Exercise for Theoretical Condensed Matter Physics

Exercise 4, April, 26th 2024. Discussion on May, 6th 2024

1) Cauchy's principal value and integrals of simple singularities

We study integrals of the form

$$\int_{-\infty}^{\infty} dx \, \frac{1}{x + i\eta} f(x), \tag{1}$$

where $f(x)$ is any function with a well behaved Taylor expansion around $x = 0$, and $\eta = 0^+$ is a positive infinitesimal.

Show that in this context $\frac{1}{x+i\eta}$ can be decomposed into the following real and imaginary parts:

$$\frac{1}{x + i\eta} = \mathcal{P}\frac{1}{x} - i\pi\delta(x). \tag{2}$$

Here $\mathcal{P}$ means Cauchy principal part:

$$\mathcal{P}\int_{-\infty}^{\infty} dx \, \frac{1}{x} f(x) \equiv \lim_{\eta \to 0^+} \left[\int_{-\infty}^{-\eta} dx \, \frac{1}{x} f(x) + \int_{\eta}^{\infty} dx \, \frac{1}{x} f(x) \right] \tag{3}$$

Hint: Start by decomposing the integrand in (1) into its real and imaginary parts. What happens to each part in the limit $\eta \to 0^+$?

2) Relations between $G^<$, $G^>$, and the spectral function

a) Show using the Lehmann representation that the spectral function defined as

$$A_\alpha(\omega) = -2\mathrm{Im}\, G^R_{\alpha\alpha}(\omega) \tag{4}$$

is connected to Fourier transform of the greater and lesser Green's functions

$$G^>_{\alpha\alpha}(t - t') = -i\langle c_\alpha(t) c^\dagger_\alpha(t')\rangle, \quad G^<_{\alpha\alpha}(t - t') = i\langle c^\dagger_\alpha(t) c_\alpha(t')\rangle \tag{5}$$

by

$$iG^>_{\alpha\alpha}(\omega) = A_\alpha(\omega)[1 - n_F(\omega)], \tag{6}$$

$$-iG^<_{\alpha\alpha}(\omega) = A_\alpha(\omega) n_F(\omega), \tag{7}$$

where $n_F(\omega)$ is the Fermi distribution function.

b) Show that the occupation number of a state α, $\bar{n}_\alpha = \langle c_\alpha^\dagger c_\alpha \rangle$ is given by

$$\bar{n}_\alpha = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A_\alpha(\omega) n_F(\omega). \tag{8}$$

3) The spectral function of the Hubbard atom

Refer back to your solution of the Hubbard atom in exercise 2. Calculate its spectral function $A(\omega)$ at half-filling and zero temperature with the help of the Lehmann representation.