



# How to Construct Swaption Volatility Surfaces



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## Swaption Volatility

An implied volatility is the volatility implied by the market price of an option based on the Black-Scholes option pricing model. An interest rate swaption volatility surface is a four-dimensional plot of the implied volatility of a swaption as a function of strike and expiry and tenor.

The term structures of implied volatilities which provide indications of the market's near- and long-term uncertainty about future short- and long-term swap rates. A crucial property of the implied volatility surface is the absence of arbitrage.



# Swaption Volatility

## Summary

- Swaption Volatility Surface Introduction
- The Summary of Volatility Surface Construction Approaches
- Arbitrage Free Conditions
- The SABR Model
- Swaption Volatility Surface Construction via The SABR Model





## Swaption Volatility

### Swaption Volatility Surface Introduction

- An implied volatility is the volatility implied by the market price of an option based on the Black-Scholes option pricing model.
- An swaption volatility surface is a four-dimensional plot of the implied volatility of a swaption as a function of strike and expiry and tenor.
- The term structures of implied volatilities provide indications of the market's near- and long-term uncertainty about future short- and long-term swap rates.
- Vol skew or smile pattern is directly related to the conditional non-normality of the underlying return risk-neutral distribution.
- In particular, a smile reflects fat tails in the return distribution whereas a skew indicates return distribution asymmetry.
- A crucial property of the implied volatility surface is the absence of arbitrage.



## Swaption Volatility

# The Summary of Volatility Surface Construction Approaches

- To construct a reliable volatility surface, it is necessarily to apply robust interpolation methods to a set of discrete volatility data. Arbitrage free conditions may be implicitly or explicitly embedded in the procedure. Typical approaches are
  - Local Volatility Model: a generalisation of the Black-Scholes model.
  - Stochastic Volatility Models: such as SABR, Heston, Levy
  - Parametric or Semi-Parametric Models: such as SVI, Omega
  - Market Volatility Model: directly modeling the implied volatility dynamics
  - Interpolation/Extrapolation Model: interpolating or extrapolating volatility data using specific function forms



# Swaption Volatility

## Arbitrage Free Conditions

- Any volatility models must meet arbitrage free conditions.
- Typical arbitrage free conditions
  - Static arbitrage free condition: Static arbitrage free condition makes it impossible to invest nothing today and receive positive return tomorrow.
  - Calendar arbitrage free condition: The cost of a calendar spread should be positive.
  - Vertical (spread) arbitrage free condition: The cost of a vertical spread should be positive.
  - Horizontal (butterfly) arbitrage free condition: The cost of a butterfly spread should be positive.



## Swaption Volatility

### Arbitrage Free Conditions (Cont)

- Vertical arbitrage free and horizontal arbitrage free conditions for swaption volatility surfaces depend on different strikes.
- There is no calendar arbitrage in swaption volatility surfaces as swaptions with different expiries and tenors have different underlying swaps and are associated with different indices. In other words, they can be treated independently.
- The absence of triangular arbitrage condition is sufficient to exclude static arbitrages in swaption surfaces.





## Swaption Volatility

### Arbitrage Free Conditions (Cont)

- Let  $Sw(t, T_s, T_e, K)$  be the present value of a swaption at time  $t$ , where  $T_s$  is the start date of the underlying swap;  $T_e$  is the end date of the underlying swap;  $K$  is the fixed rate of the underlying swap.
- The triangular arbitrage free conditions are

$$Sw(t_1, T_s, T_e, K) \leq Sw(t_2, T_s, T_e, K) \quad \text{where } t_1 \leq t_2$$

$$Sw(T_1, T_1, T_3, K) \leq Sw(T_1, T_1, T_2, K) + Sw(T_2, T_2, T_3, K)$$

$$\text{where } T_1 \leq T_2 \leq T_3$$

- At FinPricing, we use the SABR model to construct swaption volatility surfaces following the best market practice.





# Swaption Volatility

## The SABR Model

- SABR stands for “stochastic alpha, beta, rho” referring to the parameters of the model.
- The SABR model is a stochastic volatility model for the evolution of the forward price of an asset, which attempts to capture the volatility smile/skew in derivative markets.
- There is a closed-form approximation of the implied volatility of the SABR model.
- In the swaption volatility case, the underlying asset is the forward swap rate.



# Swaption Volatility

## The SABR Model (Cont)

- The dynamics of the SABR model

$$d\hat{F} = \hat{\alpha}\hat{F}^{\beta}dW_1$$

$$d\hat{\alpha} = v\hat{\alpha}dW_2$$

$$dW_1dW_2 = \rho dt$$

$$\hat{\alpha}(0) = \alpha$$

where

$\hat{F}$  the forward swap rate

$\hat{\alpha}$  the forward volatility

$W_1, W_2$  the standard Brownian motions

$\rho$  the instantaneous correlation between  $W_1$  and  $W_2$



# Swaption Volatility

## The SABR Model (Cont)

- The four model parameters ( $a$  ,  $b$  ,  $r$  ,  $n$  ) above have the following intuitive meaning:
  - $\alpha$  the volatility that is tied closely to the at-the-money volatility value.
  - $\beta$  the exponent that is related to the backbone that the ATM volatility traces when the ATM forward rate varies.
  - $\rho$  the correlation that describes the “skew” or average slope of the volatility curve across strike.
  - $v$  the volatility of volatility that describes the “smile” or curvature/convexity of the volatility curve across the strike for a given term and tenor.





## Swaption Volatility

# Constructing Swaption Volatility Surface via The SABR Model

- For each term (expiry) and tenor of the swaption, conduct the following calibration procedure.
- The  $\beta$  parameter is estimated first and typically chosen a priori according to how the market prices are to be observed.
- Alternatively  $\beta$  can be estimated by a linear regression on a time series of ATM volatilities and of forward rates.
- After  $\beta$  is set, we can obtain  $\alpha$  by using  $\sigma_{ATM}$  to solve the following equation

$$\sigma_{ATM} = \frac{\alpha}{f^{1-\beta}} \left\{ 1 + \left[ \frac{\alpha^2(1-\beta)^2}{24f^{2(1-\beta)}} + \frac{\rho\beta v\alpha}{4f^{1-\beta}} + \frac{(2-3\rho^2)v^2}{24} \right] T \right\}$$

The Viète method is used to solve this equation



# Swaption Volatility

## Constructing Swaption Volatility Surface via The SABR Model

- Given the  $\alpha$  and  $\beta$  solved above, we can find the optimized value of  $(\rho, v)$  by minimizing the distance between the SABR model output volatilities and market volatilities across all strikes for each term and tenor.

$$\min \sum_{i=1}^n [\sigma_i^{SABR}(\alpha, \beta, \rho, v) - \sigma_i^{Market}]^2$$

The Levenberg-Marquardt least-squares optimization routine is used for optimization.

- After  $\alpha, \beta, \rho, v$  calibrated, one can generate SABR volatility (swaption volatility) for any moneyness.
- Repeat the above process for each term and tenor.



# Thank You

You can find more details at

<http://www.finpricing.com/lib/lrSwnVol.html>