Intuitionistic modal logic

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- Intermediate logics
- Modal logics
- Combining logics
- **Two peculiar intuitionistic modal logics**
- A minimal setting

Two peculiar intuitionistic modal logics

Outline

- Intuitionistic Epistemic Logic
- Propositional Lax Logic

Two peculiar intuitionistic modal logics:
Intuitionistic Epistemic Logic

Study of belief and knowledge from an intuitionistic point of view

- Verifications are evidences considered sufficiently conclusive for practical purposes
- Belief and knowledge are the products of verifications

Syntax

- Atomic formulas: $p \in AF$
- Formulas: $\phi \in Fma(AF)$

$$\phi ::= p \mid \bot \mid \top \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \to \phi_2) \mid \Box\phi$$

Reading of $\Box\phi$

- It is intuitionistically believed/known that ϕ

Difference between intuitionistic and classical belief and knowledge

Relationship to their respective notions of truth

- Brouwer-Heyting-Kolmogorov semantics: an intuitionistic proposition is intuitionistically true if it is proved
- Intuitionistic belief is the product of verifications: an intuitionistic proposition is intuitionistically believed if it is intuitionistically true

Difference between intuitionistic and classical belief and knowledge

Relationship to their respective notions of truth

- Constructivity of truth, coreflection: the formula

$$\phi \to \Box\phi$$

should be accepted, seeing that all proofs are verifications

Difference between intuitionistic and classical knowledge

Relationship to their respective notions of truth

- Reflection: the formula

$$\Box\phi \to \phi$$

  should not be accepted, seeing that it is possible to have a provably verified proposition without possessing a specific proof of it

Difference between intuitionistic and classical knowledge

Relationship to their respective notions of truth

- Intuitionistic reflection: the formula

$$\Box\phi \to \neg\neg\phi$$

  should be accepted, seeing that it is not possible to produce a proof that a proposition cannot have a proof once it is verified that this proposition has a proof

Two peculiar intuitionistic modal logics: Intuitionistic Epistemic Logic

$\mathbf{IEL}^-$: Logic of intuitionistic belief

Syntax

- Atomic formulas: $p \in AF$
- Formulas: $\phi \in Fma(AF)$

$$\phi ::= p \mid \bot \mid \top \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \to \phi_2) \mid \Box\phi$$

Calculus

The calculus of $\mathbf{IEL}^-$ contains the following axioms and inference rules:

- axioms and inference rules of **IPL**
- $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
- $\phi \to \Box\phi$

Two peculiar intuitionistic modal logics: Intuitionistic Epistemic Logic

IEL: Logic of intuitionistic knowledge \ Logic of provably consistent intuitionistic beliefs

Syntax

- Atomic formulas: $p \in AF$
- Formulas: $\phi \in Fma(AF)$

$$\phi ::= p \mid \bot \mid \top \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \to \phi_2) \mid \Box\phi$$

Calculus

The calculus of **IEL** contains the following axioms and inference rules:

- axioms and inference rules of **IPL**
- $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
- $\phi \to \Box\phi$
- $\Box\phi \to \neg\neg\phi$

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Derivable inference rule

Exercise

Show that the following inference rule is derivable in $\mathbf{IEL}^-$:

- $\frac{p}{\Box p}$

Exercise

Show that the following formulas are derivable in $\mathbf{IEL}^-$:

- $\Box\top$
- $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$
- $\Box p \rightarrow \Box\Box p$
- $\neg\Box p \rightarrow \Box\neg\Box p$

Exercise

Show that the following formulas are derivable in $\mathbf{IEL}$:

- $\Box\neg p \rightarrow \neg p$
- $\Box\neg p \leftrightarrow \neg\Box p$
- $\Box\neg p \leftrightarrow \neg p$
- $\neg(\neg\Box\neg p \wedge \neg\Box p)$

Exercise

Show that the following formulas are derivable in **IEL**:

- $\neg\Box\bot$
- $\neg(\Box p \wedge \neg p)$
- $\neg p \to \neg\Box p$
- $\neg\neg(\Box p \to p)$

Exercise

Show that the following calculi are equivalent with **IEL**:

- $\mathbf{IEL}^- + \neg\Box\bot$
- $\mathbf{IEL}^- + \neg(\Box p \wedge \neg p)$
- $\mathbf{IEL}^- + \neg p \to \neg\Box p$
- $\mathbf{IEL}^- + \neg\neg(\Box p \to p)$

IEL$^-$-frames

An **IEL**$^-$-frame is a tuple $(S, \leq, R)$ where

- $S \neq \emptyset$
- $\leq$ is a partial order on S
- R is a binary relation on S such that for all s, t, u in S
  - if sRt then $s \leq t$
  - if $s \leq t$ and tRu then sRu

IEL-frames

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- R is a binary relation on S such that for all s, t, u in S
  - if sRt then $s \leq t$
  - if $s \leq t$ and tRu then sRu
  - there exists v in S such that sRv

Two peculiar intuitionistic modal logics: Intuitionistic Epistemic Logic

Models

Given a model $\mathcal{M} = (S, \leq, R, V)$ and $s \in S$

$\mathcal{M} \models_s \phi$: relation "$\phi$ is true at world s in model $\mathcal{M}$"

- $\mathcal{M} \models_s p$ iff $s \in V(p)$
- $\mathcal{M} \not\models_s \bot$
- $\mathcal{M} \models_s \top$
- $\mathcal{M} \models_s \phi \vee \psi$ iff $\mathcal{M} \models_s \phi$ or $\mathcal{M} \models_s \psi$
- $\mathcal{M} \models_s \phi \wedge \psi$ iff $\mathcal{M} \models_s \phi$ and $\mathcal{M} \models_s \psi$
- $\mathcal{M} \models_s \phi \to \psi$ iff for all $t \in S$, if $s \leq t$ and $\mathcal{M} \models_t \phi$ then $\mathcal{M} \models_t \psi$
- $\mathcal{M} \models_s \Box\phi$ iff for all $t \in S$, if sRt then $\mathcal{M} \models_t \phi$

Note: Here, $V : AF \to 2^S$ where $V(p)$ is $\leq$-upward closed, for every atomic formula p

Two peculiar intuitionistic modal logics: Intuitionistic Epistemic Logic

Exercise

Let $\mathcal{M} = (S, \leq, R, V)$ be a model and $s, t \in S$

Show that if $s \leq t$ and $\mathcal{M} \models_s \phi$ then $\mathcal{M} \models_t \phi$.

$\mathcal{M} \models \phi$: relation "ϕ is true in model $\mathcal{M} = (S, \leq, R, V)$"

$\mathcal{M} \models \phi$ iff $\mathcal{M} \models_s \phi$ for all $s \in S$

$\mathcal{F} \models \phi$: relation "ϕ is valid in frame $\mathcal{F} = (S, \leq, R)$"

$\mathcal{F} \models \phi$ iff $\mathcal{M} \models \phi$ for all models $\mathcal{M} = (S, \leq, R, V)$

$\mathcal{C} \models \phi$: relation "ϕ is valid in class $\mathcal{C}$ of frames"

$\mathcal{C} \models \phi$ iff $\mathcal{F} \models \phi$ for all frames $\mathcal{F}$ in $\mathcal{C}$

Two peculiar intuitionistic modal logics: Intuitionistic Epistemic Logic

Exercise

Show that the following inference rule is admissible in the class of all $\mathbf{IEL}^{-}$-frames and in the class of all $\mathbf{IEL}$-frames:

- $\frac{\Box p}{p}$

Two peculiar intuitionistic modal logics: Intuitionistic Epistemic Logic

Exercise

Find an **IEL**-frame in which $\Box p \to p$ is not valid.

Find an **IEL**-frame in which $\Box(p \vee q) \to \Box p \vee \Box q$ is not valid.

Show that $p \to \Box p$ is valid in the class of all $\mathbf{IEL}^-$-frames.

Show that $\Box p \to \neg\neg p$ is valid in the class of all **IEL**-frames.

Two peculiar intuitionistic modal logics: Intuitionistic Epistemic Logic

Exercise

Show that the following formulas are valid in the class of all **IEL**-frames:

- $\neg\Box\bot$
- $\neg(\Box p \wedge \neg p)$
- $\neg p \to \neg\Box p$
- $\neg\neg(\Box p \to p)$

Theorem (Soundness)

For all formulas ϕ,

- if $\vdash_{\mathbf{IEL}^-} \phi$ then ϕ is valid in the class of all $\mathbf{IEL}^-$-frames,
- if $\vdash_{\mathbf{IEL}} \phi$ then ϕ is valid in the class of all $\mathbf{IEL}$-frames.

Theorem

- $\mathbf{IEL}^- \subseteq \mathbf{IEL}$,
- $\mathbf{IEL}^- \not\supseteq \mathbf{IEL}$.

Two peculiar intuitionistic modal logics: Intuitionistic Epistemic Logic

Non-derivable formulas

Exercise

Show that the following formula is not derivable in $\mathbf{IEL}^-$:

- $\Box p \to \neg\neg p$

Exercise

Show that the following formulas are not derivable in $\mathbf{IEL}$:

- $\Box p \to p$
- $\Box(p \vee q) \to \Box p \vee \Box q$

Theorem

For all formulas ϕ,

- if ϕ is valid in the class of all $\mathbf{IEL}^{-}$-frames then $\vdash_{\mathbf{IEL}^{-}} \phi$,
- if ϕ is valid in the class of all $\mathbf{IEL}$-frames then $\vdash_{\mathbf{IEL}} \phi$.

Theorem

For all formulas ϕ, ψ,

- if $\vdash_{\mathbf{IEL}^-} \phi \vee \psi$ then $\vdash_{\mathbf{IEL}^-} \phi$ or $\vdash_{\mathbf{IEL}^-} \psi$,
- if $\vdash_{\mathbf{IEL}} \phi \vee \psi$ then $\vdash_{\mathbf{IEL}} \phi$ or $\vdash_{\mathbf{IEL}} \psi$.

Theorem

For all formulas ϕ, ψ,

- if $\vdash_{\mathbf{IEL}^-} \Box(\phi \vee \psi)$ then $\vdash_{\mathbf{IEL}^-} \Box\phi$ or $\vdash_{\mathbf{IEL}^-} \Box\psi$,
- if $\vdash_{\mathbf{IEL}} \Box(\phi \vee \psi)$ then $\vdash_{\mathbf{IEL}} \Box\phi$ or $\vdash_{\mathbf{IEL}} \Box\psi$.

Two peculiar intuitionistic modal logics:
Propositional Lax Logic

Two peculiar intuitionistic modal logics: Propositional Lax Logic

Goal

Study of the modality $\bigcirc$ characterized by

$$
\begin{aligned}
\bigcirc R &: \ \phi \to \bigcirc\phi \\
\bigcirc M &: \ \bigcirc\bigcirc\phi \to \bigcirc\phi \\
\bigcirc S &: \ \bigcirc\phi \wedge \bigcirc\psi \to \bigcirc(\phi \wedge \psi)
\end{aligned}
$$

Monotonicity: $\frac{\phi \to \psi}{\bigcirc\phi \to \bigcirc\psi}$

Syntax

- Atomic formulas: $p \in AF$
- Formulas: $\phi \in Fma(AF)$

$$\phi ::= p \mid \bot \mid \top \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \to \phi_2) \mid \bigcirc\phi$$

Two peculiar intuitionistic modal logics: Propositional Lax Logic

Intuitive readings

Reading of $\bigcirc\phi$

- for some constraint c, the formula ϕ holds under c

Reading of $\bigcirc R$, $\bigcirc M$ and $\bigcirc S$

$\phi \to \bigcirc\phi$: if ϕ holds outright then under a trivial constraint, ϕ holds

$\bigcirc\bigcirc\phi \to \bigcirc\phi$: if under some constraint, ϕ holds under another constraint then ϕ holds under an appropriately combined constraint

$\bigcirc\phi \wedge \bigcirc\psi \to \bigcirc(\phi \wedge \psi)$: if ϕ holds under a constraint and ψ holds under a constraint then $\phi \wedge \psi$ holds under an appropriately combined constraint

Two peculiar intuitionistic modal logics: Propositional Lax Logic

Intuitive readings

Reading of $\bigcirc\phi$

- the formula ϕ holds after some delay

Remarks about $\bigcirc R$, $\bigcirc M$ and $\bigcirc S$

- $\phi \to \bigcirc\phi$ involves the zero delay 0
- $\bigcirc\bigcirc\phi \to \bigcirc\phi$ involves the addition $+$ of delays
- $\bigcirc\phi \wedge \bigcirc\psi \to \bigcirc(\phi \wedge \psi)$ involves the maximum operation max on delays

Syntax

- Atomic formulas: $p \in AF$
- Formulas: $\phi \in Fma(AF)$

$$\phi ::= p \mid \bot \mid \top \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \to \phi_2) \mid \bigcirc\phi$$

Calculi

The calculus of **PLL** contains the following axioms and inference rules:

- axioms and inference rules of **IPL**
- $\phi \to \bigcirc\phi$
- $\bigcirc\bigcirc\phi \to \bigcirc\phi$
- $\bigcirc\phi \wedge \bigcirc\psi \to \bigcirc(\phi \wedge \psi)$
- $\frac{\phi \to \psi}{\bigcirc\phi \to \bigcirc\psi}$

The calculus of **PLL**$'$ contains the following axioms and inference rules:

- axioms and inference rules of **IPL**
- $(\phi \to \bigcirc\psi) \leftrightarrow (\bigcirc\phi \to \bigcirc\psi)$

Theorem

For all formulas ϕ, the following conditions are equivalent:

- $\vdash_{\mathbf{PLL}} \phi$,
- $\vdash_{\mathbf{PLL'}} \phi$.

Theorem

For all formulas ϕ,

- if $\vdash_{\mathbf{PLL}} \bigcirc\phi$ then $\vdash_{\mathbf{IPL}} \phi$.

Two peculiar intuitionistic modal logics: Propositional Lax Logic

PLL-frames

A **PLL**-frame is a tuple $(S, \leq, R, F)$ where

- $S \neq \emptyset$
- $\leq$ is a preorder on S
- R is a preorder on S such that for all s, t in S
  - if sRt then $s \leq t$
- F is a $\leq$-upward closed subset of S

Two peculiar intuitionistic modal logics: Propositional Lax Logic

Models

Given a model $\mathcal{M} = (S, \leq, R, F, V)$ and $s \in S$

$\mathcal{M} \models_s \phi$: relation "$\phi$ is true at world s in model $\mathcal{M}$"

- $\mathcal{M} \models_s p$ iff $s \in V(p)$
- $\mathcal{M} \models_s \bot$ iff $s \in F$
- $\mathcal{M} \models_s \top$
- $\mathcal{M} \models_s \phi \vee \psi$ iff $\mathcal{M} \models_s \phi$ or $\mathcal{M} \models_s \psi$
- $\mathcal{M} \models_s \phi \wedge \psi$ iff $\mathcal{M} \models_s \phi$ and $\mathcal{M} \models_s \psi$
- $\mathcal{M} \models_s \phi \to \psi$ iff for all $t \in S$, if $s \leq t$ and $\mathcal{M} \models_t \phi$ then $\mathcal{M} \models_t \psi$
- $\mathcal{M} \models_s \bigcirc\phi$ iff for all $t \in S$, if $s \leq t$ then there exists $u \in S$ such that tRu and $\mathcal{M} \models_u \phi$

Note: Here, $V : AF \to 2^S$ where $V(p)$ is $\leq$-upward closed and $F \subseteq V(p)$, for every atomic formula p

Two peculiar intuitionistic modal logics: Propositional Lax Logic

Exercise

Let $\mathcal{M} = (S, \leq, R, F, V)$ be a model and $s, t \in S$

Show that if $s \leq t$ and $\mathcal{M} \models_s \phi$ then $\mathcal{M} \models_t \phi$.

$\mathcal{M} \models \phi$: relation "ϕ is true in model $\mathcal{M} = (S, \leq, R, F, V)$"

$\mathcal{M} \models \phi$ iff $\mathcal{M} \models_s \phi$ for all $s \in S$

$\mathcal{F} \models \phi$: relation "ϕ is valid in frame $\mathcal{F} = (S, \leq, R, F)$"

$\mathcal{F} \models \phi$ iff $\mathcal{M} \models \phi$ for all models $\mathcal{M} = (S, \leq, R, F, V)$

$\mathcal{C} \models \phi$: relation "ϕ is valid in class $\mathcal{C}$ of frames"

$\mathcal{C} \models \phi$ iff $\mathcal{F} \models \phi$ for all frames $\mathcal{F}$ in $\mathcal{C}$

Two peculiar intuitionistic modal logics: Propositional Lax Logic

Exercise

Find an **PLL**-frame in which $\neg \bigcirc \perp$ is not valid.

Find an **PLL**-frame in which $\bigcirc(p \vee q) \to \bigcirc p \vee \bigcirc q$ is not valid.

Find an **PLL**-frame in which $(\bigcirc p \to \bigcirc q) \to \bigcirc(p \to q)$ is not valid.

Theorem (Soundness)

For all formulas ϕ,

- if $\vdash_{\mathbf{PLL}} \phi$ then ϕ is valid in the class of all **PLL**-frames.

Theorem (Completeness)

For all formulas ϕ,

- if ϕ is valid in the class of all **PLL**-frames then $\vdash_{\mathbf{PLL}} \phi$.

Two peculiar intuitionistic modal logics: Propositional Lax Logic

Some results

Theorem

For all formulas ϕ,

- if $\vdash_{\mathbf{PLL}} \phi$ then $\vdash_{\mathbf{IPL}} \phi^\circ$.

where ϕ° is the formula obtained from ϕ by removing all occurrences of $\bigcirc$

Theorem

For all formulas ϕ, ψ,

- if $\vdash_{\mathbf{PLL}} \phi \vee \psi$ then $\vdash_{\mathbf{PLL}} \phi$ or $\vdash_{\mathbf{PLL}} \psi$.

Theorem

The following decision problem is decidable:

- given a formula ϕ, determine whether $\vdash_{\mathbf{PLL}} \phi$.

Two peculiar intuitionistic modal logics: Propositional Lax Logic

Some results

Theorem

- **PLL** $+ \neg \bigcirc \bot$ is sound and complete with respect to the class of all frames $(S, \leq, R, F)$ such that $F = \emptyset$,
- **PLL** $+ \bigcirc(p \vee q) \rightarrow \bigcirc p \vee \bigcirc q$ is sound and complete with respect to the class of all frames $(S, \leq, R, F)$ such that $\leq$ and R are mutually confluent.

Two peculiar intuitionistic modal logics: Propositional Lax Logic

Embedding of **PLL** into classical modal logic

Let f be an arbitrary atomic formula

We define the following translation

- $\tau(p) = \Box_1(p \vee f)$
- $\tau(\bot) = \Box_1 f$
- $\tau(\top) = \top$
- $\tau(\phi \vee \psi) = \tau(\phi) \vee \tau(\psi)$
- $\tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi)$
- $\tau(\phi \to \psi) = \Box_1(\tau(\phi) \to \tau(\psi))$
- $\tau(\bigcirc\phi) = \Box_1 \Diamond_2 \tau(\phi)$

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