Conditional Probability and Independence

"$|$" $\triangleq$ "given"

$P(A|A) = 1$ AND

$P(B|A) = P(A \cap B|A)$

A is now sample space

$$\rightarrow \frac{P(A \cap B|A)}{P(A|A)} = \frac{P(A \cap B)}{P(A)}$$

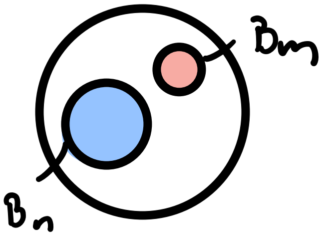
Thus we get:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

1. $P(B|A) \geq 0$
2. $P(A|A) = 1$
3. $P\left(\bigcup_{n=1}^{\infty} B_n \middle| A\right) = \sum_{n=1}^{\infty} P(B_n|A)$

($B_1, \ldots, B_n$ mutually exclusive)

It follows: $(B_n \cap A) \cap (B_m \cap A) = \varnothing, \; n \neq m$



Thus:

$$P\left(\bigcup_{n=1}^{\infty} B_n \mid A\right) = \frac{P\left(\bigcup_{n=1}^{\infty} (B_n \cap A)\right)}{P(A)}$$

$$= \sum_{n=1}^{\infty} \frac{P(B_n \cap A)}{P(A)}$$

$$= \sum_{n=1}^{\infty} P(B_n \mid A) \quad \square$$

Example

Five card hand, no replacement

↓

all spade hand (B) relative to hypothesis that four spades are in hand (A) is,

since $A \cap B = B$

$$\rightarrow P(B \mid A) = \frac{P(B)}{P(A)}$$

$\binom{13}{5}$ } full spade hands

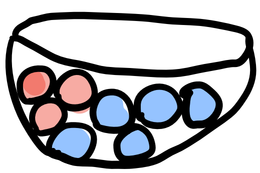
$$= \frac{\dfrac{\binom{\;}{5}}{\binom{52}{5}}}{\dfrac{\binom{13}{4}\binom{39}{1}+\binom{13}{5}}{\binom{52}{5}}} = \frac{\binom{13}{5}}{\binom{13}{4}\binom{39}{1}+\binom{13}{5}} = 0{,}0441$$

We observe

$$P(A\cap B) = P(A)\,P(B|A)$$

Example 1.4.2

Prob. of red, then blue



$$\Rightarrow \quad P(A) = \frac{3}{8}$$

$$P(B|A) = \frac{5}{7}$$

Thus: $P(A\cap B) = \frac{3}{8}\cdot\frac{5}{7} = \frac{15}{56} = 0{,}2679$

Example 1.4.3

Prob. of third spade appearing on sixth draw:

Let $A \triangleq \spadesuit\spadesuit$ in first 5 draws

$B \triangleq \spadesuit$ in 6th draw

We look for $P(A\cap B)$!

$$P(A) = \frac{\overbrace{\binom{13}{2}}^{2\spadesuit}\underbrace{\binom{39}{3}}_{\text{no more spades}}}{\binom{52}{5}} = 0{,}2743$$

...e spades

← gone

$$P(B|A) = \frac{13-2}{52-5} = \frac{11}{47} = 0{,}2340$$

↑ drawn cards

$$P(B|A) \cdot P(A) = 0{,}0642 = P(A \cap B)$$

Example 1.4.4

Drawing 4 cards, P of getting

$\spadesuit \to \clubsuit \to \heartsuit \to \diamondsuit$

$$\Leftrightarrow \frac{13}{52} \cdot \frac{13}{51} \cdot \frac{13}{50} \cdot \frac{13}{49} = 0{,}0044$$

(Multiplication rule)

Consider k mut. excl. events (exh. too)

$A_1, \dots, A_k$. $P(A_i) > 0$

Let B be another event, $P(B) > 0$

Thus B occurs with one and only one

of the events $A_1, \dots, A_k$

$$\to B = B \cap (A_1 \cup \dots \cup A_k)$$

$$= (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_k)$$

Since $A_1, \dots, A_k$ mut. excl.

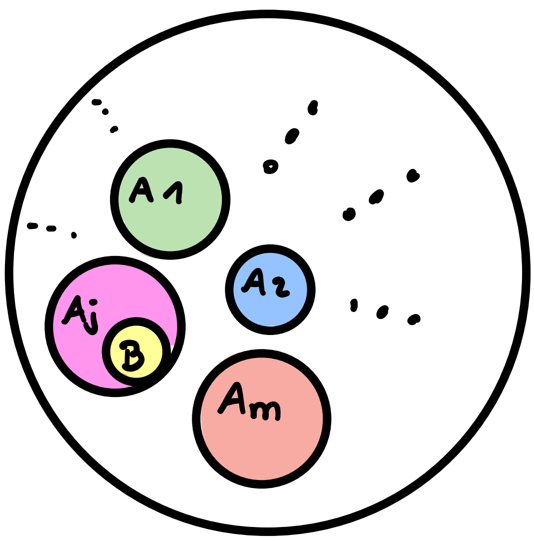
$\Rightarrow P(B) = P(B \cap A_1) + \ldots + P(B \cap A_k)$

$+ P(B \cap A_i) = P(A_i) P(B|A_i), i = 1, \ldots, k$

$\Rightarrow P(B) = P(A_1)P(B|A_1) + \ldots + P(A_k)P(B|A_k)$

$$= \sum_{i=1}^{k} P(A_i) P(B|A_i)$$

(Law of total probability)



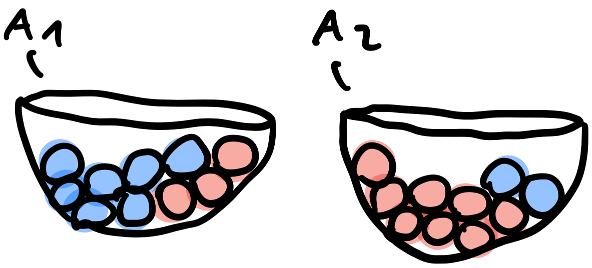
If $B \in A_n$ and A_m or a part of B even, A_n and A_m would NOT be mutually exclusive.

Theorem 1.4.1 Bayes

$A_1, \ldots, A_k$ events, $P(A_i) > 0$ AND they form a partition of Ω, let B be any event, then:

$$P(A_j|B) = \frac{P(A_j)P(B|A_j)}{\sum_{i=1}^{k} P(A_i)P(B|A_i)}$$

Example 1.4.5



Roll die: If or , A_1 is selected to draw from:

$\rightarrow P(A_1) = \frac{2}{6}$

$\rightarrow P(A_2) = \frac{4}{6}$

Then a chip gets taken $(B) \rightarrow$ red chip

$\rightarrow P(B|A_1) = \frac{3}{10}$

$\rightarrow P(B|A_2) = \frac{8}{10}$

Prob. we drew from bowl A_1, given we drew a red chip, is:

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$

$$= \frac{\frac{2}{6} \cdot \frac{3}{10}}{\frac{2}{6} \cdot \frac{3}{10} + \frac{4}{6} \cdot \frac{8}{10}} = \frac{3}{19}$$

Thus, also: $P(A_2|B)=\frac{16}{19}$

Example 1.4.6

3 Plants: A_1, A_2, A_3 produce output:
10%, 50%, 40%

A_1: 1% of products defective
A_2: 3% defective
A_3: 4% defective

B: selection of random product

$$P(A_1)=0{,}1 \qquad P(B|A_1)=0{,}01$$
$$P(A_2)=0{,}5 \quad \text{AND} \quad P(B|A_2)=0{,}03$$
$$P(A_3)=0{,}4 \qquad P(B|A_3)=0{,}04$$

Thus,

$$P(A_1|B)=\frac{P(A_1\cap B)}{P(B)}$$

$$=\frac{0{,}1\cdot 0{,}01}{0{,}1\cdot 0{,}01+0{,}5\cdot 0{,}03+0{,}4\cdot 0{,}04}$$

$$=\frac{1}{32}$$

Example 1.4.7

child abused : A

$P(A) = 0{,}01$

not abused: $N = A^c$

$P(N) = 0{,}99$

Doctor sometimes falsely classifies abuse.

N_D: Abused but D says <u>not</u> abused

A_D: nonabused falsely classified

Suppose:

$P(N_D | A) = 0{,}04$

$P(A_D | N) = 0{,}05$

Thus,

$P(A_D | A) = 0{,}96$

$P(N_D | N) = 0{,}95$

Prob. of random child being classified as abused.

Either $A \cap A_D$ or $N \cap A_D$

$$\Rightarrow P(A_D) = P(A_D|A)P(A) + P(A_D|N)P(N)$$

$$= 0{,}96 \cdot 0{,}01 + 0{,}05 \cdot 0{,}99$$

$$= 0{,}0591 \approx 6\%$$

(seems quite high)

Prob. of child being abused when doctor classified it:

$$P(A|A_D) = \frac{P(A \cap A_D)}{P(A_D)}$$

$$= \frac{0{,}96 \cdot 0{,}01}{0{,}0591} = 0{,}1624$$

$$\approx 16\,\%$$

(seems quite low)

=> implies high-error rates from doctor

Independence 1.4.1

$P(B|A) = P(B)$

→ A, B independent

=> $P(A \cap B) = P(A)P(B|A) = P(A)P(B)$

This implies:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A)P(B)}{P(B)} = P(A)$$

$P(A), P(B) > 0$

<=> $P(A \cap B) = P(A)P(B)$ (independent)

$$B = (A^c \cap B) \cup (A \cap B)$$

$$\begin{aligned} \Rightarrow P(A^c \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) \\ &= P(B)(1 - P(A)) \\ &= P(B)P(A^c) \end{aligned}$$

$\to A^c, B$ also independent

Example 1.4.8

red die, white die

$P(A) = \frac{1}{6}$, $P(B) = \frac{1}{6}$ ⇐ die colour doesn't influence $P(X)$

red = ⚃, white = ⚂

$P((4,3)) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

Suppose 3 events now, A_1, A_2, A_3

mutually independent

$\Leftrightarrow$

$$\left.\begin{aligned} P(A_1 \cap A_3) &= P(A_1)P(A_3) \\ P(A_1 \cap A_2) &= P(A_1)P(A_2) \\ P(A_2 \cap A_3) &= P(A_2)P(A_3) \end{aligned}\right\} \text{pairwise independent}$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

$A_1, \ldots, A_n$ mut. ind.

$\Leftrightarrow$

$\forall$ collections of k events, $2 \leq k \leq n$

$\wedge$ $\forall$ permutations $d_1, \ldots, d_k$:

$P(A_{d_1} \cap \ldots A_{d_k}) = P(A_{d_1}) \cdot \ldots \cdot P(A_{d_k})$

Example 1.4.9

Pairwise ind. does <u>NOT</u> imply mutual ind.

We spin a spinner with numbers 1, 2, 3, 4 twice.

A_1: sum is 5

A_2: 1^{st} number is 1

A_3: 2^{nd} number is 4

$P(A_i) = \frac{1}{4}$, $i = 1, 2, 3$ $\wedge$

For $i \neq j$

$P(A_i \cap A_j) = \frac{1}{16}$

$\rightarrow$ pairwise independence <u>BUT</u>

$P(A_1 \cap A_2 \cap A_3)$ is event, that $(1, 4)$ is spun

which is $\frac{1}{16} \neq \frac{1}{64} = P(A_1)P(A_2)P(A_3)$

$\rightarrow$ <u>NOT</u> mutually independent

Example 1.4.10

coin flipped

$A_i = H$ on i'th toss

$A_i^c = T$ on i'th toss

$P(A_i) = P(A_i^c) = \frac{1}{2}$

Thus:

$$HHTH = P(A_1 \cap A_2 \cap A_3^c \cap A_4)$$
$$= P(A_1)P(A_2)P(A_3^c)P(A_4)$$
$$= \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

Prob. of observing first head on 3rd flip:

$$P(A_1^c \cap A_2^c \cap A_3)$$
$$= P(A_1^c)P(A_2^c)P(A_3)$$
$$= \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Prob. of ≥ 1 H on four flips:

$$P(A_1 \cup A_2 \cup A_3 \cup A_4)$$
$$= 1 - P((A_1 \cup A_2 \cup A_3 \cup A_4)^c)$$

de Morgan $(A \cup B)^c = A^c \cap B^c$

$$= 1 - P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c)$$
$$= 1 - \left(\frac{1}{2}\right)^4 = \frac{15}{16}$$

Example 1.4.11

Computersystem built, s.t. if Component K_1 fails, K_2 is used. K_2 fails $\to K_3$ used...

$P(K_1 \text{ Fails}) = 0{,}01$
$P(K_2 \text{ Fails}) = 0{,}03$
$P(K_3 \text{ Fails}) = 0{,}08$

can assume that failures mutually independent

$\to \quad 0{,}01 \cdot 0{,}03 \cdot 0{,}08$

$= 0{,}000024$

P(System does not fail)

$= 1 - 0{,}000024 = 0{,}999976$

$= 99{,}99\,\%$ //

Simulations 1.4.2

Real life situation hard to model exactly $\to$ let computer run them

n simulations, proportions where A happens : $\hat{p}_n(A) \approx P(A)$

Error-Estimation: $1{,}96 \cdot \sqrt{\dfrac{\hat{p}_n(1-\hat{p}_n)}{n}}$

$$\Rightarrow \hat{p}_n = \pm\ 1{,}96 \sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}$$

(95% CONFIDENCE INTERVALS)

Example 1.4.12

Person A: tosses coin
B: rolls die

repeated until H or 1,2,3,4

Compute P(A wins):

Game completed if:

H or T {1,2,3,4} occurs

$$P(A) = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2} \cdot \frac{4}{6}} = \frac{3}{5}$$

... let R-code run...

A wins in 0,6.