Intuitionistic modal logic

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Intuitionistic modal logics

Outline

- Intermediate logics
- **Modal logics**
- Combining logics
- Two peculiar intuitionistic modal logics
- A minimal setting

Modal logics

Outline

- Syntax and semantics
- Proof theory
- Canonical models and completeness
- Multimodal languages

Modal logics: syntax and semantics

Modal logics: syntax and semantics

Modal formulas

- AF: countable set of atomic formulas
- $Fma(AF)$: set of all formulas generated from AF

- Atomic formulas: $p \in AF$
- Formulas: $\phi \in Fma(AF)$

$$\phi ::= p \mid \bot \mid (\phi_1 \to \phi_2) \mid \Box\phi$$

Modal logics: syntax and semantics

Possible readings of $\Box\phi$

- It is necessarily true that ϕ
- It will always be true that ϕ
- It ought to be that ϕ
- It is known that ϕ
- It is believed that ϕ
- It is provable in Peano Arithmetic that ϕ
- After the program terminates, ϕ

Modal logics: syntax and semantics

Other connectives

Negation: $\neg\phi ::= (\phi \to \bot)$

Verum: $\top ::= \neg\bot$

Disjunction: $(\phi_1 \lor \phi_2) ::= (\neg\phi_1 \to \phi_2)$

Conjunction: $(\phi_1 \land \phi_2) ::= \neg(\phi_1 \to \neg\phi_2)$

Equivalence: $(\phi_1 \leftrightarrow \phi_2) ::= ((\phi_1 \to \phi_2) \land (\phi_2 \to \phi_1))$

"Diamond": $\Diamond\phi ::= \neg\Box\neg\phi$

Modal logics: syntax and semantics

Exercise

Decide what $\Diamond\phi$ means under each of the above readings of $\Box$.

Which of the following should be regarded as true under the different readings of $\Box$?

- $\Box\phi \to \phi$
- $\Box\phi \to \Box\Box\phi$
- $\Diamond\top$
- $\Box\phi \to \Diamond\phi$
- $\Box\phi \vee \Box\neg\phi$
- $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
- $\Diamond\phi \wedge \Diamond\psi \to \Diamond(\phi \wedge \psi)$
- $\Box(\Box\phi \to \phi) \to \Box\phi$

Modal logics: syntax and semantics

Subformulas

$SF(\phi)$: set of all subformulas of $\phi \in Fma(AF)$

$$
\begin{aligned}
SF(p) &= \{p\} \\
SF(\bot) &= \{\bot\} \\
SF(\phi \to \psi) &= \{\phi \to \psi\} \cup SF(\phi) \cup SF(\psi) \\
SF(\Box\phi) &= \{\Box\phi\} \cup SF(\phi)
\end{aligned}
$$

Modal logics: syntax and semantics

Frames and models

A frame is a pair $\mathcal{F} = (S, R)$ where

- S is a nonempty set
- $R \subseteq S \times S$

A model on a frame is a triple $\mathcal{M} = (S, R, V)$ where

- $V : AF \to 2^S$

$V(p)$ is to be thought of as the set of points at which p is "true"

$\mathcal{M} \models_s \phi$: relation "$\phi$ is true at point s in model $\mathcal{M}$"

- $\mathcal{M} \models_s p$ iff $s \in V(p)$
- $\mathcal{M} \not\models_s \bot$
- $\mathcal{M} \models_s \phi \to \psi$ iff $\mathcal{M} \models_s \phi$ implies $\mathcal{M} \models_s \psi$
- $\mathcal{M} \models_s \Box\phi$ iff for all $t \in S$, sRt implies $\mathcal{M} \models_t \phi$

Modal logics: syntax and semantics

Exercise

Show that $\mathcal{M} \models_s \neg\phi$ iff $\mathcal{M} \not\models_s \phi$.
Work out the corresponding truth conditions for $\top$, $\phi \vee \psi$, $\phi \wedge \psi$ and $\phi \leftrightarrow \psi$.

Show that $\mathcal{M} \models_s \Diamond\phi$ iff there exists $t \in S$ with sRt and $\mathcal{M} \models_t \phi$.

Alethic logic

- sRt: t is a conceivable alternative to s
- $\Box\phi$: ϕ is necessarily true

Deontic logic

- sRt: t is a morally ideal alternative to s
- $\Box\phi$: ϕ ought to be true

Modal logics: syntax and semantics

Motivations

Temporal logic

- sRt: t is after s
- $\Box\phi$: henceforth, ϕ

Dynamic logic

- sRt: there is an execution of the program that starts in s and terminates in t
- $\Box\phi$: after the program terminates, ϕ

$\mathcal{M} \models \phi$: relation "ϕ is true in model $\mathcal{M} = (S, R, V)$"

$\mathcal{M} \models \phi$ iff $\mathcal{M} \models_s \phi$ for all $s \in S$

$\mathcal{F} \models \phi$: relation "ϕ is valid in frame $\mathcal{F} = (S, R)$"

$\mathcal{F} \models \phi$ iff $\mathcal{M} \models \phi$ for all models $\mathcal{M} = (S, R, V)$

$\mathcal{C} \models \phi$: relation "ϕ is valid in class $\mathcal{C}$ of frames"

$\mathcal{C} \models \phi$ iff $\mathcal{F} \models \phi$ for all frames $\mathcal{F}$ in $\mathcal{C}$

Modal logics: syntax and semantics

Exercise

Show that the following formulas are valid in all frames.

- $\Box\top$
- $\Box(p \to q) \to (\Box p \to \Box q)$
- $\Diamond(p \to q) \to (\Box p \to \Diamond q)$
- $\Box(p \to q) \to (\Diamond p \to \Diamond q)$
- $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$
- $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$

Modal logics: syntax and semantics

Exercise

Show that the following formulas do not have the property of being valid in all frames.

- $\Box p \to p$
- $\Box p \to \Box\Box p$
- $\Box(p \to q) \to (\Box p \to \Diamond q)$
- $\Diamond \top$
- $\Diamond p \to \Box p$
- $\Box(\Box p \to q) \vee \Box(\Box q \to p)$
- $\Box(p \vee q) \to (\Box p \vee \Box q)$
- $\Box(\Box p \to p) \to \Box p$

Modal logics: syntax and semantics

Exercise

Show that the formulas $\Diamond\top$ and $\Box p \rightarrow \Diamond p$ are valid in the same frames.

Exhibit a frame in which $\Box\bot$ is valid.

The following is a list of properties of a binary relation R that are defined by first-order sentences

1. Reflexive: $\forall s(sRs)$
2. Symmetric: $\forall s\forall t(sRt \to tRs)$
3. Serial: $\forall s\exists t(sRt)$
4. Transitive: $\forall s\forall t\forall u(sRt \wedge tRu \to sRu)$
5. Euclidean: $\forall s\forall t\forall u(sRt \wedge sRu \to tRu)$
6. Partially functional: $\forall s\forall t\forall u(sRt \wedge sRu \to t = u)$
7. Functional: $\forall s\exists! t(sRt)$
8. Weakly dense: $\forall s\forall t(sRt \to \exists u(sRu \wedge uRt))$
9. Weakly connected: $\forall s\forall t\forall u(sRt \wedge sRu \to tRu \vee t = u \vee uRt)$
10. Weakly directed: $\forall s\forall t\forall u(sRt \wedge sRu \to \exists v(tRv \wedge uRv))$

Modal logics: syntax and semantics

Conditions on R

Corresponding to this list is a list of formulas

1. $\Box p \to p$
2. $p \to \Box\Diamond p$
3. $\Box p \to \Diamond p$
4. $\Box p \to \Box\Box p$
5. $\Diamond p \to \Box\Diamond p$
6. $\Diamond p \to \Box p$
7. $\Diamond p \leftrightarrow \Box p$
8. $\Box\Box p \to \Box p$
9. $\Box(p \wedge \Box p \to q) \vee \Box(q \wedge \Box q \to p)$
10. $\Diamond\Box p \to \Box\Diamond p$

Theorem: Let $\mathcal{F} = (S, R)$ be a frame

- For each of the properties 1–10, if R satisfies the property then the corresponding formula is valid in $\mathcal{F}$

Theorem: If a frame $\mathcal{F} = (S, R)$ validates any one of the formulas 1–10 then R satisfies the corresponding property

Modal logics: syntax and semantics

Exercise

Give a property of R that is necessary and sufficient for $\mathcal{F}$ to validate the formula $p \to \Box p$.

Do the same for $\Box\bot$.

Modal logics: syntax and semantics

First-order definability

The formula

$$W : \ \Box(\Box p \to p) \to \Box p$$

is valid in frame (S, R) iff

1. R is transitive
2. there are no sequences $s_0, s_1, \ldots$ in S with $s_n R s_{n+1}$ for all $n \geq 0$

By the Compactness Theorem of first-order logic, one can prove that there can be no set of first-order sentences that defines the class of frames of W

References:

- Boolos, G.: The Unprovability of Consistency. Cambridge University Press (1979).

The class of frames of the so-called McKinsey formula

$$M: \ \Box\Diamond p \to \Diamond\Box p$$

is not defined by any set of first-order sentences

References:

- Van Benthem, J.: A note on modal formulas and relational properties. Journal of Symbolic Logic **40** (1975) 55–58.
- Goldblatt, R.: First-order definability in modal logic. Journal of Symbolic Logic **40** (1975) 35–40.

Modal logics: syntax and semantics

Undefinable conditions

There are some naturally occurring properties of a binary relation R that do not correspond to the validity of any modal formula

1. Irreflexivity: $\forall s \neg(sRs)$
2. Antisymmetry: $\forall s \forall t(sRt \wedge tRs \rightarrow s = t)$
3. Asymmetry: $\forall s \forall t(sRt \rightarrow \neg(tRs))$

Modal logics: proof theory

Given a language based on a countable set AF of atomic formulas

- a logic is defined to be any set $\mathbf{L} \subseteq Fma(AF)$ such that
  - $\mathbf{L}$ includes all tautologies
  - $\mathbf{L}$ is closed under the rule of Detachment (modus ponens), i.e.

$$\text{if } \phi \to \psi \in \mathbf{L} \text{ and } \phi \in \mathbf{L} \text{ then } \psi \in \mathbf{L}$$

  - $\mathbf{L}$ is closed under the rule of substitution, i.e. for all substitutions σ

$$\text{if } \phi \in \mathbf{L} \text{ then } \sigma(\phi) \in \mathbf{L}$$

Examples of logics

1. The set **CPL** of all tautologies
2. For any class $\mathcal{C}$ of frames, the set $\mathbf{Log}_{\mathcal{C}} = \{\phi \in Fma(AF) : \mathcal{C} \models \phi\}$
3. $Fma(AF)$

Remark: If $(\mathbf{L}_i : \ i \in I)$ is a set of logics then their intersection is a logic

The members of a logic are called its theorems

- We write $\vdash_{\mathbf{L}} \phi$ to mean that ϕ is a **L**-theorem

Modal logics: proof theory

Soundness and completeness

Let $\mathcal{C}$ be a class of frames

- The logic **L** is sound with respect to $\mathcal{C}$ if for all formulas ϕ

$$\vdash_{\mathbf{L}} \phi \Rightarrow \mathcal{C} \models \phi$$

- The logic **L** is complete with respect to $\mathcal{C}$ if for all formulas ϕ

$$\mathcal{C} \models \phi \Rightarrow \vdash_{\mathbf{L}} \phi$$

- The logic **L** is determined by $\mathcal{C}$ if it is both sound and complete with respect to $\mathcal{C}$

Modal logics: proof theory

Deducibility and consistency

A formula ϕ is deducible from a set Γ of formulas iff there exists $n \in \mathbb{N}$ and there exists $\psi_1, \ldots, \psi_n \in \Gamma$ such that

$$\vdash_{\mathbf{L}} \psi_1 \wedge \ldots \wedge \psi_n \to \phi$$

In this case, we write $\Gamma \vdash_{\mathbf{L}} \phi$

A set Γ of formulas is **L**-consistent iff $\Gamma \nvdash_{\mathbf{L}} \bot$

Modal logics: proof theory

Exercise

Show that

- $\vdash_{\mathbf{L}} \phi$ iff $\emptyset \vdash_{\mathbf{L}} \phi$
- if $\vdash_{\mathbf{L}} \phi$ then $\Gamma \vdash_{\mathbf{L}} \phi$
- if $\mathbf{L} \subseteq \mathbf{L}'$ then $\Gamma \vdash_{\mathbf{L}} \phi$ implies $\Gamma \vdash_{\mathbf{L}'} \phi$
- if $\phi \in \Gamma$ then $\Gamma \vdash_{\mathbf{L}} \phi$
- if $\Gamma \subseteq \Delta$ then $\Gamma \vdash_{\mathbf{L}} \phi$ implies $\Delta \vdash_{\mathbf{L}} \phi$
- if $\Gamma \vdash_{\mathbf{L}} \phi$ and $\{\phi\} \vdash_{\mathbf{L}} \psi$ then $\Gamma \vdash_{\mathbf{L}} \psi$
- if $\Gamma \vdash_{\mathbf{L}} \phi$ and $\Gamma \vdash_{\mathbf{L}} \phi \to \psi$ then $\Gamma \vdash_{\mathbf{L}} \psi$
- $\Gamma \cup \{\phi\} \vdash_{\mathbf{L}} \psi$ iff $\Gamma \vdash_{\mathbf{L}} \phi \to \psi$
- $\Gamma \vdash_{\mathbf{L}} \phi$ iff there exists $n \in \mathbb{N}$ and there exists $\psi_1, \ldots, \psi_n \in Fma(AF)$ such that $\psi_n = \phi$ and for all $i \leq n$, $\psi_i \in \Gamma \cup \mathbf{L}$, or there exists $j, k < i$ such that $\psi_j = \psi_k \to \psi_i$

Show that

- $\{\phi \in Fma(AF) : \Gamma \vdash_{\mathbf{L}} \phi\}$ is the smallest logic containing $\Gamma \cup \mathbf{L}$
- if $\mathcal{M} \models_s \Gamma \cup \mathbf{L}$ and $\Gamma \vdash_{\mathbf{L}} \phi$ then $\mathcal{M} \models_s \phi$
- Γ is $\mathbf{L}$-consistent iff there exists a formula ϕ such that $\Gamma \nvdash_{\mathbf{L}} \phi$
- Γ is $\mathbf{L}$-consistent iff there exists no formula ϕ having both $\Gamma \vdash_{\mathbf{L}} \phi$ and $\Gamma \vdash_{\mathbf{L}} \neg\phi$
- $\Gamma \vdash_{\mathbf{L}} \phi$ iff $\Gamma \cup \{\neg\phi\}$ is not $\mathbf{L}$-consistent
- $\Gamma \cup \{\phi\}$ is $\mathbf{L}$-consistent iff $\Gamma \nvdash_{\mathbf{L}} \neg\phi$
- if Γ is $\mathbf{L}$-consistent then for any formula ϕ, at least one of $\Gamma \cup \{\phi\}$ and $\Gamma \cup \{\neg\phi\}$ is $\mathbf{L}$-consistent

Modal logics: proof theory
Maximal sets

Given a model $\mathcal{M} = (S, R, V)$ of a logic **L**, i.e. $\mathcal{M} \models \mathbf{L}$, we associate with each $s \in S$ the set

$$\Gamma_s = \{\phi \in Fma(AF) : \mathcal{M} \models_s \phi\}$$

Then

- Γ_s is **L**-consistent
- for each formula ϕ, one of ϕ and $\neg\phi$ is in Γ_s

A set $\Gamma \subseteq Fma(AF)$ is said to be **L**-maximal if

- Γ is **L**-consistent
- for each formula ϕ, one of ϕ and $\neg\phi$ is in Γ

Modal logics: proof theory

Exercise

Suppose Γ is **L**-maximal and show that

- $\Gamma \vdash_{\mathbf{L}} \phi$ implies $\phi \in \Gamma$
- if $\phi \notin \Gamma$ then $\Gamma \cup \{\phi\}$ is not **L**-consistent
- for any formula ϕ, exactly one of ϕ and $\neg\phi$ belongs to Γ
- $\mathbf{L} \subseteq \Gamma$
- $\bot \notin \Gamma$
- $\phi \to \psi \in \Gamma$ iff ($\phi \in \Gamma$ implies $\psi \in \Gamma$)
- $\phi \wedge \psi \in \Gamma$ iff $\phi, \psi \in \Gamma$
- $\phi \vee \psi \in \Gamma$ iff $\phi \in \Gamma$, or $\psi \in \Gamma$
- $\phi \leftrightarrow \psi \in \Gamma$ iff ($\phi \in \Gamma$ iff $\psi \in \Gamma$)

Modal logics: proof theory

Existence of maximal sets

Let $S^{\mathbf{L}} = \{\Gamma \subseteq Fma(AF) : \ \Gamma \text{ is } \mathbf{L}\text{-maximal}\}$

Lindenbaum Lemma: Every **L**-consistent set of formulas is contained in a **L**-maximal set

Corollary:

1. $\{\phi : \ \Gamma \vdash_{\mathbf{L}} \phi\} = \bigcap\{\Delta \in S^{\mathbf{L}} : \ \Gamma \subseteq \Delta\}$
2. $\vdash_{\mathbf{L}} \phi$ iff ϕ belongs to every **L**-maximal set

Modal logics: proof theory

Normal logics

A logic $\mathbf{L}$ is normal if it contains all formulas of the syntactic form

$$K: \ \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

and is closed under the rule of necessitation, i.e.

$$\text{if } \vdash_{\mathbf{L}} \phi \text{ then } \vdash_{\mathbf{L}} \Box\phi$$

Examples of normal logics

1. For any class $\mathcal{C}$ of frames, the set $\mathbf{Log}_{\mathcal{C}} = \{\phi \in Fma(AF) : \ \mathcal{C} \models \phi\}$
2. $Fma(AF)$

Remark: If $(\mathbf{L}_i : i \in I)$ is a set of normal logics then their intersection is a normal logic

In particular, the intersection K of all normal logics is the smallest normal logic

Modal logics: proof theory

Exercise

Suppose **L** is a normal logic and show that

- $\vdash_{\mathbf{L}} \phi \to \psi$ implies $\vdash_{\mathbf{L}} \Box\phi \to \Box\psi$ and $\vdash_{\mathbf{L}} \Diamond\phi \to \Diamond\psi$
- $\vdash_{\mathbf{L}} \phi \leftrightarrow \psi$ implies $\vdash_{\mathbf{L}} \Box\phi \leftrightarrow \Box\psi$ and $\vdash_{\mathbf{L}} \Diamond\phi \leftrightarrow \Diamond\psi$
- $\vdash_{\mathbf{L}} \Diamond\neg\phi \leftrightarrow \neg\Box\phi$
- $\vdash_{\mathbf{L}} \Box\phi \wedge \Box\psi \leftrightarrow \Box(\phi \wedge \psi)$
- $\vdash_{\mathbf{L}} \Diamond(\phi \vee \psi) \leftrightarrow \Diamond\phi \vee \Diamond\psi$
- $\vdash_{\mathbf{L}} \Box\phi \vee \Box\psi \to \Box(\phi \vee \psi)$
- $\vdash_{\mathbf{L}} \Diamond(\phi \wedge \psi) \to \Diamond\phi \wedge \Diamond\psi$

Show that a logic **L** is normal iff for all $n \in \mathbb{N}$

- if $\vdash_{\mathbf{L}} \phi_1 \wedge \ldots \wedge \phi_n \to \psi$ then $\vdash_{\mathbf{L}} \Box\phi_1 \wedge \ldots \wedge \Box\phi_n \to \Box\psi$

Show that a logic **L** is normal iff it satisfies the following three conditions.

- $\vdash_{\mathbf{L}} \Box\top$
- $\vdash_{\mathbf{L}} \Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$
- $\vdash_{\mathbf{L}} \phi \to \psi$ implies $\vdash_{\mathbf{L}} \Box\phi \to \Box\psi$

Modal logics: proof theory

Exercise

Show that

- if a normal logic contains all formulas of the syntactic form $\Diamond\phi \to \Box\phi$ then it contains all formulas of the syntactic forms $\Box(\phi \vee \psi) \leftrightarrow (\Box\phi \vee \Box\psi)$ and $(\Box\phi \to \Box\psi) \leftrightarrow \Box(\phi \to \psi)$

Show that

- $\vdash_K \phi$ iff there exists $n \in \mathbb{N}$ and there exists $\psi_1, \ldots, \psi_n \in Fma(AF)$ such that $\psi_n = \phi$ and for all $i \leq n$, ψ_i is a tautology, or ψ_i is a formula of the syntactic form K, or there exists $j, k < i$ such that $\psi_j = \psi_k \to \psi_i$, or there exists $j < i$ such that $\psi_i = \Box\psi_j$

Modal logics: proof theory

Some standard logics

We use the notation

$$K\Sigma_1 \ldots \Sigma_n$$

to refer to the smallest normal logic containing all formulas of the syntactic forms $\Sigma_1, \ldots, \Sigma_n$

Historical names for some well-known syntactical forms are

D: $\Box\phi \rightarrow \Diamond\phi$

T: $\Box\phi \rightarrow \phi$

4: $\Box\phi \rightarrow \Box\Box\phi$

B: $\phi \rightarrow \Box\Diamond\phi$

5: $\Diamond\phi \rightarrow \Box\Diamond\phi$

L: $\Box(\phi \wedge \Box\phi \rightarrow \psi) \vee \Box(\psi \wedge \Box\psi \rightarrow \phi)$

W: $\Box(\Box\phi \rightarrow \phi) \rightarrow \Box\phi$

Modal logics: proof theory

Some standard logics

Names of some well-known logics are

$$
\begin{aligned}
S4&:\ KT4\\
S5&:\ KT4B\\
G&:\ KW\\
K4.3&:\ K4L\\
S4.3&:\ KT4L
\end{aligned}
$$

Show that KD is the smallest normal logic containing the formula

$$\Diamond\top$$

Show that

$$KB4 = KB5$$

Show that

$$S5 = KDB4 = KDB5 = KT5$$

Show that

$$K4 \subseteq G$$

Modal logics: canonical models and completeness

Modal logics: canonical models and completeness

Canonical models

$\mathcal{M}^{\mathbf{L}} = (S^{\mathbf{L}}, R^{\mathbf{L}}, V^{\mathbf{L}})$: canonical model of a normal logic $\mathbf{L}$

- $S^{\mathbf{L}} = \{s \subseteq Fma(AF) : \ s \text{ is } \mathbf{L}\text{-maximal}\}$
- $sR^{\mathbf{L}}t$ iff $\{\phi \in Fma(AF) : \ \Box\phi \in s\} \subseteq t$
- $V^{\mathbf{L}}(p) = \{s \in S^{\mathbf{L}} : \ p \in s\}$

$\mathcal{F}^{\mathbf{L}} = (S^{\mathbf{L}}, R^{\mathbf{L}})$: canonical frame of a normal logic $\mathbf{L}$

Remark: $sR^{\mathbf{L}}t$ iff $\{\neg\Box\phi : \ \phi \notin t\} \subseteq s$ iff $\{\Diamond\phi : \ \phi \in t\} \subseteq s$

Theorem: For any $s \in S^{\mathbf{L}}$ and any $\phi \in Fma(AF)$

- $\Box\phi \in s$ iff for all $t \in S^{\mathbf{L}}$, $sR^{\mathbf{L}}t$ implies $\phi \in t$

Modal logics: canonical models and completeness

Canonical models

Truth Lemma: For any $s \in S^{\mathbf{L}}$ and any $\phi \in Fma(AF)$

- $\mathcal{M}^{\mathbf{L}} \models_s \phi$ iff $\phi \in s$

Corollary: For any $\phi \in Fma(AF)$

- $\mathcal{M}^{\mathbf{L}} \models \phi$ iff $\vdash_{\mathbf{L}} \phi$

Theorem (Determination of K): For any $\phi \in Fma(AF)$

- $\vdash_K \phi$ iff ϕ is valid in all frames

Modal logics: canonical models and completeness

Completeness theorems

Theorem: If a normal logic **L** contains all formulas of the syntactic form corresponding to any one of the formulas 1–10 then $R^{\mathbf{L}}$ satisfies the corresponding first-order condition

Theorem: $S4$ is determined by the class of all reflexive and transitive frames

Modal logics: canonical models and completeness

Exercise

Show that

- KD is determined by the class of all serial frames
- $S5$ is determined by the class of all equivalence relations
- $K4.3$ is determined by the class of all transitive weakly-connected frames

Let $S4.2$ be the smallest normal logic containing $S4$ and all formulas of the syntactic form

$$2: \ \Diamond\Box\phi \to \Box\Diamond\phi$$

Show that

- $S4.2$ is determined by the class of all reflexive transitive weakly-directed frames

Modal logics: canonical models and completeness

Exercise

Axiomatise the logics determined by

- the class of all partially-functional frames
- the class of all functional frames
- the class of all weakly-dense frames

For fixed $k, l, m, n \in \mathbb{N}$, let $\mathbf{L}$ be a normal logic containing all formulas of the syntactic form

$$\Diamond^k \Box^l \phi \to \Box^m \Diamond^n \phi$$

Show that for any $s, t, u \in S^{\mathbf{L}}$

- if $s(R^{\mathbf{L}})^k t$ and $s(R^{\mathbf{L}})^m u$ then there exists $v \in S^{\mathbf{L}}$ such that $t(R^{\mathbf{L}})^l v$ and $u(R^{\mathbf{L}})^n v$

A logically necessary truth is one which is true in all possible worlds whatsoever, suggesting the semantic analysis

- $\mathcal{M} \models_s \Box\phi$ iff for all $t \in S$, $\mathcal{M} \models_t \phi$

A frame $\mathcal{F} = (S, R)$ is universal iff

- $R = S \times S$

Theorem: $S5$ is determined by the class of all universal frames

Among the theorems of $S5$ are

- $\Box\phi \rightarrow \Box\Box\phi$
- $\neg\Box\phi \rightarrow \Box\neg\Box\phi$

Modal logics: canonical models and completeness

Connectedness

A frame $\mathcal{F} = (S, R)$ is connected iff for any $s, t \in S$

- sRt, or $s = t$, or tRs

Remarks:

- any connected frame is weakly-connected
- a frame validates the formula

$$\Box(p \wedge \Box p \rightarrow q) \vee \Box(q \wedge \Box q \rightarrow p)$$

  iff it is weakly-connected

There is no formula that is valid in precisely the connected frames

Modal logics: multimodal languages

Modal logics: multimodal languages

Syntax

- AF: countable set of atomic formulas
- I: countable set of indices
- $Fma(AF, I)$: set of all formulas generated from AF and I

- Atomic formulas: $p \in AF$
- Indices: $i \in I$
- Formulas: $\phi \in Fma(AF, I)$

$$\phi ::= p \mid \bot \mid (\phi_1 \to \phi_2) \mid [i]\phi$$

"Diamond"

- $\langle i \rangle \phi ::= \neg [i] \neg \phi$

Modal logics: multimodal languages

Semantics

A frame is a pair $\mathcal{F} = (S, \{R_i : i \in I\})$ where

- S is a nonempty set
- $R_i \subseteq S \times S$ for each $i \in I$

A model on a frame is a triple $\mathcal{M} = (S, \{R_i : i \in I\}, V)$ where

- $V : AF \to 2^S$

The relation $\mathcal{M} \models_s \phi$ has the new clause

$\mathcal{M} \models_s [i]\phi$ iff for all $t \in S$, $sR_i t$ implies $\mathcal{M} \models_t \phi$

The definitions of "truth in a model" ($\mathcal{M} \models \phi$), "validity in a frame" ($\mathcal{F} \models \phi$) and "validity in a class of frames" ($\mathcal{C} \models \phi$) are unchanged

Modal logics: multimodal languages

The definitions of "logics", "**L**-deducibility", "**L**-consistency", "**L**-maximality" are unchanged

A logic **L** is normal if it contains all formulas of the syntactic form

$$K: \ [i](\phi \to \psi) \to ([i]\phi \to [i]\psi)$$

and is closed under the rules of necessitation, i.e.

$$\text{if } \vdash_{\mathbf{L}} \phi \text{ then } \vdash_{\mathbf{L}} [i]\phi$$

for every $i \in I$

The smallest normal logic will be denoted K_I

Modal logics: multimodal languages

Canonical model

For a normal logic **L**, the model

$$\mathcal{M}^{\mathbf{L}} = (S^{\mathbf{L}}, \{R_i^{\mathbf{L}} : \ i \in I\}, V^{\mathbf{L}})$$

has

$$sR_i^{\mathbf{L}}t \text{ iff } \{\phi \in Fma(AF) : \ [i]\phi \in s\} \subseteq t$$

with the definitions of $S^{\mathbf{L}}$ and $V^{\mathbf{L}}$ remaining the same

Truth Lemma for $\mathcal{M}^{\mathbf{L}}$, i.e.

$$\mathcal{M}^{\mathbf{L}} \models_s \phi \text{ iff } \phi \in s$$

continues to work as previously

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