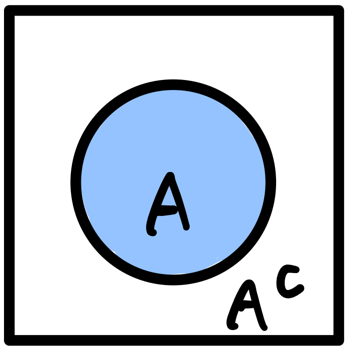
Sample Spaces

Here noted through C instead of Ω

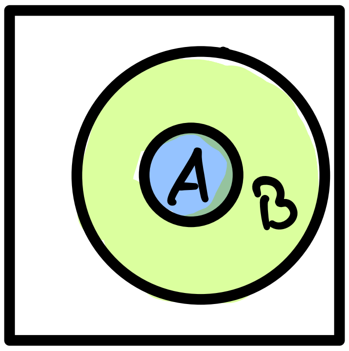
Sets

countable $\Leftrightarrow$ not infinite
or $\mathbb{N}$

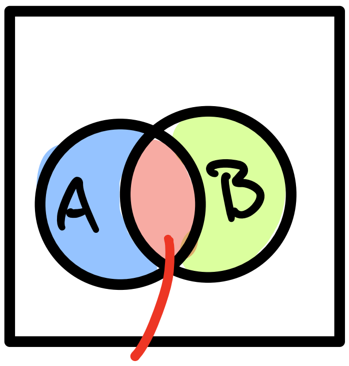
$\rightsquigarrow$ $(0,1]$ NOT countable



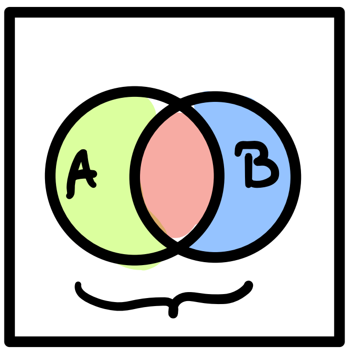
A



$A \subset B$



$A \cap B$



$A \cup B$

If $A \subset B$ and $B \subset A$, then $A = B$

$\Omega^c = \emptyset$, $\emptyset^c = \Omega$

Disjoint $\Leftrightarrow A \cap B = \emptyset$

De Morgan

$(A \cap B)^c = A^c \cup B^c$

$(A \cup B)^c = A^c \cap B^c$

We define:

$A_1 \cup .. \cup A_n = \{x \mid \exists i : x \in A_i\} = \bigcup_{i=1}^{n} A_i$

$A_1 \cap .. \cap A_n = \{x \mid \forall i : x \in A_i\} \quad \bigcap_{i=1}^{n} A_i$

Example

$\Omega = \{1, 2, 3, ..\}$, $A_n = \{1, 3, .., 2n-1\}$

$B_n = \{n, n+1, ...\}$

$$\bigcup_{n=1}^{\infty} A_n = \{1,3,5,\dots\}$$

$$\bigcap_{n=1}^{\infty} A_n = \{1\} \to \{1\} \cap \{1,\cancel{3}\} \cap \{1,\cancel{3},\cancel{5}\} \cap \dots$$

$$\bigcup_{n=1}^{\infty} B_n = \Omega \qquad \text{UND}$$

$$\bigcap_{n=1}^{\infty} B_n = \{1,2,\dots\} \cap \{2,3,\dots\} \cap \dots$$

$$= \emptyset$$

sets are non-decreasing, if nested upward. ($A_n \subset A_{n+1}$)

$$\to \lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

Ex. $A_n = \{1,3,\dots,2n-1\}$ is monotone.

Non-increasing → nested downward

$\Leftrightarrow A_n \supset A_{n+1}$

$\rightarrow \lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$

Ex. $B_n = \{n, n+1, \ldots\}$ is nonincreasing

$\rightarrow \underbrace{\{1,2,3,\ldots\}}_{n=1} \supset \underbrace{\{2,3,4,\ldots\}}_{n=2} \supset \underbrace{\{3,4,\ldots\}}_{n=3}$

Set Functions

A mapping from a set A to $\mathbb{R}$ or $\mathbb{C}$.

Ex. $\Omega = \mathbb{R}^2$, $A \subset \Omega$

$Q(A)$ be the area. If $A = \{(x,y) \mid x^2+y^2 \le 1\}$,

then $Q(A) = \pi$.

$\int_A f(x)\,dx$ AND $\iint\limits_A g(x,y)\,dx\,dy$

be the integral over a set A.

In $\iint_A$, A is two-dimensional.

Similarly: $\sum\limits_A f(x)$ AND $\sum\sum\limits_A g(x,y)$

the sum over all $(x,y) \in A$

<u>Ex.</u> $\sum\limits_{n=0}^{\infty} a^n = \frac{1}{1-a}$, $|a| < 1$

GEOMETRIC SERIES

<u>Ex.2</u> $B = \{1, 3, 5, \ldots\}$

Let $Q(A) = \sum\limits_{n \in A} \left(\frac{2}{3}\right)^n$

Then:

$$Q(B) = \sum_{n \in B} \left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{2n+1}$$

$$= \frac{2}{3} \sum_{n=0}^{\infty} \left[\left(\frac{2}{3}\right)^2\right]^n$$

$$= \frac{2}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n$$

$$= \frac{2}{3} \cdot \frac{1}{1-\frac{4}{9}} = \frac{2}{3} \cdot \frac{9}{5} = \frac{18}{15} = \frac{6}{5}$$

Ex. 3

Now, let $F(x) = e^{-x}$.

Let $\Omega = \mathbb{R}_+$ $(0, \infty)$

Let $Q(A) = \int_A e^{-x} \, dx$

$$Q[(1,3)] = \int_1^3 e^{-x}\,dx$$

$$= -e^{-x}\Big|_1^3$$

$$= e^{-1} - e^{-3} = 0{,}318$$

$$Q(\Omega) = \int_0^\infty e^{-x}\,dx = 1$$

<u>Ex.4</u> $\Omega = \mathbb{R}^n$

Let $Q(A) = \int \dots \int_A dx_1\,dx_2 \dots dx_n$

Let $A = \{(x_1, \dots, x_n) \mid 0 \leq x_1 \leq x_2,$
$0 \leq x_i \leq 1,$
for $i = 3, 4, \dots, n\}$

$$Q(A) = \int_0^1 \left[\int_0^{x_2} dx_1 \right] dx_2 \cdot \prod_{i=3}^{n} \int_0^1 dx_i$$

$$= \frac{x_2^2}{2} \Big|_0^1 \cdot 1 = \frac{1}{2}$$

Let $B = \{(x_1, \dots, x_n) \mid 0 \le x_1 \le \dots \le x_n \le 1\}$

Then:

$$Q(B) = \int_0^1 \left[\int_0^{x_n} \cdots \left[\int_0^{x_2} dx_1 \right] dx_2 \right] \cdots dx_{n-1} \Big] dx_n$$

$$= \frac{1}{n!}$$