Semantic theories of truth: Lecture 3
Fixed-point semantics - Part II

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August 5-9, 2024
35th ESSLLI – Leuven

Outline

- Summary of Lecture 2.
- Mathematical interlude: Further order-theoretic facts.
- Philosophical interlude: The supervenience of semantics.
- Fixed-point semantics and truth-theoretic puzzles.

Base semantics

Base language $\mathcal{L}(P, N)$

— Set P of propositional letters : $p, q, \ldots$
— Set N of names N: $a, b, c, \ldots$

Ground model $\mathcal{M} = (D, I^-)$

— Domain: A non-empty set D.
— Interpretation of the propositional letters: $I^- : P \to \{\mathbf{t}, \mathbf{f}\}$.
— Interpretation of the names: $I^- : N \to D$.

Base admissible valuations

- $\mathrm{Val}^{\tau}_{\mathcal{M}}(p) = I^-(p)$.
- $\mathrm{Val}^{\tau}_{\mathcal{M}}(\neg\phi) = -^{\tau}\mathrm{Val}^{\tau}_{\mathcal{M}}(\phi)$.
- $\mathrm{Val}^{\tau}_{\mathcal{M}}(\phi \wedge \psi) = \mathrm{Val}^{\tau}_{\mathcal{M}}(\phi) *^{\tau} \mathrm{Val}^{\tau}_{\mathcal{M}}(\psi)$.

A bi-valued valuation $v^- : \mathcal{L}(P) \to \{\mathbf{t}, \mathbf{f}\}$ is classically admissible iff $v^- = \mathrm{Val}^{\tau}_{\mathcal{M}}$.

Strong Kleene semantics

Truth language $\mathcal{L}_{\mathrm{T}}(P, N)$

— $\mathcal{L}(P, N)$ augmented by a unary predicate T.

Partial model $\mathcal{M} + (Z^{\mathbf{t}}, Z^{\mathbf{f}})$

— The ground model $\mathcal{M}$ expanded by a partial set $I(\mathrm{T}) = (Z^{\mathbf{t}}, Z^{\mathbf{f}})$ interpreting T.

Strong Kleene admissible valuations

- $\mathrm{Val}^{\kappa}_{\mathcal{M}+(Z^{\mathbf{t}},Z^{\mathbf{f}})}(p) = I^{-}(p)$.
- $\mathrm{Val}^{\kappa}_{\mathcal{M}+(Z^{\mathbf{t}},Z^{\mathbf{f}})}(\mathrm{T}a) = I(\mathrm{T})(I^{-}(a))$.
- $\mathrm{Val}^{\kappa}_{\mathcal{M}+(Z^{\mathbf{t}},Z^{\mathbf{f}})}(\neg\phi) = -^{\kappa}\mathrm{Val}^{\kappa}_{\mathcal{M}+(Z^{\mathbf{t}},Z^{\mathbf{f}})}(\phi)$.
- $\mathrm{Val}^{\kappa}_{\mathcal{M}+(Z^{\mathbf{t}},Z^{\mathbf{f}})}(\phi \wedge \psi) = \mathrm{Val}^{\kappa}_{\mathcal{M}+(Z^{\mathbf{t}},Z^{\mathbf{f}})}(\phi) *^{\kappa} \mathrm{Val}^{\kappa}_{\mathcal{M}+(Z^{\mathbf{t}},Z^{\mathbf{f}})}(\psi)$.

A three-valued valuation $v : \mathcal{L}(P) \to \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ is Strong Kleene admissible iff $v = \mathrm{Val}^{\kappa}_{\mathcal{M}+(Z^{\mathbf{t}},Z^{\mathbf{f}})}$.

Strong Kleene fixed-point semantics

Fix a ground model $\mathcal{M} = (D, I^-)$.

The Strong Kleene operator is the function κ on partial sets of D defined by

$$\kappa(Z^{\mathbf{t}}, Z^{\mathbf{f}}) = (W^{\mathbf{t}}, W^{\mathbf{f}}), \text{where}$$

$$W^{\mathbf{t}} = \{\phi \in \mathcal{L}_{\mathsf{T}}(N, P) \mid \mathrm{Val}^{\kappa}_{\mathcal{M}+(Z^{\mathbf{t}},Z^{\mathbf{f}})}(\phi) = \mathbf{t}\}.$$
$$W^{\mathbf{f}} = \{\phi \in \mathcal{L}_{\mathsf{T}}(N, P) \mid \mathrm{Val}^{\kappa}_{\mathcal{M}+(Z^{\mathbf{t}},Z^{\mathbf{f}})}(\phi) = \mathbf{f}\}.$$

A Strong Kleene fixed point is a partial set $(Z^{\mathbf{t}}, Z^{\mathbf{f}})$ such that $\kappa(Z^{\mathbf{t}}, Z^{\mathbf{f}}) = (Z^{\mathbf{t}}, Z^{\mathbf{f}})$. The set of fixed points of κ is denoted by $\mathrm{Fix}(\kappa)$.

A Strong Kleene fixed-point semantics is any family $S \subseteq \mathrm{Fix}(\kappa)$ of Strong Kleene fixed points.

Strong Kleene fixed points: Order-theoretic structure

Let $(Z^{\mathbf{t}}, Z^{\mathbf{f}}), (W^{\mathbf{t}}, W^{\mathbf{f}})$ be two partial sets of a domain D.

We say that $(Z^{\mathbf{t}}, Z^{\mathbf{f}})$ is included in $(W^{\mathbf{t}}, W^{\mathbf{f}})$, writing $(Z^{\mathbf{t}}, Z^{\mathbf{f}}) \subseteq (W^{\mathbf{t}}, W^{\mathbf{f}})$ if and only if $Z^{\mathbf{t}} \subseteq W^{\mathbf{t}}$ and $Z^{\mathbf{f}} \subseteq W^{\mathbf{f}}$.

Remark

1. The family of all partial sets of D ordered by inclusion is a ccpo.
2. The Strong Kleene operator κ is monotonic.
3. The set Fix(κ) of Strong Kleene fixed points, ordered by inclusion, is a ccpo.

Intrinsic fixed points

Let $(Q, \preceq)$ be a *ccpo* and let $f : Q \to Q$ be *monotonic*.

A member $x \in Q$ is f-sound iff $x \preceq f(x)$.
A fixed point x of f is f-intrinsic iff x is compatible with any other fixed point of f.

Fact (Fact 2)

1. *The set of fixed point of f has a minimum, the least fixed point, denoted by $\mathrm{lfp}(f)$.*
2. *For every f-sound element x there exists a fixed point y of f such that $x \preceq y$.*
3. *The set of all intrinsic fixed points of f forms a complete lattice. Hence, there exists the greatest intrinsic fixed point of f, denoted by $\mathrm{gifp}(f)$.*

The least fixed point

Let $(Q, \preceq)$ be a *ccpo* and let $f : Q \to Q$ be *monotonic*.

Let d denote the least element of Q and let the sequence $\langle d_\alpha \mid \alpha \in \mathrm{On} \rangle$ be defined, for every ordinal α, by

- $d_0 = d$;
- $d_{\alpha+1} = f(d_\alpha)$;
- $d_\alpha = \mathrm{lub}\{d_\beta \mid \beta < \alpha\}$, if α is limit.

Fact (Fact 3)

There exists the least ordinal α_* such that $f(d_{\alpha_*}) = d_{\alpha_*}$. Moreover, d_{α_*} is the least fixed point of f.

The supervenience of semantics

> *The interpretation of the semantical constants of the language is determined by the interpretation of the non-semantical constants. Expressed in terms of our formal theory, the intuition becomes: for any given base model there is exactly one correct interpretation of the truth predicate. I call this principle "the supervenience of semantics".*

M. Kremer, *Kripke and the logic of truth*, 1988.

Puzzle 2 - The partial Liar

Liar sentence

a: $\neg \mathrm{T}a$.

Puzzle 2

Support $X = \{\neg \mathrm{T}a\}$.

Reference list $\pi(a) = \neg \mathrm{T}a$.

Notion of solution $h : X \to \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ such that the least Strong Kleene fixed point v is such that $v \restriction X = h$.

Solution

Let $h(\neg \mathrm{T}a) = \mathbf{n}$. Let $v = \mathrm{lfp}(\kappa)$. Hence,

1. $v(\mathrm{T}a) = v(\neg \mathrm{T}a)$ [Partial truth schema]
2. $v(\neg \mathrm{T}a) = \mathbf{n} = h(\neg \mathrm{T}a)$ [Strong Kleene negation].

Example 3 - Simplified Nixon-Dean

Suppose that Nixon and Dean utter each one just two statements.

Nixon says:

- (N1) *Both statements uttered by Dean are not true.*
- (N2) *$2 + 2 = 5$. (false)*

Dean says:

- (D1) *Both statements uttered by Nixon are not true.*
- (D2) *$2 + 2 = 4$. (true)*

There is no trouble in evaluating these sentences. Since D2 is true, N1 is false. Since also N2 is false, D1 is true.

Puzzle 3 - Nixon-Dean

a: $\neg Tc \wedge \neg Td$.
b: p.
c: $\neg Ta \wedge \neg Tb$.
d: q.

Base semantics: $v^{-}(p) = \mathbf{f}$, $v^{-}(q) = \mathbf{t}$.

ϕ	$v(\phi)$	
p	**f**	Base semantics
q	**t**	Base semantics
Td	**t**	Partial truth schema
$\neg Td$	**f**	Strong Kleene negation
$\neg Tc \wedge \neg Td$	**f**	Strong Kleene conjunction
Tb	**f**	Partial truth schema
$\neg Tb$	**t**	Strong Kleene negation
Ta	**f**	Partial truth schema
$\neg Ta$	**t**	Strong Kleene negation
$\neg Ta \wedge \neg Tb$	**t**	Strong Kleene conjunction

Example 4 - Gupta's puzzle

Suppose for example that we have a situation in which speakers A and B make the assertions displayed below.

A says:

(a1) Two plus two is three. (false)
(a2) Snow is always black. (false)
(a3) Everything B says is true. (–)
(a4) Ten is a prime number. (false)
(a5) Something B says is not true. (–)

B says:

(b1) One plus one is two. (true)
(b2) My name is B. (true)
(b3) Snow is sometimes white. (true)
(b4) At most one thing A says is true. (–)

Gupta, *Truth and Paradox*, 1982.

Example 5 - Visser-Chapuis Gupta puzzle

> *Suppose that speakers A and B make the assertions displayed below.*
>
> *A says:*
>
> *(a1) Statement (b) is true.*
>
> *(a2) Statement (b) is not true.*
>
> *B says:*
>
> *(b) Statements (a1) and (a2) cannot both be true.*

There is no trouble in evaluating these sentences. (a1) and (a2) contradict each other, hence (b) is to be true. Therefore (a1) is true and (a2) is false.

Puzzle 4 - Gupta's puzzle

a: Tc.

b: $\neg$Tc.

c: $\neg$(Ta $\wedge$ Tb).

Let v be the valuation associated to the partial set $\kappa(\emptyset, \emptyset)$. Then, $v(\mathrm{T}c) = \mathbf{n}$ iff $\pi(c) \notin \emptyset \cup \emptyset$. Hence $v(\mathrm{T}c) = \mathbf{n}$.
$v(\neg\mathrm{T}c) = -v(\mathrm{T}c) = -\mathbf{n} = \mathbf{n}$.
$v(\mathrm{T}a) = v(\mathrm{T}b) = \mathbf{n}$. Thus $v(\neg(\mathrm{T}a \wedge \mathrm{T}b)) = -(\mathbf{n} * \mathbf{n}) = -\mathbf{n} = \mathbf{n}$.

By induction on the ordinals we can prove that for every ordinal α, if the three sentences do not belong to $Z^{\mathbf{t}}_{\alpha} \cup Z^{\mathbf{f}}_{\alpha}$, then they do not belong to $Z^{\mathbf{t}}_{\alpha+1} \cup Z^{\mathbf{f}}_{\alpha+1}$ too. Since at limit stages α, the sets $Z^{\mathbf{f}}_{\alpha}, Z^{\mathbf{t}}_{\alpha}$ are defined as the union of all sets $Z^{\mathbf{f}}_{\beta}, Z^{\mathbf{t}}_{\beta}$ for all $\beta < \alpha$, it follows that the three sentences do not belong to $Z^{\mathbf{t}}_{\alpha} \cup Z^{\mathbf{f}}_{\alpha}$ for any α. Since, the partial set $\mathrm{lfp}(\kappa) = (Z^{\mathbf{t}}_{\alpha_*}, Z^{\mathbf{f}}_{\alpha_*})$ is a fixed point of κ, this means that in $\mathrm{lfp}(\kappa)$ all three sentences are evaluated $\mathbf{n}$, contrary to what was expected according to the informal reasoning.