1.3.1. A positive integer from one to six is to be chosen by casting a die. Thus the elements c of the sample space $\mathcal{C}$ are $1, 2, 3, 4, 5, 6$. Suppose $C_1 = \{1, 2, 3, 4\}$ and $C_2 = \{3, 4, 5, 6\}$. If the probability set function P assigns a probability of $\frac{1}{6}$ to each of the elements of $\mathcal{C}$, compute $P(C_1)$, $P(C_2)$, $P(C_1 \cap C_2)$, and $P(C_1 \cup C_2)$.

$$P(C_1) = \frac{|\{C_1\}|}{|\Omega|} = \frac{4}{6} = P(C_2) = \frac{2}{3}$$

$$C_1 \cap C_2 = \{3, 4\}$$

$$P(C_1 \cap C_2) = \frac{2}{6} = \frac{1}{3}$$

$$C_1 \cup C_2 = \Omega \Rightarrow P(\Omega) = 1$$

1.3.2. A random experiment consists of drawing a card from an ordinary deck of 52 playing cards. Let the probability set function P assign a probability of $\frac{1}{52}$ to each of the 52 possible outcomes. Let C_1 denote the collection of the 13 hearts and let C_2 denote the collection of the 4 kings. Compute $P(C_1)$, $P(C_2)$, $P(C_1 \cap C_2)$, and $P(C_1 \cup C_2)$.

C_1 = 13 HEARTS

C_2 = 4 KINGS

$$P(C_1) = \frac{13}{52} = \frac{1}{4}$$

$$P(C_2) = \frac{4}{52} = \frac{1}{13}$$

$$C_1 \cap C_2 = \heartsuit K \Rightarrow P(C_1 \cap C_2) = \frac{1}{52}$$

$$C_1 \cup C_2 = \{\underbrace{\heartsuit A, \heartsuit 2, \ldots, \heartsuit K}_{13}, \underbrace{\spadesuit K, \clubsuit K, \diamondsuit K}_{3}\}$$

$$P(C_1 \cup C_2) = \frac{13+3}{52} = \frac{16}{52}$$

1.3.3. A coin is to be tossed as many times as necessary to turn up one head. Thus the elements c of the sample space $\mathcal{C}$ are H, TH, TTH, $TTTH$, and so forth. Let the probability set function P assign to these elements the respective probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, and so forth. Show that $P(\mathcal{C}) = 1$. Let $C_1 = \{c : c \text{ is } H, TH, TTH, TTTH, \text{ or } TTTTH\}$. Compute $P(C_1)$. Next, suppose that $C_2 = \{c : c \text{ is } TTTTH \text{ or } TTTTTH\}$. Compute $P(C_2)$, $P(C_1 \cap C_2)$, and $P(C_1 \cup C_2)$.

$P(\Omega) = 1$?

Proof

$$\frac{1}{2} + \frac{1}{4} + \ldots + \ldots \Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$S_4 = \frac{1}{2} + \ldots + \frac{1}{16} = \frac{15}{16}$$

$\vdots$

$$S_n = \frac{2^n - 1}{2^n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n} - \frac{1}{2^n}$$

$$= \lim_{n\to\infty} 1 - \frac{1}{2^n} \Rightarrow 0$$

$$= 1 \quad \square$$

$C_1 = \{c \mid H, TH, TTH, TTTH \text{ or } TTTTH\}$

$P(C_1) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$

$= \frac{16}{32} + \frac{8}{32} + \frac{4}{32} + \frac{2}{32} + \frac{1}{32}$

$= \frac{35}{36} /$

$C_2 = \{c \mid TTTTH \vee TTTTTH\}$

$P(C_2) = \frac{1}{32} + \frac{1}{64}$

$= \frac{33}{64} //$

$C_1 \cap C_2 = \{TTTTH\}$

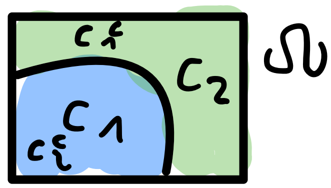
$P(C_1 \cap C_2) = \left(\frac{1}{2}\right)^5 = \frac{1}{32} //$

$C_1 \cup C_2 = \{c \mid H, TH, \ldots, TTTTTH\}$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$

$= \frac{63}{64} //$

1.3.4. If the sample space is $\mathcal{C} = C_1 \cup C_2$ and if $P(C_1) = 0.8$ and $P(C_2) = 0.5$, find $P(C_1 \cap C_2)$.



$$\Omega = C_1 \cup C_2$$

$$\Leftrightarrow (C_1 \cap C_1^c) \cup (C_2 \cap C_2^c)$$

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

$$\Leftrightarrow P(C_1 \cup C_2) + P(C_1 \cap C_2) = P(C_1) + P(C_2)$$

$$\Leftrightarrow P(C_1 \cap C_2) = \overbrace{P(C_1)}^{0{,}8} + \overbrace{P(C_2)}^{0{,}5} - \overbrace{P(C_1 \cup C_2)}^{1}$$

$$\Leftrightarrow P(C_1 \cap C_2) = 0{,}8 + 0{,}5 - P(\Omega)$$

$$= 1{,}3 - 1$$

$$= 0{,}3 \,//$$

1.3.5. Let the sample space be $\mathcal{C} = \{c : 0 < c < \infty\}$. Let $C \subset \mathcal{C}$ be defined by $C = \{c : 4 < c < \infty\}$ and take $P(C) = \int_C e^{-x}\,dx$. Show that $P(\mathcal{C}) = 1$. Evaluate $P(C)$, $P(C^c)$, and $P(C \cup C^c)$.

$$P(\Omega) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty$$

$$= -\lim_{n\to\infty} \overset{\Rightarrow 0}{\frac{1}{e^n}} - (-\overbrace{e^{-0}}^{=1})$$

$$= 0 - (-1) = 1 \,//$$

$$P(C) = \int_4^\infty e^{-x}\,dx = e^{-x}\Big|_4^\infty$$

$$\overset{\lim}{=} -\frac{1}{e^\infty} + e^{-4}$$

$$= 0 + \frac{1}{e^4}$$

$$= \frac{1}{e^4}$$

$$P(C^c) = 1 - \frac{1}{e^4}$$

$$P(C \cup C^c) = \frac{1}{e^4} + 1 - \frac{1}{e^4} = 1$$

1.3.6. If the sample space is $\mathcal{C} = \{c : -\infty < c < \infty\}$ and if $C \subset \mathcal{C}$ is a set for which the integral $\int_C e^{-|x|}\,dx$ exists, show that this set function is not a probability set function. What constant do we multiply the integrand by to make it a probability set function?

$$\int_{-\infty}^{\infty} e^{-|x|}\,dx \to \forall x \geq 0 \to \int e^{-x}\,dx$$

$$\forall x < 0 \to \int e^{-x}\,dx$$

$$P(C) \Leftrightarrow 2\int_0^\infty e^{-x}\,dx = 2\cdot(-e^{-x})\Big|_0^\infty$$

$$= 2\cdot(0-(-1)) = 2$$

Thus multiplying by $\frac{1}{2}$ would make it a Prob. Function.

1.3.7. If C_1 and C_2 are subsets of the sample space $\mathcal{C}$, show that

$$P(C_1 \cap C_2) \leq P(C_1) \leq P(C_1 \cup C_2) \leq P(C_1) + P(C_2).$$

<u>Proof</u> 1

$P(C_1 \cap C_2) \leq P(C_1)$

CASE 1: $C_1 \cap C_2 = \emptyset$

$\Rightarrow P(\emptyset) \leq P(C_1)$

$0 \leq P(C_1)$

CASE 2: $C_1 \cap C_2 = C_3$ (is max. for $C_1 = C_2$)

If $C_1 \neq C_2$, then $|C_3| < |C_1|$

thus: $P(C_1 \cap C_2) \leq P(C_1)$ $\Leftarrow$ 1

2 $P(C_1) \leq P(C_1 \cup C_2)$

CASE 1: $C_1 = C_2$, then

$P(C_1) = P(C_1 \cup C_1) = P(C_1)$

CASE 2 $C_1 \neq C_2$, thus $|C_1 \cup C_2| > C_1$, thus

$P(C_1) < P(C_1 \cup C_2)$

$\Leftarrow$ 2

③ $P(C_1 \cup C_2) \leq P(C_1) + P(C_2)$

CASE 1: $C_1 = C_2$, thus

$P(C_1 \cup C_2) = P(C_1 \cup C_1)$

$\Leftrightarrow P(C_1 \cup C_1) = \underline{P(C_1)} \leq P(C_1) + P(C_2)$

$= P(C_1) + P(C_1)$

$= \underline{2P(C_1)}$

$(P(C_1) \leq 2P(C_1))$

CASE 2: $C_1 \neq C_2$, thus excl./inclusion holds:

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - \underbrace{\overset{\delta}{P(C_1 \cap C_2)}}_{\geq 0}$$

thus

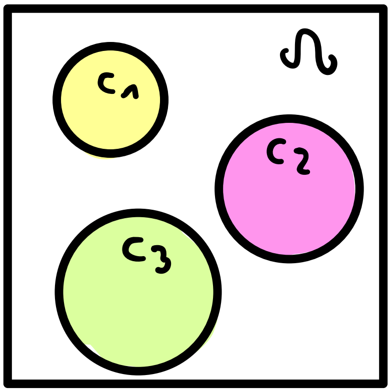
$\Leftrightarrow P(C_1 \cup C_2) - \delta = P(C_1) + P(C_2)$

$\Leftrightarrow P(C_1 \cup C_2) \leq P(C_1) + P(C_2)$

$\Leftrightarrow$ ③

From ①, ②, ③ it follows! $\square$ //

1.3.8. Let C_1, C_2, and C_3 be three mutually disjoint subsets of the sample space $\mathcal{C}$. Find $P[(C_1 \cup C_2) \cap C_3]$ and $P(C_1^c \cup C_2^c)$.



$P((C_1 \cup C_2) \cap C_3)$

Since theyre mutually excl.

$\Rightarrow$ any $x \in C_1 \cup C_2$

HAS to have : $x \notin C_3$

making the intersection $(C_1 \cup C_2) \cap C_3 = \emptyset$

thus $P(\emptyset) = 0$

$P(C_1^c \cup C_2^c) \rightarrow$ let $\Omega \setminus (C_1 \cup C_2 \cup C_3) \triangleq D$

Thus,

$C_1^c = D \cup C_2 \cup C_3$ AND

$C_2^c = D \cup C_1 \cup C_3$

$\Leftrightarrow C_1^c \cup C_2^c$

$= (D \cup C_2 \cup C_3) \cup (D \cup C_1 \cup C_2)$

$= D \cup C_2 \cup C_3 \cup \cancel{D} \cup C_1 \cup \cancel{C_2}$

$= D \cup C_1 \cup C_2 \cup C_3$

$\rightarrow P(C_1^c \cup C_2^c) = P(D \cup C_1 \cup C_2 \cup C_3)$

$\Leftrightarrow P((\Omega \setminus (C_1 \cup C_2 \cup C_3)) \cup (C_1 \cup C_2 \cup C_3))$

$\Leftrightarrow P(\Omega)$

$\Leftrightarrow 1$ //

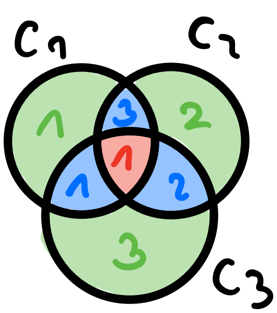
1.3.9. Consider Remark 1.3.2.

(a) If C_1, C_2, and C_3 are subsets of $\mathcal{C}$, show that

$$\begin{aligned} P(C_1 \cup C_2 \cup C_3) \quad = \quad & P(C_1) + P(C_2) + P(C_3) - P(C_1 \cap C_2) \\ & - P(C_1 \cap C_3) - P(C_2 \cap C_3) + P(C_1 \cap C_2 \cap C_3). \end{aligned}$$

(b) Now prove the general inclusion exclusion formula given by the expression (1.3.13).

a) Let the intersections of C_1, C_2, C_2 be non-empty.



Let

$P(C_1 \cup C_2 \cup C_3)$

$= P(C_1) + P(C_2) + P(C_3)$

By just adding up, we add:

$P(C_1) = 1 + 1 + 3 + 1$

$P(C_2) = 2 + 2 + 3 + 1$

$P(C_3) = 3 + 1 + 2 + 1$

we count once:

1, 2, 3

twice:

1, 2, 3

To ensure proper counting, we have to subtract: 1, 2, 3

This gives:

$$P(C_1)+P(C_2)+P(C_3)-P(C_1\cap C_2)-P(C_1\cap C_3)-P(C_2\cap C_3)$$

Here we now count:

$$1+2+3+2\cdot 1+2\cdot 2+2\cdot 3-1-2-3$$

But since 1 is element of every intersection, and now we've added it 3 times, then subtracted it 3 times, it's missing in our scheme, thus we have to add it, leaving us with

$$1+2+3-1-2-3+1$$

$\Leftrightarrow$

$$P(C_1)+P(C_2)+P(C_3)$$

$$-P(C_1\cap C_2)-P(C_1\cap C_3)-P(C_2\cap C_3)$$

$$+P(C_1\cap C_2\cap C_3) \quad \square //$$

Now:

$$P(C_1 \cup C_2 \cup \cdots \cup C_k) = p_1 - p_2 + p_3 - \cdots + (-1)^{k+1} p_k, \quad (*)$$

We can prove this by induction

<u>$n=0$</u>

$P(C_1) = p_1$, with $p_1 = C_1$ ✓

Let $(*)$ hold for any $n \in \mathbb{N}$:

<u>$n \to n+1$</u>

$p_1 = P(C_1) + P(C_2) + \ldots + P(C_n) + P(C_{n+1})$

$p_2 = P(C_1 \cap C_2) + \ldots + P(C_n \cap C_{n+1})$

$p_3 = \ldots$

$\vdots$

$p_n = P(C_1 \cap \ldots \cap C_n) + \ldots + P(C_1 \cap \ldots \cap C_{n-1} \cap C_{n+1})$

$p_{n+1} = P(C_1 \cap \ldots \cap C_{n+1})$

$\Rightarrow p_1 - p_2 + \ldots + (-1)^{n+1} p_n + (-1)^{n+2} p_{n+1}$

$\Leftrightarrow P(C_1 \cup \ldots \cup C_n \cup C_{n+1})$ $\square$ //

NOT CLEAN BUT ITS OKAY AS PROOF SKETCH I THINK!

1.3.10. A bowl contains 16 chips, of which 6 are red, 7 are white, and 3 are blue. If four chips are taken at random and without replacement, find the probability that: (a) each of the four chips is red; (b) none of the four chips is red; (c) there is at least one chip of each color.

a) $P(4\times\bullet) = \frac{6}{16}\cdot\frac{5}{15}\cdot\frac{4}{14}\cdot\frac{3}{13}$ (ONE WAY)

→ Total combinations: $\binom{16}{4} = \frac{16!}{4!\,12!} = \frac{16\cdot15\cdot14\cdot13}{4\cdot3\cdot2\cdot1} = 1820$

→ Favor. Comb.: $\binom{6}{4} = \frac{6!}{4!\,2!} = \frac{6\cdot5}{2\cdot1} = 15$ (Ways to draw 4 from 6)

$P(4\times\bullet) = \frac{15}{1820} = \frac{3}{364}$ //

b)

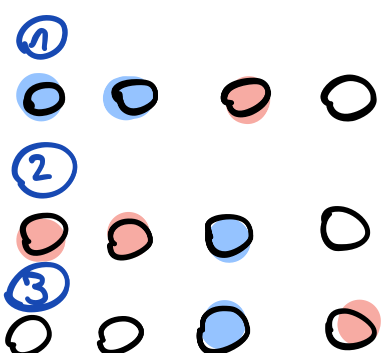
→ Tot. Comb.: $\binom{16}{4} = 1820$

# of none red balls

→ Fav. Comb. $\binom{10}{4} = \frac{10!}{4!\,6!} = \frac{10\cdot\overset{3}{\cancel{9}}\cdot\cancel{8}\cdot7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1} = 210$

$\Rightarrow \frac{210}{1820} = \frac{3}{26}$ //

c) at least one chip of each colour



① 2x blue

② 2x red

③ 2x white

$\left.\begin{matrix}\text{2x blue}\\\text{2x red}\\\text{2x white}\end{matrix}\right\}$ Fav. combinations

$\rightarrow$ 3 blue, 6 red, 7 white

1: $\binom{3}{2}\cdot\binom{6}{1}\cdot\binom{7}{1} = 126$

2: $\binom{6}{2}\cdot\binom{3}{1}\cdot\binom{7}{1} = 315$

3: $\binom{7}{2}\cdot\binom{3}{1}\cdot\binom{6}{1} = 378$

1 + 2 + 3 $= 819$ (fav. comb.)

$$\Rightarrow \frac{819}{1820} \overset{:7}{=} \frac{117}{260} \overset{:13}{=} \frac{9}{20} \quad (45\%)$$

1.3.11. A person has purchased 10 of 1000 tickets sold in a certain raffle. To determine the five prize winners, five tickets are to be drawn at random and without replacement. Compute the probability that this person wins at least one prize. *Hint:* First compute the probability that the person does not win a prize.

Total ways: $\binom{1000}{5}$

If person wins <u>NO</u> prizes, the winning tickets are among the 990 he did'nt buy.

$\Rightarrow \binom{990}{5}$

$$P(\text{No wins}) = \frac{\binom{990}{5}}{\binom{1000}{5}}$$

$\binom{990}{}$

$$P(\geq 1 \text{ win}) = 1 - \frac{\binom{990}{5}}{\binom{1000}{5}}$$

$$\rightsquigarrow \frac{\binom{990}{5}}{\binom{1000}{5}} = \frac{\frac{990!}{\cancel{5!}\,985!}}{\frac{1000!}{\cancel{5!}\,995!}} = \frac{990 \cdot 989 \cdot 988 \cdot 987 \cdot 986}{1000 \cdot 999 \cdot 998 \cdot 997 \cdot 996}$$

$$\approx 0{,}951$$

$$\Rightarrow 1 - 0{,}951 = 0{,}049 \approx 4{,}9 /\!/$$

1.3.12. Compute the probability of being dealt at random and without replacement a 13-card bridge hand consisting of: (a) 6 spades, 4 hearts, 2 diamonds, and 1 club; (b) 13 cards of the same suit.

a) Total Hands: $\binom{52}{13} \approx 6{,}35 \cdot 10^{11}$

Fav. Hand: $\underbrace{\binom{13}{6}}_{1716} \cdot \underbrace{\binom{13}{4}}_{715} \cdot \underbrace{\binom{13}{2}}_{78} \cdot \underbrace{\binom{13}{1}}_{13} = 1\,244\,725\,560$

$$\rightsquigarrow \frac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}} \approx 0{,}00196$$

$$= 0{,}196\,\% /\!/$$

b) Total: $\binom{52}{13} \approx 6{,}35 \cdot 10^{11}$

Same suit: $\underset{♠}{\binom{13}{13}} + \underset{♣}{\binom{13}{13}} + \underset{♦}{\binom{13}{13}} + \underset{♥}{\binom{13}{13}} = 4$

$$\frac{4}{\binom{52}{13}} \approx 6{,}3 \cdot 10^{-12} \,//$$

1.3.13. Three distinct integers are chosen at random from the first 20 positive integers. Compute the probability that: (a) their sum is even; (b) their product is even.

$\Omega = \{1, \dots, 20\}$, choose 3

Total: $\binom{20}{3} = 1140$

a) Sum even $\rightarrow$ even + even + even = even

even + odd + odd = even

Ways of getting only even ones: $\binom{10}{3} = 120$

Getting one even & 2 odd: $\binom{10}{1}\binom{10}{2} = 10 \cdot 45 = 450$

$$P(2|\Sigma) = \frac{\binom{10}{3} + \binom{10}{2}\binom{10}{1}}{\binom{20}{3}} = \frac{570}{1140} = \frac{1}{2} \,//$$

b) $2|\Pi$

Prod. even $\rightarrow$
$e \times e \times e \quad (2 \cdot 4 \cdot 6 = 48)$
$e \times o \times o \quad (2 \cdot 3 \cdot 5 = 30)$
$e \times e \times o \quad (2 \cdot 4 \cdot 5 = 40)$

$$\Rightarrow \binom{20}{3} = 1140$$

$$3 \text{ even} \triangleq \binom{10}{3} = 120$$

$$\begin{matrix} 1\text{even} \\ 2\text{odd} \end{matrix} \triangleq \binom{10}{1}\binom{10}{2} = 450$$

$$\begin{matrix} 2\text{even} \\ 1\text{odd} \end{matrix} \triangleq \binom{10}{2}\binom{10}{1} = 450$$

$$P(2 \mid \pi) = \frac{\binom{10}{3} + 2 \cdot \binom{10}{1}\binom{10}{2}}{\binom{20}{3}}$$

$$= \frac{1020}{1140} = \frac{17}{19} \;//$$

1.3.14. There are five red chips and three blue chips in a bowl. The red chips are numbered 1, 2, 3, 4, 5, respectively, and the blue chips are numbered 1, 2, 3, respectively. If two chips are to be drawn at random and without replacement, find the probability that these chips have either the same number or the same color.

Same number $\rightarrow$ $\left.\begin{matrix}(1,1)\\(2,2)\\(3,3)\end{matrix}\right\}$ $\left.\begin{matrix} P((1,1)) = \frac{1}{8} \cdot \frac{1}{7} \\ P((2,2)) = \frac{1}{8} \cdot \frac{1}{7} \\ P((3,3)) = \frac{1}{8} \cdot \frac{1}{7} \end{matrix}\right\} = \frac{3}{56}$

Same color $\rightarrow$ $\left.\begin{matrix}(r,r)\\(b,b)\end{matrix}\right\}$ $\begin{matrix} \frac{5}{8} \cdot \frac{4}{7} = \frac{20}{56} \\ \frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56} \end{matrix}$

$\rightsquigarrow \frac{3}{56} + \frac{20}{56} + \frac{6}{56} = \frac{29}{56}$ //

1.3.15. In a lot of 50 light bulbs, there are 2 bad bulbs. An inspector examines five bulbs, which are selected at random and without replacement.

(a) Find the probability of at least one defective bulb among the five.

(b) How many bulbs should be examined so that the probability of finding at least one bad bulb exceeds $\frac{1}{2}$?

a)

$\binom{50}{5} = 2\,118\,760$ 5 bulb combinations

48 working bulbs $\rightarrow \binom{48}{5} = 1\,712\,304$ working combs

$$P(0 \text{ defect}) = \frac{\binom{48}{5}}{\binom{50}{5}} = \frac{1\,712\,304}{2\,118\,760}$$

$$= \frac{198}{245}$$

$$\Rightarrow P(\geq 1 \text{ def.}) = 1 - P(\text{no def.})$$

$$= 1 - \frac{198}{245}$$

$$= \frac{47}{245} \text{ //}$$

b) Taking n: $\binom{50}{n} = \frac{50!}{n!(50-n)!}$

Taking n working: $\binom{48}{n} = \frac{48!}{n!(48-n)!}$

$$P(\text{no def.}) = \frac{\binom{48}{n}}{\binom{50}{n}}$$

$$= \frac{48!\,\cancel{n!}\,(50-n)!}{\cancel{n!}\,(48-n)!\cdot 50!}$$

$$= \frac{\cancel{48!}}{\underset{50\cdot 49}{\cancel{50!}}} \cdot \frac{(50-n)(49-n)\cancel{(48-n)!}}{\cancel{(48-n)!}}$$

$$= \frac{(50-n)(49-n)}{50\cdot 49}$$

$$\rightarrow P(\geq 1) = 1 - \frac{(50-n)(49-n)}{50\cdot 49} \overset{!}{>} \frac{1}{2}$$

$$= 1 - \frac{2450 - 50n - 49n + n^2}{2450}$$

$$= \frac{2\cancel{450}}{2450} - \frac{2\cancel{450} - 99n + n^2}{2450}$$

$$= -\frac{1}{2450}n^2 + \frac{99}{2450}n - \frac{1}{2}\geq 0 \quad \Big|\cdot -2450$$

$$= n^2 - 99n + 1225 \geq 0$$

$$\rightsquigarrow n_{1,2} = 49{,}5 \pm \sqrt{(-49{,}5)^2 - 1225}$$

$$= 49{,}5 \pm \sqrt{2450 - 1225}$$

$$= 49{,}5 \pm \sqrt{1225}$$

$$= 49{,}5 \pm 35$$

$$\Rightarrow n_1 = 14{,}5 \, , \quad n_2 = 84{,}5$$

Since we only take 50 light-bulbs, we can omit n_2, thus we need to take 15 light bulbs to have a 50% chance to find at least one bad one!

1.3.16. For the birthday problem, Example 1.3.3, use the given R function bday to determine the value of n so that $p(n) \geq 0.5$ and $p(n-1) < 0.5$, where $p(n)$ is the probability that at least two people in the room of n people have the same birthday.

WILL BE DONE IN R LATER ON!

1.3.17. If $C_1, \ldots, C_k$ are k events in the sample space $\mathcal{C}$, show that the probability that at least one of the events occurs is one minus the probability that none of them occur; i.e.,

$$P(C_1 \cup \cdots \cup C_k) = 1 - P(C_1^c \cap \cdots \cap C_k^c). \tag{1.3.15}$$

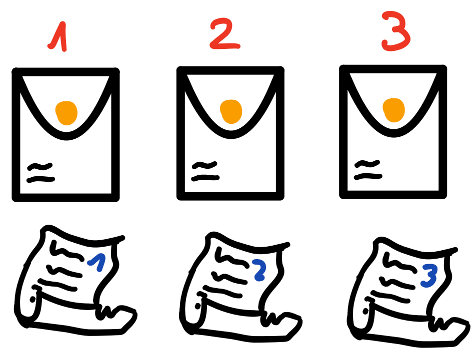
$$P(A) = 1 - P(A^c)$$

$\rightarrow$ To show: $C_1 \cup .. \cup C_k = C_1^c \cap .. \cap C_k^c$

$$\Rightarrow \bigcup_{k=1}^{n} C_k = \left(\bigcap_{k=1}^{n} C_k\right)^c \quad \text{(De Morgan)} \quad \square \, //$$

1.3.18. A secretary types three letters and the three corresponding envelopes. In a hurry, he places at random one letter in each envelope. What is the probability that at least one letter is in the correct envelope? *Hint:* Let C_i be the event that the ith letter is in the correct envelope. Expand $P(C_1 \cup C_2 \cup C_3)$ to determine the probability.

$C_i \triangleq$ i'th letter in correct envelope



(Given envelopes are ordered)

$$\sigma(\{1,2,3\}) = \{\{1,2,3\}, \{1,3,2\}, \{2,1,3\}, \{2,3,1\}, \{3,1,2\}, \{3,2,1\}\}$$

P(No letter in correct envelope)

↳ viable Permutations

$\{2,3,1\} \wedge \{1,3,2\} \rightarrow$ 2 out of 6

$\frac{2}{6} \rightarrow$ no letter fits its envelope

$\Rightarrow \frac{4}{6} \rightarrow$ At least one letter fits env.

ANOTHER CONCEPT HERE: DERANGEMENT

Derangement is perm. such that none of a sets elements appear in original position.

$$D(n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

In this case:

$$D(3) = 3! \left(\overset{1}{\frac{1}{0!}} - \overset{1}{\frac{1}{1!}} + \overbrace{\frac{1}{2!} - \frac{1}{3!}}^{\frac{3}{6} - \frac{1}{6} = \frac{1}{3}} \right)$$

$$3! \cdot \frac{1}{3} = 2$$

→ 2 ways where no envelope is in its original position.

1.3.19. Consider poker hands drawn from a well-shuffled deck as described in Example 1.3.4. Determine the probability of a full house, i.e, three of one kind and two of another.

$$P(\text{Full House}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{2}} \approx 0{,}00144$$

$\binom{13}{1} \to$ A type of card

$\binom{4}{3} \to$ choose for ex. $\underbrace{4♠, 4♣, 4♦}_{\text{3 chosen out of 4}}, 4♥$

Same for $\binom{12}{1}$ (other type) and two of those.

XXX YZ

$$P(\text{3 of a kind}) = \frac{\binom{13}{1}\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} \approx 0{,}0211$$

1.3.20. Prove expression (1.3.9).

Let $\{C_n\}$ be a decreasing sequence of events. Then

$$\lim_{n\to\infty} P(C_n) = P(\lim_{n\to\infty} C_n) = P\left(\bigcap_{n=1}^{\infty} C_n\right). \qquad (1.3.9)$$

(decreasing)

$\to C_{n+1} \subset C_n \Rightarrow C_n^c \subset C_{n+1}^c$

(Increasing)

We know: $\lim_{n\to\infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right)$

Substitute $A_n \Leftrightarrow C_n^c$

$$\Rightarrow \lim_{n\to\infty} P(C_n^c) = P\left(\bigcup_{n=1}^{\infty} C_n^c\right) \quad (*)$$

We know:

$$\Rightarrow \quad P(C_n^c) = 1 - P(C_n)$$

AND

$$\Rightarrow \bigcup_{n=1}^{\infty} C_n^c = \left(\bigcap_{n=1}^{\infty} C_n\right)^c \quad \text{(De Morgan)}$$

Thus,

$$P\left(\bigcup_{n=1}^{\infty} C_n^c\right) = P\left(\left(\bigcap_{n=1}^{\infty} C_n\right)^c\right) = 1 - P\left(\bigcap_{n=1}^{\infty} C_n\right)$$

$$\Rightarrow \lim_{n\to\infty} P(C_n^c) = \lim_{n\to\infty} 1 - P(C_n)$$

$(*)$ + De Morgan

$$\Leftrightarrow 1 - \lim_{n\to\infty} P(C_n) = 1 - P\left(\bigcap_{n=1}^{\infty} C_n\right)$$

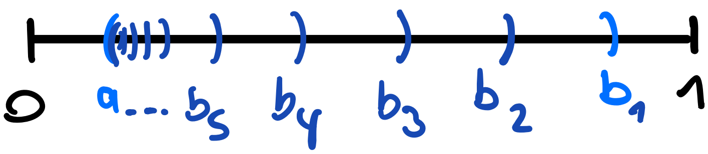
$$\Leftrightarrow \lim_{n\to\infty} P(C_n) = P\left(\bigcap_{n=1}^{\infty} C_n\right) \quad \square$$

1.3.21. Suppose the experiment is to choose a real number at random in the interval $(0, 1)$. For any subinterval $(a, b) \subset (0, 1)$, it seems reasonable to assign the probability $P[(a, b)] = b - a$; i.e., the probability of selecting the point from a subinterval is directly proportional to the length of the subinterval. If this is the case, choose an appropriate sequence of subintervals and use expression (1.3.9) to show that $P[\{a\}] = 0$, for all $a \in (0, 1)$.

$$P((a,b)) = b - a \quad (*)$$

Show: $P(\{a\}) = 0 \;\; \forall a \in (0,1)$

Using: $\lim_{n\to\infty} P(C_n) = P(\lim_{n\to\infty} C_n) = P\left(\bigcap_{n=1}^{\infty} C_n\right)$



Define b as $a+\frac{c}{n}$

$(a, \frac{c}{n})$

(*)

Now: $\lim_{n\to\infty} P((a,b)) = \lim_{n\to\infty} (\overbrace{a+\frac{c}{n}}^{b}) - a$

$\Rightarrow \lim_{n\to\infty} a - a + \frac{c}{n} \underset{n\to\infty}{=} 0$

Meaning, if b approaches a (n gets larger)

it will eventually (when $n\to\infty$) become

0.

This proves: $P(\{a\}) = 0$ $\square$//

1.3.22. Consider the events C_1, C_2, C_3.

(a) Suppose C_1, C_2, C_3 are mutually exclusive events. If $P(C_i) = p_i$, $i = 1, 2, 3$, what is the restriction on the sum $p_1 + p_2 + p_3$?

$P(C_1) = p_1$
$P(C_2) = p_2 \Rightarrow p_1 + p_2 + p_3 \leq 1$
$P(C_3) = p_3$

(b) In the notation of part (a), if $p_1 = 4/10$, $p_2 = 3/10$, and $p_3 = 5/10$, are C_1, C_2, C_3 mutually exclusive?

No, $\frac{4+3+5}{10} = \frac{12}{10} > 1$ ↯

For the last two exercises it is assumed that the reader is familiar with $\underbrace{\sigma}_{\sigma\text{-Algebra}}$-fields.

1.3.23. Suppose $\mathcal{D}$ is a nonempty collection of subsets of $\mathcal{C}$. Consider the collection of events

$$\mathcal{B} = \cap\{\mathcal{E} : \mathcal{D} \subset \mathcal{E} \text{ and } \mathcal{E} \text{ is a } \sigma\text{-field}\}.$$

Note that $\phi \in \mathcal{B}$ because it is in each σ-field, and, hence, in particular, it is in each σ-field $\mathcal{E} \supset \mathcal{D}$. Continue in this way to show that $\mathcal{B}$ is a σ-field.

1 $\emptyset \in \mathcal{B}$

By definition.

2 $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$

Let $A \in \mathcal{B} \Rightarrow \forall A : A \subset \mathcal{E}$
$\Rightarrow \forall A^c : A^c \subset \mathcal{E}$

$\Rightarrow A^c \subset \mathcal{E}$ (of every $\mathcal{E}$) $\Leftrightarrow A^c \in \mathcal{B}$

③ Let $A_1, \ldots \in \mathcal{B}$

$\Rightarrow A_i \in$ of every $\mathcal{E} \supset \mathcal{D}$

$\Rightarrow \forall \mathcal{E}: \bigcup_{i=1}^{\infty} A_i \in \mathcal{E}$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \subset \mathcal{E} \supset \mathcal{D} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$

①, ②, ③ $\Rightarrow \sigma$ $\square$ //

1.3.24. Let $\mathcal{C} = R$, where R is the set of all real numbers. Let $\mathcal{I}$ be the set of all open intervals in R. The Borel σ-field on the real line is given by

$$\mathcal{B}_0 = \cap\{\mathcal{E} : \mathcal{I} \subset \mathcal{E} \text{ and } \mathcal{E} \text{ is a } \sigma\text{-field}\}.$$

By definition, $\mathcal{B}_0$ contains the open intervals. Because $[a, \infty) = (-\infty, a)^c$ and $\mathcal{B}_0$ is closed under complements, it contains all intervals of the form $[a, \infty)$, for $a \in R$. Continue in this way and show that $\mathcal{B}_0$ contains all the closed and half-open intervals of real numbers.

① $[a, \infty)$

closed under complement

$\Rightarrow \underbrace{(-\infty, a)}_{\in \mathcal{B}_0 \text{ (open)}}{}^c = [a, \infty) \Rightarrow [a, \infty) \in \mathcal{B}_0$

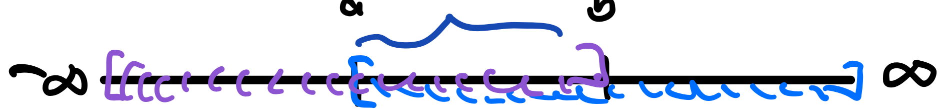
② $(-\infty, b]$

closed under comp.

$\Rightarrow (b, \infty)^c \in \mathcal{B}_0 \Rightarrow (-\infty, b] \in \mathcal{B}_0$

$[a, b]$

③ $[a,b]$



$\Rightarrow [a,b] = \underbrace{[a,\infty)}_{\in \mathcal{B}_0} \cap \underbrace{(-\infty,b]}_{\in \mathcal{B}_0}$

$\Rightarrow [a,b] \in \mathcal{B}_0$ $\leftarrow$ closed under finite intersections

④ $[a,b)$

$\Rightarrow [a,b) = \overbrace{[a,\infty)}^{\in \mathcal{B}_0} \cap \overbrace{(-\infty,b)}^{\in \mathcal{B}_0}$

$\Rightarrow [a,b) \in \mathcal{B}_0$

⑤ $(a,b]$

$\Rightarrow (a,b] = \overbrace{(a,\infty)}^{\in \mathcal{B}_0} \cap \overbrace{(-\infty,b]}^{\in \mathcal{B}_0}$

$\Rightarrow (a,b] \in \mathcal{B}_0$

①, ②, ③, ④, ⑤ $\Rightarrow \mathcal{B}_0$ contains mentioned intervals $\square$