# Intuitionistic modal logic

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# Intuitionistic modal logics

## Outline

- Intermediate logics
- Modal logics
- Combining logics
- Two peculiar intuitionistic modal logics
- **A minimal setting**

# A minimal setting

## Outline

- A word about intermediate logics
- A word about modal logics
- Then, about intuitionistic modal logics
- Conclusion and perspectives

# A minimal setting: A word about intermediate logics

## What is the language of intermediate logics ?

- Propositional letters

  $p$, $q$, etc
- Propositional connectives and parentheses

  $\rightarrow, \perp, \top, \vee, \wedge, (, )$
- Formulas

  $A ::= p|(A{\rightarrow}A)|\perp|\top|(A{\vee}A)|(A{\wedge}A)$
- Abbreviations

  $\neg A ::= (A{\rightarrow}\perp)$

  $(A{\leftrightarrow}B) ::= ((A{\rightarrow}B){\wedge}(B{\rightarrow}A))$
- Notation

  **FOR** : the set of all formulas

# A minimal setting: A word about intermediate logics

## Examples of intermediate logics

- **CPL** : Classical Propositional Logic
- **IPL** : Intuitionistic Propositional Logic

# A minimal setting: A word about intermediate logics

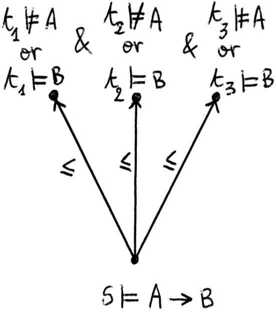
## Formal desiderata for intermediate logics

- Any $\mathbf{L}\subseteq\mathbf{FOR}$ closed for uniform substitution and such that
  $\mathbf{L}$ contains the standard axioms of $\mathbf{IPL}$
  $\mathbf{L}$ is closed with respect to the rule $\frac{p \quad p\rightarrow q}{q}$

# A minimal setting: A word about intermediate logics

## Relational semantics of intermediate logics

- Kripke frame : preordered structure $(W, \leq)$
  - $W$ : nonempty set of "worlds" $s$, $t$, etc
  - $\leq$ : reflexive transitive relation on $W$
- Valuation on $(W, \leq)$
  - $V$ : propositional letter $p \mapsto \leq$-closed subset $V(p)$ of $W$
- Truth condition for $\rightarrow$



# A minimal setting: A word about intermediate logics

## Relational semantics of intermediate logics

- Kripke frame : preordered structure $(W, \leq)$
  $W$ : nonempty set of "worlds" $s$, $t$, etc
  $\leq$ : reflexive transitive relation on $W$
- Valuation on $(W, \leq)$
  $V$ : propositional letter $p \mapsto \leq$-closed subset $V(p)$ of $W$
- Truth conditions

$$
\begin{array}{lcl}
s \models p & \Leftrightarrow & s \in V(p) \\
s \models A \rightarrow B & \Leftrightarrow & \forall t \in W,\ (s \leq t \ \Rightarrow\ t \not\models A \text{ or } t \models B) \\
s \models \bot & \Leftrightarrow & \text{never} \\
s \models \top & \Leftrightarrow & \text{always} \\
s \models A \vee B & \Leftrightarrow & s \models A \text{ or } s \models B \\
s \models A \wedge B & \Leftrightarrow & s \models A \ \&\ s \models B
\end{array}
$$

# A minimal setting: A word about intermediate logics

## Examples of intermediate logics

- **SmL** ::= **IPL** + $(\neg q \to p) \to (((p \to q) \to p) \to p)$
- **KC** ::= **IPL** + $\neg p \vee \neg\neg p$
- **LC** ::= **IPL** + $(p \to q) \vee (q \to p)$
- **SL** ::= **IPL** + $((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$
- **KP** ::= **IPL** + $(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$
- **WKP** ::= **IPL** + $(\neg p \to \neg q \vee \neg r) \to (\neg p \to \neg q) \vee (\neg p \to \neg r)$
- **CPL** ::= **IPL** + $p \vee \neg p$

# A minimal setting: A word about modal logics

## What is the language of modal logics ?

- Propositional letters

  $p$, $q$, etc

- Propositional connectives, modal connectives and parentheses

  $\to$, $\bot$, $\top$, $\vee$, $\wedge$, $\Box$, $\Diamond$, (, )

- Formulas

  $A ::= p | (A{\to}A) | \bot | \top | (A{\vee}A) | (A{\wedge}A) | \Box A | \Diamond A$

- Abbreviations

  $\neg A ::= (A{\to}\bot)$

  $(A{\leftrightarrow}B) ::= ((A{\to}B){\wedge}(B{\to}A))$

- Notation

  **FOR** : the set of all formulas

# A minimal setting: A word about modal logics

### Examples of modal logics

- $\mathbf{S}5$ : a logic of knowledge
- $\mathbf{K}$ : the least modal logic

# A minimal setting: A word about modal logics

## Formal desiderata for modal logics

- Any $\mathbf{L} \subseteq \mathbf{FOR}$ closed for uniform substitution and such that

  $\mathbf{L}$ contains the standard axioms of **CPL**

  $\mathbf{L}$ is closed with respect to the rule $\frac{p \quad p \rightarrow q}{q}$

  $\mathbf{L}$ contains the axiom $\Box p \leftrightarrow \neg \Diamond \neg p$

  $\mathbf{L}$ contains the axiom $\Diamond p \leftrightarrow \neg \Box \neg p$

  $\mathbf{L}$ contains the axiom $\Box p \wedge \Box q \rightarrow \Box (p \wedge q)$

  $\mathbf{L}$ contains the axiom $\Diamond (p \vee q) \rightarrow \Diamond p \vee \Diamond q$

  $\mathbf{L}$ contains the axiom $\Box \top$

  $\mathbf{L}$ contains the axiom $\neg \Diamond \bot$

  $\mathbf{L}$ contains the axiom $(\Diamond p \rightarrow \Box q) \rightarrow \Box (p \rightarrow q)$

  $\mathbf{L}$ contains the axiom $\Diamond (p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q)$

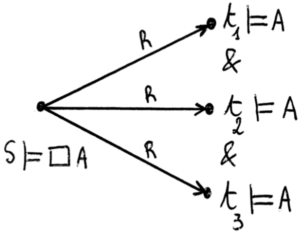
  $\mathbf{L}$ is closed with respect to the rule $\frac{p \rightarrow q}{\Box p \rightarrow \Box q}$

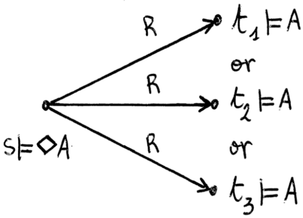
  $\mathbf{L}$ is closed with respect to the rule $\frac{p \rightarrow q}{\Diamond p \rightarrow \Diamond q}$

# A minimal setting: A word about modal logics

## Relational semantics of modal logics

- Kripke frame : relational structure $(W, R)$
  $W$ : nonempty set of "worlds" $s$, $t$, etc
  $R$ : binary relation on $W$
- Valuation on $(W, R)$
  $V$ : propositional letter $p \mapsto$ subset $V(p)$ of $W$
- Truth conditions for $\Box$ and $\Diamond$



# A minimal setting: A word about modal logics

## Relational semantics of modal logics

- Kripke frame : relational structure $(W, R)$
  $W$ : nonempty set of "worlds" $s$, $t$, etc
  $R$ : binary relation on $W$
- Valuation on $(W, R)$
  $V$ : propositional letter $p \mapsto$ subset $V(p)$ of $W$
- Truth conditions

$$
\begin{array}{lcl}
s \models p & \Leftrightarrow & s \in V(p) \\
s \models A \to B & \Leftrightarrow & s \not\models A \text{ or } s \models B \\
s \models \bot & \Leftrightarrow & \text{never} \\
s \models \top & \Leftrightarrow & \text{always} \\
s \models A \vee B & \Leftrightarrow & s \models A \text{ or } s \models B \\
s \models A \wedge B & \Leftrightarrow & s \models A \ \& \ s \models B \\
s \models \Box A & \Leftrightarrow & \forall t \in R(s),\ t \models A \\
s \models \Diamond A & \Leftrightarrow & \exists t \in R(s),\ t \models A
\end{array}
$$

# A minimal setting: A word about modal logics

## Examples of modal logics

- **KD** $::=$ **K** $+ \Box p \to \Diamond p$
- **KT** $::=$ **K** $+ \Box p \to p$
- **KB** $::=$ **K** $+ p \to \Box \Diamond p$
- **K**4 $::=$ **K** $+ \Box p \to \Box \Box p$
- **K**5 $::=$ **K** $+ \Diamond p \to \Box \Diamond p$
- **S**4 $::=$ **K** $+ \Box p \to p + \Box p \to \Box \Box p$
- **S**5 $::=$ **K** $+ \Box p \to p + \Diamond p \to \Box \Diamond p$

# A minimal setting: Then, about intuitionistic modal logics

## What is the language of intuitionistic modal logics ?

- Propositional letters

  $p$, $q$, etc

- Propositional connectives, modal connectives and parentheses

  $\rightarrow$, $\bot$, $\top$, $\vee$, $\wedge$, $\Box$, $\Diamond$, (, )

- Formulas

  $A ::= p|(A{\rightarrow}A)|\bot|\top|(A{\vee}A)|(A{\wedge}A)|\Box A|\Diamond A$

- Abbreviations

  $\neg A ::= (A{\rightarrow}\bot)$

  $(A{\leftrightarrow}B) ::= ((A{\rightarrow}B){\wedge}(B{\rightarrow}A))$

- Notations

  $\mathbf{FOR}_{\{\Box,\Diamond\}}$ : the set of all formulas

  $\mathbf{FOR}_{\{\Box\}}$ : the set of all formulas without $\Diamond$

  $\mathbf{FOR}_{\{\Diamond\}}$ : the set of all formulas without $\Box$

  $\mathbf{FOR}_{\emptyset}$ : the set of all formulas without $\Box$ and $\Diamond$

# A minimal setting: Then, about intuitionistic modal logics

### Examples of intuitionistic modal logics

- **IK** : an “intuitionistic” analogue of **K**
- **WK** : a “constructive” analogue of **K**

# A minimal setting: Then, about intuitionistic modal logics

## Informal desiderata for intuitionistic modal logics
**Simpson (1994)**

- Any $\mathbf{L} \subseteq \mathbf{FOR}_{\{\Box,\Diamond\}}$ closed for uniform substitution and such that
  - — $\mathbf{L} \cap \mathbf{FOR}_{\emptyset}$ is $\mathbf{IPL}$
  - — $\mathbf{L}$ is closed with respect to the rule $\frac{p \quad p \to q}{q}$
  - — $\mathbf{L} + p \vee \neg p$ is a modal logic
  - — $\mathbf{L}$ possesses the disjunction property
  - — $\Box$ and $\Diamond$ are independent in $\mathbf{L}$
  - — Given an arbitrary model $(W, \leq, R, V)$ and $s, t \in W$, if $s \models A$ and $s \leq t$ then $t \models A$

# A minimal setting: Then, about intuitionistic modal logics

## Historical perspective

- Pioneers

  Fitch (1948)

  Bull (1965, 1966)

- “Intuitionistic” founders

  **Fischer Servi (1977, 1978, 1984)**

  Plotkin and Stirling (1986)

- “Constructive” founders

  **Wijesekera (1990)**

  Alechina, Mendler, de Paiva and Ritter (2001)

# A minimal setting: Then, about intuitionistic modal logics

## Relational semantics of intuitionistic modal logics

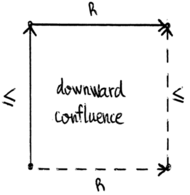
- Kripke frame : preordered relational structure $(W, \leq, R)$
  - $W$ : nonempty set of "worlds" $s$, $t$, etc
  - $\leq$ : reflexive transitive relation on $W$
  - $R$ : binary relation on $W$
- Valuation on $(W, \leq, R)$
  - $V$ : propositional letter $p \mapsto \leq$-closed subset $V(p)$ of $W$
- Truth conditions

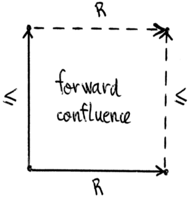
$$
\begin{array}{lcl}
s \models p & \Leftrightarrow & s \in V(p) \\
s \models A \to B & \Leftrightarrow & \forall t \in W,\ (s \leq t \ \Rightarrow\ t \not\models A \text{ or } t \models B) \\
s \models \bot & \Leftrightarrow & \text{never} \\
s \models \top & \Leftrightarrow & \text{always} \\
s \models A \vee B & \Leftrightarrow & s \models A \text{ or } s \models B \\
s \models A \wedge B & \Leftrightarrow & s \models A \ \&\ s \models B \\
s \models \Box A & \Leftrightarrow & \forall t \in R(s),\ t \models A\ ? \\
s \models \Diamond A & \Leftrightarrow & \exists t \in R(s),\ t \models A\ ?
\end{array}
$$

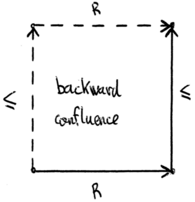
# A minimal setting: Then, about intuitionistic modal logics

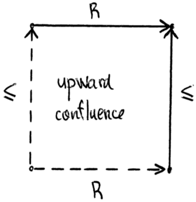
## Relational semantics of intuitionistic modal logics

- Kripke frame : preordered relational structure $(W, \leq, R)$
  $W$ : nonempty set of "worlds" $s$, $t$, etc
  $\leq$ : reflexive transitive relation on $W$
  $R$ : binary relation on $W$



# A minimal setting: Then, about intuitionistic modal logics

## Relational semantics of intuitionistic modal logics

- Kripke frame : preordered relational structure $(W, \leq, R)$
  - $W$ : nonempty set of "worlds" $s$, $t$, etc
  - $\leq$ : reflexive transitive relation on $W$
  - $R$ : binary relation on $W$
- Interactions between $\leq$ and $R$ : 4 possibilities !
  - $(W, \leq, R)$ is **downward confluent** if $\leq \circ R \subseteq R \circ \leq$
  - $(W, \leq, R)$ is **forward confluent** if $\geq \circ R \subseteq R \circ \geq$
  - $(W, \leq, R)$ is **backward confluent** if $R \circ \leq \subseteq \leq \circ R$
  - $(W, \leq, R)$ is **upward confluent** if $R \circ \geq \subseteq \geq \circ R$
- All in all
  - $2 \times 2 \times 2 \times 2$ choices at the level of the class of frames !

# A minimal setting: Then, about intuitionistic modal logics

## Relational semantics of intuitionistic modal logics

- Kripke frame : preordered relational structure $(W, \leq, R)$
- Valuation on $(W, \leq, R)$

  $V$ : propositional letter $p \mapsto \leq$-closed subset $V(p)$ of $W$
- Truth conditions for $\Box$ : 2 possibilities !

  $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$

  $s \models \Box A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \forall u \in R(t),\ u \models A)$
- Truth conditions for $\Diamond$ : 2 possibilities !

  $s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$

  $s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$
- All in all
  - $2 \times 2$ choices at the level of the truth conditions !

# A minimal setting: Then, about intuitionistic modal logics

## Restricting the language to $\mathbf{FOR}_{\{\Box\}}$

- Semantics : all downward confluent $(W, \leq, R)$
  - Truth conditions for $\Box$

$$
\begin{aligned}
s \models \Box A \quad & \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A) \\
s \models \Box A \quad & \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \forall u \in R(t),\ u \models A) \\
s \models \Box A \quad & \Leftrightarrow \forall t \in R(s),\ t \models A
\end{aligned}
$$

# A minimal setting: Then, about intuitionistic modal logics

## Restricting the language to $\mathbf{FOR}_{\{\Box\}}$

**Božić and Došen (1984)** — all down. conf. $(W, \leq, R)$

- Defined logic $\mathbf{HK}\Box$
  - Least $\mathbf{L} \subseteq \mathbf{FOR}$ closed for uniform substitution and such that
    - $\mathbf{L}$ contains the standard axioms of $\cancel{\mathbf{CPL}}$ $\mathbf{IPL}$
    - $\mathbf{L}$ is closed with respect to the rule $\frac{p \quad p \to q}{q}$
    - ~~$\mathbf{L}$ contains the axiom $\Box p \leftrightarrow \neg \Diamond \neg p$~~
    - ~~$\mathbf{L}$ contains the axiom $\Diamond p \leftrightarrow \neg \Box \neg p$~~
    - $\mathbf{L}$ contains the axiom $\Box p \wedge \Box q \to \Box (p \wedge q)$
    - ~~$\mathbf{L}$ contains the axiom $\Diamond (p \vee q) \to \Diamond p \vee \Diamond q$~~
    - $\mathbf{L}$ contains the axiom $\Box \top$
    - ~~$\mathbf{L}$ contains the axiom $\neg \Diamond \bot$~~
    - ~~$\mathbf{L}$ contains the axiom $(\Diamond p \to \Box q) \to \Box (p \to q)$~~
    - ~~$\mathbf{L}$ contains the axiom $\Diamond (p \to q) \to (\Box p \to \Diamond q)$~~
    - $\mathbf{L}$ is closed with respect to the rule $\frac{p \to q}{\Box p \to \Box q}$
    - ~~$\mathbf{L}$ is closed with respect to the rule $\frac{p \to q}{\Diamond p \to \Diamond q}$~~

# A minimal setting: Then, about intuitionistic modal logics

## Restricting the language to $\mathbf{FOR}_{\{\Diamond\}}$

- Semantics : all forward confluent $(W, \leq, R)$
  - Truth conditions for $\Diamond$

$$
\begin{array}{ll}
s \models \Diamond A & \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A) \\
s \models \Diamond A & \Leftrightarrow \exists t \in W,\ (s \geq t\ \&\ \exists u \in R(t),\ u \models A) \\
s \models \Diamond A & \Leftrightarrow \exists t \in R(s),\ t \models A
\end{array}
$$

# A minimal setting: Then, about intuitionistic modal logics

Restricting the language to $\mathbf{FOR}_{\{\Diamond\}}$
**Božić and Došen (1984)** — all for. conf. $(W, \leq, R)$

- Defined logic **HK**$\Diamond$
  - Least **L**$\subseteq$**FOR** closed for uniform substitution and such that
    **L** contains the standard axioms of ~~**CPL**~~ **IPL**
    **L** is closed with respect to the rule $\frac{p \quad p \to q}{q}$
    ~~**L** contains the axiom $\Box p \leftrightarrow \neg\Diamond\neg p$~~
    ~~**L** contains the axiom $\Diamond p \leftrightarrow \neg\Box\neg p$~~
    ~~**L** contains the axiom $\Box p \wedge \Box q \to \Box(p \wedge q)$~~
    **L** contains the axiom $\Diamond(p \vee q) \to \Diamond p \vee \Diamond q$
    ~~**L** contains the axiom $\Box\top$~~
    **L** contains the axiom $\neg\Diamond\bot$
    ~~**L** contains the axiom $(\Diamond p \to \Box q) \to \Box(p \to q)$~~
    ~~**L** contains the axiom $\Diamond(p \to q) \to (\Box p \to \Diamond q)$~~
    ~~**L** is closed with respect to the rule $\frac{p \to q}{\Box p \to \Box q}$~~
    **L** is closed with respect to the rule $\frac{p \to q}{\Diamond p \to \Diamond q}$

# A minimal setting: Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a first possibility

- Semantics : all forward confluent $(W, \leq, R)$
  - Truth conditions for $\Box$
    $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t\ \Rightarrow\ \forall u \in R(t),\ u \models A)$
  - Truth conditions for $\Diamond$
    $s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t\ \Rightarrow\ \exists u \in R(t),\ u \models A)$
    $s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t\ \&\ \exists u \in R(t),\ u \models A)$
    $s \models \Diamond A \quad \Leftrightarrow \exists t \in R(s),\ t \models A$

# A minimal setting: Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a first possibility
**Fischer Servi (1984)** — all back. and for. conf. $(W, \leq, R)$

- Defined logic **IK**
  - Least $\mathbf{L}\subseteq\mathbf{FOR}$ closed for uniform substitution and such that

    $\mathbf{L}$ contains the standard axioms of ~~**CPL**~~ **IPL**

    $\mathbf{L}$ is closed with respect to the rule $\frac{p \quad p\to q}{q}$

    ~~$\mathbf{L}$ contains the axiom $\Box p\leftrightarrow\neg\Diamond\neg p$~~

    ~~$\mathbf{L}$ contains the axiom $\Diamond p\leftrightarrow\neg\Box\neg p$~~

    $\mathbf{L}$ contains the axiom $\Box p\wedge\Box q\to\Box(p\wedge q)$

    $\mathbf{L}$ contains the axiom $\Diamond(p\vee q)\to\Diamond p\vee\Diamond q$

    $\mathbf{L}$ contains the axiom $\Box\top$

    $\mathbf{L}$ contains the axiom $\neg\Diamond\bot$

    $\mathbf{L}$ contains the axiom $(\Diamond p\to\Box q)\to\Box(p\to q)$

    $\mathbf{L}$ contains the axiom $\Diamond(p\to q)\to(\Box p\to\Diamond q)$

    $\mathbf{L}$ is closed with respect to the rule $\frac{p\to q}{\Box p\to\Box q}$

    $\mathbf{L}$ is closed with respect to the rule $\frac{p\to q}{\Diamond p\to\Diamond q}$

# A minimal setting: Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a first possibility

**B., Gao, Gencer and Olivetti** — all for. conf. $(W, \leq, R)$

- Defined logic **FIK**
  - Least $\mathbf{L} \subseteq \mathbf{FOR}$ closed for uniform substitution and such that

    **L** contains the standard axioms of ~~**CPL**~~ **IPL**

    **L** is closed with respect to the rule $\frac{p \quad p \to q}{q}$

    ~~**L** contains the axiom $\Box p \leftrightarrow \neg\Diamond\neg p$~~

    ~~**L** contains the axiom $\Diamond p \leftrightarrow \neg\Box\neg p$~~

    **L** contains the axiom $\Box p \wedge \Box q \to \Box(p \wedge q)$

    **L** contains the axiom $\Diamond(p \vee q) \to \Diamond p \vee \Diamond q$

    **L** contains the axiom $\Box\top$

    **L** contains the axiom $\neg\Diamond\bot$

    ~~**L** contains the axiom $(\Diamond p \to \Box q) \to \Box(p \to q)$~~

    ~~**L** contains the axiom $\Diamond(p \to q) \to (\Box p \to \Diamond q)$~~

    **L** is closed with respect to the rule $\frac{p \to q}{\Box p \to \Box q}$

    **L** is closed with respect to the rule $\frac{p \to q}{\Diamond p \to \Diamond q}$

    — **L** contains the axiom $\Box(p \vee q) \to ((\Diamond p \to \Box q) \to \Box q)$

    — **L** contains the axiom $\Box(p \to q) \to (\Diamond p \to \Diamond q)$

# A minimal setting: Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a second possibility

- Semantics : all $(W, \leq, R)$
  - Truth conditions for $\Box$

    $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t\ \Rightarrow\ \forall u \in R(t),\ u \models A)$
  - Truth conditions for $\Diamond$

    $s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t\ \Rightarrow\ \exists u \in R(t),\ u \models A)$

# A minimal setting: Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a second possibility
**Wijesekera (1990)** — all $(W, \leq, R)$

- Defined logic **WK**
  - Least $\mathbf{L} \subseteq \mathbf{FOR}$ closed for uniform substitution and such that
    - **L** contains the standard axioms of ~~**CPL**~~ **IPL**
    - **L** is closed with respect to the rule $\frac{p \quad p \to q}{q}$
    - ~~**L** contains the axiom $\Box p \leftrightarrow \neg \Diamond \neg p$~~
    - ~~**L** contains the axiom $\Diamond p \leftrightarrow \neg \Box \neg p$~~
    - **L** contains the axiom $\Box p \wedge \Box q \to \Box (p \wedge q)$
    - ~~**L** contains the axiom $\Diamond (p \vee q) \to \Diamond p \vee \Diamond q$~~
    - **L** contains the axiom $\Box \top$
    - **L** contains the axiom $\neg \Diamond \bot$
    - ~~**L** contains the axiom $(\Diamond p \to \Box q) \to \Box (p \to q)$~~
    - **L** contains the axiom $\Diamond (p \to q) \to (\Box p \to \Diamond q)$
    - **L** is closed with respect to the rule $\frac{p \to q}{\Box p \to \Box q}$
    - **L** is closed with respect to the rule $\frac{p \to q}{\Diamond p \to \Diamond q}$

# A minimal setting: Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a third possibility

- Semantics : all $(W, \leq, R)$
  - Truth conditions for $\Box$
    $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$
  - Truth conditions for $\Diamond$
    $s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$

# A minimal setting: Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a third possibility
**Plotkin and Stirling** — all $(W, \leq, R)$

- Defined logic ?
  - **Simpson (1994):** "axiomatization is rather complicated"

# A minimal setting: Then, about intuitionistic modal logics

## Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a third possibility

## **B. and Gencer** — all $(W, \leq, R)$

- Defined logic $\mathbf{L}_{\min}$
  - Least $\mathbf{L}\subseteq\mathbf{FOR}$ closed for uniform substitution and such that

    **L** contains the standard axioms of ~~**CPL**~~ **IPL**

    **L** is closed with respect to the rule $\frac{p \quad p\to q}{q}$

    ~~**L** contains the axiom~~ $\cancel{\Box p\leftrightarrow\neg\Diamond\neg p}$

    ~~**L** contains the axiom~~ $\cancel{\Diamond p\leftrightarrow\neg\Box\neg p}$

    **L** contains the axiom $\Box p\wedge\Box q\to\Box(p\wedge q)$

    **L** contains the axiom $\Diamond(p\vee q)\to\Diamond p\vee\Diamond q$

    **L** contains the axiom $\Box\top$

    **L** contains the axiom $\neg\Diamond\bot$

    ~~**L** contains the axiom~~ $\cancel{(\Diamond p\to\Box q)\to\Box(p\to q)}$

    ~~**L** contains the axiom~~ $\cancel{\Diamond(p\to q)\to(\Box p\to\Diamond q)}$

    **L** is closed with respect to the rule $\frac{p\to q}{\Box p\to\Box q}$

    **L** is closed with respect to the rule $\frac{p\to q}{\Diamond p\to\Diamond q}$

    — **L** contains the axiom $\Box(p\vee q)\to((\Diamond p\to\Box q)\to\Box q)$

    — **L** is closed with respect to the rule $\frac{\Diamond p\to q\vee\Box(p\to r)}{\Diamond p\to q\vee\Diamond r}$

# A minimal setting: Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a fourth possibility

- Semantics : all downward and forward confluent $(W, \leq, R)$
  - Truth conditions for $\Box$

$$
\begin{aligned}
s \models \Box A &\Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A) \\
s \models \Box A &\Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \forall u \in R(t),\ u \models A) \\
s \models \Box A &\Leftrightarrow \forall t \in R(s),\ t \models A
\end{aligned}
$$

  - Truth conditions for $\Diamond$

$$
\begin{aligned}
s \models \Diamond A &\Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A) \\
s \models \Diamond A &\Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A) \\
s \models \Diamond A &\Leftrightarrow \exists t \in R(s),\ t \models A
\end{aligned}
$$

# A minimal setting: Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a fourth possibility
**B. and Gencer** — all down. and for. conf. $(W, \leq, R)$

- Defined logic $\mathbf{HK}\Box\Diamond = \mathbf{L}_{\mathsf{min}} + \Box(p\vee q)\to\Diamond p\vee\Box q + \Diamond(p\to q)\to(\Box p\to\Diamond q)$

# A minimal setting: Conclusion and perspectives

## Fischer Servi (1984)

- Semantics : all forward and backward confluent $(W, \leq, R)$
  - Truth conditions for $\Box$
    $$s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$$
  - Truth conditions for $\Diamond$
    $$s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$$
    $$s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$$
    $$s \models \Diamond A \quad \Leftrightarrow \exists t \in R(s),\ t \models A$$
- Defined logic **IK**
- Extensions of **IK**
  - **IT** $::=$ **IK** $+\ \Box p \to p\ +\ p \to \Diamond p$
  - **I4** $::=$ **IK** $+\ \Box p \to \Box\Box p\ +\ \Diamond\Diamond p \to \Diamond p$
  - **IB** $::=$ **IK** $+\ p \to \Box\Diamond p\ +\ \Diamond\Box p \to p$
  - **I5** $::=$ **IK** $+\ \Box p \to p\ +\ p \to \Diamond p\ +\ \Diamond p \to \Box\Diamond p\ +\ \Diamond\Box p \to \Box p$

# A minimal setting: Conclusion and perspectives

## B., Gao, Gencer and Olivetti

- Semantics : all forward confluent $(W, \leq, R)$
  - Truth conditions for $\Box$

    $$s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t\ \Rightarrow\ \forall u \in R(t),\ u \models A)$$
  - Truth conditions for $\Diamond$

    $$s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t\ \Rightarrow\ \exists u \in R(t),\ u \models A)$$
    $$s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t\ \&\ \exists u \in R(t),\ u \models A)$$
    $$s \models \Diamond A \quad \Leftrightarrow \exists t \in R(s),\ t \models A$$
- Defined logic **FIK**

# A minimal setting: Conclusion and perspectives

## Wijesekera (1990)

- Semantics : all $(W, \leq, R)$
  - Truth conditions for $\Box$
    $$s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$$
  - Truth conditions for $\Diamond$
    $$s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$$
- Defined logic **WK**
- Extension of **WK**
  - **WK** $+\ \neg\Diamond\top \to \Box\bot\ +\ (\Diamond\top \to \Box p) \to \Box p$

# A minimal setting: Conclusion and perspectives

## B. and Gencer

- Semantics : all $(W, \leq, R)$
  - Truth conditions for $\Box$
    $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$
  - Truth conditions for $\Diamond$
    $s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$
- Defined logic $\mathbf{L}_{\mathrm{min}}$
- Extensions of $\mathbf{L}_{\mathrm{min}}$
  - $\mathbf{L}_{\mathbf{dc}} ::= \mathbf{L}_{\mathrm{min}} + \Box(p \vee q) \to \Diamond p \vee \Box q$
  - $\mathbf{L}_{\mathbf{fc}} ::= \mathbf{L}_{\mathrm{min}} + \Diamond(p \to q) \to (\Box p \to \Diamond q)$
  - $\mathbf{L}_{\mathbf{bc}} ::= \mathbf{L}_{\mathrm{min}} + (\Diamond p \to \Box q) \to \Box(p \to q)$
  - $\mathbf{L}_{\mathbf{dfc}} ::= \mathbf{L}_{\mathrm{min}} + \Box(p \vee q) \to \Diamond p \vee \Box q + \Diamond(p \to q) \to (\Box p \to \Diamond q)$
  - $\mathbf{L}_{\mathbf{dbc}} ::= \mathbf{L}_{\mathrm{min}} + \Box(p \vee q) \to \Diamond p \vee \Box q + (\Diamond p \to \Box q) \to \Box(p \to q)$
  - $\mathbf{L}_{\mathbf{fbc}} ::= \mathbf{L}_{\mathrm{min}} + \Diamond(p \to q) \to (\Box p \to \Diamond q) + (\Diamond p \to \Box q) \to \Box(p \to q)$
  - $\mathbf{L}_{\mathbf{dfbc}} ::= \mathbf{L}_{\mathrm{min}} + \Box(p \vee q) \to \Diamond p \vee \Box q + \Diamond(p \to q) \to (\Box p \to \Diamond q) + (\Diamond p \to \Box q) \to \Box(p \to q)$

# A minimal setting: Conclusion and perspectives

## B. and Gencer

- Semantics : all down. and for. conf. $(W, \leq, R)$
  - Truth conditions for $\Box$

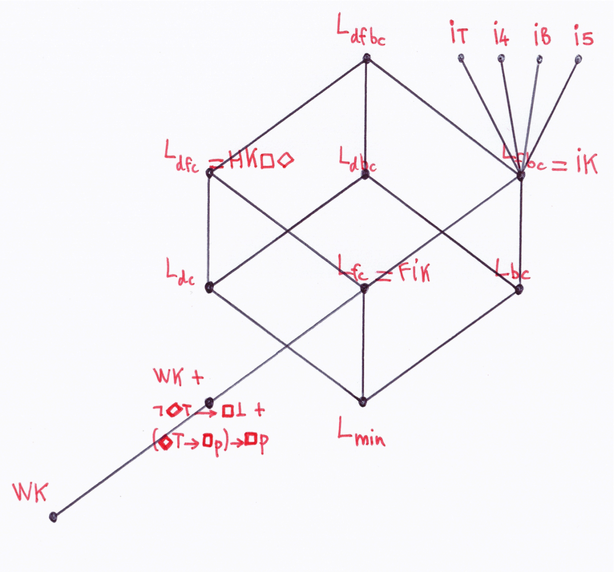
    $$\begin{array}{lll} s \models \Box A & \Leftrightarrow & \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A) \\ s \models \Box A & \Leftrightarrow & \exists t \in W,\ (s \geq t\ \&\ \forall u \in R(t),\ u \models A) \\ s \models \Box A & \Leftrightarrow & \forall t \in R(s),\ t \models A \end{array}$$

  - Truth conditions for $\Diamond$

    $$\begin{array}{lll} s \models \Diamond A & \Leftrightarrow & \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A) \\ s \models \Diamond A & \Leftrightarrow & \exists t \in W,\ (s \geq t\ \&\ \exists u \in R(t),\ u \models A) \\ s \models \Diamond A & \Leftrightarrow & \exists t \in R(s),\ t \models A \end{array}$$

- Defined logic $\mathbf{HK}\Box\Diamond$

# A minimal setting: Conclusion and perspectives



# A minimal setting: Conclusion and perspectives

## Computability and complexity

**IS**5 is decidable

- Mints (1971)

**WK** is decidable

- Wijesekera (1990)

**IK** is decidable

- Grefe (1998)

**IS**4 is decidable

- Girlando, Kuznets, Marin, Morales, Straßburger (2023)

**FIK** is decidable

- B., Gao, Gencer, Olivetti (2024)

**HK**$\Box\Diamond$ is decidable

- B., Gao, Gencer, Olivetti (2024)

# A minimal setting: Conclusion and perspectives

### Correspondence Theory à la Fischer Servi

For all forward confluent frames $\mathcal{F} = (W, \leq, R)$,

1. $\mathcal{F} \models \Box\bot \to \bot$ if and only if $\leq\circ R$ is serial,
2. $\mathcal{F} \models \top \to \Diamond\top$ if and only if $R$ is serial,
3. $\mathcal{F} \models \Box p \to p$ if and only if $\leq\circ R\circ\leq$ is reflexive,
4. $\mathcal{F} \models p \to \Diamond p$ if and only if $R\circ\geq$ is reflexive,
5. $\mathcal{F} \models \Box p \to \Box\Box p$ if and only if $R\circ\leq\circ R$ is included in $\leq\circ R\circ\leq$,
6. $\mathcal{F} \models \Diamond\Diamond p \to \Diamond p$ if and only if $R\circ R$ is included in $R\circ\geq$,
7. $\mathcal{F} \models \Diamond\Box p \to p$ if and only if $R^{-1}$ is included in $\leq\circ R\circ\leq$,
8. $\mathcal{F} \models p \to \Box\Diamond p$ if and only if $R^{-1}$ is included in $R\circ\geq$,
9. $\mathcal{F} \models \Diamond\Box p \to \Box p$ if and only if $R^{-1}\circ R$ is included in $\leq\circ R\circ\leq$,
10. $\mathcal{F} \models \Diamond p \to \Box\Diamond p$ if and only if $R^{-1}\circ R$ is included in $R\circ\geq$.

# A minimal setting: Conclusion and perspectives

## Correspondence Theory à la Fischer Servi

For all forward and backward confluent frames $\mathcal{F} = (W, \leq, R)$,

1. $\mathcal{F} \models \Box\bot \to \bot$ if and only if $\leq \circ R$ is serial,
2. $\mathcal{F} \models \top \to \Diamond\top$ if and only if $R$ is serial,
3. $\mathcal{F} \models \Box p \to p$ if and only if $\leq \circ R$ is reflexive,
4. $\mathcal{F} \models p \to \Diamond p$ if and only if $R \circ \geq$ is reflexive,
5. $\mathcal{F} \models \Box p \to \Box\Box p$ if and only if $R \circ R$ is included in $\leq \circ R$,
6. $\mathcal{F} \models \Diamond\Diamond p \to \Diamond p$ if and only if $R \circ R$ is included in $R \circ \geq$,
7. $\mathcal{F} \models \Diamond\Box p \to p$ if and only if $R^{-1}$ is included in $\leq \circ R$,
8. $\mathcal{F} \models p \to \Box\Diamond p$ if and only if $R^{-1}$ is included in $R \circ \geq$,
9. $\mathcal{F} \models \Diamond\Box p \to \Box p$ if and only if $R^{-1} \circ R$ is included in $\leq \circ R$,
10. $\mathcal{F} \models \Diamond p \to \Box\Diamond p$ if and only if $R^{-1} \circ R$ is included in $R \circ \geq$.

# A minimal setting: Conclusion and perspectives

## Correspondence Theory à la Fischer Servi

For all forward and downward confluent frames $\mathcal{F} = (W, \leq, R)$,

1. $\mathcal{F} \models \Box\bot \to \bot$ if and only if $R$ is serial,
2. $\mathcal{F} \models \top \to \Diamond\top$ if and only if $R$ is serial,
3. $\mathcal{F} \models \Box p \to p$ if and only if $R \circ \leq$ is reflexive,
4. $\mathcal{F} \models p \to \Diamond p$ if and only if $R \circ \geq$ is reflexive,
5. $\mathcal{F} \models \Box p \to \Box\Box p$ if and only if $R \circ R$ is included in $R \circ \leq$,
6. $\mathcal{F} \models \Diamond\Diamond p \to \Diamond p$ if and only if $R \circ R$ is included in $R \circ \geq$,
7. $\mathcal{F} \models \Diamond\Box p \to p$ if and only if $R^{-1}$ is included in $R \circ \leq$,
8. $\mathcal{F} \models p \to \Box\Diamond p$ if and only if $R^{-1}$ is included in $R \circ \geq$,
9. $\mathcal{F} \models \Diamond\Box p \to \Box p$ if and only if $R^{-1} \circ R$ is included in $R \circ \leq$,
10. $\mathcal{F} \models \Diamond p \to \Box\Diamond p$ if and only if $R^{-1} \circ R$ is included in $R \circ \geq$.

# A minimal setting: Conclusion and perspectives

## Correspondence Theory à la Wijesekera

For all frames $\mathcal{F} = (W, \leq, R)$,

1. $\mathcal{F} \models \Box\bot \to \bot$ if and only if ???
2. $\mathcal{F} \models \top \to \Diamond\top$ if and only if ???
3. $\mathcal{F} \models \Box p \to p$ if and only if ???
4. $\mathcal{F} \models p \to \Diamond p$ if and only if ???
5. $\mathcal{F} \models \Box p \to \Box\Box p$ if and only if ???
6. $\mathcal{F} \models \Diamond\Diamond p \to \Diamond p$ if and only if ???
7. $\mathcal{F} \models \Diamond\Box p \to p$ if and only if ???
8. $\mathcal{F} \models p \to \Box\Diamond p$ if and only if ???
9. $\mathcal{F} \models \Diamond\Box p \to \Box p$ if and only if ???
10. $\mathcal{F} \models \Diamond p \to \Box\Diamond p$ if and only if ???

# A minimal setting: Conclusion and perspectives

## Correspondence Theory

Modal formulas vs. sentences

- A modal formula is first-order definable in a class of frames if there exists a sentence that is true exactly in the frames of that class validating the modal formula
- A sentence is modally definable in a class of frames if there exists a modal formula that is valid exactly in the frames of that class satisfying the sentence
- A modal formula and a sentence correspond in a class of frames if they are respectively valid and true in the same frames of that class

# A minimal setting: Conclusion and perspectives

## In the case of classical modal logic

Sahlqvist's Correspondence and Completeness Theorems:

- Large set of modal formulas which guarantee completeness with respect to first-order definable classes of frames
- The first-order conditions corresponding to Sahlqvist formulas are effectively computable

Chagrova's Undecidability Theorems: The following decision problems are undecidable:

- Given a modal formula, determine whether it is first-order definable with respect to the class of all frames
- Given a sentence, determine whether it is modally definable with respect to the class of all frames
- Given a modal formula and a sentence, determine whether they correspond with respect to the class of all frames

## In the case of intuitionistic modal logic?

# A minimal setting: Conclusion and perspectives

## Intuitionistic dynamic epistemic logics

- Formulas

$$A \ ::= \ p | (A \to A) | \bot | \top | (A \vee A) | (A \wedge A) | \Box A | \Diamond A | [A]A | \langle A \rangle A$$

Sadrzadeh, Palmigiano and Ma (2011)
Maffezioli and Negri (2015)
Nomura, Sano and Tojo (2015)
B. and Galmiche (2016)

In the case of intuitionistic modal logic?

# A minimal setting: Conclusion and perspectives

## Extended language of intuitionistic modal logics

- Formulas

$$A \ ::= \ p|(A{\rightarrow}A)|\bot|\top|(A{\vee}A)|(A{\wedge}A)|\Box_1 A|\Diamond_1 A|\Box_2 A|\Diamond_2 A$$

## Extended relational semantics of intuitionistic modal logics

- Truth conditions

$$\begin{array}{lcl}
s{\models}p & \Leftrightarrow & s{\in}V(p) \\
s{\models}A{\rightarrow}B & \Leftrightarrow & \forall t{\in}W,\ (s{\leq}t \ \Rightarrow \ t{\not\models}A \text{ or } t{\models}B) \\
\ldots & & \\
s{\models}\Box_1 A & \Leftrightarrow & \forall t{\in}W,\ (s{\leq}t \ \Rightarrow \ \forall u{\in}R(t),\ u{\models}A) \\
s{\models}\Diamond_1 A & \Leftrightarrow & \exists t{\in}W,\ (s{\geq}t \ \&\ \exists u{\in}R(t),\ u{\models}A) \\
s{\models}\Box_2 A & \Leftrightarrow & \exists t{\in}W,\ (s{\geq}t \ \&\ \forall u{\in}R(t),\ u{\models}A) \\
s{\models}\Diamond_2 A & \Leftrightarrow & \forall t{\in}W,\ (s{\leq}t \ \Rightarrow \ \exists u{\in}R(t),\ u{\models}A)
\end{array}$$

# A minimal setting: Conclusion and perspectives

## Extended language of intuitionistic modal logics

- Formulas

$$A \ ::= \ p|(A \rightharpoonup A)|(A \leftharpoondown A)|\bot|\top|(A\vee A)|(A\wedge A)|\Box_1 A|\Diamond_1 A|\Box_2 A|\Diamond_2 A$$

## Extended relational semantics of intuitionistic modal logics

- Truth conditions

$$\begin{array}{lcl}
s\models p & \Leftrightarrow & s\in V(p) \\
s\models A\rightharpoonup B & \Leftrightarrow & \forall t\in W,\ (s\leq t \ \Rightarrow \ t\not\models A \text{ or } t\models B) \\
s\models A\leftharpoondown B & \Leftrightarrow & \exists t\in W,\ (s\geq t \ \&\ t\not\models A \ \&\ t\models B) \\
\ldots & & \\
s\models \Box_1 A & \Leftrightarrow & \forall t\in W,\ (s\leq t \ \Rightarrow \ \forall u\in R(t),\ u\models A) \\
s\models \Diamond_1 A & \Leftrightarrow & \exists t\in W,\ (s\geq t \ \&\ \exists u\in R(t),\ u\models A) \\
s\models \Box_2 A & \Leftrightarrow & \exists t\in W,\ (s\geq t \ \&\ \forall u\in R(t),\ u\models A) \\
s\models \Diamond_2 A & \Leftrightarrow & \forall t\in W,\ (s\leq t \ \Rightarrow \ \exists u\in R(t),\ u\models A)
\end{array}$$

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