Intuitionistic modal logic

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- **Intermediate logics**
- Modal logics
- Combining logics
- Two peculiar intuitionistic modal logics
- A minimal setting

Intermediate logics

Outline

- Classical Propositional Logic
- Intuitionistic Propositional Logic
- Examples of intermediate logics

Intermediate logics: Classical Propositional Logic

Intermediate logics: Classical Propositional Logic

Syntax and semantics

- AF: countable set of atomic formulas
- $Fma(AF)$: set of all formulas generated from AF

- Atomic formulas: $p \in AF$
- Formulas: $\phi \in Fma(AF)$

$$\phi ::= p \mid \bot \mid \top \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \to \phi_2)$$

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Possible readings

- $\perp$: "false"
- $\top$: "true"
- $\phi_1 \vee \phi_2$: "ϕ_1 or ϕ_2"
- $\phi_1 \wedge \phi_2$: "ϕ_1 and ϕ_2"
- $\phi_1 \to \phi_2$: "if ϕ_1 then ϕ_2"

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Other connectives

Negation: $\neg\phi ::= (\phi \to \bot)$

Equivalence: $(\phi_1 \leftrightarrow \phi_2) ::= ((\phi_1 \to \phi_2) \wedge (\phi_2 \to \phi_1))$

For all formulas ϕ

$\|\phi\|$ denotes the number of symbols in ϕ

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Examples of formulas

- $(((p \to \bot) \vee q) \to ((q \wedge r) \wedge s))$
- $\neg p \vee q \to q \wedge r \wedge s$

$SF(\phi)$: set of all subformulas of $\phi \in Fma(AF)$

$$\begin{aligned}
SF(p) &= \{p\} \\
SF(\bot) &= \{\bot\} \\
SF(\top) &= \{\top\} \\
SF(\phi \vee \psi) &= \{\phi \vee \psi\} \cup SF(\phi) \cup SF(\psi) \\
SF(\phi \wedge \psi) &= \{\phi \wedge \psi\} \cup SF(\phi) \cup SF(\psi) \\
SF(\phi \to \psi) &= \{\phi \to \psi\} \cup SF(\phi) \cup SF(\psi)
\end{aligned}$$

Lemma

$\mathtt{Card}(SF(\phi)) = \mathcal{O}(\|\phi\|)$

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Semantic assumptions

- each atomic formula is either true ($\mathtt{T}$) or false ($\mathtt{F}$)
- $\bot$ is always false ($\mathtt{F}$)
- $\top$ is always true ($\mathtt{T}$)
- the truth values of $\phi_1 \vee \phi_2$, $\phi_1 \wedge \phi_2$ and $\phi_1 \to \phi_2$ are uniquely determined by the truth values of ϕ_1 and ϕ_2

ϕ_1	ϕ_2	$\phi_1 \vee \phi_2$	$\phi_1 \wedge \phi_2$	$\phi_1 \to \phi_2$
F	F	F	F	T
F	T	T	F	T
T	F	T	F	F
T	T	T	T	T

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Truth values of the other connectives

As a result

- the truth values of $\neg\phi$ and $\phi_1 \leftrightarrow \phi_2$ are uniquely determined by the truth values of ϕ, ϕ_1 and ϕ_2

ϕ	$\neg\phi$
F	T
T	F

ϕ_1	ϕ_2	$\phi_1 \leftrightarrow \phi_2$
F	F	T
F	T	F
T	F	F
T	T	T

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Exercise

Show that $(\phi_1 \leftrightarrow \phi_2) \leftrightarrow \phi_3$, $(\phi_2 \leftrightarrow \phi_3) \leftrightarrow \phi_1$ and $(\phi_3 \leftrightarrow \phi_1) \leftrightarrow \phi_2$ have the same truth values.

Show that $\neg(\phi_1 \leftrightarrow \phi_2)$, $\neg\phi_1 \leftrightarrow \phi_2$ and $\phi_1 \leftrightarrow \neg\phi_2$ have the same truth values.

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Models

A model is a function $\mathcal{M} = V$ where

- $V : AF \to \{\mathtt{F}, \mathtt{T}\}$

$\mathcal{M} \models \phi$: relation "ϕ is true in model $\mathcal{M}$"

- $\mathcal{M} \models p$ iff $V(p) = \mathtt{T}$
- $\mathcal{M} \not\models \bot$
- $\mathcal{M} \models \top$
- $\mathcal{M} \models \phi \vee \psi$ iff $\mathcal{M} \models \phi$ or $\mathcal{M} \models \psi$
- $\mathcal{M} \models \phi \wedge \psi$ iff $\mathcal{M} \models \phi$ and $\mathcal{M} \models \psi$
- $\mathcal{M} \models \phi \to \psi$ iff $\mathcal{M} \not\models \phi$ or $\mathcal{M} \models \psi$

Intermediate logics: Classical Propositional Logic

Exercise

Let $\mathcal{M} = V$ be a model where $V : AF \to \{\mathtt{F}, \mathtt{T}\}$

Show that $\mathcal{M} \models \neg\phi$ iff $\mathcal{M} \not\models \phi$.

Work out the corresponding truth condition for $\phi \leftrightarrow \psi$.

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Validities

A formula is valid if it is true in all models

- $p \vee (p \to \bot)$ is valid
- $(p \wedge q \to \bot) \to (p \to \bot)$ is not valid

Let **CPL** be the set of all ϕ such that ϕ is valid

p	$\bot$	$p \to \bot$	$p \vee (p \to \bot)$
F	F	T	T
T	F	F	T

p	q	$\bot$	$p \to \bot$	$p \wedge q$	$p \wedge q \to \bot$	$(p \wedge q \to \bot) \to (p \to \bot)$
F	F	F	T	F	T	T
F	T	F	T	F	T	T
T	F	F	F	F	T	F
T	T	F	F	T	F	T

Intermediate logics: Classical Propositional Logic

Frames and models

A frame is a set $\mathcal{F} = S$ where

- S is a nonempty set of "worlds"

A model on a frame is a pair $\mathcal{M} = (S, V)$ where

- $V : AF \to 2^S$

$V(p)$: set of worlds where p is "true", for every atomic formula p

$\mathcal{M} \models_s \phi$: relation "$\phi$ is true at world s in model $\mathcal{M}$"

- $\mathcal{M} \models_s p$ iff $s \in V(p)$
- $\mathcal{M} \not\models_s \bot$
- $\mathcal{M} \models_s \top$
- $\mathcal{M} \models_s \phi \vee \psi$ iff $\mathcal{M} \models_s \phi$ or $\mathcal{M} \models_s \psi$
- $\mathcal{M} \models_s \phi \wedge \psi$ iff $\mathcal{M} \models_s \phi$ and $\mathcal{M} \models_s \psi$
- $\mathcal{M} \models_s \phi \to \psi$ iff $\mathcal{M} \not\models_s \phi$ or $\mathcal{M} \models_s \psi$

Intermediate logics: Classical Propositional Logic

Exercise

Let $\mathcal{F} = S$ be a frame where S is a nonempty set of "worlds" and $\mathcal{M} = (S, V)$ be a model on $\mathcal{F}$ where $V : AF \to 2^S$

Show that for all $s \in S$, $\mathcal{M} \models_s \neg\phi$ iff $\mathcal{M} \not\models_s \phi$.
Work out the corresponding truth condition for $\phi \leftrightarrow \psi$.

Let ϕ be a formula

Show that the following conditions are equivalent:

- ϕ is valid,
- ϕ is true at all worlds in all models.

Determining whether a given formula is valid

Our wish: design a systematic procedure

Semantic tableaux

A semantic tableau is a pair $t = (\Gamma, \Delta)$ with $\Gamma, \Delta \subseteq Fma(AF)$

Disjoint tableaux

The tableau $t = (\Gamma, \Delta)$ is disjoint if

- $\Gamma \cap \Delta = \emptyset$

Saturated tableaux

The tableau $t = (\Gamma, \Delta)$ is saturated if for all formulas ϕ, ψ

- $\bot \in \Delta$
- $\top \in \Gamma$
- if $\phi \vee \psi$ is in Γ then $\psi \in \Gamma$ or $\phi \in \Gamma$
- if $\phi \vee \psi$ is in Δ then $\psi \in \Delta$ and $\phi \in \Delta$
- if $\phi \wedge \psi$ is in Γ then $\psi \in \Gamma$ and $\phi \in \Gamma$
- if $\phi \wedge \psi$ is in Δ then $\psi \in \Delta$ or $\phi \in \Delta$
- if $\phi \to \psi$ is in Γ then $\psi \in \Gamma$ or $\phi \in \Delta$
- if $\phi \to \psi$ is in Δ then $\psi \in \Delta$ and $\phi \in \Gamma$

Extensions of tableaux

The tableau $t' = (\Gamma', \Delta')$ is an extension of the tableau $t = (\Gamma, \Delta)$ if

- $\Gamma \subseteq \Gamma'$
- $\Delta \subseteq \Delta'$

Realizable tableaux

The tableau $t = (\Gamma, \Delta)$ is realizable in model $\mathcal{M} = (S, V)$ where S is a nonempty set of "worlds" and $V : AF \to 2^S$ if for all formulas ϕ, ψ,

- if $\phi \in \Gamma$ then $\mathcal{M} \models \phi$
- if $\psi \in \Delta$ then $\mathcal{M} \not\models \psi$

Proposition

A tableau is realizable if and only if it can be extended to a disjoint saturated tableau.

Proposition

A finite tableau t is realizable if and only if there exists a finite sequence $t_1, \ldots, t_n$ of tableaux such that

- $t_1 = t$,
- for all $1 \leq i < n$, t_{i+1} is obtained from t_i by applying to it one of the saturation rules,
- t_n is a disjoint saturated tableau.

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Exercise

By using the two previous propositions, prove that the following formulas are valid:

idempotency $p \vee p \leftrightarrow p$

idempotency $p \wedge p \leftrightarrow p$

commutativity $p \vee q \leftrightarrow q \vee p$

commutativity $p \wedge q \leftrightarrow q \wedge p$

$p \vee \bot \leftrightarrow p$

$p \wedge \bot \leftrightarrow \bot$

$p \vee \top \leftrightarrow \top$

$p \wedge \top \leftrightarrow p$

$\bot \rightarrow p$

$p \rightarrow \top$

Intermediate logics: Classical Propositional Logic

Exercise

By using the two previous propositions, prove that the following formulas are valid:

	$q \to p \vee \neg p$
	$p \wedge \neg p \to q$
associativity	$p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$
associativity	$p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r$
absorption	$(p \wedge q) \vee q \leftrightarrow q$
absorption	$(p \vee q) \wedge q \leftrightarrow q$
distributivity	$(p \wedge q) \vee r \leftrightarrow (p \vee r) \wedge (q \vee r)$
distributivity	$(p \vee q) \wedge r \leftrightarrow (p \wedge r) \vee (q \wedge r)$

By using the two previous propositions, prove that the following formulas are valid:

simplification	$p \to (q \to p)$
syllogism	$(p \to q) \to ((q \to r) \to (p \to r))$
Frege's law	$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
	$p \to p \vee q$
	$q \to p \vee q$
	$p \wedge q \to p$
	$p \wedge q \to q$
De Morgan's law	$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$
De Morgan's law	$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$

Intermediate logics: Classical Propositional Logic

Exercise

By using the two previous propositions, prove that the following formulas are valid:

$$(p \to q) \leftrightarrow \neg p \vee q$$

$$(p \to q) \leftrightarrow \neg(p \wedge \neg q)$$

$$(p \vee q) \wedge (p \vee \neg q) \leftrightarrow p$$

$$(p \wedge q) \vee (p \wedge \neg q) \leftrightarrow p$$

Pierce's law $((p \to q) \to p) \to p$

excluded middle $p \vee \neg p$

contraposition $(p \to q) \leftrightarrow (\neg q \to \neg p)$

double negation $p \leftrightarrow \neg\neg p$

Intermediate logics: Classical Propositional Logic

Exercise

Prove that the following inference rules preserve validity:

modus ponens $\frac{\phi \quad \phi\to\psi}{\psi}$

substitution $\frac{\phi}{\sigma(\phi)}$, where σ is a substitution

Intermediate logics: Classical Propositional Logic

Calculus

The calculus of **CPL** contains the following axioms and inference rules

Axioms

- $p \to (q \to p)$
- $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
- $p \to p \vee q$ and $q \to p \vee q$
- $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$
- $p \wedge q \to p$ and $p \wedge q \to q$
- $p \to (q \to p \wedge q)$
- $\bot \to p$ and $p \to \top$
- $p \vee (p \to \bot)$

Inference rules

modus ponens $\frac{\phi \quad \phi \to \psi}{\psi}$

substitution $\frac{\phi}{\sigma(\phi)}$, where σ is a substitution

Owing to the importance of modus ponens and substitution

A logic is a set of formulas which is closed under the inference rules of modus ponens and substitution

Let ϕ be a formula

A derivation of ϕ is a sequence $\phi_1, \ldots, \phi_n$ of formulas such that

- for all $1 \leq i \leq n$, ϕ_i is an axiom or is obtained from some of the preceding formulas in the sequence by one of the inference rules,
- $\phi_n = \phi$.

If ϕ is derivable then we write $\vdash_{\mathbf{CPL}} \phi$

CPL

The set of all derivable formulas

Lemma

CPL is a logic.

Intermediate logics: Classical Propositional Logic

Examples

Let ϕ be a formula

Derivation of $\phi \to \phi$

1. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
2. $(\phi \to ((\phi \to \phi) \to \phi)) \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi))$
3. $p \to (q \to p)$
4. $\phi \to ((\phi \to \phi) \to \phi)$
5. $(\phi \to (\phi \to \phi)) \to (\phi \to \phi)$
6. $\phi \to (\phi \to \phi)$
7. $\phi \to \phi$

Let ϕ be a formula and Γ be a set of formulas

A derivation of ϕ from the set Γ of assumptions is a sequence $\phi_1, \ldots, \phi_n$ of formulas such that

- for all $1 \leq i \leq n$, ϕ_i is an axiom or an assumption in Γ or is obtained from some of the preceding formulas in the sequence by one of the inference rules, the inference rule of substitution being applied only to axioms,
- $\phi_n = \phi$.

If ϕ is derivable from the set Γ of assumptions then we write $\Gamma \vdash_{\mathbf{CPL}} \phi$

Proposition

If $\Gamma \vdash_{\mathbf{CPL}} \phi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_{\mathbf{CPL}} \phi$.

Proposition

If $\Gamma \vdash_{\mathbf{CPL}} \phi \to \psi$ and $\Delta \vdash_{\mathbf{CPL}} \phi$ then $\Gamma, \Delta \vdash_{\mathbf{CPL}} \psi$.

Theorem (Deduction Theorem)

If $\Gamma, \phi \vdash_{\mathbf{CPL}} \psi$ then $\Gamma \vdash_{\mathbf{CPL}} \phi \to \psi$.

Intermediate logics: Classical Propositional Logic

Exercise

Let ϕ, ψ, χ be formulas and Γ be a set of formulas

By using the Deduction Theorem, prove that

- $\vdash_{\mathbf{CPL}} (\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$,
- if $\Gamma \vdash_{\mathbf{CPL}} \phi \vee \psi$ and $\Gamma, \phi \vdash_{\mathbf{CPL}} \psi$ then $\Gamma \vdash_{\mathbf{CPL}} \psi$,
- $\vdash_{\mathbf{CPL}} \phi \vee (\phi \to \psi)$.

Theorem

For all formulas ϕ, the following conditions are equivalent:

- $\vdash_{\mathbf{CPL}} \phi$,
- ϕ is valid.

A logic **L** is consistent if $\phi \notin \mathbf{L}$ for some formula ϕ

Theorem

CPL is consistent.

A logic is Post-complete if it is consistent and it has no proper consistent extension

Theorem

CPL is Post-complete.

A logic **L** is 0-reducible if for all formulas ϕ, if $\phi \notin \mathbf{L}$ then there exists a variable-free instance ϕ' of ϕ such that $\phi' \notin \mathbf{L}$

Theorem

CPL is 0-reducible.

A logic **L** is independently axiomatizable by a set Γ of formulas if the closure of Γ under modus ponens and substitution is **L** but no proper subset of Γ possesses this property

Theorem

CPL is independently axiomatizable.

A rule of inference $\frac{\phi_1,\ldots,\phi_n}{\psi}$ is derivable in a logic **L** if there is a derivation of ϕ in **L** from the assumptions $\phi_1, \ldots, \phi_n$

A rule of inference $\frac{\phi_1,\ldots,\phi_n}{\psi}$ is admissible in a logic **L** if for all substitutions σ, if $\sigma(\phi_1), \ldots, \sigma(\phi_n) \in \mathbf{L}$ then $\sigma(\psi) \in \mathbf{L}$

Lemma

Every **CPL**-derivable inference rule is **CPL**-admissible.

A logic **L** is structurally complete if every **L**-admissible inference rule is **L**-derivable

Theorem

CPL is structurally complete.

Corollary

The admissibility problem for inference rules in **CPL** is decidable.

A logic **L** has the Craig interpolation property if for all formulas ϕ, ψ, if $\phi \to \psi \in$ **L** then there exists a formula χ whose variables occur both in ϕ and ψ and such that $\phi \to \chi \in$ **L** and $\chi \to \psi \in$ **L**

Theorem

CPL has the Craig interpolation property.

A logic **L** is locally tabular if for all $n \geq 0$, **L** contains only a finite number of pairwise nonequivalent formulas built from variables $p_1, \ldots, p_n$

Theorem

CPL is locally tabular.

A logic **L** is Halldén-complete if for all formulas ϕ, ψ containing no common variables, $\phi \vee \psi \in$ **L** if and only if $\phi \in$ **L** or $\psi \in$ **L**

Theorem

CPL is Halldén-complete.

A logic **L** possesses the disjunction property if for all formulas ϕ, ψ, $\phi \vee \psi \in$ **L** if and only if $\phi \in$ **L** or $\psi \in$ **L**

Theorem

CPL does not possess the disjunction property.

Intermediate logics: Classical Propositional Logic

Exercise

A formula ϕ is in disjunctive (respectively conjunctive) normal form if there exists $n \geq 1$ and there exists formulas $\psi_1, \ldots, \psi_n$ such that $\phi = \psi_1 \vee \ldots \vee \psi_n$ (respectively $\phi = \psi_1 \wedge \ldots \wedge \psi_n$) and each ψ_i is a conjunction (respectively disjunction) of atoms and negations of atoms. Show that every formula is equivalent to a formula in disjunctive (respectively conjunctive) normal form.

Show that each of the sets $\{\neg, \vee\}$, $\{\neg, \wedge\}$, $\{\bot, \rightarrow\}$ and $\{\neg, \rightarrow\}$ of connectives is truth-functionally complete in the sense that for all $n \geq 1$, every Boolean function from $\{\mathtt{F}, \mathtt{T}\}^n$ to $\{\mathtt{F}, \mathtt{T}\}$ can be represented by a formula containing only connectives in that set.

Intermediate logics: Intuitionistic Propositional Logic

The law of excluded middle

It allows proof of disjunctions $\phi \vee \psi$ such that neither ϕ, nor ψ is provable

- $p \vee (p \to \bot)$

Proofs of that sort are known as non-constructive

Brouwer-Heyting-Kolmogorov reading

The intended meaning of the intuitionistic connectives is given in terms of proofs and constructions

- A proof of $\phi \vee \psi$ is given by presenting either a proof of ϕ, or a proof of ψ
- A proof of $\phi \wedge \psi$ consists of a proof of ϕ and a proof of ψ
- A proof of $\phi \to \psi$ is a construction which, given a proof of ϕ, produces a proof of ψ

Intermediate logics: Intuitionistic Propositional Logic

Motivation

Classical Propositional Logic

- each atomic formula is either true (T) or false (F)
- the truth values of $\phi_1 \vee \phi_2$, $\phi_1 \wedge \phi_2$ and $\phi_1 \to \phi_2$ are uniquely determined by the truth values of ϕ_1 and ϕ_2

Intuitionistic Propositional Logic

- it is natural to regard an atomic formula established at a world s to be true at s and to remain true at all further possible worlds
- an atomic formula which is not true at a world s cannot be in general regarded as false for it may become true at one of the subsequent worlds

Intermediate logics: Intuitionistic Propositional Logic

Motivation

Classical Propositional Logic

- $\phi \vee \psi$ is true when either ϕ or ψ is true
- $\phi \wedge \psi$ is true when both ϕ and ψ are true
- $\phi \to \psi$ is true when ϕ is true only if ψ is true

Intuitionistic Propositional Logic

- $\phi \vee \psi$ is true at world *s* when either ϕ or ψ is true at *s*
- $\phi \wedge \psi$ is true at world *s* when both ϕ and ψ are true at *s*
- $\phi \to \psi$ is true at world *s* when for every subsequent possible world *t*, in particular *s* itself, ϕ is true at *t* only if ψ is true at *t*

Intermediate logics: Intuitionistic Propositional Logic

Syntax and semantics

- AF: countable set of atomic formulas
- $Fma(AF)$: set of all formulas generated from AF

- Atomic formulas: $p \in AF$
- Formulas: $\phi \in Fma(AF)$

$$\phi ::= p \mid \bot \mid \top \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \to \phi_2)$$

Frames

A frame is a pair $\mathcal{F} = (S, \leq)$ where

- S is a nonempty set of "worlds"
- $\leq$ is a partial order on S, i.e.
  - $\leq$ is reflexive: $\forall s \in S,\ s \leq s$
  - $\leq$ is transitive: $\forall s, t, u \in S,\ s \leq t \;\&\; t \leq u \;\Rightarrow\; s \leq u$
  - $\leq$ is antisymmetric: $\forall s, t \in S,\ s \leq t \;\&\; t \leq s \;\Rightarrow\; s = t$

Models

A model on a frame is a triple $\mathcal{M} = (S, \leq, V)$ where

- $V : \ AF \to 2^S$

$V(p)$ is $\leq$-upward closed, for every atomic formula p

- $\forall s, t \in S,\ s \in V(p) \;\&\; s \leq t \;\Rightarrow\; t \in V(p)$

Intermediate logics: Intuitionistic Propositional Logic

Frames and models

Given a model $\mathcal{M} = (S, \leq, V)$ and $s \in S$

$\mathcal{M} \models_s \phi$: relation "$\phi$ is true at world s in model $\mathcal{M}$"

- $\mathcal{M} \models_s p$ iff $s \in V(p)$
- $\mathcal{M} \not\models_s \bot$
- $\mathcal{M} \models_s \top$
- $\mathcal{M} \models_s \phi \vee \psi$ iff $\mathcal{M} \models_s \phi$ or $\mathcal{M} \models_s \psi$
- $\mathcal{M} \models_s \phi \wedge \psi$ iff $\mathcal{M} \models_s \phi$ and $\mathcal{M} \models_s \psi$
- $\mathcal{M} \models_s \phi \to \psi$ iff for all $t \in S$, if $s \leq t$ and $\mathcal{M} \models_t \phi$ then $\mathcal{M} \models_t \psi$

Let $\mathcal{M} = (S, \leq, V)$ be a model and $s \in S$

Show that $\mathcal{M} \models_s \neg\phi$ iff for all $t \in S$, if $s \leq t$ then $\mathcal{M} \not\models_t \phi$.
Work out the corresponding truth condition for $\phi \leftrightarrow \psi$.

Let $\mathcal{M} = (S, \leq, V)$ be a model and $s, t \in S$

Show that if $s \leq t$ and $\mathcal{M} \models_s \phi$ then $\mathcal{M} \models_t \phi$.

Intermediate logics: Intuitionistic Propositional Logic

Truth and validity

$\mathcal{M} \models \phi$: relation "ϕ is true in model $\mathcal{M} = (S, \leq, V)$"

$\mathcal{M} \models \phi$ iff $\mathcal{M} \models_s \phi$ for all $s \in S$

$\mathcal{F} \models \phi$: relation "ϕ is valid in frame $\mathcal{F} = (S, \leq)$"

$\mathcal{F} \models \phi$ iff $\mathcal{M} \models \phi$ for all models $\mathcal{M} = (S, \leq, V)$

$\mathcal{C} \models \phi$: relation "ϕ is valid in class $\mathcal{C}$ of frames"

$\mathcal{C} \models \phi$ iff $\mathcal{F} \models \phi$ for all frames $\mathcal{F}$ in $\mathcal{C}$

Show that the following formulas are valid in the class of all frames:

- $\phi \to (\psi \to \phi)$,
- $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$,
- $\phi \to \phi \vee \psi$ and $\psi \to \phi \vee \psi$,
- $(\phi \to \chi) \to ((\psi \to \chi) \to ((\phi \vee \psi) \to \chi))$,
- $\phi \wedge \psi \to \phi$ and $\phi \wedge \psi \to \psi$,
- $\phi \to (\psi \to \phi \wedge \psi)$,
- $\bot \to \phi$ and $\phi \to \top$.

Show that $\phi \to \neg\neg\phi$ is valid in the class of all frames.

Intermediate logics: Intuitionistic Propositional Logic

Exercise

Find a frame in which $p \vee (p \to \bot)$ is not valid.

Find a frame in which $\neg\neg p \to p$ is not valid.

Find a frame in which $((p \to q) \to p) \to p$ is not valid.

Intermediate logics: Intuitionistic Propositional Logic

IPL

Let **IPL** be the set of all ϕ such that ϕ is valid in the class of all frames

Lemma

IPL is a logic.

Proposition

IPL $\subseteq$ **CPL**.

Proposition

Since $p \vee (p \to \bot)$ is in **CPL** and $p \vee (p \to \bot)$ is not in **IPL**, then the above inclusion is strict.

$\mathcal{M}^t = (S^t, \leq^t, V^t)$: submodel of $\mathcal{M} = (S, \leq, V)$ generated by $t \in S$

- $S^t = \{u \in S : \ t \leq u\}$
- $\leq^t = \leq \cap (S^t \times S^t)$
- $V^t(p) = V(p) \cap S^t$

$\mathcal{F}^t = (S^t, \leq^t)$: subframe of $\mathcal{F} = (S, \leq)$ generated by $t \in S$

Rooted frame

A frame $\mathcal{F} = (S, \leq)$ is rooted if there exists $t \in S$ such that $\mathcal{F}^t = \mathcal{F}$

Show that S^t is the smallest subset of S that contains t and is closed under $\leq$.

Intermediate logics: Intuitionistic Propositional Logic

Generated submodels

Submodel Lemma: For any $u \in S^t$

- $\mathcal{M}^t \models_u \phi$ iff $\mathcal{M} \models_u \phi$

Corollary:

1. $\mathcal{M} \models \phi$ implies $\mathcal{M}^t \models \phi$
2. $\mathcal{M} \models \phi$ iff ϕ is true in all generated submodels of $\mathcal{M}$
3. $\mathcal{F} \models \phi$ iff ϕ is valid in all generated subframes of $\mathcal{F}$

Intermediate logics: Intuitionistic Propositional Logic

Bounded morphisms

A function $f: S_1 \to S_2$ is called a bounded morphism from $\mathcal{M}_1 = (S_1, \leq_1, V_1)$ to $\mathcal{M}_2 = (S_2, \leq_2, V_2)$ iff

- $s_1 \leq_1 t_1$ implies $f(s_1) \leq_2 f(t_1)$
- $f(s_1) \leq_2 t_2$ implies there exists $t_1 \in S_1$ such that $s_1 \leq_1 t_1$ and $f(t_1) = t_2$
- $s_1 \in V_1(p)$ iff $f(s_1) \in V_2(p)$

Bounded Morphism Lemma: For any $s_1 \in S_1$

- $\mathcal{M}_1 \models_{s_1} \phi$ iff $\mathcal{M}_2 \models_{f(s_1)} \phi$

Intermediate logics: Intuitionistic Propositional Logic

Bounded morphisms

A function $f : S_1 \to S_2$ is called a bounded morphism from $\mathcal{F}_1 = (S_1, \leq_1)$ to $\mathcal{F}_2 = (S_2, \leq_2)$ iff

- $s_1 \leq_1 t_1$ implies $f(s_1) \leq_2 f(t_1)$
- $f(s_1) \leq_2 t_2$ implies there exists $t_1 \in S_1$ such that $s_1 \leq_1 t_1$ and $f(t_1) = t_2$

If there is a bounded morphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ that is surjective then $\mathcal{F}_2$ is called a bounded morphic image of $\mathcal{F}_1$

Bounded Morphism Lemma: If $\mathcal{F}_2$ is a bounded morphic image of $\mathcal{F}_1$ then for any formula ϕ

- $\mathcal{F}_1 \models \phi$ implies $\mathcal{F}_2 \models \phi$

Theorem

Every rooted frame $\mathcal{F}$ is a bounded morphic image of some tree, which is finite if $\mathcal{F}$ is finite.

Corollary

IPL is the set of all ϕ such that ϕ is valid in the class of all trees.

About trees

For all $n \geq 2$, let $\mathcal{T}_n$ be the full n-ary tree

Theorem

Let $n \geq 2$. Every finite tree is a bounded morphic image of $\mathcal{T}_n$.

Corollary

Let $n \geq 2$. Every finite rooted frame is a bounded morphic image of $\mathcal{T}_n$.

Proposition

For all variable-free formulas ϕ, either $\phi \leftrightarrow \bot$ is in **IPL**, or $\phi \leftrightarrow \top$ is in **IPL**.

Corollary

For all variable-free formulas ϕ, ϕ is in **IPL** if and only if ϕ is in **CPL**.

Corollary

IPL is not 0-reducible.

Determining whether a given formula is valid in the class of all frames

Our wish: design a systematic procedure

Semantic tableaux

A semantic tableau is a pair $t = (\Gamma, \Delta)$ with $\Gamma, \Delta \subseteq Fma(AF)$

Disjoint tableaux

The tableau $t = (\Gamma, \Delta)$ is disjoint if

- $\Gamma \cap \Delta = \emptyset$

Saturated tableaux

The tableau $t = (\Gamma, \Delta)$ is saturated if for all formulas ϕ, ψ

- $\bot \in \Delta$
- $\top \in \Gamma$
- if $\phi \vee \psi$ is in Γ then $\psi \in \Gamma$ or $\phi \in \Gamma$
- if $\phi \vee \psi$ is in Δ then $\psi \in \Delta$ and $\phi \in \Delta$
- if $\phi \wedge \psi$ is in Γ then $\psi \in \Gamma$ and $\phi \in \Gamma$
- if $\phi \wedge \psi$ is in Δ then $\psi \in \Delta$ or $\phi \in \Delta$
- if $\phi \to \psi$ is in Γ then $\psi \in \Gamma$ or $\phi \in \Delta$

Hintikka systems

A Hintikka system is a pair $(S, \leq)$ where

- S is a nonempty set of disjoint saturated tableaux
- $\leq$ is a partial order on S such that
  - if $t = (\Gamma, \Delta)$ and $t' = (\Gamma', \Delta')$ are in S and $t \leq t'$ then $\Gamma \subseteq \Gamma'$
  - if $t = (\Gamma, \Delta)$ is in S and $\phi \to \psi$ is in Δ then there exists $t' = (\Gamma', \Delta')$ in S such that $t \leq t'$, $\psi \in \Delta'$ and $\phi \in \Gamma'$

Hintikka systems for tableaux

A Hintikka system $(S, \leq)$ is a Hintikka system for the tableau t if there exists t' in S such that $t \leq t'$

Realizable tableaux

The tableau (Γ, Δ) is realizable in model $\mathcal{M} = (S, \leq, V)$ if there exists s in S such that for all formulas ϕ, ψ,

- if $\phi \in \Gamma$ then $\mathcal{M} \models_s \phi$
- if $\psi \in \Delta$ then $\mathcal{M} \not\models_s \psi$

Proposition

A tableau t is realizable in some model if and only if there exists a Hintikka system for t.

Theorem

A tableau t is realizable in some model if and only if there exists a Hintikka system $(S, \leq)$ for t such that $\|S\| \leq 2^{\mathtt{Card}(SF(\phi))}$

Intermediate logics: Intuitionistic Propositional Logic

Exercise

By using Hintikka systems, show that

- $\neg\neg\neg p \to \neg p$

is in **IPL**.

By using Hintikka systems, show that

Dummett formula $(p \to q) \vee (q \to p)$,

Weak excluded middle $\neg p \vee \neg\neg p$,

Pierce formula $((p \to q) \to p) \to p$,

Scott formula $((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$,

are not in **IPL**.

Intermediate logics: Intuitionistic Propositional Logic

The following is a list of properties of a partial order $\leq$ that are defined by first-order sentences

1. Strongly connected: $\forall s \forall t \forall u (s \leq t \wedge s \leq u \rightarrow t \leq u \ \vee \ u \leq t)$
2. Strongly directed: $\forall s \forall t \forall u (s \leq t \wedge s \leq u \rightarrow \exists v (t \leq v \ \& \ u \leq v))$
3. n-bounded depth: $\forall s_0 \ldots \forall s_n (\bigwedge \{ s_i \leq s_{i+1} : \ 0 \leq i < n \} \rightarrow \bigvee \{ s_i = s_j : \ 0 \leq i < j \leq n \})$ where $n \geq 1$
4. n-bounded width: $\forall s \forall t_0 \ldots \forall t_n (\bigwedge \{ s \leq t_i : \ 0 \leq i \leq n \} \rightarrow \bigvee \{ t_i = t_j : \ 0 \leq i < j \leq n \})$ where $n \geq 1$

Corresponding to this list is a list of formulas

da $(p \to q) \vee (q \to p)$

wem $\neg p \vee \neg\neg p$

$\mathbf{bd}_1$ $p_1 \vee (p_1 \to \bot)$

$\mathbf{bd}_{n+1}$ $p_{n+1} \vee (p_{n+1} \to \mathbf{bd}_n)$

$\mathbf{bw}_n$ $\bigvee\{p_i \to \bigvee\{p_j : \ 0 \leq j \leq n \ \& \ i \neq j\} : \ 0 \leq i \leq n\}$

Theorem: Let $\mathcal{F} = (S, \leq)$ be a frame.

- $\leq$ is strongly connected if and only if $\mathcal{F} \models \mathbf{da}$,
- $\leq$ is strongly directed if and only if $\mathcal{F} \models \mathbf{wem}$,
- $\leq$ has n-bounded depth if and only if $\mathcal{F} \models \mathbf{bd}_n$, where $n \geq 1$,
- $\leq$ has n-bounded width if and only if $\mathcal{F} \models \mathbf{bw}_n$, where $n \geq 1$.

The calculus of **IPL** contains the following axioms and inference rules

Axioms

- $p \to (q \to p)$
- $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
- $p \to p \vee q$ and $q \to p \vee q$
- $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$
- $p \wedge q \to p$ and $p \wedge q \to q$
- $p \to (q \to p \wedge q)$
- $\bot \to p$ and $p \to \top$

Inference rules

modus ponens $\frac{\phi \quad \phi \to \psi}{\psi}$

substitution $\frac{\phi}{\sigma(\phi)}$, where σ is a substitution

Let ϕ be a formula

A derivation of ϕ is a sequence $\phi_1, \ldots, \phi_n$ of formulas such that

- for all $1 \leq i \leq n$, ϕ_i is an axiom or is obtained from some of the preceding formulas in the sequence by one of the inference rules,
- $\phi_n = \phi$.

If ϕ is derivable then we write $\vdash_{\mathbf{IPL}} \phi$

IPL

The set of all derivable formulas

Lemma

IPL is a logic.

Intermediate logics: Intuitionistic Propositional Logic

Derivations from assumptions

Let ϕ be a formula and Γ be a set of formulas

A derivation of ϕ from the set Γ of assumptions is a sequence $\phi_1, \ldots, \phi_n$ of formulas such that

- for all $1 \leq i \leq n$, ϕ_i is an axiom or an assumption in Γ or is obtained from some of the preceding formulas in the sequence by one of the inference rules, the inference rule of substitution being applied only to axioms,
- $\phi_n = \phi$.

If ϕ is derivable from the set Γ of assumptions then we write $\Gamma \vdash_{\mathbf{IPL}} \phi$

Proposition

If $\Gamma \vdash_{\mathbf{IPL}} \phi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_{\mathbf{IPL}} \phi$.

Proposition

If $\Gamma \vdash_{\mathbf{IPL}} \phi \to \psi$ and $\Delta \vdash_{\mathbf{IPL}} \phi$ then $\Gamma, \Delta \vdash_{\mathbf{IPL}} \psi$.

Theorem (Deduction Theorem)

If $\Gamma, \phi \vdash_{\mathbf{IPL}} \psi$ then $\Gamma \vdash_{\mathbf{IPL}} \phi \to \psi$.

Theorem

For all formulas ϕ, the following conditions are equivalent:

- $\vdash_{\mathbf{IPL}} \phi$,
- ϕ is valid.

Theorem (Glivenko Theorem)

For all formulas ϕ, ϕ is in **CPL** if and only if $\neg\neg\phi$ is in **IPL**.

Corollary

For all formulas ϕ, $\neg\phi$ is in **CPL** if and only if $\neg\phi$ is in **IPL**.

Corollary

For all formulas ϕ, ψ, $\phi \to \neg\psi$ is in **CPL** if and only if $\phi \to \neg\psi$ is in **IPL**.

Corollary

For all formulas ϕ based on $\wedge$ and $\neg$, ϕ is in **CPL** if and only if ϕ is in **IPL**.

Intermediate logics: Intuitionistic Propositional Logic

Basic properties of **IPL**

Theorem

IPL is consistent.

Theorem

IPL is decidable.

A logic **L** is tabular if there exists a finite frame $\mathcal{F} = (S, \leq)$ such that $\mathbf{L} = \{\phi : \mathcal{F} \models \phi\}$

Theorem

IPL is not tabular.

A logic **L** is finitely approximable if there exists a class $\mathcal{C}$ of finite frames such that $\mathbf{L} = \{\phi : \mathcal{C} \models \phi\}$

Theorem

IPL is finitely approximable.

Theorem
CPL is the only Post-complete extension of **IPL**.

Theorem
IPL is independently axiomatizable.

Theorem
IPL is not locally tabular.

Theorem
IPL is Halldén-complete.

Theorem
IPL possesses the disjunction property.

Lemma

Every **IPL**-derivable inference rule is **IPL**-admissible.

Proposition

The following rules are **IPL**-admissible but not **IPL**-derivable:

$$\text{Harrop rule} \quad \frac{\neg p \to q \vee r}{(\neg p \to q) \vee (\neg p \to r)},$$

$$\text{Mints rule} \quad \frac{(p \to q) \to p \vee r}{((p \to q) \to p) \vee ((p \to q) \to r)},$$

$$\text{Scott rule} \quad \frac{(\neg\neg p \to p) \to p \vee \neg p}{\neg p \vee \neg\neg p}.$$

Lemma

Every **IPL**-admissible inference rule is **CPL**-derivable.

Intermediate logics: examples of intermediate logics

- **SmL** $::=$ **IPL** $+$ $(\neg q \rightarrow p) \rightarrow (((p \rightarrow q) \rightarrow p) \rightarrow p)$
- **KC** $::=$ **IPL** $+$ $\neg p \vee \neg\neg p$
- **LC** $::=$ **IPL** $+$ $(p \rightarrow q) \vee (q \rightarrow p)$
- **HT** $::=$ **IPL** $+$ $p \vee (p \rightarrow q) \vee \neg q$
- **SL** $::=$ **IPL** $+$ $((\neg\neg p \rightarrow p) \rightarrow p \vee \neg p) \rightarrow \neg p \vee \neg\neg p$
- **KP** $::=$ **IPL** $+$ $(\neg p \rightarrow q \vee r) \rightarrow (\neg p \rightarrow q) \vee (\neg p \rightarrow r)$
- **WKP** $::=$ **IPL** $+$ $(\neg p \rightarrow \neg q \vee \neg r) \rightarrow (\neg p \rightarrow \neg q) \vee (\neg p \rightarrow \neg r)$
- **CPL** $::=$ **IPL** $+$ $p \vee \neg p$

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