Exercises

1.2.1. Find the union $C_1 \cup C_2$ and the intersection $C_1 \cap C_2$ of the two sets C_1 and C_2, where

(a) $C_1 = \{0, 1, 2, \}$, $C_2 = \{2, 3, 4\}$.

(b) $C_1 = \{x : 0 < x < 2\}$, $C_2 = \{x : 1 \leq x < 3\}$.

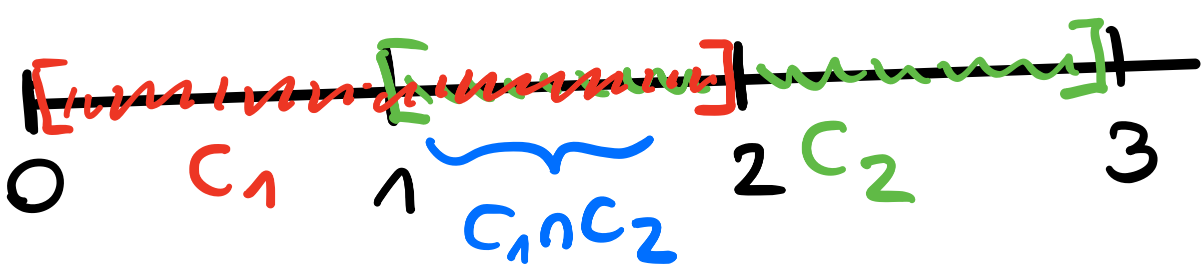
(c) $C_1 = \{(x, y) : 0 < x < 2, 1 < y < 2\}$, $C_2 = \{(x, y) : 1 < x < 3, 1 < y < 3\}$.

a) $C_1 \cup C_2 = \{0, 1, 2, 3, 4\}$

$C_1 \cap C_2 = \{2\}$

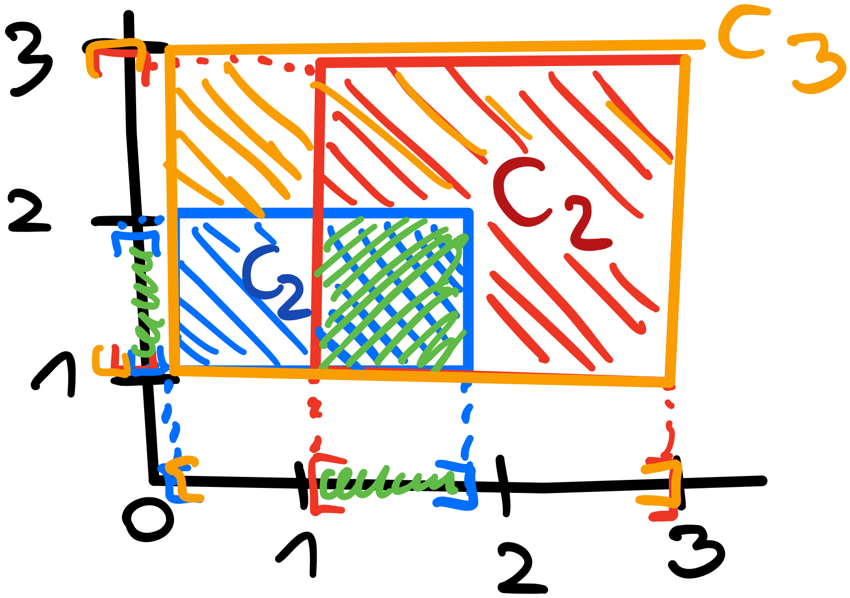
b) $C_1 \cup C_2 = \{x \mid 0 < x < 3\}$

$C_1 \cap C_2 = \{1 \leq x < 2\}$



(c) $C_1 = \{(x, y) : 0 < x < 2, 1 < y < 2\}$, $C_2 = \{(x, y) : 1 < x < 3, 1 < y < 3\}$.

c)



$c_1 \cup c_2 = \{(x,y) \mid 0 < x < 3, 1 < y < 3\}$

$c_1 \cap c_2 = \{(x,y) \mid 1 < x < 2, 1 < y < 2\}$

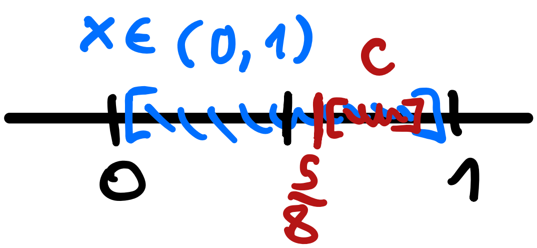
1.2.2. Find the complement C^c of the set C with respect to the space $\mathcal{C}$ if

(a) $\mathcal{C} = \{x : 0 < x < 1\}$, $C = \{x : \frac{5}{8} < x < 1\}$.

(b) $\mathcal{C} = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$, $C = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$.

(c) $\mathcal{C} = \{(x, y) : |x| + |y| \leq 2\}$, $C = \{(x, y) : x^2 + y^2 < 2\}$.

a) Ω:



$C^c = \{x \leq \frac{5}{8} \}$

b) $C^c = \{(x,y,z) \mid x^2+y^2+z^2 < 1\}$

c) $\Omega = \{(x,y) \mid |x|+|y| \leq 2\}$

HELP!

1.2.3. List all possible arrangements of the four letters $m, a, r,$ and y. Let C_1 be the collection of the arrangements in which y is in the last position. Let C_2 be the collection of the arrangements in which m is in the first position. Find the union and the intersection of C_1 and C_2.

$\Omega = \{m, a, r, y\}$

Perm. von $\{a,r,y\}$ = ary rya yar
ayr ray yra

Perm. von $\{m,r,y\}$ = mry rmy ymr
myr rym yrm

Perm. von $\{m,a,y\}$ = may amy yma
mya aym yam

Perm. von $\{mar\}$ = mar arm ram
mra amr rma

This plus mξ, aξ, rξ, yξ gives all 24!

$\sigma(\Omega) = \{\{m,a,r,y\}, \{m,a,y,r\},$
$\{m,r,y,a\}, \{m,r,a,y\}, \{m,y,a,r\}, \{m,y,r,a\},$
$\{a,m,r,y\}, \{a,m,y,r\}, \{a,r,m,y\}, \{a,r,y,m\},$
$\{a,y,m,r\}, \{a,y,r,m\}, \{r,m,a,y\}, \{r,m,y,a\},$
$\{r,a,m,y\}, \{r,a,y,m\}, \{r,y,m,a\}, \{r,y,a,m\},$
$\{y,m,a,r\}, \{y,m,r,a\}, \{y,a,r,m\}, \{y,a,m,r\},$
$\{y,r,a,m\}, \{y,r,m,a\}\}$

$\rightarrow$ Der Rest ist Fleißarbeit aber easy!

1.2.4. Concerning DeMorgan's Laws (1.2.6) and (1.2.7):

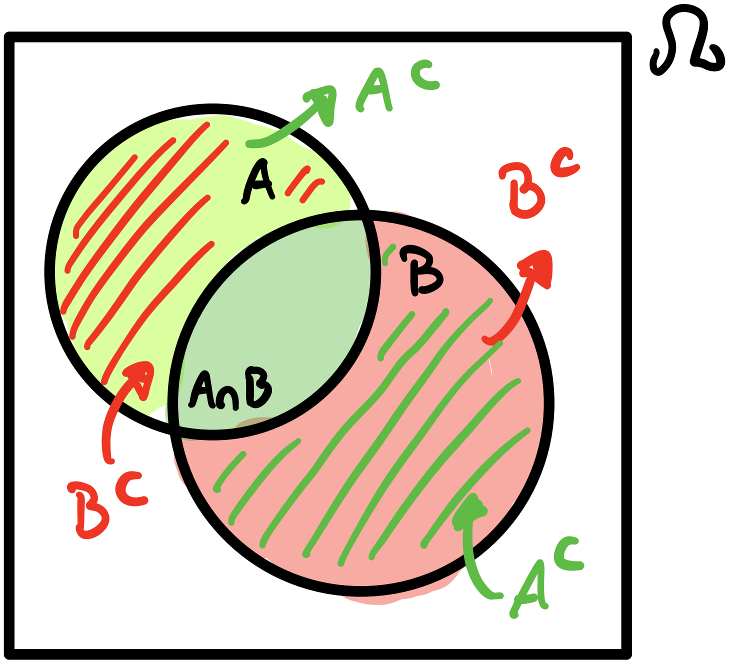
(a) Use Venn diagrams to verify the laws.

(b) Show that the laws are true.

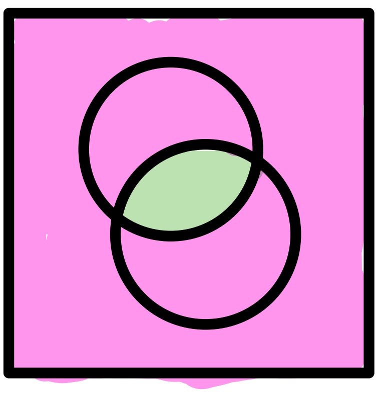
(c) Generalize the laws to countable unions and intersections.

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c.$$

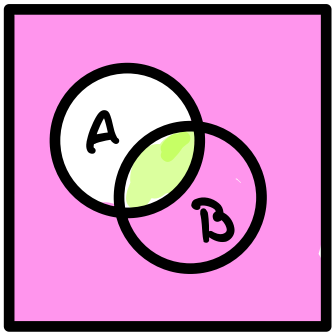
a)



$(A \cap B)^c =$



$A^c \cup B^c =$



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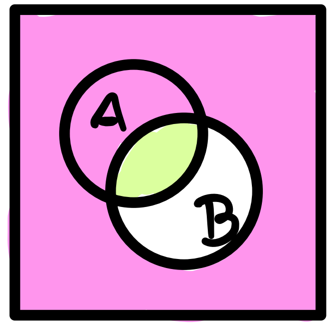
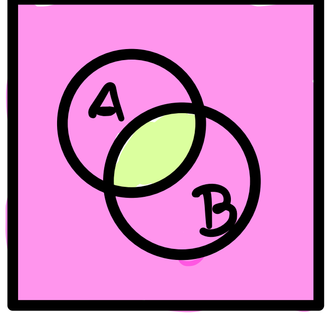


$=$ 

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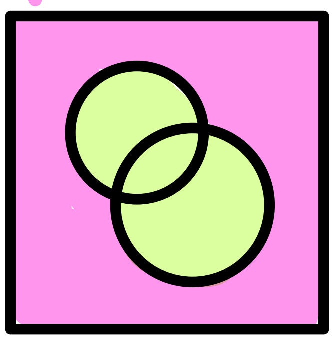
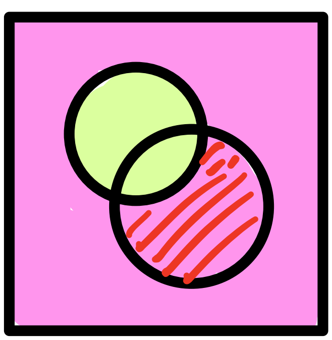
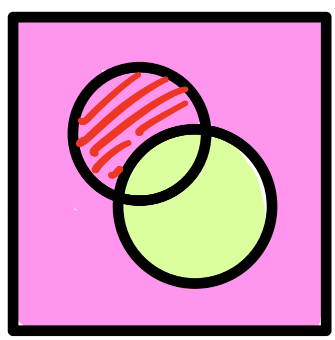
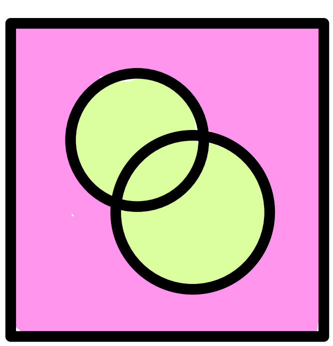
$(A \cup B)^c =$ 

$A^c \cap B^c =$  $+$ 

// : disappears in "$\cap$"

leaving us with: 

$\square$ //

$$(A \cap B)^c = A^c \cup B^c \quad ①$$
$$(A \cup B)^c = A^c \cap B^c. \quad ②$$

Proof:

① Let $x \in (A \cap B)^c$, thus $x \notin A \cap B$

$\Leftrightarrow \neg(x \in A \cap B)$

$\Leftrightarrow \neg(x \in A \wedge x \in B)$

$\Leftrightarrow x \in \neg A \vee x \in \neg B$

$\Leftrightarrow x \in A^c \vee x \in B^c$

$\Leftrightarrow x \in A^c \cup B^c \quad \square //$

② Let $x \in (A \cup B)^c$, thus $x \notin A \cup B$

$\Leftrightarrow \neg(x \in A \cup B)$

$\Leftrightarrow \neg(x \in A \cup x \in B)$

$\Leftrightarrow x \in \neg A \wedge x \in \neg B$

$\Leftrightarrow x \notin A \vee x \notin B$

$\Leftrightarrow x \in A^c \vee x \in B^c$

$\Leftrightarrow x \in A^c \cup B^c$ $\square$ //

c)

(Proof similar to ①)

① $\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$ ② $\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$

Proof:

Let $x \in \left(\bigcap_{i=1}^{n} A_i\right)^c$, thus $x \notin \bigcap_{i=1}^{n} A_i$

$\Leftrightarrow \neg \left(x \in \bigcap_{i=1}^{n} A_i\right)$

$\Leftrightarrow \neg (x \in A_1 \wedge \ldots \wedge x \in A_n)$

$\Leftrightarrow x \notin A_1 \vee \ldots \vee x \notin A_n$

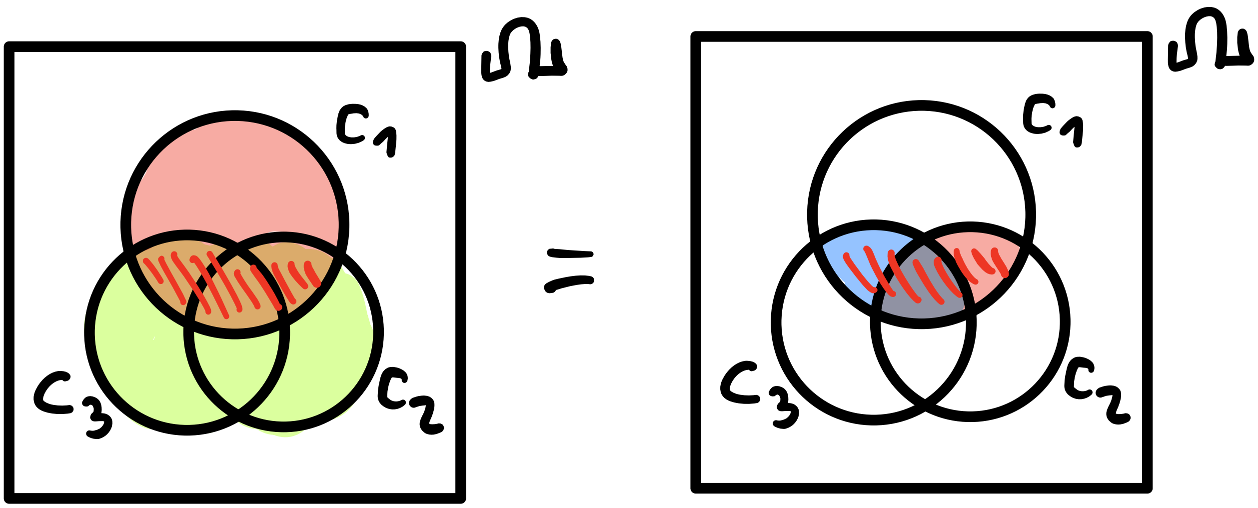
$\Leftrightarrow x \in A_1^c \vee \ldots \vee x \in A_n^c$

$\Leftrightarrow x \in (A_1^c \vee \ldots \vee A_n^c) \Leftrightarrow x \in \bigcup_{i=1}^{n} A_i^c$ $\square$ //

1.2.5. By the use of Venn diagrams, in which the space $\mathcal{C}$ is the set of points enclosed by a rectangle containing the circles C_1, C_2, and C_3, compare the following sets. These laws are called the **distributive laws**.

(a) $C_1 \cap (C_2 \cup C_3)$ and $(C_1 \cap C_2) \cup (C_1 \cap C_3)$.

(b) $C_1 \cup (C_2 \cap C_3)$ and $(C_1 \cup C_2) \cap (C_1 \cup C_3)$.



$C_2 \cup C_3$, C_1 $\qquad$ $C_1 \cap C_2$, $C_2 \cap C_3$

b) prove similarly!

DISTRIBUTIVE LAWS

1.2.6. Show that the following sequences of sets, $\{C_k\}$, are nondecreasing, (1.2.16), then find $\lim_{k\to\infty} C_k$.

(a) $C_k = \{x : 1/k \le x \le 3 - 1/k\}$, $k = 1, 2, 3, \ldots$.

(b) $C_k = \{(x, y) : 1/k \le x^2 + y^2 \le 4 - 1/k\}$, $k = 1, 2, 3, \ldots$.

a) $C_k = \{x \mid \frac{1}{k} \leq x \leq 3 - \frac{1}{k}\}, k \in \mathbb{N}$

$C_1 = \{x \mid 1 \leq x \leq 2\}$

z.z

$\hookrightarrow C_1 \subset C_2 \subset \ldots \subseteq C_n \subset \ldots$

$C_2 = \{x \mid \frac{1}{2} \leq x \leq \frac{5}{2}\}$

$\vdots$

$C_n = \{x \mid \frac{1}{n} \leq x \leq 3 - \frac{1}{n}\}$

$C_{n+1} = \{x \mid \frac{1}{n+1} \leq x \leq 3 - \frac{1}{n+1}\}$

z.z: Intervallänge (Abstand)/(Betrag)

von $C_{n+1} > C_n$

$\rightarrow \left|3 - \underbrace{\frac{1}{n} - \frac{1}{n}}\right| < \left|3 - \underbrace{\frac{1}{n+1} - \frac{1}{n+1}}\right|$

$\Leftrightarrow \left|3 - \frac{2}{n}\right| < \left|3 - \frac{2}{n+1}\right|$, da $n \in \mathbb{N}$

$\Rightarrow 3 - \frac{2}{n} < 3 - \frac{2}{n+1} \quad |-3$

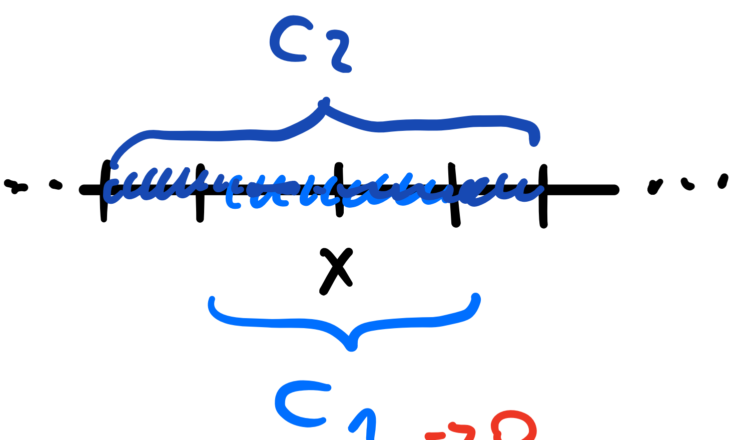
$\Leftrightarrow -\frac{2}{n} < -\frac{2}{n+1} \overset{\cdot(-1)}{\Leftrightarrow} \frac{2}{n} > \frac{2}{n+1} \quad |\cdot(n+1) \quad |:2$

$\Leftrightarrow \frac{n+1}{n} > 1 \quad | \cdot n$

$\Leftrightarrow n+1 > n \quad | -n$

$\Leftrightarrow 1 > 0 \Rightarrow$ nondecreasing $\square$ //

Thus, every $x \in C_1$ will be element of C_2,
since it expands from x further than C_1.



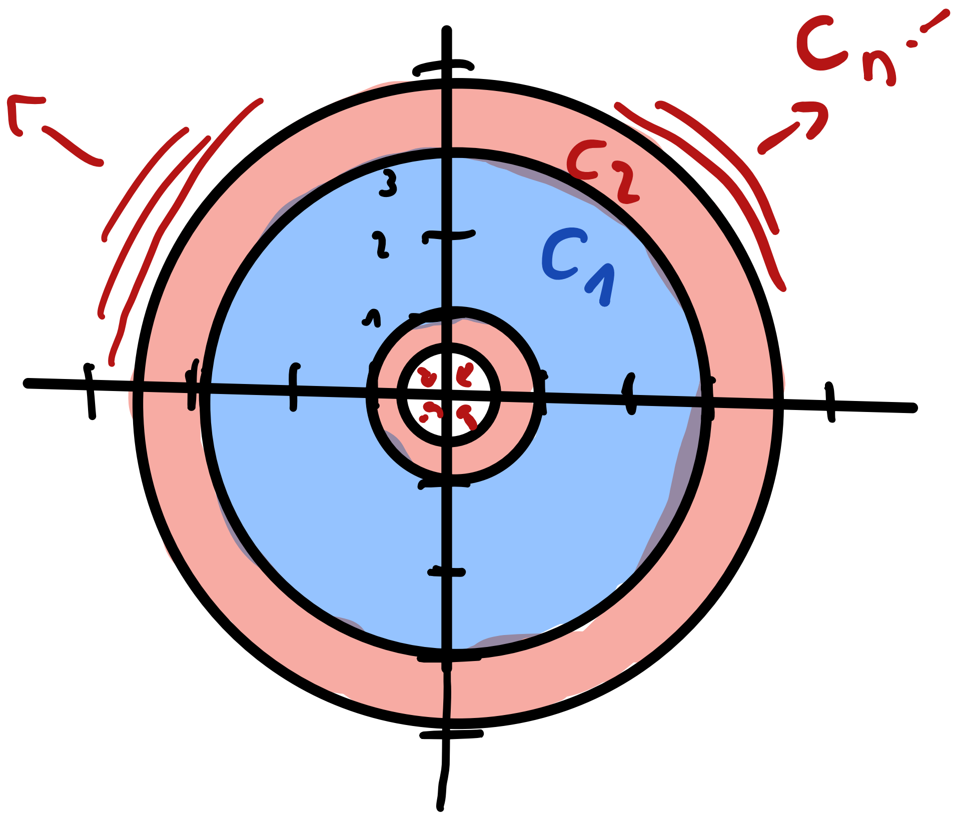
$$\lim_{k\to\infty} C_k := \left\{x \;\middle|\; \overbrace{\lim_{k\to\infty} \frac{1}{k}}^{\Rightarrow 0} \le x \le \lim \overbrace{3-\frac{1}{k}}^{\Rightarrow 3} \right\}$$

$$= \{x \mid 0 \le x \le 3\}$$ //

(b) $C_k = \{(x,y) : 1/k \le x^2 + y^2 \le 4 - 1/k\}$, $k = 1, 2, 3, \ldots$.

$C_1 = \{(x,y) \mid 1 \le x^2 + y^2 \le 3\}$

$C_2 = \{(x,y) \mid \frac{1}{2} \le x^2 + y^2 \le \frac{7}{2}\}$ ...



$\rightarrow C_n = \{(x,y) \mid \frac{1}{n} \le x^2 + y^2 \le 4 - \frac{1}{n}\}$

z.z $\quad x \in C_n \Leftrightarrow x \in C_{n+1}$

Induction $(a = x^2+y^2)$

Start: Let $a \in C_1$. This set is defined on $[1, 3]$. C_2 is defined on $[\frac{1}{2}, \frac{7}{2}]$.

Thus $a \in C_2$. ✓

Hypothesis

Suppose for any set, that its surpassing set contains it.

Step: $n \to n+1$

Let $a \in C_n$, thus $\overbrace{x^2+y^2}^{a} \in [\frac{1}{n}, 4-\frac{1}{n}]$

Applying the strategy from a):

$$\left|4-\frac{1}{n}-\frac{1}{n}\right| > \left|4-\frac{1}{n+1}-\frac{1}{n+1}\right|$$

$$\Leftrightarrow \left|\underbrace{4-\frac{2}{n}}_{<4}\right| > \left|\underbrace{4-\frac{2}{n+1}}_{<4}\right|$$

$$\Leftrightarrow 4-\frac{2}{n} > 4-\frac{2}{n+1} \quad \Big| -4 \quad \Big| \cdot \frac{n+1}{2}$$

$$\frac{n+1}{n} > 1 \quad \Big| \cdot n$$

$$n+1 > n \quad \Big| -n$$

$1 > 0 \Rightarrow$ nondecreasing $\square$

$$\lim_{k\to\infty} C_k := \left\{(x,y) \,\Big|\, \underbrace{\lim_{k\to\infty} \frac{1}{k}}_{\Rightarrow 0} \le x^2+y^2 \le \underbrace{\lim_{k\to\infty} 4-\frac{1}{k}}_{\Rightarrow 4}\right\}$$

thus, $x^2+y^2 \in [0,4]$.

So,

$$\lim_{k\to\infty} C_k = \left\{(x,y) \,\Big|\, 0 \le x^2+y^2 \le 4\right\}$$

1.2.7. Show that the following sequences of sets, $\{C_k\}$, are nonincreasing, (1.2.17), then find $\lim_{k\to\infty} C_k$.

(a) $C_k = \{x : 2 - 1/k < x \leq 2\}$, $k = 1, 2, 3, \ldots$.

(b) $C_k = \{x : 2 < x \leq 2 + 1/k\}$, $k = 1, 2, 3, \ldots$.

To show: $C_n \supset C_{n+1} \supset \ldots$

a) $C_1 = \{x \mid 1 < x \leq 2\}$

$C_2 = \{x \mid \frac{3}{2} < x \leq 2\}$ ...

n=1:

$\Leftrightarrow x \in (1, 2]$

$C_2 \Leftrightarrow (\frac{3}{2}, 2]$

It immediately follows, that since

$x \in C_2 \Rightarrow x \in C_1$ ✓

Hypothesis

Suppose C_k is nonincreasing for all $n \in \mathbb{N}$.

Step: $n \to n+1$

$C_{n+1} = \{x \mid 2 - \frac{1}{n+1} < x \leq 2\}$

$\Leftrightarrow \left(2-\frac{1}{n+1}, 2\right]$ AND

$C_n \Leftrightarrow \left(2-\frac{1}{n}, 2\right]$

Let $x \in C_{n+1}$.

$$2-\frac{1}{n+1} > 2-\frac{1}{n} \quad \Big| -2$$

$$-\frac{1}{n+1} > -\frac{1}{n} \quad \Big| \cdot(-1)$$

$$\frac{1}{n+1} < \frac{1}{n} \quad \Big| \cdot (n+1)$$

$$1 < \frac{n+1}{n} \quad \Big| \cdot n$$

$$n < n+1 \quad \Big| -n$$

$$0 < 1 \checkmark$$

Thus, if $x \in C_{n+1}$, $x \in C_n$. $\square$ //

Similarly for b) and c)

1.2.8. For every one-dimensional set C, define the function $Q(C) = \sum_C f(x)$, where $f(x) = (\frac{2}{3})(\frac{1}{3})^x$, $x = 0, 1, 2, \ldots$, zero elsewhere. If $C_1 = \{x : x = 0, 1, 2, 3\}$ and $C_2 = \{x : x = 0, 1, 2, \ldots\}$, find $Q(C_1)$ and $Q(C_2)$.
Hint: Recall that $S_n = a + ar + \cdots + ar^{n-1} = a(1 - r^n)/(1 - r)$ and, hence, it follows that $\lim_{n\to\infty} S_n = a/(1 - r)$ provided that $|r| < 1$.

$Q(C_1)$, $C_1 = \{x \mid x = 0, 1, 2, 3\}$

$$\Rightarrow Q(C_1) = \sum_{C_1} f(x) = \sum_{C_1} \left(\frac{2}{3}\right)\cdot\left(\frac{1}{3}\right)^x$$

$$= 2\sum_{C_1}\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^x = \frac{2}{3}\sum_{C_1}\left(\frac{1}{3}\right)^x, \quad x = 0, 1, 2, \ldots$$

0 1 2 3

$$= \frac{2}{3}\cdot\left(\underbrace{\left(\frac{1}{3}\right)^0}_{1} + \underbrace{\left(\frac{1}{3}\right)^1}_{1/3} + \underbrace{\left(\frac{1}{3}\right)^2}_{1/9} + \underbrace{\left(\frac{1}{3}\right)^3}_{1/27}\right)$$

$$= \frac{2}{3}\cdot\left(\underbrace{1}_{27} + \frac{9}{27} + \frac{3}{27} + \frac{1}{27}\right)$$

$27 + 9 + 3 + 1$

$$= \frac{2}{3}\cdot\frac{40}{27} = \frac{80}{81} \quad //$$

$$\Rightarrow Q(C_2),\ C_2 = \mathbb{N}_0$$

geom. Series

$$\to \frac{2}{3}\sum_{C_2}\left(\frac{1}{3}\right)^x = \underbrace{\frac{2}{3}\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k}$$

$$= Q(C_2) = \frac{2}{3}\lim_{k\to\infty}\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k$$

$$= \frac{2}{3}\cdot\left(\frac{1}{1-\frac{1}{3}}\right)$$

$$= \frac{2}{3}\cdot\left(\frac{1}{\frac{2}{3}}\right)$$

$$= \frac{2}{3}\cdot\frac{3}{2}$$

$$= 1 \mathbin{/\!/}$$

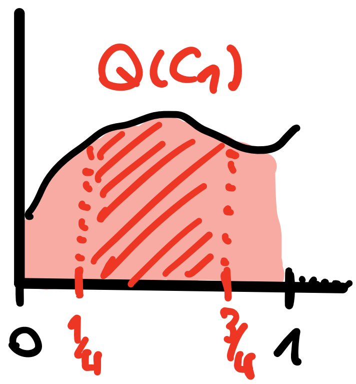
1.2.9. For every one-dimensional set C for which the integral exists, let $Q(C) = \int_C f(x)\,dx$, where $f(x) = 6x(1-x)$, $0 < x < 1$, zero elsewhere; otherwise, let $Q(C)$ be undefined. If $C_1 = \{x : \frac{1}{4} < x < \frac{3}{4}\}$, $C_2 = \{\frac{1}{2}\}$, and $C_3 = \{x : 0 < x < 10\}$, find $Q(C_1), Q(C_2)$, and $Q(C_3)$.

$$Q(C) = \int_C f(x)\,dx$$

$$f(x) = 6x(1-x),\ 0 < x < 1$$

$$= -6x^2 + 6x$$

$$C_1 = \{x \mid \tfrac{1}{4} < x < \tfrac{3}{4}\} \rightarrow$$



$$\rightarrow \int_{1/4}^{3/4} -6x^2 + 6x \, dx$$

$$= -2x^3 + 3x^2 \Big|_{1/4}^{3/4}$$

$$= \left(-2\cdot\frac{\overset{27}{3^3}}{\underset{64}{4^3}} + 3\cdot\frac{\overset{9}{3^2}}{\underset{16}{4^2}}\right) - \left(-2\cdot\frac{1^3}{\underset{64}{4^3}} + 3\cdot\frac{1^2}{\underset{16}{4^2}}\right)$$

$$= \left(-2\cdot\frac{27}{64} + 3\cdot\frac{9}{16}\right) - \left(-2\cdot\frac{1}{64} + 3\cdot\frac{1}{16}\right)$$

$$= -\frac{54}{64} + \frac{27}{16} + \frac{2}{64} - \frac{3}{16}$$

$$= -\frac{52}{64} + \frac{24}{16}$$

$$= -\frac{52}{64} + \frac{96}{64}$$

$$= \frac{44}{64}$$

$$= \frac{11}{16} //$$

$\rightarrow C_2 = \{\frac{1}{2}\}$

$\int_{1/2}^{1/2} f(x)dx \rightarrow \int_a^a f(x)dx = 0$ per definition //

$\rightarrow C_3 = \{x \mid 0 < x < 1\}$

Since $f(x) \Big|_0^1$, we only need :

$$\int_0^1 -6x^2+6x\,dx$$

$$= -2x^3+3x^2 \Big|_0^1$$

$$= (-2\cdot 1+3\cdot 1)-\underbrace{(-2\cdot 0+3\cdot 0)}_{=0}$$

$$= -2+3 = 1$$

1.2.10. For every two-dimensional set C contained in R^2 for which the integral exists, let $Q(C) = \int\int_C (x^2+y^2)\,dxdy$. If $C_1 = \{(x,y): -1 \le x \le 1, -1 \le y \le 1\}$, $C_2 = \{(x,y): -1 \le x = y \le 1\}$, and $C_3 = \{(x,y): x^2+y^2 \le 1\}$, find $Q(C_1), Q(C_2)$, and $Q(C_3)$.

$$Q(C_1) := \int_{-1}^{1}\int_{-1}^{1} x^2+y^2\,dy\,dx$$

$$= \int_{-1}^{1} \frac{1}{3}y^3 + x^2y \Big|_{-1}^{1} dx$$

$$= \int_{-1}^{1} \left(\left(\frac{1}{3}+x^2\right)-\left(-\frac{1}{3}-x^2\right)\right) dx$$

$$= \int_{-1}^{1} \frac{1}{3}+x^2+\frac{1}{3}+x^2\,dx$$

$$= \int_{-1}^{1} 2x^2 + \frac{2}{3} \, dx$$

$$= \frac{2}{3}x^3 + \frac{2}{3}x \Big|_{-1}^{1}$$

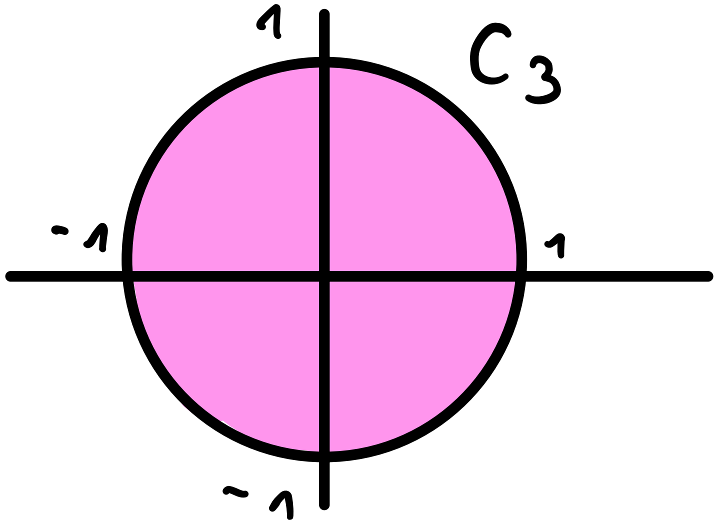
$$= \left(\frac{2}{3} + \frac{2}{3}\right) - \Big(\underbrace{-\frac{2}{3} - \frac{2}{3}}_{-\frac{4}{3}}\Big)$$

$$= \frac{4}{3} + \frac{4}{3}$$

$$= \frac{8}{3} \;/\!/ \qquad \big(Q(C_2) = Q(C_1), \text{ da } C_1 = C_2\big)$$

$$Q(C_3), \quad C_3 = \{(x,y) \mid x^2 + y^2 \leq 1\}$$

$$\iint_{C_3} x^2 + y^2 \, dy \, dx$$



$\rightarrow r = 1, \quad r^2 \pi = \pi$ //

HELP NEEDED!

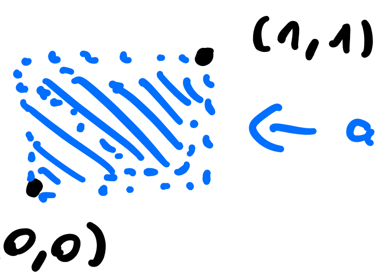
NOT SURE IF THIS IS THE RIGHT APPROACH

1.2.11. Let $\mathcal{C}$ denote the set of points that are interior to, or on the boundary of, a square with opposite vertices at the points $(0,0)$ and $(1,1)$. Let $Q(C) = \int\int_C dy\,dx$.

(a) If $C \subset \mathcal{C}$ is the set $\{(x,y) : 0 < x < y < 1\}$, compute $Q(C)$.

(b) If $C \subset \mathcal{C}$ is the set $\{(x,y) : 0 < x = y < 1\}$, compute $Q(C)$.

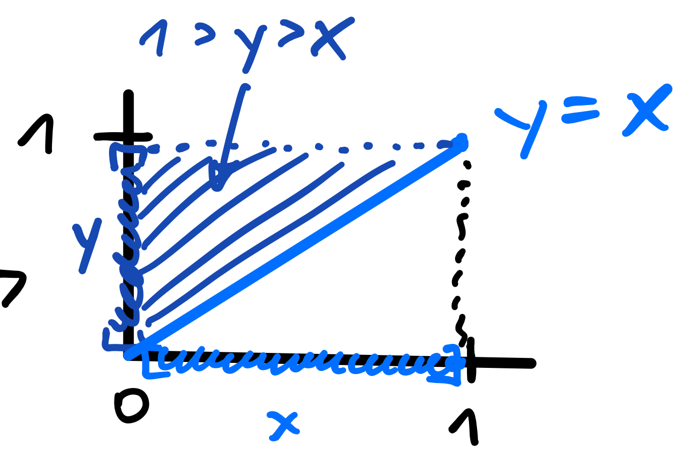
(c) If $C \subset \mathcal{C}$ is the set $\{(x,y) : 0 < x/2 \leq y \leq 3x/2 < 1\}$, compute $Q(C)$.

$\Omega =$ 

$\leftarrow$ all points $\in \Omega$

$$Q(C) = \iint_C dy\,dx$$

a) $0 < x < y < 1 \Rightarrow$



$$Q(C_1) = \frac{1}{2} \iint_{\Omega} dy\,dx$$

$$= \frac{1}{2} \int_0^1 \int_0^1 dy\,dx$$

$$= \frac{1}{2} \int_0^1 y \Big|_0^1 dx \Rightarrow (1-0) = 1$$

$$= \frac{1}{2} \int_0^1 1\,dx$$

$$= \frac{1}{2} \cdot x \Big|_0^1$$

$$= \frac{1}{2} (1-0)$$

$$= \frac{1}{2}$$

b) $x = y$

$0 < x, y < 1$

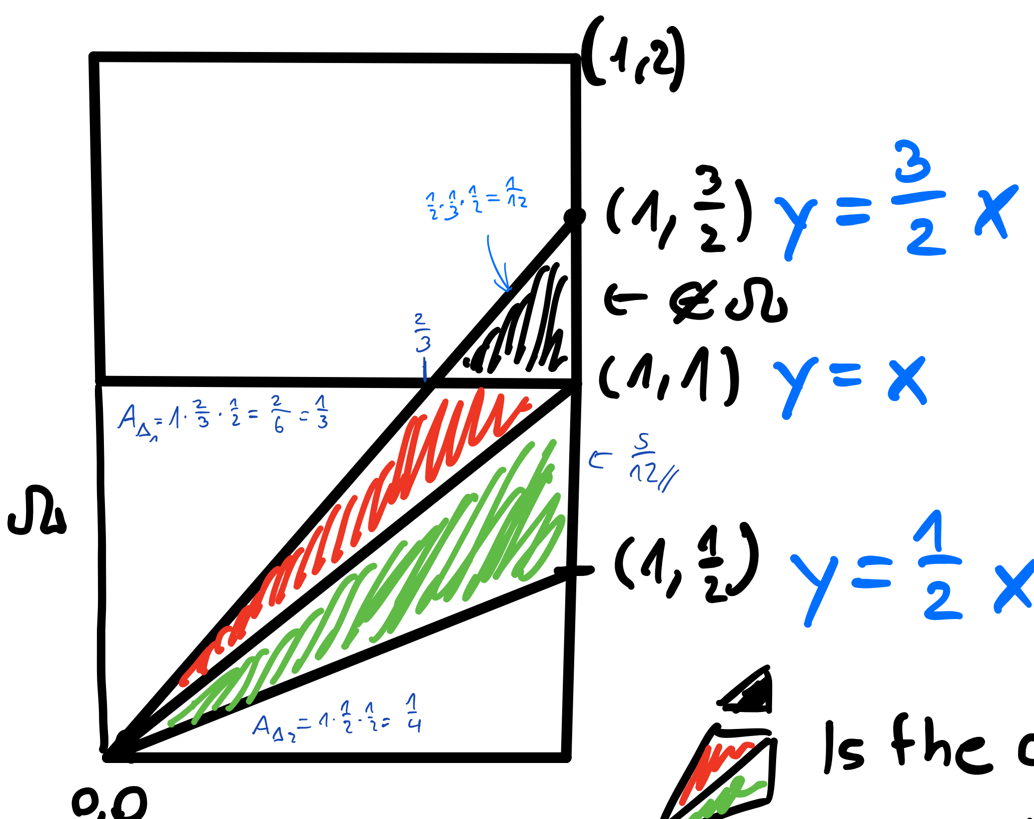
$$\Rightarrow Q(C_2) = \int_0^1 \int_0^1 dy\,dx$$

$$= \int_0^1 y \Big|_0^1 dx$$

$$= \int_0^1 dx = x \Big|_0^1 = 1$$

(c) If $C \subset \mathcal{C}$ is the set $\{(x, y) : 0 < x/2 \le y \le 3x/2 < 1\}$, compute $Q(C)$.

$$0 < \frac{x}{2} \le y \le \frac{3x}{2} < 1$$



Is the area we are integrating over!

$$1 - (A_{\Delta_1} + A_{\Delta_2}) = 1 - (\frac{1}{3} + \frac{1}{4})$$
$$= 1 - (\frac{4}{12} + \frac{3}{12})$$
$$= 1 - \frac{7}{12}$$
$$= \frac{5}{12}$$

CHECK IF THE INT BORDERS

ARE CORR

B ARE CORRECT!

$$\iint_{C_3} dy\,dx = \int_0^1 \int_{x/2}^{3x/2} dy\,dx - \blacktriangle$$

$$= \int_0^1 y \Big|_{\frac{x}{2}}^{\frac{3x}{2}} dx$$

$$= \int_0^1 \frac{3x}{2} - \frac{x}{2}\, dx$$

$$= \frac{1}{2} \int_0^1 2x$$

$$= \frac{1}{2} \cdot x^2 \Big|_0^1$$

$$= \frac{1}{2} \cdot (1-0)$$

$$= \frac{1}{2} - \blacktriangle$$

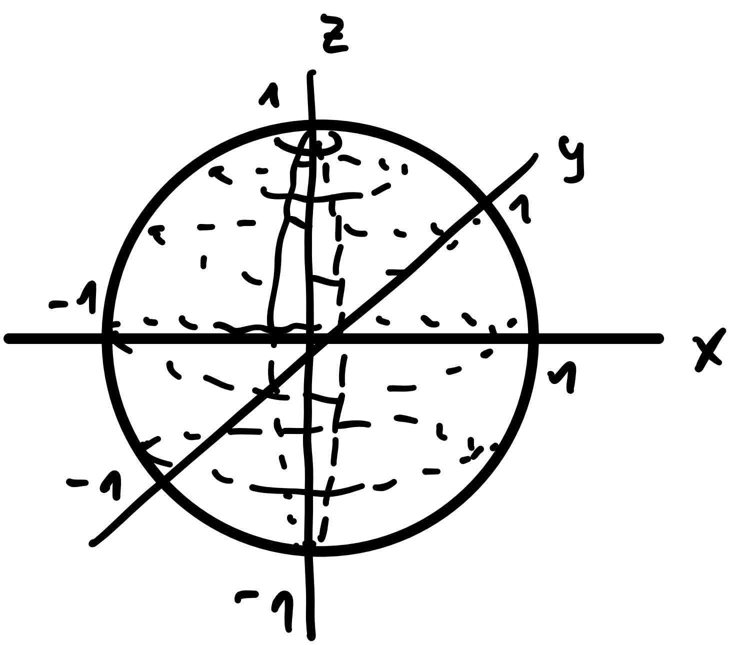
$$\blacktriangle = \frac{1}{2} \cdot \left(\frac{1}{3} \cdot \frac{1}{2}\right) = \frac{1}{12}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{12} = \frac{6}{12} - \frac{1}{12} = \frac{5}{12}$$

Since x goes from $0 < x < 1$ and $y \leq \frac{3x}{2}$, at some point $(\frac{2}{3}, 1)$, we leave Ω by its definition, thus when integrating we need to subtract the area of ◢.

1.2.13. Let C denote the set $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$. Using spherical coordinates, evaluate

$$Q(C) = \int\int\int_C \sqrt{x^2 + y^2 + z^2}\, dxdydz.$$



Spherical coordinates give: radius $= \rho$

$x = \rho \cos\theta \sin\phi$

$y = \rho \sin\theta \sin\phi$

$z = \rho \cos\phi$

$$\sqrt{x^2 + y^2 + z^2}$$

$$\Leftrightarrow \sqrt{p^2\cos^2\theta\sin^2\phi + p^2\sin^2\theta\sin^2\phi + p^2\cos^2\phi}$$

$$\Leftrightarrow \sqrt{p^2(\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi}$$

$$\Leftrightarrow p\sqrt{\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi}$$

$$\Leftrightarrow p\sqrt{\sin^2\phi(\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) + \cos^2\phi}$$

$$\Leftrightarrow p\sqrt{\underbrace{\sin^2\phi + \cos^2\phi}_{=1}}$$

$$\Leftrightarrow p\sqrt{1}$$

$= p$, since $p=1$, we obtain

$$\iiint 1 \, dV$$

$$\Leftrightarrow \int_0^1 \int_0^{2\pi} \int_0^{\pi} p^2 \sin(\phi) \, d\phi \, d\theta \, dp$$

$$\Leftrightarrow \int_0^1 \int_0^{2\pi} p^2 \cdot (-\cos(\phi)) \Big|_0^{\pi} \, d\theta \, dp$$

$$\Leftrightarrow \int_0^1 \int_0^{2\pi} p^2 \cdot \Big(\overbrace{-\overbrace{\cos\pi}^{-1}}^{1} + \overbrace{\cos 0}^{1}\Big) \, d\theta \, dp$$

$$\Leftrightarrow \int_0^1 \int_0^{2\pi} 2p^2 \, d\theta \, dp$$

$$\Leftrightarrow \int_0^1 2p^2 \int_0^{2\pi} d\theta \, dp$$

$$\Leftrightarrow \int_0^1 2p^2 \cdot \theta \Big|_0^{2\pi} \, dp$$

$$\Leftrightarrow \int_0^1 2p^2(2\pi - 0)\,dp$$

$$\Leftrightarrow \int_0^1 4\pi p^2\,dp$$

$$\Leftrightarrow 4\pi \int_0^1 p^2\,dp$$

$$\Leftrightarrow 4\pi \cdot \left(\frac{1}{3}p^3\right)\Big|_0^1$$

$$\Leftrightarrow 4\pi \cdot \frac{1}{3}$$

$$\Leftrightarrow \frac{4\pi}{3}$$

1.2.14. To join a certain club, a person must be either a statistician or a mathematician or both. Of the 25 members in this club, 19 are statisticians and 16 are mathematicians. How many persons in the club are both a statistician and a mathematician?

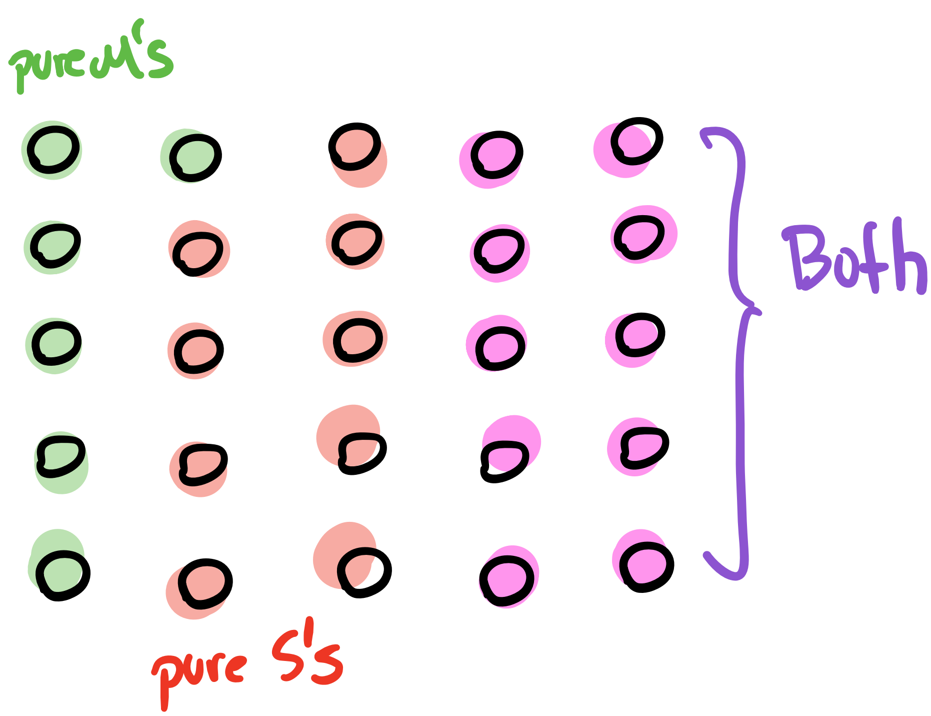
Stat. = S

Math. = M $\rightarrow$ To join: $S \vee M \vee (S \wedge M)$

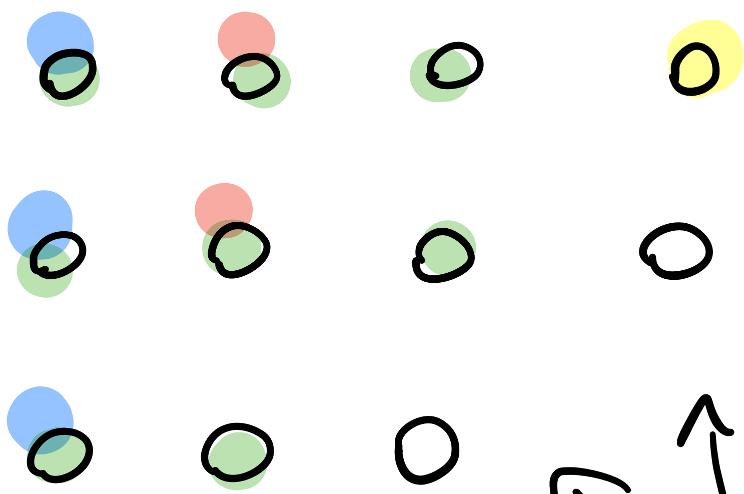
$S = \frac{19}{25}$, $M = \frac{16}{25}$

since 19 are S's, 6 people are <u>NO</u> S's but M's, pure M's

Since 16 are M's, 9 are S's but <u>NO</u> M's pure S's



1.2.15. After a hard-fought football game, it was reported that, of the 11 starting players, 8 hurt a hip, 6 hurt an arm, 5 hurt a knee, 3 hurt both a hip and an arm, 2 hurt both a hip and a knee, 1 hurt both an arm and a knee, and no one hurt all three. Comment on the accuracy of the report.



2 arms to be Filled

2 knees to be Filled

The report cant be occurate sinc 2 arms and 2 knees remain, but only two slots are available, otherwise catgories would OVERLAP!