1.4.1. If $P(A_1) > 0$ and if $A_2, A_3, A_4, \ldots$ are mutually disjoint sets, show that

$$P(A_2 \cup A_3 \cup \cdots | A_1) = P(A_2|A_1) + P(A_3|A_1) + \cdots .$$

$A_2, \ldots$ mutually disjoint

$\Leftrightarrow P(A_2 \cup .. \cup ..) = P(A_2) + \ldots$

Thus:

$P(A_2 \cup .. \cup .. | A_1)$

$= P(A_2|A_1) + P(A_3|A_1) + \ldots \quad \square //$

1.4.2. Assume that $P(A_1 \cap A_2 \cap A_3) > 0$. Prove that

(*)

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)P(A_4|A_1 \cap A_2 \cap A_3).$$

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)}$$

$$P(A_3|A_1 \cap A_2) = \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)}$$

$$P(A_4|A_1 \cap A_2 \cap A_3) = \frac{P(A_1 \cap A_2 \cap A_3 \cap A_4)}{P(A_1 \cap A_2 \cap A_3)}$$

$$(*) \Rightarrow \cancel{P(A_1)} \cdot \frac{P(A_1 \cap \cancel{A_2})}{\cancel{P(A_1)}} \cdot \frac{P(A_1 \cap \cancel{A_2} \cap A_3)}{P(\cancel{A_1} \cap A_2)} \cdot \frac{P(A_1 \cap A_2 \cap A_3 \cap A_4)}{P(A_1 \cancel{\cap A_2} \cap A_3)}$$

$$\Leftrightarrow P(A_1 \cap A_2 \cap A_3 \cap A_4) \quad \square \; //$$

1.4.3. Suppose we are playing draw poker. We are dealt (from a well-shuffled deck) five cards, which contain four spades and another card of a different suit. We decide to discard the card of a different suit and draw one card from the remaining cards to complete a flush in spades (all five cards spades). Determine the probability of completing the flush.

5 in hand → 47 left in deck

13 − 4 ♠ = 9 ♠ left in deck

Thus: $\frac{9}{47}$ //

1.4.4. From a well-shuffled deck of ordinary playing cards, four cards are turned over one at a time without replacement. What is the probability that the spades and red cards alternate?

♦, ♥ : $\frac{26}{52} = \frac{1}{2}$

♠ : $\frac{13}{52} = \frac{1}{4}$

$$P(A) = \frac{13}{52} \cdot \frac{26}{51} \cdot \frac{12}{50} \cdot \frac{25}{49}$$

$$= 0{,}0156 \approx 15{,}6\% \; //$$

1.4.5. A hand of 13 cards is to be dealt at random and without replacement from an ordinary deck of playing cards. Find the conditional probability that there are at least three kings in the hand given that the hand contains at least two kings.

A = at least two kings

$\frac{4}{52} = P(\text{King})$

$$\to P(\geq 3 \mid \geq 2) = \frac{P(3) + P(4)}{P(2) + P(3) + P(4)}$$

$$= \frac{\binom{4}{3}\binom{48}{10} + \binom{4}{4}\binom{48}{9}}{\binom{4}{2}\binom{48}{11} + \binom{4}{3}\binom{48}{10} + \binom{4}{4}\binom{48}{9}}$$

$$= \frac{913}{5359} \approx 17{,}04\,\%$$

1.4.6. A drawer contains eight different pairs of socks. If six socks are taken at random and without replacement, compute the probability that there is at least one matching pair among these six socks. *Hint:* Compute the probability that there is not a matching pair.

1.4.6 $\frac{111}{143}$.

At least one matching pair!

$\rightsquigarrow$ 16 Socks
8 pairs

Total: $\binom{16}{6} = \frac{16!}{6!\,10!} = 8008$

$P(\overline{A}) \rightarrow$ <u>no</u> matching pairs

=> There are 8 pairs and we're selecting 6 coming from a different pair each time.

Choose 6 pairs out of 8.

$\binom{8}{6} = \frac{8!}{2!\,6!} = \frac{8 \cdot 7}{2} = 28$

From each of these 6 pairs chose only 1 Sock!

$2^6 = 64$

Total outcomes:

$\binom{8}{6} \cdot 2^6 = 28 \cdot 64 = 1792$

$$\to \frac{1792^{:56}}{8008_{:56}} = \frac{32}{143} \to \text{no pairs}$$

$$P(\geq 1 \text{ Pair}) = 1 - \frac{32}{143} = \frac{111}{143}$$

1.4.7. A pair of dice is cast until either the sum of seven or eight appears.

(a) Show that the probability of a seven before an eight is 6/11.

(b) Next, this pair of dice is cast until a seven appears twice or until each of a six and eight has appeared at least once. Show that the probability of the six and eight occurring before two sevens is 0.546.

$\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$$P(\Sigma=7) = \frac{6}{36} = \frac{1}{6}$$

$$P(\Sigma=8) = \frac{5}{36}$$

$\{(2,6),(3,5),(4,4),(5,3),(6,2)\}$

$$\text{"7" before "8"} = \frac{6}{11}$$

$$P(7 \mid 7 \text{ or } 8) = \frac{P(7)}{P(7)+P(8)} = \frac{\frac{6}{36}}{\frac{6}{36}+\frac{5}{36}}$$

$$= \frac{6}{6+5}$$

$$= \frac{6}{11}$$

(since probs. are independent here)

(b) Next, this pair of dice is cast until a seven appears twice or until each of a six and eight has appeared at least once. Show that the probability of the six and eight occurring before two sevens is 0.546.

$P(7) = \frac{6}{36}$ $\boxed{⚅} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$

$P(8) = \frac{5}{36}$

$P(6) = \frac{5}{36}$

Three ways:

1

$7 \rightarrow$ keep rolling $\rightarrow 7$

$P(1) = \frac{6}{16} \cdot \frac{6}{16}$

6 ways to roll 7
5+5 ways to roll 6 v 8
6+5+5 = 16

2

$7 \rightarrow$ keep rolling $\rightarrow 6 \vee 8 \rightarrow$ keep rolling $\rightarrow 7$

$P(2) = \frac{6}{16} \cdot \underbrace{\frac{10}{16}}_{\frac{5+5}{16}} \cdot \underbrace{\frac{6}{11}}_{\frac{6}{16-5}}$ bec. 6 v8 already appeared once

3

$6 \vee 8 \rightarrow$ keep rolling $\rightarrow 7 \rightarrow$ keep rolling $\rightarrow 7$

$P(3) = \frac{10}{16} \cdot \frac{6}{11} \cdot \frac{6}{11}$

Here we calculated 7 appearing first,

thus:

$P(6 \cup 8 \text{ appear first}) = 1 - (P(1) + P(2) + P(3))$

$= 1 - \left(\frac{6}{16} \cdot \frac{6}{16} + \frac{6}{16} \cdot \frac{10}{16} \cdot \frac{6}{11} + \frac{10}{16} \cdot \frac{6}{11} \cdot \frac{6}{11} \right)$

$= 1 - \left(\frac{36}{256}_{\cdot 121}^{\cdot 121} + \frac{360}{2816}_{\cdot 11}^{\cdot 11} + \frac{360}{1936}_{\cdot 16}^{\cdot 16} \right)$

$= 1 - \left(\frac{4356}{30976} + \frac{3960}{30976} + \frac{5760}{30976} \right)$

$= 1 - \frac{14076}{30976}$

$\approx 0{,}45442$

$= 1 - 0{,}45442$

$= 0{,}5455 \approx 0{,}546$ //

1.4.8. In a certain factory, machines I, II, and III are all producing springs of the same length. Machines I, II, and III produce 1%, 4%, and 2% defective springs, respectively. Of the total production of springs in the factory, Machine I produces 30%, Machine II produces 25%, and Machine III produces 45%.

(a) If one spring is selected at random from the total springs produced in a given day, determine the probability that it is defective.

$M_1 = 0{,}01$ Total: $X_{M_1} = 0{,}3$

$M_2 = 0{,}04$ $X_{M_2} = 0{,}25$

$M_3 = 0{,}02$ $X_{M_3} = 0{,}45$

$P(\text{defective}) = P(d)$

a) Law of total probability

$$P(d) = P(d|M_1)P(M_1) + P(d|M_2)P(M_2) + P(d|M_3)P(M_3)$$

$$\Rightarrow 0{,}01 \cdot 0{,}3 + 0{,}04 \cdot 0{,}25 + 0{,}02 \cdot 0{,}45$$

$$= 0{,}022 = 2{,}2\,\%$$

(b) Given that the selected spring is defective, find the conditional probability that it was produced by Machine II.

$$P(d) = 0{,}022$$

$$P(M_2|d) = \frac{P(M_2 \cap d)}{P(d)} = \frac{P(d|M_2)P(M_2)}{P(d)}$$

$$\Rightarrow \frac{0{,}04 \cdot 0{,}25}{0{,}022} \approx 0{,}454$$

1.4.9. Bowl I contains six red chips and four blue chips. Five of these 10 chips are selected at random and without replacement and put in bowl II, which was originally empty. One chip is then drawn at random from bowl II. Given that this chip is blue, find the conditional probability that two red chips and three blue chips are transferred from bowl I to bowl II.

B = blue chip

A = 2r, 3b chosen

$$P(A) = \frac{\binom{6}{2} + \binom{4}{3}}{\binom{10}{5}} = \frac{\overbrace{\frac{6!}{2!4!}}^{\frac{\cancel{6}^{3}\cdot 5}{\cancel{2}} = 15} + \overbrace{\frac{4!}{3!1!}}^{=4}}{\underbrace{\frac{10!}{5!5!}}} = \frac{19}{252}$$

$$\frac{\cancel{10}^{2}\cdot \cancel{9}^{3}\cdot \cancel{8}^{2}\cdot 7 \cdot \cancel{6}^{3}}{\cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 4 \cdot 9 \cdot 7 = 63 \cdot 4 = 252$$

$$P(B) = \frac{4}{10}$$

$$P(B|A) = \frac{3}{5}$$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$\Leftrightarrow \frac{\frac{3}{5} \cdot \frac{19}{252}}{\frac{4}{10}} = \frac{3}{5} \cdot \frac{19}{252} \cdot \frac{10}{4} = \frac{570}{5040} \begin{matrix} :5 \\ :5 \end{matrix}$$

$$= \frac{115}{1008} \approx 0{,}1141 \approx 11{,}41\,\%$$

1.4.10. In an office there are two boxes of thumb drives: Box A_1 contains seven 100 GB drives and three 500 GB drives, and box A_2 contains two 100 GB drives and eight 500 GB drives. A person is handed a box at random with prior probabilities $P(A_1) = \frac{2}{3}$ and $P(A_2) = \frac{1}{3}$, possibly due to the boxes' respective locations. A drive is then selected at random and the event B occurs if it is a 500 GB drive. Using an equally likely assumption for each drive in the selected box, compute $P(A_1|B)$ and $P(A_2|B)$.

1.4.10 $\frac{3}{7}, \frac{4}{7}$.

B = 500GB drive

$P(A_1) = \frac{2}{3}$

$P(A_2) = \frac{1}{3}$

$P(B|A_1) = \frac{3}{10}$

$P(B|A_2) = \frac{8}{10}$

(LAW OF TOTAL PROBABILITY)

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$$

$$= \frac{3}{10} \cdot \frac{2}{3} + \frac{8}{10} \cdot \frac{1}{3}$$

$$= \frac{6}{30} + \frac{8}{30} = \frac{14}{30} = \frac{7}{15}$$

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B)}$$

$$= \frac{\frac{3}{10}\cdot\frac{2}{3}}{\frac{7}{15}}$$

$$= \frac{\cancel{6}^{\,3}}{\cancel{30}_{\,2}}\cdot\frac{\cancel{15}}{7}$$

$$= \frac{3}{7} \;//$$

$$\Rightarrow P(A_2|B) = \frac{4}{7} \;//$$

1.4.11. Suppose A and B are independent events. In expression (1.4.6) we showed that A^c and B are independent events. Show similarly that the following pairs of events are also independent: (a) A and B^c and (b) A^c and B^c.

$$\overset{(*)}{P(A\cap B) = P(A)P(B)} \quad \text{(INDEPENDENCE)}$$

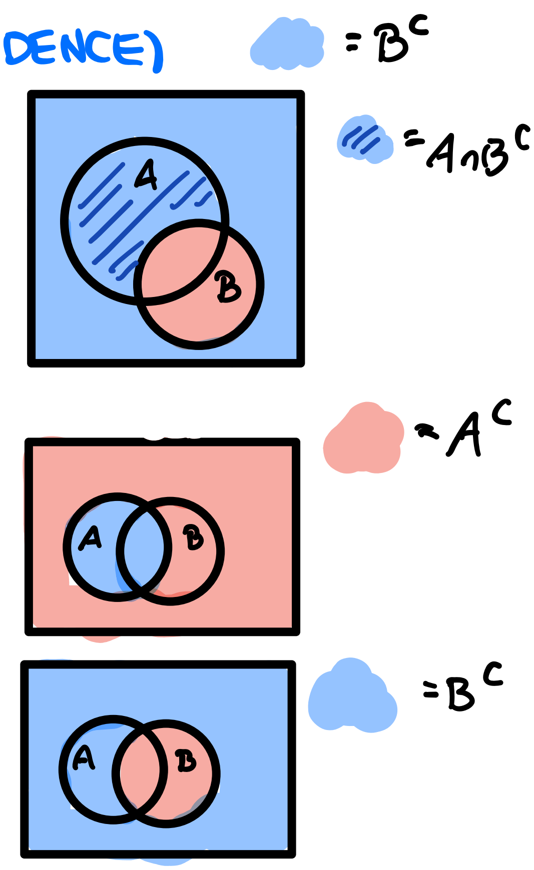
a)

$$\rightarrow P(A\cap B^c) = P(A) - P(A\cap B)$$

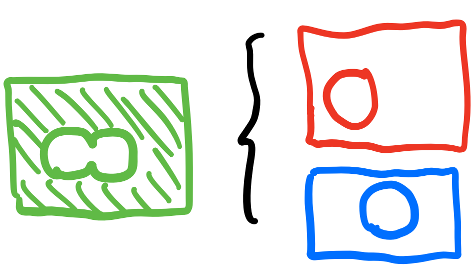
$$\overset{(*)}{=} P(A) - P(A)P(B)$$

$$= P(A)\,(1-P(B))$$

$$= P(A)P(B^c) \;\square //$$



b) $P(A^c\cap B^c) = P((A\cup B)^c)$



NOTE:

$P(A^c) = 1 - P(A)$

$P(B^c) = 1 - P(B)$

$\leadsto P(A^c \cap B^c) = P((A \cup B)^c)$

$= 1 - P(A \cup B)$

(INCLUSION – EXCLUSION)

$= 1 - P(A) - P(B) + P(A \cap B)$

$= 1 - P(A) - P(B) + P(A)P(B)$

$= (1 - P(A))(1 - P(B))$

$= P(A^c)P(B^c) \quad \square //$

1.4.12. Let C_1 and C_2 be independent events with $P(C_1) = 0.6$ and $P(C_2) = 0.3$. Compute (a) $P(C_1 \cap C_2)$, (b) $P(C_1 \cup C_2)$, and (c) $P(C_1 \cup C_2^c)$.

1.4.12 (c) 0.88.

a) $P(C_1 \cap C_2) = P(C_1)P(C_2)$

$= 0{,}6 \cdot 0{,}3$

$= 0{,}18 /$

b) $P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$

$= 0{,}6 + 0{,}3 - 0{,}18$

$= 0{,}72 /$

$0{,}6 \cdot 0{,}7 = 0{,}42$

c) $P(C_1 \cup C_2^c) = 0{,}6 + 0{,}7 - 0{,}6 \cdot 0{,}7$

$= 1{,}3 - 0{,}42 = 0{,}88 /$

1.4.13. Generalize Exercise 1.2.5 to obtain

$$(C_1 \cup C_2 \cup \cdots \cup C_k)^c = C_1^c \cap C_2^c \cap \cdots \cap C_k^c. \quad (*)$$

Say that $C_1, C_2, \ldots, C_k$ are independent events that have respective probabilities $p_1, p_2, \ldots, p_k$. Argue that the probability of at least one of $C_1, C_2, \ldots, C_k$ is equal to

$$1 - (1 - p_1)(1 - p_2) \cdots (1 - p_k).$$

(*)

Generalizing → easy by induction → skip it

<u>Proof</u>

$P(\geq 1) = P(C_1 \cup \ldots \cup C_k)$

$P(A) = 1 - P(A^c)$

$\Rightarrow P(C_1 \cup .. \cup C_k) = 1 - P((C_1 \cup .. \cup C_k)^c)$

De Morgan

$\Rightarrow 1 - P(C_1^c \cap .. \cap C_k^c)$

Compl. of independent also independent

$\Rightarrow 1 - (P(C_1^c) \cdot \ldots \cdot P(C_k^c))$

Prob. of complement:

$p_i^c = 1 - p_i$

Thus:

$P(C_1^c) \ldots$

$1 - (1 - p_1) \cdot \ldots \cdot (1 - p_k)$ $\square$ //

1.4.14. Each of four persons fires one shot at a target. Let C_k denote the event that the target is hit by person k, $k = 1, 2, 3, 4$. If C_1, C_2, C_3, C_4 are independent and if $P(C_1) = P(C_2) = 0.7$, $P(C_3) = 0.9$, and $P(C_4) = 0.4$, compute the probability that (a) all of them hit the target; (b) exactly one hits the target; (c) no one hits the target; (d) at least one hits the target.

a) C_i independent

$\Rightarrow 0{,}7^2 \cdot 0{,}9 \cdot 0{,}4 = 0{,}1764$ //

b) ① $\overset{0,7}{P(C_1)}\overset{1-0,7}{P(C_2^c)}\overset{1-0,9}{P(C_3^c)}\overset{1-0,4}{P(C_4^c)} = 0{,}0126$ //

② $P(C_1^c)P(C_2)P(C_3^c)P(C_4^c) = 0{,}0126$ //

③ $P(C_1^c)P(C_2^c)P(C_3)P(C_4^c) = 0{,}0486$ //

④ $P(C_1^c)P(C_2^c)P(C_3^c)P(C_4) = 0{,}0036$ //

① + ② + ③ + ④ $= 0{,}0774$ //

c) $P(C_1^c)P(C_2^c)P(C_3^c)P(C_4^c) = 0{,}0054$

d) $P(\Omega) - P(\text{no one hits}) = 1 - 0{,}0054$

$= 0{,}9946$ //

1.4.15. A bowl contains three red (R) balls and seven white (W) balls of exactly the same size and shape. Select balls successively at random and with replacement so that the events of white on the first trial, white on the second, and so on, can be assumed to be independent. In four trials, make certain assumptions and compute the probabilities of the following ordered sequences: (a) WWRW; (b) RWWW; (c) WWWR; and (d) WRWW. Compute the probability of exactly one red ball in the four trials.

$P(R) = \frac{3}{10}$

$P(W) = \frac{7}{10}$

a) $P(WWRW) = \frac{\overset{1}{\cancel{7}}}{\underset{\cancel{2}}{\cancel{10}}} \cdot \frac{\overset{\cancel{2}}{\cancel{6}}}{\underset{\cancel{3}}{\cancel{9}}} \cdot \frac{\cancel{3}}{8} \cdot \frac{\overset{1}{\cancel{5}}}{\cancel{7}_{1}}$

$$= \frac{2 \cdot 3}{2 \cdot 3 \cdot 8} = \frac{1}{8}$$ //

b) $P(RWWW) = \frac{3}{10} \cdot \frac{7}{9} \cdot \frac{6}{8} \cdot \frac{5}{7} = \frac{1}{8}$

c) identical $\frac{1}{8}$

d) identical $\frac{1}{8}$

$\rightarrow$ P(exactly 1 R in 4 Pulls)

$$= P((RWWW),(WRWW),(WWRW),(WWWR))$$

$$= P(RWWW) + \ldots + P(WWWR)$$

$$= \frac{1}{8} + \ldots + \frac{1}{8}$$

$$= \frac{4}{8} = \frac{1}{2}$$ //

1.4.16. A coin is tossed two independent times, each resulting in a tail (T) or a head (H). The sample space consists of four ordered pairs: TT, TH, HT, HH. Making certain assumptions, compute the probability of each of these ordered pairs. What is the probability of at least one head?

$P(TT) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\Rightarrow P(TT) = P(TH) = P(HT) = P(HH)$

$$P(\geq 1\ H) = 1 - P(TT)$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$ //

1.4.17. For Example 1.4.7, obtain the following probabilities. Explain what they mean in terms of the problem.

(a) $P(N_D)$.

(b) $P(N \mid A_D)$.

(c) $P(A \mid N_D)$.

(d) $P(N \mid N_D)$.

A = abused

$A^c = N$ = not abused

N_D = class. as NOT by doctor

a) $P(N_D)$ = Prob. of a child being classified as "not abused" by a doctor
(condition)

b) $P(N/A_D)$ = Prob. of a child not being abused even tho it got classified as such.
(False Positives)

c) $P(A/N_D)$ = Prob. of a child being a victim of abuse even tho it has'nt been classified in such manner.
(False Negatives)

d) $P(N/N_D)$ = Prob. of a child being classified correctly as non-abused.
(True Negatives)

1.4.18. A die is cast independently until the first 6 appears. If the casting stops on an odd number of times, Bob wins; otherwise, Joe wins.

(a) Assuming the die is fair, what is the probability that Bob wins?

Prob. of 6 appearing on k'th roll:

$$p_k = \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6}$$

Event of Bob winning: $2 \nmid k$

$$P(B) = \sum_{2 \nmid k} p_k = \sum_{j=1}^{\infty} p_{2j-1}$$

$$\Rightarrow P(B) = \sum_{j=1}^{\infty} \left(\frac{5}{6}\right)^{2j-2} \cdot \frac{1}{6}$$

$$= \frac{1}{6} \sum_{j=1}^{\infty} \left(\frac{25}{36}\right)^{j-1} \Big\} \text{ Geom. series} \Rightarrow \frac{1}{1-q}$$

$$= \frac{1}{6} \cdot \frac{1}{1 - \frac{25}{36}}$$

$$= \frac{6}{11} \;//$$

(b) Let p denote the probability of a 6. Show that the game favors Bob, for all p, $0 < p < 1$.

Here: $p_k = (1-p)^{k-1} p$

Bob

$$\Rightarrow Pr(B) = \sum_{j=1}^{\infty} (1-p)^{2j-2} p$$

$$= p \sum_{j=1}^{\infty} (\underbrace{(1-p)^2}_{<1 \Rightarrow \text{geom. series}})^{j-1}$$

$$= p \cdot \frac{1}{1-(1-p)^2}$$

$$= \frac{1}{2-p}$$

$\Rightarrow$ For $0 < p < 1$:

$$\Rightarrow \quad 2-p < 2 \quad \Big| \text{ Kehrwert}$$

$$\Rightarrow \quad \frac{1}{2-p} > \frac{1}{2}$$, thus favors Bob. //

1.4.19. Cards are drawn at random and with replacement from an ordinary deck of 52 cards until a spade appears.

(a) What is the probability that at least four draws are necessary?

→ We don't draw ♠ within the first three:

$$\frac{13}{52} = \spadesuit$$

$$\Rightarrow \frac{39}{52} \cdot \frac{39}{52} \cdot \frac{39}{52}$$

$$= \left(\frac{3}{4}\right)^3 = P(\times\spadesuit) \;//$$

$\rightsquigarrow$ ♠

(b) Same as part (a), except the cards are drawn without replacement.

$$\Rightarrow \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50} = P(\spadesuit\times) \;//$$

1.4.20. A person answers each of two multiple choice questions at random. If there are four possible choices on each question, what is the conditional probability that both answers are correct given that at least one is correct?

Let $E_1 :=$ First question answered correctly

$E_2 :=$ second...

$\Rightarrow E_1, E_2$ independent

$\Rightarrow P(E_1) = P(E_2) = \frac{1}{4}$

Let $A = E_1 \cup E_2$ (at least one correct answer)

$B = E_1 \cap E_2$ (both correct)

$$P(A) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{1}{16}$$

$$= \frac{7}{16}$$

$$P(B) = P(E_1)P(E_2)$$

$$= \frac{1}{16}$$

$$\Rightarrow P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$= \frac{P(B)}{P(A)}$$

$$= \frac{1}{\cancel{16}} \cdot \frac{\cancel{16}}{7}$$

$$= \frac{1}{7} //$$

1.4.21. Suppose a fair 6-sided die is rolled six independent times. A match occurs if side i is observed on the ith trial, $i = 1, \ldots, 6$.

(a) What is the probability of at least one match on the six rolls? *Hint:* Let C_i be the event of a match on the ith trial and use Exercise 1.4.13 to determine the desired probability.

Let $C_i :=$ i on i'th roll,

at least one match : $(C_1 \cup .. \cup C_6)$

$P(C_i) = \frac{1}{6}$

REMEMBER 1.4.13

Say that $C_1, C_2, \ldots, C_k$ are independent events that have respective probabilities $p_1, p_2, \ldots, p_k$. Argue that the probability of at least one of $C_1, C_2, \ldots, C_k$ is equal to

$$1 - (1 - p_1)(1 - p_2) \cdots (1 - p_k).$$

$$P(C_1 \cup .. \cup C_6) = 1 - \left(1 - \frac{1}{6}\right)^6$$

$$= 1 - \left(\frac{5}{6}\right)^6$$

$$\approx 0{,}665 //$$

(b) Extend part (a) to a fair n-sided die with n independent rolls. Then determine the limit of the probability as $n \to \infty$.

$P(C_1 \cup .. \cup C_n)$ with $n \to \infty$

$$\Rightarrow P(C_1 \cup .. \cup C_n) = 1 - \left(1 - \frac{1}{n}\right)^n$$

with,

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$$

thus,

$$\lim_{n \to \infty} \left(\bigcup_{j=1}^{n} C_j\right) = 1 - \frac{1}{e}$$

$$\approx 0{,}632 \; // \; \square$$

1.4.22. Players A and B play a sequence of independent games. Player A throws a die first and wins on a "six." If he fails, B throws and wins on a "five" or "six." If he fails, A throws and wins on a "four," "five," or "six." And so on. Find the probability of each player winning the sequence.

3 ways for A to win.

1

A wins 1st round (⚅)

2

A fails 1st (⚀, ⚁, ⚂, ⚃, ⚄)
B fails 2nd (⚀, ⚁, ⚂, ⚃)
A wins 3rd (⚃, ⚄, ⚅)

3

A fails 1st (⚀, ⚁, ⚂, ⚃, ⚄)
B fails 2nd (⚀, ⚁, ⚂, ⚃)
A fails 3rd (⚀, ⚁, ⚂)
B fails 4th (⚀, ⚁)

A wins 5th (⚁, ⚂, ⚃, ⚄, ⚅)

$$\Rightarrow P(1) = \frac{1}{6}$$

$$P(2) = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6}$$

$$P(3) = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{5}{6}$$

$$\Rightarrow P(1) + P(2) + P(3) = \frac{169}{324} \approx 52{,}16\,\% \,//$$

1.4.23. Let C_1, C_2, C_3 be independent events with probabilities $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, respectively. Compute $P(C_1 \cup C_2 \cup C_3)$.

(INCLUSION-EXCLUSION)

$$P(C_1 \cap C_2) = \frac{1}{6}$$
$$P(C_1 \cap C_3) = \frac{1}{8}$$
$$P(C_2 \cap C_3) = \frac{1}{12}$$

$$P(C_1 \cap C_2 \cap C_3) = \frac{1}{2 \cdot 3 \cdot 4} = \frac{1}{24}$$

$$P(C_1 \cup C_2 \cup C_3) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{12} + \frac{1}{24}$$

$$= \frac{12 + 8 + 6 - 4 - 3 - 2 + 1}{24}$$

$$= \frac{18}{24} = \frac{3}{4} \,//$$

1.4.24. From a bowl containing five red, three white, and seven blue chips, select four at random and without replacement. Compute the conditional probability of one red, zero white, and three blue chips, given that there are at least three blue chips in this sample of four chips.

$\bigcirc = \frac{5}{15} = \frac{1}{3}$

$\bigcirc = \frac{3}{15} = \frac{1}{5}$

$\bigcirc = \frac{7}{15}$

$P(\bigcirc\bigcirc\bigcirc\bigcirc) = ?$

Let $A :=$ selecting 1 red, 3 blue, 0 white

Let $B :=$ selecting at least 3 blue chips (so 3 ∨ 4)

$$\Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)}$$

($\cancel{*}$ $A \cap B = A$)

$$\Rightarrow P(A|B) = \frac{P(A)}{P(B)}$$

$$P(A) = \frac{\binom{5}{1}\binom{3}{0}\binom{7}{3}}{\binom{15}{4}} = \frac{5}{39}$$ ($P(A)$)

$$P(B) = P(3 \times \bigcirc) + P(4 \times \bigcirc)$$

$$P(3 \times \bigcirc) = \frac{\binom{7}{3}\binom{8}{1}}{\binom{15}{4}}$$ ← other non blues

$$= \frac{280}{1365}$$

$$P(4 \times \bigcirc) = \frac{\binom{7}{4}}{\binom{15}{4}} = \frac{35}{1365} /$$

$$\Rightarrow \frac{280 + 35}{1365} = \frac{3}{13} / \quad (P(B))$$

$$\Rightarrow \frac{P(A)}{P(B)} = \frac{5}{\cancel{39}_{3}} \cdot \frac{\cancel{13}^{1}}{3}$$

$$= \frac{5}{9} //$$

1.4.25. Let the three mutually independent events C_1, C_2, and C_3 be such that $P(C_1) = P(C_2) = P(C_3) = \frac{1}{4}$. Find $P[(C_1^c \cap C_2^c) \cup C_3]$.

$$\Rightarrow P(C_1^c \cap C_2^c)$$

$$\Leftrightarrow P(C_1^c) P(C_2^c)$$

$$\Leftrightarrow (1 - P(C_1))(1 - P(C_2))$$

$$= \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$$

(USE $P(A \cup B) = P(A) + P(B) - P(A \cap B)$)

$$\Rightarrow P(C_1^c \cap C_2^c) + P(C_3) - P((C_1^c \cap C_2^c) \cap C_3) \quad (*)$$

$(*) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4}$

$= \frac{9}{64}$

$\Rightarrow \underbrace{\frac{9}{16}}_{P(A)} + \underbrace{\frac{1}{4}}_{P(B)} - \underbrace{\frac{9}{64}}_{P(A\cap B)}$

$= \frac{36+16-9}{64}$

$= \frac{43}{64}$ //

1.4.26. Each bag in a large box contains 25 tulip bulbs. It is known that 60% of the bags contain bulbs for 5 red and 20 yellow tulips, while the remaining 40% of the bags contain bulbs for 15 red and 10 yellow tulips. A bag is selected at random and a bulb taken at random from this bag is planted.

(a) What is the probability that it will be a yellow tulip?

$P(B_1) = 0{,}6$

$P(B_2) = 0{,}4$

$P(Y|B_1) = \frac{20}{25} = 0{,}8$

$P(Y|B_2) = \frac{10}{25} = 0{,}4$

a)

1 $P(Y) = P(Y|B_1)P(B_1) + P(Y|B_2)P(B_2)$

$\Rightarrow 0{,}8 \cdot 0{,}6 + 0{,}4 \cdot 0{,}4$

$= 0{,}64 = 64\%$ //

(b) Given that it is yellow, what is the conditional probability it comes from a bag that contained 5 red and 20 yellow bulbs?

b) To find: $P(B_1|Y)$

$P(Y) = 0{,}64$

$P(B_1) = 0{,}6$

$P(Y|B_1) = 0{,}8$

(BAYES)

$$P(B_1|Y) = \frac{P(Y|B_1)\,P(B_1)}{P(Y)}$$

$$= \frac{0{,}8 \cdot 0{,}6}{0{,}64}$$

$$= 0{,}75 = 75\,\% \;//$$

1.4.27. The following game is played. The player randomly draws from the set of integers $\{1, 2, \ldots, 20\}$. Let x denote the number drawn. Next the player draws at random from the set $\{x, \ldots, 25\}$. If on this second draw, he draws a number greater than 21 he wins; otherwise, he loses.

(a) Determine the sum that gives the probability that the player wins.

$P(X=k) = \frac{1}{20}$

$$\text{Winning } y \text{ for } x = \begin{cases} 4, & x \leq 21 \\ 25-x+1, & x \geq 22 \end{cases}$$

(if $x = 23 \rightarrow \{23, 24, 25\} = 25 - 23 + 1 = 3$ ways of winning)

$$P(WIN \mid X) = \frac{4}{25-X+1}, \quad X \leq 21$$

AND 0, $X > 21$ (Since $X \in \{1, \dots, 20\}$)

$$P(WIN) = P(WIN \mid X) P(X)$$

$$\Rightarrow \sum_{X=1}^{20} \frac{1}{20} \cdot \frac{4}{25-X+1}$$

$$= \frac{1}{20} \sum_{X=1}^{20} \frac{4}{26-X}$$

(b) Write and run a line of R code that computes the probability that the player wins.

$P(WIN) \approx 0{,}0899$

1.4.28. A bowl contains 10 chips numbered $1, 2, \dots, 10$, respectively. Five chips are drawn at random, one at a time, and without replacement. What is the probability that two even-numbered chips are drawn and they occur on even-numbered draws?

Draw 1 2 3 4 5

X_1 Y_1 X_2 Y_2 X_3

Y_1, Y_2 even

X_1, X_2, X_3 odd

5 chips are even 2, 4, 6, 8, 10

5 are odd 1, 3, 5, 7, 9

odds evens

$$\Rightarrow \frac{\binom{5}{1}}{\binom{10}{1}} \cdot \frac{\binom{5}{1}}{\binom{9}{1}} \cdot \frac{\binom{4}{1}}{\binom{8}{1}} \cdot \frac{\binom{4}{1}}{\binom{7}{1}} \cdot \frac{\binom{3}{1}}{\binom{6}{1}}$$

all

$$\Rightarrow \frac{\cancel{5} \cdot 5 \cdot \cancel{4} \cdot \cancel{4} \cdot \cancel{3}}{{}_2\cancel{10} \cdot 9 \cdot \cancel{8}_{\cancel{2}} \cdot 7 \cdot \cancel{6}_{\cancel{2}}}$$

$$\Rightarrow \frac{5}{9 \cdot 7 \cdot 2} = \frac{5}{126} \;//$$

1.4.29. A person bets 1 dollar to b dollars that he can draw two cards from an ordinary deck of cards without replacement and that they will be of the same suit. Find b so that the bet is fair.

$\binom{52}{2} = 26 \cdot 51$ ways of drawing 2 cards

$\underbrace{\binom{13}{2} = 13 \cdot 6}$ ways of drawing same suit

• 4 cause there are 4 suits

$$\Rightarrow \frac{1\cancel{3} \cdot 6 \cdot \cancel{4}^{2}}{\cancel{2}\cancel{2}6 \cdot 51}$$

$$\Rightarrow \frac{12}{51} = \frac{4}{17} /$$

Thus,

The odds are: $\frac{4}{17} : \frac{13}{17}$

$\Rightarrow 1 : 3{,}25$, thus, $b = 3{,}25$ //

1.4.30 (Monte Hall Problem)**.** Suppose there are three curtains. Behind one curtain there is a nice prize, while behind the other two there are worthless prizes. A contestant selects one curtain at random, and then Monte Hall opens one of the other two curtains to reveal a worthless prize. Hall then expresses the willingness to trade the curtain that the contestant has chosen for the other curtain that has not been opened. Should the contestant switch curtains or stick with the one that she has? To answer the question, determine the probability that she wins the prize if she switches.

W= contestant wins

S= contestant switches

$P(W|S)$ isto compute.

If contestant switches, she will win only if she picket one of the "bad" curtains.

Being: $P(W|S) = \frac{2}{3}$

$P(W|S^c) = \frac{1}{3}$

So unintuitively, switching resembles the better option. //

1.4.31. A French nobleman, Chevalier de Méré, had asked a famous mathematician, Pascal, to explain why the following two probabilities were different (the difference had been noted from playing the game many times): (1) at least one six in four independent casts of a six-sided die; (2) at least a pair of sixes in 24 independent casts of a pair of dice. From proportions it seemed to de Méré that the probabilities should be the same. Compute the probabilities of (1) and (2).

1

$$P(\geq 1 ⚅) = 1 - P(0 \times ⚅)$$

$$= 1 - \left(\frac{5}{6}\right)^4 \approx 0{,}5177 \;//$$

② $P(\geq 1$ ⚅⚅$)$

$\Leftrightarrow 1 - P(0\times$ ⚅⚅$)$

$= 1 - \left(\frac{35}{36}\right)^{24}$

$\approx 0{,}4914$ //

1.4.32. Hunters A and B shoot at a target; the probabilities of hitting the target are p_1 and p_2, respectively. Assuming independence, can p_1 and p_2 be selected so that

$$P(\text{zero hits}) = P(\text{one hit}) = P(\text{two hits})?$$

Suppose it is possible:

$P(0) = P(1) = P(2) \overset{!}{=} \frac{1}{3}$ (by independence)

$\Rightarrow 0 = P(2) - P(0)$

$= \underbrace{p_1 p_2}_{\text{hit}\cdot\text{hit}} - \underbrace{(1-p_1)}_{\text{no hit}}\cdot\underbrace{(1-p_2)}_{\text{no hit}}$

$= \cancel{p_1 p_2} - (1 - p_2 - p_1 + \cancel{p_1 p_2})$

$= p_1 + p_2 - 1 \qquad | -p_2, \cdot(-1)$

$\Rightarrow p_2 = 1 - p_1 \quad (*)$

From $\underbrace{p_1 p_2}_{P(2)} = \frac{1}{3}$, follows:

(*)

$$p_1(1-p_1)=\frac{1}{3}$$

$$\Rightarrow p_1-p_1^2-\frac{1}{3}=0 \quad |\cdot(-1)$$

$$\Rightarrow p_1^2-p_1+\frac{1}{3}=0$$

$\boxed{pq}$ $\left(\frac{3}{12}-\frac{4}{12}=-\frac{1}{12}\right)$

$$p_{1_{1,2}}=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{1}{3}}$$

$$=\frac{1}{2}\pm\sqrt{-\frac{1}{12}}$$

$\Rightarrow$ has no solutions in $\mathbb{R}$

$\Rightarrow$ ↯ $\square$ //

1.4.33. At the beginning of a study of individuals, 15% were classified as heavy smokers, 30% were classified as light smokers, and 55% were classified as nonsmokers. In the five-year study, it was determined that the death rates of the heavy and light smokers were five and three times that of the nonsmokers, respectively. A randomly selected participant died over the five-year period: calculate the probability that the participant was a nonsmoker.

Death rate of non smoker = r

$P(N)=0{,}55$

$P(L)=0{,}3$

$P(H)=0{,}15$

also:

$P(D|N) = r$

$P(D|L) = 3r$

$P(D|H) = 5r$ (BAYES)

$$P(N|D) = \frac{P(D|N)P(N)}{P(D|N)P(N) + P(D|L)P(L) + P(D|H)P(H)}$$

$$\Rightarrow \frac{r \cdot 0{,}55}{r \cdot 0{,}55 + 3r \cdot 0{,}3 + 5r \cdot 0{,}15}$$

$$= \frac{\cancel{r} \cdot 0{,}55}{\cancel{r} \cdot (0{,}55 + 3 \cdot 0{,}3 + 5 \cdot 0{,}15)}$$

$$= \frac{0{,}55}{0{,}55 + 0{,}9 + 0{,}75}$$

$$= \frac{0{,}55}{2{,}20} \quad \left(\frac{55}{220} = \frac{11}{44}\right)$$

$$= \frac{1}{4} \approx 25\%$$

1.4.34. A chemist wishes to detect an impurity in a certain compound that she is making. There is a test that detects an impurity with probability 0.90; however, this test indicates that an impurity is there when it is not about 5% of the time. The chemist produces compounds with the impurity about 20% of the time. A compound is selected at random from the chemist's output. The test indicates that an impurity is present. What is the conditional probability that the compound actually has the impurity?

I = Impurity

+ = positive Test

$P(+|I) = 0{,}9$

$P(+|I^c) = 0{,}05$

$P(I) = 0{,}2$

(BAYES)

$$\Rightarrow P(I|+) = \frac{P(+|I)\,P(I)}{P(+|I)P(I) + P(+|I^c)P(I^c)}$$

$$= \frac{0{,}9 \cdot 0{,}2}{0{,}9 \cdot 0{,}2 + 0{,}05 \cdot 0{,}8}$$

$$= \frac{9}{11} \approx 0{,}818$$

$$= 81{,}8\,\% \;/\!/$$