Exercise for Theoretical Condensed Matter Physics

Exercise 5, May, 27th 2024. "Präsenzübung"

2) The general Bogoliubov transformation for fermions

We study the Bogoliubov transformation in second quantization by considering a bilinear Hamiltonian of the general form

$$H = E_0(a^\dagger a + b^\dagger b) + E_1(a^\dagger b^\dagger + ba), \tag{1}$$

where a, b ($a^\dagger, b^\dagger$) denote annihilation (creation) operators. We introduce a new set of Fermi operators denoted $\hat{\alpha}$ and $\hat{\beta}$, and seek a linear transformation of the form

$$a = u\hat{\alpha} - v\hat{\beta}^\dagger, \qquad b = u\hat{\beta} + v\hat{\alpha}^\dagger \tag{2}$$

that diagonalizes the Hamiltonian (here u and v are both real).

a) Show that by inserting these expressions in the anticommutation relations for a, b, $a^\dagger$, and $b^\dagger$ one obtains

$$u^2 + v^2 = 1. \tag{3}$$

b) By inserting (2) into (1), show that the unwanted cross-terms $\hat{\alpha}\hat{\beta}$ and $\hat{\alpha}^\dagger\hat{\beta}^\dagger$ vanish if

$$E_1(u^2 - v^2) - 2E_0 uv = 0 \tag{4}$$

c) Using the parametrization $u = \cos t$, $v = \sin t$ (*cf.* (3) show that the Hamiltonian (1) is diagonalized by (2) and becomes:

$$H = \lambda(\hat{\alpha}^\dagger\hat{\alpha} + \hat{\beta}^\dagger\hat{\beta}) + \text{const}, \tag{5}$$

with

$$\lambda = \sqrt{E_0^2 + E_1^2}. \tag{6}$$

d) Check that the particle hole transformation for one of the fermion species (*e.g.* $b \to c^\dagger$) yields a Hamiltonian that conserves particle number. Discuss on this basis a connection between a superconductor and an excitonic insulator.