Exercise for Theoretical Condensed Matter Physics

Exercise 8, June, 24th 2024. Discussion on July, 1st 2024

1) Matsubara sums I

a) Using the definition of the Matsubara Green's functions $\mathcal{C}_{AB}(\tau) \equiv -\langle T_\tau A(\tau)B(0)\rangle$ and of its Fourier transform, evaluate the following sum over the Matsubara frequencies

$$\frac{1}{\beta}\sum_n e^{i\omega_n 0^+}\mathcal{C}_{AB}(\omega_n), \tag{1}$$

where the sum runs over bosonic or fermionic frequencies depending on the type of operators A and B. 0^+ is a infinitesimal positive number.

b) Apply the result from part a) to the Matsubara Green's functions of free fermions (bosons) with dispersion ϵ_k ($\mathcal{G}_k(\omega_n) = \frac{1}{i\omega_n - \epsilon_k}$).

c) What changes if instead of 0^+ we take an infinitesimal negative number 0^-?

2) **Phonon Matsubara Green's function**

Phonons are coherent oscillations of lattice atoms. The displacements of atomic nuclei can be viewed as a real field.

In the simplest harmonic approximation, the phonon Hamiltonian reads:

$$H_{\text{ph}} = \sum_{q,\lambda} \omega_{q,\lambda} \left(b^{\dagger}_{q,\lambda} b_{q,\lambda} + \frac{1}{2} \right) . \tag{2}$$

Here λ denotes the phonon polarization (eigenmodes of oscillations, number of those modes is equal to the number of atoms in the unit cell multiplied by the dimensionality of space, which is usually 3), q is the quasimomentum of the phonon, $\omega_{q,\lambda}$ is the phonon dispersion, and $b, b^{\dagger}$ are the anihilation and creation operators of the phonons. The latter are introduced very similar to the way those operators are introduced in a quantum mechanical harmonic oscillator. As in that case the Fourier transform of the displacement operator of the particle is proportional to $b^{\dagger}_{-q,\lambda} + b_{q,\lambda}$. It is convenient to use the displacement fields (up to a a q-dependent factor that is a question of definition and does not affect the calculations)

$$\mathcal{A}_{q,\lambda} \equiv b_{q,\lambda} + b^{\dagger}_{-q,\lambda} \tag{3}$$

to define the phonon Green's function: (note that $\mathcal{A}^{\dagger}_{q,\lambda} = \mathcal{A}_{-q,\lambda}$)

$$\mathcal{D}_{q,\lambda}(\tau) = - \left\langle T_{\tau} \mathcal{A}_{q,\lambda}(\tau) \mathcal{A}^{\dagger}_{q,\lambda}(0) \right\rangle . \tag{4}$$

While operators $b, b^{\dagger}$ annihilate or create a phonon with a given momentum, the operator $\mathcal{A}_q$ rather removes the momentum q from the phonon system by either annihilating a phonon with momentum q *or* creating a phonon with momentum $-q$.

- **a)** Using definition (3) find the explicit form for $\mathcal{D}_{q,\lambda}(\tau)$. The phonon system is to be considered a system of bosons at thermal equilibrium. Also notice that the phonon dispersion is an even function of q: $\omega_{q,\lambda} = \omega_{-q,\lambda}$.
- **b)** Do the Fourier transform of the result of part a) and derive the phonon Green's function in the frequency domain. Keep in mind that phononic operators are bosonic!

Hint: Follow the argumentation in Bruus & Flensberg's book chapter 17.1.