames: $(S, \leq)$ where $S \neq \emptyset$ and $\leq$ partial order on S
- Models: $(S, \leq, V)$ where V intuitionistic valuation on S
- Truth conditions:
  - $\mathcal{M} \models_s p$ iff $s \in V(p)$
  - $\mathcal{M} \not\models_s \bot$
  - $\mathcal{M} \models_s \top$
  - $\mathcal{M} \models_s \phi \vee \psi$ iff $\mathcal{M} \models_s \phi$ or $\mathcal{M} \models_s \psi$
  - $\mathcal{M} \models_s \phi \wedge \psi$ iff $\mathcal{M} \models_s \phi$ and $\mathcal{M} \models_s \psi$
  - $\mathcal{M} \models_s \phi \rightarrow_i \psi$ iff for all $t \in S$, if $s \leq t$ then $\mathcal{M} \not\models_t \phi$ or $\mathcal{M} \models_t \psi$
  - $\mathcal{M} \models_s \phi \rightarrow_c \psi$ iff $\mathcal{M} \not\models_s \phi$ or $\mathcal{M} \models_s \psi$

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Exercise

Let $\mathcal{M} = (S, \leq, V)$ be a model and $s \in S$.
Show that $\mathcal{M} \models_s \neg_i \phi$ iff for all $t \in S$, if $s \leq t$ then $\mathcal{M} \not\models_t \phi$.
Show that $\mathcal{M} \models_s \neg_c \phi$ iff $\mathcal{M} \not\models_s \phi$.

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Exercise

Find a formula ϕ, a model $\mathcal{M}$ and s, t in $\mathcal{M}$ such that

- $s \leq t$
- $\mathcal{M} \models_s \phi$
- $\mathcal{M} \not\models_t \phi$

Exercise

Show that although $p \rightarrow_i (q \rightarrow_i p)$ is valid, $\neg_c p \rightarrow_i (q \rightarrow_i \neg_c p)$ is not valid

Exercise

Show that the rule of uniform substitution does not preserve validity

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As a result

We cannot get an axiomatization of the set of all valid formulas by simply putting together

- an axiomatization of **IPL**
- an axiomatization of **CPL**

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Persistent formulas

A formula is persistent if every occurrence of $\rightarrow_c$ is in the scope of an occurrence of $\rightarrow_i$

Examples

- $p \rightarrow_i (q \rightarrow_c p)$ is persistent
- $p \rightarrow_c (q \rightarrow_i p)$ is not persistent

Lemma

For all persistent formulas ϕ, for all models $\mathcal{M}$ and for all s, t in $\mathcal{M}$,

- if $s \leq t$ and $\mathcal{M} \models_s \phi$ then $\mathcal{M} \models_t \phi$

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Axiomatization

- all theorems of **CPL**
- $(\phi \rightarrow_i (\psi \rightarrow_c \chi)) \rightarrow_c ((\phi \rightarrow_i \psi) \rightarrow_c (\phi \rightarrow_i \chi))$
- $\phi \rightarrow_i \phi$
- $(\phi \rightarrow_i \psi) \rightarrow_c (\phi \rightarrow_c \psi)$
- $\phi \rightarrow_c (\psi \rightarrow_i \phi)$ when ϕ is persistent
- $\frac{\phi \quad \phi \rightarrow_i \psi}{\psi}$
- $\frac{\phi}{\psi \rightarrow_i \phi}$

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Proposition

For all formulas ϕ, the following conditions are equivalent:

- ϕ is a theorem of the above axiomatization,
- ϕ is valid in the class of all frames.

Proposition

The unrestricted acceptance of one of the following axioms would make the above axiomatization collapse into **CPL**

- $\phi \rightarrow_i (\psi \rightarrow_i \phi)$,
- $\phi \rightarrow_c (\psi \rightarrow_i \phi)$.

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combining intuitionistic connectives and modal connectives

Syntax of intermediate logics

$$\phi ::= p \mid \bot \mid \top \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \to \phi_2)$$

Syntax of modal logics

$$\phi ::= p \mid \bot \mid \top \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \rightarrow \phi_2) \mid \Box\phi \mid \Diamond\phi$$

Combining logics: combining intuitionistic connectives and modal connectives

Syntax of intuitionistic modal logics

$$\phi ::= p \mid \bot \mid \top \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \to \phi_2) \mid \Box\phi \mid \Diamond\phi$$

Brouwer-Heyting-Kolmogorov reading

The intended meaning of the intuitionistic connectives is given in terms of proofs and constructions

- A proof of $\phi \to \psi$ is a construction which, given a proof of ϕ, produces a proof of ψ

Combining logics: combining intuitionistic connectives and modal connectives

Alethic readings of $\Box$ and $\Diamond$

- $\Box\phi$: It is necessarily true that ϕ
- $\Diamond\phi$: It is possibly true that ϕ

Deontic readings of $\Box$ and $\Diamond$

- $\Box\phi$: It ought to be that ϕ
- $\Diamond\phi$: It is permitted that ϕ

What about the intended meaning of the modal connectives?

- A proof of $\Box\phi$ is ...
- A proof of $\Diamond\phi$ is ...

Combining logics: combining intuitionistic connectives and modal connectives

Kripke semantics of intermediate logics

- $\phi \rightarrow \psi$ is true at world s when for every subsequent possible world t, in particular s itself, ϕ is true at t only if ψ is true at t

Kripke semantics of modal logics

- $\Box\phi$ is true at world s when for every accessible possible world t, ϕ is true at t
- $\Diamond\phi$ is true at world s when for some accessible possible world t, ϕ is true at t

What about the Kripke semantics of intuitionistic modal logics?

- $\Box\phi$ is true at world s when ...
- $\Diamond\phi$ is true at world s when ...

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Expected properties of an intuitionistic modal logic **IML**

Property $\mathbf{P_1}$: closure with respect to modus ponens and substitution

IML is closed with respect to the following inference rules:

modus ponens $\frac{\phi \quad \phi \to \psi}{\psi}$

substitution $\frac{\phi}{\sigma(\phi)}$, where σ is a substitution

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Expected properties of an intuitionistic modal logic **IML**

Property $\mathbf{P_2}$: conservativity

IML is a conservative extension of **IPL**

- For all $\{\Box, \Diamond\}$-free formulas ϕ, ϕ is in **IML** if and only if ϕ is in **IPL**

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Expected properties of an intuitionistic modal logic **IML**

Property $\mathbf{P_3}$: adding Pierce's law or excluded middle

The addition of one of the following formulas to **IML** yields a standard classical modal logic

- $p \vee (p \to \perp)$
- $\neg\neg p \to p$
- $((p \to q) \to p) \to p$

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Expected properties of an intuitionistic modal logic **IML**

Property $\mathbf{P_4}$: disjunction property

For all formulas ϕ, ψ,

- if $\phi \vee \psi$ is in **IML** then ϕ is in **IML** or ψ is in **IML**

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Expected properties of an intuitionistic modal logic **IML**

Property $\mathbf{P_5}$: independence of $\Box$ and $\Diamond$

$\Box$ and $\Diamond$ are not interdefinable in **IML**, i.e.

- there is no $\Box$-free formula ϕ such that $\Box p \leftrightarrow \phi$ is in **IML**
- there is no $\Diamond$-free formula ϕ such that $\Diamond p \leftrightarrow \phi$ is in **IML**

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Expected properties of an intuitionistic modal logic **IML**

Property $\mathbf{P_6}$: heredity of truth

For all formulas ϕ, for all models $\mathcal{M}$ and for all s, t in $\mathcal{M}$,

- If $s \leq t$ and $\mathcal{M} \models_s \phi$ then $\mathcal{M} \models_t \phi$

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Early approaches to intuitionistic modal logic

Fitch (1948)

First-order intuitionistic version of the modal logic **KT** with the Barcan formula

- Hilbert-style axiomatization
- Gentzen-style sequent calculus formulation

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Early approaches to intuitionistic modal logic

Prior (1957)

Propositional intuitionistic version **MIPQ** of the modal logic **S**5

Bull (1965, 1966)

- Algebraic semantics of **MIPQ** in terms of Heyting algebras with additional structure
- Finite model property of **MIPQ**
- Faithfulness of a translation into first-order intuitionistic logic

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Early approaches to intuitionistic modal logic

Prawitz (1965)

Intuitionistic analogues of the modal logics **S**4 and **S**5

- Natural deduction systems
- Normalization theorem for the $\Diamond$-free fragment of the intuitionistic analogue of the modal logic **S**4

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Early approaches to intuitionistic modal logic

Ono (1977)

Intuitionistic analogues of the modal logics **S**4 and **S**5

- Algebraic semantics
- Kripke-style semantics
- Finite model property

Font (1986)

- Classification of the combinations of $\neg$ and $\Box$ in the Intuitionistic analogues of the modal logic **S**4

Fischer Servi (1977, 1978, 1984)

Determination of the correct intuitionistic analogues of classical modal logics

- The intuitionistic analogue of **S**5 determined by a Gödel-like translation into a classical bimodal logic is none other than Prior's **MIPQ**
- Algebraic and Kripke semantics of some intuitionistic modal logics
- Axiomatizations of some intuitionistic modal logics

Ewald (1986)

Models for intuitionistic temporal logics based on Kripke's models of first-order intuitionistic logic

- Axiomatizations of some intuitionistic temporal logics corresponding to various conditions on the temporal ordering
- Finite model property

Other approaches

- Gabbay (1975)
- Božić and Došen (1984)
- Sotirov (1984)
- Wijesekera (1990)

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Various applications in Computer Science of intuitionistic modal logic

Stirling (1987)

- An intuitionistic modal logic is used to capture a bisimulation preorder on diverging processes

Nerode and Wijesekera (1990)

- An intuitionistic version of dynamic logic is used to build a logic on top of transition systems between partial states

Pitts (1990)

- An intuitionistic modal logic is used to reason about functional programs with side-effects

Combining logics: combining intuitionistic connectives and modal connectives

Two peculiar intuitionistic modal logics

Artemov and Protopopescu (2016)

- **IEL**: Intuitionistic Epistemic Logic

Fairtlough and Mendler (1997)

- **PLL**: Propositional Lax Logic

References:

- Artemov, S., Protopopescu, T.: Intuitionistic Epistemic Logic. The Review of Symbolic Logic **9** (2016) 266–298.
- Božić, M., Došen, K.: Models for normal intuitionistic modal logics. Studia Logica **43** (1984) 217–245.
- Bull, R.: A modal extension of intuitionistic logic. Notre Dame Journal of Formal Logic **VI** (1965) 142–146.
- Bull, R.: **MIPC** as the formalisation of an intuitionistic concept of modality. Journal of Symbolic Logic **31** (1966) 609–616.
- Ewald, W.: Intuitionistic tense and modal logic. The Journal of Symbolic Logic **51** (1986) 166–179.
- Fairtlough, M., Mendler, M.: Propositional Lax Logic. Information and Computation **137** (1997) 1–33.

References:

- Fariñas del Cerro, L., Herzig, A.: Combining classical and intuitionistic logic. Or: intuitionistic implication as a conditional. In: Frontiers of Combining Systems. Springer (1996) 93–102.
- Fischer Servi, G.: On modal logic with an intuitionistic base. Studia Logica **36** (1977) 141–149.
- Fischer Servi, G.: Semantics for a class of intuitionistic modal calculi. Bulletin of the Section of Logic **7** (1978) 26–29.
- Fischer Servi, G.: Axiomatizations for some intuitionistic modal logics. Rendiconti del Seminario Matematico Università e Politecnico di Torino **42** (1984) 179–194.
- Fitch, F.: Intuitionistic modal logic with quantifiers. Portugaliae Mathematica **7** (1948) 113–118.
- Font, J.: Modality and possibility in some intuitionistic modal logics. Notre Dame Journal of Formal Logic **27** (1986) 533–546.

References:

- Gabbay, D.: Decidability results in non-classical logics. Part I. Annals of Mathematical Logic **8** (1975) 237–295.
- Gabbay, D., Kurucz, A., Wolter, F., Zakharyaschev, M.: Many-Dimensional Modal Logics: Theory and Applications. Elsevier (2003).
- Gabbay, D., Shehtman, V.: Products of modal logics, part 1. Logic Journal of the IGPL **6** (1998) 73–146.
- Gabbay, D., Shehtman, V.: Products of modal logics. Part 2: relativized quantifiers in classical logic. Logic Journal of the IGPL **8** (2000) 165–210.
- Kracht, M., Wolter, F.: Properties of independently axiomatizable bimodal logics. Journal of Symbolic Logic **56** (1991) 1469–1485.
- Kurucz, A.: Combining modal logics. In: Handbook of Modal Logic. Elsevier (2007) 869–924.

References:

- Ono, H.: On some intuitionistic modal logics. Publications of the Research Institute for Mathematical Sciences **13** (1977) 687–722.
- Prawitz, D.: Natural Deduction – A Proof Theoretical Study. Almqvist and Wiksell (1965).
- Prior, A.: Time and Modality. Oxford University Press (1957).
- Simpson, A.: The Proof Theory and Semantics of Intuitionistic Modal Logic. Doctoral thesis at the University of Edinburgh (1994).
- Sotirov, V.: Modal theories with intuitionistic logic. In *Mathematical Logic.* Publishing House of the Bulgarian Academy of Sciences (1984) 139–171.
- Wijesekera, D.: Constructive modal logics I. Annals of Pure and Applied Logic **50** (1990) 271–301.

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