Set Functions

Collection $\mathcal{B}$ of subsets (events) also called

σ- Field

Remark

$A \subset \Omega$, we repeat experiments N times

$\to$ rel. Frequency $f_A = \frac{|\{A\}|}{N}$ ← # of times A occured

$\to f_A \geq 0$ AND $f_\Omega = 1$

Let A_1, A_2 with $A_1 \cap A_2 = \emptyset$, then

$f_{A_1 \cup A_2} = f_{A_1} + f_{A_2}$

Definition

Ω : Sample Space

$\mathcal{B}$: set of events

P: real-valued Functions on $\mathcal{B}$

P must satisfy:

1. $P(A) \geq 0, \forall \in \mathcal{B}$
2. $P(\Omega) = 1$
3. If $\{A_n\}$ sequence of events in $\mathcal{B}$ with

$A_m \cap A_n = \emptyset \ \forall m \neq n$, then:

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

A collection $\mathbb{B} = \{A_n\}, n \in \mathbb{N}$ is exhaustive if $\bigcup_{i=1}^{\infty} A_i = \Omega$. Mutually exclusive AND exh. collections of events form a partition on Ω.

Theorem 1.3.1

$A \in \mathbb{B}$, $P(A) = 1 - P(A^c)$

Proof:

$\Omega = A \cup A^c$, $A \cap A^c = \emptyset$

$\Rightarrow P(\Omega) = P(A) + P(A^c) = 1$

$\Leftrightarrow P(A) = 1 - P(A^c)$ $\square$ //

Theorem 1.3.2

$P(\emptyset) = 0$

Proof:

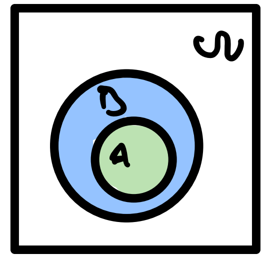
Let $A = \emptyset$, thus $A^c = \Omega$

From $P(\Omega) = P(A) + P(A^c) = 1$ follows, that

$P(\Omega) = P(\emptyset) + P(\Omega)$

$1 = 1 + P(\emptyset)$

$\Leftrightarrow 1 - 1 = P(\emptyset) = 0$ □//



Theorem 1.3.3

A, B events s.t. $A \subset B$, then $P(A) \leq P(B)$

Proof

$B = A \cup (A^c \cap B)$

$A = A \cap (A^c \cap B) = \emptyset$

$\Rightarrow P(B) = P(A) + \overbrace{P(A^c \cap B)}^{= \delta \geq 0}$, thus

$P(B) = P(A) + \delta$

$\Rightarrow P(B) \geq P(A)$ □//

Theorem 1.3.4

Foreach $A \in \mathcal{B}$, $0 \leq P(A) \leq 1$

Proof $\emptyset \subset A \subset \Omega$ is trivial

$\Rightarrow \emptyset \subset A \subset \Omega \Leftrightarrow P(\emptyset) \leq P(A) \leq P(\Omega)$

$\Leftrightarrow 0 \leq P(A) \leq 1$ □//

Theorem 1.3.5

Let $A, B \in \Omega$, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof

$$A \cup B = A \cup (A^c \cap B)$$

$$B = (A \cap B) \cup (A^c \cap B)$$

$$\Rightarrow P(A \cup B) = P(A) + P(A^c \cap B) \;, \quad ①$$

$$P(B) = P(A \cap B) + P(A^c \cap B) \quad ②$$

$$② \; P(A^c \cap B) = P(B) - P(A \cap B)$$

In ①

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \square$$

Examples

1.3.1

Ω, two dies are thrown $P((x,y)) = \frac{1}{36}$

$C_1 = \{(1,1), (2,1), (3,1), (4,1), (5,1)\}$

$C_2 = \{(1,2), (2,2), (3,2)\}$

$P(C_1) = \frac{5}{36}$, $P(C_2) = \frac{3}{36}$, $P(C_1 \cup C_2) = \frac{8}{36}$

AND $P(C_1 \cap C_2) = P(\emptyset) = 0$

1.3.2

Two coins tossed

$\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$

P assigns $\frac{1}{4}$ to each outcome.

$C_1 = \{(H,H),(H,T)\}$

$C_2 = \{(H,H),(T,H)\}$

$P(C_1) = P(C_2) = \frac{1}{2}$

$P(C_1 \cap C_2) = \frac{1}{4}$

$P(C_1 \cup C_2) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

Finite sample space

$\Omega = \{x_1, \dots, x_m\}$

$0 \leq p_i \leq 1, i \in \{1, 2, \dots, m\}$ AND $\sum_{i=1}^{m} p_i = 1$

$$P(A) = \sum_{x_i \in A} p_i$$

Definition 1.3.2

Equilikely case, $\Omega = \{x_1, \dots, x_m\}$

$$P(A) = \sum_{x_i \in A} \frac{1}{m} = \frac{|\{A\}|}{m}$$

(for "fair" experiments)

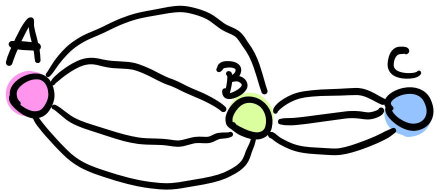
Counting Rules 1.3.1

mn-rule

$A = \{x_1, \dots, x_m\}$

$B = \{y_1, \dots, y_n\}$

There exist $m \cdot n$ ordered pairs



4 ways $A \to B$

3 ways $B \to C$

By mn-rule: $4 \cdot 3 = 12$ ways $A \to C$

Example

Drivers licence = 3 capital letters & 3 Numbers

gives $26^3 \cdot 10^3$ possibilities.

k-tuples of A:

$n \cdot n \cdot \ldots \cdot n = n^k$

k-tuples (no repetition)

1st 2nd kth

$n \cdot (n-1) \cdot \ldots \cdot (n-(k-1))$

these tuples: PERMUTATIONS

$\to P_k^n$ (# of k-perm's from an n Element set)

$$P_k^n = n \cdot (n-1) \cdot \ldots \cdot (n-(k-1))$$
$$= \frac{n!}{(n-k)!}$$

Example (Birthday Problem)

n people, $n<365$

A = at least two people share birthday

365^n possible n-tuples with a probability of $365^{-n} = \frac{1}{365^n}$

A^c = all Birthdays are distinct

$\to$ number of n tuples in A^c, which is P_n^{365}.

Thus, $P(A) = 1 - \frac{P_n^{365}}{365^n}$

If $n=2$, $P(A) = 1 - \frac{365 \cdot 364}{365^2} = 0{,}0027$
$= 0{,}27\%$ //

Now suppose order is irrelevant

$\to$ DONT COUNT PERMUTATIONS

→ COUNT NUMBER OF SUBSETS
(of k elements)

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

also: $\binom{n}{k} = C^n_k$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Example (Poker)

$|\Omega| = 52$ we draw spades card

$E_1 = ♠$

$P(E_1) = \frac{13}{52} = \frac{1}{4}$

$E_2 = \text{KING}$

$P(E_2) = \frac{4}{52} = \frac{1}{13}$

A "hand" consist of 5 cards, thus

$\binom{52}{5}$ "hands" are playable.

Prob. of a hand: $P(H_i) = \frac{1}{\binom{52}{5}}$

$E_1 = \text{FLUSH}$

$\binom{4}{1} = 4$ Suits AND

$\binom{13}{5} = 1287$ hands for each suit

Thus, we obtain:

$$P(E_1) = \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} = \frac{4 \cdot 1287}{2598960} = 0{,}00198$$

(straight flush prob. is included here of course)

E_2 = THREE OF A KIND (only 3 of a kind)

- The "kind" can have $\binom{13}{1} = 13$ kinds
- The "three" has $\binom{4}{3}$ ways.

(From ♠3 ♣3 ♦3 ♥3) ← we take 3

- The other "two cards" $\binom{12}{2}$ and one from each of the last two kinds: $\binom{4}{1}\binom{4}{1}$ ways

$\underbrace{\{♠ ♣ ♦ ♥\} \times \{♠ ♣ ♦ ♥\}}_{4}$

Thus $P(E_2)$: (exact 3 of a kind)

$$P(E_2) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}} = 0{,}0211$$

$\approx 2{,}11\%$

E_3 = exactly 3 KINGS and 2 QUEENS

$$P(E_3) = \frac{\binom{4}{3}\binom{4}{2}}{\binom{52}{5}} = 0{,}0000093$$

Additional Properties of Probability

Theorem 1.3.6

$\{C_n\}$ nondecreasing event sequence :

$$\lim_{n\to\infty} P(C_n) = P(\lim_{n\to\infty} C_n) = P\left(\bigcup_{n=1}^{\infty} C_n\right)$$

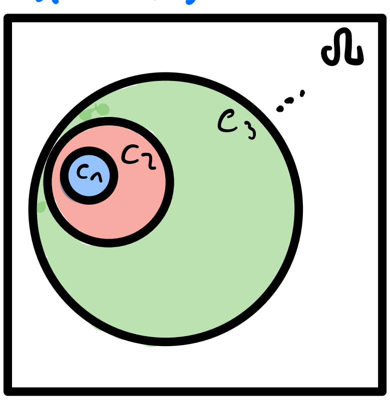
$\{C_n\}$ increasing :

$$\lim_{n\to\infty} P(C_n) = P(\lim_{n\to\infty} C_n) = P\left(\bigcap_{n=1}^{\infty} C_n\right)$$

Proof

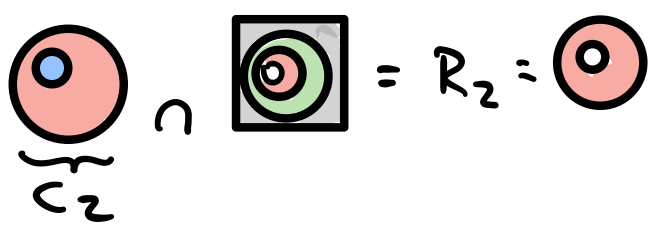
Sets are called rings. Define:

$R_1 = C_1$, $n>1$ $R_n = C_n \cap C_{n-1}^c$



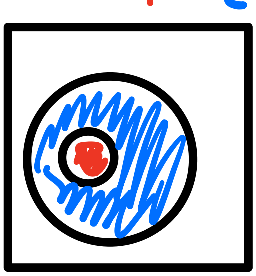
$C_1 = R_1$

$R_2 := C_2 \cap C_1^c$

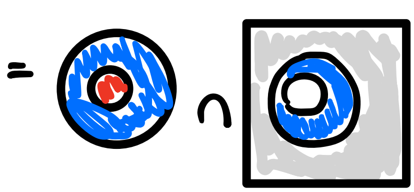


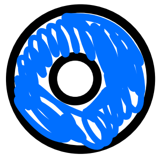
$$\Rightarrow \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} R_n \wedge R_m \cap R_n = \emptyset, m \neq n$$

$C_1 \subset C_2$



$R_2 = C_2 \cap C_1^c$



$=$  Ring R_2

$\Rightarrow R_1 \not\subset R_2$

Thus $R_m \cap R_n = \emptyset$, also

$$P(R_n) = P(C_n) - P(C_{n-1}) \quad (*)$$

C_n

$P($  $C_{n-1}) = P($  $) - P($  $)$

This yields:

$$P\left(\lim_{n\to\infty} C_n\right) = P\left(\bigcup_{n=1}^{\infty} C_n\right)$$

$$= P\left(\bigcup_{n=1}^{\infty} R_n\right)$$

$$= \sum_{n=1}^{\infty} P(R_n)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} P(R_j)$$

$$\downarrow (*)$$

$$= \lim_{n\to\infty} \left(P(C_1) + \sum_{j=2}^{n} P(C_j) - P(C_{j-1}) \right)$$

$$= \lim_{n\to\infty} P(C_n) \qquad \square \, //$$

Theorem 1.3.7 (Boole's Inequality)

$\{C_n\}$ sequence of events :

$$P\left(\bigcup_{n=1}^{\infty} C_n\right) \leq \sum_{n=1}^{\infty} P(C_n)$$

Proof

Let $D_n = \bigcup_{i=1}^{n} C_i$ (*)

$\Rightarrow \{D_n\}$ is increasing sequence

upto $\bigcup_{n=1}^{\infty} C_n$.

Also:

$\forall j : D_j = D_{j-1} \cup C_j$ (*)

Hence :

$$P(D_j) \leq P(D_{j-1}) + P(C_j)$$

$$\Leftrightarrow P(D_j) - P(D_{j-1}) \leq P(C_j)$$

Using $P(C_1) = P(D_1)$ gives

$$P\left(\bigcup_{n=1}^{\infty} C_n\right) = P\left(\bigcup_{n=1}^{\infty} D_n\right)$$

$$= \lim_{n\to\infty} \left(P(D_1) + \sum_{j=2}^{n} P(D_j) - P(D_{j-1})\right)$$

$$\leq \lim_{n\to\infty} \sum_{j=1}^{n} P(C_j) = \sum_{n=1}^{\infty} P(C_n) \quad \square$$

Remark 1.3.2 (Inclusion - Exclusion)

$$P(C_1 \cup C_2 \cup C_3) = p_1 - p_2 + p_3$$

$$p_1 = P(C_1) + P(C_2) + P(C_3)$$

$$p_2 = P(C_1 \cap C_2) + P(C_1 \cap C_3) + P(C_2 \cap C_3)$$

$$p_3 = P(C_1 \cap C_2 \cap C_3)$$

Generalized:

$$P(C_1 \cup .. \cup C_k) = p_1 - p_2 + \ldots + (-1)^{k+1} p_k$$

with p_i being the sum of the P's of all possible intersections involving i sets.

It holds $p_1 \geq \ldots \geq p_k$

1.3.7

$$p_1 = P(C_1) + \ldots + P(C_k) \geq P(C_1 \cup \ldots \cup C_k)$$

For $k=2$

$$1 \geq P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

this gives

Bonferroni's inequality

$$P(C_1 \cap C_2) \geq P(C_1) + P(C_2) - 1$$

Other useful inequalities come from this:

$$p_1 \geq P(C_1 \cup \dots \cup C_k) \geq p_1 - p_2$$

AND

$$p_1 - p_2 + p_3 \geq P(C_1 \cup \dots \cup C_k) \geq p_1 - p_2 + p_3 - p_4$$

//