**Exercise for Theoretical Condensed Matter Physics**

Exercise 7, June, 10th 2024. Discussion on June, 17th 2024

## Polarizability of free electrons.

Consider a system of free electrons $H_0 = \sum_k \epsilon_k c_k^\dagger c_k$.

The polarizability $\chi_\mathrm{d}$ of an electronic system gives the change in electron density *i.e.* the induced charge density $\rho_\mathrm{ind}$, in linear order of the applied potential $\phi_\mathrm{ext}$:

$$\rho_\mathrm{ind}(r,t) = \int \mathrm{d}r \int_{t_0}^{t} \mathrm{d}t' \chi_\mathrm{d}(rt, r't') \phi_\mathrm{ext}(r', t').$$

$\chi_\mathrm{d}$ is given by the density-density correlation function and should be calculated here as a function of the wave vector $q$ and frequency $\omega$ using the Wick's theorem and Matsubara summation.

The starting point is the density-density correlation function in imaginary time

$$\chi_\mathrm{d}(q,\tau) = -\frac{1}{V}\langle T_\tau \rho(q,\tau)\rho(-q)\rangle,$$

where $V$ is the system volume and $\rho(q) = \sum_k c_k^\dagger c_{k+q}$ is the Fourier transform of the density operator. Thus

$$\chi_\mathrm{d}(q,\tau) = -\frac{1}{V}\sum_{k,k'} \langle T_\tau c_k^\dagger(\tau + 0^+) c_{k+q}(\tau) c_{k'}^\dagger(0^+) c_{k'-q}\rangle.$$

**a)** Using the Wick's theorem show that for $q \neq 0$

$$\chi_\mathrm{d}(q,\tau) = \frac{1}{V}\sum_k G_0(k+q,\tau) G_0(k,-\tau).$$

**b)** Let $\chi_\mathrm{d}(q, i\Omega_n) = \int_0^\beta \mathrm{d}\,\tau \chi_\mathrm{d}(q,\tau) e^{i\Omega_n \tau}$, with $\Omega_n = 2n\pi/\beta$ being the bosonic Matsubara frequencies and $\beta$ the inverse temperature. Show that

$$\chi_\mathrm{d}(q, i\Omega_n) = \frac{1}{V\beta}\sum_{\omega_n', k} G_0(k+q, i\Omega_n + i\omega_n') G_0(k, i\omega_n')$$

with the sum being taken over the fermionic Matsubara frequencies $\omega_n'$.

**c)** Use the method of the complex contour integral to evaluate the sum and show that

$$\chi_\mathrm{d}(q, i\Omega_n) = \frac{1}{V}\sum_k \frac{n_F(\epsilon_k) - n_F(\epsilon_{k+q})}{i\Omega_n + \epsilon_k - \epsilon_{k+q}},$$

with $n_F(\epsilon) = 1/(e^{\beta\epsilon} + 1)$ being the Fermi function distribution.

We attempt now to evaluate the remaining $k$-sum in two limiting cases, where at sufficiently low temperatures major contributions to $\chi_\mathrm{d}(q, i\Omega_n)$ are determined by the vicinity of the Fermi surface. In these cases a linearization of the dispersion around the Fermi energy $\mu$ allows to obtain meaningful results. Take $\epsilon_k \approx v_F(|k| - k_F)$ ($v_F$ and $k_F$ being the Fermi velocity and momentum respectively) and evaluate $\chi_d$ in two important cases

**d)** $q \longrightarrow 0$ and

**e)** in 1D case $q = 2k_F + p$ with $p \ll k_F$.